TASK 2: Discretization of convection/diffusion

Derive a proper variational formulation of the convection/diffusion problem. Derive sucient conditions that make the problem well-posed. Discuss why oscillations appear for standard Galerkin methods and show how SUPG methods resolve these problems. Discuss also approximation properties in light of Cea's lemma.

The convection–diffusion equation is a combination of the diffusion and convection (advection) equations, and describes physical phenomena where particles, energy, or other physical quantities are transferred inside a physical system due to two processes

$$\mu\nabla^2 u + v \cdot \nabla u = f \ \in \Omega$$
$$u = g \ \in \partial\Omega$$

- u = Is the
- $\mu\nabla^2 u$ = Diffusion term, distribution of consentration due to consentration differene (Ink in glass..)
- μ = Dynamic viscoucity
- $v \cdot \nabla u$ = Convection(Advection), distribution of consentration due to fluid flow
- v = Can bee fluid flow average, etc..

Singular Pertubation problem

Consequence $\mu \to 0$, boundary conditions can't be satisfied. Changes the very nature of problem. Practical situations $\mu > 0$, but small in the sence that $\mu << |v|$. This results in a overdetermined problem. (Also if μ goes to 0). For such problem where $\mu << |v|$, solution will be similar to $\mu = 0$, EXCEPT close to **NON-inflow boundary**. Here we will typically have a boundary layer. We will also observe that the discretized problem, will result in unphysical oscillations starting at this boundary layer.

0.1 1D con-diff problem

$$u_x - \mu u_{xx} = 0$$
$$u(0) = 0 \quad u(1) = 1$$
$$u_e(x) = \frac{e^{\frac{-x}{\mu}} - 1}{e^{\frac{-1}{\mu}} - 1}$$

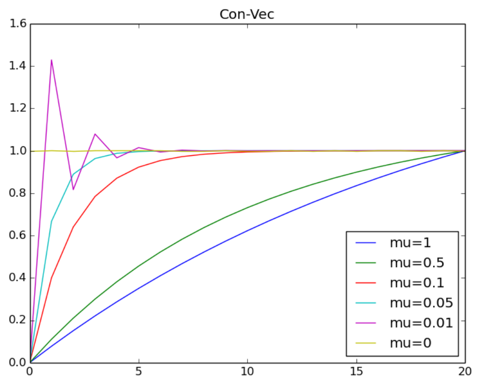
For $\mu \to 0$ both $e^{\frac{-x}{\mu}}$ and $e^{\frac{-1}{\mu}}$ and $u(x) \approx 1$ unless $x \approx 0$. Close to the inflow boundary x $= 0$, there will be a boundary layer where u has exponential growth.

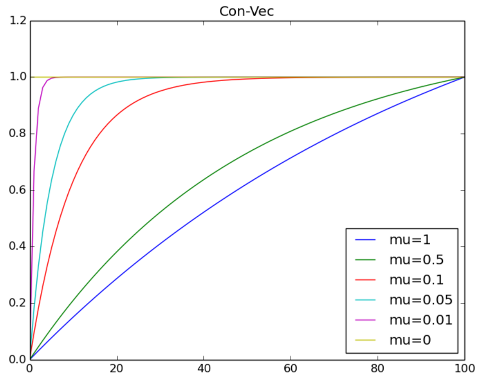
Galerkin method

Find $u \in H^1_{0,1}$ such that

$$\int_0^1 \big(u_x v - \mu u_{xx} v\big)\, dx = 0$$
$$\langle -u_x \,, v\rangle \ \ \mu\langle \nabla u \,, \nabla v\rangle = 0 \ \ \forall v \in H^1_{0,0}$$

CONVEC N $=$ 20 and N $=$ 100





Observe oscillations for low choice of N. WHY DO THIS HAPPEND? Explain with FDM.

Using central difference

$$\frac{\mu}{h^2}\Big[u_{i+1} - 2u_i + u_{i-1}\Big] - \frac{v}{2h}\Big[u_{i+1} - u_{i-1}\Big] = 0$$
$$u_0 = 0 \quad u_N = 1$$

for $\mu = 0$

$$\frac{v}{2h}\Big[u_{i+1} - u_{i-1}\Big] = 0 \quad u_{i+1} = u_{i-1} \quad \textbf{SOURCE OSCILLATIONS}$$

THE CURE Introduce and artifical diffusion term.
drop central difference scheme, use upwind such that

$$\frac{\partial u}{\partial x}(x_i) = \frac{u_{i+1} - u_i}{h} \quad v < 0$$
$$\frac{\partial u}{\partial x} = \frac{u_i - u_{i-1}}{h} \quad v > 0$$

- **PROS** = Oscillations dissapear
- **CONS** = 1 order convergence

POINT! Show relation to **upwind** and **artifical diffusion** Observe

$$\frac{u_{i+1} - u_{i-1}}{2h} \quad \text{Central Scheme, 1. order convergence}$$