INF 5620
Fenics Project

Andreas Slyngstad

30. november 2015

In this project we are going to work with the nonlinear diffusion model

$$\varrho u_t = \nabla \cdot (\alpha(u)\nabla u) + f(\mathbf{x}, t) \qquad \mathbf{x} \in \Omega, t \in (0, T] \tag{1}$$

With ϱ is an constant, and $\alpha(u)$ is a known function of u .The following initial and boundary condition apply

$$u(\mathbf{x}, 0) = I(x) \qquad \mathbf{x} \in \Omega, t \in (0, T] \tag{2}$$

$$\frac{\partial u}{\partial n} = 0 \qquad \mathbf{x} \in \partial\Omega_D, t \in (0, T] \tag{3}$$

Exercise a

Choosing the Backward Euler discretization, I get the following implicit scheme.

$$\varrho \frac{u^n - u^{n-1}}{\Delta t} = \nabla \cdot (\alpha(u^n)\nabla u^n) f(u^n) \tag{4}$$

Where $u(\mathbf{x})^n$ denotes the solution of u at timestep n Using the finite element method we want to approximate u in space $V = span \psi_0(x), \psi_1(x), .., \psi(x)_N$, where ψ_j denotes the basis functions of V. To derive a variational formulation of the initial condition, we simply multiply or initial condition function *I(x)* with a testfunction from our space V, and integrate over our domain Ω

$$\int_\Omega Iv \;\; dx = (I, v) = 0 \tag{5}$$

For notation, we redefine the current and last timestep u^n, u^{n-1}, as u and $u^{(1)}$. Regarding the total system we first reorganize our equation as

$$u - \frac{\Delta t}{\varrho} \nabla \cdot (\alpha(u)\nabla u) = u^{(1)} + \frac{\Delta t}{\varrho} f(\mathbf{x}, t) \tag{6}$$

We now multiply the system with a testfunction v, $\forall v \in V$ and integrate over the domain Ω. Utilizing Green's first identity, we can rewrite the second term as follows.

$$-\int_\Omega \nabla \cdot (\alpha(u)\nabla u)v \; dx = \int_\Omega (\alpha(u)\nabla u) \cdot \nabla v \; dx - \int_\Omega \alpha(u)\frac{\partial u}{\partial n} v \; ds \tag{7}$$

Using the boundarycondition, the boundary integral equals zero. Using the inner product notation as (,), we can write the variationalform as follows

$$(u, v) + \frac{\Delta t}{\varrho}(\alpha(u)\nabla u), \nabla v) = (u^{(1)}, v) + \frac{\Delta t}{\varrho}(f(\mathbf{x}, t), v) \tag{8}$$

Exercise b

To solve the implicit system we will implement Picard iteration, using the recently computed u in the $\alpha(u)$ function. Our goal is to linearize the system. Letting $u^{n,k}$ be the calculated u after k Picard iterations. Our system yields

$$u^{n,k+1} - \frac{\Delta t}{\varrho}\nabla\cdot(\alpha(u^{n,k}\nabla u^{n,k+1}) = u^{(1)} + \frac{\Delta t}{\varrho}f(\mathbf{x},t) \tag{9}$$

Our initial guess $u^{n,k=0}$ which starts the iteration, will be the previous time step which we denote as u^{-}. Replacing $u^{n,k+1}$ as u which is the unknown we want to solve for, and using the variational form we can write the system in a simble form

$$(u,v) + \frac{\Delta t}{\varrho}(\alpha(u^{-})\nabla u), \nabla v) = (u^{(1)},v) + \frac{\Delta t}{\varrho}(f(\mathbf{x},t),v) \tag{10}$$

Exercise c

```
def Piccard(self, iterations, getstep = None):
  #Define Functionspace
    mesh = self.mesh

  V = FunctionSpace(mesh, 'CG', 1); self.V = V
  u = TrialFunction(V)
  v = TestFunction(V)

  #Initial Condition
  u0 = self.u0
  u_1 = interpolate(u0, V)
  #Define variatonal problem
    dt = self.dt; q = self.q; T = self.T
  a = u*v*dx + dt/q*self.alpha_func(u_1)*inner(nabla_grad(u), nabla_grad(v))*dx
  L = (u_1 + dt/q*self.f_func() )*v*dx
  A = assemble(a)
  u = Function(V)
   Nt = int(round(T/dt)+1)
    E_list = np.zeros(Nt+1)
  error = 1
  k = 0

    for i in range(1, Nt+1):
        #self.f_func().t = i*dt
        self.f.t = i*dt
        while error > 1E-5 and k < iterations:
            b = assemble(L)
            solve(A, u.vector(), b)
            k += 1
            u_1.assign(u)
            error = errornorm(u,u_1)

        if getstep == i*dt:
            u0.t = i*dt
            ue = interpolate(u0, V)
            e = ue.vector().array() - \
                u_1.vector().array()
            E = np.sqrt(np.sum(e*e)/u_1.vector().array().size)
            self.u = u_1
            return E
    self.u = u
```

```
        return u
```

Exercise d

For our first verification of the FEniCS implementation, we will try to reproduce a constant solution $u(\mathbf{x},t) = C$ where C is some arbitary constant. Implementing this solution into our equation, we observe the following relations

- $u_t = 0$ implies ϱ can be chosen arbitary
- $\nabla u = 0$ implies α can be chosen arbitary
- $f(\mathbf{x},t) = 0$
- $I(\mathbf{x}) = C$

```
    for i in range(1,3): #Testing P1 and P2 elements
            while dim <= 3: #Increasing dimention of solutionspace
                sol.define_mesh(Dim[dim-1])
                u = self.sol.Piccard(1, P_element = i)
                u_vector = u.vector().array()
                #u_vector[0]-1E-5 because of small round off errors
                test = all(x == u_vector[0]-1E-5 for x in u_vector)
                assert(test == True) #Test if all values are equal
                dim += 1
            dim = 1
```

For constant solution the assert test will not be raised, which is my output in my coderun.

Exercise e

We know want to take a step furhter, and try to reproduce an analytical solution. We will assume the following

- $\alpha(u) = 1$
- $f(\mathbf{x},t) = 0$
- $\Omega = [0,1] \times [0,1]$ with P1 elements
- $I(\mathbf{x}) = cos(\pi x)$

These assumptions will result in the manufactured solution $u(x,y,t) = e^{-\pi^2 t} cos(\pi x)$
Error in space is $\mathcal{O}(\Delta x^2) + (\Delta y^2)$, while for time $\mathcal{O}(\Delta t^p)$.
We can then write a model for the error such that
$E = K_t \Delta t^p + K_x \Delta x^2 + K_y \Delta y^2 = Kh$, where $h = \Delta t^p = \Delta x^2 = \Delta y^2$
By reducing h, we are to show that $\frac{E}{h}$ remains approximately constant as h is redfined. By dividing h by 2 for each run I get the following result.

```
E/h = 1.0827 for t=0.01000 seconds
E/h = 1.0861 for t=0.00500 seconds
E/h = 1.0901 for t=0.00250 seconds
E/h = 1.0838 for t=0.00125 seconds
E/h = 1.0849 for t=0.00063 seconds
```

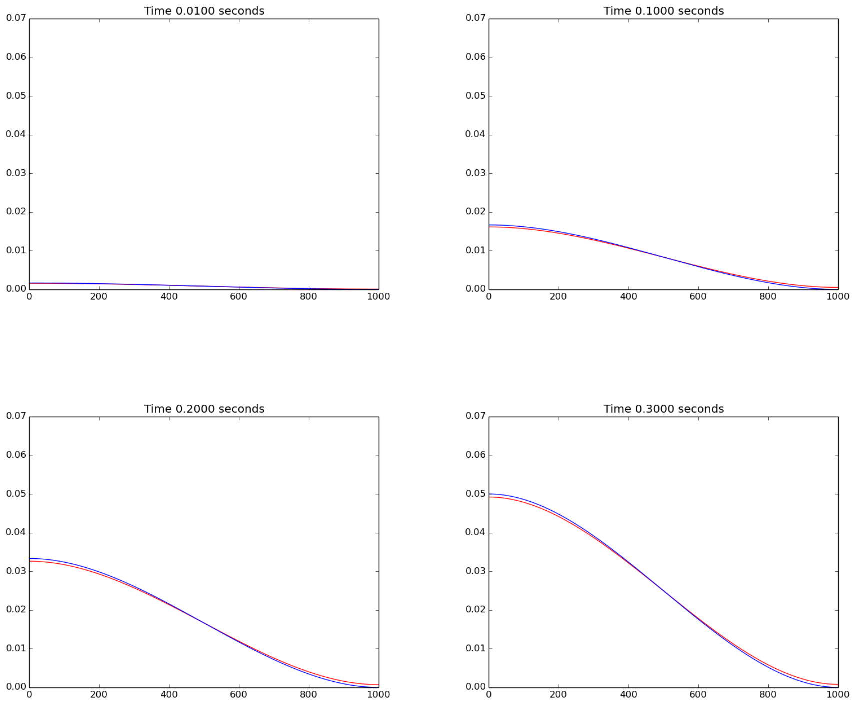
We observe that this ratio remains approximately constant.

Exercise f

In this exercise we will test our solver with a manufactured solution. Restricting us to one space dimension, define $\alpha(u) = 1 + u^2$ we choose u(x,t) such that

$$u(x,t) = t\int_0^x q(1-q)\ dq = tx^2(\frac{1}{2} - \frac{x}{3}) \tag{11}$$

Using the presented sympycode to calculated f, we compare the numerical solution with the manufactured solution.



Which are some satisfying results.

Exercise g

Errors contributing to errors in the numerical calculations.

- Numerical Integration in the variational form
- Choosing a finite number of functions in our functionspace to represent the solution.
- The values of dt and dx, chosen to high
- Choosing only one Piccard iteration

Exercise h

Using the presented sympy code we will try to eliminate the error due to a single Picard iteration. This by producing a manufactured solution for $\alpha(u^{(1)})$. We keep the solution from exercise *f*, and $\alpha(u) = 1 + u^2$. Using the relation $h = \Delta t = \Delta x^2$ we will make a convergence test by assuming $E \sim h^r$, showing that r tends to converge to 1. My numerical solver produce the following

```
1.00082805   1.00553296   1.00521814   1.00479102   1.0026296]
```

Exercise i

Finally we will simulate the nonlinear diffusion equation of a Gaussian function.

$$I(x, y) = exp(-\frac{1}{2\sigma^2}(x^2 + y^2)) \qquad (x, y) \in \Omega = [0, 1] \times [0, 1] \tag{12}$$

For my solution I chose $\sigma = 0.09, \beta = 100.0$, Check out the gaussian.gif file, to see animation of the solution.