3 Functions in the Sobolev space H^1

The Sobolev space H^1 is of fundamental importance for the formulation of finite element methods. In this exercise, we examine for several functions whether they are in H^1 or not.

The definition of H^1 is based on the inner product

$$(f, g)_{H^1} = \int_\Omega f(\mathbf{x})\ g(\mathbf{x}) + \nabla f(\mathbf{x}) \cdot \nabla g(\mathbf{x})\ dV \tag{1}$$

where $\Omega \subset R^n$ and $f, g : \Omega \to R$. Note that for $n = 1$, the inner product becomes simply

$$(f, g)_{H^1} = \int_a^b f(x)\ g(x) + f'(x)\ g'(x)\ dx. \tag{2}$$

Based on the inner product (2), the H^1-norm

$$\|f\|_{H^1} = \sqrt{(f, f)_{H^1}} \tag{3}$$

is defined and we obtain the Sobolev space

$$H^1(\Omega) = \{f : \Omega \to R \mid \|f\|_{H^1} < \infty\} \tag{4}$$

of all functions on Ω for which the H^1-norm is finite.

3.1 A piecewise function on R

The space H^1 contains functions which are continuous and piecewise continually differentiable. In this example, it is shown that functions including jumps in the function values are not included in this space.

Tasks:

1. Show that the piecewise function $f : [-1, 1] \to R$,

$$f(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{otherwise} \end{cases} \tag{5}$$

is not in H^1.

Hints:

1. The problem can not be solved by simply computing

$$\sqrt{\int_a^b f(x)^2 + f'(x)^2\ dx} = \sqrt{2} \tag{6}$$

since the integral does not "see" the discontinuity at $x = 0$. This simple considerations is only suited if the origin $x = 0$ is excluded from the domain on which the function is defined. In other words: By evaluating the integral (6), it is shown that the function $g : [-1, 1]\backslash 0 \to R$,

$$g(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{otherwise} \end{cases} \tag{7}$$

is in H^1. If the point $x = 0$ should be included in the domain, a regularization step is required.

2. In order to show that f is not in H^1, define the continous function $\tilde{f} : [-1, 1] \to R$,

$$\tilde{f}(x) = \begin{cases} -1 & \text{if} \quad x < -\epsilon \\ \frac{x}{\epsilon} & \text{if} \quad -\epsilon \le x < \epsilon \\ 1 & \text{otherwise} \end{cases} . \tag{8}$$

Since $\tilde{f}$ becomes f if ϵ goes to zero,

$$\|f\|_{H^1} = \lim_{\epsilon \to 0} \|\tilde{f}\|_{H^1}. \tag{9}$$

Use (9) to show that $f \notin H^1$.

3. Piecewise functions can be defined in Mathematica using `Piecewise`.

3.2 Crack tip functions

Crack tip functions are used in mechanics to model displacements on domains including a crack. The crack tip functions are defined in polar coordinates by

$$\hat{f}_1(r, \theta) = \sqrt{r} \sin\left(\frac{\theta}{2}\right) \tag{10}$$

$$\hat{f}_2(r, \theta) = \sqrt{r} \cos\left(\frac{\theta}{2}\right) \tag{11}$$

$$\hat{f}_3(r, \theta) = \sqrt{r} \sin\left(\frac{\theta}{2}\right) \sin(\theta) \tag{12}$$

$$\hat{f}_4(r, \theta) = \sqrt{r} \cos\left(\frac{\theta}{2}\right) \sin(\theta). \tag{13}$$

Using the relations

$$r(x, y) = \sqrt{x^2 + y^2} \tag{14}$$

$$\theta(x, y) = \arctan\left(\frac{x}{y}\right) \tag{15}$$

$$\tag{16}$$

the crack tip functions can be easily expressed in terms of cartesian coordinates. In this example, we consider the functions $f_i : [-1, 1]^2 \backslash \{(x, 0) \mid x < 0\} \to R$,

$$f_i(x, y) = \hat{f}_i(r(x, y), \theta(x, y)),\ i = 1, \ldots, 4. \tag{17}$$

In the definition of the domain of f_i, points on the negative x-axis are excluded in order to incorporate the crack shown in Figure 1.

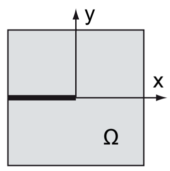

Figure 1: Domain with crack

Tasks:

1. Define and plot the crack tip functions f_i.
2. The integrand in the H^1 norm is $f^2(x, y) + \nabla f(x) \cdot \nabla f(x)$. In order to gain insight, what the H^1 norm evaluates, define and plot the functions

$$g_i(x, y) = f_i^2(x, y) + \nabla f_i(x) \cdot \nabla f_i(x),\ i = 1, \ldots, 4 \tag{18}$$

3. Show that

$$f_i \in H^1(\Omega) \tag{19}$$

with $\Omega = [-1, 1]^2 \backslash \{(x, 0) \in R^2 \mid x < 0\}$.

Hints:

1. The gradient of a function can be computed in Mathematica using `D[f,{{x, y}}]`.
2. Since the crack is excluded from the domain, it is not necessary to consider the discontinuity in the functions for the evaluation of the integral.