MEK 4300
Mandatory Assignment

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Taylor-Green Vortex

Abstract

Physical problem

The Taylor-Green vortex is a unsteady flow were we observe the flow of a decaying vortex. This flow has an exact solution for the incrompessible Navier-Stokes equation in 2D, while for the 3D case there are several numerical results for comparison. Assuming the fluid is incompressible, the exact solution for velocity and pressure are given as ($t = 0$)

$$u(x, y, t) = (V_0 cos(\pi x) sin(\pi y) e^{-2t\nu\pi^2},\ \ V_0 cos(\pi y) sin(\pi x) e^{-2t\nu\pi^2}) \tag{1}$$

$$p(x, y, t) = -0.25(cos(2\pi x) + cos(2\pi y)) e^{-4t\nu\pi^2} \tag{2}$$

For the 3D case we will consider the kinetic energy E_k and kinetic energy dissipation rate ϵ for the system. These quantities are explored thoroughly by other authors, and will be used as comparison. The initial field set up in the 3D Taylor-Green vortex is defined as

$$u(x, y, z) = (V_0 sin(\frac{x}{L}) cos(\frac{y}{L}) cos(\frac{z}{L}),\ \ -V_0 cos(\frac{x}{L}) sin(\frac{y}{L}) cos(\frac{z}{L}),\ \ 0) \tag{3}$$

$$p(x, y, z) = \rho_0 + \frac{\rho_0 V_0^2}{16}(cos(\frac{2x}{L}) + cos(\frac{2y}{L}))(cos(\frac{2z}{L}) + 2) \tag{4}$$

Exploring the incompressible flow condition we define $\rho_0 = \rho$, and for simplicity we let $V_0 = 1$

Governing Equation and Computations

The incompressible Naiver-Stokes equation describes the flow motion, from the principles of conservation of momentum and continuum.

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \nu \nabla^2 \mathbf{v} \tag{5}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{6}$$

There is a open sea full of different approaches to solve this non-linear equation. We will for this time explore Chorins method, and the incremental pressure correction scheme ($IPCS$).

We will use the FEniCS project, a open-source PDE solver using the finite element method approach. As verification of our schemes, the known analytical solution for the 2D Taylor-Green Vortex will be used as comparison. Finally we will move on to the 3D Taylor-Green vortex, using ... as reference.

The Reynolds number, discovered Osborne Reynolds as the relation between inertial and viscous forces, is defined as

$$Re = \frac{\rho V_0 L}{\mu} = \frac{V_0 L}{\nu} \tag{7}$$

Where ν denotes the kinematic viscosity, while U_0 and D is some characteristic velocity and lenght.

We define the kinetic energy as $E_k = \frac{1}{2}||u||^2_{L^2}$, and the kinetic energy dissipation rate $\epsilon = \frac{-dE_k}{dt}$

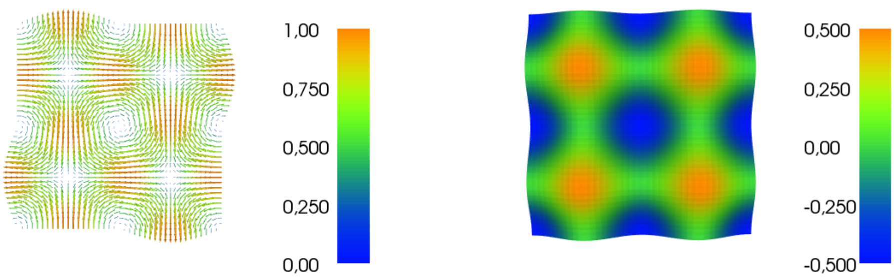
Setting up 2D problem

For the 2D case, the computational domain is set as $\Omega \in [-1, 1]^2$. Further on we will set the flow conditions as

Physical Quantity	Value
Reynolds Number, Re	1000
Characteristic length, L	2
Characteristic velocity, V_0	1
Time step Δt	0.001
End time, T	1.0

Using our analytical solution for time $t = 0$, we set up the initial condition for the domain Ω

Initial velocity field



Results

2D case Oasis Runtime: 3.8886

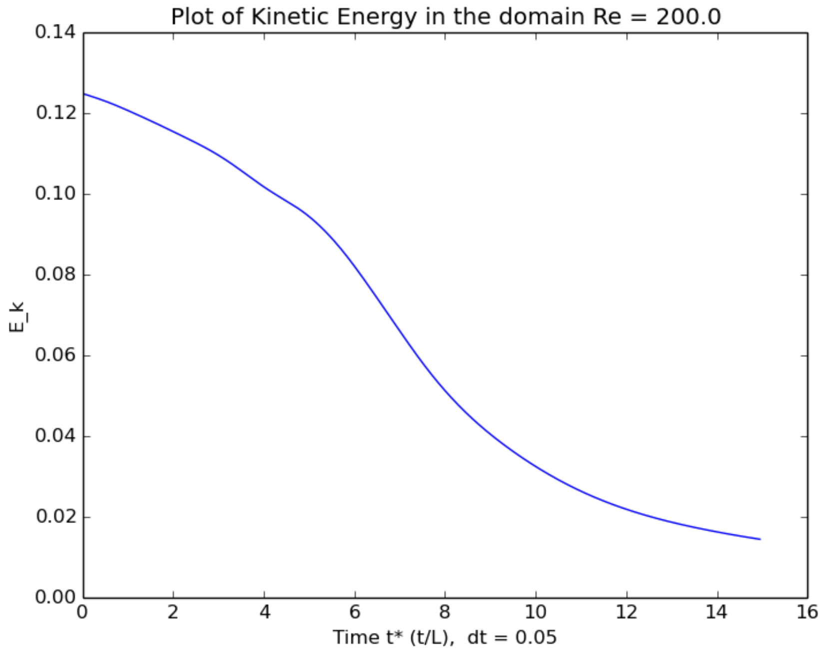
N	dt = 0.1	Runtime	dt = 0.01	Runtime	dt = 0.001	Runtime
8	0.539835	0.164657	0.180778	0.519237	0.178371	4.76148
16	2.04897	0.179117	0.0183534	1.0009	0.0184022	9.40705
32	0.104095	0.400865	0.00130196	2.85381	0.0012364	53.0239
64	0.0210904	1.26767	0.00484368	10.0555	0.000161193	160.618

Chorin own implementation 7.039

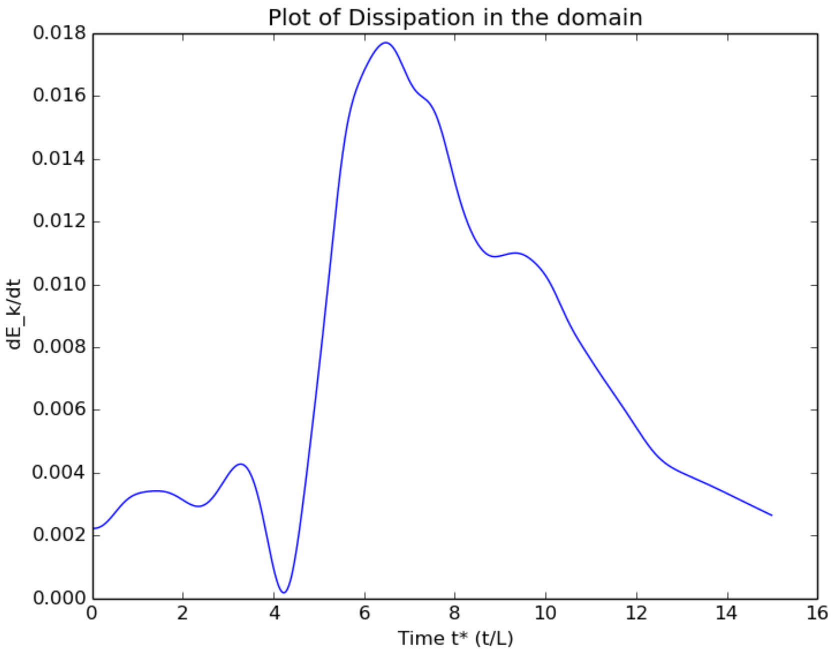
Setting up 3D problem

The computational domain is defined as a cube with sides of length $2\pi L$, $-\pi L \leq x, y, z \leq \pi L$

Kinetic Energy



Dissipation Energy



Physical Quantity	Value
Reynolds Number, Re	1000
Characteristic length, L	1
Characteristic velocity, V_0	1
Time step Δt	0.001
End time, T	1.0