Buffon's Needle Problem

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1 Introduction

The Buffon's Needle Problem [3] seems to have been the beginning of the study of "Geometric Probability", where probabilities are determined by comparison of measurements, rather than counting possible outcomes [2]. This approach comes to be useful when dealing with continuous variables, considering that it is a tool to solve the problem of infinite outcomes by measuring them in terms of length, area, or volume [4]. From a geometric solution to find the better rules for a gambling game [1], it turned to be a seed for numerical integration techniques using random numbers.

2 History

Monte Carlo methods have this name because of the Monte Carlo Casinos. The methods essential idea is using randomness to solve problems that might be deterministic in principle. Today, is told that the Needle Problem can help estimate π through a Monte Carlo method, and it's very common to know the first method as the Buffon idea. Note that a Monte Carlo method is exactly the opposite process of what Buffon did with geometry and needles.

Figure 1: La *Rue Buffon*

Then, in November 2013, the EMS[1] planned an action to promote mathematics: the "Buffon's experience" would be symbolically reproduced in *Rue Buffon*, at Paris. To better tell the history, they consulted a historian:

" In May 1733, Buffon submitted to the Académie Royale des Sciences (which he would become a member in 1734) an article in which, among other geometrical problems, correctly calculated the probability of a filiform object of length $2r$, thrown randomly, intersect any of the parallel lines separated by a constant distance a (with $2r < a$):

'Sur un plancher qui n'est formé que de planches égales & parallèles, on jette une Baguette d'une certaine longueur, & qu'on suppose sans largeur. Quand tombera-t-elle franchement sur une seule planche?' [2]

After deducing its formula, Buffon points out that it can be used to determine the value for which the probability of the stick falling within a single board is 50%:

'Il y a donc une certaine largeur de la planche qui rendroit le pari ou le jeu égal, & c'est ce que M. le Clerc a déterminé par une aire de Cycloïde avec beaucoup d' élégance au jugement de l'Académie' [3]

More than forty years later, Buffon returned to this problem more extensively in 1777, in his "Éssai d'Arithmétique morale", contained in "Histoire naturelle". Reading it clearly reveals that the main motivation for his investigations lay in the calculation of probabilities for players:

'Je suppose que, dans une chambre dont le parquet est simplement divisé par des points parallèles, on jette en l'air une baguette, et que l'un des joueurs parie que la baguette ne croisera aucune des parallèles du parquet, et que l'autre au contraire parie que la baguette croisera quelques-unes des ces parallèles; on demande le sort de ces deux joueurs. (On peut jouer ce jeu sur un damier avec une aiguille à coudre ou une épingle sans tête).' [4]

In the 40 years interval between these two written references, the only ones known to be written by him on this question, Buffon did *de facto* an experimental investigation - not on the problem of the needle, but on what is now known as the *St. Petersburg Paradox*. Briefly, this is based on a game in which

[1]European Mathematical Society

[2]"On a board floor, with equal and parallel boards, is dropped a stick with a certain length, and that will be assumed of negligible thickness. When will it fall on a single board?"

[3]"There is, therefore, a certain width of the boards which makes this bet, or game, fair, and that is what M. LeClerc determined through the area of a cycloid very elegantly, in the opinion of this Academy"

[4]"Suppose now that, in a room whose floor is simply divided by parallel lines, one throws a stick in the air, and a player bets that the stick will not cross none of the parallel lines of the floor, while the other, on the contrary, bet that the stick will cross one of these parallel lines. It is then asked the probability of success of each player (you can play this game on a checkerboard with a sewing needle or with pins without a head)."

a balanced coin is tossed until it falls with the chosen face up. If this event occurs in the $k-th$ release, the player gains $2k$ ducats. It is easy to verify that the expected value of the reward is infinite, and expect that the fair value that a player has to pay to play is, also, infinite. Buffon describes the game in [3], at the top of page 394. On page 399, he claims to have conducted experimental studies related to this problem:

'J'ai donc fait deux mille quarante-huit expériences sur cette question, c'est-à-dire j'ai joué deux mille quarante-huit fois ce jeu, en faisant jeter la pièce par un enfant.'[5]

Laplace considered, in his 1812 treatise on Probability Theory, the needle problem - not attributing its origin to Buffon, but explicitly referring, as far as is known for the first time, the possibility of using the theoretical calculations to determine an experimental approximation for π. In fact, after determining the probability of a line being touched by the needle, writes Laplace:

'Si l'on projette un grand nombre de fois ce cylindre, le rapport du nombre de fois où le cylindre rencontrera l'une des divisions du plan au nombre total des projections sera, à très peu près, la valeur de $4r/a\pi$, ce qui fera connaître la valeur de la circonférence 2π.' [6]

The Buffon's needle problem seems to have aroused interest in real experiences from the mid-nineteenth century. The first documented experimental study dated from 1850 and was performed by Rudolf Wolf (1816-1893), then a professor at the University of Bern. Curiously, Wolf discovered the result indirectly through the encyclopedia *Un million de faits de Lallane*(1843), which did not refer to its origin. Wolf was not aware at the time that the problem was due to Buffon. Augustus de Morgan refers, in 1859, that a certain Mr. Ambrose Smith performed the experiment in 1855, with 3204 launches, and one student of his, with 600 launches. The American astronomer Asaph Hall (who, curiously, gives the problem to Laplace) describes a set of experiments with more than half a thousand launches each, carried out in 1864 by his friend, Captain O. C. Fox, immobilized by a war wound."(free translation from [1])

Because of the knowledge of these facts, the EMS cancelled the action. The documents showed that the probability of Buffon had tried to estimate π with his experiments was minimum.

[5]"I have done 2048 experiments on this subject, that is, I have played this game 2048 times, using a child to toss the coin into the air."

[6]"If this cylinder is thrown a large number of times, the quotient between the total number of times the cylinder touches one of the lines and the total number of launches will have approximately the value $4r/a\pi$, which will allow to determine the value of the circumference 2π."

3 Calculating the probability

The Buffon's Needle Problem is: suppose a needle of length l, and a board floor with equal and parallel lines t units apart. Dropping the needle on the floor, what is the probability that it will lie across a line upon landing? At first, we have to define the sample space. Then, the necessary condition to the needle cross a line. Dividing the number of events which satisfy the condition by the total number of events in the sample space gives us the probability. Because of the subject addressed in this text, we'll try to demonstrate the solution in a geometrical way.

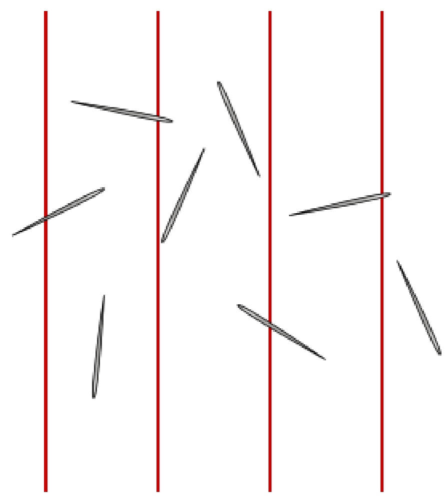

Figure 2: A lot of needles

To set the sample space, we can imagine the needle tip within a perpendicular to the bordering lines. The angle between the needle and the perpendicular will be θ [**]. Choosing a border, the farthest the needle tip can move away from it is t. Let x be an aleatory position in $[0, t]$. The uniform probability density distribution for the needle tip positions is:

$$\begin{cases} \frac{1}{t} & \text{for } 0 \le x \le t \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

For each point over the length t, the needle head can describe an arc of length $\pi/2$. Again, we have an uniform distribution:

$$\begin{cases} \frac{2}{\pi} & \text{for } 0 \le \theta \le \pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

[**] To simplify, we will use a quarter of the circumference. Besides this, we will not consider its radius, as they appear both in the numerator and the denominator, and the calculations cancel it.

The random variables x and θ are independent. Then, we can represent our sample space as: (3) the volume drawn by the summation of all the t lengths through $[0, \pi/2]$, or; (4) the volume drawn by the summation of all the $\pi/2$ arcs through $[0, t]$, or; (Figure 3) geometrically:

$$\begin{cases} 1/\int_0^{\pi/2} t d\theta = 2/t\pi & \text{for } 0 \leq \theta \leq \pi/2 \text{ and } 0 \leq x \leq t \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$\begin{cases} 1/\int_0^{t} \pi/2 dt = 2/t\pi & \text{for } 0 \leq \theta \leq \pi/2 \text{ and } 0 \leq x \leq t \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

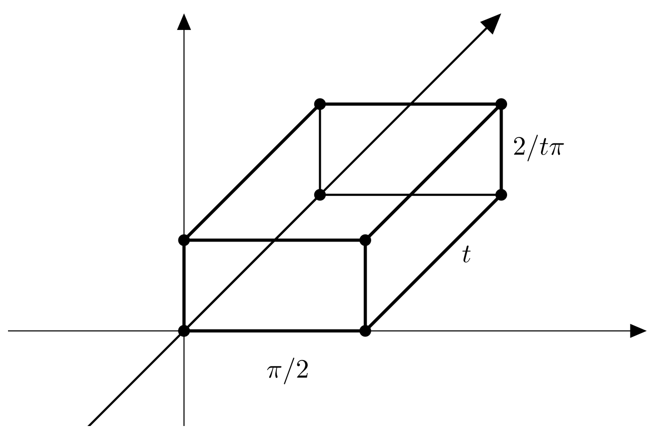

Figure 3: the sample space, geometrically

Set the sample space, we need to define our event. The distance $l \cos\theta$ from the chosen border is needed in order for the needle to cross it. The event that the needle tip is away from the border no more than a distance $l \cos\theta$ is:

$$0 \leq l \cos\theta \leq l \Leftrightarrow 0 \leq \cos\theta \leq 1 \Leftrightarrow -\pi/2 \leq \theta \leq \pi/2^{\dagger\dagger} \tag{5}$$

Finally, our probability is given by:

$$P = \frac{\int_0^{\pi/2} l \cos\theta d\theta}{\int_0^{\pi/2} t d\theta} = \frac{l \int_0^{\pi/2} \cos\theta d\theta}{t \int_0^{\pi/2} 1 d\theta} = \frac{l}{t\pi/2} = \frac{2l}{t\pi} \tag{6}$$

[††] Remember that, because we using a quarter of the circumference, we will use the interval $0 \leq l \cos\theta \leq l \Leftrightarrow 0 \leq \cos\theta \leq 1 \Leftrightarrow 0 \leq \theta \leq \pi/2$

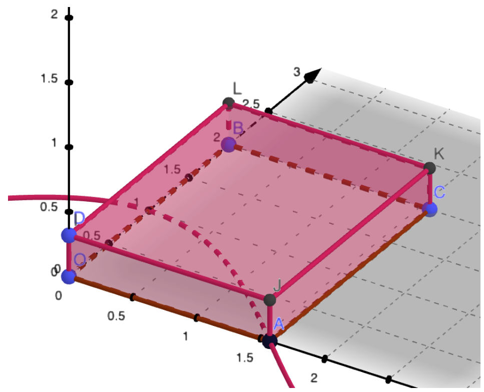

Figure 4: Extruding the curve of $\cos \pi/2$ inside the sample space, for $l = 1$ and $t = 2$, we have the boudaries of the event

References

[1] Ehrhard Behrends and Jorge Buescu. "Terá Buffon realmente lançado agulhas?" In: *Boletim da Sociedade Portuguesa de Matemática* 71 (2014), pp. 123–132.

[2] Scott E. Brodie. *Buffon's Needle Problem*. URL: `https://www.cut-the-knot.org/fta/Buffon/buffon9.shtml`. (last accessed: 16.03.2019).

[3] G. L. Leclerc de Buffon. *Essai d'Arithmétique morale*. Histoire naturelle, générale er particulière, Supplément 4. de l'imprimerie royale, 1777, pp. 46–123. URL: `https://books.google.com.br/books?id=AjhYD1vsVAIC`.

[4] Pranshu Gaba et al. *Geometric Probability*. URL: `https://brilliant.org/wiki/1-dimensional-geometric-probability/`. (last accessed: 16.03.2019).