FEniCS Course

Lecture 8: The Stokes problem

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The Stokes equations

$$\begin{aligned} -\Delta u + \nabla p &= f && \text{in } \Omega && \text{Momentum equation} \\ \nabla \cdot u &= 0 && \text{in } \Omega && \text{Continuity equation} \\ u &= g_D && \text{on } \partial\Omega_D && \\ \frac{\partial u}{\partial n} - pn &= g_N && \text{on } \partial\Omega_N && \end{aligned}$$

- u is the fluid velocity and p is the pressure
- f is a given body force per unit volume
- g_{D} is a given boundary flow
- g_{N} is a given function for the natural boundary condition

Variational problem

Multiply the momentum equation by a test function v and integrate by parts:

$$\int_\Omega \nabla u : \nabla v \,\mathrm{d}x - \int_\Omega p \nabla \cdot v \,\mathrm{d}x = \int_\Omega f \cdot v \,\mathrm{d}x + \int_{\partial\Omega_N} g_N \cdot v \,\mathrm{d}s$$

Short-hand notation:

$$\underbrace{\langle \nabla u, \nabla v \rangle}_{a(u,v)} \underbrace{- \langle p, \nabla \cdot v \rangle}_{b(v,p)} = \underbrace{\langle f, v \rangle + \langle g_N, v \rangle_{\partial\Omega_N}}_{L(v)}$$

Multiply the continuity equation by a test function q:

$$\underbrace{\pm \langle \nabla \cdot u, q \rangle}_{b(u,q)} = 0$$

Definitions of $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are meaningful if $u \in H^1(\Omega)$ and $p \in L^2(\Omega)$

Saddle point formulation of the Stokes problem

The Stokes problem is an example of a saddle point problem:
Find $(u, p) \in V \times Q$ such that for all $(v, q) \in \widehat{V} \times \widehat{Q}$

$$
\begin{aligned}
a(u, v) + b(v, p) &= L(v) \\
b(u, q) &= 0
\end{aligned}
$$

Sum up: $A(u, p; v, q) := a(u, v) + b(v, p) + b(u, q) = L(v)$
Mixed spaces:

$$
\begin{aligned}
V &= [H^1_{g_D, \Gamma_D}(\Omega)]^d \qquad & \widehat{V} &= [H^1_{0, \Gamma_D}(\Omega)]^d \\
Q &= L^2(\Omega) \qquad & \widehat{Q} &= L^2(\Omega)
\end{aligned}
$$

The inf-sup condition

$$
\inf_{q \in Q} \sup_{v \in V} \frac{b(v, q)}{\|v\|_V \|q\|_Q} \geqslant C
$$

is critical for the unique solvability of the saddle point problem

Discrete variational problem

Find $(u_h, p_h) \in V_h \times Q_h$ such that for all $(v_h, q_h) \in \widehat{V_h} \times \widehat{Q_h}$

$A_h(u_h, p_h; v_h, q_h) := a_h(u_h, v_h) + b_h(v_h, p_h) + b_h(u_h, q_h) = L_h(v_h)$

A stable mixed element $V_h \times Q_h \subset V \times Q$ should satisfy a uniform inf-sup condition

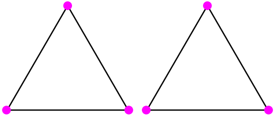
$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b_h(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \geqslant c_b$$

with c_b independent of the mesh $\mathcal{T}_h$!

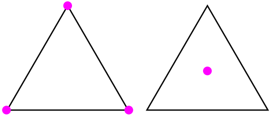
$\Rightarrow$ The right "mixture" of elements is critical for stability and convergence!

Unstable and stable Stokes elements

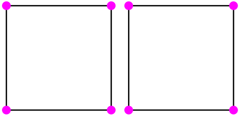
Unstable elements



$\mathbb{P}_1/\,\mathbb{P}_1$

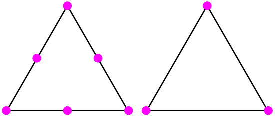


$\mathbb{P}_1/\,\mathbb{P}_0$

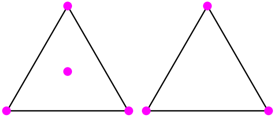


$\mathbb{P}_1/\,\mathbb{P}_1$

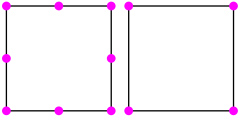
Stable elements



$\mathbb{P}_2/\,\mathbb{P}_1$



$\mathbb{P}_1^b/\,\mathbb{P}_1$



$\mathbb{Q}_2/\,\mathbb{Q}_1$

Taylor-Hood elements: $\mathbb{P}_{k+1}/\,\mathbb{P}_k$, $\mathbb{Q}_{k+1}/\,\mathbb{Q}_k$ for $k \geqslant 1$
Mini-element: $\mathbb{P}_1^b/\,\mathbb{P}_1$

Useful FEniCS tools (I)

Mixed elements:

Python code

```
V = VectorFunctionSpace(mesh, "Lagrange", 2)
Q = FunctionSpace(mesh, "Lagrange", 1)
W = V*Q
```

Defining functions, test and trial functions:

Python code

```
up = Function(W)
(u,p) = split(up)
```

Shortcut:

Python code

```
(u, p) = Functions(W)
# similar for test and trial functions
(u, p) = TrialFunctions(W)
(v, q) = TestFunctions(W)
```

Useful FEniCS tools (II)

Access subspaces:

Python code

```
W.sub(0) #corresponds to V
W.sub(1) #corresponds to Q
```

Splitting solution into components:

Python code

```
w = Function(W)
solve(a == L, w, bcs)
(u, p) = w.split()
```

Rectangle mesh:

Python code

```
mesh = RectangleMesh(0.0, 0.0, 5.0, 1.0, 50, 10)
```

Python code

```
h = CellSize(mesh)
```

Exercise: Spurious pressure modes

Compute the finite element approximation for Couette flow on the unit square. Use the boundary data

$$u = 1 \text{ on } y = 1, \quad u = 0 \text{ on } y = 0, \quad g_N = 0 \text{ on } x = 0 \text{ or } x = 1$$

Use $\mathbb{P}_1/\mathbb{P}_1$ and $\mathbb{P}_1/\mathbb{P}_0$ elements. The exact solution is given by

$$u = (y, 0), \qquad p = 0$$

What do you observe? Why?

Exercise: A stabilized $\mathbb{P}_1/\mathbb{P}_1$ method

Define the bilinear forms

$$\begin{aligned} a_h(u_h, v_h) &= (\nabla u_h, \nabla v_h) \\ b_h(v_h, q_h) &= -(\nabla \cdot v_h, q_h) \\ c_h(p_h, q_h) &= \sum_{T \in \mathcal{T}_h} \mu_T (\nabla p_h, \nabla q_h) \end{aligned}$$

and solve: find $(u_h, p_h) \in V_h \times Q_h$ such that $\forall\ (v_h, q_h) \in \widehat{V}_h \times \widehat{Q}_h$

$$\begin{aligned} A(u_h, p_h; v_h, q_h) &:= a(u_h, v_h) + b(v_h, p_h) + b(u_h, q_h) - c(p_h, q_h) \\ &= (f, v_h) - \sum_{T \in \mathcal{T}_h} \mu_T (f, \nabla q_h) \end{aligned}$$

Exercise: Implement this scheme for the Couette flow example using $\mu_T = \beta h_T^2$, $\beta = 0.2$.