SMSTC 2021
Variational Methods of PDEs

1 User's Guide to Sobolev Spaces

1.1 Basic definitions

Let $\Omega \subset \mathbb{R}^N$ be an open set, N a positive integer, and $p \in \mathbb{R}$ be such that $1 \leq p \leq \infty$.

If we recall from Multi-variable Calculus, we know that if u is of class $\mathcal{C}^1(\Omega)$, then integration by parts yields

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = -\int_\Omega \frac{\partial u}{\partial x_i} \varphi \, \mathrm{d}x,$$

whenever $\varphi \equiv 0$ on $\partial\Omega$. This gives us an alternative definition for the derivative of u, which we can extend for measurable functions: Let $u \in L^p(\Omega)$ and $i \in \{1, \ldots, N\}$. If there exists $g_i \in L^p(\Omega)$ such that

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = -\int_\Omega g_i \varphi \, \mathrm{d}x, \tag{1}$$

for all $\varphi \in \mathcal{C}_c^1(\Omega)^\dagger$, then we say that g_i is the *weak partial derivative of u with respect to x_i*, and for convenience we denote $\frac{\partial u}{\partial x_i} := g_i$. Likewise, if there exist $g_1, \ldots g_N \in L^p(\Omega)$ such that (1) is satisfied for all $i \in \{1, \ldots, N\}$, then we say that the vector $\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right)$ is the *weak derivative of u*.

The set of $L^p(\Omega)$ functions that have weak derivatives form the *Sobolev space* $W^{1,p}(\Omega)$ defined by

$$W^{1,p}(\Omega) := \left\{ u \in L^p(\Omega) : \begin{array}{l} \exists \{g_i\}_{i \in \{1,\ldots,N\}} \text{ such that} \\ \displaystyle\int_\Omega u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = -\int_\Omega g_i \varphi \, \mathrm{d}x \quad \forall \varphi \in \mathcal{C}_c^1(\Omega) \quad \forall i \in \{1, \ldots, N\} \end{array} \right\}.$$

This space is equipped with the norm*

$$\|u\|_{W^{1,p}} := \|u\|_p + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_p.$$

†We denote $\mathcal{C}_c^\infty(\Omega)$ as the set of compactly supported functions of class $\mathcal{C}^1$ defined over Ω:

$$\mathcal{C}_c^1(\Omega) = \left\{ \varphi \in \mathcal{C}^1(\Omega) : \Omega \supset \operatorname{supp} \varphi = \{x : \varphi(x) \neq 0\} \text{ is compact} \right\}.$$

*When there is no confusion, we will write $W^{1,p}$ instead of $W^{1,p}(\Omega)$.

We further note $H^1(\Omega) := W^{1,2}(\Omega)$, which has the associated scalar product

$$(u,v)_{H^1} = (u,v)_{L^2} + \sum_{i=1}^{N} \left(\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_i} \right)_{L^2}.$$

Notice that by the Cauchy–Bunyakovsky–Schwarz inequality the associated norm in H^1 is equivalent to the $W^{1,2}$ norm.

Proposition 1.0.1. *The weak derivative is unique (almost everywhere).*

Proof. Let $u \in W^{1,p}(\Omega)$, and suppose that there exists g_i and h_i such that both satisfy equation (1), then

$$0 = \int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x - \int_\Omega u \frac{\partial \varphi}{\partial x_i} \,\mathrm{d}x = \int_\Omega (g_i - h_i)\varphi \,\mathrm{d}x$$

for all $\varphi \in \mathcal{C}_c^1(\Omega)$. By the Fundamental lemma of the calculus of variations (which we will prove in the next subsection), we have that $g_i - h_i = 0$ almost everywhere, that is $g_i = h_i$ (a.e). $\square$

As weak derivatives are unique, there is no ambiguity in writing $g_i = \frac{\partial u}{\partial x_i}$; i.e., our notation is consistent. Finally, we will see an important example of this theory, namely a function that is not globally $\mathcal{C}^1(\Omega)$ but has a weak derivative.

Example 1.1. *Let $\Omega = (-1,1) \subset \mathbb{R}$ and consider the function $u = |\cdot|$. Notice that u is not differentiable at 0; however we will determine its derivative. We have that integration by parts with any $\varphi \in \mathcal{C}_c^1(\Omega)$ yields*

$$\int_\Omega u\varphi' \,\mathrm{d}x = \int_0^1 x\varphi' \,\mathrm{d}x - \int_{-1}^0 x\varphi' \,\mathrm{d}x = -\int_0^1 \varphi \,\mathrm{d}x + \int_{-1}^0 \varphi \,\mathrm{d}x = -\int_\Omega g\varphi \,\mathrm{d}x,$$

where g is defined as

$$g(x) = \begin{cases} +1 & \text{if } x \in (0,1), \\ -1 & \text{if } x \in (-1,0). \end{cases}$$

Notice that by definition $u \in L^p(\Omega)$ for every $1 \le p \le \infty$, and it is clear that $g \in L^p(\Omega)$ as well.

On the other hand, notice that g does not *belong to $W^{1,p}(\Omega)$ for any $1 \le p \le \infty$. To see this, note that from Fundamental theorem of calculus and the fact that $\varphi \in \mathcal{C}_c^1(\Omega)$:*

$$\int_\Omega g\varphi' \,\mathrm{d}x = \int_0^1 \varphi' \,\mathrm{d}x - \int_{-1}^0 \varphi' \,\mathrm{d}x = \varphi(1) - \varphi(0) - \varphi(0) + \varphi(-1) = -2\varphi(0).$$

As a result, we would have that $g = 2\delta_0$, with δ_0 being Dirac's delta, which is a tempered distribution. On the other hand, from calculus we would need $\frac{\mathrm{d}g}{\mathrm{d}x} = 0$ for $x \in (-1,0) \cup (0,1)$, which would imply that $\varphi(0) = 0$ for all $\varphi \in \mathcal{C}_c^1(\Omega)$, which is a contradiction.

Let $\Omega \subset \mathbb{R}^N$ be an open set, N a positive integer, $1 \leq p < \infty$, and $u \in L^p(\Omega)$. We would like see whether we can approximate u by a sequence of smooth functions.

From the previous subsection, we saw that $\mathcal{C}_c^1(\Omega)$ is the space of $\mathcal{C}^1$ functions with compact support in Ω. Recall that the support of a function φ is a set $K \subset \Omega$ such that is the smallest compact set where $\varphi(x) = 0$ for $x \in \Omega \setminus K$, and we note it $\operatorname{supp} \varphi$. On a similar fashion, we denote $\mathcal{C}_c^k(\Omega)$ as the space of $\mathcal{C}^k$ functions with compact support in Ω, and likewise $\mathcal{C}_c^\infty(\Omega)$ as the space of $\mathcal{C}^\infty$ functions with compact support in Ω.

Note that a nonzero function $\varphi \in \mathcal{C}_c^\infty(\Omega)$ cannot be analytic, as its Taylor coefficients are zero for $x \notin \operatorname{supp} \varphi$. An example of a nonzero function of this kind is

$$e(x) := \begin{cases} \lambda \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x| < 1, \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where λ is conveniently chosen such that $\int e = 1$. It can be shown that the n-th derivative of e is given by $\frac{\mathrm{d}^n e}{\mathrm{d}x^n} = \frac{P_n(x)}{(x^2-1)^{2n}} e(x)$ for $|x| < 1$, with P_n a polynomial [2]; furthermore, $e \in \mathcal{C}_c^\infty(\mathbb{R}^N)$. It turns out that the function e defined above is part of a set of functions named *mollifiers*.

Definition 1.2. *A function $\rho \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ is called a* mollifier *if it is non-negative and $\int\limits_{\mathbb{R}^N} \rho(x)\,\mathrm{d}x = 1$.*

It is easy to generate a *sequence of mollifiers* starting with a single function. For instance, if we consider the starndard mollifier e and let $\varepsilon > 0$, then the functions

$$e_\varepsilon(x) := \varepsilon^{-N} e(x/\varepsilon)$$

determine a sequence of mollifiers as clearly they are non-negative and

$$\int\limits_{\mathbb{R}^N} e_\varepsilon(x)\,\mathrm{d}x = \int\limits_{\mathbb{R}^N} e(x)\,\mathrm{d}x = 1.$$

Moreover, $\operatorname{supp} e_\varepsilon \subset B(0, \varepsilon)$. In Figure 1 we see the shape of the sequence e_ε for different values of ε.

1.2.1 Convolution

As we can see from Figure 1, e_ε approximates Dirac's delta function, for small values of ε. As a result, we can approximate $u \in L^p(\Omega)$ by its *mollification* given by

$$u_\varepsilon(x) := e_\varepsilon \star u(x) = \int\limits_{\mathbb{R}^N} e_\varepsilon(x-y) u(y)\,\mathrm{d}y.$$

The mollification operation has the following properties:

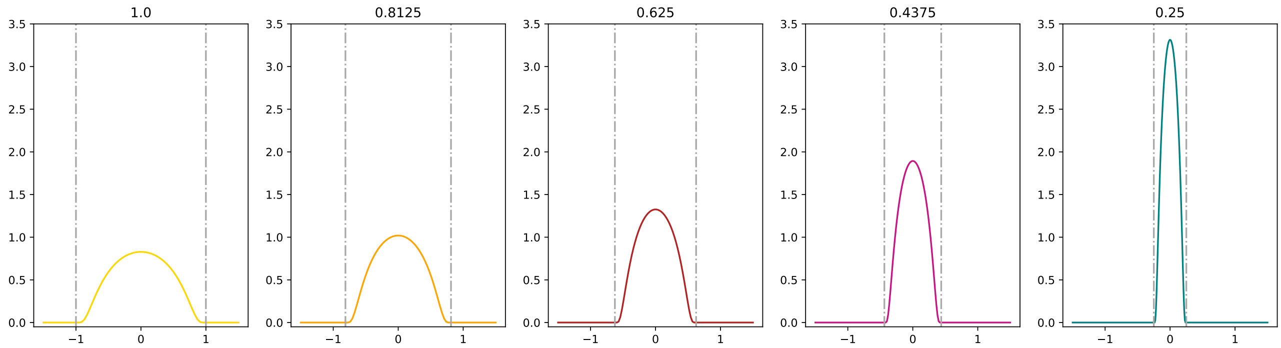

Figure 1: Mollifier sequence generated from the standard mollifier e.
Each panel plots e_ε for a decreasing value of ε. The vertical lines represent the cuts $x = -\varepsilon$ and $x = \varepsilon$.

Lemma 1.3. *Let $1 \leq p < \infty$ and $u \in L^p(\Omega)$. If we extende u to be zero outside Ω, then*

- $u_\varepsilon \in C_c^\infty(\mathbb{R}^N)$,
- $\lim_{\varepsilon \to 0} \|u - u_\varepsilon\|_{L^p} = 0.$

Moreover, if $u \in W^{1,p}(\mathbb{R}^N)$, then $\lim_{\varepsilon \to 0} \|u - u_\varepsilon\|_{W^{1,p}} = 0$.

As we can expect, when $\varepsilon \to 0$, we have that e_ε becomes δ_0 and we expect $\delta_0 \star u_\varepsilon(x) = u(x)$, for we have approximated L^p with smooth functions. In other words, the set C_c^∞ is dense in $L^p(\Omega)$ for any $1 \leq p < \infty$.

Proposition 1.3.1 (Differentiation of product)**.** *Let $u, v \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ with $1 \leq p \leq \infty$. Then $uv \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$ and*

$$\frac{\partial}{\partial x_i}(uv) = \frac{\partial u}{\partial x_i} v + u \frac{\partial v}{\partial x_i},$$

for all $i \in \{1, \ldots, N\}$.

We omit the proof of this proposition and refer the reader to Proposition 9.4 in [3].

Proposition 1.3.2 (Differentiation of composition)**.** *Let $G \in C^1(\mathbb{R})$ be such that $G(0) = 0$ and $|G'(x)| \leq M$ for all $x \in \mathbb{R}$ and for some constant M. Let $u \in W^{1,p}(\Omega)$ with $1 \leq p \leq \infty$. Then $G \circ u W^{1,p}(\Omega)$ and*

$$\frac{\partial}{\partial x_i}(G \circ u) = (G' \circ u) \frac{\partial u}{\partial x_i}$$

for all $i \in \{1, \ldots, N\}$.

We omit the proof of this proposition and refer the reader to Proposition 9.5 in [3].

1.2.2 Higher order Sobolev Spaces

All of the above discussion can be extended for higher order derivatives. For this end, it is convenient to have a compact notation for expressing mixed partial derivatives. A *multi-index* α is an N-tuple $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$, and we write $|\alpha| = \alpha_1 + \ldots + \alpha_N$.

Let $\varphi \in \mathcal{C}_c^\infty(\Omega)$. We define

$$D^\alpha \varphi = \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\right) \cdots \left(\frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}\right) u = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_N^{\alpha_N}}.$$

For example, if $\beta = (2, 0, 1)$ and $N = 3$, we get that $|\beta| = 3$ and $D^\beta \varphi = \frac{\partial^3 \varphi}{\partial x_1^2 \partial x_3}$.

Using this notation, we can say that a function $v \in L^p(\Omega)$ is the α^{th} *weak derivative* of a function $u \in L^p(\Omega)$ if it satisfies

$$\int_\Omega u D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega v \varphi \, \mathrm{d}x,$$

for all $\varphi \in \mathcal{C}_c^{|\alpha|}(\Omega)$. As a result, we can now define the general Sobolev Spaces:

Definition 1.4. *Let m be a non-negative integer and let $1 \leq p \leq \infty$. The Sobolev Space $W^{m,p}(\Omega)$ is the linear space of functions $u \in L^p(\Omega)$ such that for each α, $0 \leq |\alpha| \leq m$, the weak derivative D^α exists and belongs to $L^p(\Omega)$. Namely,*

$$W^{m,p}(\Omega) := \left\{ u \in L^p(\Omega) : \begin{array}{c} \forall \alpha \text{ with } |\alpha| \leq m, \exists g_\alpha \in L^p(\Omega) \text{ such that} \\ \displaystyle\int_\Omega u D^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_\Omega g_\alpha \varphi \, \mathrm{d}x \quad \forall \varphi \in \mathcal{C}_c^\infty(\Omega) \end{array} \right\}.$$

Just as with $m = 1$, this space is also equipped with the norm

$$\|u\|_{W^{m,p}} := \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p,$$

and it is a Banach space, see Theorem 8 [2]. Moreover, the space $W^{m,2}(\Omega)$ is often noted as $H^m(\Omega)$ with the scalar product

$$(u, v)_{H^m} := \sum_{0 \leq |\alpha| \leq m} (D^\alpha u, D^\alpha v)_{L^2}.$$

Notice that $W^{0,p}(\Omega) = L^p(\Omega)$. Finally, the uniqueness of the derivative also follows from the Fundamental lemma of the calculus of variations, which we present now:

Lemma 1.5 (Fundamental lemma of the calculus of variations)**.** *Let $u \in L^1_{\mathrm{loc}}(\Omega)$ satisfy*

$$\int_\Omega u \varphi \, \mathrm{d}x = 0$$

for all $\varphi \in \mathcal{C}_c^k(\Omega)$, then $u = 0$ a.e. on Ω.

Proof. Without loss of generality, we can consider the limit case $k = \infty$. To see this, let $g \in L^\infty(\mathbb{R}^N)$ be a function such that $\operatorname{supp} g$ is compact and contained in Ω. We define $g_\varepsilon = \rho_\varepsilon \star g$ with ρ_ε a mollifier. Let K be compact such that $K \subset \Omega$. If $\varepsilon < \operatorname{dist}(K, \partial\Omega)$, then for each x the function $\rho_\varepsilon(x - y)$ belongs

to $C_c^\infty(\Omega)$. As a result
$$\int_\Omega u g_\varepsilon \, \mathrm{d}x = 0$$
for all $x \in E$. However, we have that $g_\varepsilon \to g$ in $L^1(\Omega)$, from where, via a subsequence, $g_\varepsilon \to g$ almost everywhere on $\mathbb{R}^N$. Moreover, as $\|g_\varepsilon\|_\infty \leq \|g\|_\infty$, we can use Lebesgue's dominated convergence theorem to conclude $\int_\Omega ug \, \mathrm{d}x = 0$. Now, we can pick g as the function

$$g(x) = \begin{cases} \text{sign}(u) & \text{on } K, \\ 0 & \text{on } \mathbb{R}^N \setminus K. \end{cases}$$

As a result, we have that $\int_K |u| \, \mathrm{d}x = 0$ and thus $u = 0$ a.e. on K. Since this hold for any compact set $K \subset \Omega$, we conclude that $u = 0$ a.e. on Ω. □

1.3 Extension operator

It is often convenient to establish properties of functions in $W^{1,p}(\Omega)$ starting with the case $\Omega = \mathbb{R}^N$. As a result, it is useful to be able to extend a function $u \in W^{1,p}(\Omega)$ to a function in $W^{1,p}(\mathbb{R}^N)$.

We start with some notation. Given $x \in \mathbb{R}^N$, we will write

$$x = (x', x_N) \text{ with } x' \in \mathbb{R}^{N-1}, \qquad x' = (x_1, \dots, x_{N-1}).$$

Moreover, we will notate $\mathbb{R}^N_+$ as the set

$$\mathbb{R}^N_+ := \{x = (x', x_N) : \, x_N > 0\}.$$

Lemma 1.6. *Given $u \in W^{1,p}(\mathbb{R}^N_+)$ with $1 \leq p \leq \infty$, one defines the function u^* on $\mathbb{R}^N$ to be the* extension by reflection*, that is,*

$$u^*(x', x_N) = \begin{cases} u(x', x_N) & \text{if } x_N > 0, \\ u(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Then $u^* \in W^{1,p}(\mathbb{R}^N)$ and

$$\|u^*\|_{L^p(\mathbb{R}^N)} \leq 2\|u\|_{L^p(\mathbb{R}^N_+)}, \qquad \|u^*\|_{W^{1,p}(\mathbb{R}^N)} \leq 2\|u\|_{W^{1,p}(\mathbb{R}^N_+)}.$$

In fact, this comes from

$$\frac{\partial u^*}{\partial x_i} = \left(\frac{\partial u}{\partial x_i}\right)^*$$

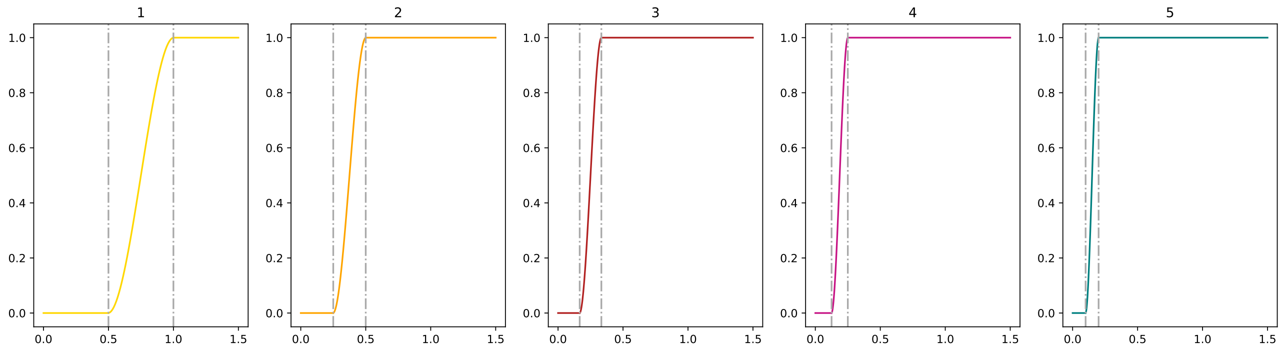

Figure 2: A representation of η and the sequence η_k for the first values of $k \in \mathbb{N}^*$.
The vertical lines correspond to $x = 1/k$ and $x = 1/2k$.

for $i \in \{1, \dots, N-1\}$ *and*

$$\frac{\partial u^*}{\partial x_N} = \left(\frac{\partial u}{\partial x_i}\right)^{\square} = \begin{cases} \frac{\partial u}{\partial x_N}(x', x_N) & \text{if } x_N > 0, \\ -\frac{\partial u}{\partial x_N}(x', -x_N) & \text{if } x_N < 0. \end{cases}$$

Proof. Let $\eta \in \mathcal{C}^\infty$ be such that

$$\eta(t) = \begin{cases} 0 & \text{if } t < 1/2 \\ 1 & \text{if } t > 1. \end{cases}$$

We shall use a sequence $(\eta_k)_{k \in \mathbb{N}^*}$ of functions in $\mathcal{C}^\infty(\mathbb{R})$ defined by

$$\eta_k(t) = \eta(kt) = \begin{cases} 0 & \text{if } t < 1/2k \\ 1 & \text{if } t > 1/k. \end{cases}$$

Let $\varphi \in \mathcal{C}_c^1(\mathbb{R}^N)$ and $1 \leq i \leq N-1$. By definition we have that

$$\int_{\mathbb{R}^N} u^* \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = \int_{\mathbb{R}^N_+} u \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x + \int_{\mathbb{R}^N_-} u(x', -x_N) \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = \int_{\mathbb{R}^N_+} u \frac{\partial \psi}{\partial x_i} \, \mathrm{d}x,$$

where $\phi(x', x_N) := \varphi(x', x_N) + \varphi(x', -x_N)$. The function ψ does not in general belong to $\mathcal{C}_c^1(\mathbb{R}^N)_+$, and thus it cannot be used as a test function in the definition of $W^{1,p}$. On the other hand, we note that $\eta_k(x_N)\psi(x', x_N) \in C_c^1(\mathbb{R}^N_+)$ and thus

$$\int_{\mathbb{R}^N_+} u \frac{\partial}{\partial x_i}(\eta_k \psi) \, \mathrm{d}x = -\int_{\mathbb{R}^N_+} \frac{\partial u}{\partial x_i} \eta_k \psi \, \mathrm{d}x.$$

Since $\frac{\partial}{\partial x_i}(\eta_k \psi) = \eta_k \frac{\partial \psi}{\partial x_i}$, we have

$$\int_{\mathbb{R}^N_+} u \eta_k \frac{\partial}{\partial x_i} \psi \, \mathrm{d}x = -\int_{\mathbb{R}^N_+} \frac{\partial u}{\partial x_i} \eta_k \psi \, \mathrm{d}x.$$

Passing to the limit as $k \to \infty$ (by Lebesgue's dominated convergence theorem), we obtain

$$\int_{\mathbb{R}^N_+} u \frac{\partial}{\partial x_i} \psi \, \mathrm{d}x = - \int_{\mathbb{R}^N_+} \frac{\partial u}{\partial x_i} \psi \, \mathrm{d}x.$$

As a result, we have gotten

$$\int_{\mathbb{R}^N} u^* \frac{\partial \varphi}{\partial x_i} \, \mathrm{d}x = - \int_{\mathbb{R}^N_+} \frac{\partial u}{\partial x_i} \psi \, \mathrm{d}x = - \int_{\mathbb{R}^N} \left(\frac{\partial u}{\partial x_i} \right)^* \psi \, \mathrm{d}x.$$

Now, for the case $i = N$, we have

$$\int_{\mathbb{R}^N} u^* \frac{\partial \varphi}{\partial x_N} \, \mathrm{d}x = \int_{\mathbb{R}^N_+} u \frac{\partial \chi}{\partial x_N} \, \mathrm{d}x,$$

where $\chi(x', x_N) := \varphi(x', x_N) - \varphi(x', -x_N)$. Note that $\chi(x', 0) = 0$, and thus there exist a constant M such that $|\chi(x', x_N)| \le M|x_N|$ on $\mathbb{R}^N$. Since $\eta_k \chi \in C^1_c(\mathbb{R}^N_+)$, we have

$$\int_{\mathbb{R}^N_+} u \frac{\partial}{\partial x_N} (\eta_k \chi) \, \mathrm{d}x = - \int_{\mathbb{R}^N_+} \frac{\partial u}{\partial x_N} \eta_k \chi \, \mathrm{d}x.$$

But $\frac{\partial}{\partial x_N}(\eta_k \chi) = \eta_k \frac{\partial \chi}{\partial x_N} + k\eta'(kx_N)\chi$. Notwithstanding, we further have

$$\left| \int_{\mathbb{R}^N_+} uk\eta'(kx_N)\chi \, \mathrm{d}x \right| \le kMC \int_{\mathbb{R}^N_+ \cap \{|x_N| < 1/k\}} |u| x_N \, \mathrm{d}x \le MC \int_{\mathbb{R}^N_+ \cap \{|x_N| < 1/k\}} |u| \, \mathrm{d}x,$$

thus

$$\int_{\mathbb{R}^N_+} uk\eta'(kx_N)\chi \, \mathrm{d}x \to 0$$

as $k \to \infty$. As a result, we have that

$$\int_{\mathbb{R}^N_+} u \frac{\partial}{\partial x_N} (\chi) \, \mathrm{d}x = - \int_{\mathbb{R}^N_+} \frac{\partial u}{\partial x_N} \psi \, \mathrm{d}x.$$

Finally, we have

$$\int_{\mathbb{R}^N_+} \frac{\partial u}{\partial x_N} \chi \, \mathrm{d}x = \int_{\mathbb{R}^N} \left(\frac{\partial u}{\partial x_N} \right)^{\square} \varphi \, \mathrm{d}x.$$ □

It is worth mentioning that this construct can be carried out for $W^{1,p}(\Omega)$, when Ω is a $\mathcal{C}^1$ domain, see Theorem 9.7 in [3].

1.4 Sobolev Inequalities

The Sobolev Inequalities, also known as Sobolev embeding theorems, are a group of results that allow to recover some regularity from $W^{1,p}$ spaces.

Theorem 1.7. *Let $\Omega \subseteq \mathbb{R}^N$ be $\mathbb{R}^N$, open of class $\mathcal{C}^1$ with $\partial\Omega$ bounded, or $\mathbb{R}^N_+$. For $1 \leq p \leq \infty$, we have*

$$W^{1,p}(\Omega) \subset L^{p^*}(\Omega) \qquad \text{where} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} \qquad \text{if } p < N, \tag{2}$$

$$W^{1,p}(\Omega) \subset L^{q}(\Omega) \qquad \text{for all} \quad q \in [p, +\infty) \qquad \text{if } p = N, \tag{3}$$

$$W^{1,p}(\Omega) \subset L^{\infty}(\Omega) \qquad \text{if } p > N; \tag{4}$$

and all the injections are continuous. Moreover, if $p > N$ we have, for all $u \in W^{1,p}(\Omega)$,

$$|u(x) - u(y)| \leq C\|u\|_{W^{1,p}}|x-y|^\alpha \qquad \text{a.e. } x, y \in \Omega,$$

with $\alpha = 1 - N/p$ and C depends only on Ω, p, and N. In particular, $W^{1,p}(\Omega) \subset C(\bar{\Omega})$.

Regarding the last case, if m is a positive integer and $m - N/p$ is not an integer then $W^{m,p}(\Omega) \subset C^k(\bar{\Omega})$, where $k = m - n/p$.

Consider the case $p = 2$. We then have that

- $N = 1$: $H^1(\Omega) \subset \mathcal{C}^{0,1/2}(\bar{\Omega})$.
- $N = 2$: $H^1(\Omega) \subset L^q(\Omega)$ for all $q \in [2, +\infty)$.
- $N = 3$: $H^1(\Omega) \subset L^6(\Omega)$.

The proof of Theorem 1.7 can be found in [3], pages 278–285. We will not include it here. However, we will include a proof for $N = 1$ and $p \in \{1, 2\}$ which will use the following result:

Theorem 1.8 (Theorem 8.2 in [3])**.** *Let $u \in W^{1,p}(I)$ with $1 \leq p \leq \infty$, and $I \subseteq \mathbb{R}$. Then there exists a function $\tilde{u} \in \mathcal{C}(\bar{I})$ such that $u = \tilde{u}$ almost everywhere on I and*

$$\tilde{u}(x) - \tilde{u}(y) = \int_y^x u'(t)\,\mathrm{d}t \qquad \forall x, y \in \bar{I}.$$

Notice that this theorem asserts that every function $u \in W^{1,p}$ admits one, and only one, continuous representative on $\bar{I}$; i.e., there exists a continuous function on $\bar{I}$ that belongs to the equivalence class of u. To see this in $p = 2$, notice that

$$\Big|u(x) - u(y)\Big| \leq \left|\int_y^x u'(t)\cdot 1\,\mathrm{d}t\right| \leq \left(\int_y^x u'(t)\,\mathrm{d}t\right)^{1/2} |x-y|^{1/2} \leq \|u\|_{H^1}|x-y|^{1/2},$$

thus $u \in \mathcal{C}^{0,1/2}(\bar{\Omega})$.

There is a result that only occurs in $N = 1$: if $p = 1$ and $I = (0,1)$, we have

$$\int_0^1 u(x) - u(y)\,\mathrm{d}y = \int_0^1 \int_y^x u'(t)\,\mathrm{d}t\,\mathrm{d}y \le \int_0^1 |u'|(z)\,\mathrm{d}z.$$

Thus

$$|u(x)| \le \int_0^1 |u'|(z)\,\mathrm{d}z + \int_0^1 |u|(y)\,\mathrm{d}y,$$

and we have that $u \in L^\infty(0,1)$. This can be further extended to more general domains as a characterisation of $W^{1,p}(\mathbb{R})$, see Proposition 8.3 in [3]; notwithstanding, it is only applicable for $N = 1$. For $N > 1$, we can consider the function

$$u(x) = \left(\log 1/|x|\right)^{\alpha},$$

defined over $\Omega = B(0, 1/2)$ with $\alpha \in (1, 1 - 1/N)$. This function belongs to $W^{1,N}(\Omega)$ but it is not bounded due to its singularity at $x = 0$.

Theorem 1.9. *Suppose that Ω is bounded, open, and of class $\mathcal{C}^1$. Then we have the following compact injections:*

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega) \qquad \textit{for all } q \in [1, p^*), \quad \textit{where} \quad \frac{1}{p^*} = \frac{1}{p} - \frac{1}{N} \qquad \textit{if } p < N, \tag{5}$$

$$W^{1,p}(\Omega) \subset\subset L^q(\Omega) \qquad \textit{for all} \quad q \in [p, +\infty) \qquad \textit{if } p = N, \tag{6}$$

$$W^{1,p}(\Omega) \subset\subset \mathcal{C}(\bar{\Omega}) \qquad \textit{if } p > N; \tag{7}$$

where with compact injection we mean that every bounded sequence in $W^{1,p}$ has a convergent subsequence in L^q.

The proof of this theorem can be found in [3], page 285. Here it is noteworthy to see that the property for the case $p > N$ follows from Theorem 1.7 and Ascoli–Arzelà's theorem.

1.5 Notion of traces

Now we turn into the question of how can we define the boundary values of a function $u \in W^{1,p}(\Omega)$. This is not a trivial task even if $\partial\Omega$ were smooth. On the one hand, we have that u is in principle defined only in Ω. On the other hand, even if u could be extended, the values of $\tilde{u}$ on $\partial\Omega$ would have no meaning since $\partial\Omega$ has measure zero, and $\tilde{u}$ may be altered at will on sets of measure zero.

Let $1 \le p < \infty$. We begin with a fundamental lemma:

Lemma 1.10. *Let $\Omega = \mathbb{R}^N_+$. There exists a constant C such that*

$$\int_{\mathbb{R}^{N-1}} |u(x',0)|^p\,\mathrm{d}x' \le C\|u\|^p_{W^{1,p}(\Omega)} \qquad \forall u \in \mathcal{C}^1_c(\mathbb{R}^N).$$

Proof. We have that

$$|u|^p(x',0) = -\int_0^\infty \frac{\partial}{\partial x_N}|u|^p \,\mathrm{d}x_N = -p\int_0^\infty |u|^{p-1}\frac{\partial u}{\partial x_N}\,\mathrm{d}x_N.$$

Young's inequality states the following: if $a, b \geq 0$ and p, q are such that $1 = 1/p + 1/q$, then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

A simple proof of this inequality by convexity is found in [4], Appendix B2. Using this inequality above we get

$$|u|^p(x',0) \leq C\int_0^\infty |u|^{(p-1)q} + \left|\frac{\partial u}{\partial x_N}\right|^p \mathrm{d}x_N,$$

and by definition $1/q = (p-1)/p$, thus

$$|u|^p(x',0) \leq C\int_0^\infty |u|^p + \left|\frac{\partial u}{\partial x_N}\right|^p \mathrm{d}x_N,$$

and the conclusion follows by integration in $x' \in \mathbb{R}^{N-1}$. $\square$

As $\mathcal{C}_c^1(\mathbb{R}^N)$ is dense in $W^{1,p}(\mathbb{R}_+^N)$, then $u \to u\Big|_{\mathbb{R}^{N-1}\times\{x_N=0\}}$ extends in a linear and continuous way from $W^{1,p}(\mathbb{R}_+^N)$ to $L^p\Big(\mathbb{R}^{N-1}\times\{x_N=0\}\Big)$. This operator is, by definition, the *trace* of u on $\partial\Omega$.

For a domain of class $\mathcal{C}^1$, there is also a *trace* operator from $W^{1,p}(\Omega)$ to $L^p(\partial\Omega)$. This operator has the following properties:

- It is bounded; i.e., there exist a constant $C \geq 0$ such that

$$\|u\|_{L^p(\partial\Omega)} \leq C\|u\|_{L^{1,p}(\Omega)}.$$

- Its kernel is the set $W_0^{1,p}(\Omega)$, which is the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$ with respect to the $W^{1,p}$ norm.

As a result, we have the following property:

Proposition 1.10.1. *A function u belongs to $W_0^{1,p}(\Omega)$ if and only if $u\Big|_{\partial\Omega} = 0$.*

The following is a direct application of this result:

Example 1.11. *Consider a domain Ω as shown in Figure 3, which is split into the domains Ω^+ and Ω^-. Assume that P is of class $\mathcal{C}^1$. Moreover, assume that*

$$\begin{cases} u^- \in W^{1,p}(\Omega^-), \\ u^+ \in W^{1,p}(\Omega^+). \end{cases}$$

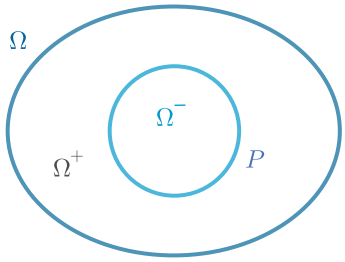

Figure 3: Domain Ω.

If we define

$$u = \begin{cases} u^- \text{ in } \Omega^-, \\ u^+ \text{ in } \Omega^+; \end{cases}$$

Then, we have the following:

> *A function u belongs to $W^{1,p}(\Omega)$ if an only if $u^- = u^+$ on P.*

As a corollary, this property yields another proof for the Heaviside function not belonging to any $W^{1,p}(\mathbb{R})$:

Proof. The Heaviside function is defined as

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Notice that $\lim_{x \to 0^-} H(x) = 0$ and $\lim_{x \to 0^+} H(x) = 1$, thus H cannot belong to $W^{1,p}(\Omega)$. □

Further references

Most of the notes here can be traced back to the book of Brezis [3]. A comprehensive reference on Sobolev spaces is Sobolev spaces, second edition by Adams and Fournier [1]. A nice review on inequalities used in functional analysis is available in Evans [4]. Finally, a quick summary on Sobolev spaces is also available in Ball [2].

Availability of data, material, and code

All the files and this document are available as in the following repository:

`https://github.com/andresrmt/Variational-Methods-for-PDEs`

References

[1] Adams, R. A. and Fournier, J. J. (2003). *Sobolev spaces*. Academic Press, Amsterdam Boston.

[2] Ball, J. (2019). Variational methods - lectures 1-8. `http://people.maths.ox.ac.uk/~ball/Teaching/MIGSAA2019.pdf`.

[3] Brezis, H. (2010). *Functional Analysis, Sobolev Spaces and Partial Differential Equations*. Springer New York, ii edition.

[4] Evans, L. (2010). *Partial differential equations*. American Mathematical Society, Providence, R.I.