

# Von Karman Viscous Pump

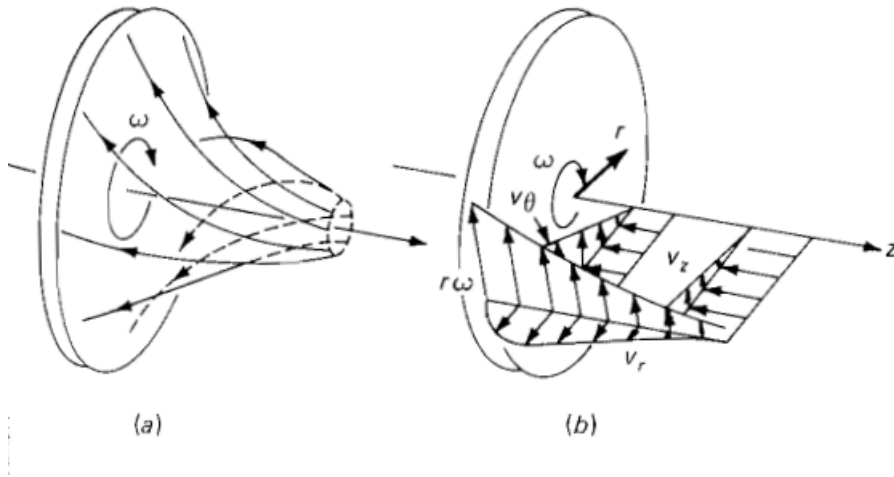
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## Introduction of problem

Consider an infinite rotating disk placed at  $z=0$  under a newtonian viscous fluid initially at rest, viscous forces will set up a rotating velocity field in the fluid. von Karman(1) showed that the steady state flow of this problem could be reduced to a set of ordinary differential equations. He solved them by approximate integration method, which will be used as reference for the CFD results in this test.

Spinning circle



## Theory

Let the velocity vector components be defined as  $\mathbf{v} = (v_r, v_\theta, v_z)$ . By the assumption of radial symmetry, we expect our components to be independent of the angle  $\theta$ . The continuity equation for a polar coordinate system in accordance with our assumption is defined by

$$\nabla \cdot \mathbf{v} = \frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{\partial v_z}{\partial z} = 0 \quad (1)$$

Moving on to the equation of momentum, the familiar Navier-Stokes equation yields.

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \frac{\mu}{\rho} \nabla^2 \mathbf{v} \quad (2)$$

Considering steady-flow we end up with the following component equations.

$$r : \quad v_r \frac{\partial v_r}{\partial r} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \frac{\mu}{\rho} \left( \frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r} \frac{\partial v_r}{\partial r} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} \right) \quad (3)$$

$$\theta : \quad v_r \frac{\partial v_\theta}{\partial r} + v_z \frac{\partial v_\theta}{\partial z} + \frac{1}{r} v_\theta v_r = \frac{\mu}{\rho} \left( \frac{\partial^2 v_\theta}{\partial r^2} + \frac{1}{r} \frac{\partial v_\theta}{\partial r} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2} \right) \quad (4)$$

$$z : \quad v_r \frac{\partial v_z}{\partial r} + \frac{\partial v_z}{\partial z} = \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{\mu}{\rho} \left( \frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r} \frac{\partial v_z}{\partial r} + \frac{\partial^2 v_z}{\partial z^2} \right) \quad (5)$$

For our problem we define the following boundary conditions.

- On disk surface

$$v_r = 0, v_\theta = r\Omega, v_z = 0$$

- Disk circumference

$$\frac{\partial}{\partial r} \left( \frac{v_r}{r} \right) = 0, \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) = 0, \frac{\partial v_z}{\partial r} = 0$$

- End of integration domain

$$v_r = 0, v_\theta = 0, \frac{\partial v_z}{\partial z} = 0$$

Introducing von Karman's substitutions of the velocity components,

$$v_r = r\omega F(\zeta) \quad v_\theta = r\omega G(\zeta) \quad v_z = (\nu\omega)^{\frac{1}{2}} H(\zeta) \\ p = \rho\nu\omega, \quad \zeta = (\omega/\nu)^{\frac{1}{2}} z$$

we can rewrite (3-5) as a system of ODE's as follows

$$F^2 - G^2 + HF' = F'' \quad (6)$$

$$2FG + HG' = G'' \quad (7)$$

$$2F + H' = 0 \quad (8)$$

$$P' + 2F' = -HH' \quad (9)$$

$$(10)$$

The corresponding boundary conditions yields

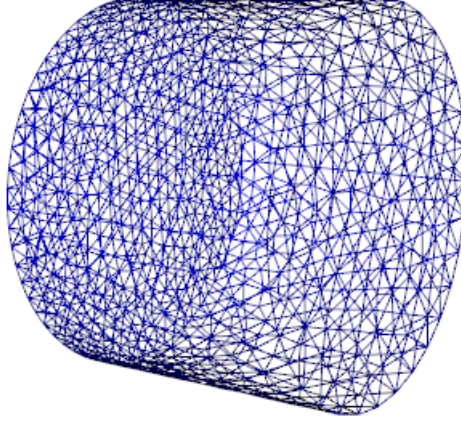
- $F = 0, \quad G = 1, \quad H = -a \quad \text{at } \zeta = 0$
- $F = 0, \quad G = 0 \quad \text{at } \zeta = \infty$

## Solving the problem

For the presented problem, the Navier-Stokes solver Oasis based on the FEniCS software will be used to solve the PDE system using finite element method.

For the computation we have to construct a mesh to do our calculations upon. In this problem we will use Gmsh, a free 3D finite element grid generator. The problem will be solved on a pipe with  $r = 1$  and height  $z = 3$ . The pipe will be constructed with denser elements at the spinning boundary, as seen in the presented figure.

## Pipe mesh



Continuing on the reference solution, we derive Karman's method of approximate solution. Firstly we want to observe how the differential equations behave at the limit  $\zeta \rightarrow \infty$ . Assuming  $H$  approaches a limit  $-c$  as while  $F, G \rightarrow 0$  as  $\zeta \rightarrow \infty$ , we end up with the following relations for large  $\zeta$ .

$$-cG' = G'' \quad -cF' = F'' \quad (11)$$

$$F = Ae^{-c\zeta}, \quad G = Be^{-c\zeta}, \quad H = -c + \frac{2A}{c}e^{-c\zeta} \quad (12)$$

As we can see,  $F$  and  $G$  approaches 0 exponentially, and we can assume they are 0 for some value of  $\zeta$  which we will call  $\zeta_0$ . This will be exploited in the following integration scheme.

Starting by integrating equation (6)-(7) from 0 to  $\infty$ , and using relation (8).

$$\begin{aligned} \int_0^\infty HF' d\zeta &= [HF]_0^\infty - \int_0^\infty H' F d\zeta = 2 \int_0^\infty F^2 d\zeta \\ \int_0^\infty HG' d\zeta &= [HG]_0^\infty - \int_0^\infty H' G d\zeta = 2 \int_0^\infty FG d\zeta \end{aligned}$$

Combining these results with (12), we end up with the following result

$$-F'(0) = \int_0^\infty (3F^2 - G^2) d\zeta \quad (13)$$

$$-G'(0) = 4 \int_0^\infty FG d\zeta \quad (14)$$

As Karman, we assume that due to the exponential growth that  $F$  and  $G$  are zero for values of  $\zeta$  greater than  $\zeta_0$ . As a result

$$F(\zeta_0) = 0, \quad F'(\zeta_0) = 0, \quad G(\zeta_0) = 0, \quad G'(\zeta_0) = 0 \quad (15)$$

We can now also find  $F''(0)$  and  $G''(0)$  by setting  $\zeta = 0$  in the system of ODE's

$$F''(0) = -1 \quad G''(0) = 0 \quad (16)$$

Now if we let  $F'(0)$  is some constant  $a$ , the following functions fulfills equation (15)-(16) and the boundary conditions.

$$F = (1 - \frac{\zeta}{\zeta_0})^2 \left( a\zeta + (\frac{2a}{\zeta_0})\zeta^2 \right) \quad (17)$$

$$G = (1 - \frac{\zeta}{\zeta_0})^2 \left( 1 + \frac{\zeta}{2\zeta_0} \right) \quad (18)$$

This yields  $G'(0) = -\frac{3}{2\zeta_0}$ . Inserting (17)-(18) in (13)-(14), we get a system of equations to solve  $a$  and  $\zeta_0$ .

## Computations

$\zeta$	F	F'	G	G'	H	-P
0.0	0.0	0.51023	1.00	-0.61592	0.0	0.0
0.1	0.0462	0.4163	0.9386	-0.6112	-0.0048	0.0924
0.2	0.0836	0.3338	0.8780	-0.5987	-0.0179	0.1674
0.3	0.1133	0.2620	0.8190	-0.5803	-0.0377	0.2274
0.4	0.1364	0.1999	0.7621	-0.5577	-0.0628	0.2747
0.5	0.1536	0.1467	0.7075	-0.5321	-0.0919	0.3115
0.6	0.1660	0.1015	0.6557	-0.5047	-0.1239	0.3396
0.7	0.1742	0.0635	0.6067	-0.4763	-0.1580	0.3608
0.8	0.1789	0.0317	0.5605	-0.4476	-0.1933	0.3764
0.9	0.1807	0.0056	0.5171	-0.4191	-0.2293	0.3877
1.0	0.1801	-0.0157	0.4766	-0.3911	-0.2655	0.3955

## Comments