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Continuum Mechanics

When studying the dynamics of a mediums with fluid or structure properties under the influence of forces, we need in some sense a good description of how these forces act and alter the system itself.
Any medium on a microscopic scale is built up of a structure of atoms, meaning we can observe empty spaces between each atom or discontinuities in the medium. Discribing any phsycial phenomen on larger scales in such a way are tedious and most often out of bounds due to the high number of particles. Instead we consider the medium to be continously distributed throughout the entire reagion it occupies. Hence we want to study some phsyical properties of the complete volume and not down on atomic scale.
We consider the medium with continuum properties. By a continuum we mean a volume $V(t) \subset \mathbb{R}^3$ consiting of particles, which we observe for some properties. One property of interest could be the veloctity $\mathbf{v}(x,t)$ for some point $x \in V(t)$ in time $t \in (0,T]$, which would mean the average velocity of the particles occupying this point x at time t

1.1 Coordinate system

We assume that our medium is continiously distributed throughout its own volume, and we start our observation of this medium at som time t_0. As this choice is arbitary, we often choose to observe a medium in a stress free initial state. We call this state $V(t_0)$ of the medium as the *reference configuration.* We let $V(t)$ for $t \geq t_0$ denote the *current configuration.*

1.1.1 Lagrangian

As the medium is act upon by forces, one of the main properties of interest is the deformation. Let $\hat{\mathrm{x}}$ be a particle in the reference cofiguration $\hat{\mathrm{x}} \in \hat{\mathrm{V}}$. Further let $\mathrm{x}(\hat{\mathrm{x}},\, \mathrm{t})$ be the new location of a particle $\hat{\mathrm{x}}$ for time t such that $x \in V(t)$. We assume that no two particles $\hat{\mathrm{x}}_a, \hat{\mathrm{x}}_b \in \hat{\mathrm{V}}$ occupy the same location for some time $V(t)$. Hence the map $\hat{\mathrm{T}}(\hat{\mathrm{x}},t) = x(\hat{\mathrm{x}},t)$ maps a particle $\hat{\mathrm{x}}$ from the *reference configuration* $\hat{\mathrm{V}}$ to the *current configuration* $V(t)$ Assuming that the path for some $\hat{\mathrm{x}}$ is continious in time, we can define the inverse mapping $\hat{\mathrm{T}}^{-1}(x,t) = \hat{\mathrm{x}}(x,t)$, which maps $x(\hat{\mathrm{x}},t)$ back to its initial location at time $t = t_0$.

We now have enough background to define the *deformation*

$$\hat{\mathrm{T}}(\hat{\mathrm{x}},t) = \hat{\mathbf{u}}(\hat{\mathrm{x}},t) = x(\hat{\mathrm{x}},t) - \hat{\mathrm{x}} \tag{1.1}$$

and the *deformation velocity*

$$\frac{\partial \hat{\mathrm{T}}(\hat{\mathrm{x}},t)}{\partial t} = \hat{\mathbf{v}}(\hat{\mathrm{x}},t) = d_t x(\hat{\mathrm{x}},t) = d_t \hat{\mathbf{u}}(\hat{\mathrm{x}},t) \tag{1.2}$$

Such a description of tracking each particle $\hat{x} \in \hat{V}$ is often denoted the *Lagrangian Framework* and is a natural choice of describing structure mechanics.

1.1.2 Eulerian

Considering a flow of fluid particles in a river, a *Lagrangian* description of the particles would be tidious as the number of particles entring and leaving the domain quickly rise to a immense number. Instead consider defining a view-point V fixed in time, and monitor every fluid particle passing the coordinate $x \in V(t)$ as time elapses. Such a description is defined as the *Eulerian framework.* It is important to mention that the we are not interested in which particle is occupying a certain point in our domain, but only its properties. Such a description falls natural for describing fluid dynamics.
We can describe the particles occupying the *current configuration* $V(t)$ for some time $t \geq t_0$

$$x = \hat{x} + \hat{\mathbf{u}}(\hat{x}, t)$$

Since our domain is fixed can define the deformation for a particle occupying position $x = x(\hat{x}, t)$ as

$$\mathbf{u}(x, t) = \hat{\mathbf{u}}(\hat{x}, t) = x - \hat{x}$$

and its velocity

$$\mathbf{v}(x, t) = \partial_t u(x, t) = \partial_t \hat{\mathbf{u}}(\hat{x}, t) = \hat{\mathbf{v}}(\hat{x}, t)$$

1.2 Deformation gradients

When studying continuum mechanics we observe continious mediums as they are deformed over time. These deformations results in relative changes of positions due to external and internal forces acting.. These relative changes of postition is called *strain*, and is the primary property that causes *stress* within a medium of interest [1]. We define stress as the internal forces that particles within a continuous material exert on each other.

The equations of mechanics can be derived with respect to either a deformed or undeformend configuration of our medium of interest. The choice of refering our equations to the current or reference configuration is indifferent from a theoretical point of view. In practice however this choice can have a severe impact on our strategy of solution methods and physical of modelling. [2]. We will therefore define the strain measures for both configurations of our medium.

Definition 1.2.1. Deformation gradient.

$$\hat{\mathbf{F}} = I + \hat{\nabla} \hat{\mathbf{u}} \tag{1.3}$$

Mind that deformation gradient of $\hat{\mathbf{u}}$ is which respect to the reference configuration. From the assumption that no two particles $\hat{x}_a, \hat{x}_b \in \hat{V}$ occupy the same location for some time $V(t)$, the presented transformation must be linear. As a consequence from the invertible matrix theorem found in linear algebra, the linear operator $\mathbf{F}$ cannot be a singuar. We define the *determinant of the deformation gradient* as J, which denotes the local change of volume of our domain.

Definition 1.2.2. Determinant of the deformation gradient

$$J = det(\hat{\mathbf{F}}) = det(I + \hat{\nabla} \hat{\mathbf{u}}) \neq 0 \tag{1.4}$$

By the assumption that the medium can't be selfpenetrated, we must limit J to be greater than 0 [2]

1.3 Measures of Strain and Stress

The equations describing forces on our domain can be derived in accordinance with the current or reference configuration. With this in mind, different measures of strain can be derived accordning to which configuration we are interested in. We will here by [1] show the most common measures of strain. We will first introduce the right *Cauchy-Green* tensor $\mathbf{C}$, which is one of the most used strain measures [2].
Uttrykk 1.3 fra Godboka, LAG TEGNING

Let $\hat{\mathrm{x}}, \hat{\mathrm{y}} \in \hat{\mathrm{V}}$ be two points in our referemce configuration and let $\hat{\mathrm{a}} = \hat{\mathrm{y}} - \hat{\mathrm{x}}$ denote the length of the line bewtween these two points. As our domain undergoes deformation let $x = \hat{\mathrm{x}} + \hat{\mathbf{u}}(\hat{\mathrm{x}})$ and $x = \hat{\mathrm{y}} + \hat{\mathbf{u}}(\hat{\mathrm{y}})$ be the position of our points in the current configuration, and let $a = y - x$ be our new line segment. By [1] we have by first order Taylor expansion

$$
\begin{aligned}
&y - x = \hat{\mathrm{y}} + \hat{\mathbf{u}}(\hat{\mathrm{y}}) - \hat{\mathrm{x}} - \hat{\mathbf{u}}(\hat{\mathrm{x}}) = \ \hat{\mathrm{y}} - \hat{\mathrm{x}} + \hat{\nabla}\hat{\mathrm{u}}(\hat{\mathrm{x}})(\hat{\mathrm{y}} - \hat{\mathrm{x}}) + \mathcal{O}(|\hat{\mathrm{y}} - \hat{\mathrm{x}}|^2) \\
&\frac{y - x}{|\hat{\mathrm{y}} - \hat{\mathrm{x}}|} = [I + \hat{\nabla}\hat{\mathbf{u}}(\hat{\mathrm{x}}]\frac{\hat{\mathrm{y}} - \hat{\mathrm{x}}}{|\hat{\mathrm{y}} - \hat{\mathrm{x}}|} + \mathcal{O}(|\hat{\mathrm{y}} - \hat{\mathrm{x}}|)
\end{aligned}
$$

This detour from [1] we have that

$$
\begin{aligned}
&a = y - x = \hat{\mathbf{F}}(\hat{\mathrm{x}})\hat{\mathrm{a}} + \mathcal{O}(|\hat{\mathrm{a}}|^2) \\
&|a| = \sqrt{(\hat{\mathbf{F}}\hat{\mathrm{a}}, \hat{\mathbf{F}}\hat{\mathrm{a}}) + \mathcal{O}(|\hat{\mathrm{a}}^3|)} = \sqrt{(\hat{\mathrm{a}}^T, \hat{\mathbf{F}}^T\hat{\mathbf{F}}\hat{\mathrm{a}})} + \mathcal{O}(|\hat{\mathrm{a}}^2|)
\end{aligned}
$$

We let $\hat{\mathrm{C}} = \hat{\mathrm{F}}^T\hat{\mathrm{F}}$ denote the right *Cauchy-Green tensor*. By observation the Cauchy-Green tensor is not zero at the reference configuration

$$
\hat{\mathrm{C}} = \hat{\mathrm{F}}^T\hat{\mathrm{F}} = (I + \hat{\nabla}\hat{\mathrm{u}})^T(I + \hat{\nabla}\hat{\mathrm{u}}) = 1
$$

Hence it is convenient to introduce a tensor which is zero at the reference configuration. We define the *Grenn-Lagrange strain tensor*, which arises from the squard rate of change of the linesegment $\hat{\mathrm{a}}$ and a. By using the definition of the Cauchy-Green tensor we have the relation

$$
\begin{aligned}
&\frac{1}{2}(|a|^2 + |\hat{\mathrm{a}}|^2) = \frac{1}{2}(\hat{\mathrm{a}}^T\hat{\mathbf{C}}\hat{\mathrm{a}} - \hat{\mathrm{a}}^T\hat{\mathrm{a}}) + \mathcal{O}(|\hat{\mathrm{a}}^3| = \hat{\mathrm{a}}^T\big(\frac{1}{2}(\hat{\mathbf{F}}^T\hat{\mathbf{F}} - I)\big)\hat{\mathrm{a}} + \mathcal{O}(\hat{\mathrm{a}}^3) \\
&\hat{\mathbf{E}} = \frac{1}{2}(\hat{\mathbf{C}} - I)
\end{aligned}
$$

Both the *right Cauchy-Green tensor* $\hat{\mathbf{C}}$ and the *Green-Lagrange* $\hat{\mathbf{E}}$

1.4 Transformations of conservation laws

To be clear later, transformation of conservation principles from Eulerian coordinates to other coordinate systems are essential for deriving the equations of fluid-structure interaction. These conservation laws often has some spatial or temporal derivaties, and the essence is how to we transform these operators between our coordinate systems.
With this in mind, we will derive these transformations with the help of a new arbitary fixed reference system $\hat{\mathrm{W}}$. We will throughout this subsection follow the ideas and approaches found in [1]. Further we denote its deformation gradient as $\hat{\mathbf{F}}_w$ and its determinant $\hat{\mathrm{J}}_w$. Then the invertibale mapping $\hat{\mathrm{T}}_w : \hat{\mathrm{W}} \to V(t)$ exists.
For $\hat{\mathrm{V}} = \hat{\mathrm{W}}$, $\hat{\mathrm{W}}$, simply denotes the familiar Lagrangian description. In the case $\hat{\mathrm{V}} \neq \hat{\mathrm{W}}$, $\hat{\mathrm{W}}$ have no direct physical meaning. Hence it is important to notice that the physical velocity $\frac{\partial \hat{\mathrm{V}}}{\partial t}\hat{\mathbf{v}}$ and the velocity of arbitary domain $\frac{\partial \hat{\mathrm{W}}_w}{\partial t}$ doesn't neseccary coincide. This observation is essential, as we will soon see.
We will first consider the transformation of spatial derivatives from $V(t)$ to $\hat{\mathrm{W}}$

Lemma 1.4.1. Let f be a function $f : V(t) \to \mathbb{R}$ and $\mathbf{w} \to \mathbb{R}^d$ be a vectorfield. Then the spatial transformation from from $V(t)$ to Ŵ is defined as

$$\nabla f = \hat{\mathbf{F}}_W^{-T} \hat{\nabla} \hat{\mathrm{f}}, \quad \nabla \mathbf{w} = \hat{\nabla} \hat{\mathrm{f}} \hat{\mathbf{F}}_W^{-1}$$

Lemma 1.4.2. Let f be a function $f : V(t) \to \mathbb{R}$. Then the temporal transformation from from $V(t)$ to Ŵ is defined as

$$\frac{\partial f}{\partial t} = \frac{\partial \hat{\mathrm{f}}}{\partial t} - (\hat{\mathbf{F}}_W^{-1} \hat{\mathbf{v}} \cdot \hat{\nabla}) \hat{\mathrm{f}} + \hat{\mathbf{F}}_W^{-T} \hat{\nabla} \hat{\mathrm{f}} \cdot \hat{\mathbf{v}}$$

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1.5 Conservationlaws

1.5.1 Conservation of continuum

1.5.2 Conservation of momentum

1.6 The fluid

1.7 The solid

1.7.1 Material models, St. Venant Kirchhoff materia, incomp neo-Hookean materiall

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