# ON THE EFFECTS OF UNIFORM SUCTION ON THE STEADY FLOW DUE TO A ROTATING DISK

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SUMMARY

The exact ordinary differential equations of von Kármán for the flow due to a rotating disk of infinite radius are integrated for the case of uniform suction through the disk. In the analysis a suction parameter $a$ is introduced, where $a\sqrt{(\nu\omega)}$ is the velocity of suction, $\nu$ being the kinematic viscosity and $\omega$ the angular velocity of the disk. For $a = 1$ the equations are integrated numerically, but for higher values of $a$ a series solution in descending powers of $a$ is obtained.

The magnitude of the radial component of flow is found to decrease rapidly as the suction increases, while at the disk the derivative of the tangential component—with respect to distance from the disk—increases. If the ratio of distance from disk to displacement thickness is used as the dimensionless independent variable, the change of the radial component with suction is seen to occur mainly in the velocity scale, with little change of shape, while the tangential component of flow changes very little.



## Symbols

| | | | |
|---|---|---|---|
| $a$ | suction parameter. | $\zeta$ | $z(\omega/\nu)^{\frac{1}{2}}$. |
| $r, \phi, z$ | cylindrical polar coordinates. | $\eta$ | $a\zeta$. |
| $u, v, w$ | corresponding velocity components | $\delta^*$ | displacement thickness. |
| $F, G, H$ | reduced velocity components. | $\theta$ | momentum thickness. |
| $\omega$ | angular velocity. | $H^*$ | $\delta^*/\theta$. |
| $\nu$ | kinematic viscosity. | $H_\infty$ | limiting value of $H(\zeta)$ at infinity. |
| $\rho$ | density. | $\lambda$ | angle of yaw in flow. |
| $p$ | pressure. | $\lambda_\infty$ | limiting value of $\lambda$ at infinity. |
| $P$ | reduced pressure. | $\mu$ | $z/\delta^*$. |
| | | $F_{\max}$ | maximum value of $F$. |
| | | $M$ | $4F_{\max}$. |

## 1. Introduction

It has been shown by von Kármán (1) that the equations of steady flow of a viscous, incompressible fluid due to an infinite rotating disk can be reduced to a set of ordinary differential equations. He solved them by an approximate integral method, while later Cochran (2) integrated the equations numerically for the case of zero normal velocity at the disk. The inverse problem, the case of an infinite plane at rest in fluid rotating with uniform angular velocity at large distances from the plane, has been solved numerically by Bödewadt (3) (see also Schlichting (4), p. 148). The solutions mentioned above are particular cases of a general class of rotationally symmetric flows which has been discussed by Batchelor (5).

The simplified Navier–Stokes equations derived by von Kármán are applicable also to the case when there is a uniform flow through the surface of the disk, as has been pointed out by Batchelor (5). In the present paper the case of suction is examined, but the case of blowing through the disk is omitted. For low values of the suction parameter $a$ the solution may be obtained numerically, but for higher values of $a$ a series solution in descending powers of the parameter is derived. In the present paper a numerical solution is obtained for one particular value of the suction parameter ($a = 1$), while for the other values considered the series is used. The results are discussed in section 5.

## 2. Statement of the problem

Cylindrical polar coordinates are used, $r$ being the radial distance from the axis of rotation of the disk, $\phi$ the polar angle in the direction of rotation, and $z$ the normal distance from the disk. The respective velocity components are $u$, $v$, $w$, while $\omega$ is the angular velocity. In addition, $p$ denotes pressure, $\rho$ density, and $\nu$ kinematic viscosity. Then the Navier–Stokes equations of motion (4) in these coordinates are

$$u\frac{\partial u}{\partial r}+\frac{v}{r}\frac{\partial u}{\partial\phi}+w\frac{\partial u}{\partial z}-\frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left(\nabla^2 u-\frac{u}{r^2}-\frac{2}{r^2}\frac{\partial v}{\partial\phi}\right), \tag{2.1}$$

$$u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial\phi}+w\frac{\partial v}{\partial z}+\frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\phi}+\nu\left(\nabla^2 v+\frac{2}{r^2}\frac{\partial u}{\partial\phi}-\frac{v}{r^2}\right), \tag{2.2}$$

$$u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial\phi}+w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\nabla^2 w, \tag{2.3}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(ru)+\frac{1}{r}\frac{\partial v}{\partial\phi}+\frac{\partial w}{\partial z} = 0, \tag{2.4}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}+\frac{\partial^2}{\partial z^2}. \tag{2.5}$$

With von Kármán's substitutions, namely

$$\left.\begin{aligned} u = r\omega F(\zeta), \qquad v = r\omega G(\zeta), \qquad w = (\nu\omega)^{\frac{1}{2}}H(\zeta) \\ p = \rho\nu\omega P(\zeta), \qquad \zeta = (\omega/\nu)^{\frac{1}{2}}z \end{aligned}\right\}, \tag{2.6}$$

equations (2.1) to (2.4) become

$$\left.\begin{aligned} F^2-G^2+HF' &= F'' \\ 2FG+HG' &= G'' \\ 2F+H' &= 0 \end{aligned}\right\}, \tag{2.7}$$

$$-HH'-2F' = P', \tag{2.8}$$

where primes denote differentiation with respect to $\zeta$.

Since $H$ is independent of $r$, these equations are applicable to the case of uniform flow ($w_0$) through the disk, $H$ then taking a constant non-zero value at $\zeta = 0$. The boundary conditions for uniform suction through the rotating disk are

$$\left.\begin{aligned} F = 0, \qquad G = 1, \qquad H = -a \quad \text{at } \zeta = 0 \\ F = G = 0 \quad \text{at } \zeta = \infty \end{aligned}\right\}, \tag{2.9}$$

where $a$ is a positive constant.

In suction problems it is usual to introduce a suction parameter $w_0 R^{\frac{1}{2}}/U_0$, where $w_0$ is the velocity of suction, $U_0$ a representative velocity, and $R$ the Reynolds number. In our case, since

$$w_0 = a\sqrt{(\nu\omega)}, \qquad U_0 = r\omega, \qquad R = r^2\omega/\nu, \tag{2.10}$$

$a$ is the appropriate suction parameter.

Equation (2.8) is immediately integrable and yields

$$P = \text{constant} - \tfrac{1}{2}H^2 - 2F, \tag{2.11}$$

while equations (2.7) have to be integrated subject to the five conditions (2.9).

For small values of $a$ a numerical solution may be obtained, the value $a = 1$ being selected for illustration in section 4. A solution valid for fairly high values of $a$, of order 2 or greater, is developed in section 3.

## 3. Solution in inverse powers of the suction parameter

Let us first of all examine the form which the solution takes for very large values of $a$. On physical grounds, we expect $H$ to be nearly constant for large values of $a$, and therefore put $H = -a$, with its consequences, $H' = F = 0$, in equations (2.7).

To a first approximation, then, these equations become

$$F'' = -aF' - G^2, \tag{3.1}$$

$$G'' = -aG', \tag{3.2}$$

the boundary conditions being the relevant ones of (2.9). The solution of (3.2) which satisfies (2.9) is

$$G = e^{-a\zeta}. \tag{3.3}$$

After substituting (3.3) in (3.1), we obtain an equation for $F$, the solution of which is

$$F = \frac{1}{2a^2}(e^{-a\zeta}-e^{-2a\zeta}). \tag{3.4}$$

The zero value of $F$ assumed initially is seen to be an approximation to (3.4) for large $a$.

The functions (3.3) and (3.4), together with $H = -a$, may be regarded as a first approximation to the solution of (2.7) for large $a$. The form of this solution suggests two things: (1) that a solution may be obtained in descending powers of $a$; and (2) that $a\zeta$ is more suitable than $\zeta$ as the independent variable.

We therefore define $$\eta \equiv a\zeta \tag{3.5}$$

and transform equations (2.3) to

$$\left.\begin{aligned} a^2F'' &= F^2+aF'H-G^2 \\ a^2G'' &= 2FG+aG'H \\ 0 &= 2F+aH' \end{aligned}\right\}, \tag{3.6}$$

*where primes denote differentiation with respect to* $\eta$ *(and not* $\zeta$*) in the remainder of this section only.*

A solution is assumed of the form

$$\left.\begin{aligned} H &= -a+\sum_{r=0}^{\infty} a^{-r}H_r(\eta) \\ F &= \sum_{r=0}^{\infty} a^{-r}F_r(\eta) \\ G &= \sum_{r=0}^{\infty} a^{-r}G_r(\eta) \end{aligned}\right\}. \tag{3.7}$$

After substituting these series in (3.6) and equating successive powers of $a$ to zero, we obtain the following set of equations:

$$F_0''+F_0' = 0,$$

$$F_1''+F_1' = F_0'H_0,$$

$$F_2''+F_2' = F_0^2-G_0^2+F_0'H_1+F_1'H_0,$$

$$F_{2n}''+F_{2n}' = F_{n-1}^2-G_{n-1}^2+2\sum_{r=0}^{n-2}(F_rF_{2n-2-r}-G_rG_{2n-2-r})+$$
$$+\sum_{r=0}^{2n-1}F_r'H_{2n-1-r} \quad (n \geqslant 2),$$

$$F''_{2n+1}+F'_{2n+1} = 2\sum_{r=0}^{n-1}(F_r F_{2n-1-r}-G_r G_{2n-1-r})+\sum_{r=0}^{2n} F'_r H_{2n-r} \quad (n \geqslant 1),$$

$$G''_0+G'_0 = 0,$$

$$G''_1+G'_1 = H_0 G'_0,$$

$$G''_n+G'_n = 2\sum_{r=0}^{n-2} F_r G_{n-2-r}+\sum_{r=0}^{n-1} H_r G'_{n-1-r} \quad (n \geqslant 2),$$

$$H'_0 = 0,$$

$$H'_n = -2F_{n-1} \quad (n \geqslant 1). \tag{3.8}$$

The relevant boundary conditions are

$$\left.\begin{array}{l} F_\nu = 0, \quad G_0 = 1, \quad G_\mu = 0, \quad H_\nu = 0 \quad \text{at } \eta = 0 \\ F_\nu = 0, \quad G_\nu = 0 \quad \text{at } \eta = \infty \\ \text{where} \quad \nu = 0, 1, 2, 3, 4,..., n,... \\ \mu = 1, 2, 3, 4,..., n,... \end{array}\right\}. \tag{3.9}$$

After the above equations have been solved in succession, we obtain the following series for $F$, $G$, and $H$:

$$F = \frac{1}{2a^2}(e^{-\eta}-e^{-2\eta})+$$
$$+\frac{1}{a^6}[(-\tfrac{1}{4}\eta-\tfrac{59}{144})e^{-\eta}+(\tfrac{1}{2}\eta+\tfrac{7}{24})e^{-2\eta}+\tfrac{1}{8}e^{-3\eta}-\tfrac{1}{144}e^{-4\eta}]+$$
$$+\frac{1}{a^{10}}[(\tfrac{1}{16}\eta^2+\tfrac{319}{576}\eta+\tfrac{28369}{34560})e^{-\eta}+(-\tfrac{1}{4}\eta^2-\tfrac{95}{96}\eta-\tfrac{197}{384})e^{-2\eta}+$$
$$+(-\tfrac{3}{16}\eta-\tfrac{173}{576})e^{-3\eta}+(\tfrac{1}{72}\eta-\tfrac{11}{864})e^{-4\eta}+\tfrac{13}{2304}e^{-5\eta}-\tfrac{7}{17280}e^{-6\eta}]+$$
$$+O\left(\frac{1}{a^{14}}\right), \tag{3.10}$$

$$G = e^{-\eta}+\frac{1}{a^4}[\tfrac{1}{12}e^{-\eta}-\tfrac{1}{2}\eta e^{-\eta}-\tfrac{1}{12}e^{-3\eta}]+$$
$$+\frac{1}{a^8}[(\tfrac{1}{8}\eta^2+\tfrac{21}{32}\eta-\tfrac{173}{1152})e^{-\eta}+\tfrac{1}{8}(1+\eta)e^{-3\eta}+\tfrac{1}{36}e^{-4\eta}-\tfrac{1}{384}e^{-5\eta}]+$$
$$+O\left(\frac{1}{a^{12}}\right), \tag{3.11}$$

$$
\begin{aligned}
H = {} & -a+\frac{1}{a^3}\left(-\tfrac{1}{2}+e^{-\eta}-\tfrac{1}{2}e^{-2\eta}\right)+ \\
& +\frac{1}{a^7}\left[\tfrac{201}{288}+\left(-\tfrac{1}{2}\eta-\tfrac{95}{72}\right)e^{-\eta}+\left(\tfrac{1}{2}\eta+\tfrac{13}{24}\right)e^{-2\eta}+\tfrac{1}{12}e^{-3\eta}-\tfrac{1}{288}e^{-4\eta}\right]+ \\
& +\frac{1}{a^{11}}\left[-\tfrac{31073}{12960}+\left(\tfrac{1}{8}\eta^2+\tfrac{201}{288}\eta+\tfrac{51829}{17280}\right)e^{-\eta}+\left(-\tfrac{1}{4}\eta^2-\tfrac{119}{96}\eta-\tfrac{145}{128}\right)e^{-2\eta}+\right. \\
& \qquad \left.+\left(-\tfrac{1}{8}\eta-\tfrac{209}{864}\right)e^{-3\eta}+\left(\tfrac{1}{144}\eta-\tfrac{1}{216}\right)e^{-4\eta}+\tfrac{13}{5760}e^{-5\eta}-\tfrac{7}{51840}e^{-6\eta}\right]+ \\
& +O\left(\frac{1}{a^{15}}\right).
\end{aligned} \tag{3.12}
$$

The numerical results for particular values of $a$ as given by these formulae are discussed in section 5.

## 4. Numerical solution for small values of the suction parameter

For zero suction, Cochran (2) integrated equations (2.7) by the Adams–Bashforth method (6). In order to bridge the gap between this solution for $a = 0$ and that of the previous section for $a$ of order 2, equations (2.7) have been solved for $a = 1$ by Cochran's method. The equations are first of all converted to a set of five first-order non-linear equations, which have to satisfy the five conditions (2.9). The solution near to $\zeta = 0$ can then be obtained in the form of series containing two arbitrary constants $a_1$ and $b_1$:

$$
\begin{aligned}
F &= a_1\zeta+\dots, \\
G &= 1+b_1\zeta+\dots, \\
H &= -a-a_1\zeta^2+\dots.
\end{aligned} \tag{4.1}
$$

For particular values of $a_1$ and $b_1$ the solution is then developed by forward integration in the Adams–Bashforth manner.

There is also an asymptotic solution (2) for large $\zeta$ of the form

$$
\begin{aligned}
F &\sim Ae^{-c\zeta}+\dots, \\
G &\sim Be^{-c\zeta}+\dots, \\
H &\sim -c+(2A/c)e^{-c\zeta}+\dots,
\end{aligned} \tag{4.2}
$$

where $A$, $B$, and $c$ are constants, and $-c$ represents the limiting value of $H$ for large $\zeta$. The constants $a_1$, $b_1$, $A$, $B$, $c$ are chosen by trial and error so that the solution obtained by forward integration links properly with the asymptotic solution (4.2).

For the case of $a = 1$ the following values of the constants are applicable:

$$a_1 = 0{\cdot}389,\ b_1 = -1{\cdot}175,\ A = 0{\cdot}334,\ B = 1{\cdot}034,\ c = 1{\cdot}259. \quad (4.3)$$

The method gives the functions $F$, $G$, $H$, $F'$, and $G'$ directly in numerical form. The results for $F$ and $G$ are given graphically and discussed later.

It was mentioned in the introduction that von Kármán devised an integral (or momentum) method of solving equations (2.7). Formulae analogous to his can be used to check the above calculations for $a = 1$. By integrating the first equation of (2.7) by parts and using the third with the conditions (2.9), we obtain

$$F'(0) = a_1 = \int\limits_0^\infty (G^2 - 3F^2)\, d\zeta. \quad (4.4)$$

And, similarly, we derive

$$G'(0) = b_1 = -a - \int\limits_0^\infty 4FG\, d\zeta. \quad (4.5)$$

For the case $a = 1$ integration of $(G^2 - 3F^2)$ by Simpson's rule yields $a_1 = 0{\cdot}3897$, showing a difference of $0{\cdot}18$ per cent. compared with the value of (4.3). Integration of $4FG$ gives $b_1 = -1{\cdot}174$, again showing a very small difference compared with the value of (4.3).

## 5. The nature of the fluid motion

A rotating disk acts as a centrifugal fan, the radial flow being balanced by an induced axial flow towards the rotating disk. When suction is applied the radial flow is decreased, while the axial flow at infinity towards the disk is larger. The effect of suction on the tangential component is much as for two-dimensional flow, the boundary layer being thinned.

For a range of values of $a$, the functions $F$ and $G$ are plotted against $\zeta$ in Figs. 1 and 2, Cochran's curves also being given for comparison. For $a = 1$ the curves are those obtained numerically by the method of section 4. It may be mentioned here that (3.10) gives $F$ accurately to four decimal places at $a = 2$, while (3.11) gives $G$ accurately to three decimal places. The considerable reduction in the three-dimensional character of the flow is shown in Fig. 1. The radial component shows a rapid decrease in magnitude with increase of suction, the reduction compared with $a = 0$ being more than 50 per cent. for $a = 1$ and even larger for higher values of $a$. The maximum values of $F$ are given below in Table 2. The thinning of the boundary layer is shown by both Fig. 1 and Fig. 2.

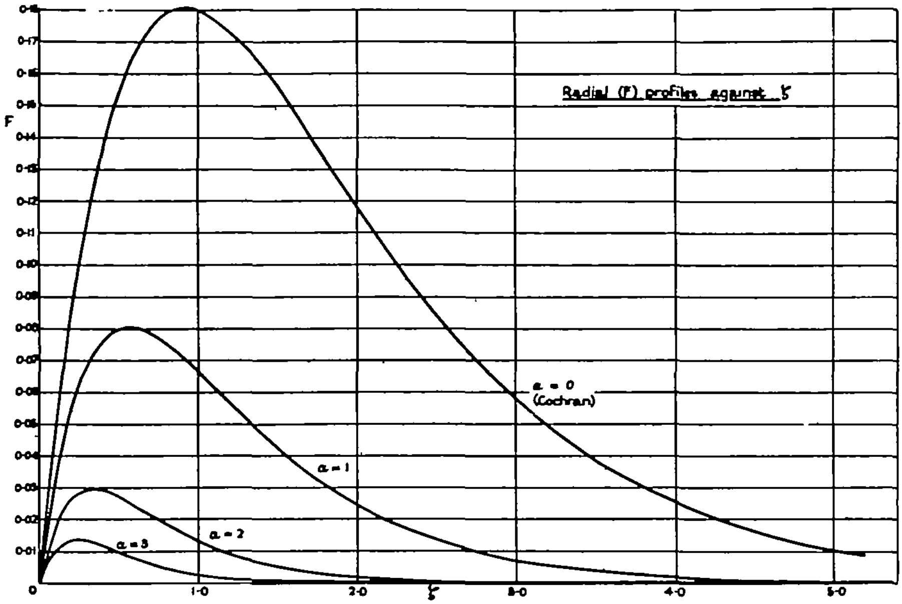


FIG. 1. Radial ($F$) profiles against $\zeta$.

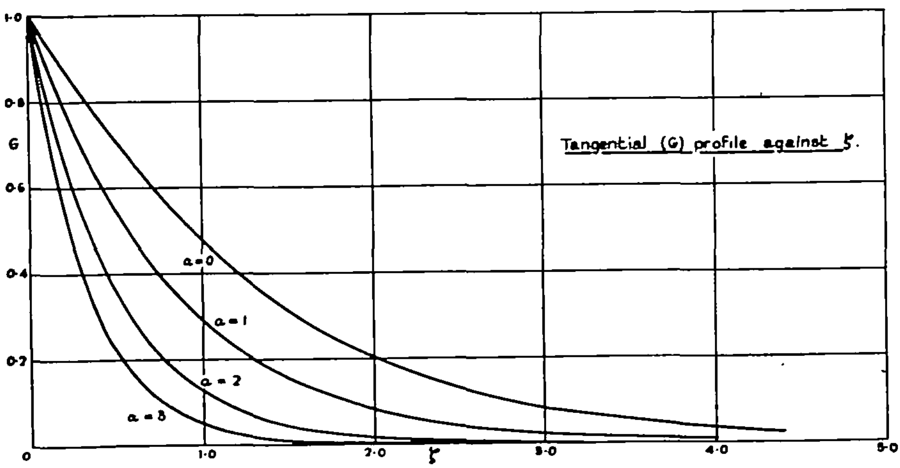


FIG. 2. Tangential ($G$) profiles against $\zeta$.

*Standard boundary-layer lengths and parameters*

A more precise measure of the reduction in boundary-layer thickness may be obtained by consideration of a displacement thickness. In three-dimensional boundary-layer theory it is the practice to define two 'displacement thicknesses', corresponding to two perpendicular directions (7). In our case the obvious directions are along the radius and tangential to it; but since the radial flow is zero both at the disk and at infinity, the radial 'displacement thickness' is meaningless. The tangential component of flow gives

$$\delta^* = \int_0^\infty (v/\omega r)\,dz, \tag{5.1}$$

in the notation of Fig. 1. With the transformation (2.6) this becomes

$$\delta^* = (\nu/\omega)^{\frac{1}{2}} \int_0^\infty G\,d\zeta. \tag{5.2}$$

Similarly, we have a momentum thickness

$$\theta = (\nu/\omega)^{\frac{1}{2}} \int_0^\infty G(1-G)\,d\zeta. \tag{5.3}$$

We define also
$$H^* = \delta^*/\theta, \tag{5.4}$$

the superscript being introduced to avoid a clash with the notation for the flow function $H(\zeta)$.

Within the range of validity of (3.11), (5.2) and (5.3) can be integrated to give

$$\delta^*(\omega/\nu)^{\frac{1}{2}} = \frac{1}{a}\left(1-\frac{4}{9a^4}+\frac{589}{720a^8}+\dots\right), \tag{5.5}$$

$$\theta(\omega/\nu)^{\frac{1}{2}} = \frac{1}{2a}\left(1-\frac{5}{12a^4}+\dots\right), \tag{5.6}$$

while the ratio yields
$$H^* = 2\left(1-\frac{1}{36a^4}+\dots\right). \tag{5.7}$$

For $a = 0$ and 1, the $G$ function can be integrated numerically to give $\delta^*$, $\theta$, and $H^*$. The results are given in Table 1.

TABLE 1

| $a$ | 0 | 1 | 2 | 3 | 4 | $\infty$ |
|---|---|---|---|---|---|---|
| $\delta^*(\omega/\nu)^{\frac{1}{2}}$ | 1·271 | 0·811 | 0·488 | 0·331 | 0·250 | 0 |
| $\theta(\omega/\nu)^{\frac{1}{2}}$ | 0·599 | 0·401 | 0·244 | 0·166 | 0·125 | 0 |
| $H^*$ | 2·122 | 2·022 | 2·000 | 2·000 | 2·000 | 2 |

*Further precise properties of the fluid motion*

The axial flow $H(\zeta)$ is not plotted, but its limiting value at infinite distances from the disk, $H_\infty$, is given in Table 2 for a series of values of $a$. It is noted that $H_\infty + a$, the change of $H$ between $\zeta = 0$ and $\zeta = \infty$, tends to zero as $a$ tends to infinity. Within the range of $a$ greater than 2, $H_\infty$ is given by (3.12), namely

$$H_\infty = -a - \tfrac{1}{2}a^{-3} + \tfrac{201}{288}a^{-7} - \tfrac{21023}{12960}a^{-11} + \dots. \tag{5.8}$$

Let us define $\lambda$ as the angle between the direction of motion of the disk and the stream line projected onto the plane of the disk. Then $\lambda$ varies with the distance from the disk; it is zero at the disk, but has a finite limiting value $\lambda_\infty$ at large $\zeta$ given by

$$\lambda_\infty = \lim_{\zeta \to \infty} \tan^{-1}(F/G). \tag{5.9}$$

For $a = 1$ the value of $\lambda_\infty$ is derivable from (4.2) with (4.3), while a similar procedure suffices for Cochran's (2) case of $a = 0$. And within the range of (3.10) and (3.11) the series are applicable. These results are given in Table 2, where it is seen that $\lambda_\infty$ decreases rapidly with increase of suction and eventually tends to zero.

TABLE 2

| $a$ | 0 | 1 | 2 | 3 | 4 | $\infty$ |
|---|---|---|---|---|---|---|
| $F_{max}$ | 0·181 | 0·080 | 0·0295 | 0·0136 | 0·0078 | 0 |
| $-H_\infty$ | 0·886 | 1·259 | 2·057 | 3·019 | 4·008 | 0 |
| $-(H_\infty + a)$ | 0·886 | 0·259 | 0·057 | 0·019 | 0·008 | 0 |
| $\lambda_\infty$ | 37° 42′ | 17° 54′ | 6° 47′ | 3° 10′ | 1° 48′ | 0° |

In many respects $\mu = z/\delta^*$ is more suitable than $\zeta$ for comparing velocity profiles. Thus in Fig. 3 the function $F$ is plotted against $\mu$, while in Fig. 4 the $G$ profiles for zero and infinite suction are plotted. The maxima of $F$ are at about the same value of $\mu$, and a large part of the change of $F$ with suction is seen to be one of scale. To show this the $F$ profile for large $a$, namely

$$F = \frac{1}{2a^2}(e^{-\mu} - e^{-2\mu}), \tag{5.10}$$

is scaled up to lie near the $F$ curve for $a = 0$; the closeness is remarkable. Similarly, the $F$ curves for $a = 1$, 2, 3 do not differ much from suitably scaled versions of (5.10). In deriving (5.10) one notes from (5.5) that $\eta$ tends to $\mu$ for large $a$.

The curve for $G$ against $\mu$ at $a = \infty$, which is $G = \exp(-\mu)$, is not very different from that for $a = 0$, only one suction curve being plotted as they

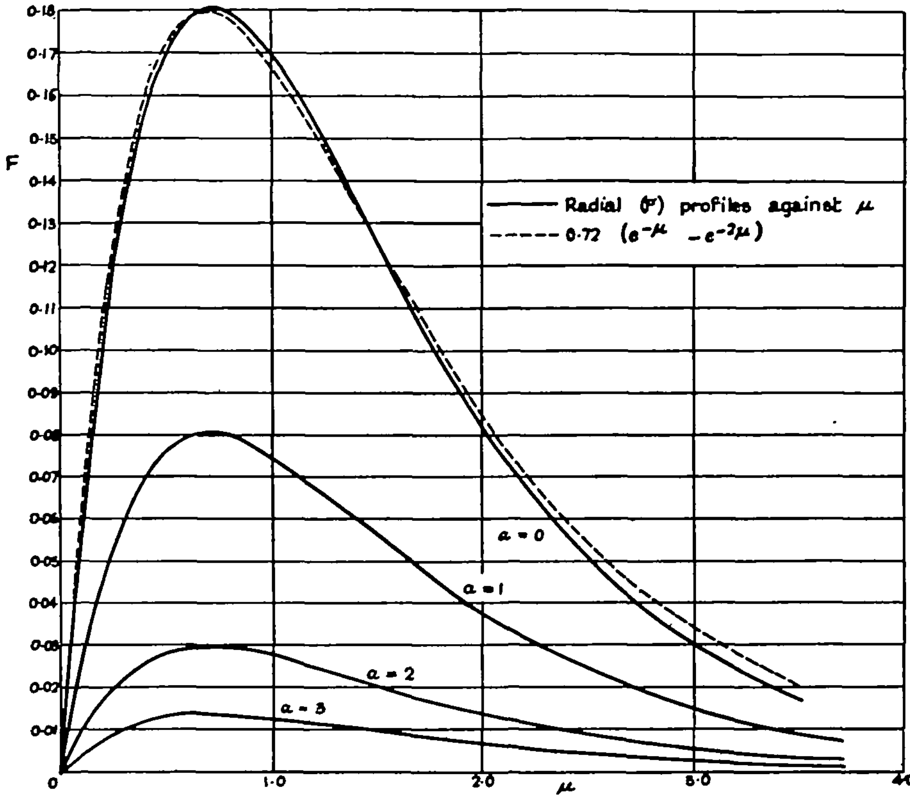


FIG. 3. Radial ($F$) profiles against $\mu$.

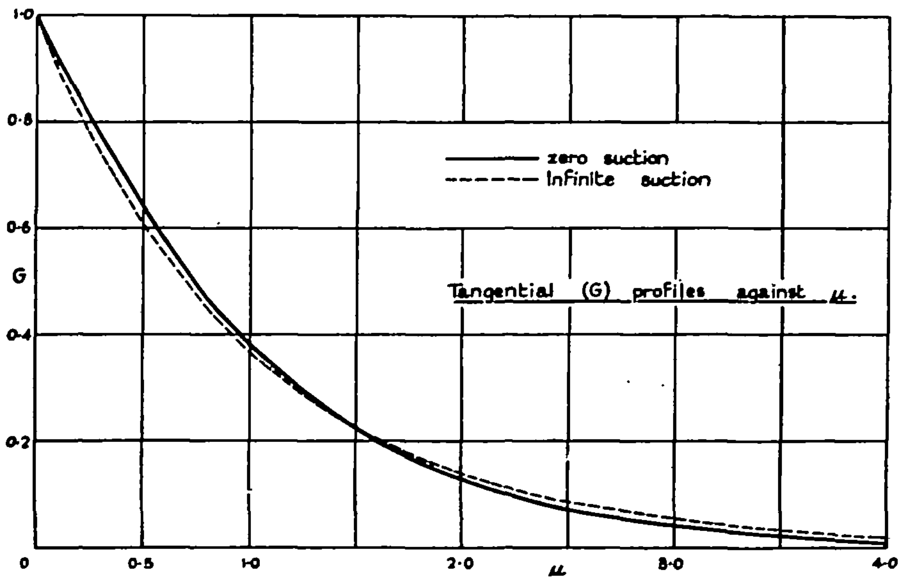


FIG. 4. Tangential ($G$) profiles against $\mu$.

all lie close to that for $a = \infty$. But the qualitative change from $a = 0$ to $a = \infty$ is similar to that for two-dimensional flow. For a Blasius distribution the parameter $H^*$ changes from about 2·6 for zero suction to 2 for the asymptotic suction profile, whereas here $H^*$ changes from 2·122 to 2. This small change of $H^*$ is seen to be in agreement with the closeness of the curves of Fig. 4, since the value of $H^*-2$ is a measure of the closeness of a velocity profile to the corresponding asymptotic suction profile. It is shown by Figs. 3 and 4 that, in the variable $\mu$, the velocity distribution with or without suction can be represented approximately by

$$F = M(e^{-\mu} - e^{-2\mu}),\quad G = e^{-\mu}, \tag{5.11}$$

where

$$M = 4F_{\max}. \tag{5.12}$$

These formulae become increasingly accurate as $a$ increases. But, of course, these approximate formulae do not hold as closely for derivatives of $F$ and $G$ as they do for the actual functions $F$ and $G$.

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