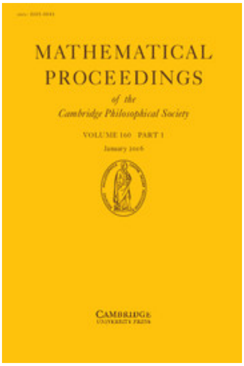

The flow due to a rotating disc

W. G. Cochran

Mathematical Proceedings of the Cambridge Philosophical Society / Volume 30 / Issue 03 / July 1934, pp 365 - 375
DOI: 10.1017/S0305004100012561, Published online: 24 October 2008

The flow due to a rotating disc. By W. G. COCHRAN, B.A., St John's College. (Communicated by Dr S. GOLDSTEIN.)

[*Received* 21 May, *read* 28 May 1934.]

1. The steady motion of an incompressible viscous fluid, due to an infinite rotating plane lamina, has been considered by Kármán *. If r, θ, z are cylindrical polar coordinates, the plane lamina is taken to be $z=0$; it is rotating with constant angular velocity ω about the axis $r=0$. We consider the motion of the fluid on the side of the plane for which z is positive; the fluid is infinite in extent and $z=0$ is the only boundary. If u, v, w are the components of the velocity of the fluid in the directions of r, θ and z increasing, respectively, and p is the pressure, then Kármán shows that the equations of motion and continuity are satisfied by taking

$$u=rf(z),\quad v=rg(z),\quad w=h(z),\quad p=p(z). \tag{1}$$

The boundary conditions are $u=0$, $v=\omega r$, $w=0$ at $z=0$, and $u=0$, $v=0$ at $z=\infty$. w does not vanish at $z=\infty$, but must tend to a finite negative limit. For the rotating lamina acts as a kind of centrifugal fan; the fluid moves radially outwards, especially near the lamina, and there must, therefore, be an axial motion towards the lamina in order to preserve continuity.

If $\mathbf{v}$ is the velocity of the fluid, ρ its density and ν its kinematic viscosity, the vector equation of motion of the incompressible fluid is (ρ being constant)

$$\operatorname{grad} \tfrac{1}{2}\mathbf{v}^2 - \mathbf{v} \wedge \operatorname{curl} \mathbf{v} = -\operatorname{grad} p/\rho - \nu \operatorname{curl}\operatorname{curl} \mathbf{v}, \tag{2}$$

and the equation of continuity is

$$\operatorname{div} \mathbf{v} = 0. \tag{3}$$

With components as in equations (1) these give the four equations

$$\left.\begin{aligned} f^2 - g^2 + h\frac{df}{dz} &= \nu\frac{d^2f}{dz^2}, \\ 2fg + h\frac{dg}{dz} &= \nu\frac{d^2g}{dz^2}, \\ h\frac{dh}{dz} &= -\frac{1}{\rho}\frac{dp}{dz} - 2\nu\frac{df}{dz}, \\ 2f + \frac{dh}{dz} &= 0, \end{aligned}\right\} \tag{4}$$

* *Zeitschrift für angewandte Mathematik u. Mechanik*, **1** (1921), 244–7.

and the boundary conditions are

$$f(0)=0,\ g(0)=\omega,\ h(0)=0,\ f(\infty)=0,\ g(\infty)=0. \quad (5)$$

To obtain the equations in a non-dimensional form, put

$$f=\omega F,\ g=\omega G,\ h=(\nu\omega)^{\frac{1}{2}} H,\ p=\rho\nu\omega P,\ z=(\nu/\omega)^{\frac{1}{2}}\zeta. \quad (6)$$

The equations become

$$\left.\begin{aligned} F^2-G^2+HF'&=F'',\\ 2FG+HG'&=G'',\\ HH'+2F'&=P',\\ 2F+H'&=0,\end{aligned}\right\} \quad (7)$$

and the boundary conditions become

$$F(0)=0,\ G(0)=1,\ H(0)=0,\ F(\infty)=0,\ G(\infty)=0. \quad (8)$$

H must tend to a finite negative limit, say $-c$, as ζ tends to infinity. The first two and the fourth of equations (7) give F, G and H. The third gives P.

If F, $G\to 0$ at infinity, and $H\to -c$, then, to a first approximation, we have, for large values of ζ,

$$-cF'=F'',\quad -cG'=G'', \quad (9)$$

giving $$F=Ae^{-c\zeta},\quad G=Be^{-c\zeta},\quad H=-c+\frac{2A}{c}e^{-c\zeta}. \quad (10)$$

There are, in fact, formal expansions of F, G, H in powers of $e^{-c\zeta}$ (see below). Thus F and G tend to zero exponentially, and are practically zero for some finite value of ζ. Hence, if ν/ω is small, $u/\omega r$ and $v/\omega r$ are appreciable only in a thin layer near the lamina whose thickness is of order $(\nu/\omega)^{\frac{1}{2}}$. Also, from the third of equations (7), if P_0 is the value of P at the plate,

$$P-P_0=\tfrac{1}{2}H^2+2F, \quad (11)$$

and the differences of the pressure p, throughout the layer in which $u/\omega r$ and $v/\omega r$ are sensible, are of order $\rho\nu\omega$. These results agree with the general results of the boundary layer theory for large Reynolds numbers, but for this particular case have been obtained from the equation of motion without any approximation. It is in this that the importance of the solution lies. We may note also that the axial inflow velocity is of the order $(\nu\omega)^{\frac{1}{2}}$.

If we neglect edge effect, we can find the frictional moment on a rotating disc of radius a. The shearing stress is given by

$$p_{z\theta}=\rho\nu\frac{dv}{dz}=\rho(\nu\omega^3)^{\frac{1}{2}}rG'(\zeta), \quad (12)$$

and hence the moment is

$$-\int_0^a 2\pi r^2 p_{z\theta}\, dr = -\tfrac{1}{2}\pi a^4 \rho\, (\nu\omega^3)^{\frac{1}{2}}\, G'(0). \qquad (13)$$

This is the moment for one side only. For both sides the result must be doubled. Hence, in terms of the Reynolds number

$$R = a^2\omega/\nu, \qquad (14)$$

we have for the moment

$$M = -\pi G'(0)\, \rho a^5 \omega^2 / R^{\frac{1}{2}}, \qquad (15)$$

and for the non-dimensional moment coefficient, with $S = \pi a^2$,

$$k_M = \frac{M}{\rho a^3 \omega^2 S} = -G'(0)/R^{\frac{1}{2}}. \qquad (16)$$

The neglect of edge effect is probably justified if the radius is large compared with the thickness of the boundary layer, or with $(\nu/\omega)^{\frac{1}{2}}$. The results of this section were all obtained by Kármán.

2. *Kármán's method of approximate solution.* The system of equations (7) and (8) does not appear to have been integrated accurately. Kármán uses an approximate process which he had invented for the integration of boundary layer equations (cf. Kármán, *loc. cit.* pp. 235, 236; Pohlhausen, *loc. cit.* pp. 252–68). Integrating the first two of the equations (7) between 0 and ∞, and using the relations

$$\left.\begin{aligned} \int_0^\infty HF'd\zeta &= \Big[HF\Big]_0^\infty - \int_0^\infty H'F\,d\zeta = 2\int_0^\infty F^2 d\zeta, \\ \int_0^\infty HG'd\zeta &= \Big[HG\Big]_0^\infty - \int_0^\infty H'G\,d\zeta = 2\int_0^\infty FG\,d\zeta, \end{aligned}\right\} \qquad (17)$$

together with $F'(\infty) = 0$, $G'(\infty) = 0$, we get

$$\left.\begin{aligned} -F'(0) &= \int_0^\infty (3F^2 - G^2)\, d\zeta, \\ -G'(0) &= 4\int_0^\infty FG\,d\zeta. \end{aligned}\right\} \qquad (18)$$

Kármán now assumes that F^2, G^2 and FG fall off sufficiently rapidly so that a good approximation, particularly to $G'(0)$, which occurs in the expression for the moment, can be found by replacing the upper limit in the integrals by some finite value ζ_0 of ζ, or rather by supposing F and G zero for values of ζ greater than ζ_0. We must then have

$$F(\zeta_0) = 0, \quad F'(\zeta_0) = 0, \quad G(\zeta_0) = 0, \quad G'(\zeta_0) = 0. \qquad (19)$$

F and G must also satisfy the conditions at the plate, namely

$$F(0)=0, \quad G(0)=1, \tag{20}$$

and by putting $\zeta=0$ in (7), we obtain the conditions

$$F''(0)=-1, \quad G''(0)=0. \tag{21}$$

Further boundary conditions, obtained by differentiating the equations (7) before putting $\zeta=0$, are ignored, and so are the conditions that the second and higher derivatives of F and G must vanish at $\zeta=\zeta_0$. Then if $F'(0)=a$ $(=\alpha/\xi_0$ in Kármán's notation), expressions satisfying (19), (20) and (21) are

$$\left.\begin{aligned} F &= \left(1-\frac{\zeta}{\zeta_0}\right)^2\left[a\zeta+\left(\frac{2a}{\zeta_0}-\tfrac{1}{2}\right)\zeta^2\right],\\ G &= \left(1-\frac{\zeta}{\zeta_0}\right)^2\left(1+\frac{\zeta}{2\zeta_0}\right), \end{aligned}\right\} \tag{22}$$

so that $G'(0)=-\dfrac{3}{2\zeta_0}$. These expressions are then substituted in (18), with the upper limits in the integrals replaced by ζ_0. Thus we get a pair of simultaneous equations for a and ζ_0, which are solved to give the approximate solution.

There is, however, an error in Kármán's working. In the first of equations (22) he writes the last term

$$\left(\frac{2a}{\zeta_0}-\frac{1}{2\zeta_0^2}\right)\zeta^2$$

instead of

$$\left(\frac{2a}{\zeta_0}-\tfrac{1}{2}\right)\zeta^2.$$

[In addition, the first of Kármán's equations (28) (*loc. cit.* p. 246) should read, in Kármán's notation, even without making the change noticed above,

$$\int_0^{\xi_0} f^2 d\xi=\xi_0[0{\cdot}0301\alpha^2-0{\cdot}00675\alpha+0{\cdot}00040],$$

i.e. the numbers 0·00326 and 0·00159 as given by Kármán should be 0·00675 and 0·00040 respectively.]

Thus the results at which Kármán arrived,

$$\zeta_0=2{\cdot}58, \quad F'(0)=0{\cdot}40, \quad G'(0)=-0{\cdot}58, \quad H(\infty)=-0{\cdot}71,$$

are inaccurate even according to the approximate method*. In fact,

* These results are given, as Kármán stated them, in the *Handbuch der Physik*, 7 (1927), 158, 159; the *Handbuch der Experimental-Physik*, 4 (1931), Part I, 255–7; *Bulletin No.* 84 of the National Research Council; Lamb, *Hydrodynamics* (1932), 280–2; Müller, *Einführung in die Theorie der zähen Flüssigkeiten* (1932), 226–9. A comparison with experiment was given by Kempf, *Vorträge aus dem Gebiete der Hydro- und Aerodynamik* (Innsbruck, 1922) edited by Kármán and Levi-Cività (1924), 168–70.

If $\omega a^2/\nu$ is too large (greater than about 5×10^5), the motion is turbulent.

Kármán's equations (29) should read

$$\left.\begin{aligned} 0{\cdot}09048\alpha^2 - 0{\cdot}02024\alpha + 0{\cdot}00119 &= \frac{1}{\zeta_0^4}(0{\cdot}23571 - \alpha), \\ 0{\cdot}24286\alpha - 0{\cdot}02262 &= \frac{3}{2\zeta_0^4}. \end{aligned}\right\} \quad (23)$$

These give a quadratic equation for α, whose roots are $+0{\cdot}09648$ and $+0{\cdot}19466$. The larger root gives

$$\zeta_0 = 2{\cdot}79, \quad F'(0) = 0{\cdot}54, \quad G'(0) = -0{\cdot}54, \quad H(\infty) = -0{\cdot}55. \quad (24)$$

The smaller root need not be considered since it leads to a function F which in $(0, \zeta_0)$ rises to a maximum, decreases to zero and crosses the ζ-axis to a negative minimum, increasing again to zero at ζ_0. Such a graph is inadmissible as an approximation. The graphs of F, G, H, as obtained from the larger root, are shown by the broken lines in Fig. 1, and those of F', G' and P by the broken lines in Fig. 2.

If we write

$$F = \left(1 - \frac{\zeta}{\zeta_0}\right)^2 F_1, \quad G = \left(1 - \frac{\zeta}{\zeta_0}\right)^2 G_1,$$

so that F and G satisfy the conditions

$$F(\zeta_0) = F'(\zeta_0) = G(\zeta_0) = G'(\zeta_0) = 0,$$

then the above approximations have been obtained by making F_1 quadratic and G_1 linear in ζ, and satisfying the additional conditions

$$F(0) = 0, \quad F'(0) = \alpha, \quad F''(0) = -1, \quad G(0) = 1, \quad G''(0) = 0.$$

The following alternative assumptions were also investigated:

(i) F_1 linear, G_1 quadratic, chosen to satisfy

$$F(0) = 0, \quad F''(0) = -1, \quad G(0) = 0, \quad G'(0) = b, \quad G''(0) = 0.$$

These give

$$\left.\begin{aligned} F &= \frac{\zeta_0\zeta}{4}\left(1 - \frac{\zeta}{\zeta_0}\right)^2, \\ G &= \left(1 - \frac{\zeta}{\zeta_0}\right)^2 \left[1 + \left(b + \frac{2}{\zeta_0}\right)\zeta + \left(\frac{2b}{\zeta_0} + \frac{3}{\zeta_0^2}\right)\zeta^2\right] \end{aligned}\right\} \quad (25)$$

and the equations corresponding to (22) are, if $y = b\zeta_0$,

$$\left.\begin{aligned} 0{\cdot}03016y^2 + 0{\cdot}21190y + 0{\cdot}23571 &= 0{\cdot}00179\zeta_0^4, \\ 0{\cdot}01668y + 0{\cdot}06308 &= -\frac{y}{\zeta_0^4}. \end{aligned}\right\} \quad (26)$$

We thus obtain a cubic equation in y, of which only one root, $y = -1{\cdot}074$, gives a real value of ζ_0. From this we get

$$\zeta_0 = 2{\cdot}21, \quad F'(0) = 0{\cdot}55, \quad G'(0) = -0{\cdot}49, \quad H(\infty) = -0{\cdot}45.$$

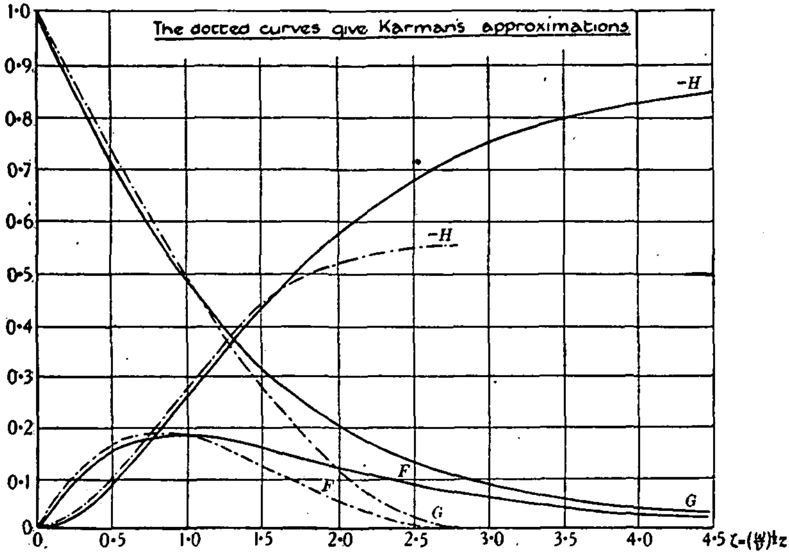
The dotted curves give Karman's approximations
1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0
0
0·5
1·0
1·5
2·0
2·5
3·0
3·5
4·0
4·5 ζ=(ω/ν)½z
-H
-H
F
F
G
G

Fig. 1.

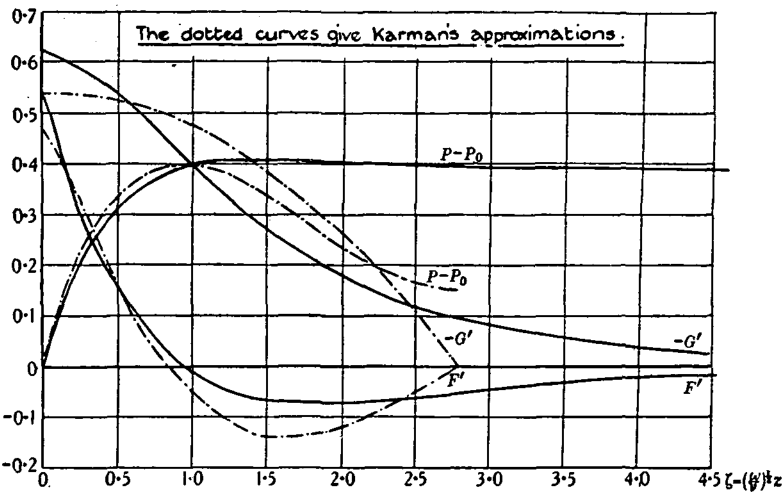
The dotted curves give Karman's approximations.
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0
-0·1
-0·2
0
0·5
1·0
1·5
2·0
2·5
3·0
3·5
4·0
4·5 ζ=(ω/ν)½z
P−P0
P−P0
-G′
-G′
F′
F′

Fig. 2.

(ii) F_1, G_1 quadratic, chosen to satisfy

$$F(0)=0,\quad F''(0)=-1,\quad F'''(0)=-2b,$$
$$G(0)=1,\quad G'(0)=b,\quad G''(0)=0,$$

the third of these equations being obtained by differentiating the first of equations (7) and putting $\zeta=0$. This leads by the same process to a quartic equation in $y=b\zeta_0$. The equation has two real roots, but the graphs for F corresponding to these are both inadmissible, being similar in shape to the inadmissible graph already mentioned.

(iii) F_1, G_1 quadratic, chosen to satisfy

$$F(0)=0,\quad F'(0)=a,\quad F''(0)=-1,$$
$$G(0)=1,\quad G''(0)=0,\quad G'''(0)=2a,$$

the last condition being obtained by differentiating the second of equations (7), and putting $\zeta=0$. These lead eventually to a quintic equation in $(a\zeta_0)^3$, only one of whose roots, $(a\zeta_0)^3=31{\cdot}073$, gives ζ_0 real. The corresponding F, however, has again a negative minimum in $(0, \zeta_0)$ and is therefore inadmissible.

One might perhaps have expected that by satisfying the extra conditions in (ii) and (iii) above, better approximations would be obtained, whereas Kármán's method does not yield an approximation at all in either case. As a check, the results of this section were also obtained independently by Mr L. Howarth, B.A., of Gonville and Caius College, to whom my thanks are due.

3. *Numerical integration of the equations.* By substituting in equations (7) and equating coefficients, we obtain formal expressions for F, G and H in powers of $e^{-c\zeta}$, as mentioned above. The first few terms are

$$\left.\begin{aligned}
F &= Ae^{-c\zeta}-\frac{(A^2+B^2)}{2c^2}e^{-2c\zeta}+\frac{A(A^2+B^2)}{4c^4}e^{-3c\zeta}\\
&\quad -\frac{1}{144c^6}(A^2+B^2)(17A^2+B^2)e^{-4c\zeta}+\dots,\\
G &= Be^{-c\zeta}-\frac{B(A^2+B^2)}{12c^4}e^{-3c\zeta}+\frac{1}{18c^6}AB(A^2+B^2)e^{-4c\zeta}+\dots,\\
H &= -c+\frac{2A}{c}e^{-c\zeta}-\frac{(A^2+B^2)}{2c^3}e^{-2c\zeta}+\frac{A(A^2+B^2)}{6c^5}e^{-3c\zeta}\\
&\quad -\frac{1}{288c^7}(A^2+B^2)(17A^2+B^2)e^{-4c\zeta}+\dots,\\
P &= \text{constant}+\frac{(3A^2-B^2)}{2c^2}e^{-2c\zeta}-\frac{2A(A^2+B^2)}{3c^4}e^{-3c\zeta}\\
&\quad +\frac{1}{96c^6}(A^2+B^2)(27A^2+11B^2)e^{-4c\zeta}+\dots,
\end{aligned}\right\}\quad(27)$$

the last being obtained from (11). These give solutions, valid for large ζ, of a differential system I, consisting of equations (7) together with the boundary conditions

$$F(\infty)=0, \quad G(\infty)=0, \quad H(\infty)=-c. \tag{28}$$

Now by numerical integration in the direction of ζ increasing, we can obtain a solution of a differential system II consisting of equations (7) together with the boundary conditions

$$F(0)=0, \quad G(0)=1, \quad H(0)=0, \quad F'(0)=a, \quad G'(0)=b. \tag{29}$$

Our aim is to find a, b, A, B, c so that systems I and II have a common solution, for such a solution will satisfy the equations (7) and (8) and therefore be a solution of the physical problem which we are considering.

I began with $a=0{\cdot}54$, $b=-0{\cdot}54$, the approximations given by equations (24), and integrated the first, second and last of equations (7), with a view to joining F, G, H, F' and G' at a suitable point to the asymptotic expressions (27). It was soon clear that these were not sufficiently good approximations to make a join. F and G appeared to be tending to $-\infty$ and H to $+\infty$. By trial and error I was led to obtain the sets of solutions corresponding to

$$a=0{\cdot}54,\ b=-0{\cdot}62; \quad a=0{\cdot}50,\ b=-0{\cdot}62; \quad a=0{\cdot}54,\ b=-0{\cdot}60.$$

From these three sets it appeared likely on inspection that the correct value of a lay between 0·50 and 0·54, and that the correct b was not far from $-0{\cdot}62$. I made a join with the set $a=0{\cdot}54$, $b=-0{\cdot}62$ at $\zeta=2{\cdot}5$, and by an application of Newton's rule, in a form suitable for five variables, obtained the values $a=0{\cdot}513$, $b=-0{\cdot}618$, from the three sets.

The method was as follows. Let F_0, G_0, H_0, F_0', G_0' be values of F, G, H, F', G' for some value of ζ obtained from the numerical integration of the system II and let F_1, G_1, H_1, F_1', G_1' be values obtained from the asymptotic series (27) for the same value of ζ, and let

$$\lambda_1=F_0-F_1, \quad \lambda_2=G_0-G_1, \quad \lambda_3=H_0-H_1,$$

$$\lambda_4=F_0'-F_1', \quad \lambda_5=G_0'-G_1',$$

so that, for fixed ζ, λ_i is a function of a, b, c, A and B ($i=1, 2, \ldots, 5$). We want to find a, b, c, A and B so that λ_i vanish over a range of ζ. For $a=0{\cdot}54$, $b=-0{\cdot}62$, $\zeta=2{\cdot}5$, A_1, B_1 and c_1 were chosen to give the smallest values of the set λ_i. The results from the integrations for $a = 0{\cdot}50$, $b = -0{\cdot}62$; $a = 0{\cdot}54$, $b = -0{\cdot}60$ enable us to get estimates of

$$\frac{\partial \lambda_i}{\partial a} \text{ and } \frac{\partial \lambda_i}{\partial b},$$

$(i = 1, \ldots, 5)$, for $a = 0{\cdot}54$, $b = -0{\cdot}62$, $A = A_1$, $B = B_1$, $c = c_1$, $\zeta = 2{\cdot}5$. From the asymptotic series we can find

$$\frac{\partial \lambda_i}{\partial A}, \quad \frac{\partial \lambda_i}{\partial B}, \quad \frac{\partial \lambda_i}{\partial c},$$

for $a = 0{\cdot}54$, $b = -0{\cdot}62$, $A = A_1$, $B = B_1$, $c = c_1$, $\zeta = 2{\cdot}5$. We then have sufficient material to apply Newton's rule, which gives the neighbouring values of a, b, A, B and c so that $\lambda_1, \ldots, \lambda_5$ vanish at $\zeta = 2{\cdot}5$.

The join for $a = 0{\cdot}513$, $b = -0{\cdot}618$ was still not satisfactory, H and F' being about 10 per cent. out, but it was quite clear in what direction variation was necessary. With $a = 0{\cdot}510$, $b = -0{\cdot}616$, $c = 0{\cdot}886$, $A = 0{\cdot}934$, $B = 1{\cdot}208$, the values of F_0, G_0, H_0, F_0', G_0', and F_1, G_1, H_1, F_1', G_1' are compared in the range $\zeta = 1{\cdot}9$ to $\zeta = 2{\cdot}5$ in the tables below.

ζ	F_0	F_1	G_0	G_1	H_0	H_1
1·9	0·126	0·127	0·222	0·222	−0·548	−0·548
2·1	0·111	0·112	0·186	0·187	−0·596	−0·596
2·3	0·097	0·098	0·156	0·157	−0·637	−0·638
2·5	0·084	0·085	0·131	0·131	−0·674	−0·675

ζ	F_0'	F_1'	G_0'	G_1'
1·9	−0·075	−0·075	−0·194	−0·194
2·1	−0·072	−0·073	−0·164	−0·163
2·3	−0·067	−0·068	−0·138	−0·138
2·5	−0·061	−0·062	−0·116	−0·116

The graphs of F, G and H for these values of the constants are shown by the continuous curves in Fig. 1 from $\zeta = 0$ to $\zeta = 4{\cdot}5$, and similarly for F', G' and P in Fig. 2; in both cases they are compared with Kármán's approximations, shown by the dotted curves. The approximate value, -0.54, of $G'(0)$ given by the correct application of Kármán's method is about 12 per cent. out, though, owing to errors, the value given by Kármán himself was about 6 per cent. out. The graph of $P - P_0$ rises to a maximum at about $\zeta = 1.45$, and then decreases slowly and steadily to $\frac{1}{2}c^2 = 0.3925$. This maximum at a finite distance from the plate would persist even with absolutely correct values of the constants a, b, c, A, B, for it can be seen from the fourth of equations (27) that P will increase steadily from 0 to infinity only if $B > \sqrt{3}A$. Now for the values above $B/A = 1.29$, and for absolutely correct values the above condition, $B > \sqrt{3}A$, is certainly not satisfied.

The first, third and fourth of equations (7), which were those integrated numerically, are equivalent to five first order linear equations. By taking H as independent variable, the system can be reduced to four first order linear equations, but these are more complicated and no time would be saved by the transformation. The equations were therefore integrated numerically as they stand. The method of Adams* was used, proceeding from $\zeta = 0$ by intervals of 0.1 and working to third differences. The four values of F, G, H, F' and G' required to start the process were obtained by finding, by means of substitution in equations (7), series for F, G, H, F' and G' in ascending powers of ζ. The terms necessary to give these values to four decimal places, which was the figure used, for $\zeta = 0.1, 0.2, 0.3$ are

$$F = a\zeta - \tfrac{1}{2}\zeta^2 - \tfrac{1}{3}b\zeta^3 - \tfrac{1}{12}b^2\zeta^4 - \tfrac{1}{60}a\zeta^5 + \left(\frac{1}{360} - \frac{ab}{90}\right)\zeta^6 + \left(\frac{b}{315} + \frac{ab^2}{1260}\right)\zeta^7 + \dots,$$

$$G = 1 + b\zeta + \tfrac{1}{3}a\zeta^3 + \tfrac{1}{12}(ab-1)\zeta^4 - \tfrac{1}{15}b\zeta^5 - \left(\frac{a^2}{90} + \frac{b^2}{45}\right)\zeta^6 + \left(\frac{a}{315} - \frac{b^3}{315} - \frac{a^2b}{252}\right)\zeta^7 + \dots,$$

$$H = -[a\zeta^2 - \tfrac{1}{3}\zeta^3 - \tfrac{1}{6}b\zeta^4 - \tfrac{1}{30}b^2\zeta^5 - \tfrac{1}{180}a\zeta^6 + \dots].$$

F' and G' are obtained, up to the term in ζ^6, by differentiation.

The table below gives F, G, H, F', G' and P to three decimal places from zero to 2.6 by intervals of 0.1, and from 2.6 to 4.4 by intervals of 0.2. Above $\zeta = 4.4$, all are given to within one unit in the third decimal place by the term in $e^{-c\zeta}$ in the asymptotic series.

* Cf. Whittaker and Robinson, *Calculus of Observations*, 2nd ed., 363–7.

ζ	F	G	H	F'	G'	$P - P_0$
0	0	1·000	0	0·510	−0·616	0
0·1	0·046	0·939	−0·005	0·416	−0·611	0·092
0·2	0·084	0·878	−0·018	0·334	−0·599	0·167
0·3	0·114	0·819	−0·038	0·262	−0·580	0·228
0·4	0·136	0·762	−0·063	0·200	−0·558	0·275
0·5	0·154	0·708	−0·092	0·147	−0·532	0·312
0·6	0·166	0·656	−0·124	0·102	−0·505	0·340
0·7	0·174	0·607	−0·158	0·063	−0·476	0·361
0·8	0·179	0·561	−0·193	0·032	−0·448	0·377
0·9	0·181	0·517	−0·230	0·006	−0·419	0·388
1·0	0·180	0·468	−0·266	−0·016	−0·391	0·395
1·1	0·177	0·439	−0·301	−0·033	−0·364	0·400
1·2	0·173	0·404	−0·336	−0·046	−0·338	0·403
1·3	0·168	0·371	−0·371	−0·057	−0·313	0·405
1·4	0·162	0·341	−0·404	−0·064	−0·290	0·406
1·5	0·156	0·313	−0·435	−0·070	−0·268	0·406
1·6	0·148	0·288	−0·466	−0·073	−0·247	0·405
1·7	0·141	0·264	−0·495	−0·075	−0·228	0·404
1·8	0·133	0·242	−0·522	−0·076	−0·210	0·403
1·9	0·126	0·222	−0·548	−0·075	−0·193	0·402
2·0	0·118	0·203	−0·572	−0·074	−0·177	0·401
2·1	0·111	0·186	−0·596	−0·072	−0·163	0·399
2·2	0·104	0·171	−0·617	−0·070	−0·150	0·398
2·3	0·097	0·156	−0·637	−0·067	−0·137	0·397
2·4	0·091	0·143	−0·656	−0·065	−0·126	0·396
2·5	0·084	0·131	−0·674	−0·061	−0·116	0·395
2·6	0·078	0·120	−0·690	−0·058	−0·106	0·395
2·8	0·068	0·101	−0·721	−0·052	−0·089	0·395
3·0	0·058	0·083	−0·746	−0·046	−0·075	0·395
3·2	0·050	0·071	−0·768	−0·040	−0·063	0·395
3·4	0·042	0·059	−0·786	−0·035	−0·053	0·394
3·6	0·036	0·050	−0·802	−0·030	−0·044	0·394
3·8	0·031	0·042	−0·815	−0·025	−0·037	0·393
4·0	0·026	0·035	−0·826	−0·022	−0·031	0·393
4·2	0·022	0·029	−0·836	−0·019	−0·026	0·393
4·4	0·018	0·024	−0·844	−0·016	−0·022	0·393

This work was suggested to me by Dr S. Goldstein, who has helped me at every stage. It is a pleasure to express my thanks to him.