

MAT-INF4130

Mandatory Assignment 1

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As a strategy to approximate a function f with a function g ..

$$\|f - g\| = \sqrt{\int_0^1 (f(t) - g(t))^2 dt} \quad (1)$$

Further we define $N_1(x), N_2(x), \dots, N_n(x)$ to be a basis for the set of polynomials of degree at most $n-1$. For a chosen set of basis, we want to find a best fit polynomial to f , such that $\|\sum_{j=1}^n c_j N_j - f\|$ is minimized with respect to the coefficients c_j . From the least squares approximation, we end up with the system

$$\mathbf{c}^T \mathbf{N} \mathbf{c} - 2\mathbf{b}^T \mathbf{c} + \|f\|^2 \quad (2)$$

Where \mathbf{N} is the matrix with entries $\langle N_i, N_j \rangle$ and \mathbf{b} is the vector with entries $\langle N_j, f \rangle$

Problem 1

REWRITE AS HERMETIAN MATRIX!! First we want to show that matrix \mathbf{N} is positive definite. A $n \times n$ real matrix M defined as positive definite if the scalar $x^T M x$ is positive for every choice of a non-zero column vector x of dimension n . A natural extension of the definition would be to consider the system $\mathbf{c}^T \mathbf{N} \mathbf{c}$.

$$\mathbf{c}^T \mathbf{N} \mathbf{c} = c_i \langle N_i, N_j \rangle c_j = \int_0^1 \sum_{i=1}^n c_i N_i \sum_{j=1}^n c_j N_j dx = \int_0^1 p(x)^2 dx \geq 0 \quad (3)$$

Problem 2

Computing the gradient of the expression $\mathbf{c}^T \mathbf{N} \mathbf{c} - 2\mathbf{b}^T \mathbf{c} + \|f\|^2$ with respect to c_i we get

$$\frac{\partial}{\partial c_i} \mathbf{c}^T \mathbf{N} \mathbf{c} - 2\mathbf{b}^T \mathbf{c} + \|f\|^2 = 2\mathbf{N} \mathbf{c} - 2\mathbf{b}^T \quad (4)$$

(5)

From simple observations of the least squares method, one can see that the trend of $\|f - g\| = \sqrt{\int_0^1 (f(t) - g(t))^2 dt}$ will follow some sort of second order polynomial. Hence the minimal extreme point for the smallest error will be found for $\frac{\partial}{\partial c_i} \mathbf{c}^T \mathbf{N} \mathbf{c} - 2\mathbf{b}^T \mathbf{c} + \|f\|^2 = 0$ hence $\mathbf{N} \mathbf{c} = \mathbf{b}$

Problem 3

Defining $N_j(x) = x^{j-1}$ $1 \leq j \leq n$, we are ought to show that the matrix \mathbf{N} really is the Hilbert matrix with entries $\frac{1}{i+j-1}$

$$\mathbf{N} = \langle N_i, N_j \rangle = \int_0^1 x^{i-1} x^{j-1} dx = \int_0^1 x^{i+j-2} dx = \left[\frac{1}{i+j-1} x^{i+j-1} \right]_0^1 = \frac{1}{i+j-1} \quad (6)$$

Problem 4

Showing that $P_n(x) = x^n + \sum_{k=0}^{n-1} c_k x^k$ is orthogonal to $1, x, \dots, x^{n-1}$ on $[-1, 1]$, we require that $\langle P_n, N_j \rangle = 0$

Since $\langle P_n, N_j \rangle = 0$, then using the same arguments as in problem 3, we have to solve the linear system $\mathbf{Nc} = \mathbf{b}$, where \mathbf{b} in this will be a contribution from the x^n term.

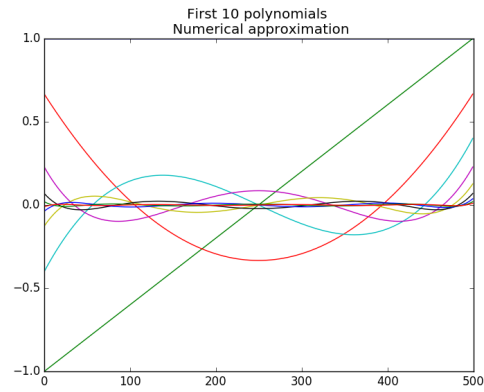
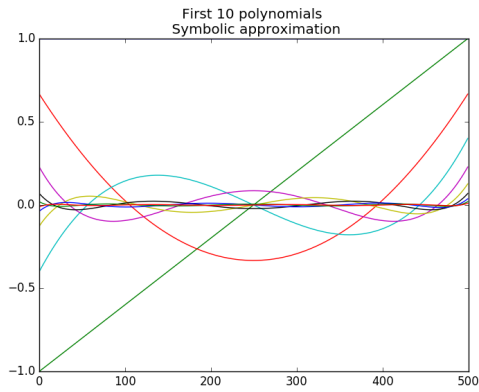
$$\langle P_n, N_j \rangle = \int_{-1}^1 \left[x^n + \sum_{k=0}^{n-1} c_k x^k \right] x^{j-1} dx = 0 \quad (7)$$

$$\int_{-1}^1 \sum_{k=0}^{n-1} c_k x^k x^{j-1} dx = - \int_{-1}^1 x^n x^{-j} dx = -\langle x^n, x^{j-1} \rangle \quad (8)$$

$$-\langle x^n, x^{j-1} \rangle = -\left[\frac{x^{n+j}}{n+j} \right]_{-1}^1 = \begin{cases} 0 & \text{if } n+j \text{ is even} \\ -\frac{2}{n+j} & \text{if } n+j \text{ is odd} \end{cases} \quad (9)$$

Problem 6

First 10 Legendre polynomials



Problem 7

First indication of trouble

