Chapter 2

Experiments with photons

A Mach-Zehnder interferometer includes two beam splitters, two mirrors, and two detectors. A photon in the interferometer can take either of two paths and we use a two-component wavefunction to describe its behavior. The effect of a beam splitter is encoded in a two-by-two matrix that acts on the wavefunction. The curious properties of quantum interference give the possibility of an interaction-free measurement, which we illustrate in the context of Elitzur-Vaidman bombs.

2.1 Mach-Zehnder interferometer

We have discussed before the Mach-Zehnder interferometer, which we show again in Figure 2.1. It contains two beam-splitters BS1 and BS2 and two mirrors. Inside the interferometer we have two beams, one going over the upper branch and one going over the lower branch. This extends beyond BS2: the upper branch continues to D0 while the lower branch continues to D1.

Imagine vertical cuts in the above figure, lying between BS1 and BS2. Any such cut intersects the two beams and we can ask what is the probability to find a photon in each of the two beams, at that cut. For this we need two probability *amplitudes*, or two complex numbers, whose norm-squared would give probabilities. We can encode this information in a two component vector as

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} . \tag{2.1.1}$$

Here α is the probability amplitude to be in the upper beam and β the probability amplitude to be in the lower beam. Therefore, $|\alpha|^2$ would be the probability to find the photon in the upper beam and $|\beta|^2$ the probability to find the photon in the lower beam. Since the photon must be found in either one of the beams we must have

$$\left|\alpha\right|^2 + \left|\beta\right|^2 = 1 \, . \tag{2.1.2}$$

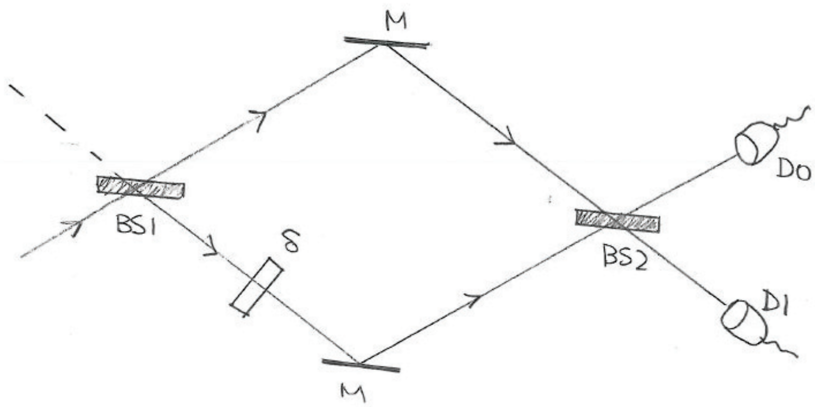

Figure 2.1: The Mach-Zehnder Interferometer

Following this notation, we would have for the states when the photon is definitely in one or the other beam:

$$\text{photon on upper beam: } \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \text{photon on bottom beam : } \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.1.3}$$

We can view the state (2.1.1) as a superposition of these two simpler states using the rules of vector addition and multiplication:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.1.4}$$

In the interferometer shown in Figure 2.1 we included in the lower branch a 'phase shifter',

Figure 2.2: A phase shifter of phase factor $e^{i\delta}$. The amplitude gets multiplied by the phase.

a piece of material whose only effect is to multiply the probability *amplitude* by a fixed phase $e^{i\delta}$ with $\delta \in \mathbb{R}$. As shown in Figure 2.2, the probability amplitude α to the left of the device becomes $e^{i\delta}\alpha$ to the right of the device. Since the norm of a phase is one, the phase-shifter does not change the probability to find the photon in the beam. When the phase δ is equal to π the effect of the phase shifter is to change the sign of the wavefunction since $e^{i\pi} = -1$.

Let us now consider the effect of beam splitters in detail. If the incoming photon hits a beam-splitter from the top, we consider this photon to belong to the upper branch and represent it by $\begin{pmatrix}1\\0\end{pmatrix}$. If the incoming photon hits the beam-splitter from the bottom, we consider this photon to belong to the lower branch, and represent it by $\begin{pmatrix}0\\1\end{pmatrix}$. We show the two cases in Figure 2.3. The effect of the beam splitter is to give an output wavefunction for each of the two cases:

$$\text{Left BS:} \quad \begin{pmatrix}1\\0\end{pmatrix} \to \begin{pmatrix}s\\t\end{pmatrix}, \qquad \text{Right BS:} \quad \begin{pmatrix}0\\1\end{pmatrix} \to \begin{pmatrix}u\\v\end{pmatrix}. \tag{2.1.5}$$

As you can see from the diagram, for the photon hitting from above, s can be viewed as

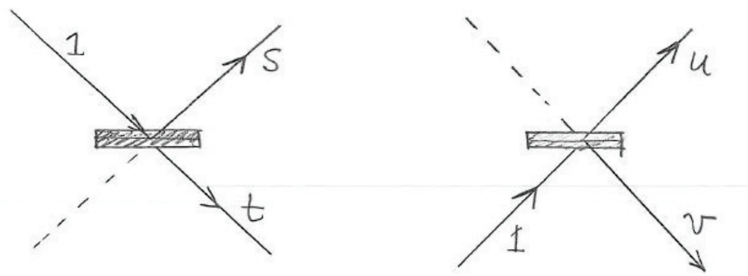

Figure 2.3: Left: A photon incident from the top; s and t are the reflected and transmitted amplitudes, respectively. Right: A photon incident from the bottom; v and u are the reflected and transmitted amplitudes, respectively.

a reflection amplitude and t as a transmission coefficient. Similarly, for the photon hitting from below, v can be viewed as a reflection amplitude and u as a transmission coefficient. The four numbers s, t, u, v, by linearity, characterize completely the beam splitter. While they are not completely arbitrary, for a given beam splitter they can be used to predict the output given any incident photon, which may have amplitudes to hit both from above and from below. Indeed, an incident photon state $\begin{pmatrix}\alpha\\\beta\end{pmatrix}$ would give

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} = \alpha\begin{pmatrix}1\\0\end{pmatrix} + \beta\begin{pmatrix}0\\1\end{pmatrix} \quad \to \quad \alpha\begin{pmatrix}s\\t\end{pmatrix} + \beta\begin{pmatrix}u\\v\end{pmatrix} \; = \; \begin{pmatrix}\alpha s + \beta u\\ \alpha t + \beta v\end{pmatrix} \; = \; \begin{pmatrix}s & u\\ t & v\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}. \tag{2.1.6}$$

In summary, we see that the BS produces the following effect

$$\begin{pmatrix}\alpha\\\beta\end{pmatrix} \quad \to \quad \begin{pmatrix}s & u\\ t & v\end{pmatrix}\begin{pmatrix}\alpha\\\beta\end{pmatrix}. \tag{2.1.7}$$

We can therefore represent the action of the beam splitter as matrix multiplication on the incoming wavefunction column vector, with the two-by-two matrix

$$\begin{pmatrix}s & u\\ t & v\end{pmatrix}. \tag{2.1.8}$$

We must now figure out the constraints on s, t, u, v. Because probabilities must add up to one, equation (2.1.5) implies that

$$\begin{aligned} \big|s\big|^2 + \big|t\big|^2 &= 1\,, \qquad (2.1.9)\\ \big|u\big|^2 + \big|v\big|^2 &= 1\,. \qquad (2.1.10)\end{aligned}$$

The kind of beam splitters we use are called balanced, which means that reflection and transmission probabilities are the same. So all four constants must have equal norm-squared:

$$\big|s\big|^2 = \big|t\big|^2 = \big|u\big|^2 = \big|v\big|^2 = \tfrac{1}{2}\,. \qquad (2.1.11)$$

This means all coefficients are determined up to phases. Let's try a guess for the values. Could we have

$$\begin{pmatrix} s & u \\ t & v \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\ ? \qquad (2.1.12)$$

This fails if acting on normalized wavefunctions (or column vectors) does not yield normalized wavefunctions. So we try with a couple of wavefunctions

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, \qquad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \qquad (2.1.13)$$

While the first example works out, the second one does not, as $|1|^2 + |1|^2 = 2 \neq 1$. An easy fix is achieved by changing the sign of v:

$$\begin{pmatrix} s & u \\ t & v \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \qquad (2.1.14)$$

Let's check that this matrix works in general. Consider acting on a normalized state $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$, so that $|\alpha|^2 + |\beta|^2 = 1$. We then find

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \alpha + \beta \\ \alpha - \beta \end{pmatrix}. \qquad (2.1.15)$$

Indeed the resulting state is well normalized:

$$\begin{aligned} \tfrac{1}{2}\big|\alpha + \beta\big|^2 + \tfrac{1}{2}\big|\alpha - \beta\big|^2 &= \tfrac{1}{2}\big(\big|\alpha\big|^2 + \big|\beta\big|^2 + \alpha\beta^* + \alpha^*\beta\big) + \tfrac{1}{2}\big(\big|\alpha\big|^2 + \big|\beta\big|^2 - \alpha\beta^* - \alpha^*\beta\big) \\ &= \big|\alpha\big|^2 + \big|\beta\big|^2 = 1\,. \end{aligned} \qquad (2.1.16)$$

As you can quickly verify, the minus sign in the bottom right entry of (2.1.14) means that a photon incident from below, as it is reflected, will have its amplitude changed by a sign or equivalently a phase shift by π. This effect is in fact familiar in electromagnetic theory. A typical beam splitter consists of a glass plate with a reflective dielectric coating on one side. The refractive index of the coating is chosen to be bigger than that of air and smaller than that of glass. In a medium, a reflected electromagnetic wave is phase shifted by π when the wave reflects off a material of higher refractive index. This is the case as a wave hits the coating from the air, but not when the wave hits the coating from glass. Thus the beam splitter represented by (2.1.14) would have its coating on the bottom side. Transmitted waves have no phase shift. The phase shift in the electromagnetic wave picture implies the phase shift for the quantum amplitudes.

Another possibility for a beam splitter matrix is

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}, \tag{2.1.17}$$

which would be realized by a dielectric coating on the top side. You can quickly check that, like the previous matrix, its action also conserves probability. We will call the left beam-splitter BS1 and the right beam splitter BS2 and their respective matrices will be

$$\text{BS1}:\ \frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}, \quad \text{BS2}:\ \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}. \tag{2.1.18}$$

The two beam splitters are combined to form the interferometer shown in Figure 2.4. If we now assume an input photon wavefunction $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ from the left, the output wavefunction that goes into the detectors is obtained by acting first with the BS1 matrix and then with the BS2 matrix:

$$\text{input}:\ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \qquad \text{output:}\ \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}0 & 2\\ -2 & 0\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}\beta\\ -\alpha\end{pmatrix}. \tag{2.1.19}$$

With the help of this result, for any input photon state we can write immediately the output photon state that goes into the detectors.

If the input photon beam is $\begin{pmatrix}0\\ 1\end{pmatrix}$, the output from the interferometer is $\begin{pmatrix}1\\ 0\end{pmatrix}$, and therefore a photon will be detected at D0. This is shown in Figure 2.5. We can make a very simple table with the possible outcomes and their respective probabilities P:

Outcome	P
photon at D0	1
photon at D1	0

(2.1.20)

Let us now block the lower path, as indicated in Figure 2.6. What happens then? It is best to track down things systematically. The input beam, acted by BS1 gives

$$\frac{1}{\sqrt{2}}\begin{pmatrix}-1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}0\\ 1\end{pmatrix} = \tfrac{1}{\sqrt{2}}\begin{pmatrix}1\\ 1\end{pmatrix}. \tag{2.1.21}$$

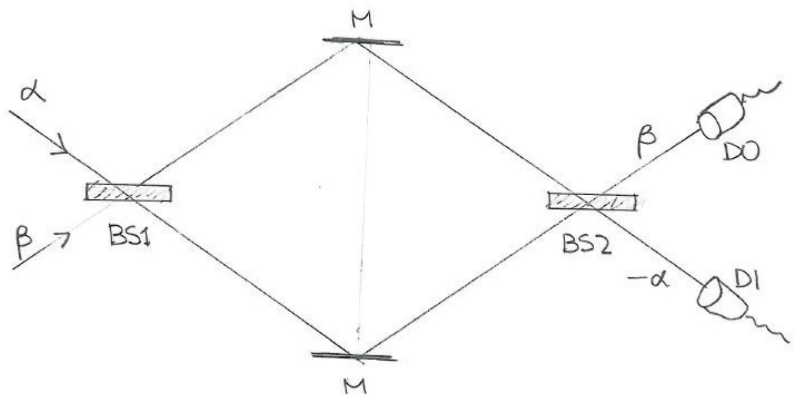

Figure 2.4: The Mach-Zehnder interferometer with input and output wavefunctions indicated.

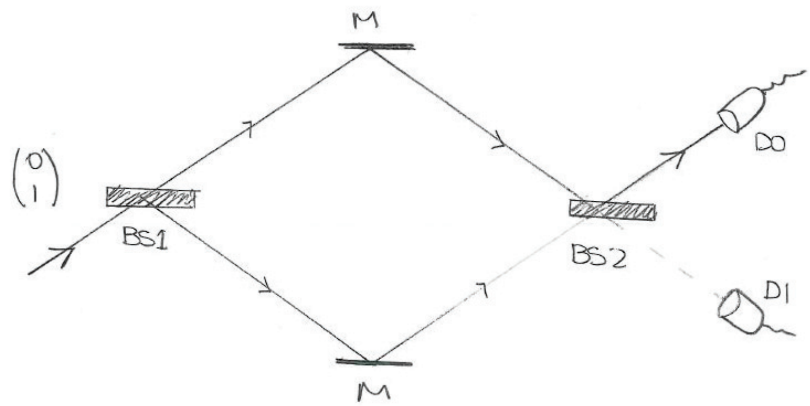

Figure 2.5: Incident photon from below will go into D0.

This is indicated in the figure, to the right of BS1. Then the lower branch is stopped, while the upper branch continues. The upper branch reaches BS2, and here the input is $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$, because nothing is coming from the lower branch. We therefore get an output

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} . \tag{2.1.22}$$

In this experiment there are three possible outcomes: the photon can be absorbed by the

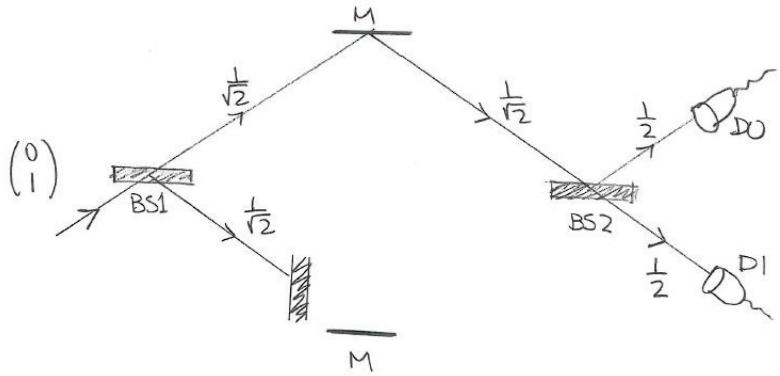

Figure 2.6: The probability to detect the photon at D1 can be changed by blocking one of the paths.

block, or can go into any of the two detectors. Looking the diagram we read the probabilities:

Outcome	P
photon at block	$\frac{1}{2}$
photon at D0	$\frac{1}{4}$
photon at D1	$\frac{1}{4}$

(2.1.23)

Note that before blocking the lower path we could not get a photon to D1. The probability to reach D1 is now 1/4 and was in fact *increased* by blocking a path.

2.2 Elitzur-Vaidman bombs

To see that allowing the photon to reach D1 by blocking a path is very strange, we consider an imaginary situation proposed by physicists Avshalom Elitzur and Lev Vaidman, from Tel-Aviv University, in Israel. They imagined bombs with a special type of trigger: a photon detector. A narrow tube goes across each bomb and in the middle of the tube there is a photon detector. To detonate the bomb one sends a photon into the tube. The photon is then detected by the photon detector and the bomb explodes. If the photon detector is defective, however, the photon is not detected at all. It propagates freely through the tube and comes out of the bomb. The bomb does not explode.

Here is the situation we want to address. Suppose we have a number of Elitzur-Vaidman (EV) bombs and we suspect some of them have become defective. How could we tell if a bomb is operational without detonating it? Assume, for the sake of the problem, that we are unable to examine the detector without destroying the bomb.

We seem to be facing an impossible situation. If we send a photon into the detector tube and nothing happens we know the bomb is defective, but if the bomb is operational it would simply explode. It seems impossible to confirm that the photon detector in the bomb is working without testing it. Indeed, it is impossible in classical physics. It is not impossible in quantum mechanics, however. As we will see, we can perform what can be called an interaction-free measurement!

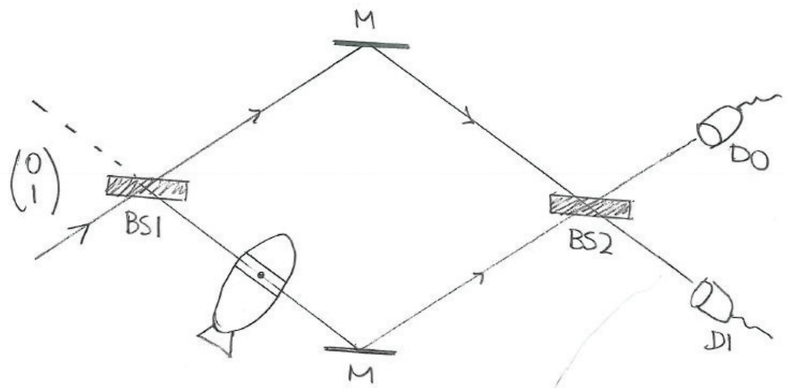

Figure 2.7: A Mach-Zehnder interferometer and an Elitzur-Vaidman bomb inserted on the lower branch, with the detector tube properly aligned. If the bomb is faulty all incident photons will end up at D0. If a photon ends up at D1 we know that the bomb is operational, even though the photon never went into the bomb detector!

We now place an EV bomb on the lower path of the interferometer, with the detector tube properly aligned. Suppose we send in a photon as pictured in Figure 2.7 . If the bomb is defective it is as if there is no detector, the lower branch of the interferometer is free and all the photons that we send in will end up in D0, just as they did in Figure 2.5.

Outcome	P
photon at D0 no explosion	1
photon at D1 no explosion	0
bomb explodes	0

(2.2.24)

If the bomb is working, on the other hand, we have the situation we had in Figure 2.6, where we placed a block in the lower branch of the interferometer:

Outcome	P
bomb explodes	$\frac{1}{2}$
photon at D0 no explosion	$\frac{1}{4}$
photon at D1 no explosion	$\frac{1}{4}$

(2.2.25)

Assume the bomb is working. Then 50% of the times the photon will hit it and it will explode, 25% of the time the photon will end in D0 and we can't tell if it is defective or not. But 25% of the time the photon will end in D1, and since this was impossible for a faulty bomb, we have learned that the bomb is operational! We have learned this fact even though the photon *never made it* through the bomb; it ended on D1. If you think about this you will surely realize it is extremely surprising and counterintuitive. But it is true, and experiments (without using bombs!) have confirmed that this kind of interaction-free measurement is indeed possible.

The above setup is not optimal, after all, 50% of the operational bombs will explode, but you will consider in the exercises more refined setups in which the probability of exploding an operational bomb can be made arbitrarily small.