FEniCS Course

Lecture 9: Incompressible Navier–Stokes

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The incompressible Navier–Stokes equations

$$
\begin{aligned}
\rho(\dot{u} + u \cdot \nabla u) - \nabla \cdot \sigma(u, p) &= f && \text{in } \Omega \times (0, T] \\
\nabla \cdot u &= 0 && \text{in } \Omega \times (0, T] \\
u &= g_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} \times (0, T] \\
\sigma \cdot n &= g_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}} \times (0, T] \\
u(\cdot, 0) &= u_0 && \text{in } \Omega
\end{aligned}
$$

- u is the fluid velocity and p is the pressure
- ρ is the fluid density
- $\sigma(u, p) = 2\mu\epsilon(u) - pI$ is the Cauchy stress tensor
- $\epsilon(u) = \frac{1}{2}(\nabla u + \nabla u^{\top})$ is the strain rate tensor
- f is a given body force per unit volume
- g_{D} is a given boundary velocity
- g_{N} is a given boundary traction
- u_0 is a given initial velocity

Mixed variational formulation of Navier–Stokes

Multiply the momentum equation by a test function v and integrate by parts:

$$\int_\Omega \rho(\dot{u}+u\cdot\nabla u)\cdot v\,\mathrm{d}x+\int_\Omega \sigma(u,p):\epsilon(v)\,\mathrm{d}x=\int_\Omega f\cdot v\,\mathrm{d}x+\int_{\Gamma_\mathrm{N}} g_\mathrm{N}\cdot v\,\mathrm{d}s$$

Short-hand notation: $\langle\cdot,\cdot\rangle$ is L^2-inner product

$$\langle\rho\dot{u},v\rangle+\langle\rho u\cdot\nabla u,v\rangle+\langle\sigma(u,p),\epsilon(v)\rangle=\langle f,v\rangle+\langle g_\mathrm{N},v\rangle_{\Gamma_\mathrm{N}}$$

Multiply the continuity equation by a test function q and sum up: find $(u,p)\in V$ such that

$$\langle\rho\dot{u},v\rangle+\langle\rho u\cdot\nabla u,v\rangle+\langle\sigma(u,p),\epsilon(v)\rangle+\langle\nabla\cdot u,q\rangle=\langle f,v\rangle+\langle g_\mathrm{N},v\rangle_{\Gamma_\mathrm{N}}$$

for all $(v,q)\in\hat{V}$

Discrete mixed variational form of Navier–Stokes

Time-discretization leads to a *saddle-point* problem on each time step:

$$\begin{bmatrix} M + \Delta t A + \Delta t N(U) & \Delta t B \\ \Delta t B^\top & 0 \end{bmatrix} \begin{bmatrix} U \\ P \end{bmatrix} = \begin{bmatrix} b \\ 0 \end{bmatrix}$$

- Efficient solution of the saddle-point problem relies on the efficiency of special-purpose preconditioners (Uzawa iteration, Schur complement preconditioners, ...)
- We will use another approach (simpler and often more efficient)

A spliting scheme for Navier–Stokes

Use a Crank-Nicolson approximation with explicit convection for the time-discretization of the momentum equation:

$$\rho D_t u^n + \rho u^{n-1} \cdot \nabla u^{n-1} - \nabla \cdot \sigma(u^{n-1/2}, p^{n-1/2}) = f^{n-1/2}$$

Step 1: Compute the *tentative velocity* $u^\star$ using the approximation

$$\rho D_t u^\star + \rho u^{n-1} \cdot \nabla u^{n-1} - \nabla \cdot \sigma(u^{n-1/2}, p^{n-3/2}) = f^{n-1/2}$$

$D_t u = (u^n - u^{n-1})/k_n$ and $k_n = t_n - t_{n-1}$

A splitting scheme for N-S (contd.)

Subtract the equation for the tentative velocity from the equation for the *real* velocity:

$$\rho(D_t u^n - D_t u^\star) - \nabla \cdot \sigma(0, p^{n-1/2} - p^{n-3/2}) = 0$$

Expanding the D_t, using the equations, and rearranging give

$$\rho u^n = \rho u^\star - k_n \nabla (p^{n-1/2} - p^{n-3/2}) \tag{1}$$

Take the divergence and set $\nabla \cdot u^n = 0$:

$$-k_n \Delta p^{n-1/2} = -k_n \Delta p^{n-3/2} - \rho \nabla \cdot u^\star \tag{2}$$

- **Step 2**: Compute $p^{n-1/2}$ by solving the Poisson problem (2)
- **Step 3**: Compute u^n by solving the projection problem (1)

NB: Which boundary conditions for the Poisson problem (2)?

Boundary conditions

- For outflow boundary conditions, corresponding to so-called "do-nothing" boundary conditions for the Laplacian formulation, we take $\partial_n u = 0$:

$$\sigma(u,p)\cdot n = (2\mu\epsilon(u) - pI)\cdot n = \mu\nabla u\cdot n + \mu(\nabla u)^\top\cdot n - pn$$
$$= \mu\nabla u\cdot n - pn \approx \mu\nabla u^{n-1/2}\cdot n - p^{n-3/2}n$$

- Boundary conditions for the pressure Poisson problem:

$$\partial_n \dot{p} = 0$$

on the pressure Neumann boundary

Incremental pressure correction scheme (IPCS)

1. Compute the tentative velocity $u^\star$ by

$$\langle \rho D_t^n u^\star, v\rangle + \langle \rho u^{n-1}\cdot\nabla u^{n-1}, v\rangle + \langle \sigma(u^{n-\frac{1}{2}}, p^{n-3/2}), \epsilon(v)\rangle$$
$$- \langle \mu\nabla u^{n-\frac{1}{2}}\cdot n, v\rangle_{\partial\Omega} + \langle p^{n-3/2} n, v\rangle_{\partial\Omega} = \langle f^{n-1/2}, v\rangle$$

2. Compute the corrected pressure $p^{n-1/2}$ by

$$k_n\langle \nabla p^{n-1/2}, \nabla q\rangle = k_n\langle \nabla p^{n-3/2}, \nabla q\rangle - \langle \rho\nabla\cdot u^\star, q\rangle$$

3. Compute the corrected velocity u^n by

$$\langle \rho u^n, v\rangle = \langle \rho u^\star, v\rangle - k_n\langle \nabla(p^{n-1/2} - p^{n-3/2}), v\rangle$$

Useful FEniCS tools (I)

Note grad vs. ∇:

Python code

```
dot(grad(u), u)
dot(u, nabla_grad(u))
```

Defining operators:

Python code

```
def sigma(u, p):
    return 2.0*mu*sym(grad(u))-p*Identity(len(u))
```

The facet normal n:

Python code

```
n = FacetNormal(mesh)
```

Useful FEniCS tools (II)

Assembling matrices and vectors:

Python code

```
A = assemble(a)
b = assemble(L)
```

Solving linear systems:

Python code

```
solve(A, x, b)
solve(A, x, b, "gmres", "ilu")
solve(A, x, b, "cg", "amg")
```

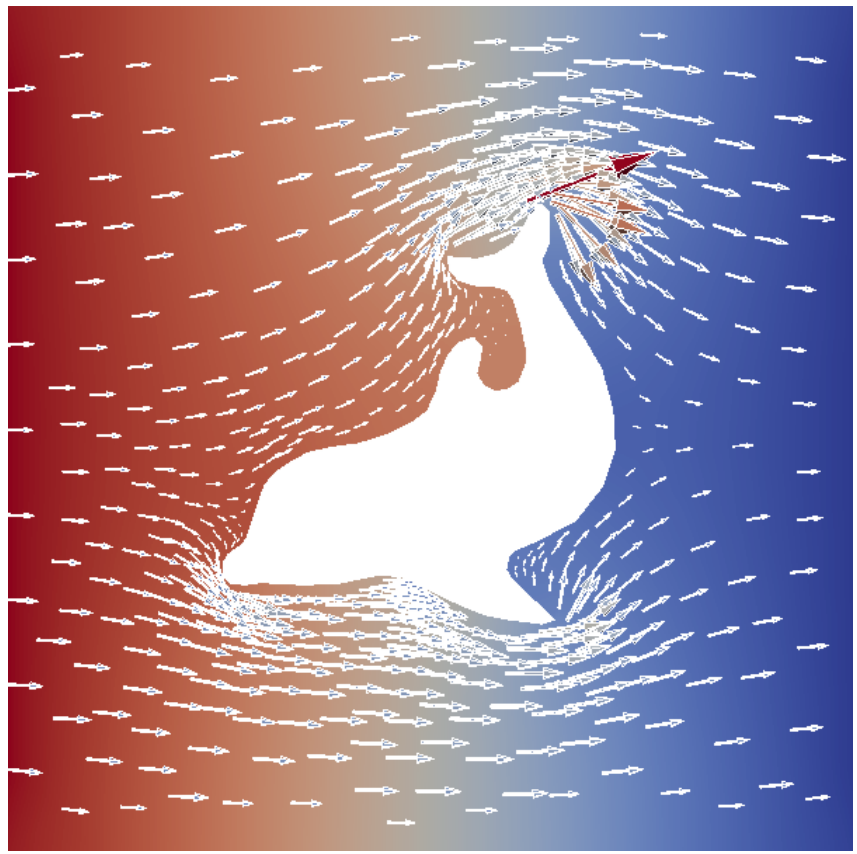
Extracting left- and right-hand sides:

Python code

```
F = <complicated expression>
a = lhs(F)
L = rhs(F)
```

The FEniCS challenge!

Solve the incompressible Navier–Stokes equations for the flow of water around a dolphin. The water is initially at rest and the flow is driven by a pressure gradient.



The FEniCS challenge!

- Use the mesh `dolfin_channel.xml.gz`.
- Compute the solution on the time interval $[0, 0.1]$ with time steps of size $k = 0.0005$
- Set $p = 1$ kPa at the inflow and $p = 0$ at the outflow
- The density of water is $\rho = 1000$ kg/m^3 and the viscosity is $\mu = 0.001002$ kg/(m $\cdot$ s)
- To check your answer, compute the average velocity in the x-direction:

$$\bar{u}_x = \frac{1}{|\Omega|} \int_\Omega u \cdot (1, 0) \, \mathrm{d}x$$

The student(s) who first produce the right answer will be rewarded with an exclusive FEniCS surprise!