Vagueness at every order

Andrew Bacon

March 16th

I will assume that we understand the locution ‘it’s borderline whether P’, and represent it with a propositional operator ∇P. We may then introduce the following notions by definition.

- Say that *it’s determinate that* P when P and it’s not borderline whether P. $\Delta P := P \wedge \neg\nabla P$
- Say that it’s determinate* that P when P is determinate at all orders: $\Delta^* P := P \wedge \Delta P \wedge \Delta\Delta P \wedge ....$

Key point: determinacy as I have defined it is not a theory specific notion, it is available to anyone with the concept of borderlineness.

Some principles:

CL The principles of classical logic (including the obvious infinitary analogues of $\wedge$-intro and elim for infinite conjunction).

K $\Delta(A \rightarrow B) \wedge \Delta A \rightarrow \Delta B$

B $A \rightarrow \Delta\neg\Delta\neg A$

Nec If $\vdash A$ then $\vdash \Delta A$.

We then have the following theorem:

Theorem From the above principles we can derive:

Δ^+ $\Delta^* A \rightarrow \Delta\Delta^* A$

Δ^- $\neg\Delta^* A \rightarrow \Delta\neg\Delta^* A$

Corollary We can thus derive $\Delta\Delta^* A \vee \Delta\neg\Delta^* A$, or given our definitions, $\neg\nabla\Delta^* A$.

I think this is bad news, when combined with the claim that I used to be a determinate* child, and now I’m not: it entails that there is a nanosecond at which I stopped being a determinate* child, and it is not a vague matter when that nanosecond occurred.

1 Is a conjunction of determinate truths determinate?

Some people (e.g. Keefe, Field) have thought that Δ^+ is already problematic. There is a straightforward argument in its favour from the following plausible principle stating that a conjunction of determinate truths is determinate.

$\Delta\wedge$-Dist $\Delta A_1 \wedge \Delta A_2 \wedge ... \to \Delta(A_1 \wedge A_2 \wedge ...)$

Field has sought to solve the paradoxes of higher-order vagueness by denying this principle. (As well as revenge versions of the semantic paradoxes.)

But:

(i) Borderlineness in a conjunction has to come from somewhere: if it's not coming from the conjuncts, it must be coming from the operation of conjunction — but that is surely precise. (We can make this argument explicit by appealing to the principle that combining precise things via application yields o precise result. I give more detailed treatment of this in *Vagueness and Thought*.)

(ii) This line of resistance is beside the point: you must already deny one of my assumptions, CL, K, B or Nec. You can actually prove $\Delta\wedge$-Dist (and thus Δ^+) from these assumptions alone. (See also Prior's derivation of the Barcan formula from B.)

2 Nihilism* and propositional vagueness

What about the assumption that there are *any* determinate* children? Perhaps nothing is determinately* true, not even logical truths, so we cannot even help ourselves to the rule of necessitation, Nec.

There might be a plausible story to be told here if you think vagueness is fundamentally a linguistic property. Iterating determinacy predicates forces one to consider distant worlds in which, perhaps, even words like 'or' and 'not' have deviant meanings. But if you believe in propositional vagueness I think this picture looks less plausible.

3 Revenge problems

Unfortunately if we substitute B with any of the following principles, we can still prove our theorem and its corollary.

B^n $A \to \Delta\neg\Delta^n\neg A$

B^* $\Delta(A \to \Delta A) \to \neg\Delta\neg A \to A$

B^n corresponds to the condition that if x sees y, then you can get back to x from y in n hops. B^* to the claim that you can get back in some finite number of hops or other.

4 The logic of determinacy

Definition 1. *A v-frame is a triple* $\langle W, d(\cdot,\cdot), f(\cdot)\rangle$ *where* $\langle W, d\rangle$ *is a metric space, and* $f : W \to \mathbb{R}^+$ *obeys the following:*

$$\forall w, v \in W, |f(w) - f(v)| \leq d(w, v)$$

A formula of propositional modal logic is valid on a v-frame $\langle W, d, f\rangle$ iff it is valid on the Kripke frame $\langle W, R\rangle$ where Rxy iff $d(x, y) \leq f(x)$.

Theorem 2. *The logic of v-frames is* KT*.*

KT is also the logic of reflexive Kripke frames. It's not true that every reflexive Kripke frame can be endowed with a metric and radius function that turns it into a v-frame. In a v-frame every cycle contains a 2-cycle, so v-frames impose extra conditions on the Kripke frame. But these extra constraints do not impose extra validities. There is a standard technique for ironing out things like cycles, and the ironed out models can be straightforwardly turned into v-frames.

Theorem 3. *The modal logic of v-frames based on* $\mathbb{R}^n$ *in which* $f(x) > 0$ *contains* KTB^**, but refutes each* B^n*.*

5 Realistic models

Given the completeness theorem we can refute B^* in the class of v-frames. But the v-frame delivered by the completeness theorem are somewhat unnatural: we would ideally like to be able to find a more realistic model of our sorites puzzle.

In general the set of worlds, W, ought to correspond to broadly possible worlds (as opposed to, e.g., metaphysically possible worlds). But for simplicity we will assume that all worlds agree about everyone's age in nanoseconds, and that the only propositions concern whether people are children. So the worlds may be thought of as cutoff points for being a child that aren't determinately* incorrect.

- $W = \{(n, k) \mid N - k < n < N + k, k \leq K\}$ where N is the cutoff for childhood in nanoseconds, and K is a number $< N$ where $N \pm K$ represents the largest/smallest broadly possible cutoff for childhood.
- $R(n, k)(n', k')$ iff $|n - n'| \leq 1$ and $k' = k - 1$
- $w \in W$ represents a broadly possible world in which the cutoff for being a child is w nanoseconds.