Algebra II Notes

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Contents

1 Functions (8/12 to 8/27)

This unit will cover linear equations, solving for their inverses, and making compositions of functions.

1.1 Unit Vocabulary

- **Function:** A relation between two sets such that each value of one set maps to exactly one value of the other set
- **Domain:** The x-axis, or the first set of a function
- **Range:** The y-axis, or the second set of a function
- **Inverse Function:** A function with the domain and range swapped, usually notated with a superscript like $f^{-1}(x)$
- **Composition Function:** Using the output of one function as the input of a second

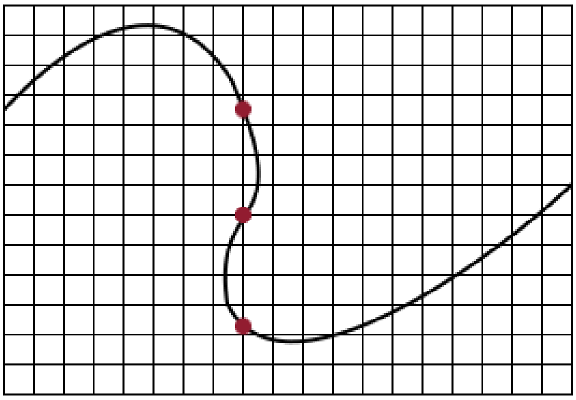
1.2 Intro to Linear Equations

A linear equation is one that can be expressed in the form $y = ax + b$, where a is a non-zero constant and b is a numerical constant. The important thing to note for a linear function is that the x variable does not have an exponent, and is simply raised to the first power. When graphed, linear equations look like lines. They can be expressed in three different forms: Slope-Intercept Form, Point-Slope Form, and Standard Form.

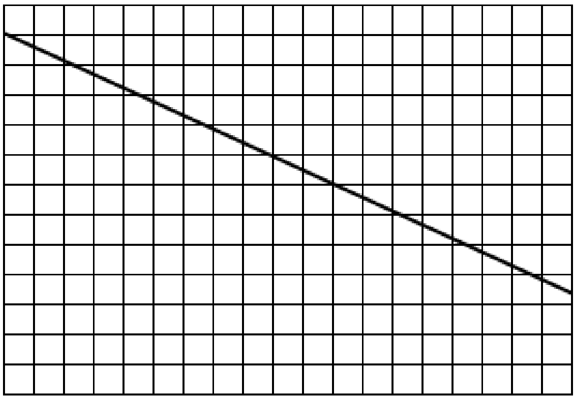
1. **Slope-Intercept Form:** $y = ax + b$
   - `Slope:` a
   - `Y-Intercept:` b
   - `X-Intercept:` $\frac{-b}{a}$
2. **Point-Slope Form:** $y - y_1 = m(x - x_1)$
   - `Slope:` m
   - `Point on Line:` (x_1, y_1)
   - `Y-Intercept:` $-mx_1 + y_1$
   - `X-Intercept:` $-\frac{y_1}{m} + x_1$
3. **Standard Form:** $Ax + By = C$
   - `Important Note:` A B and C must all be integers, and A must be positive.
   - `Slope:` $-\frac{A}{B}$
   - `Y-Intercept:` $\frac{C}{B}$
   - `X-Intercept:` $\frac{C}{A}$

1.3 What's a Function?

In the previous subsection I covered linear equations, which are an example of a function. A function is defined as a relation between two sets—the domain (x-axis) and the range (y-axis)—such that every domain value pairs with only one range value. We can test to see if a relation is a function by analyzing a table/function, or using the vertical line test. For a table, the relation is NOT a function if we can find an x-value that has multiple y outputs. With a graph, we can use what's called the Vertical Line Test. If we're able to draw a vertical line down the graph such that it hits the relation at least twice, it's not a function.



Not a function



Function

As you can see from above, the first graph has an instance where a line would hit the graph 3 times, hence not being a function. For the linear equation below that, however, no vertical line can disprove it's functionality.

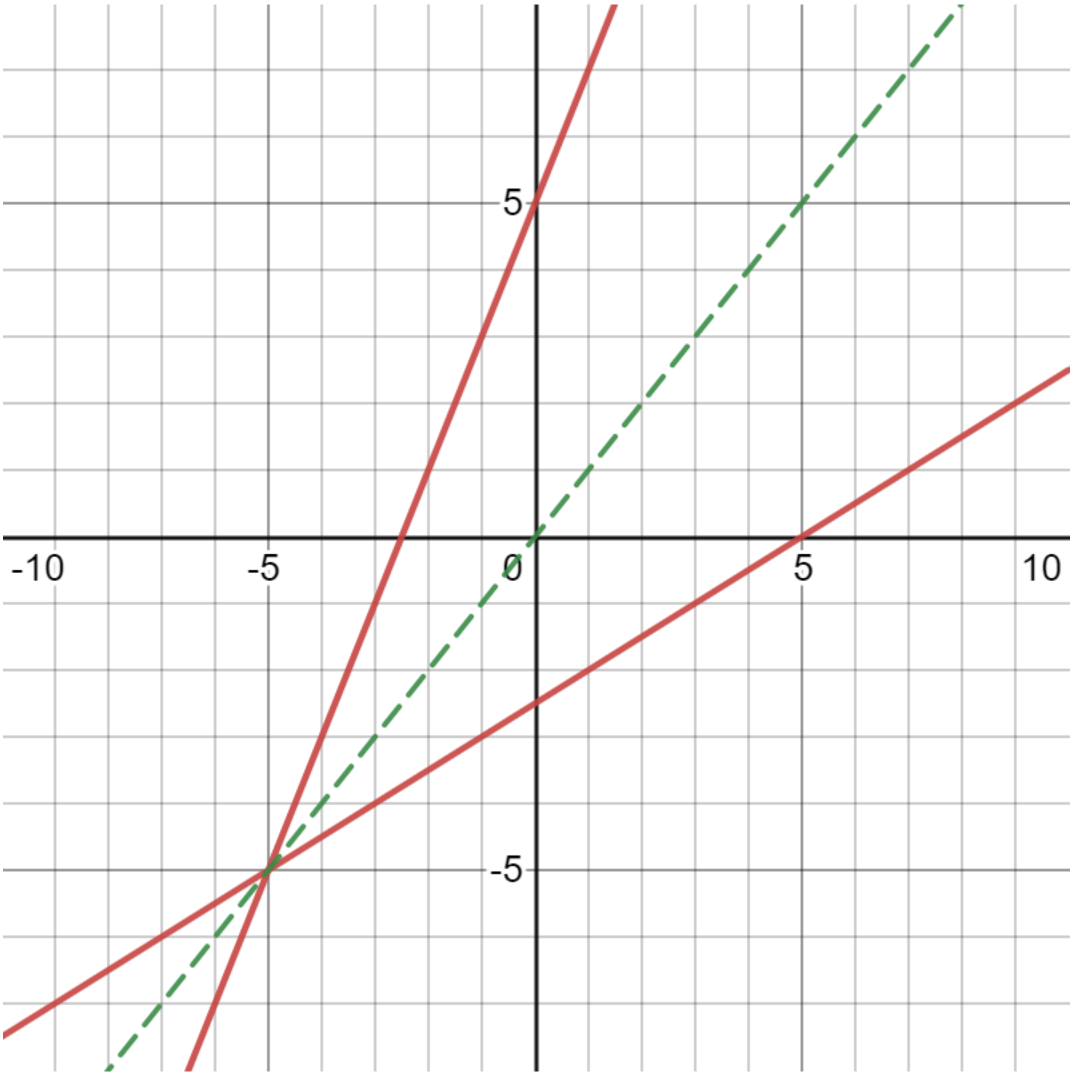
1.4 Function Composition

With functions, we can notate operations with them in different ways. Remembering that $f(2)$ means plugging in 2 for the variable in the function $f(x)$, we can rewrite $f(2) + g(2)$ as $(f + g)(2)$. We can do the same process for other operations, getting $(f - g)(2)$, $(f \times g)(2)$, and $\left(\frac{f}{g}\right)(2)$.

Additionally, we can write what are called composite functions. This simply means taking the output of a function to plug in as the input to another. For instance, we would calculate $f(g(2))$ by first figuring out what $g(2)$ is, then plugging that output into $f(x)$. The composite function $f(g(2))$ can be rewritten as $(f \circ g)(2)$.

1.5 Inverse Functions

When we swap the domain and range of a function, we can get a relation known as an inverse relationship. For instance, if the old function had $\{(2,0);(3,2);(10,1)\}$, the inverse relationship would have $\{(0,2);(2,3);(1,10)\}$. It's important to note that the inverse of a function will not necessarily be a function itself. For instance, the equation $y = x^2$ has both $(-2,4)$ and $(2,4)$ in its relationship, so the inverse would have 4 matched up with 2 ranges. When a function's inverse is also a function, it's known as a **one-to-one function**, since every domain is paired with one range value, and every range is paired with one domain value. Every linear function is a one-to-one function. It's also interesting to note that a function and its inverse are reflections over the equation $y = x$, because reflecting the point (α, β) over $y = x$ returns (β, α).



To solve for an inverse relation, simply switch the x and y variables and solve for y once again. An example is shown below.

What is the inverse of $y = 2x + 5$?

Swapping the x and y values, we get $x = 2y + 5$.

$$x - 5 = 2y$$

$$y = \frac{x-5}{2}$$

$$\boxed{y = \frac{1}{2}x - \frac{5}{2}}$$

2 System of Equations (8/28 to 9/15)

This section will go over different methods to solve a system of equations.

2.1 Unit Vocabulary

- **System of Equations:** A group of equations that share variables, usually used to solve for a specific set of solutions
- **Inequality:** Rather than setting two sets equal to each other (equations), inequalities set one set to be greater or less than the second set. Additionally we can add the set itself, so set 1 is greater/less than <u>or equal to</u> Set 2 ($2x + 4 \geq 7x - 1$)

2.2 What is a System?

Especially with word problems, we sometimes have to use multiple equations to figure out a solution to a problem. When we're solving for a common solution in multiple equations, we're solving a system of equations. There's three main ways to solve a system of equations: Substitution, Elimination, and Reduced-Row Echelon Form. I'm going to go through each of the methods one at a time, all of them using the common system below:

$$\begin{cases} x - 4y = 2 \\ 2x + 4y = 16 \end{cases}$$

2.2.1 Substitution Method

For the substitution method, we want to rewrite one of our equations solving for a specific variable. We typically use the substitution method when the coefficient of a variable is 1. In this case, the first part of our system has a 1 coefficient for x, so we can solve as the following

$$x - 4y = 2 \implies x = 4y + 2$$

$$2x + 4y = 16 \implies 2(4y + 2) + 4y = 16$$
$$2(4y + 2) + 4y = 8y + 4 + 4y = 12y + 4 = 16$$
$$12y = 12 \implies \boxed{y = 1}$$

Now that we know y, we can simply plug this into either equation to find x.

$$x - 4(1) = 2$$

$$x - 4 = 2 \implies \boxed{x = 6}$$

2.2.2 Elimination Method

For the elimination method, we can add or subtract the two equations to cancel out variables. This is especially easy when the coefficients of the same variable are opposites (the first equation has $-\mathbf{4}y$ and the second has $\mathbf{4}y$). If necessary, we can also multiply one/both of the equations by a constant to make inverse coefficients. If we add $x - 4y = 2$ with $2x + 4y = 16$ we get the following.

$$\begin{cases} x - 4y = 2 \\ 2x + 4y = 16 \end{cases}$$

$$3x = 18 \implies \boxed{x = 6}$$

Just like the substitution method, we can just plug in our x value to find out $\boxed{y = 1}$.

2.2.3 Reduced-Row Echelon Form (RREF)

This is probably the most confusing of the three methods, but is a good last resort if both substitution and elimination become drastically complicated. For this to work, we need to rewrite our equation in a <u>matrix</u>. Each row of the matrix will be for a different equation, and each column will contain the coefficients/constant for the respective variables. If we write our matrix in column order (x-coefficient, y-coefficient, constant), we get the following:

$$\begin{bmatrix} 1 & -4 & 2 \\ 2 & 4 & 16 \end{bmatrix}$$

We can now plug this into our calculator (handheld or online) to get our result. If you're working online, the Desmos matrix calculator `https://www.desmos.com/matrix` works really well. For a handheld, I can explain the TI-84 instructions below.

First, we want to hit our keys in the order 2ND (BLUE KEY), MATRIX (ABOVE THE x^{-1}), RIGHT ARROW, ALPHA (GREEN KEY), B (ABOVE THE APPS). Additionally, you can also just scroll down until you find RREF rather than click ALPHA → B. We then want to add our matrix by doing ALPHA (GREEN KEY, F3 (ABOVE THE ZOOM BUTTON) and then setting our dimensions. Since we have two equations and three slots in total—(x-coefficient, y-coefficient, constant)—we select 2 rows and 3 columns. After inputting data into the matrix like previously stated, we get an interesting resulting matrix that results in the following:

$$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 1 \end{bmatrix}$$

Remember how we had to input this in the equation order x y coefficient? If we rewrite in the same way we get two equations: $x + 0y = 6$ and $0x + y = 1$. This simply means $x = 6$ and $y = 1$!

2.3 Three-Variable Matrices (Uh-oh)

Three-Variable matrices can also be solved in the same way as the above mentioned two-variable matrices. Because of their complications, the rref method becomes a lot more common for problem solving. The below system of equations can be rewritten in the following matrix, noting the new $3x4$ dimensions.

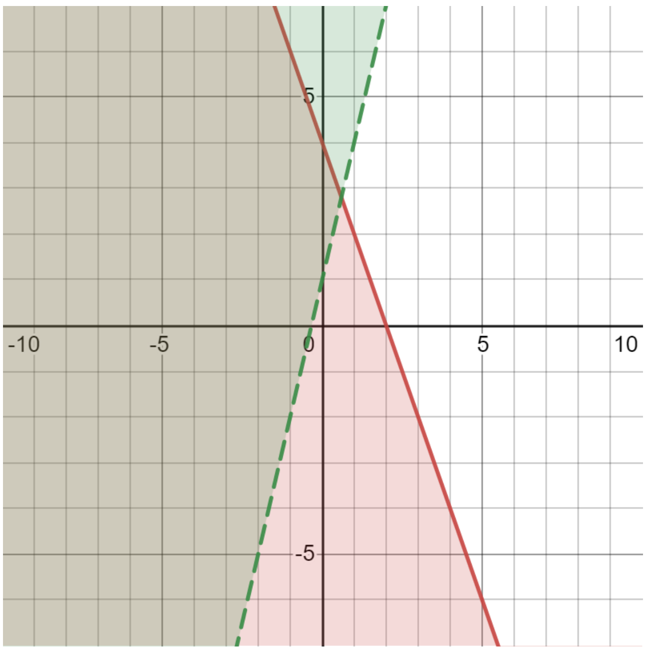
$$\begin{cases} 2x + 4y - z = 7 \\ 7x - y + z = -4 \\ 3x + z = 5 \end{cases}$$

$$\begin{bmatrix} 2 & 4 & -1 & 7 \\ 7 & -1 & 1 & -4 \\ 3 & 0 & 1 & 5 \end{bmatrix}$$

It's important to note the 0 that appears in the third row. Because we don't have a y variable, the coefficient is 0 ($0y = 0$).

2.4 Graphing and Inequalities

At times, we necessarily don't need a solution set to be equal for both equations. Maybe we want one part of our equation to be greater than another part, or less than it. For this, we can still use graphing to figure out these solution sets! To make this easy, we can convert our equations into slope-intercept form. For instance if our original equation was $4x + 2y \leq 8$, we can rewrite this as $y \leq -2x + 4$. Obviously to graph this, we want to mark a few points on the equation and then use a ruler or straightedge to draw a line. However for inequalities, we have to consider two additional factors. If our inequality has an "equal" part in it ($\leq$ or $\geq$), our line is a regular bold one. If there is no "equal" part ($<$ or $>$), we use a dashed line instead (since our solution can't directly be on that line). The second part to consider is our shading. If the equation is $y \geq$ [expression] or $y >$ [expression], we shade above our line. If the equation is $y \leq$ [expression] or $y <$ [expression], we shade below our line. An example for $y \leq -2x + 4$ (red) and $y > 3x - 1$ (green) is shown below.



3 Absolute Value (9/16 to 10/1)

This section will cover the absolute value function, characteristics of its graphing, and how to solve these equations.

3.1 Unit Vocabulary

- **Absolute Value:** A function that measures a point's distance from zero, such that the absolute value of a positive number is that number, and the absolute value of a negative number is its additive inverse
- **Parent Function:** The simplest equation possible that preserves the original meaning of a function (the parent absolute function is $y = |x|$)
- **Extraneous Solution:** A solution that is found when solving an equation, but does not work within the original equation

3.2 Parent and Template

The absolute value of a given number always returns a non-negative number. If the number is already non-negative to begin with, the same input is returned. If the number is negative, its additive inverse is returned (removing the negative sign). In piecewise function terms:

$$|x| = \begin{cases} x < 0, & -x \\ x \geq 0, & x \end{cases}$$

The most basic absolute value function is $y = |x|$, which is known as the parent function. When graphed, equations can change a lot from the original parent function. The equation of a function's transformation is $y = a|b(x - h)| + k$. Each variable means something important, explained below:

1. **a**
   - $0 < |a| < 1$ = Vertical Shrink/Compression
   - $|a| > 1$ = Vertical Stretch
   - $a < 0$ = Reflection over x-axis
2. **b**
   - $0 < |b| < 1$ = Horizontal Stretch
   - $|b| > 1$ = Horizontal Shrink/Compression
   - $b < 0$ = Reflection over y-axis
3. **h**
   - $h > 0$ = Vertical Shift right h units
   - $h < 1$ = Vertical Shift left h units
4. **k**
   - $k > 0$ = Vertical Shift up k units
   - $k < 1$ = Vertical Shift down k units

3.3 Solving Absolute Value Equations

To solve absolute value equations, we have to remember what an absolute value function actually does. When solving the equation $2x + 1 = 7$, there's only one value that satisfies this. However for $|2x + 1| = 7$, the quantity $2x + 1$ can be 7 OR -7. This is because if the previous sentence is true, then $|7| = |-7| = 7$. Therefore to solve absolute value equations, we have to consider when the parts inside the absolute value bars are positive or negative values. To solve $|2x + 1| = 7$, we can display our work as the following:

```
|2x + 1| = 7

2x + 1 = 7
   OR
-(2x + 1) = 7
```

```
2x + 1 =    7
   - 1    - 1
_______________
2x     =    6
--          -
2           2
_______________
x         = 3
```

```
-(2x + 1) =   7
-2x - 1   =   7
    + 1     + 1
________________
-2x       =   8
---          --
-2           -2
________________
x        =   -4
```

Therefore, we can conclude that our solution set is $x = \{-4, 3\}$. The only problem with solving absolute value equations is that *extraneous* solutions exist. This means that even though we found an x value that works, it may not actually be a solution to the ORIGINAL equation due to the nature of the absolute value function. For instance, if we were to solve $|x + 6| = 2x$ we would find a solution set $x = \{-2, 6\}$. If we plug in each x value back into the equation, we find something interesting.

$$|-2+6| = 2(-2) \implies 4 = -4$$

$$|6+6| = 2(6) \implies 12 = 12$$

As we can see, -2 doesn't work in the original equation, even though we found it as a solution in our process. Thus, we found an extraneous solution. It's super important to check for these by plugging into the original equation.

3.4 Systems and Inequalities

Solving for systems with/without inequalities is very similar to the information covered in Section 2.4. We use dashed and straight lines depending on if our sign is $\geq/\leq$ versus $>/<$, and shade above or below depending on if our sign is $\geq/>$ versus $\leq/<$. We would graph each equation in a given system and then find a point (or shaded range) in which the equations are satisfied.

4 Quadratics (10/2 to 12/14)

This section (which in school covered the majority of the first semester) will cover quadratic standard form and factoring, complex roots and different methods to solve a quadratic.

4.1 Unit Vocabulary

- **Quadratic:** A function of the second degree, such that it contains an x^2 term (along with optional x and constant terms). Examples include $2x^2 + 3x + 1$ and $7x^2 - 2$.
- **Standard Form:** An arrangement of the terms in a quadratic such that they are in descending degree order; in the form $ax^2 + bx + c$ such that a b and c are coefficient values.
- **Vertex Form:** A way of writing a quadratic equation such that the vertex is clearly highlighted; $a(x - h)^2 + k$
- **Factoring:** A way to rewrite a quadratic (or other polynomial) such that it contains individual binomial factors ($2x^2 + 3x + 1$ can be factored as $(x + 1)(2x + 1)$)
- **Completing the Square:** A method to solve quadratics by rewriting it as the square of a binomial plus a constant, in other words turning $x^2 + bx + c$ into $(x - \alpha)^2 + \beta$, where b c α and β are all constants

- **Quadratic Formula:** Just like Spongebob's fishing net Ol' Reliable, the quadratic formula is an expression guaranteed to solve for the roots of a quadratic (often used as a last resort)

- **Discriminant:** A constant that gives information about the roots of a quadratic; for a quadratic $ax^2 + bx + c$ the discriminant is $b^2 - 4ac$

4.2 Vertex vs Standard Form

For quadratics, we can write them in two ways that give us insight to some properties. Both ways are listed below.

1. **Standard Form**
   - Equation: $y = ax^2 + bx + c$
   - $a > 0$ implies the equation opens up (like a U)
   - $a < 0$ implies the equation opens down (like an upside down U)
   - Sum of Roots $= -\frac{b}{a}$
   - Product of Roots $= \frac{c}{a}$
   - x-coordinate of vertex: $-\frac{b}{2a}$
   - y-intercept: c
2. **Vertex Form**
   - Equation: $y = a(x-h)^2 + k$
   - $a > 0$ implies the equation opens up (like a U)
   - $a < 0$ implies the equation opens down (like an upside down U)
   - Vertex: (h, k)
   - Sum of Roots: $2h$
   - Product of Roots: $h^2 + \frac{k}{a}$
   - Roots: $h \pm \sqrt{-\frac{k}{a}}$
   - y-intercept: $ah^2 + k$

To convert from vertex to standard, simply expand the binomial squared and simplify any like terms. For instance in $2(x+1)^2 + 5$, the square expands to $2(x^2 + 2x + 1) + 5$ which simplifies to $2x^2 + 4x + 7$.
For vice versa, we can solve for the vertex of an the equation and solve for a if necessary. Let's say we start out with the equation $x^2 + 4x + 1$. Using an expression previously mentioned, the x-coordinate of our vertex is -2, and plugging this into the equation results in a vertex of $(-2, -3)$. This means we already have h and k, so plugging these into vertex form we get $y = a(x+2)^2 - 3$. We can then plug in another coordinate for x and y (such as $(0, 1)$) in order to solve for a.

4.3 The Quadratic Formula

To solve for the roots of a quadratic, we can proceed in a couple of different ways. The most straightforward way is plugging in coefficients into the quadratic formula. If the equation is written in $ax^2 + bx + c$ standard form, we can plug these values into the equation below.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If you paid attention to the vocabulary, you would notice that the quadratic formula has the discriminant in it. This is super helpful because we can use logic to analyze the roots without directly plugging into the formula.

1. **Case 1: Discriminant is positive** $\rightarrow$ If the discriminant is a positive integer, this would mean one root would have a numerator of $-b+\sqrt{\text{Some Value}}$ and another $-b-\sqrt{\text{Some Value}}$. These would result in two different values, thus TWO DIFFERENT ROOTS.
2. **Case 2: Discriminant is zero** $\rightarrow$ We know from common sense that $a+0=a$ and $a-0=a$ for any value a. As a result in the quadratic formula, the value of the numerator would not change regardless of whether the $\pm$ was representing $+$ or $-$. Thus, WE ONLY HAVE ONE ROOT.
3. **Case 3: Discriminant is negative** $\rightarrow$ In case the discriminant is negative, we run into a little complication. Our quadratic formula takes the square root of the discriminant, meaning we're taking the square root of a negative. Since this doesn't give us a real number, we would say that this equation HAS NO REAL SOLUTIONS. However, the imaginary constant $i=\sqrt{-1}$ can help us solve for imaginary roots. For instance if the discriminant was -4, the value in the quadratic formula can be rewritten as $\sqrt{-4}=\sqrt{4\cdot-1}=\sqrt{4}\cdot\sqrt{-1}=\boxed{2i}$.

While there are other ways to solve quadratics that could potentially be easier, the quadratic formula is always a good last resort due to how easy it is to plug in variables and simplify (at least, most of the time).

4.4 Completing the Square

To complete the square is to basically convert a quadratic into vertex form. The easiest way to do this is to start by moving the c constant to the right-hand side, so $x^2+6x-11=0$ would become $x^2+6x=11$. We then want to add a constant on each side such that the left-hand side becomes a binomial. If we expand the binomial to the right we get the following useful info: $\left(x+\left(\frac{b}{2}\right)\right)^2=x^2+bx+\left(\frac{b}{2}\right)^2$. Therefore, the constant we want to add to each side is the square of $\frac{b}{2}$, where b represents the coefficient of the x term. In this case, $b=6$, meaning that the constant we want to add is $\left(\frac{6}{2}\right)^2=9$. We then make our new equation $x^2+6x+9=11+9$, making sure to add 9 to the right-hand side to balance the equation. This then becomes $x^2+6x+9=\boxed{(x+3)^2=20}$.
Another example is worked below:

$$x^2-8x+1=0$$

$$x^2-8x=-1$$

$$x^2-8x+16=-1+16$$

$$(x-4)^2=15$$

Even though the coefficient b is negative in the above example (-8), the number we add to both sides is still positive because we're squaring.

4.5 Factoring

Even though this was probably covered in your Algebra I class, I decided to make this a section as a refresher before the next unit. To factor a quadratic ax^2+bx+c (in most beginning scenarios $a=1$), we want to find two numbers that multiply to ac and add to b. An example of such a quadratic would be $3x^2+4x-4$, and after some work we can see that our two numbers are 6 and -2 $(6+(-2)=4,\ 6\cdot(-2)=3\cdot(-4)=-12)$. We can use these numbers to split the linear term, so we can rewrite $3x^2+4x-4$ as $3x^2+6x-2x-4$. We then split this expression in half, and factor each side as much as possible. Our first half is $3x^2+6x$, and each term has a factor $3x$. We can therefore rewrite this as $3x^2+6x=3x(x+2)$. Using similar logic, we can rewrite the second half as $-2x-4=-2(x+2)$. Therefore, $3x^2+4x-4=3x(x+2)-2(x+2)$. As these each share a common factor $(x+2)$, we can finally rewrite this in factorized form as $(3x-2)(x+2)$.
Now the question is, how exactly are we going to find the roots of this quadratic? Recall that a root is an x value such that the end result plugging this value in is 0. Since our quadratic is now the product of two binomials, we can figure out what x value will make each of these equal to 0.

$$3x-2=0$$

$$3x=2$$

$$x = \boxed{\frac{2}{3}}$$

$$x + 2 = 0$$

$$x = \boxed{-2}$$

In conclusion, the solution set for $3x^2 + 4x - 4 = 0$ is $x = \{-2, \frac{2}{3}\}$. In case the equation is unfactorable (we couldn't find two numbers that multiply to ac and add to b), we would then use the quadratic formula or complete the square.

5 The Square Root (1/5 to 1/12, 3/1 to 3/4)

This section will cover transformations of the square root function, inverses, and extraneous solutions when solving.

5.1 Unit Vocabulary

- **Square Root:** If the square root of x was α, then $\alpha^2 = x$. The square root function has a domain of $x \geq 0$, meaning that the square root of a negative function cannot be plotted (it does exist, however, as an imaginary number).

5.2 Template and Transformations

The most basic absolute value function is $y = \sqrt{x}$, which is known as the parent function. When graphed, equations can change a lot from the original parent function. The equation of a function's transformation is $y = a\sqrt{b(x-h)} + k$. Each variable means something important, explained below:

1. **a**
   - $0 < |a| < 1$ = Vertical Shrink/Compression
   - $|a| > 1$ = Vertical Stretch
   - $a < 0$ = Reflection over x-axis
2. **b**
   - $0 < |b| < 1$ = Horizontal Stretch
   - $|b| > 1$ = Horizontal Shrink/Compression
   - $b < 0$ = Reflection over y-axis
3. **h**
   - $h > 0$ = Vertical Shift right h units
   - $h < 1$ = Vertical Shift left h units
4. **k**
   - $k > 0$ = Vertical Shift up k units
   - $k < 1$ = Vertical Shift down k units
5. **Domain**

$$\begin{cases} b > 0 & [h, \infty) \\ b < 0 & (-\infty, h] \end{cases}$$

6. **Range**

$$\begin{cases} a > 0 & [k, \infty) \\ a < 0 & (-\infty, k] \end{cases}$$

5.3 Solving Equations

To solve square root equations, it's a good idea to leave any square roots on a side by themselves. For instance if we have $\sqrt{2x+5}-4=1$, it's best to move the 4 to the right-hand side. Since square roots and squares are inverses, we can square each side to remove the square roots and work with something easier. Because square roots are only defined as real numbers for $x \geq 0$, we have to remember to check for extraneous solutions. A worked example is shown below:

$$\sqrt{2-x} = x$$
$$(\sqrt{2-x})^2 = x^2$$
$$2-x = x^2$$
$$0 = x^2+x-2$$
$$(x+2)(x-1) = 0$$
$$x = \{-2, 1\}$$

Even though we found two solutions for x, we have to plug these in to make sure we don't have extraneous solutions.

$$\sqrt{2-(-2)} \stackrel{?}{=} -2$$
$$\sqrt{4} \stackrel{?}{=} -2$$
$$2 \neq 2$$
$$\sqrt{2-1} \stackrel{?}{=} 1$$
$$\sqrt{1} \stackrel{?}{=} 1$$
$$1 \stackrel{\checkmark}{=} 1$$

Thus, our only solution is $\boxed{x=1}$.

5.4 Inverses

For inverses, we still follow the same process by switching x and y and solving for y. There's two important things we have to note, however. The first is that we have to remember the negative scenario for square roots, such that the square root of α is $\pm\sqrt{\alpha}$ rather than just positive $\sqrt{\alpha}$. The second is that we have to consider domains. If our domain is restricted (like a square root function), our inverse will also have a restricted domain in order to match every point without adding additional ones. Examples are worked below:

What is the inverse of $y = x^2 - 3$?

$$x = y^2 - 3$$
$$x + 3 = y^2$$
$$y = \pm\sqrt{x+3}$$

What is the inverse of $y = \sqrt{x+1}$?

$$x = \sqrt{y+1}$$
$$x^2 = y+1$$
$$y = x^2 - 1$$

For the latter case, since the vertex point (farthest left point, in this case) of the original equation is $(-1, 0)$. our inverse is going to only have a domain of $[0, \infty)$. It's not going to be -1 since we're swapping the values.

5.5 Cubes and Cube Roots

Due to my extremely lazy nature, I decided to put cubes and cube roots in the square root section. Why? Because a lot of the solving process is the same. The template function for a cubic equation is $y = a(b(x-h))^3+k$, and the template for a cube root equation is $y = a\sqrt[3]{b(x-h)} + k$. The transformation logic is also the same, which I'll still list below:

1. **a**
   - $0 < |a| < 1$ = Vertical Shrink/Compression
   - $|a| > 1$ = Vertical Stretch
   - $a < 0$ = Reflection over x-axis
2. **b**
   - $0 < |b| < 1$ = Horizontal Stretch
   - $|b| > 1$ = Horizontal Shrink/Compression
   - $b < 0$ = Reflection over y-axis
3. **h**
   - $h > 0$ = Vertical Shift right h units
   - $h < 1$ = Vertical Shift left h units
4. **k**
   - $k > 0$ = Vertical Shift up k units
   - $k < 1$ = Vertical Shift down k units

6 Polynomials (1/15 to 2/2)

This section will cover classification of polynomials, end behavior, factoring a few special polynomial cases, and some important theorems.

6.1 Unit Vocabulary

- **Polynomial:** An expression with variables and coefficients that utilize the four operations.
- **Term:** A constant, variable, or product of a number and a variable raised to a positive power
- **Degree:** The highest exponent of a variable in a polynomial
- **End Behavior:** A description of how a polynomial changes as x approaches $-\infty$ and ∞ respectively
- **Irrational Root Theorem:** If a polynomial has a root $a + \sqrt{b}$, then $a - \sqrt{b}$ is also a root
- **Imaginary Root Theorem:** If a polynomial has a root $a + bi$, then $a - bi$ is also a root
- **Fundamental Theorem of Algebra:** A polynomial of the nth degree will have a total of n roots, real and imaginary combined
- **Rational Root Theorem:** For a polynomial with constant p and leading coefficient q, the possible roots for the polynomial are $\frac{\text{Factors of } p}{\text{Factors of } q}$

6.2 Classifying Polynomials

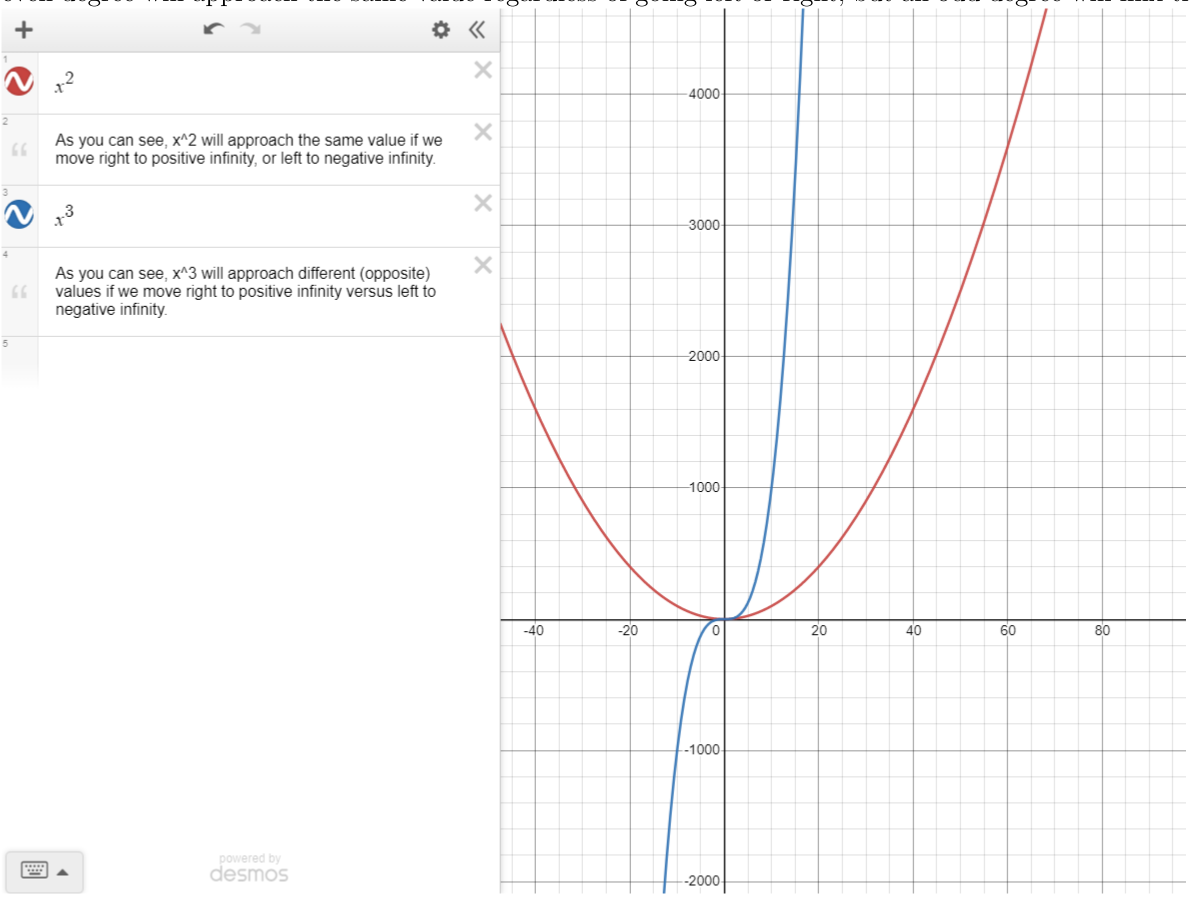
We can classify polynomials in one of two ways. The first way is using the degree of the polynomial, or the highest exponent found on a variable. The second is to count the number of terms. Below are two tables that show how naming these polynomials go:

# of Terms	Example	Name
1	x	Monomial
2	x^2+3	Binomial
3	x^2+2x-1	Trinomial

Degree #	Example	Name
0	5	Constant
1	x	Linear
2	x^2+3	Quadratic
3	x^3-2x^2+1	Cubic
4	$3x^4+1$	Quartic
5	$-x^5+3x^3-2x^2-1$	Quintic

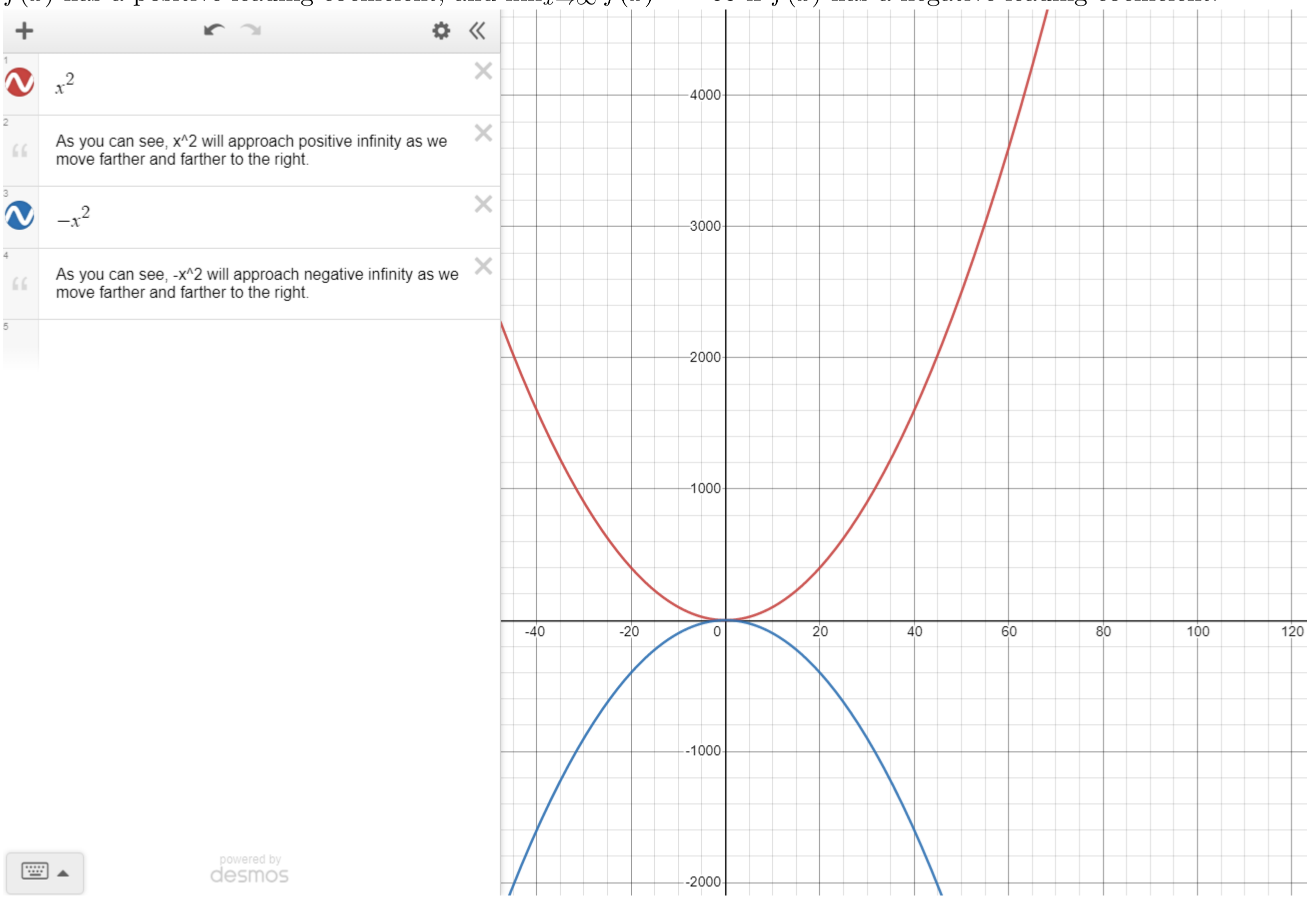
6.3 End Behavior

By analyzing the first term of a polynomial, we can figure out some interesting information. If the degree of the polynomial (the exponent of the first term in standard form) is even, the equation will approach the same number (∞ or $infty$) as we approach either side of the x-axis. If the degree is odd, the equation will approach opposite numbers. Since that probably made zero sense (I don't even understand what I said), examples make things a lot easier. Let's take the most basic even-degree expression: x^2. As our x value gets bigger and bigger, the value of x^2 goes towards infinity ($1^2=1$, $10^2=100$, $87^2=7569$, etc). If we approach the left-side (moving towards $-\infty$), we also have a value that gets bigger and bigger. In limit terms, we can say that $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x)$ if our degree is even, and $\lim_{x\to\infty} f(x) = -\left(\lim_{x\to-\infty} f(x)\right)$. In other words, an even degree will approach the same value regardless of going left or right, but an odd degree will mix things up.



The second thing to consider is the leading coefficient, or the one that is in front of the highest-exponent term.

If this coefficient is positive, then the equation will go towards ∞ as we move to the right. If this coefficient is negative, then this equation will go towards $-\infty$ as we move to the right. In limit terms: $\lim_{x\to\infty} f(x) = \infty$ if $f(x)$ has a positive leading coefficient, and $\lim_{x\to\infty} f(x) = -\infty$ if $f(x)$ has a negative leading coefficient.



Below is a table that will hopefully clarify this:

Leading Coefficient	Degree	Value Approaching $-\infty$	Value Approaching ∞
Positive	Even	Infinity ∞	Infinity ∞
Positive	Odd	Negative Infinity $-\infty$	Infinity ∞
Negative	Even	Negative Infinity $-\infty$	Negative Infinity $-\infty$
Negative	Odd	Infinity ∞	Negative Infinity $-\infty$

6.4 Theorem Time!

In Algebra II, there's four main theorems we should cover regarding polynomials: the Irrational Root Theorem, the Imaginary Root Theorem, the Fundamental Theorem of Algebra, and the Rational Root Theorem.

6.4.1 Irrational & Imaginary Root Theorem

I decided to cover these in the same sub-sub-section due to being similar, and under the same category: The Conjugate Root Theorems. The Irrational Root Theorem states that if $a + \sqrt{b}$ is a root of a polynomial, so is $a - \sqrt{b}$. The Imaginary Root Theorem states that of $a + bi$ is a root, so is $a - bi$. This is helpful because we can solve for a polynomial despite not being given all its roots. For instance, let's figure out a polynomial that has roots 2 and $3 + i$. Since we know $3 + i$ is a root, $3 - i$ must also be a root. Thus, our polynomial is $(x-2)(x-(3+i))(x-(3-i))$. If you plug in each of the roots we just discovered, you will see why this works. Expanding, we get a cubic function $x^3 - 8x^2 + 22x - 20$.

6.4.2 Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra states that a function of degree α has a total of α roots. For instance, any cubic equation will have 3 roots. Looking at the previous subsection, $x^3 - 8x^2 + 22x - 20$ does not have 3 real roots nor 3 imaginary roots. However, it has a total of 3 (1 real & 2 imaginary) and thus proves our point. As an additional note to this section, a polynomial with degree α will have a maximum of α turning points (which you will later refer to as local minima and maxima), which is when the direction of the polynomial changes from going up to down and vice versa.

6.4.3 Rational Root Theorem

This was unarguably the most confusing of the theorems when I learned these in class. This theorem gives us the ability to list out all the potential <u>real</u> roots of a polynomial. We have to start by listing out the factors of the constant term (which I'll call p) and the leading coefficient (q). For instance if our equation was $2x^2-3x+4$, the set for p's factors would be $\{\pm1, \pm2, \pm4\}$, and the set for q's factors would be $\{\pm1, \pm2\}$. For each value in sets p and q, we will divide the former by the latter, or divide each factor of p by each factor of q. This would lead us to $\frac{\pm1}{\pm1}, \frac{\pm2}{\pm1}, \frac{\pm4}{\pm1}, \frac{\pm1}{\pm2}, \frac{\pm2}{\pm2}, \frac{\pm4}{\pm2}$, or a set of $\{\pm\frac{1}{2}, \pm1, \pm2, \pm4\}$. We can then test these values and figure out 2 is indeed a factor of $x^3-8x^2+22x-20$, and then do polynomial division to make it easier to find further factors ($(x^3-8x^2+22x-20) \div (x-2) = x^2-6x+10$, and we can use the quadratic formula to find roots $3+i$ and $3-i$).

6.5 Bonus: Special Factoring

I realized that I forgot to cover special factorings, so I'll list those main three below here.

- **Difference of Squares:** $a^2-b^2=(a+b)(a-b)$
- **Sum of Cubes:** $a^3+b^3=(a+b)(a^2-ab+b^2)$
- **Difference of Cubes:** $a^3-b^3=(a-b)(a^2+ab+b^2)$

For instance if we had to factor the equation x^2-25, we could notice that this could be rewritten as x^2-5^2. Substituting $a=x$ and $b=5$, we get $x^2-25=(x+5)(x-5)$.

7 Roots and Radicals (2/5 to 2/26)

This section will cover non-integral exponents and simplifying roots and radicals in equations.

7.1 Unit Vocabulary

- **Roots:** If the nth root of a number ($n=2 \implies$ square, $n=3 \implies$ cube, $n=4 \implies$ fourth, $n=5 \implies$ fifth, etc) is γ, then $\gamma^n =$ the original number.
- **Radical:** A symbol used to denote the nth root, such that the nth root of γ would be written as $\sqrt[n]{\gamma}$ (you're probably familiar with this from square roots)

7.2 Reviewing Exponent Rules

- $\beta^0=1$
- $\beta^{-m}=\frac{1}{\beta^m}$
- $\beta^m \cdot \beta^n = \beta^{m+n}$
- $\beta^m \div \beta^n = \beta^{m-n}$
- $(\beta^m)^n = \beta^{mn}$
- $\left(\frac{\alpha}{\beta}\right)^m = \frac{\alpha^m}{\alpha^m}$
- $\left(\frac{\alpha}{\beta}\right)^{-m} = \left(\frac{\beta}{\alpha}\right)^m$
- $(\alpha\beta)^m = \alpha^m\beta^m$

7.3 Non-Integral Exponents

When our exponents are integers, it's pretty easy to figure out what the resulting number is. If we were calculating 7^4, we could just do $7 \cdot 7 \cdot 7 \cdot 7 = 2401$ and be done. However our process changes a bit when we have non-integral exponents, usually fractions. For this, we have to go back to our exponent rules. We know that $(\beta^m)^n = \beta^{mn}$. Let's pretend that the product mn was equal to a fraction, such that the numerator is m and the denominator is $\frac{1}{n}$. Since examples are easier to work with, let's say we're calculating $4^{\frac{3}{2}}$. If we split this into an integer times a unit fraction, we get $\frac{3}{2} = 3 \cdot \frac{1}{2}$. Therefore, $4^{\frac{3}{2}} = (4^3)^{\frac{1}{2}}$ by our exponent properties. This simplifies to $64^{\frac{1}{2}}$, and this is where our new knowledge will be applied. The nth root of a number is equal to raising that number to the reciprocal of n, or $\beta^{\frac{1}{n}} = \sqrt[n]{\beta}$. This means that $64^{\frac{1}{2}} = \sqrt{64} = 8$. The easiest way to think about this is this simple equation: $\beta^{\frac{m}{n}} = \sqrt[n]{\beta^m} = (\sqrt[n]{\beta})^m$. As another example, $4^{\frac{5}{2}} = (\sqrt[2]{4})^5 = 2^5 = \boxed{32}$.

7.4 Solving Equations with Rational Exponents

The trick for these is to get any part with a rational exponent to one side, and then raise each side to a common power. To determine this common power, we have to once again look at our rules. If $(\beta^m)^n = \beta^{mn}$ and $mn = 1$, then $(\beta^m)^n = \beta$. This means that if an expression is being raised to a power m, then raising this to the RECIPROCAL of m will cancel out ($m \cdot \frac{1}{m} = 1$ for all $m \neq 0$). An example problem is worked below.

$$(2x+5)^{\frac{1}{3}} + 3 = 5$$

$$(2x+5)^{\frac{1}{3}} = 2$$

To make $mn = 1$ knowing $m = \frac{1}{3}$, we must raise each side to the 3rd power, as $\frac{1}{3} \cdot 3 = 1$.

$$((2x+5)^{\frac{1}{3}})^3 = (2x+5)^{\frac{1}{3} \cdot 3} = 2^3$$

$$2x + 5 = 8$$

$$2x = 3$$

$$\boxed{x = 1.5}$$

8 Rational Functions (3/17 to 4/8)

This will cover different types of variations, reciprocal transformations, and rational function asymptotes.

8.1 Unit Vocabulary

- **Continuous Graph:** One that doesn't have any breaks
- **Discontinuous Graph:** A graph that has breaks at asymptotes or undefined x-values
- **Branch:** Each part of a discontinuous graph
- **Rational Function:** A function that can be written in the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are functions with integral coefficients
- **Reciprocal Function:** A function with parent $y = \frac{1}{x}$ and undefined at $x = 0$

8.2 Variations

To describe data with x and y values (independent and dependent, domain and range, etc), we have two main ways to describe how they're related: direct and inverse variation (I have zero idea why it isn't called indirect variation).

When our data has direct variation, it can be plotted as a linear equation, and x and y change proportionally (they either both go up or both go down). For direction variation, the value $\frac{y}{x}$ will always equal some value k no matter what x or y values are plugged in (given they're the same point of course, if we have points $(2,5)$ and $(3,4)$ it wouldn't make sense to plug in $x = 2$ and $y = 4$). If this is true, we can say that x varies directly with y. For inverse variation, the values of x and y will be curved when plotted. This means that for any paired x and y values, $x \cdot y$ will always equal the same constant k. If x is increasing and y is decreasing, or vice versa, you may be working with inverse variation. If neither of these prove to be true, we say the equation is simply neither directly varying or inversely varying.

There are also combined variations, which is when we have more than 2 sets of data varying with each other. The three main types of *Combined* Variation you may run into are displayed below:

- x **varies jointly with** x **and** $y \implies z = kxy$
- x **varies jointly with** x **and** y **and inversely with** $w \implies z = \frac{kxy}{w}$
- x **varies directly with** x **and inversely with the product** $wy \implies z = \frac{kx}{wy}$

8.3 The Reciprocal Function

The parent reciprocal function is $y = \frac{1}{x}$ for all $x \neq 0$ (because $1 \div 0$ is undefined, but this was hopefully obvious). The template function is $f(x) = \frac{a}{b(x-h)} + k$, and the transformation effects are shown below.

1. **a**
   - $0 < |a| < 1$ = Vertical Shrink/Compression
   - $|a| > 1$ = Vertical Stretch
   - $a < 0$ = Reflection over x-axis
2. **b**
   - $0 < |b| < 1$ = Horizontal Stretch
   - $|b| > 1$ = Horizontal Shrink/Compression
   - $b < 0$ = Reflection over y-axis
3. **h**
   - $h > 0$ = Vertical Shift right h units
   - $h < 1$ = Vertical Shift left h units
4. **k**
   - $k > 0$ = Vertical Shift up k units
   - $k < 1$ = Vertical Shift down k units
5. **Vertical Asymptote:** $x = h$
6. **Horizontal Asymptote:** $y = k$
7. **Domain:** $(-\infty, h) \cup (h, \infty)$
8. **Range:** $(\infty, k) \cup (k, \infty)$

8.4 Asymptotes of Rational Functions

For rational functions, we have removable and non-removable discontinuity. If the discontinuity is removable, it means there's a hole in the graph such that the point is undefined. This occurs when the numerator and denominator have a common factor. To solve for the x-coordinate of this point, simply find out when the common factor equal 0. Then, plug this into the simplified function (cancelling out common factor) to get the y-coordinate. For instance, let's say we had the equation $y = \frac{(x-5)(x-3)}{(x-3)}$. The common factor is $(x-3)$, which is equal to zero when $\boxed{x = 3}$. After cancelling this out we're left with $x - 5$, and plugging in 3 gets us $3 - 5 = \boxed{-2}$. Thus, the point is $(3, -2)$. If we were to graph this equation, it would look exactly like $y = x - 5$, except this isn't defined at $x = 3$. If the function does NOT have a removable discontinuity, we simplify figure out what x-values make the denominator equal 0.

Solving for the vertical asymptote is the same process as looking for non-removable discontinuity: simplify any common factors and solve for denominator = 0. For the horizontal asymptote, we can use the (to say the least, *interesting*) acronym: BOB0 BOTN EATS-DC. I'll break this down one by one, but note that these all revolve around the degree of the top and bottom parts of the rational function (in the Unit Vocabulary, $P(x)$ and $Q(x)$).

- BOB0 stands for "Bigger on Bottom 0". If the exponent on top is bigger than the exponent on bottom, like $\frac{2x^{\mathbf{2}} + 5x}{3x^{\mathbf{3}} - x^2 + 6}$, the asymptote is just $y = 0$.
- BOTN stands for "Bigger on Top Nothing". If the exponent on top is bigger than the bottom, like $\frac{2x^{\mathbf{4}} + 3x^2 - x}{3x^{\mathbf{2}} - 1}$, there is no horizontal asymptote.
- EATS-DC stands for "Exponents are the Same - Divide Coefficients". If the degree of the numerator and denominator is the same, divide the leading coefficients. For instance, $\frac{2x^2 + 3x}{3x^2 - 1}$ has a horizontal asymptote of $y = \frac{2}{3}$

9 Logarithms and Exponential Functions (4/19 to 5/4)

This section will cover the attributes of exponential and logarithmic functions, and logarithmic properties.

9.1 Unit Vocabulary

- **Exponential:** An equation in the form $y = ab^{x-h} + k$ where a b h and k are constants, usually associated with a curve
- **Logarithm:** The inverse of an exponent function, such that if $x^y = z$, then $\log_x(z) = y$
- **Base (of a log):** The base of a logarithm is the subscript that may appear below the log (the number x in the above definition). If this is not present, the base is 10.
- **Natural Log:** A term denoting log base e, or $\ln(x) = \log_e(x)$

9.2 Exponential Function

The template for an exponential function is $y = ab^{x-h} + k$. Each of the transformations are listed below.

1. **a**
   - $0 < |a| < 1$ = Vertical Shrink/Compression
   - $|a| > 1$ = Vertical Stretch
   - $a < 0$ = Reflection over x-axis
2. **h**
   - $h > 0$ = Vertical Shift right h units
   - $h < 1$ = Vertical Shift left h units

3. **k**
    - $k > 0$ = Vertical Shift up k units
    - $k < 1$ = Vertical Shift down k units
4. **Horizontal Asymptote:** $y = k$
5. **Domain:** $(-\infty, \infty)$
6. **Range:** (k, ∞)
7. **As x approaches ∞...**

$$\begin{cases} b > 0 & y \text{ approaches } \infty \\ b < 0 & y \text{ approaches } 0 \end{cases}$$

9.3 Logarithmic Function

The template for an exponential function is $y = a\log_b(x-h)+k$. Each of the transformations are listed below.

1. **a**
    - $0 < |a| < 1$ = Vertical Shrink/Compression
    - $|a| > 1$ = Vertical Stretch
    - $a < 0$ = Reflection over x-axis
2. **h**
    - $h > 0$ = Vertical Shift right h units
    - $h < 1$ = Vertical Shift left h units
3. **k**
    - $k > 0$ = Vertical Shift up k units
    - $k < 1$ = Vertical Shift down k units
4. **Horizontal Asymptote:** $x = h$
5. **Domain:** (h, ∞)
6. **Range:** $(-\infty, \infty)$

9.4 List of Logarithmic Rules/Properties

- $\log_x(y) = z \implies x^z = y$
- $\log_b(1) = 0$
- $\log_b(b) = 1$
- $b^{\log_b M} = M$
- $\log_b(b^M) = M$
- $\log_b(m) + \log_b(n) = \log_b(mn)$
- $\log_b(m) - \log_b(n) = \log_b\left(\frac{m}{n}\right)$
- $\log_b(m^n) = n \cdot \log_b(m)$
- $\log_b(x) = \log_b(y) \implies x = y$
- $\log_b(a) = \frac{\log(b)}{\log(a)}$, where $\log(b)$ is base 10

9.5 Bonus: Interest!

You're probably already familiar with simple and compound interest, where the latter uses exponent properties. There's also continuously compounded interest, which uses exponent properties with e as a base. The equations are listed below.

1. **Simple Interest:** $A = P(1 + rt)$
    - P = Principal (starting amount)
    - r = Rate as a decimal ($50\% = 0.5$)
    - t = Time in years
    - A = Final amount
2. **Compound Interest:** $A = P\left(1 + \frac{r}{n}\right)^{nt}$
    - P = Principal (starting amount)
    - r = Rate as a decimal ($50\% = 0.5$)
    - t = Time in years
    - n = How many times a year compound is occurring (Compounded Monthly = 12, Quarterly = 4, etc)
    - A = Final amount
3. **Continuous Interest:** $A = Pe^{rt}$
    - P = Principal (starting amount)
    - e = Euler's Number ≈ 2.71828
    - r = Rate as a decimal ($50\% = 0.5$)
    - t = Time in years
    - A = Final amount

10 Final Note

Wow I'm finally done with this, and you've reached the end of this long PDF of notes! I hope you enjoyed using it, and that it provided helpful and accurate information. If you have any feedback about anything, especially informational errors on this, let me know at warithr21@gmail.com. ~~Also this is based on Allen ISD's curriculum so don't sue me if some of this stuff does not come up in your class, or if these notes are low quality.~~ Also join the Allen Discord Server because why not!