Network reconstruction from indirect measurements and dynamics

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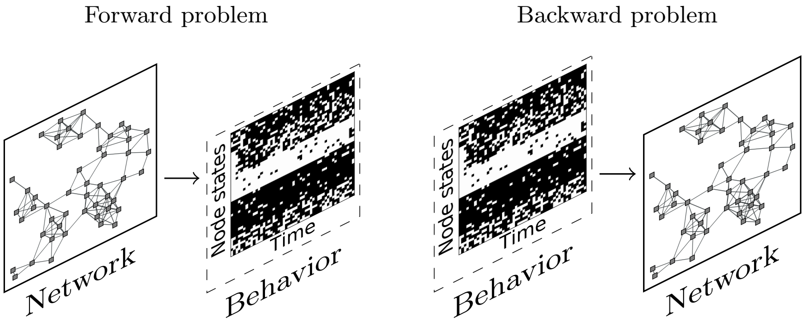
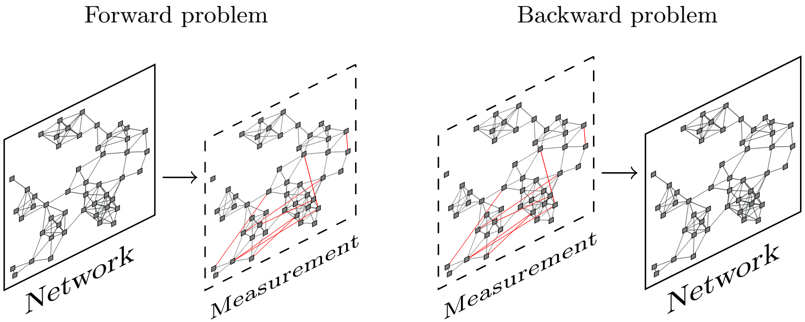
Network reconstruction from dynamics

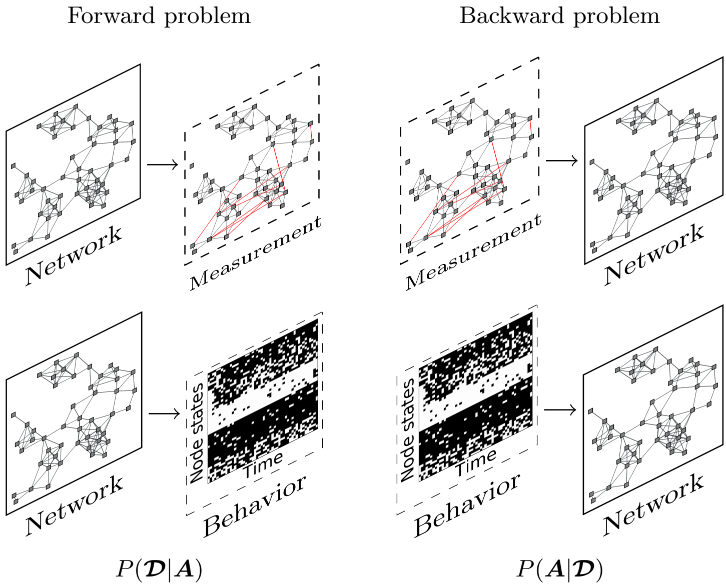


Network reconstruction from noisy measurements



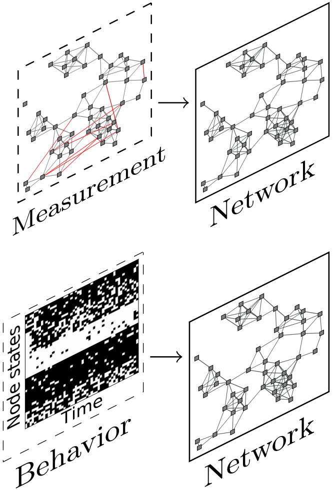
Network reconstruction

$\boldsymbol{A} \rightarrow$ Network $\boldsymbol{\mathcal{D}} \rightarrow$ Data (Measurement or Behavior)



Bayesian network reconstruction

Backward problem



$$\boxed{P(\boldsymbol{A}|\boldsymbol{\mathcal{D}}) = \frac{P(\boldsymbol{\mathcal{D}}|\boldsymbol{A})P(\boldsymbol{A})}{P(\boldsymbol{\mathcal{D}})}}$$

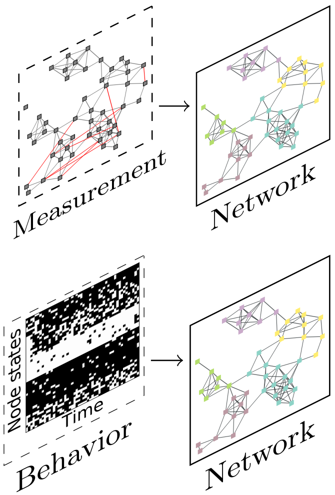
$P(\boldsymbol{\mathcal{D}}|\boldsymbol{A}) \to$ Likelihood

$P(\boldsymbol{A}) \to$ Prior

$$P(\boldsymbol{\mathcal{D}}) = \sum_{\boldsymbol{A}} P(\boldsymbol{\mathcal{D}}|\boldsymbol{A})P(\boldsymbol{A})$$

Joint reconstruction and community detection

Backward problem



$\boldsymbol{b} \rightarrow$ Partition of the nodes into groups

$$\boxed{P(\boldsymbol{A}, \boldsymbol{b}|\boldsymbol{\mathcal{D}}) = \frac{P(\boldsymbol{\mathcal{D}}|\boldsymbol{A})P(\boldsymbol{A}|\boldsymbol{b})P(\boldsymbol{b})}{P(\boldsymbol{\mathcal{D}})}}$$

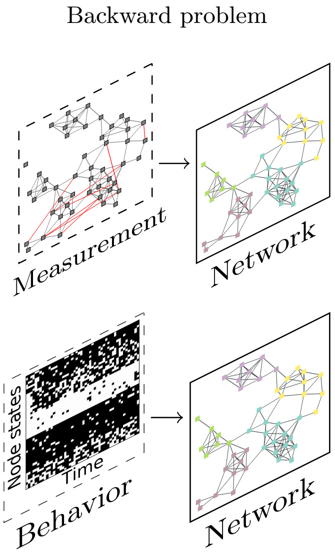
$P(\boldsymbol{\mathcal{D}}|\boldsymbol{A}) \rightarrow$ Likelihood

$P(\boldsymbol{A}|\boldsymbol{b}) \rightarrow$ Prior w/ latent communities

$P(\boldsymbol{b}) \rightarrow$ Partition prior

$$P(\boldsymbol{\mathcal{D}}) = \sum_{\boldsymbol{A},\boldsymbol{b}} P(\boldsymbol{\mathcal{D}}|\boldsymbol{A})P(\boldsymbol{A}|\boldsymbol{b})P(\boldsymbol{b})$$

Joint reconstruction and community detection



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$P(\boldsymbol{b}) \rightarrow$ Partition prior

$$P(\boldsymbol{\mathcal{D}}) = \sum_{\boldsymbol{A},\boldsymbol{b}} P(\boldsymbol{\mathcal{D}}|\boldsymbol{A})P(\boldsymbol{A}|\boldsymbol{b})P(\boldsymbol{b})$$

"Strict" reconstruction:

$$P(\boldsymbol{A}|\boldsymbol{\mathcal{D}}) = \sum_{\boldsymbol{b}} P(\boldsymbol{A},\boldsymbol{b}|\boldsymbol{\mathcal{D}})$$

"Strict" community detection:

$$P(\boldsymbol{b}|\boldsymbol{\mathcal{D}}) = \sum_{\boldsymbol{A}} P(\boldsymbol{A},\boldsymbol{b}|\boldsymbol{\mathcal{D}})$$

Noisy network reconstruction task

$$\underbrace{\boldsymbol{A}^*}_{\text{True network}} \longrightarrow \boxed{\text{Noisy measurement}} \longrightarrow \underbrace{\boldsymbol{A}}_{\text{Observed network}}$$

$$\underbrace{\boldsymbol{A}}_{\text{Observed network}} \longrightarrow \boxed{\text{Reconstruction}} \longrightarrow \underbrace{\hat{\boldsymbol{A}}}_{\text{Reconstructed network}}$$

So that $\hat{\boldsymbol{A}}$ is as close as possible to $\boldsymbol{A}^*$.

Caveats:

- With a single copy of $\boldsymbol{A}$.
- Without knowing how strong the noise is (i.e. the number of missing or spurious edges).

How is reconstruction possible?

(a)

(b)

How is reconstruction possible?


(a)


(b)

We need:

- A model for structure.
- A model for noise.

How is reconstruction possible?

(a) (b)

We need:

- A model for structure.
- A model for noise.

(but for networks)

Nonparametric Bayesian inference

- A model for structure, $P(\boldsymbol{A}|\theta)$
- A model for noise, $P(\boldsymbol{\mathcal{D}}|\boldsymbol{A},\phi)$

$\boldsymbol{A} \to$ Network, $\boldsymbol{\mathcal{D}} \to$ Measured data, $(\theta,\phi) \to$ Parameters

Posterior distribution:

$$\boxed{P(\boldsymbol{A}|\boldsymbol{\mathcal{D}}) = \frac{P(\boldsymbol{\mathcal{D}}|\boldsymbol{A})P(\boldsymbol{A})}{P(\boldsymbol{\mathcal{D}})}}$$

Marginal probabilities:

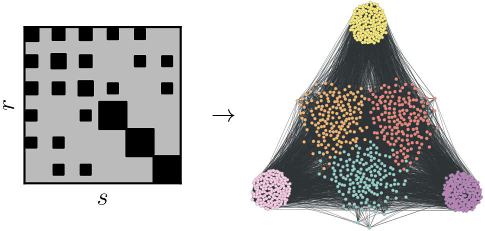
$$P(\boldsymbol{\mathcal{D}}|\boldsymbol{A}) = \int P(\boldsymbol{\mathcal{D}}|\boldsymbol{A},\phi)P(\phi)\mathrm{d}\phi$$

$$P(\boldsymbol{A}) = \int P(\boldsymbol{A}|\theta)P(\theta)\mathrm{d}\theta$$

Network structure: The stochastic block model (SBM)

Planted partition: N nodes divided into B groups.

Parameters: $b_i \to$ group membership of node i
$\omega_{rs} \to$ edge probability from group r to s.



Degree-corrected: Arbitrary degree sequence: $\{\kappa_i\}$

- Not restricted to assortative structures ("communities").
- Easily generalizable (edge direction, overlapping groups, etc.)

Bayesian SBM

$$P(\boldsymbol{A}|\boldsymbol{\lambda},\boldsymbol{\kappa},\boldsymbol{b}) = \prod_{i<j} \frac{(\kappa_i\kappa_j\lambda_{b_ib_j})^{A_{ij}} e^{-\kappa_i\kappa_j\lambda_{b_ib_j}}}{A_{ij}!} \times \prod_i \frac{(\kappa_i^2\lambda_{b_ib_i}/2)^{A_{ii}/2} e^{-\kappa_i^2\lambda_{b_ib_i}/2}}{(A_{ii}/2)!}$$

Noninformative priors:

$$P(\boldsymbol{\lambda}|\boldsymbol{b}) = \prod_{r\le s} e^{-\lambda_{rs}/(1+\delta_{rs})\bar{\lambda}}/(1+\delta_{rs})\bar{\lambda}$$

$$P(\boldsymbol{\kappa}|\boldsymbol{b}) = \prod_r (n_r-1)!\delta(\textstyle\sum_i \kappa_i\delta_{b_i,r}-1)$$

Bayesian SBM

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$$P(\boldsymbol{\kappa}|\boldsymbol{b}) = \prod_r (n_r-1)!\,\delta(\textstyle\sum_i \kappa_i\delta_{b_i,r}-1)$$

Marginal likelihood:

$$\begin{aligned}
P(\boldsymbol{A}|\boldsymbol{b}) &= \int P(\boldsymbol{A}|\boldsymbol{\lambda},\boldsymbol{\kappa},\boldsymbol{b})P(\boldsymbol{\lambda}|\boldsymbol{b})P(\boldsymbol{\kappa}|\boldsymbol{b})\,\mathrm{d}\boldsymbol{\lambda}\mathrm{d}\boldsymbol{\kappa}\\
&= \frac{\bar\lambda^E}{(\bar\lambda+1)^{E+B(B+1)/2}} \times \frac{\prod_{r<s} e_{rs}!\prod_r e_{rr}!!}{\prod_{i<j} A_{ij}!\prod_i A_{ii}!!} \times \prod_r \frac{(n_r-1)!}{(e_r+n_r-1)!} \times \prod_i k_i!\\
&= P(\boldsymbol{A}|\boldsymbol{k},\boldsymbol{e},\boldsymbol{b})P(\boldsymbol{k}|\boldsymbol{e},\boldsymbol{b})P(\boldsymbol{e})
\end{aligned}$$

Bayesian SBM

$$P(\boldsymbol{A}|\boldsymbol{\lambda},\boldsymbol{\kappa},\boldsymbol{b}) = \prod_{i<j} \frac{(\kappa_i\kappa_j\lambda_{b_ib_j})^{A_{ij}} e^{-\kappa_i\kappa_j\lambda_{b_ib_j}}}{A_{ij}!} \times \prod_i \frac{(\kappa_i^2\lambda_{b_ib_i}/2)^{A_{ii}/2} e^{-\kappa_i^2\lambda_{b_ib_i}/2}}{(A_{ii}/2)!}$$

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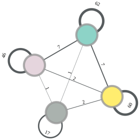
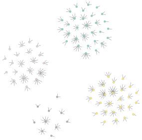
Marginal likelihood:

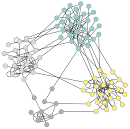
$$\begin{aligned} P(\boldsymbol{A}|\boldsymbol{b}) &= \int P(\boldsymbol{A}|\boldsymbol{\lambda},\boldsymbol{\kappa},\boldsymbol{b})P(\boldsymbol{\lambda}|\boldsymbol{b})P(\boldsymbol{\kappa}|\boldsymbol{b})\,\mathrm{d}\boldsymbol{\lambda}\mathrm{d}\boldsymbol{\kappa} \\ &= \frac{\bar{\lambda}^E}{(\bar{\lambda}+1)^{E+B(B+1)/2}} \times \frac{\prod_{r<s} e_{rs}!\prod_r e_{rr}!!}{\prod_{i<j} A_{ij}!\prod_i A_{ii}!!} \times \prod_r \frac{(n_r-1)!}{(e_r+n_r-1)!} \times \prod_i k_i! \\ &= P(\boldsymbol{A}|\boldsymbol{k},\boldsymbol{e},\boldsymbol{b})P(\boldsymbol{k}|\boldsymbol{e},\boldsymbol{b})P(\boldsymbol{e}) \end{aligned}$$



Edge counts $\boldsymbol{e}$.

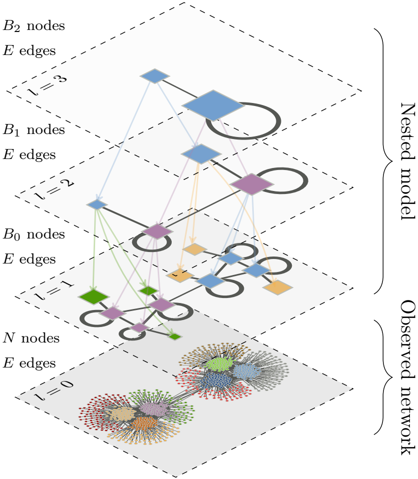


Degrees, $\boldsymbol{k}$.



Network, $\boldsymbol{A}$.

Nested SBM: Group hierarchies



Deeper Bayesian hierarchy:

- Prevents underfitting.
- Multiple scales of description.

Measurement model

$\boxed{\text{Edge-o-meter}}$

$p \to$ probability of a missing edge $(1 \to 0)$
$q \to$ probability of a spurious edge $(0 \to 1)$.

$n_{ij} \to$ number of measurements of pair (i, j)
$x_{ij} \to$ number of edges recorded

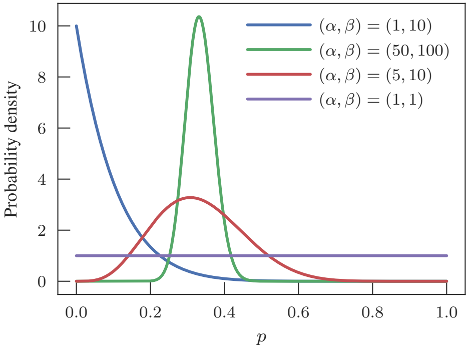
$$P(x_{ij}|n_{ij}, A_{ij}, p, q) = \binom{n_{ij}}{x_{ij}} \left[(1-p)^{x_{ij}} p^{n_{ij}-x_{ij}}\right]^{A_{ij}} \left[q^{x_{ij}} (1-q)^{n_{ij}-x_{ij}}\right]^{1-A_{ij}}$$

$$P(\boldsymbol{x}|\boldsymbol{n}, \boldsymbol{A}, \alpha, \beta, \mu, \nu) = \int P(\boldsymbol{x}|\boldsymbol{n}, \boldsymbol{A}, p, q) P(p|\alpha, \beta) P(q|\mu, \nu)\, \mathrm{d}p\, \mathrm{d}q$$

$$P(p|\alpha, \beta), P(q|\mu, \nu) \to \text{Beta priors}$$

The Edge-o-meter

$$P(p|\alpha,\beta) = \frac{p^{\alpha-1}(1-p)^{\alpha-1}}{\mathcal{B}(\alpha,\beta)} \qquad P(q|\mu,\nu) = \frac{q^{\mu-1}(1-q)^{\mu-1}}{\mathcal{B}(\mu,\nu)}$$



- $(\alpha,\beta) = (1,10) \rightarrow$ accurate measurement (low noise)
- $(\alpha,\beta) = (50,100) \rightarrow$ high noise, good calibration
- $(\alpha,\beta) = (5,10) \rightarrow$ high noise, bad calibration
- $(\alpha,\beta) = (1,1) \rightarrow$ <u>noninformative</u> (i.e. uniform distribution)

The full reconstruction method

Posterior distribution:

$$P(\boldsymbol{A},\boldsymbol{b}|\boldsymbol{n},\boldsymbol{x},\alpha,\beta,\mu,\nu) = \frac{P(\boldsymbol{x}|\boldsymbol{n},\boldsymbol{A},\alpha,\beta,\mu,\nu)P(\boldsymbol{A}|\boldsymbol{b})P(\boldsymbol{b})}{P(\boldsymbol{x}|\alpha,\beta,\mu,\nu)}.$$

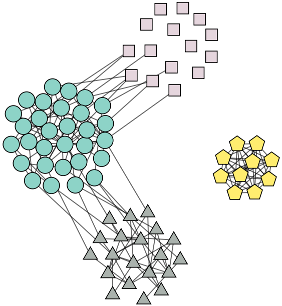
We infer both the network $\boldsymbol{A}$ as well as the SBM latent variables $\boldsymbol{b}$ via MCMC:

Move proposals $P(\boldsymbol{b}'|\boldsymbol{A},\boldsymbol{b})$ and $P(\boldsymbol{A}'|\boldsymbol{A},\boldsymbol{b})$, accept with probability

$$\min\left(1, \frac{P(\boldsymbol{A}',\boldsymbol{b}'|\boldsymbol{\mathcal{D}})P(\boldsymbol{A}|\boldsymbol{A}',\boldsymbol{b}')P(\boldsymbol{b}|\boldsymbol{A}',\boldsymbol{b}')}{P(\boldsymbol{A},\boldsymbol{b}|\boldsymbol{\mathcal{D}})P(\boldsymbol{A}'|\boldsymbol{A},\boldsymbol{b})P(\boldsymbol{b}'|\boldsymbol{A},\boldsymbol{b})}\right).$$

(Efficient, scales to very large networks.)

How does it work?



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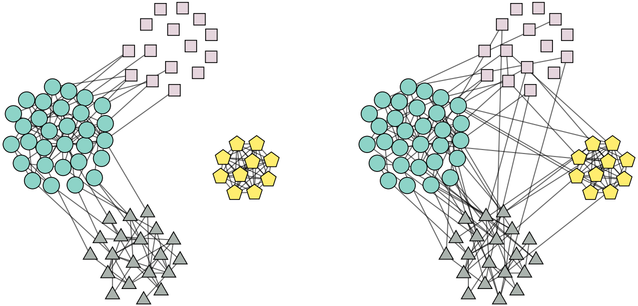
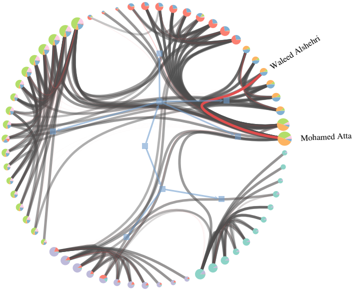
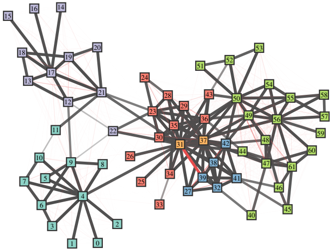
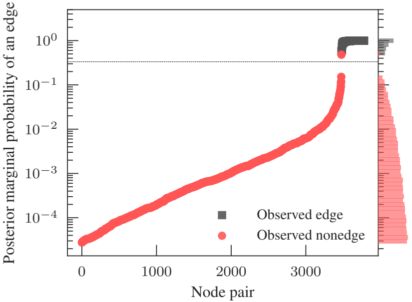


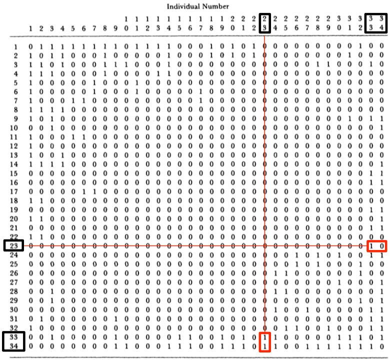
Example: Terrorist associations







Example: Zachary's karate club



Individual Number

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
1	0	1	1	1	1	1	1	1	1	0	1	1	1	1	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	1	0	0
2	1	0	1	1	0	0	0	1	0	0	0	0	0	1	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	1	0	0	0
3	1	1	0	1	0	0	0	1	1	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	1	0
4	1	1	1	0	0	0	0	1	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
6	1	0	0	0	0	0	1	0	0	0	1	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
7	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
8	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
9	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	1
10	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
11	1	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
12	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
13	1	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
14	1	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
15	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
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17	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
18	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
20	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1
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22	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
24	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	1	1
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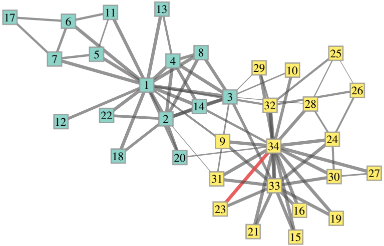
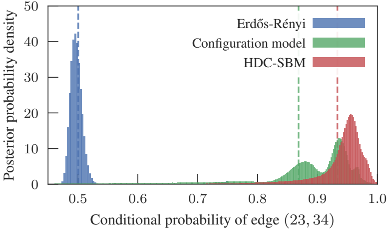
26	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	1	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1
28	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	0	0	0	0	0	0	1
29	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1
30	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	0	0	0	0	0	1	1
31	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1
32	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0	0	1	0	0	0	1	1
33	0	0	1	0	0	0	0	0	1	0	0	0	0	0	1	1	0	0	1	0	1	0	1	1	0	0	0	0	0	1	1	1	0	1
34	0	0	0	0	0	0	0	0	1	1	0	0	0	1	1	1	0	0	1	1	1	0	1	1	0	0	1	1	1	1	1	1	1	0

(credit: Aaron Clauset)





Wait! Is this just edge prediction?

It *is* edge prediction, but it yields a full posterior distribution $P(\boldsymbol{A}|\boldsymbol{n},\boldsymbol{x})$ that is **nonparametric**.

We can:

- Perform maximum marginal posterior estimation,

$$\hat{A}_{ij} = \begin{cases} 1 & \text{if } \pi_{ij} > 1/2 \\ 0 & \text{if } \pi_{ij} < 1/2, \end{cases}$$

  where $\pi_{ij} = \sum_{\boldsymbol{A}} A_{ij} P(\boldsymbol{A}|\boldsymbol{n},\boldsymbol{x})$ is the marginal posterior edge probability.

- Estimate network properties $y(\boldsymbol{A})$ and their error estimates:

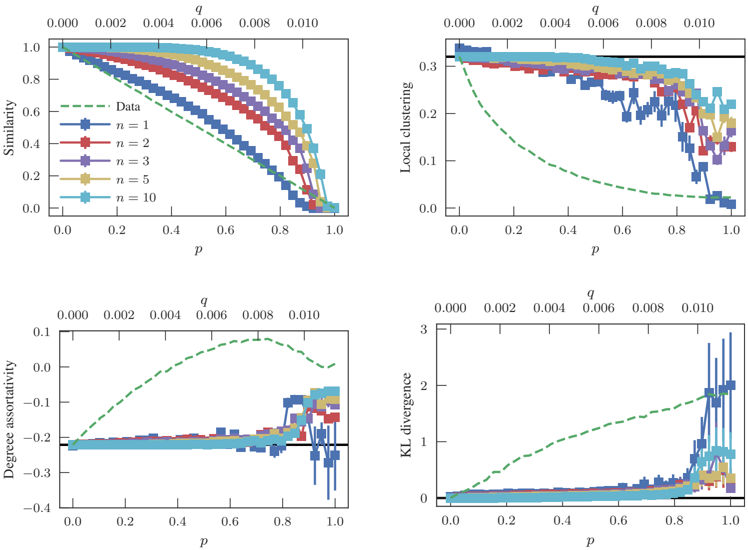
$$\hat{y} = \sum_{\boldsymbol{A}} y(\boldsymbol{A}) P(\boldsymbol{A}|\boldsymbol{n},\boldsymbol{x})$$

$$\sigma_y^2 = \sum_{\boldsymbol{A}} (\hat{y} - y(\boldsymbol{A}))^2 P(\boldsymbol{A}|\boldsymbol{n},\boldsymbol{x}).$$

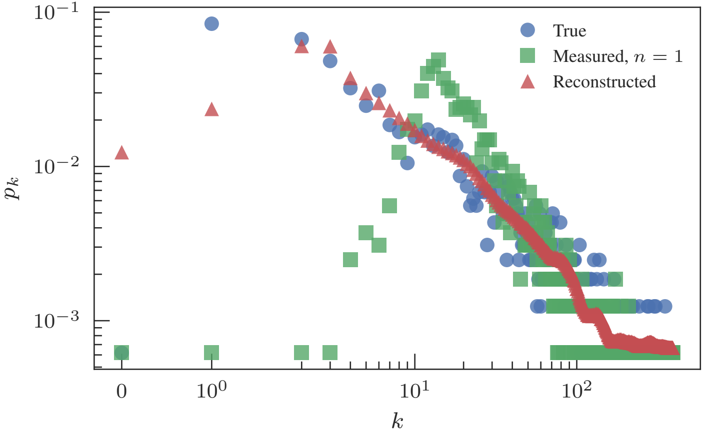
Reconstruction performance

Real network (political blogs) + simulated noise:

$$p \in [0,1],\ q = pE/[\binom{N}{2} - E]$$

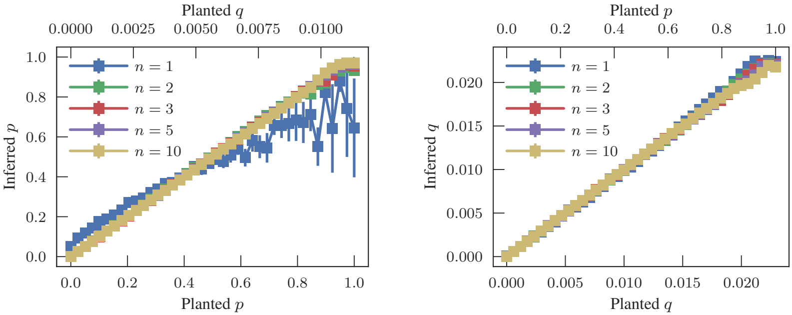


Reconstruction performance: Degrees



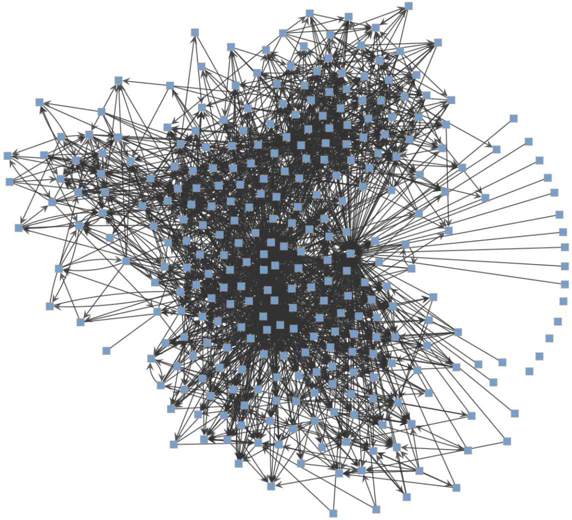
$(p, q) = (0.41, 0.0094)$

Inferring the noise



Empirical reconstruction redux

C. elegans neural network



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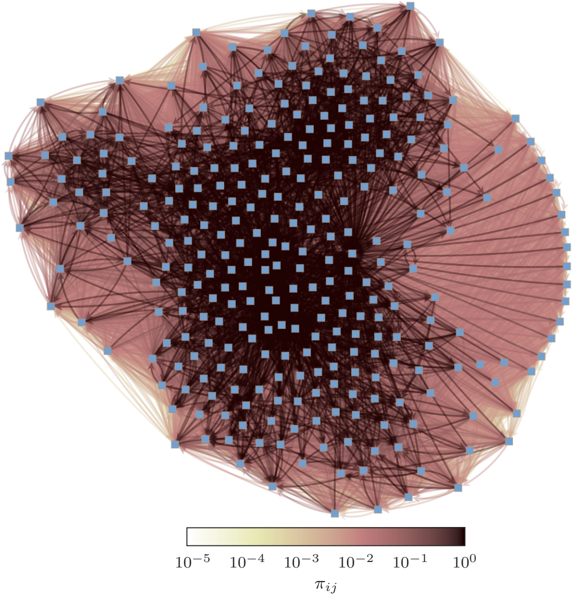

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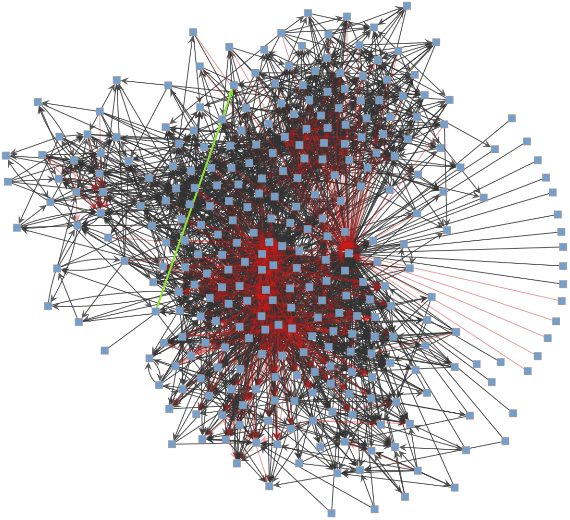
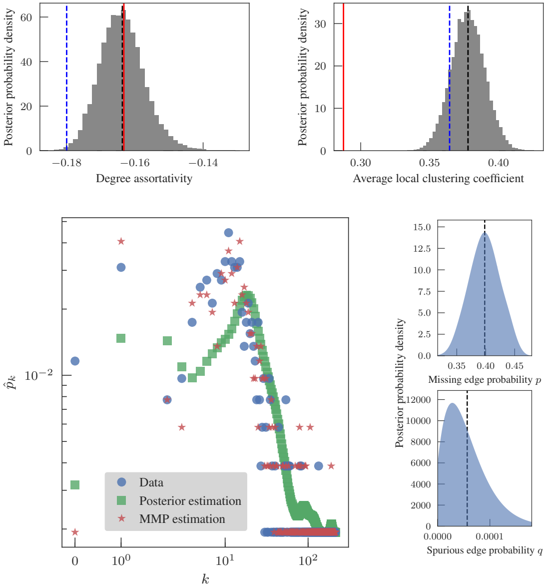


C. elegans NEURAL NETWORK



Multiple measurements and heterogeneous errors

Observational error does not need to be uniform for every pair (i,j).
Non-uniform model, w/ pair-specific error rates: p_{ij} and q_{ij}

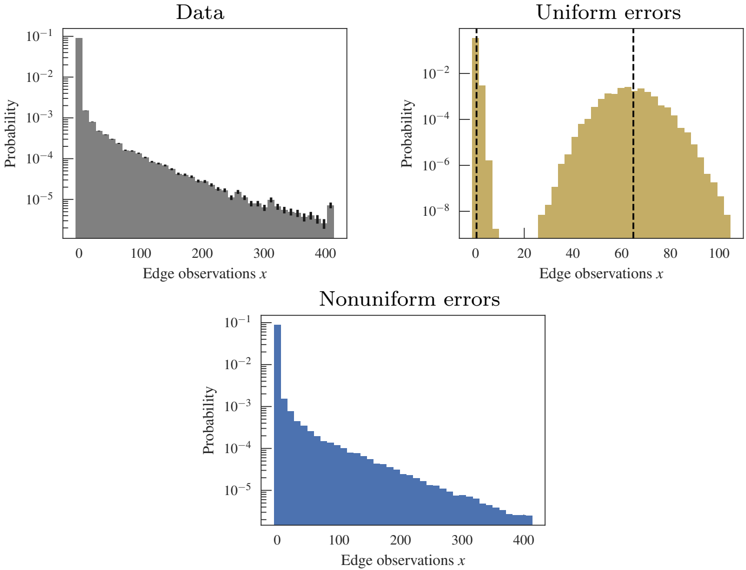
$$P(x_{ij}|n_{ij},A_{ij},p_{ij},q_{ij}) = \binom{n_{ij}}{x_{ij}} \left[(1-p_{ij})^{x_{ij}} p_{ij}^{n_{ij}-x_{ij}}\right]^{A_{ij}} \left[q_{ij}^{x_{ij}}(1-q_{ij})^{n_{ij}-x_{ij}}\right]^{1-A_{ij}}$$

Marginal probability,

$$\begin{aligned} &P(x_{ij}|n_{ij},A_{ij},\alpha,\beta,\mu,\nu) \\ &\quad= \int P(x_{ij}|n_{ij},A_{ij},p_{ij},q_{ij}) P(p_{ij}|\alpha,\beta) P(q_{ij}|\mu,\nu)\, \mathrm{d}p_{ij} \mathrm{d}q_{ij} \\ &\quad= \binom{n_{ij}}{x_{ij}} \left[\frac{\mathcal{B}(n_{ij}-x_{ij}+\alpha, x_{ij}+\beta)}{\mathcal{B}(\alpha,\beta)}\right]^{A_{ij}} \times \\ &\qquad\qquad \left[\frac{\mathcal{B}(x_{ij}+\mu, n_{ij}-x_{ij}+\nu)}{\mathcal{B}(\mu,\nu)}\right]^{1-A_{ij}}. \end{aligned}$$

Human connectome

418 individuals



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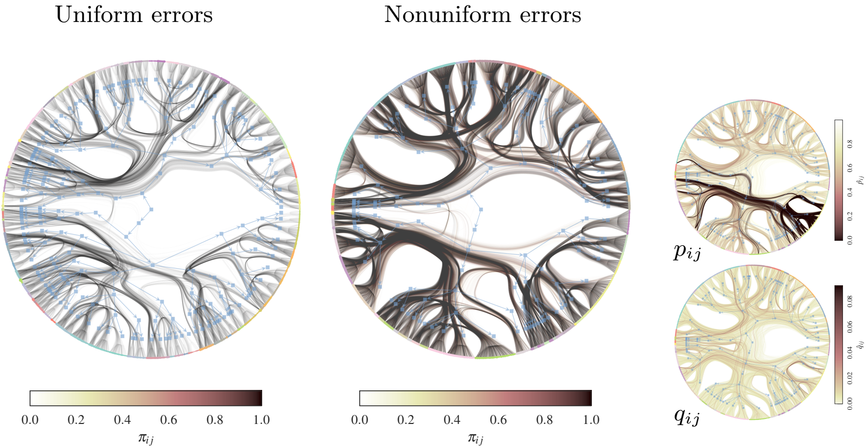


Extrinsic error estimates

$Q_{ij} \in [0,1] \to$ experimentally determined uncertainties

$$P_Q(\boldsymbol{A}|\boldsymbol{Q}) = \prod_{i<j} Q_{ij}^{A_{ij}} (1 - Q_{ij})^{1-A_{ij}}.$$

Example:

STRING Protein-Protein interaction network database, Szklarczyk et al, Nucleic Acids Research 45, D362–D368 (2017).

Errors are estimated via a combination of: (i) direct experiments, (ii) database curation, (iii) publication text-mining, (iv) co-expression data, (v) genome proximity, (vi) ortholog fusion, (vii) phylogenetic co-ocurrence.

Extrinsic error estimates

The distribution $P_Q(\boldsymbol{A}|\boldsymbol{Q})$ implies the following noisy measurement process,

$$P(\boldsymbol{Q}|\boldsymbol{A}) = \frac{P_Q(\boldsymbol{A}|\boldsymbol{Q})P_Q(\boldsymbol{Q})}{P_Q(\boldsymbol{A})},$$

with prior

$$P_Q(\boldsymbol{Q}) = \prod_{i<j} P(Q_{ij}),$$

and normalization constant

$$P_Q(\boldsymbol{A}) = \int P_Q(\boldsymbol{A}|\boldsymbol{Q})P_Q(\boldsymbol{Q})\,\mathrm{d}\boldsymbol{Q} = \prod_{i<j} \bar{Q}^{A_{ij}}(1-\bar{Q})^{1-A_{ij}},$$

with $\bar{Q} = \int_0^1 QP(Q)\mathrm{d}Q$. Combining these together we have

$$P(\boldsymbol{Q}|\boldsymbol{A}) = P_Q(\boldsymbol{Q}) \prod_{i<j} \left(\frac{Q_{ij}}{\bar{Q}}\right)^{A_{ij}} \left(\frac{1-Q_{ij}}{1-\bar{Q}}\right)^{1-A_{ij}},$$

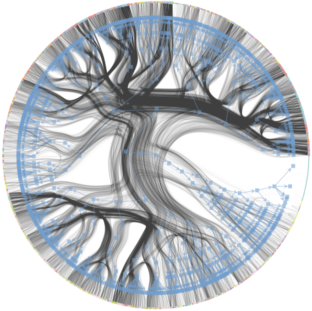
$$P(\boldsymbol{A}|\boldsymbol{Q}) = \frac{P(\boldsymbol{Q}|\boldsymbol{A})P(\boldsymbol{A})}{P(\boldsymbol{Q})}, \qquad \bar{Q} = \frac{\sum_{i<j} Q_{ij}}{\binom{N}{2}}.$$

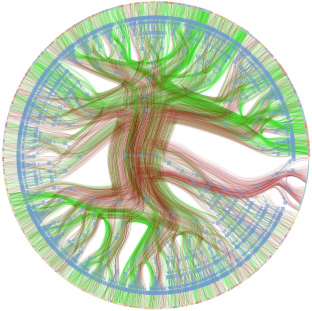
Extrinsic error estimates

$$P(\boldsymbol{A}|\boldsymbol{Q}) = \frac{P(\boldsymbol{Q}|\boldsymbol{A})P(\boldsymbol{A})}{P(\boldsymbol{Q})}, \qquad P(\boldsymbol{A}|\boldsymbol{Q}) \neq P_Q(\boldsymbol{A}|\boldsymbol{Q})!$$

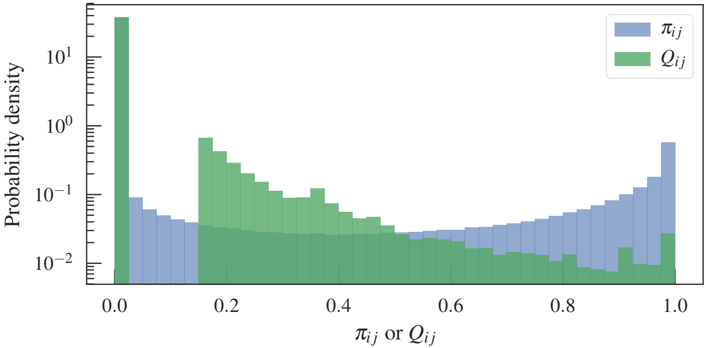
We are keeping the same noise generating process, but changing our prior assumption about the data.

E. coli proteins:





Extrinsic error estimates



Reconstruction from epidemic dynamics

$\boxed{\text{SIS epidemic model}}$

$$\sigma_i(t) \in \{0,1\} \rightarrow \text{ State of node } i \text{ at time } t$$

$$\tau_{ij} \rightarrow \text{ Transmission probability via edge } i \leftarrow j$$

$$\gamma \rightarrow \; 1 \rightarrow 0 \text{ Recovery probability}$$

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$$P(\boldsymbol{\sigma}|\boldsymbol{A}, \boldsymbol{\tau}, \gamma) = \prod_t \prod_i P(\sigma_i(t)|\boldsymbol{\sigma}(t-1))$$

$$P(\sigma_i(t)|\boldsymbol{\sigma}(t-1)) = f(\mathrm{e}^{m_i(t-1)}, \sigma_i(t))^{1-\sigma_i(t-1)} \times f(\gamma, \sigma_i(t))^{\sigma_i(t-1)}$$

$$f(p, \sigma) = (1-p)^\sigma p^{1-\sigma}, \qquad m_i(t) = \sum_j A_{ij} \ln(1 - \tau_{ij}) \sigma_j(t)$$

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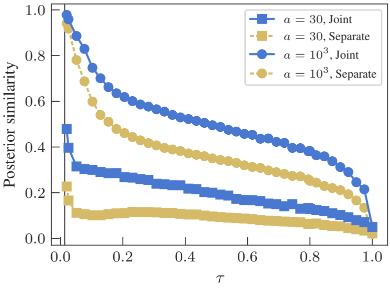
Posterior:

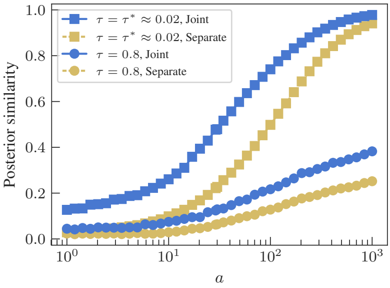
$$\boxed{P(\boldsymbol{A}, \boldsymbol{\tau}, \boldsymbol{b}|\boldsymbol{\sigma}) = \frac{P(\boldsymbol{\sigma}|\boldsymbol{A}, \boldsymbol{\tau}, \gamma)P(\boldsymbol{A}|\boldsymbol{b})P(\boldsymbol{b})P(\boldsymbol{\tau})}{P(\boldsymbol{\sigma}|\gamma)}}$$

Reconstruction from epidemic dynamics

Joint reconstruction with community detection improves accuracy.

Example: Simulated SIS for the worldwide directed network of $N = 3,286$ airports with $E = 39,430$ edges.





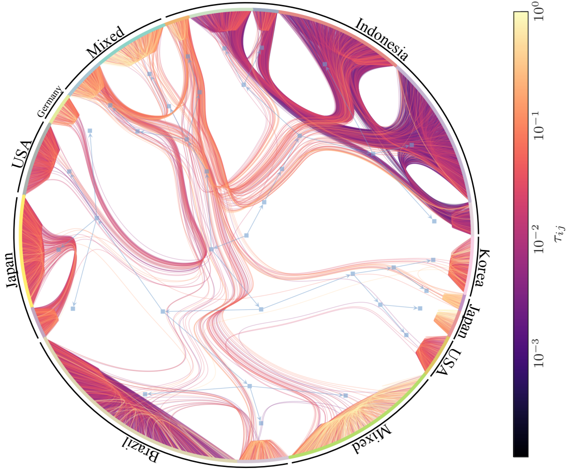
$a \to$ activity, i.e. number of $0 \to 1$ transitions per node

"Separate" means sampling from the posterior

$$P(\boldsymbol{A}|\boldsymbol{\mathcal{D}}) = \frac{P(\boldsymbol{\mathcal{D}}|\boldsymbol{A})P(\boldsymbol{A})}{P(\boldsymbol{\mathcal{D}})}, \qquad P(\boldsymbol{A}) \to \text{Erdős-Rényi}$$

Example: Reconstruction from twitter cascades

Reconstruction of the directed network of influence between $N = 1,833$ twitter users from $58,224$ re-tweets, using a SI infection model.



The inverse Ising model

Spins $s_i \in \{-1, 1\}$ sampled with probability

$$P(\boldsymbol{s}|\boldsymbol{A}, \beta, \boldsymbol{J}, \boldsymbol{h}) = \frac{\exp\left(\beta \sum_{i<j} J_{ij} A_{ij} s_i s_j + \sum_i h_i s_i\right)}{Z(\boldsymbol{A}, \beta, \boldsymbol{J}, \boldsymbol{h})}$$

Partition function:

$$Z(\boldsymbol{A}, \beta, \boldsymbol{J}, \boldsymbol{h}) = \sum_{\boldsymbol{s}} \exp\left(\beta \sum_{i<j} J_{ij} A_{ij} s_i s_j + \sum_i h_i s_i\right)$$

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Unfortunately computing $Z(\boldsymbol{A}, \beta, \boldsymbol{J}, \boldsymbol{h})$ is intractable. Fortunatelly, we can resort to an accurate approximation called ***pseudo-likelihood***:

$$P(\boldsymbol{s}|\boldsymbol{A}, \beta, \boldsymbol{J}, \boldsymbol{h}) = \prod_i P(s_i|\boldsymbol{s}\setminus s_i, \boldsymbol{A}, \beta, \boldsymbol{J}, \boldsymbol{h}) = \prod_i \frac{\exp(\beta s_i \sum_j J_{ij} A_{ij} s_j + h_i s_i)}{2\cosh(\beta \sum_j J_{ij} A_{ij} s_j + h_i)}$$

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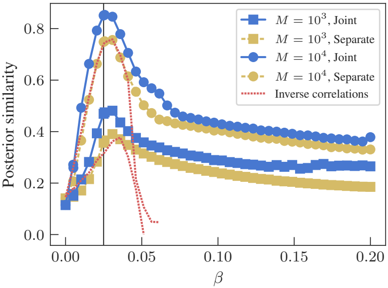
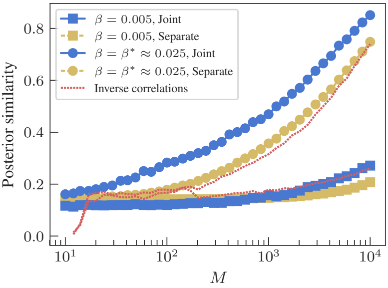
We observe a set of M microstates $\bar{\boldsymbol{s}} = \{\boldsymbol{s}_1, \ldots, \boldsymbol{s}_M\}$, with likelihood $P(\bar{\boldsymbol{s}}|\boldsymbol{A}, \beta, \boldsymbol{J}, \boldsymbol{h}) = \prod_l P(\boldsymbol{s}_l|\boldsymbol{A}, \beta, \boldsymbol{J}, \boldsymbol{h})$, and perform reconstruction with the posterior

$$\boxed{P(\boldsymbol{A}, \boldsymbol{b}, \beta, \boldsymbol{J}, \boldsymbol{h}|\bar{\boldsymbol{s}}) = \frac{P(\bar{\boldsymbol{s}}|\boldsymbol{A}, \beta, \boldsymbol{J}, \boldsymbol{h}) P(\beta) P(\boldsymbol{h}) P(\boldsymbol{J}|\boldsymbol{A}) P(\boldsymbol{A}|\boldsymbol{b}) P(\boldsymbol{b})}{P(\bar{\boldsymbol{s}})}}$$

The inverse Ising model

Simulated Ising model on a food web from Little Rock Lake, containing $N = 183$ nodes and $E = 2,434$ edges.





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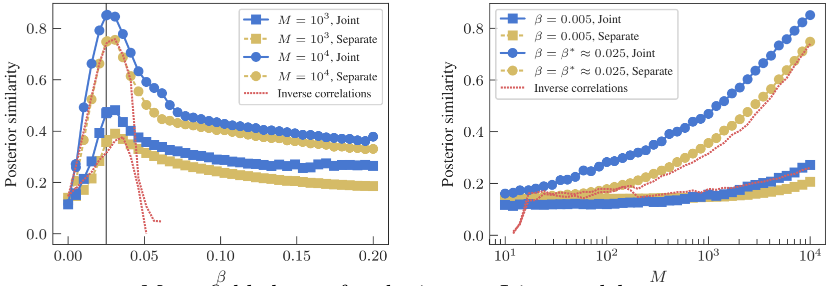

Mean-field theory for the inverse Ising model:

$$P(\boldsymbol{s}) = \prod_i \frac{1 + m_i s_i}{2}, \quad m_i = \langle s_i \rangle$$

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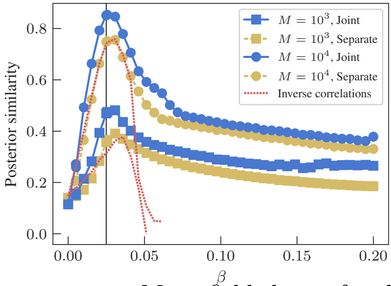

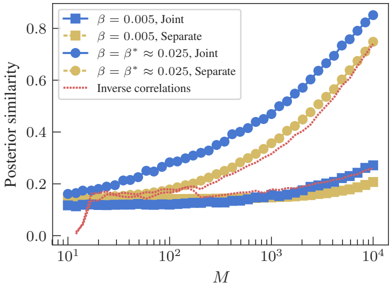

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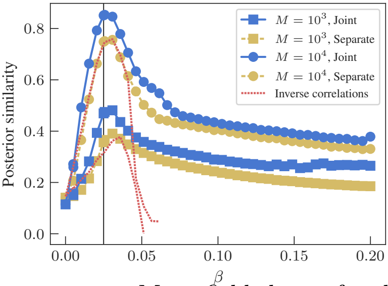

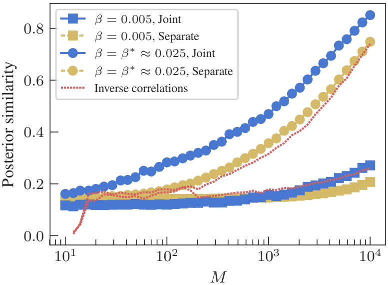

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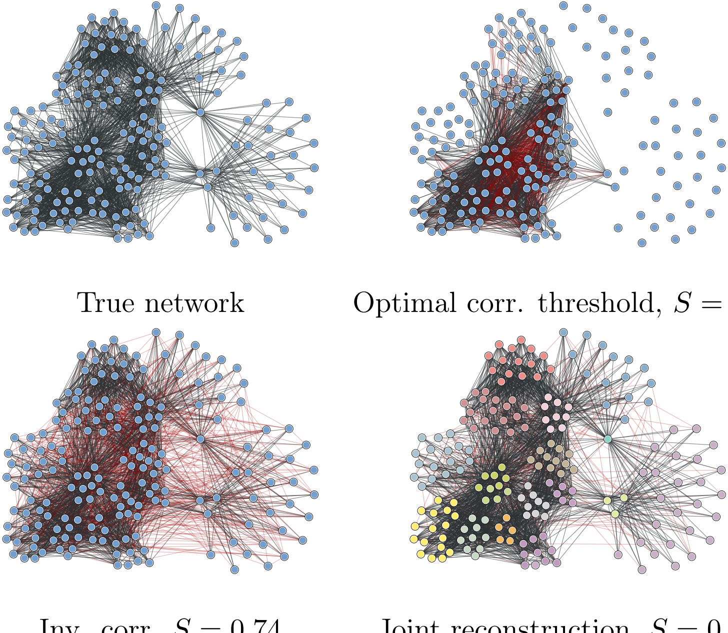
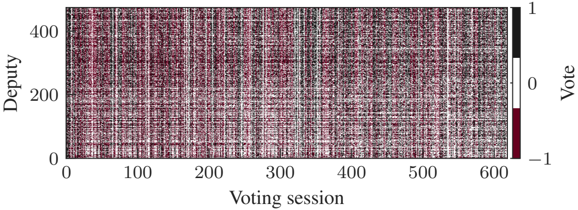
$$\frac{\partial^2 G(\boldsymbol{J}, \boldsymbol{m})}{\partial m_i \partial m_j} = [\boldsymbol{C}^{-1}]_{ij}, \quad C_{ij} = \langle s_i s_j \rangle - m_i m_j \quad \rightarrow \quad \boxed{J_{ij} = -[\boldsymbol{C}^{-1}]_{ij}}$$

Inverse Ising model



Interactions between members of congress



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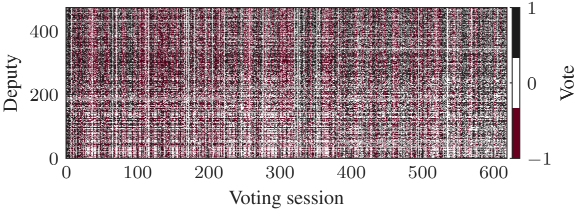

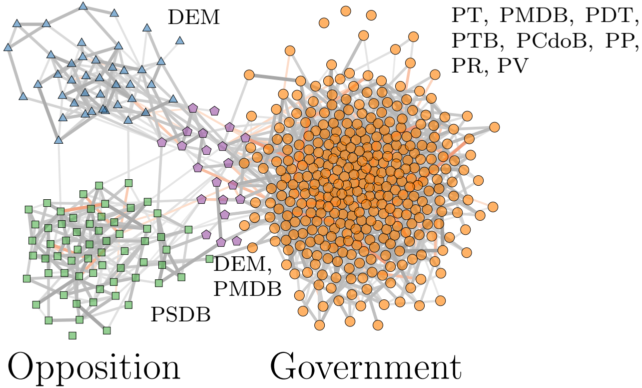

Synergy in joint reconstruction

> Finding communities helps reconstruction. But the joint approach also helps to find communities in the first place.

"Planted partition" SBM with

$$\lambda_{rs} = \lambda_{\text{in}}\delta_{rs} + \lambda_{\text{out}}(1 - \delta_{rs}),$$

where

$$\epsilon = N(\lambda_{\text{in}} - \lambda_{\text{out}})/\langle k\rangle B$$

Detectability transition exists at $\epsilon^* = 1/\sqrt{\langle k\rangle}$.

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Point estimate:

$$\hat{A}_{ij} = \begin{cases} 1 & \text{if } \pi_{ij} > 1/2 \\ 0 & \text{if } \pi_{ij} < 1/2, \end{cases}$$

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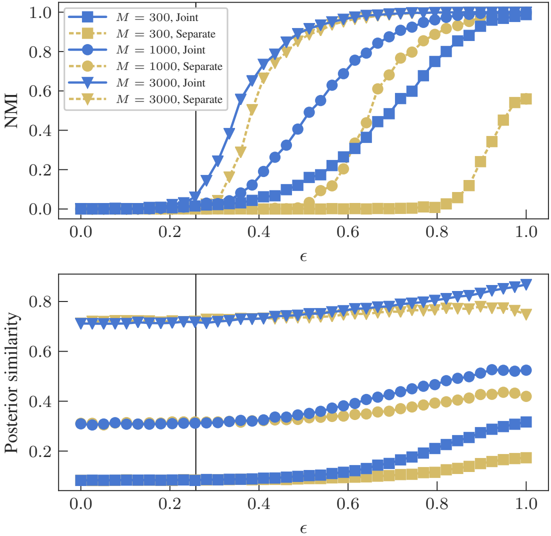
SBM inference:

$$P(\boldsymbol{b}|\hat{\boldsymbol{A}}) = \frac{P(\hat{\boldsymbol{A}}|\boldsymbol{b})P(\boldsymbol{b})}{P(\hat{\boldsymbol{A}})}$$

Synergy in joint reconstruction

$N = 1000$, $\langle k \rangle = 15$, $B = 10$, Ising model at critical temperature.



T. P. P., Phys. Rev. X 8 041011 (2018)

T. P. P., Phys. Rev. Lett. 123 128301 (2019)

For code, see:

`https://graph-tool.skewed.de`

(See also HOWTO at: `https://graph-tool.skewed.de/static/doc/demos/inference/inference.html`)