Pre-Calc Notes

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1 Complex Numbers

Anything related to complex numbers, plotting them, performing operations, and conversion between polar and rectangular form.

1.1 Unit Vocabulary

- **Imaginary number:** A number that contains i, the square root of -1, such as $7i$
- **Real number:** A number that does not contain i, such as 13
- **Complex Number:** A number that is either only real, only imaginary, or has both parts (like $4 + 2i$)
- **Argand Diagram:** Similar to the coordinate plane, this diagram uses real and imaginary axes instead of x and y axes. This is used for plotting complex numbers to graphically visualize them.
- **Magnitude (of a Complex Number):** The distance between the origin of the Argand Diagram and the complex number, calculated using the Pythagorean Theorem ($a^2 + b^2 = c^2$)
- **Polar Form:** A way to describe complex numbers using two main factors: a magnitude measurement and an angle measurement
- **Rectangular Form:** A way to describe complex numbers using a real and imaginary part, similar to the (x, y) format of a coordinate plane.
- **Euler's Formula (for Polar Coordinates):** $re^{i\theta} = r(\cos\theta + i\sin\theta)$, given θ is in radians

1.2 The imaginary number i

The imaginary number was originally coined in the 17th century as derogatory and pointless by René Descartes, but further work by mathematicians like Euler and Gauss have since then proven the importance of the concept. The imaginary unit i is equal to $\sqrt{-1}$. The applications of it are most often seen in solving for roots of an equation, as not all equations will have solely real roots. The powers of the imaginary constant rotate in groups of four, so $i^1 = i^5 = i^9 = \ldots$. The rotation goes as follows:

$$i^1 = i$$

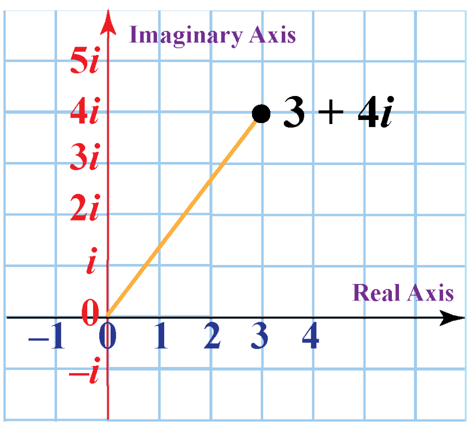
$$i^2 = -1$$

$$i^3 = -i$$

$$i^4 = -1$$

1.3 Visualizing Complex Numbers

To visualize complex numbers, one of the most common ways is to use what's called an **Argand Diagram.** Similar to how the coordinate plane has an x and y axis, the Argand Diagram contains a Real (Re) and Imaginary (Im) axis. The most basic way to plot on the Argand Diagram is to use **Rectangular Form,** which is composed of the real and imaginary parts of a complex number. For instance, to plot the number $3 + 4i$, we would move 3 to the right on the real axis and 4 up on the imaginary axis.



1.4 Math with Complex Numbers

Performing addition and subtraction is easy with complex numbers: simply add/subtract the real parts to each other, and add/subtract the imaginary parts to each other. For instance, adding $2+4i$ with $3-7i$ would result in the following:

$$(2+4i)+(3-7i)=(2+3)+(4i-7i)=\boxed{5-3i}$$

To multiply complex numbers, we have to implement a method known as **FOILing.** The letters in the acronym stand for **F**irst **O**utside-**I**nside **L**ast, which represent the order to multiply the parts of the complex numbers in. Using a similar example to the one above, the following process would used to multiply $(2+4i)\cdot(3-7i)$.

$$(2+4i)(3-7i)=(2\cdot 3)+(2\cdot -7i+4i\cdot 3)+(4i\cdot -7i)=6-2i+28=\boxed{34-2i}$$

1.5 Complex Conjugates

The complex conjugate of a number contains an equal real part, but an opposite imaginary part. For instance, the conjugate of $4+3i$ is $4-3i$. These conjugates are very important in dividing complex numbers. In order to divide two complex numbers, we must multiply both parts of the fraction (the divided and divisor) by the conjugate of the denominator (divisor). By doing this, the denominator will result in an integral number. The reason behind this is because $(a+bi)(a-bi)=a^2+b^2$, which can be proven by either FOILing or using the difference of squares. To divide $34-2i$ by $3-7i$, we would use the following approach:

$$\frac{34-2i}{3-7i}=\frac{34-2i}{3-7i}\cdot\frac{3+7i}{3+7i}=\frac{(34-2i)(3+7i)}{(3-7i)(3+7i)}=\frac{116+232i}{58}=\boxed{2+4i}$$

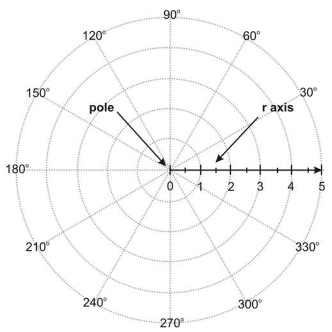
1.6 Polar Form

The **Polar Form** of a complex number uses a magnitude and angle in order to describe a number. The **magnitude** of a complex number describes the distance between the origin and a complex number on the Argand Diagram. In other words, it's the length of a ray with an endpoint at the origin and ends at the plotted point. For a complex number $a+bi$, the magnitude is equal to $\sqrt{a^2+b^2}$, which is simply application of the Pythagorean Theorem.

To calculate the angle for a given complex number, we can use the arctan function in order to calculate it, remembering that $0°$ is the eastward direction (like $30+0i$). Because the range for the arctan is only $-90°$ to $90°$, or $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, we have to perform additional operations depending on what quadrant our complex number is in. Given a number $a+bi$ we have to do the following... (assuming degrees, replace any given numbers with their respective radian conversions if necessary)

- If $a>0$ and $b>0$ (first quadrant): $\theta=\arctan\left(\frac{b}{a}\right)$
- If $a>0$ and $b<0$ (fourth quadrant): $\theta=360+\arctan\left(\frac{b}{a}\right)$
- If $a<0$ (second/third quadrant): $\theta=180+\arctan\left(\frac{b}{a}\right)$

An example of the degree visualization ($0°$ = far east, $180°$ = far west) is shown in an Argand diagram image below. This image also has concentric circles to depict magnitude, as a point with magnitude 1 would be on the innermost circle (since it has radius 1).



Polar form can be notated in different ways. Given magnitude r and angle θ in radians, it can be notated as (r,θ), $r\angle\theta$, or $re^{i\theta}$. The last method is known as **Euler's Formula** (well, for polar representation).
To multiply or divide numbers in polar form, we can simply use exponential properties. Recall the fact that $a^b \cdot a^c = a^{bc}$, and $a^b \div a^c = a^{\frac{b}{c}}$.

$$3e^{i\pi} \cdot 3e^{\frac{i\pi}{2}} = (3\cdot 3)\left(e^{i\pi + \frac{i\pi}{2}}\right) = \boxed{9e^{\frac{3i\pi}{2}}}$$

$$3e^{i\pi} \div 3e^{\frac{i\pi}{2}} = (3\div 3)\left(e^{i\pi - \frac{i\pi}{2}}\right) = \boxed{e^{\frac{i\pi}{2}}}$$

To convert from polar form to rectangular form, we can simply write it as the following: $(r,\theta) \to r\cos\theta + i\sin\theta$.

2 Polynomials

This section is a review of polynomials from Algebra II. This unit will cover things like operations with polynomials, binomial expansion, and solving equations.

2.1 Unit Vocabulary

- **Polynomial Term:** A part of a polynomial that is either a number (4), a variable (z), or a product of a number and at least one variable ($3x^3$)
- **Like Terms:** Two terms that share the same variables and exponents, or lack of variables (3 and 4, or $2x$ and $-7x$)
- **Pascal's Triangle:** A triangular diagram of numbers named after French mathematician Blaise Pascal. The first row consists of **1**, the second rows consists of 2 **1**s, and every following number is equal to the sum of the two numbers above it.
- **Synthetic Division:** A strategy for polynomial-by-binomial division that is an alternative to polynomial long division
- **Binomial Expansion:** Describes the expansion of raising a binomial (two-termed polynomial) to a given nth power, such as $(2x+1)^3$

2.2 Operations with Polynomials

To perform addition and subtraction with polynomials, we simply combine **like terms**. Like terms are terms that share the same variables as well as variable exponents, such as $3x^4y^2$ and $-x^4y^2$.

$$(4x^2 + 4y) - (2x^2 - 7y + 1) = (4x^2 - 2x^2) + (4y - -7y) + (0 - 1) = \boxed{2x^2 + 11y - 1}$$

To multiply polynomials, we have to multiply each term of each polynomial by one another. With binomial multiplication, we can use FOILing. For polynomials with more terms, however, it's better to write out each individual term or create a diagram to simplify the process.

$$(2x^2+2x-1)(3x^2-x) = (2x^2 \cdot 3x^2) + (2x^2 \cdot -x) + (2x \cdot 3x^2) + (2x \cdot -x) + (-1 \cdot 3x^2) + (-1 \cdot -x) = \boxed{6x^4 + 4x^3 - 5x^2 + x}$$

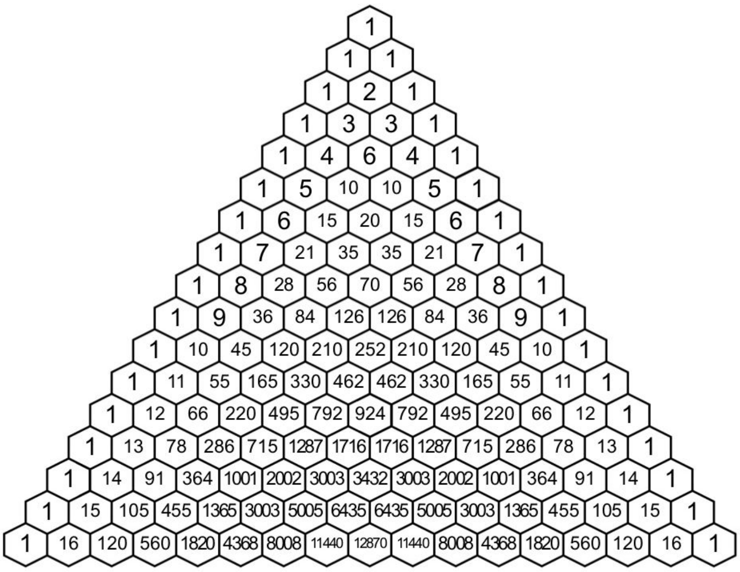
The best way to divide polynomials depends a lot on the divisor, or denominator of the fraction...

- **Polynomial divided by Monomial:** Simply divide each term in the polynomial by the monomial, and recombine as a finalized quotient (with a potential denominator)
- **Polynomial divided by Binomial:** For this we can either use brute force long division, or use **synthetic division** assuming the binomial has a leading term with one variable (like $3x$ or $7y$)
- **Polynomial divided by Trinomial (or more terms):** All you can do here is use long division, assuming you don't have an online resource like Symbolab to help

2.3 Binomial Expansion

Whenever you have to expand a binomial by raising it to the nth power, we can use what's called **Binomial Expansion.** To do so, we first need to take the nth row of the **Pascal Triangle,** a special diagram with many different mathematical properties. The complicated combinatoric formula for binomial expansion is shown below, along with a visual of Pascal's Triangle.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$



To better understand the concept, I'll use the example of expanding $(2x+1)^3$. Looking at the third row of Pascal's Triangle, we can see it consists of the following numbers: 1 3 3 1. For each value from 0 to n, the variable part of each term in the expansion will be $x^{n-k}y^k$. For instance here, we start off with x^3 and move on to x^2y, xy^2, and y^3. For each term we'll also perform the respective operation, and then multiply by the respective number in Pascal's triangle (the sequence 1 3 3 1). For the first term (x^3), the coefficient will be equal to $2^3 \cdot 1^0 \cdot 1 = 8$. For the second term ($x^2y$), the coefficient is $2^2 \cdot 1^1 \cdot 3 = 12$. Eventually, this forms the final expansion of $\boxed{8x^3 + 12x^2 + 6x + 1}$. `Fun fact: If you take each coefficient as place values and add, the numbers actually form a perfect cube!` $(2\cdot 10+1)^3 = 8\cdot 10^3 + 12\cdot 10^2 + 6\cdot 10^1 + 1\cdot 10^0 = 9261$.

2.4 Solving a System of Equations

To solve polynomial equations, we can either solve them graphically or algebraically. To solve a polynomial-polynomial equation graphically, we can graph these equations and look for any intersection points. To solve algebraically (especially helpful if solutions contain large numbers), we can cancel like terms out and eventually use the quadratic formula if necessary.

$$2x^2 + 4x + 1 = -3x + 7$$

$$2x^2 + 4x + 1 + 3x - 7 = -3x + 7 + 3x - 7$$

$$2x^2 + 7x - 6 = 0$$

$$x = \frac{-7 \pm \sqrt{7^2 - 4(2)(-6)}}{2 \cdot 2}$$

$$\boxed{x = \frac{-7 \pm \sqrt{97}}{4}}$$

3 Composite and Inverse Functions

This section covers composite functions and how to evaluate them, as well as what an inverse function is and how they work.

3.1 Unit Vocabulary

- **Composite Function:** A function made up of one function being plugged into another. For instance, if there were functions $f(x)$ and $g(x)$, then a composite function would be $f(g(x))$ which is plugging the value of $g(x)$ into the x of $f(x)$.
- **Function:** A relationship between a domain and range, such that every domain value is connected to only 1 range value (a range value can have multiple paired domain values, however)
- **Inverse Function:** A relationship that swaps the domain and range of a given function. In formulaic terms, a function $f(x)$ has an inverse of $f(y) = x$.
- **Invertible:** If a function is invertible, the domain and range can be swapped such that each new domain value still corresponds with one new range value.
- **One-to-One Function / Bijective Function:** A function that maps every value in the domain to exactly one range value. If a function is a One-to-One function, it is also by definition invertible.
- **Line Tests:** The vertical line test is a graphical way of testing if a given graph is a function. If a vertical line can be drawn such that it intersects the graph in 2 or more places, it's NOT a function. Similarly, the horizontal line test checks to see if a given graph is an invertible function.

3.2 What are Functions?

A function is described as the relationship between two sets of numerical data: namely a domain and range. Any correspondence between two sets of numbers can be considered a relation, but there are specific requirements for a set to be a function. If a group of data is considered a function, every domain value is paired up with ONLY one range value. Multiple domains can be part of the same range, but no domain can have two ranges. An example of a functional group of points is $\{(1, 2); (2, 3); (5, 3)\}$, whereas an example of a non-function is $\{(0, 1); (3, 4); (3, 5)\}$ (notice how the domain value 3 is paired with both 4 and 5).

3.3 What is a Composite Function?

A composite function is defined as simply plugging in a function inside another function. To illustrate this example, let's define two functions $f(x) = 2x + 1$ and $g(x) = 3x - 4$, and attempt to evaluate the complex function $f(g(3))$. This equation is equal to evaluating $g(3)$, and the substituting the result into $f(x)$.

$$f(g(3)) =?$$

$$g(3) = 3(3) - 4 = 9 - 4 = 5$$

$$f(5) = 2(5) + 1 = \boxed{11}$$

Another way to evaluate this is by actually plugging one equation into another (noting that this method would not work for graphical or tabular examples). For instance...

$$f(g(x)) =?$$

$$g(x) = 3x - 4$$

$$f(g(x)) = f(3x - 4) = 2(3x - 4) + 1 = 6x - 8 + 1 = \boxed{6x - 7}$$

3.4 What is an Inverse Function?

An inverse function is defined as a relation that swaps the domain and range of the original function. To figure out the inverse of a function, simply swap the y and x variables and re-arrange in terms of y. These functions may just be re-declared (as in making a new function $g(x)$ the inverse of $f(x)$), or denoting the inverse with a -1 superscript (such as $f^{-1}(x)$).

$$y = 2x - 4$$

$$\text{Inverse: } x = 2y - 4$$

$$x + 4 = 2y$$

$$\boxed{y = \frac{x}{2} + 2}$$

While I was able to find the inverse of the above example, not every function is invertible! A function is only invertible if it's a **one-to-one function**. This means that every value in the domain is paired with one and ONLY one range value. The reason behind this is because of the definition of a one-to-one function, as it satisfies the requirements for a function even after being inverted.

3.5 The Line Tests

The line tests are two graphical ways of determining if a graph, or series of points, is indeed a function. The **Vertical** Line Test is simply drawing vertical lines onto the graph. If any vertical line can be drawn such that it intersects 2+ points, the resulting series is NOT a function. Similarly, the **Horizontal** Line Test is a method to see if a function is invertible, and works similarly to the vertical test.

4 Trigonometry

This unit deals with the primary trig functions, as well as their inverses. Processes to help solve sinusoidal equations will also be covered, as well as various trig identities.

4.1 Unit Vocabulary

- **The Trigonometric Functions:** These functions are sine, cosine, and tangent (denoted typically as sin(), cos(), and tan(). The handy mnemonic for remembering these functions and their ratios is SOH CAH TOA. The inverses for each of these are arcsine, arccosine, and arctangent (notice the arcs).
- **Sinusoidal:** A sinusoidal function is one that appears in a wave-like shape when graphed. The two sinusoidal trig functions are the sine and cosine.
- **Trigonometric Identities:** These identities are equations that use trigonometric formulas, that are true no matter what values are plugged in. These are especially helpful when simplifying long expressions.

4.2 The Inverse Functions

Since the trigonometric functions output a ratio given an angle, inverse functions are defined so we can work vice versa. In other terms:

$$\sin\theta = r \rightarrow \sin^{-1} r = \theta$$

These inverse functions are either notated with a -1 superscript (like how we named inverse functions in the previous unit), or simply by their names (arcsin, arccos, and arctan). Because the trigonometric formulas have multiple domain values assigned to the same range, the inverses are restricted to a small range. Because of this, $\arccos(\cos\theta)$ may not necessarily be equal to the original θ value. Specifically, the limitations are below (given in degrees).

$$\arcsin \rightarrow [-90, 90]$$

$$\arccos \rightarrow [0, 180]$$

$$\arctan \rightarrow (-90, 90)$$

Because of the small range of the inverses, we have to do a little more thinking if we want to think of other solutions to an equation. For instance, plugging $\arccos(0.75)$ into a calculator would result in about $41.4°$, but

other values satisfy the same equation (like 401.4° and 318.6°). How do we discover these new values? We use properties of the trig ratios!
For both sine and cosine waves, it's true that they're overlapping every 360° period. It's also important to note that supplementary angles have equal sines, and angles that add up to 360° have equal cosines. In other words:

$$\sin\theta = \sin(\theta + 360°)$$
$$\sin\theta = \sin(180° - \theta)$$
$$\cos\theta = \cos(\theta + 360°)$$
$$\cos\theta = \cos(360° - \theta)$$

For instance, if we plug arcsin(0.154) into a calculator, the result is approximately 8.86°. Using the properties we listed above, we can find other results such as $8.8588 + 360 = \boxed{368.9°}$ and $180 - 8.588 = \boxed{171.1°}$.

4.3 Trigonometric Identities

There are a lot of different identities that we can use to simplify problems. It's encouraged to memorize as many of these as possible (specifically the first 3 sets), but advanced knowledge can be used to derive some of these if absolutely necessary.

1. Reciprocals
   - $\csc\theta = \dfrac{1}{\sin\theta}$
   - $\sec\theta = \dfrac{1}{\cos\theta}$
   - $\cot\theta = \dfrac{1}{\tan\theta}$
2. Pythagorean Identities
   - $\sin^2\theta + \cos^2\theta = 1$
   - $\tan^2\theta + 1 = \sec\theta$
   - $1 + \cot^2\theta = \csc^2\theta$
3. Addition/Subtraction Identities
   - $\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \sin(\beta)\cos(\alpha)$
   - $\sin(\alpha-\beta) = \sin(\alpha)\cos(\beta) - \sin(\beta)\cos(\alpha)$
   - $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$
   - $\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$
   - $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan(\alpha)\tan(\beta)}$
   - $\tan(\alpha-\beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan(\alpha)\tan(\beta)}$
4. Double Angle Identities
   - $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
   - $\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1$
   - $\tan(2\theta) = \dfrac{2\tan\theta}{1-\tan^2\theta}$
5. Half Angle Identities
   - $\sin\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1-\cos\theta}{2}}$
   - $\cos\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1+\cos\theta}{2}}$
   - $\tan\left(\dfrac{\theta}{2}\right) = \dfrac{1-\cos\theta}{\sin\theta}$

5 Vectors

This unit covers vector operations. It'll go over the components of a vector, and how to perform operations in the different ways they can be described.

5.1 Unit Vocabulary

- **Initial Point:** The starting point for a vector; in ray terms, the endpoint
- **Terminal Point:** The ending point of a vector; in ray terms, the point the arrow ends at
- **Component Form:** Assuming the initial point is the origin, component form is just the final terminal point (like $(2, 4)$)
- **Magnitude:** The length of a vector (go back to the first unit, with polar form notation
- **Unit Vector:** A vector with magnitude 1
- **Scalar Multiplication:** Multiplying a vector by a quantity to change its terminal point and as a result change the vector

5.2 Parts of a Vector

A vector consists of two points, an **initial** and **terminal point**. Sometimes the initial point is the origin of the coordinate plane, so a vector may sometimes be notated by solely the terminal point. Vectors are considered equivalent if they have the same **magnitude** and angle/slope, so vectors can be equal despite having different initial and/or terminal points. Alternatively, a vector can be written with the magnitude and degree it covers, exactly like polar form for complex numbers from Unit 1.

5.3 The Unit Vector

The unit vector of a given vector is a vector in the same direction but only of magnitude 1. In order to calculate this vector, we divide each part of the component form by the magnitude. For instance, the point $(6, 3)$ has a magnitude of $\sqrt{45}$, so the unit vector would divide each original part by the magnitude. In equation form:

$$(a, b) \to \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$$

5.4 Scalar Multiplication

Scalar Multiplication is multiplying each part of a vector (in component form) by a quantity in order to alter it. Simply, multiplying a vector $\vec{v} = (x, y)$ by a quantity λ results in a new vector $(\lambda x, \lambda y)$ If the quantity λ is negative, the resulting vector would then be pointing in the opposite direction of the original $\vec{v}$.

5.5 Vector Operations

With component form, adding/subtracting vectors is just working with the parts of the component form themselves.

$$\vec{v} = (x, y); \vec{\gamma} = (\chi, \psi)$$
$$\vec{v} + \vec{\gamma} = (x + \chi, y + \psi)$$
$$\vec{v} - \vec{\gamma} = (x - \chi, y - \psi)$$

For vectors in the form (magnitude, angle) or (r, θ), we would have to convert to component form and then perform the operations as shown above, of course converting back to original form if necessary. To convert from (r, θ) to component form, simply use the same strategy as converting from polar to rectangular forms of complex numbers.

6 Matrices

This unit will cover what a matrix is, how to represent linear equations as matrices, operations with matrices, and polygonal transformations.

6.1 Unit Vocabulary

- **Matrix:** A rectangular arrangement of numbers consisting of rows and columns; an array of arrays, for those familiar with the concept in programming languages like Java and C++
- **Scalar:** A specific quantity that can be used to alter the elements in a matrix via division.
- **Dot Product:** Multiplying respective elements by each other and then adding, used to multiply matrices
- **Determinant:** A specific value that can show the properties of a matrix, represented by the equation $ad - bc$ for 2x2 matrices.
- **Zero Matrix:** A matrix solely consisting of zeroes
- **Identity Matrix:** A square matrix consisting of almost only zeroes, but containing a diagonal of ones from the top-left to bottom-right

6.2 Matrix Dimensions

A matrix is a rectangular arrangement of numerical values aligned in rows and columns. To describe the dimensions of an array, we say it is a "row" by "column" matrix. For instance, the below matrix A has dimensions 2 by 3.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

To describe a specific element in the matrix, we denote it by using the row and column number. For instance, $A_{1,2}$ describes the element in the 1^{st} row and 2^{nd} column of matrix A, or 2.

6.3 Linear Systems of Equations

To solve linear system of equations, there are many approaches such as graphing and solving by hand. We can additionally solve these with matrices! For instance, let's say we had equations $2x+4y+z=7$, $2x+4y-2z=8$, and $3x-z=4$. We must first write these out in a matrix of dimensions 3 by 4. Here, the 3 denotes the number of equations, and the number 4 is one more than the total amount of variables we have (think x y z). Each column of the matrix will represent the variable in the equation, and the final one will represent the values on the right side of the equal side. As a result we can get the following matrix:

$$\begin{bmatrix} 2 & 4 & 1 & 7 \\ 2 & 4 & -2 & 8 \\ 3 & 0 & -1 & 4 \end{bmatrix}$$

Notice how we have to use a 0 for the third equation, since there's no y value present (this is the same as just writing $3x + 0y - z = 4$). We can then take the **Reduced Row Echelon Form** of this matrix (or rref), which can simply be found on a calculator. After plugging this in we get the following interesting matrix (with approximated values):

$$\begin{bmatrix} 1 & 0 & 0 & 1.22 \\ 0 & 1 & 0 & 1.22 \\ 0 & 0 & 1 & -0.33 \end{bmatrix}$$

If we look at the first row and recall how we set up the original matrices, we notice it states $1x+0y+0z = 1.22$. This can simply be rewritten as $x = 1.22$. Expanding this to the other rows, we can deduce that each row is just the solution set to our problem ($x = 1.22, y = 1.22, z = -0.33$)!

6.4 Operations with Matrices

6.4.1 Adding, Subtracting, Scalar Multiplication

To add and subtract two matrices, it's mandatory that they share the exact same dimensions. Assuming they have the same dimensions, simply just add across for each respective element.

$$\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 7 & 5 \\ 9 & 2 \end{bmatrix} = \begin{bmatrix} 1+7 & 2+5 \\ 4+9 & 3+2 \end{bmatrix} = \begin{bmatrix} 8 & 7 \\ 13 & 5 \end{bmatrix}$$

To multiply a matrix by a single scalar quantity, just multiply every value in the matrix by the given scalar value (regardless of matrix size).

$$7\begin{bmatrix}-1 & 4\\ 3 & 6\end{bmatrix} = \begin{bmatrix}-7 & 28\\ 21 & 42\end{bmatrix}$$

6.4.2 Dot Product

To multiply two matrices together, we sadly cannot just multiply elements across like addition and subtraction. We have to use a concept called the **Dot Product**. The dot product does require a restriction, however: The columns of the first matrix has to be equal to the rows of the second matrix. In other words, if a matrix has dimensions r_1 by c_1 and another has dimensions r_2 and c_2, then $r_2 = c_1$ must be satisfied. If it is satisfied, the product matrix will have dimensions r_1 by c_2. Pay attention to the wording and emphasis of first and second matrix. This is because for any two matrices A and B, $AB \neq BA$. Multiplication is NOT commutative in matrices, however it is still associative $(A \cdot (BC) = (AB) \cdot C)$.
Below I'm going to list out two matrices in order to help explain matrix multiplication as best as I can (labelling given matrices as A and B). As the topic is very confusing, I suggest talking with a teacher or using resources like mathisfun in order to clear up any confusion I may cause.

$$\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\ 5 & 4\end{bmatrix}$$

For each value in the row of the first matrix, and each value in the column of the second matrix, multiply each respective element. For instance, let's start with the first row of matrix A, and the first column of matrix B. The A row consists of 2 and 4, whereas the B column consists of 1 and 5. Multiplying each set of numbers respectively and adding, we get $2 \cdot 1 + 4 \cdot 5 = 22$. Therefore, the element in the first row (we used A's first row) and first column (we used B's first column) is 22. Multiplying the rest of the elements, we get the following solution:

$$\begin{bmatrix}2 & 4\\ 1 & 2\end{bmatrix}\begin{bmatrix}1 & 3\\ 5 & 4\end{bmatrix} = \begin{bmatrix}22 & 22\\ 11 & 11\end{bmatrix}$$

6.5 Transformations

In the past, you've probably just been used to transforming things with actual words, such as translating 7 to the right or dilating a polygon by a scale factor of 2. With matrices, we can also represent transformations, as well as perform them. For a polygon, we can convert the points in it into a 2 by α matrix, where α is the number of vertices. The first row would represent the x coordinates, and the second row the y coordinates. For instance, a polygon with vertices $\{(0, 2); (3, 4), (8, -1)\}$ would be notated as the following:

$$\begin{bmatrix}0 & 3 & 8\\ 2 & 4 & -1\end{bmatrix}$$

The transformation matrix (at a basic level) is a 2 by 2 matrix that shows how a set of two points is being changed. The starting transformation matrix is an identity matrix, with each column representing a point: $(1, 0)$ and $(0, 1)$. Transformation matrices will be rewritten in a form such to highlight how those starting two points are changed. For instance, the below matrix T shows the point $(3, 0)$ instead of $(1, 0)$, and $(0, 3)$ instead of $(0, 1)$. If we visualize this, we can notice that this matrix would dilate a polygon and expand in by a factor of 3.

$$\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$$

While we can just analyze the matrix like we did above, we can also just multiply the point matrix by the transformation matrix. This is especially helpful if the transformation is not as obvious as the above matrix. Like mentioned previously, dimensions and order is super important in matrix multiplication, so make sure that the order is point matrix by transformation matrix `(fun fact: I actually messed this up typing it the first time, and only realized when I plugged it into my calculator)`. An example with the matrices we state above is as follows:

$$\begin{bmatrix}0 & 3 & 8\\ 2 & 4 & -1\end{bmatrix} \cdot \begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix} = \begin{bmatrix}0 & 9 & 24\\ 6 & 12 & -3\end{bmatrix}$$

6.6 Determinants and Inverses

Spoiler Note: Everything in this section will just cover basic 2 by 2 matrices, as Pre-Calculus doesn't go into bigger matrices according to my knowledge. If necessary, I'll update this document. The determinant of a matrix is a special value that we can use for ONLY square matrices. To calculate the determinant of a 2 by 2 matrix, we use the formula $ad - bc$, with the variable layout shown below. This multiplies the top-left and bottom-right values, and subtracts the product of the other two.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

One important usage of determinants is inverting a matrix. Like functions and trig ratios and practically everything in Pre-Calculus, matrices have inverses. Not all matrices are invertible, however. The first requirement is that a matrix has to be a square one, as in dimensions α by α. The second requirement is that the determinant cannot be equal to zero. For example, matrix C below is invertible with a determinant of 24, whereas D has a determinant of zero and is therefore non-invertible.

$$C = \begin{bmatrix} 7 & 3 \\ 13 & 9 \end{bmatrix}, D = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$$

To find the inverse of a 2 by 2 matrix, simply divide each element by the determinant. Using the above matrix C as an example, we get the following (rounding to 3 significant digits due to fractions looking bad on LaTeX):

$$C^{-1} = \begin{bmatrix} 0.375 & -0.125 \\ -0.542 & 0.292 \end{bmatrix}$$

It's important to note that while the original matrix had order $a\ b\ c\ d$ from left to right and up to down, the inverse matrix has new order $d\ {-b}\ {-c}\ a$.

7 Sequences

This section will cover arithmetic and geometric sequences, notating them, and using summation formulas for geometric sequences.

7.1 Unit Vocabulary

- **Arithmetic Sequence:** A sequence that is created by adding or subtracting a constant factor
- **Geometric Sequence:** A sequence that is created by multiplying or dividing a constant factor
- **Recursive Formula:** A way of notating a series by stating an initial value, followed by the pattern afterwards
- **Explicit Formula:** A way of notating a series that states a value based on its position in the sequence

7.2 Recursive vs Explicit formulas

For arithmetic and geometric sequences, there are two different ways to write down the series (other than just writing down all the elements, of course).

7.2.1 Recursive

The first way is a **recursive** formula, which defines the first element and then the transformation for every element afterwards. Take the following example below:

$$\begin{cases} d(1) = 3 \\ d(n) = d(n-1) + 4 \end{cases}$$

The above recursive formula first states that the first element is equal to 3. If we plug in 2 for n in the second part of the formula, we get $d(2) = d(2-1) + 4 = 3 + 4 = \boxed{7}$. If we continue plugging increasing values of n in, we realize that this is a sequence of just adding 4.

7.2.2 Explicit

The other way is an **explicit** formula, which defines terms solely off of their positioning. Let's use the following example to demonstrate how this works:

$$d(n) = 3 \cdot 5^{n-1}$$

For each of the values of the sequence, we can just plug in the position and the formula above will give us the respective value. The 1^{st} value is $3 \cdot 5^{1-1} = 3$, the 2^{nd} value being $3 \cdot 5^{2-1} = 15$, and the 3^{rd} equal to $3 \cdot 5^{3-1} = 75$. Looking carefully, we see the next term is made by multiplying the previous by 5.

7.3 Summations

To take the sum of a certain number of elements in an arithmetic sequence, we can use the following formula:

$$S_n = \frac{n(a_1 + a_n)}{2}$$

Above, S_n represents the sum of the first n terms, a_1 is the first term, and a_n is the nth term. We're basically taking the average of the first and last terms, and multiplying by the number of terms we have.
For an *infinite* geometric sequence, we can use the following formula:

$$S = \frac{a}{1-r}$$

Here, S is the sum, a is the first term, and r is the ratio that we're multiply every term by. To get the infinite sum, we divide the initial term by the difference between 1 and the ratio. It's SUPER important to note that this formula does not work if $|r| > 1$, as the series will go on to infinity and diverge.
For a *finite* geometric sequence, we can modify our above formula slightly and get the following:

$$S_n = \frac{a(r^n - 1)}{1-r}$$

Here, S_n is the sum of the first n elements, a is the first term, and r is the common ratio. Unlike the infinite sequence formula, the finite series can be found for ANY value of r.

8 Conics (my least favorite yay)

This unit will cover the four conic sections, as well as their features.

8.1 Unit Vocabulary

- **Conic Section:** A curve or figure obtained through a cross section of an hourglass-like polyhedron, or two cones stacked on to each other through the vertex
- **Focus (plural foci):** Special points that are the basis of constructing most curved figures
- **Directrix:** A special line such that every point on a given parabola is equidistant from the directrix compared to the focus
- **Circle:** A 2-D curved figure consisting of all the points a distance n from a center
- **Ellipse:** A 2-D curved figure centralized around two foci, such that the summed distance between an elliptical point and each of the two foci is a constant n
- **Parabola:** A symmetrical curve that is U-shaped; the shape formed with a quadratic equation
- **Hyperbola:** A symmetrical set of two U-shaped curves (branches) that are facing opposite directions and equidistant from a centralized point

8.2 The Circle

A circle is defined as a curve connecting all the points equidistant from a center on the same plane. The standard form for a circle is given below, as well as the more uncommon expanded form:

$$(x-h)^2 + (y-k)^2 = r^2$$

$$ax^2 + bx + cy^2 + dy + e = 0$$

The top equation is typical standard form, with the coordinates (h, k) being the center of the circle, and r being the radius. The expanded form is a lot more uncommon, but when seen should be simplified with the **Completing the Square** method. The method works like the following:

$$x^2 + ax = \left(x + \frac{a}{2}\right)^2 - a^2$$

For instance, $x^2 + 6x$ would be rewritten as $(x+3)^2 - 9$. Do this for the y terms and simplify into the standard form.

8.3 Ellipses

An ellipse is simply an oval, defined as all the points a constant distance away from two foci. The formula for an ellipse is given below:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

The center of the ellipse is (h, k), the horizontal radius is a, and the vertical radius is b. The bigger of the two diameters is often referred to as the major axis, and the smaller one the minor axis. The foci are written in the format $(h, k \pm c)$ or $(h \pm c, k)$ if the major axis is the vertical or horizontal respectively. The c value actually forms a Pythagorean triple with the axis lengths: namely $a^2 = b^2 + c^2$ if the major axis is horizontal, or $b^2 = a^2 + c^2$ if the major axis is vertical. In other words, the longer radius (or semi-major axis) squared equals the sum of the variable c squared and the shorter radius (or semi-minor axis) squared.

8.4 Parabolas

Parabolas, also known as quadratic equations, are U-shaped that are symmetrical over a vertical axis. Quadratics were covered a lot in previous years, like Algebra I/II, but they go even further in depth for conic-related math. You're probably familiar with two ways of writing quadratics: standard form ($y = ax^2 + bx + c$) and vertex form ($y = a(x-h)^2 + k$). The latter form is important in conics, as it helps reveal the focus and **directrix** (a line such that every point on the parabola is equidistant from the focus as it is to the directrix). For vertical and horizontal parabolas, the information is given below as follows:

1. **Vertical Parabolas:**
   - Equation: $y = a(x-h)^2 + k$
   - Focus: $\left(h, k + \frac{1}{4a}\right)$
   - Directrix: $y = k - \frac{1}{4a}$
2. **Horizontal Parabolas:**
   - Equation: $x = (y-h)^2 + k$
   - Focus: $\left(h + \frac{1}{4a}, k\right)$
   - Directrix: $x = k - \frac{1}{4a}$

8.5 Hyperbolas

Hyperbolas are similar to parabolas, but they instead have two U-shaped curves facing away from each other in opposite directions. The vertices of each U-shaped curve are equidistant from a central point, and face vertically or horizontally depending on how the equation is written. While our elliptical function was wider horizontally or vertically depending on the a and b values, the direction of the hyperbola depends on whether the y or x term comes first. For the below equations, the constant term c is equal to $\sqrt{a^2 + b^2}$.

1. **Vertical Hyperbolas:**
   - Equation: $\dfrac{(y-k)^2}{a^2} - \dfrac{(x-h)^2}{b^2} = 1$
   - Center: (h, k)
   - Vertices: $(h, k \pm a)$
   - Co-Vertices: $(h \pm b, k)$
   - Foci: $(h, k \pm c)$
   - Asymptotes: $y = \pm\dfrac{a}{b}(x-h) + k$
2. **Horizontal Hyperbolas:**
   - Equation: $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$
   - Center: (h, k)
   - Vertices: $(h \pm a, k)$
   - Co-Vertices: $(h, k \pm b)$
   - Foci: $(h \pm c, k)$
   - Asymptotes: $y = \pm\dfrac{a}{b}(x-a) + k$

Note: For all the equations above (in both hyperbolas and ellipses), it's safe to assume that h and k will simply be 0 *so the equations are centered at the origin. The equations above are for more advance use.*

9 Combinatorics and Probability (C&P)

This unit will cover calculating probability of different events, and introductions to solving combinatoric problems.

9.1 Unit Vocabulary

- **Simple Probability:** Calculating the probability of one event happening in a given scenario
- **Compound Events:** Multiple events whose probabilities are calculated at the same time, such as drawing a red ball from a bag and then drawing a blue one afterwards
- **Independent Probability:** Two events such that the outcome of one does not affect the other
- **Dependent Probability:** Two events such that the outcome of one has an impact on the other
- **Permutations:** The mathematical arrangement of a set of values, such that order matters (ABC $\neq$ BAC)
- **Combinations:** The mathematical arrangement of a set of values, such that order does not matter, but instead the values themselves matter (ABC = BAC, but ABC $\neq$ BCD)

9.2 Simple & Compound Probability

Because not everything is guaranteed in life, we often have to calculate probability in order to get a better idea on the likelihood of a given scenario. Simple probability is simply dividing the favorable options by the total options. For instance, let's calculate the probability of picking a red ball from a bag. The bag has 6 red balls and 4 blue balls. We have 6 favorable red balls, and a total of 10 to choose from, so our probability would be $\frac{6}{10} = \frac{3}{5}$ or 60%.
For compound probability, we have to look at the probability of multiple things happening at a similar time-frame. For instance, let's find out the probability that I roll an even number on a die and also pick a blue ball from the example above? We figured out earlier that there's 4 blue balls out of a total ten, so that probability is $\frac{2}{5}$ simplified. The probability that I get an even number is $\frac{1}{2}$, since there are three even numbers (2 4 6) out of a total six numbers (1 2 3 4 5 6). As a result, we multiply the probabilities and get $\frac{2}{5} \cdot \frac{1}{2} = \boxed{\frac{1}{5}}$. In other terms, the probability of an event occurring (simple or compound) is the probability of each event occurring in the scenario multiplied. Order is super important due to dependent probability, which will be covered below...

$$P(x) = \prod \frac{\text{Favorable Outcomes}}{\text{Total Outcomes}}$$

9.3 (In)dependent Probability

You're probably familiar with the definition of the words "independent" and "dependent". The easiest way to explain the difference in probability is probably using clauses from English (yeah English and math... who would've guessed).
A dependent clause is one that need another part in order to be a complete sentence. For instance, the clause "Although he was tall" is not complete, and depends on another clause to complete it ("Although he was tall, he could not reach the ball"). Dependent probability follows a similar concept, as events rely on one another to change the final outcome. For instance, let's take the probability of selecting two red balls from a bag of 4 red and 6 blue balls. The twist to this is that balls aren't replaced, so once we pick a ball it's never put back in. The probability of picking a red the first time is $\frac{2}{5}$. Now that we've taken a red ball out, the number of favorable outcomes AND total outcomes both change. As a result, our second event has probability $\frac{3}{9} = \frac{1}{3}$ instead of the same $\frac{2}{5}$. Multiplying these, we get $\frac{2}{5} \cdot \frac{1}{3} = \boxed{\frac{2}{15}}$
An independent clause is one that can stand alone, and does not require a new clause in order to actually be a sentence. The clause "the bird can fly" is an example, because it's a sentence by itself (along with another clause of course, such as "the bird can fly, even though he's very young"). Independent probability occurs when the events in a given scenario do not have an impact on each other. For instance, let's calculate the probability of rolling an even number on a die and getting at least a 10 on a spinner from 1 to 20. Notice how the probability won't be affected no matter what number I get on the roll or what number I get when spinning. Calculating the probability, we would get $\frac{1}{2} \cdot \frac{11}{20} = \boxed{\frac{11}{40}}$.

9.4 Combinations and Permutations

This sub-section will be split into two parts to cover the concepts.

9.4.1 Permutations

Permutations are a group of items such that the order matters. For instance, the code 314 may unlock a locker, but not 413 despite having the same values. If the permutation we're using can have repetition, we just keep multiplying the number of options (in equation terms, n^r). For instance, let's figure out the number of locker combinations we can make, if each number is a digit 0 to 9 and made up of three digits. We have a total of ten numbers and three spots, so the answer is $10^3 = \boxed{1000}$.
If we're not allowed to have repetition, we reduce the amount of options for each scenario. Using the same example, what if we can't have repeated digits (so 334 or 222 are both prohibited)? This means we have 10 options the first time, $10 - 1$ the second time, and $10 - 2$ the third time. Multiplying, we get $10 \cdot 9 \cdot 8 = 720$. To write this in other terms, we can use the factorial function to write the permutation calculation.

$${}_nP_r = \frac{n!}{(n-r)!}$$

In this case, n is the size of our choosable set (10 digits), r is the amount we're choosing (we're picking 3 numbers), and ! is the factorial function. Note the numerous ways to represent a permutation, such as $P(n,r)$, ${}^nP_r$, and ${}_nP_r$.

9.4.2 Combinations

Previously, I stated that permutations are very strict on order. Combinations, however, do not care about order whatsoever. For example, a stack with a blue book above a red book is the same as a red book on top of a blue book. To calculate this, we can modify our permutation formula to do the following:

$$C(n,r) =^n C_r =_n C_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

The above formula only works with lack of repetition, however. If we switch our focus so we CAN have repeats (so having two red books is allowed), the formula gets a bit more confusing. The formula is shown below, along with the expansion to get there (shockingly it's not magic that makes zero sense).

$$\binom{r+n-1}{r} = \binom{r+n-1}{n-1} = \frac{(r+n-1)!}{r!(n-1)!}$$

Let's use two examples so this might make a bit more sense. Let's pretend we went to a cool ice cream store with 15 flavors. We want to pick a sundae that contains three of these flavors. Below shows the substitution for BOTH formulas, so taking into account both repetitions and lack of repetitions.

$$\binom{10}{3} = \frac{10!}{3!(10-3)!} = \frac{10!}{7!3!} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 10}{6(1 \cdot 2 \cdot 3 \cdot \ldots \cdot 7)} = \frac{8 \cdot 9 \cdot 10}{6} = \boxed{120}$$

$$\binom{3+10-1}{10+1} = \binom{12}{9} = \frac{12!}{9!3!} = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot 12}{6(1 \cdot 2 \cdot 3 \cdot \ldots \cdot 9)} = \frac{10 \cdot 11 \cdot 12}{6} = \boxed{220}$$

It's also a cool fact and important thing to note that we can figure out these combinations using Pascal's Triangle! Just go to the nth row and rth value of the row, where the top-most row (the solely 1 row) is the 0th, and the left-most value of a given row is also the 0th.

10 Final Note

Wow you've reached the end of this long PDF of notes! I hope you enjoyed using it, and that it provided helpful and accurate information. If you have any feedback about anything, especially informational errors on this, let me know at warithr21@gmail.com. ~~Also this is based on Khan Academy's curriculum so don't sue me if some of this stuff does not come up in your class, or if your class covers material not on here.~~