MAT-INF4130
Mandatory Assignment 1

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As a strategy to approximate a function f with a function g ..

$$||f - g|| = \sqrt{\int_0^1 (f(t) - g(t))^2 dt} \tag{1}$$

Further we define $N_1(x), N_2(x), ..., N_n(x)$ to be a basis for the set of polynomials of degree at most n-1. For a chosen set of basis, we want to find a best fitpolynomial to f, such that $|| \sum_{j=1}^{n} c_j N_j - f||$ is minimized with respect to the coefficients c_j From the least squares approximation, we end up with the system

$$\mathbf{c^T N c} - 2\mathbf{b^T c} + ||f||^2 \tag{2}$$

Where $\mathbf{N}$ is the matrix with entries $\langle N_i, N_j \rangle$ and $\mathbf{b}$ is the vector with entries $\langle N_j, f \rangle$

Problem 1

REWRITE AS HERMETIAN MATRIX!! First we want to show that matrix $\mathbf{N}$ is positive definite. A n $\times$ n real matrix M defined as positive definite if the scalar $x^T M x$ is positive for every choice of a non-zero column vector x x of dimention n. A natural extention of the definition would be to consider the system $\mathbf{c^T N c}$.

$$c^T N c = c_i \langle N_i, N_j \rangle c_j = \int_0^1 \sum_{i=1}^{n} c_i N_i \sum_{j=1}^{n} c_j N_j dx = \int_0^1 p(x)^2 dx \geq 0 \tag{3}$$

Problem 2

Computing the gradient of the expression $\mathbf{c^T N c} - 2\mathbf{b^t c} + ||f||^2$ with respect to c_i we get

$$\frac{\partial}{\partial c_i} \mathbf{c^T N c} - 2\mathbf{b^t c} + ||f||^2 = 2\mathbf{N c} - 2\mathbf{b^t} \tag{4}$$

$$\tag{5}$$

From simple observations of the least squares method, one can see that the trend of $||f - g|| = \sqrt{\int_0^1 (f(t) - g(t))^2 dt}$ will follow some sort of second order polynomial. Hence the minimal extreme point for the smallest error will be found for $\frac{\partial}{\partial c_i} \mathbf{c^T N c} - 2\mathbf{b^t c} + ||f||^2 = 0$ hence $\mathbf{Nc} = \mathbf{b}$

Problem 3

Defining $N_j(x) = x^{j-1} \;\; 1 \leq j \leq n$, we are ought to show that the matrix $\mathbf{N}$ really is the Hilbert matrix with entries $\frac{1}{i+j-1}$

$$\mathbf{N} = \langle N_i, N_j \rangle = \int_0^1 x^{i-1} x^{j-1} dx = \int_0^1 x^{i+j-2} dx = \Big[\frac{1}{i+j+1} x^{i+j+1} \Big]_0^1 = \frac{1}{i+j+1} \tag{6}$$

Problem 4

Showing that $P_n(x) = x^n + \sum_{k=0}^{n-1} c_k x^k$ is orthogonal to $1, x, ..., x^{n-1}$ on [-1, 1], we require that $\langle P_n, N_j \rangle = 0$
Since $\langle P_n, N_j \rangle = 0$, then using the same arguments as in problem 3, we have to solve the linear system $\mathbf{Nc} = \mathbf{b}$, where b in this will be a contribution from the x^n.

$$\langle P_n, N_j \rangle = \int_{-1}^{1} \left[x^n + \sum_{k=0}^{n-1} c_k x^k \right] x^{j-1} dx = 0 \tag{7}$$

$$\int_{-1}^{1} \sum_{k=0}^{n-1} c_k x^k x^{j-1} dx = -\int_{-1}^{1} x^n x^{-j} dx = -\langle x^n, x^{j-1} \rangle \tag{8}$$

$$-\langle x^n, x^{j-1} \rangle = -\left[\frac{x^{n+j}}{n+j} \right]_{-1}^{1} = \begin{cases} 0 & \text{if n + j is even} \\ -\frac{2}{n+j} & \text{if n + j is odd} \end{cases} \tag{9}$$