Von Karman Viscous Pump

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Abstract

Consider a tube filled with a newtonian viscous fluid initially at rest. Upon applying a pressure-gradient, a pressuredriven flow will occur known as Hagen–Poiseuille flow. Limiting ourselves to steady state case, we will derive an analytical solution for this problem. Our goal is to validate the Oasis software, a high-performance Navier-Stokes solver based on the opensoure Fenics software. The solver is based on the finite element method and implemented in python.

Navier-Stokes Equation

Let the velocity vector components be defined as $\mathbf{v} = (v_r, v_\theta, v_z)$ The balance of momentum within the fluid introduces the famous Navier-Stokes eqations. By assuming that fluid fluid is incompressible, we get the Navier-Stokes eqations as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \frac{\mu}{\rho} \nabla^2 \mathbf{v} \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

Limiting ourselves to the steady-state case of a fully developed fluid, following component equations.

$$r: \qquad v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2}\right) \tag{3}$$

$$\theta: \qquad v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{1}{r} v_\theta v_r = \frac{\mu}{\rho}\left(\frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial v_\theta}{\partial r} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2}\right) \tag{4}$$

$$z: \qquad v_r \frac{\partial v_z}{\partial r} + \frac{\partial v_z}{\partial z} = \frac{1}{\rho}\frac{\partial p}{\partial z} + \frac{\mu}{\rho}\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2}\right) \tag{5}$$

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Laminar flow

We will further simplify the Navier-Stokes equations by specifying certain conditions for the fluid flow. We will for simplicity look at a constructed test case for a straight tube with constant diameter, with a fully developed laminar flow. From this we can conclude that we have a fluid flow with the property of constant axial velocity, and no radial or angular velocity. Further we will assume that the fluid is driven by a pressure gradient parallel to the axial direction. As a result we have the following conditions.

$$\frac{\partial v}{\partial t} = 0 \tag{6}$$

$$v \cdot \nabla(v) = 0 \tag{7}$$

$$v_r = v_\theta = 0 \tag{8}$$

Now lets imagine extracting a ring-shaped control volume from the fluid flow and assess the forces acting on this volume. Let this control volume have thickness dr and width dx. SETT INN BILDE Due to the balance of momentum the following forces must balance each other. As observed the only acting forces are the pressure and viscous forces, denoted as p and τ

$$2\pi r P_x dr - 2\pi r P_{x+\delta x} dr + 2\pi r dx \tau_r - 2\pi r dx \tau_{r+\delta r} = 0 \quad * \left(\frac{1}{2\pi dr dx}\right) \tag{9}$$

$$r\frac{P_{x+\delta x} - P_x}{dx} + \frac{r(\tau_{r+\delta r} - \tau_r)}{dr} = 0 \tag{10}$$

In the limit dr, dx $\to 0$, we get the following equation

$$r\frac{dP}{dx} + \frac{d(r\tau)}{dr} = 0 \tag{11}$$

Now by replacing τ with $-\mu\frac{\mathrm{u}}{dx}$ where the constant μ denotes the dynamic viscosity. Now, this means that choice of x or r, the relation (10) must be fulfilled. Hence we must conclude that $\frac{dP}{dx}$ must be some constant. This constant can be derived by changing our control volume with a slice at any point in the tube. Using the same relations it can be shown that

$$\frac{dP}{dx} = \frac{-2\tau_W}{R} \tag{12}$$

Where τ_W denotes the wall shear stress.

We now solve the second order differential equation(10), using double integration

$$u(r) = \frac{1}{4\mu}\frac{dP}{dx} + C_1 ln(r) + C_2 \tag{13}$$

To remove the non-physical consequence of $\lim_{r\to 0} ln(r) \to \infty$, we choose $C_1 = 0$. By using our second condition $u(R) = 0$ where R denotes the tube of the radius, the final analytical result yields

$$u(r) = -\frac{1}{4\mu}\frac{dP}{dx}\left(R^2 - r^2\right) \tag{14}$$

1 Computation

Using Oasis, we can choose between two sets of solver methods, coupled and fracstep. **Coupled** is a steady-state solver solving

For the computation we have to construct a mesh to do our calculations upon. In this problem we will use Gmsh, a free 3D finite element grid generator. The problem will be solved on a tube with radius = 1 and length z = 3.

Setting up the problem

Continuing on the reference solution, we derive Karman's method of approximate solution. Firstly we want to observe how the differential equations behave at the limit $\zeta \to \infty$. Assuming H approaches a limit $-c$ as while $F, G \to$ as $\zeta \to \infty$, we end up with the following relations for large ζ.

$$-cG^{'} = G^{''} \quad -cF^{'} = F^{''} \tag{15}$$

$$F = Ae^{-c\zeta}, \quad G = Be^{-c\zeta}, \quad H = -c + \frac{2A}{c}e^{-c\zeta} \tag{16}$$

As we can see, F and G approaches 0 exponentially, and we can assume they are 0 for some value of ζ which we will call ζ_0. This will be exploited in the following integration scheme.

Starting by integrating equation (6)-(7)from 0 to ∞, and using relation (8).

$$\int_0^\infty HF^{'}d\zeta = \left[HF\right]_0^\infty - \int_0^\infty H^{'}Fd\zeta = 2\int_0^\infty F^2 d\zeta$$
$$\int_0^\infty HG^{'}d\zeta = \left[HG\right]_0^\infty - \int_0^\infty H^{'}Gd\zeta = 2\int_0^\infty FGd\zeta$$

Combining these results with (12), we end up with the following result

$$-F^{'}(0) = \int_0^\infty (3F^2 - G^2)d\zeta \tag{17}$$
$$-G^{'}(0) = 4\int_0^\infty FGd\zeta \tag{18}$$

As Karman, we assume that due to the expoential growth that F and G are zero for values of ζ greater than ζ_0. As a result

$$F(\zeta_0) = 0, \quad F^{'}(\zeta_0) = 0, \quad G(\zeta_0) = 0, \quad G'(\zeta_0) = 0 \tag{19}$$

We can now also find $F''(0)$ and $G^{''}(0)$ by setting $\zeta = 0$ in the system of ODE's

$$F^{''}(0) = -1 \quad G^{''}(0) = 0 \tag{20}$$

Now if we let $F^{'}(0)$ is some constant a, the following functions fulfills equation (15)-(16) and the boundary conditions.

$$F = (1 - \frac{\zeta}{\zeta_0})^2 \ \left(a\zeta + (\frac{2a}{\zeta_0})\zeta^2\right) \tag{21}$$
$$G = (1 - \frac{\zeta}{\zeta_0})^2 \ (1 + \frac{\zeta}{2\zeta_0}) \tag{22}$$

This yields $G^{'}(0) = -\frac{3}{2\zeta_0}$. Inserting (17)-(18) in (13)-(14), we get a system of equations to solve a and ζ_0.

Computations

ζ	F	F'	G	G'	H	-P
0.0	0.0	0.51023	1.00	-0.61592	0.0	0.0
0.1	0.0462	0.4163	0.9386	-0.6112	-0.0048	0.0924
0.2	0.0836	0.3338	0.8780	-0.5987	-0.0179	0.1674
0.3	0.1133	0.2620	0.8190	-0.5803	-0.0377	0.2274
0.4	0.1364	0.1999	0.7621	-0.5577	-0.0628	0.2747
0.5	0.1536	0.1467	0.7075	-0.5321	-0.0919	0.3115
0.6	0.1660	0.1015	0.6557	-0.5047	-0.1239	0.3396
0.7	0.1742	0.0635	0.6067	-0.4763	-0.1580	0.3608
0.8	0.1789	0.0317	0.5605	-0.4476	-0.1933	0.3764
0.9	0.1807	0.0056	0.5171	-0.4191	-0.2293	0.3877
1.0	0.1801	-0.0157	0.4766	-0.3911	-0.2655	0.3955

Comments