MEK 4250 Elementmethod
Mandatory Assignment

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1 Exercise 1

In these set of exercises we will study the Stokes problem defined as

$$
\begin{aligned}
-\Delta u + \nabla p &= f \ \text{ in } \ \Omega \\
\nabla \cdot v &= 0 \ \text{ in } \ \partial\Omega \\
u &= g \ \text{ in } \ \partial\Omega_N \\
\frac{\partial u}{\partial x} - pn &= h \ \text{ in } \ \partial\Omega_N
\end{aligned}
$$

First off we will define the weak formulation for the stokes problem. Let u $\in H^1_{D,g}$ and $p \in L^2$. Then the stokes problem can be defined as

$$
\begin{aligned}
a(u,v) + b(p,v) &= f(v) \ \ v \in H^1_{D,0} \\
b(q,u) &= 0 \ \ q \in L^2
\end{aligned}
$$

Where a and b defines the bilinear form, and f defines the linear form as

$$
\begin{aligned}
a(u,v) &= \int \nabla u : \nabla v \ dx \\
b(p,v) &= \int p \nabla \cdot v \ dx \\
f(v) = \int f v \ dx &+ \int_{\Omega_N} hv \ ds
\end{aligned}
$$

Further we will define to properties which will be usefull for solving the exercises
Cauchy-Schwarts inequality
Let V be a inner product space, then

$$
|\langle u \,, v \rangle| \ \leq \ ||u|| \cdot ||w|| \ \ \forall \ v, q \in V
$$

Poincare's Inequality Let $v \in H^1_0(\Omega)$

$$
||v||_{L^2(\Omega)} \leq C|v|_{H^1}(\Omega)
$$

Exercise 7.1

In this section we are to prove the conditions (7.14-7.16) from the course lecturenotes. Starting off with the first **Condition 7.14**

$$a(u_h, v_h) \leq C_1 ||u_n||_{V_n} ||v_n||_{V_n} \ \ \forall u_n, v_n \in V_n$$

As for in all of these conditions we will assume that $V_h \in H_0^1$, and for later conditions that$Q_h \in L^2$. First off we write out the term $a(u_h, v_h)$, and we observe we can use the Cauchy-Schwarts inequality since V is an inner product space.

$$a(u_h, v_h) = \int_\Omega \nabla u_h : \nabla v_h \; dx = \langle \nabla u_h \, , \nabla v_h \rangle$$
$$\langle \nabla u_h \, , \nabla v_h \rangle \leq |\langle \nabla u_h \, , \nabla v_h \rangle|_0 \leq ||\nabla u_h|| \cdot ||\nabla v_h||$$

Now since we have defined that $u_h, v_h \in V_h \in H_0^1$ we can use the Poincare inequality. First we observe that

$$||\nabla u_h||_{L^2}^2 \leq ||u_h||_{H^1}^2 = ||u_h||_{L^2}^2 + ||\nabla u_h||_{L^2}^2 \leq C_1 |u_h|_{H_1}^2 + ||\nabla u_h||_{L^2}^2 = C_1 ||\nabla u_h||_{L2}^2 + ||\nabla u_h||_{L^2}^2$$
$$(C_1 + 1)||\nabla u_h||_{L2}^2 \leq D ||u_h||_{H^1}^2$$

Condition 7.14

$$b(u_h, q_h) \leq C_2 ||u_h||_{V_h} ||q_h||_{Q_h} \ \ V_h \in H_0^1, \ \ Q_h \in L^2,$$

By direct insertion we get

$$b(u_h, q_h) = \int p \nabla \cdot v \; dx = \langle p \, , \nabla \cdot u \rangle \leq |\langle p \, , \nabla \cdot u \rangle|_0$$

Using the Cauchy-Schwarts inequality we can show that

$$|\langle q \, , \nabla \cdot u \rangle| \leq ||q||_0 \cdot ||\nabla \cdot u||_0$$

Hence, it holds to show that

$$||q||_0 \cdot ||\nabla \cdot u||_0 \leq C_2 ||u_h||_1 ||q_h||_0$$
$$||\nabla \cdot u||_0 \leq C_2 ||u_h||_1$$

We choose square the left side of the inequality and expand the norm, in hope of finding a term to determine the upper bound of b. Applying the poincare inequality on line 2, we can determine that the bound must be determined by some constant C_2

$$||\nabla \cdot u||_0^2 \leq ||u||_1^2 = ||u||_0^2 + ||\nabla u||_0^2$$
$$||u||_0^2 + ||\nabla u||_0^2 \leq C_2^2 |u|_1 + ||\nabla u||_0^2 = (C_2^2 + 1)||\nabla u||_0^2 = C_2^2 ||\nabla u||_0^2$$

Hence, to show the implied boundedness of b, it holds to show $||\nabla \cdot u||_0^2 \leq (C_2^2 + 1)||\nabla u||_0^2$ For simplicity we write out the terms for the R^2 case, but the same proof can be showed for the general case R^n.

$$||\nabla \cdot u||_0^2 = \int (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})^2 \; dx$$
$$||\nabla u||_0^2 = \int (\frac{\partial^2 u}{\partial x^2})^2 + (\frac{\partial^2 u}{\partial y^2})^2 + (\frac{\partial^2 v}{\partial x^2})^2 + (\frac{\partial^2 v}{\partial y^2})^2 \; dx$$

Remembering that since we are in a normed vector space V, the triangle inequality holds.

$$||x + y|| \leq ||x|| + ||y|| \ \ \forall x, y \in V$$

Rearrangeing the terms we observe that

$$
\begin{aligned}
||\nabla \cdot u||_0^2 &= \int (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})^2 \, dx \\
||\nabla u||_0^2 = \int (\frac{\partial^2 u}{\partial x^2})^2 + (\frac{\partial^2 v}{\partial y^2})^2 \, dx &+ \int (\frac{\partial^2 v}{\partial x^2})^2 + (\frac{\partial^2 u}{\partial y^2})^2 \, dx \\
\sqrt{\int (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2})^2 \, dx} &\leq \sqrt{\int (\frac{\partial^2 u}{\partial x^2})^2} + \sqrt{(\frac{\partial^2 v}{\partial y^2})^2 \, dx}
\end{aligned}
$$

Hence $||\nabla \cdot u||_0^2 \leq C_2^2 ||\nabla u||_0^2$, and the inequality holds.
Condition 7.16, Coersivity of a

$$a(u_h, v_h) \geq C_3 ||u_h||_{V_h}^2 \quad \forall u_h \in V_h$$

Using the Cauchy-Schwarts inequality as in the other proofs, we get similar result

$$
\begin{aligned}
a(u_h, v_h) = \int \nabla u : \nabla v \, dx &= \langle \nabla u_h \, , \nabla v_h \rangle \\
|\langle \nabla u_h \, , \nabla v_h \rangle|_0 &\leq ||\nabla u||_0 \cdot ||\nabla v||_0
\end{aligned}
$$