# Vagueness at every order: the prospects of denying B

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**Abstract**

A number of arguments purport to show that vague properties determine sharp boundaries at higher orders. That is, although we may countenance vagueness concerning the location of boundaries for vague predicates, every predicate can instead be associated with a precise knowable cut-off points deriving from precision in its higher order boundaries.

I argue that this conclusion is indeed paradoxical, and identify the assumption responsible for the paradox as the Brouwerian principle B for vagueness: that if $p$ then it's completely determinate that either it's vague whether $p$, or $p$. Other paradoxes which do not appear to turn on B turn instead on some subtle issues concerning the relation between assertion, belief and higher order vagueness.

In this paper a B-free picture of assertion, knowledge and logic is outlined which is completely free of higher order precision. A class of realistic models containing counterexamples to B and a number of weakenings of B is introduced and its logic is shown to support vagueness at every order. A novel framework for thinking about the semantic apparatus in the presence of metalinguistic vagueness is also developed, in which the metalinguistic apparatus is fully contained in the object language.

As is well known, classical logicians are committed to the existence of a last small number. But don't worry, they say, it's vague which number that is. The boundary between the small and non-small isn't sharp. There is a boundary, but it's *vague* where it lies, and it is the existence sharp boundaries, not vague ones, that we should be worried about.[1]

This, I take it, is the classicists answer to the Sorites paradox in a very schematic form. To illustrate why sharp boundaries (but not vague ones) are problematic consider the following example. There is something very bad about asserting that the total length of your childhood was 378432178928476829 nanoseconds. Vagueness prevents you from ever discovering this, and similar precise facts about the length of your childhood. If the boundary between your childhood and the rest of your life was not vague, however, there would have been no reason you couldn't have discovered the length of your childhood in nanoseconds, just as, perhaps, one could find out the number of nanoseconds in a year, and no reason to refrain from going about asserting it. Everyone, epistemicist, contextualist, supervaluationist or what have you, must agree that the above assertion is bad, and presumably, that this badness is due to its being vague.

This reasoning extends. Say that it's determinate that $p$ just in case $p$ and it's not vague whether $p$. Is there a sharp boundary between the determinate children, i.e. non-borderline children, and everyone else? It seems we should not be any happier about assigning sharp numbers to ones determinate childhood than to ones childhood. There is a completely analogous Sorites for 'determinate child' as there is for 'child'. To be sure, there is a last child and a last determinate child in any Sorites sequence, but it is always vague which that last child is. Similar comments apply to the further iterations: it's vague which the last determinately determinate child is in the sequence, and so on and so forth through the finite orders.

Indeed, there are some persons which are determinately$^n$ children for any amount of iterations, $n$, and some which are not. Surely it is vague where that boundary lies as well? In other words, there are some children such that it's not vague, or even higher order vague, whether they're children, and others such that it is vague or higher order vague whether they are children, and it's indeterminate where the boundary between the two lies. To see this note that:

> The period of my childhood during which it was neither vague nor higher order vague whether I was a child was 378432178928476829 nanoseconds in length. (1)

sounds just as terrible, and would sound terrible no matter what number one used. However, if neither this sentence, nor any sentence like it, were vague, what possible reason could prevent us from finding out whether or not (1) was true? Thus to deny the existence of vagueness in (1) is just as problematic for

[1]It should be noted that every classicist can say this, including epistemicists. If you want these claims to assuage the initial intuition that there can't be a last small number, know that you let the epistemicist off the hook too.

the epistemicist as anyone else. Linguistic accounts of vagueness should also take offence to the thought that the way English speakers use the word 'child' somehow safely determines where this boundary is located to this degree of precision.[2] The location of this boundary is no more determined by the practices of English speakers than the boundary between children and non-children.

The subject of this paper concerns a number of arguments that for any predicate, $F$, there must be a sharp boundary between the things which are determinately $F$ at every order (henceforth "determinately* $F$") and the rest. That being vaguely $F$ at some order or other and not being vague at any are precise distinctions. If this argument succeeds we should expect to be seeing exact numbers associated with vague predicates all over the place. Indeed numbers that are in principle discoverable; thus one should not be surprised to hear things like 'my determinate* childhood lasted exactly 378432178928476829 nanoseconds' or 'I became determinately* bald after I lost my 1451$st$ hair' and so on. It is tempting to sweep the infinitary version of the Sorites paradox under the carpet - to say that predicates of the form 'determinately*-$F$' are indeed precise but they're so esoteric we shouldn't worry about them. I think that recognising that this response involves the possibility of finding out propositions like (1) is a cost many would not be willing to pay.

Let me introduce some notation. I shall write $\Delta p$ to mean that $p$ and it's not vague whether $p$, and $\Delta^* p$ to mean $p$ and it's neither vague nor higher order vague whether $p$. The problems considered can be generated without iterating into the transfinite ordinals, so by 'higher order vague' I shall just mean $n$th-order vague for some finite order $n$. Accordingly these operators can be connected in terms of an infinite conjunction as follows: $\Delta^* p \equiv \bigwedge_{n\in\omega} \Delta^n p$.

The claim that it is never vague whether something is determinate* is formally represented by the schema

$$\Delta\Delta^* p \vee \Delta\neg\Delta^* p \tag{2}$$

which can be split up into two claims:

$$\Delta^* p \rightarrow \Delta\Delta^* p. \tag{3}$$

$$\neg\Delta^* p \rightarrow \Delta\neg\Delta^* p. \tag{4}$$

The former principle is introduced and its plausibility suggested it in Williamson [17]. Whilst some have rejected it, notably Field [7], it is not our intention to do so here. In fact an argument for (3) is given in §3.3. The latter principle has received less attention, although it is crucial for the problem. One way to prove (4) is to argue for the Brouwerian principle for $\Delta$

$$\mathsf{B}\text{: } p \rightarrow \Delta\neg\Delta\neg p \tag{5}$$

Although $\mathsf{B}$ is the most obvious candidate (see [17], [4] footnote 11, [5]) (2) is also entailed by a class of weaker principles

$$\mathsf{B}^n\text{: } p \rightarrow \Delta(q \rightarrow \phi_n) \tag{6}$$

[2]I take this apply equally to Williamson's account of vagueness in terms of safety from changes in usage that would disrupt a sentences truth value.

where $\phi_1 := \neg\Delta\neg p$; $\phi_{n+1} := \neg\Delta\neg(q \wedge \phi_n)$.[3] I shall discuss the motivations behind these principles in §3. To this list I would also like to add a principle, which I dub $\mathsf{B}^*$:[4]

$$\mathsf{B}^*\text{: } \Delta(p \rightarrow \Delta p) \rightarrow (\neg p \rightarrow \Delta\neg p) \tag{7}$$

In this paper I shall explain how rejecting these principles one can resist the conclusion that there is a sharp boundary between the determinately* $F$'s and those which are not. I generalize some recent considerations (Mahtani [12], Dorr [3]) that show that $\mathsf{B}$ fails to show that $\mathsf{B}^n$ can fail for any $n$. Finally I consider $\mathsf{B}^*$ and provide tentative reasons against and in favour of it. I hope to draw attention to $\mathsf{B}^*$ as a central and important principle in the debate about higher order vagueness. Finally I discuss some technical issues concerning the logic of vagueness.

The structure of the paper is as follows. In §1 I introduce the Kripke semantics for modelling vagueness. I offer an interpretation of the framework and defend it from a number of objections. I explain how vagueness can arise in a seemingly precise metalanguage like this and how this vagueness connects up with higher order vagueness and iterations of 'determinately'. In §2 I outline the problem of higher order vagueness, the principles which generate it, and argue that it is $\mathsf{B}$ and it's weakenings that are responsible. In §3 I outline a picture of the structure of higher order vagueness in which $\mathsf{B}$ fails, consider the relation between vagueness, assertability and knowledge and address some other paradoxes which appear not to rely on $\mathsf{B}$. Finally I answer some technical questions to do with the proposed logic of vagueness, and provide a simple model for a language which can talk about, and in particular talk about the vagueness of, its own metatheory.

# 1 Higher order vagueness and and metalinguistic vagueness

Theorists of vagueness have long been in the business of providing formal semantics for vague languages. Typically these involve some kind of set theoretic construction which assigns semantic values to sentences. There is a recurring question concerning how the semantic status of a vague proposition according to the model, say Fred's being bald, relates to Fred's actual baldness, and moreover how these semantic statuses relate to vagueness in general. Without a good account of this relation it is quite tempting to think there can be no answer to a

[3] The weakenings of $\mathsf{B}$ sometimes considered are the principles $\mathsf{B}^{n\prime}$: $p \rightarrow \Delta\neg\Delta^n\neg p$. These are slightly weaker (see footnote 4 for the frame conditions) but we shall see that the strengthening is more motivated in this context.

[4] In terms of Kripke frames, $\mathsf{B}^n$ corresponds to the condition that if $x$ can see $y$, then there is a path back to $x$ from $y$ with at most $n$ steps each step of which $x$ can see, whereas $\mathsf{B}^*$ entails that $x$ can see a finite path back but places no bounds on the number of steps it might take. $\mathsf{B}^{n\prime}$ is weaker that $\mathsf{B}^n$ stating only that there is a path with at most $n$-steps back, but that $x$ needn't see any of these steps.

kind of metalinguistic Sorites that purports to show that there are sharp cutoff's in the assignment of semantic values. Part of the problem is that the semantic notions such as 'having an intermediate semantic value' belong to a different language from the target language in which the vagueness is found.

In what follows I shall provide formal framework that has been proved very useful in the study of classical theories of vagueness, namely the frame semantics due to Kripke (see for example [8], [16].) The basic ingredients of a frame involve a set of points and a two-place relation between those points. Contrast this with, for example, the central notions of a supervaluationist account of vagueness: a set of interpretations of the language, and the one-place property of the admissibility of an interpretation. Vagueness is then characterised as truth according to some but not all admissible interpretations. However, with suitable substitutions for 'admissible' and 'interpretation' in this analysis, any classical account of vagueness (in which $\Delta$ obeys the modal logic $\mathsf{K}$) can brought in line with this definition.[5] An admissible interpretation could be an interpretation of the language compatible with the linguistic practices of its speakers. However it could just as easily understood as one compatible with what the speakers know about the correct interpretation. The points may represent interpretations, precisifications, maximal epistemically consistent propositions, contexts, impossible worlds or any number of things depending on the application. I think the question of the correct analysis of vagueness is quite independent of the logical issues I am concerned with in this paper.

To interpret richer languages these frames are extended to models consisting of a set of points, a domain, a relation between these points, a designated point and an interpretation function assigning every atomic formula a truth value relative to each world and variable assignment $\langle W, D, R, v^*, [\![\cdot]\!]\rangle$. This apparatus is sufficient for describing a vague first-order language with an operator, $\Delta$, for stating when something is vague. There are a number of objections that have been raised against these models: (i) that since we are dealing with some kind of set theoretic construction our model must be precise (ii) that no precise characterisation of a vague notion can be determinately adequate (iii) these models specify a designated interpretation, $v^*$, which effectively eliminates all vagueness since $v^*$ settles the truth value of every sentence.

In what follows I shall offer an interpretation of these models. Since we are interested in the - possibly vague - language the meta-theoretician is speaking we need to be able to talk about the vagueness of the theoretical vocabulary continuously with the vagueness of the statements of the more humdrum vague language. Thus I shall be speaking a first-order fragment of English that contains the means to talk about: vagueness - an operator 'it's vague whether ...', interpretations, a satisfaction relation, $\Vdash$, or in other words, an operator $v \Vdash$ for each interpretation $v$, a predicate 'admissible' applying to interpretatons and a special name $v^*$ for the intended interpretation. This will allow us to get clear on the issues surrounding 'metalinguistic vagueness', and also links it up with

[5] For example, if we substituted 'interpretation' for 'maximal logically consistent proposition' and 'admissible interpretation' for 'maximal logically consistent proposition that is not determinately false' (where 'determinately $p$' is defined as '$p$ and it's not vague whether $p$'.)

the literature on iterations of the determinately operator (see the appendix §4.3 for a model of this.)

I shall stick with the supervaluationist terminology in which the points are thought of as interpretations - sometimes called 'precisifications' - of the language we are interested in, in our case, a fragment of English. These are classical bivalent interpretations that assign each lexical item of the language a sharp semantic value. In the simple case we are considering these are just assignments of extensions to predicates and referents to names. Not all of these interpretations have to accurately represent English. There might be, for example, interpretations according to which 1000 is small even though this is determinately not the case. If an admissible interpretation is one which gets nothing determinately wrong, then we might say our models includes inadmissible as well as admissible interpretations.

Let us build up, within our language, some useful definitions. Say that it's determinate that $p$ iff $p$ and it's not vague whether $p$. We can define what it is for an interpretation, $v$, to be 'correct' or 'intended' by the following contextual definition, the disquotational schema:

$$v \Vdash \phi \text{ if and only if } \phi. \tag{8}$$

Thus our use of the name $v^*$, for the intended interpretation, is completely governed by the above schema. If we had chosen to include propositional quantification in our language this could have been made into an explicit definition. We say that an interpretation is *admissible* iff it is intended or it's vague whether $v$ is intended, which is to say:

$$v \text{ is admissible just in case } v \text{ is not determinately incorrect.} \tag{9}$$

Finally we may define the accessibility relation

$$v \text{ is accessible to } u \text{ just in case } u \Vdash v \textsf{ is admissible.} \tag{10}$$

Note that these definitions have all be carried out in what is essentially a single reasonably rich object language. If one has any doubts as to the consistency of these definitions, a model is given in the appendix §4.3 based on Prior's hybrid modal logic.

Notice that it does not follow from our definitions that it's determinate that $\phi$ iff $v \Vdash \phi$ for every admissible $v$. What we've said leaves it open whether it could fail to be determinately false that $\phi$ even though global penumbral connections ensure there's no complete interpretation $v$ such that $v \Vdash \phi$ and it's not determinately false that $v$'s correct. This is an independent and substantive claim.[6] Notice that even with this substantive claim, our proposed connection between determinacy and admissibility does not distinguish classical theories such as epistemicism, contextualism or supervaluationism. This is seen by the fact that the technical notions are defined in terms of theory neutral vocabulary such as 'it's vague whether $p$' and $v \Vdash p$. Somewhat surprisingly the claim that

[6]Compare Fine's condition of 'completeability' [8].

vagueness is truth on some but not all admissible interpretations is validated even in some non-classical theories such as Łukasiewicz logic.[7]

We said earlier that points can interpret sentences very differently from English. That said, to get a sensible logic we shall assume that every interpretation agrees with the identification of determinacy with truth on all admissible interpretations. Also we assume every interpretation agrees about the properties of conjunction, negation and $\Vdash$, because these are not only precise vocabulary, but determinately precise, determinately determinately precise, and so on and so forth. Putting this all together we get the following facts about truth-at-an-interpretations in $W$. I shall use $\mathsf{this}$ typeface to indicate the scope of an $v \Vdash$ operator:

- $v \Vdash \phi\ \mathsf{and}\ \psi$ iff $v \Vdash \phi$ and $v \Vdash \psi$
- $v \Vdash \mathsf{it's\ not\ the\ case\ that}\ \phi$ iff $v \nVdash \phi$
- For any $x$, $v \Vdash \phi(x)$ iff $v \Vdash \mathsf{for\ any}\ x,\ \phi(x)$.
- $v \Vdash x\ \mathsf{is\ accessible\ to}\ y$ iff $x$ is accessible to $y$.
- $v \Vdash u \Vdash \phi$ iff $u \Vdash \phi$.
- $v \Vdash \mathsf{it's\ determinate\ that}\ \phi$ iff $u \Vdash \phi$ for every $u \in W$ such that $v \Vdash u\ \mathsf{is\ admissible}$

The effect the last clause has is to essentially allow us to assimilate the above model to a Kripke frame of the form $\langle W, D, R, v^* \rangle$ by defining $Rxy$ to hold just in case $x \Vdash y\ \mathsf{is\ admissible}$. 'Determinately' thus plays the role 'necessarily' would in the modal analogue of this framework. The interpretation of 'it's vague whether' and hence 'determinately' and 'admissible' are not assumed to be precise and fixed by the model (in the way that 'and' is) and may vary from interpretation to interpretation. This means that unlike the modal case we cannot do away with $R$ and talk about truth at every member of $W$ instead.

Here are a few relevant facts that follow from what we have said so far:

- $v \Vdash v\ \mathsf{is\ intended}$ for every $v \in W$.
- For all $v$, either it's determinate that $v \Vdash \phi$ or it's determinate that $v \Vdash \neg\phi$.
- $v \Vdash \mathsf{it's\ vague\ whether}\ \phi$ iff there's an $x$ and a $y$ such that $Rvx$, $Rvy$, $x \Vdash \phi$ and $y \nVdash \phi$.
- $v \Vdash \mathsf{determinately}\ \phi\ \mathsf{iff}\ u \Vdash \phi\ \mathsf{for\ every\ admissible}\ u$ for every $v$].

[7] One can show this with $|\Delta p| := 2.|p| - 1$, or indeed, any continuous function with $f(x) = 0$ for $x \leq \frac{1}{2}$. By a precise interpretation for a propositional language I mean any sharp function from sentences of the language to 2, including functions that don't necessarily obey the truth functional constraints on the truth functional connectives.

Notice that everything I've said so far about the model $\langle W, R\rangle$ can be said precisely. The notion of an interpretation is precise, as is the relation $x \Vdash y$ is admissible. Truth, vagueness and unrelativised admissibility, however, are not precise notions. And indeed, we have thus far said nothing about how these models relate to truth, vagueness or admissibility in English.

To do this we pick out a point, $v^*$ in our frame, $\langle W, R, v^*\rangle$, which is intended in the disquotational sense of (8). Assuming no two points satisfy exactly the same formulae there can be at most one intended interpretation. Clearly satisfaction of (8) guarantees that truth in English just is truth according to the intended interpretation. Furthermore, since a sentence is determinate-in-English iff it's true at all admissible interpretations, it follows that it's determinate iff it's true at all interpretations $u$ such that $v^* \Vdash u$ is admissible. Truth-in-our-model is just $v^* \Vdash$. Once we have added the correct interpretation to our set and our relation, we have completed our account of vagueness in English.

At this point one might quite justifiably object that by specifying a single interpretation as the correct one we have lost sight of vagueness altogether! For each interpretation in our set, $x$, it is a completely precise matter whether $x \Vdash \phi$ (see the second bulletted point above.) Isn't this just to deny there is vagueness in English - or at least - to assert that there is a determinate matter of fact about seemingly vague matters? This might be so if we had succeeding in specifying the correct interpretation precisely, but we haven't. Whenever it's vague whether $\phi$ it's at best vague whether $x$ satisfies the $\phi$-instance of the disquotational schema above, and thus, it is at best vague whether $x$ is intended. Determinately there is a unique intended interpretation but it's vague which one it is. $v^*$ must be thought of as a is a vague name, analogous to 'Princeton'; it refers but it's vague which object it refers to.

This is my preferred interpretation of the framework, and the interpretation that I shall be working with in this paper. There are other views that accept my description of a model up until the specification of a set of interpretations and a relation between them, but disagree on how the model relates to truth in English. For example this way of understanding the metatheory is distinct from Keefe's [11], who takes the paradoxes of higher order vagueness to suggest that the description of the frame itself might be vague, and that there might be notions of vagueness not expressible relative to a given model. Timothy Williamson also offers a different interpretation of the semantics on the basis that "for the supervaluationist no sharpening is uniquely correct and actual-point validity is not well defined." For this reason, he claims, it is better for a supervaluationist to identify truth-in-a-model as truth at every interpretation in the model (including interpretations that aren't even indeterminately correct.) As noted above, I simply disagree with the claim that 'no sharpening is uniquely correct'. One may accept this without endorsing the false claim that what holds according to the intended interpretation is a precise matter. Truth simpliciter is just truth on the correct interpretation, although it is vague which one that is. In fact classical logic practically guarantees the uniqueness of a correct interpretation (given the disquotational definition of correctness.) For example, imagine a very simple language that can only say whether or not Fred is bald.

This language has only two precisifications, $v_1$ and $v_2$; according to $v_1$ Fred is bald, and according to $v_2$ he isn't. However, given that either he's bald or he isn't, it follows by classical logic that either $v_1$ or $v_2$ is correct, i.e. that one of these says that Fred is bald just in case he is.

Let us work through a couple of questions that might arise on this set up.

> "Doesn't this interpretation commit us to there being two distinct ways a sentence can be vague? It could be vague whether the correct interpretation is a $p$-interpretation or the intended interpretation could see a $p$-interpretation and a $\neg p$-interpretation?"

While these both constitute ways that $p$ can be vague, they are not distinct ways.

Suppose it is vague whether $v^* \Vdash p$. Since, determinately, $v^* \Vdash p$ if and only if $p$, it follows that it's vague whether $p$. Since $v^*$ satisfies the T-schema it follows that $v^* \Vdash$ it's vague whether $p$. From the definition of $R$ in terms of vagueness-at-an-interpretation it follows, by the facts established above, that for some $u$ and $w$ such that $Rv^*u$ and $Rv^*w$, $u \Vdash p$ and $w \nVdash p$ as required. The converse follows by reversing this reasoning.

> "According to this view it's determinate that either there are admissible interpretations in which the extension of 'is red' includes non-red things, or there are admissible interpretations in which the extension of 'is red' excludes red things. How can these interpretations really be 'admissible' if they are clearly getting something wrong."

While (determinately) there are admissible interpretations getting things wrong, there is no interpretation which is determinately getting things wrong. Any object which is red but not in the extension of 'red' on some admissible precisification is at best vaguely red. Really this is just to say that there are some things that are red but not determinately red or things that are not red but not determinately not red. There is nothing puzzling about this: one *has* to accept this once one accepts there are objects which are borderline red, and one accepts excluded middle for 'red'. This is thus something that every classical logician must accept. The further claim about admissible interpretations are just consequences of the definition of admissibility.

## 1.1 Internal model theory

In this section we defined within a object language of colloquial English metatheoretic notions such as 'admissible', 'accessible' and 'correct' which we defined internally from the notion of an interpretation that satisfied the T-schema, and the object language operator 'it's vague whether $p$'. Alternatively one could have started with a Kripke frame of the kind described and shown how to add these notions to the language conservatively. This would also serve as a consistency proof for the kinds of things we were saying in that section (the consistency proof may not be difficult, but is worth highlighting since the semantic paradoxes are a salient worry in this area.)

Let us suppose that we have a Kripke model: $\langle W, D, R, v^*, [\![\cdot]\!]\rangle$. Recall that $W$ could be a set of interpretations/precisifications, maximal epistemically consistent propositions, contexts, impossible worlds or any number of things depending on the application. $D$ is the domain of individuals. For simplicity it is a fixed domain, i.e. there is no vague existence, but that is not essential to our argument. We do however stipulate that $W \subseteq D$ so that we may quantify over precisifications in the object language. $R$ a relation between elements of $W$and $v^*$ the correct (or intended) interpretation.

Let us add to our language the following predicates with the following interpretations:

- A special predicate $Ax$, read as '$x$ is an admissible interpretation', with $[\![A]\!]_w = R(w) = \{x \mid Rwx\}$.
- A semi-operator $v \Vdash p$ with $[\![v \Vdash p]\!]_w = T$ iff $[\![p]\!]_{[\![v]\!]} = T$.
- A non-rigid designator $v^*$ with $[\![v^*]\!]_w = w$.

With these notions at hand we may also define:

- $Rxy$: $x \Vdash Ay$.
- $x$ is correct: $x = v^*$.

These models are closely related to Arthur Prior's 'hybrid modal logic' in which one is able to talk about and quantify over worlds within the modal object language using a series of indexed actuality operators. Let me end by listing a number of sentences that come out true on this model (including some important ones from §1):

1. $v^* \Vdash \phi \leftrightarrow \phi$, for any $\phi$.
2. $\Delta\exists! v Correct(v)$
3. $\neg\exists v \Delta Correct(v)$
4. $\forall v v \Vdash Correct(v)$
5. $Ax \leftrightarrow \neg\Delta\neg Correct(x)$
6. $\nabla\phi \leftrightarrow \exists xy(Ax \wedge Ay \wedge x \Vdash \phi \wedge \neg y \Vdash \phi)$
7. $x \Vdash \nabla\phi \leftrightarrow \exists yz(Rxy \wedge Rxz \wedge y \Vdash \phi \wedge \neg z \Vdash \phi)$
8. $\nabla\exists xy(Ax \wedge Ay \wedge x \Vdash \phi \wedge \neg y \Vdash \phi) \leftrightarrow \nabla\nabla\phi$

As can be seen, 1. demonstrates that $v^*$ really is a (vague) name for the correct interpretation. The validity of 2. demonstrates the uniqueness of the intended interpretation, and 3. the vagueness over which that interpretation is.

5. confirms our definition of admissibility in terms of not being determinately incorrect. 6. demonstrates the equivalence of vagueness with truth according to some but not all admissible precisifications, while 7. says when a proposition is vague at a point in terms of the metatheoretic notion of accessibility. 8. demonstrates how second order vagueness corresponds to vagueness in a metatheoretic statement about admissibility.

# 2 The problem of higher order vagueness

## 2.1 What is higher order vagueness?

The term 'higher order vagueness' is thrown about a lot in the literature and it's often not clear the different uses of this term form a natural kind. For example higher order vagueness is often used to describe the phenomenon of vagueness in the metalanguage (see for example [13] and [14].) The problem of higher order vagueness, under this reading, is the observation that most theorists of vagueness attempt to model vagueness in a precise metatheory, which, for example, assign sentences truth degrees like 0.82912 or similar intermediate truth values. These theorists are committed to sharp cut-off points between full truth, and anything less than full truth (falsity, truth to degree less than 1, superfalsity or gappiness, etc...)

Another use refers to the fact that 'is vague' is a vague predicate (for example, [15], [10].) According to this use higher order vagueness corresponds to the existence of predicates such that it is borderline whether 'is vague' applies to them.

Finally there are uses according to which the structure of higher order vagueness is determined by the logic of the determinacy operator, $\Delta$, and the way it iterates ([4], [17], [7].)

I think these notions do form a natural kind. The first problem of higher order vagueness is not really a problem for anyone who embraces a vague metalanguage. Indeed, the homophonic metalanguage described in the last section embraces such vagueness: the description of the intended model $\langle W, D, R, v^* \rangle$ was vague because the name $v^*$ was vague, which meant the corresponding notion of admissibility was vague too. Furthermore, we saw that it was vague whether $v^* \Vdash \phi$ just in case it's vague whether $\phi$, and also iff $v^* \Vdash \textsf{it's vague whether } \phi$. Vagueness in the metalanguage is nothing more than simple first order vagueness. This point should hold generally for anyone who holds that the T-schema is determinate and a minimal logic of vagueness: it's vague whether $p$ just in case it's vague whether $p$ is true according to the intended model.

At any rate, vague statements about the semantic theory correspond directly to object linguistic statements involving $\nabla$ so it should be clear how the first and third notions line up. What about the second and the third? Assume that '$F$ is vague' just means '$F$ posesses borderline cases'.[8] Notice that this property,

[8] Actually it would be more accurate to identify vagueness with the possibility of having a borderline case: $\lambda F \Diamond \exists x \nabla Fx$. The following argument is much more controversial if we pay

$\lambda F\exists x\nabla Fx$, is vague iff it has a borderline case: $\exists F\nabla\exists x\nabla Fx$. Since existence is not vague we may assume the Barcan formula and its converse: $\forall x\Delta\phi \leftrightarrow \Delta\forall x\phi$. You can then check that from this it follows that $\exists F\exists x\nabla\nabla Fx$. So there are true claims of the form $\nabla\nabla p$.[9]

## 2.2 The problem of higher order vagueness

Imagine that we are talking about the natural numbers less than 100 and we want to know which ones are small. Obviously, there's no sharp boundary between small and non-small numbers. So there will be the numbers which are definitely small, the numbers which are definitely not small, and the borderline cases in between. Of course, there's no sharp boundary between the definitely small numbers and the rest either: there are numbers for which it's vague whether they're definitely small or borderline small. To put it another way, there are numbers which are definitely definitely small, and those which are definitely not definitely small, but there's a range of borderline cases between the two in this case as well. Similar points apply to the boundary between the definitely definitely small numbers and the numbers which are not definitely definitely small, and so on and so forth.

One can see that the set of small numbers, the definitely small numbers, the definitely definitely small numbers, and so on, gradually shrinks as the number of 'definitely's' increases; after all, being definitely $F$ is generally a more stringent condition than being $F$. But this set can't shrink forever! The first set in this sequence clearly starts off with less than 10,000 members, so in this most generous case it can shrink at most 10,000 times before it becomes empty. For some $N$ being definitely$^N$ small is the same as being definitely$^m$ small for any $m \geq N$.

Does this result mean there is some kind of sharp boundary between the determinately$^N$ small numbers and the rest? Fortunately this doesn't follow from what we have said so far. We may, and indeed must, accept that there is a set of numbers which are determinately$^n$ small for every $n$, and a largest such set, but still maintain that *it's vague which set that is.* The starting point of this shrinking process - the set of small numbers - was vague, so there is no reason to think that it won't be vague which set you end up with after the shrinking process is complete. In keeping with the denial of sharp boundaries, we may still hold that it's vague which number is the last determinately$^N$ small number.

In the following sections I shall be considering various proposals that supplement this argument to show there must be sharp boundaries. First let us make this a little bit more formal. The toy argument above made an essential

attention to this distinction.

[9] Compare this with Hyde's argument [10]: that $\lambda F\exists x\nabla Fx$ is vague implies that the relation $\lambda F\lambda x\nabla Fx$ is vague since the existential quantifier is not vague and vagueness is hereditary. Thus this relation has a borderline case: $\exists F\exists x\nabla\nabla Fx$. The argument in the text has the benefit that it does not need to assume that vagueness is hereditary, something that some theorists of higher order vagueness may have to reject. See the discussion of Fields' view in §2.3.

appeal to the fact that the Sorites sequence considered was discrete. Not all Sorites sequences are discrete, however, for example, smallness over the rational numbers, or redness over a spectrum. A completely general argument can be found in Williamson [16] which shows that if something is definitely$^n$ $F$ for any number of iterations $n$, then it definitely is definitely$^n$ $F$ for every $n$. We may define this strong notion of definiteness using infinitary conjunction:

- $\Delta^*\phi := \bigwedge_{n<\omega} \Delta^n\phi$

To be completely transparent about the logic being used I'll list the principles.

C1. $\bigwedge_{i<\omega} \phi_i \to \phi_n$ for each $n < \omega$.

C2. $\bigwedge_{i<\omega}(\phi \to \psi_i) \to (\phi \to \bigwedge_{i<\omega} \psi_i)$.

C3. If $\vdash \phi_i$ for each $i < \omega$, $\vdash \bigwedge_{i<\omega} \phi$.

D1. $\bigwedge_{i<\omega} \Delta\phi_i \to \Delta \bigwedge_{i<\omega} \phi_i$.

C1, C2 and C3 guarantee the obvious fact that $\vdash \bigwedge_{n<\omega} \Delta^n\phi \to \bigwedge_{n<\omega} \Delta^{n+1}\phi$.[10] This is obvious since we just eliminated the first conjunct of this conjunction. But from D1 we can immediately infer $\vdash \bigwedge_{n<\omega} \Delta^n\phi \to \Delta \bigwedge_{n<\omega} \Delta^n\phi$, i.e.

$$\vdash \Delta^*\phi \to \Delta\Delta^*\phi. \tag{11}$$

It should also be noted that none of these principles are characteristically classical, so this result extends to a number of non-classical logics.

## 2.3 Is a conjunction of determinate truths determinate?

The most natural place to block Williamson's argument is to deny D1. This is Fields strategy in [7]. But is this denial at all plausible? I claim it isn't if we accept the following principle concerning vagueness:

> If each semantic constistuent of a sentence is precise then the sentence itself is precise. (12)

Suppose for reductio that $\bigwedge_{i\leq\omega} \Delta\phi_i$ but that $\neg\Delta \bigwedge_{i\leq\omega} \phi_i$. Since each $\phi_i$ and infinitary conjunction are precise, it follows that $\bigwedge_{i\leq\omega} \phi_i$ is precise by (12). By assumption it's not true that $\Delta \bigwedge_{i\leq\omega} \phi_i$ so $\Delta\neg \bigwedge_{i\leq\omega} \phi_i$ must hold. By factivity we have $\phi_i$ for each $i$ and $\neg \bigwedge_{i\leq\omega} \phi_i$ - this is a contradiction by C3.

## 2.4 Sharp boundaries from B

To show that 'determinately* small' is a sharp predicate we must show that it can never be vague whether something is determinately* small. We have so far shown that if a number is determinately* small then it's not vague whether it's

[10] $\vdash \bigwedge_{n<\omega} \Delta^n\phi \to \Delta^i\phi$ for each $0 < i < \omega$ by C1. Thus $\vdash \bigwedge_{i<\omega}(\bigwedge_{n<\omega} \Delta^n\phi \to \Delta^{i+1}\phi)$ by C3. So finally $\vdash \bigwedge_{n<\omega} \Delta^n\phi \to \bigwedge_{i<\omega} \Delta^{i+1}\phi$ by C2.

determinately* small. If we could show that if a number is not determinately* small then it's not vague whether it's determinately* small we would be able to get our conclusion from an instance of excluded middle and reasoning by cases. More generally, if we can show that $\vdash \neg\Delta^* p \to \Delta\neg\Delta^* p$ then it would follow that $\vdash \Delta\Delta^* p \vee \Delta\neg\Delta^* p$ meaning it could never be vague whether something was determinately* true.

A simple way to close this gap would be to introduce the principle $\mathsf{B}$:

$$\mathsf{B}\text{: } \neg p \to \Delta\neg\Delta p \tag{13}$$

From $\mathsf{B}$ one can prove $\neg\Delta^* p \to \Delta\neg\Delta^* p$, and hence from (3) and excluded middle $\Delta\Delta^* p \vee \Delta\neg\Delta^* p$: We have $\neg\Delta^* p \to \Delta\neg\Delta\Delta^* p$ by $\mathsf{B}$, and since we already have $\Delta(\Delta^* p \to \Delta\Delta^* p)$ we have $\Delta\neg\Delta\Delta^* p \to \Delta\neg\Delta^* p$ by contraposition inside the scope of $\Delta$ and $\mathsf{K}$. So by transitivity we have $\neg\Delta^* p \to \Delta\neg\Delta^* p$.[11] Indeed, it is exactly this principle which is assumed in Williamsons discussion of these issues in [17].

The axiom $\mathsf{B}$ is motivated by Williamson's fixed margin models described in [16]. Although he is understanding truth with respect to a Kripke model slightly differently than we are here the considerations mostly transfer to our setting. Williamson describes the class of Kripke frames $\mathcal{C}$ which contains frames, $\langle W, R\rangle$, for which there is some metric over $W$, $d(\cdot,\cdot)$, and $\alpha \in \mathbb{R}$ such that $Rxy$ iff $d(x,y) \leq \alpha$. The motivation for this semantics is roughly the same whether we are epistemicist or some kind of supervaluationist. We should think of the members of $W$ as precise interpretations of the language with the metric representing the degree of similarity between two interpretations. For example, suppose two interpretations, $x$ and $y$, agree on how to interpret every expression, except $x$ interprets 'small for a number less than 100' as the numbers less than 14 and $y$ as the numbers less than 13. These two interpretations should be considered quite close by the intended metric, i.e. $d(x,y)$ will be a relatively small number. A formula, $\phi$, is determinately true according to an interpretion, $x$, if $\phi$ is true at all the interpretations similar enough to $x$. 'Similar enough' means the measure of their differences does no exceed $\alpha$. At the interpretation $x$ described above, '2 is small for a number less than 100' is determinately true because interpretations that disagree with $x$ on this sentence are quite far away.[12]

It is clear that the accessibility relation of each such frames is symmetric, guaranteeing that the axiom $\mathsf{B}$ is valid: if the distance between $x$ and $y$ is less than $\alpha$ then, obviously, the distance between $y$ and $x$ is less than $\alpha$, so for any frame $\langle W, R\rangle$ in $\mathcal{C}$ the axiom $\mathsf{B}$ is valid.

[11] It is interesting to note that $\mathsf{B}$ is not so central for the non-classical theorist. The problematic principle is the rule: if $\vdash \phi \to \Delta\phi$ then $\vdash \phi \vee \neg\phi$. For example both Łukasiewicz and Field's recent logic have this principle. Once one has this principle and (3) all the problematic classical theorems concerning $\Delta^*$ statements are provable. Things would be different, however, if we adopted the weaker rule: if $\vdash \phi \to \Delta\phi$ and $\vdash \neg\phi \to \Delta\neg\phi$ then $\vdash \phi \vee \neg\phi$.

[12] The closest interpretation to $x$ which disagrees interprets 'is small for a number less than 100' as the numbers less than 2, whereas $x$ interprets that as the numbers less than 14. In this context this constitutes a fairly big difference.

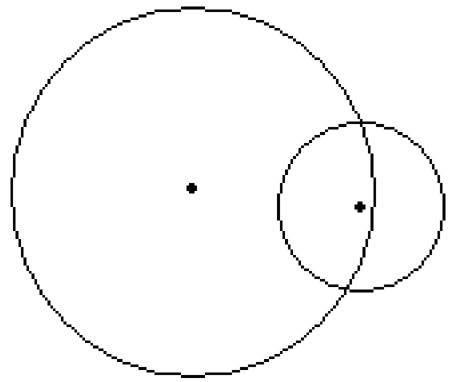


Figure 1: Mahtani's counterexample to $\mathsf{B}$

### 2.5 Mahtani on failures of B

In [12] Mahtani argues that since the term 'determinately' is itself vague, its interpretation ought to vary from point to point and that Williamson's models fail to capture this fact. Of course, the interpretation of 'determinately' technically does vary from interpretation to interpretation on Williamson's semantics - being determinate at $x$ depends essentially on which interpretations are closest to $x$. However they do not vary in all the salient respects. In particular, all interpretations agree about how close an interpretation has to be to be 'close enough' in the relevant sense; $\alpha$ is a fixed quantity throughout the model. In Mahtani's terminology the 'accessibility range' does not vary, when it should.

If each point, $x$, has its own accessibility range, $f(x)$, symmetricity is no longer guaranteed. The distance between $x$ and $y$ maybe be less than $f(x)$ but not less than $f(y)$ (see figure 1.)

### 2.6 Sharp boundaries from other principles

Our initial concern was whether it must always be a precise matter whether something is determinate*. We noted that $\mathsf{B}$ was one way, but not the only way, to close the gap between Williamson's argument for (1) and this claim.

Unfortunately there are an infinite chain of weaker principles, all stated in the finitary language, that also prove that 'determinate*' is precise.

$$\mathsf{B}^n\text{: } p \to \Delta(q \to \phi_n) \tag{14}$$

where $\phi_1 := \neg\Delta\neg p$; $\phi_{n+1} := \neg\Delta\neg(q \wedge \phi_n)$. And the principle

$$\mathsf{B}^*\text{: } \Delta(p \to \Delta p) \to (\neg p \to \Delta\neg p) \tag{15}$$

For example, adding $\mathsf{B}^*$ to the infinitary language allows us to prove the problematic $\neg\Delta^*\phi \to \Delta\neg\Delta^*\phi$. By applying necessitation to (3), which we can already prove, we have: $\vdash \Delta(\Delta^*\phi \to \Delta\Delta^*\phi)$. However, $\Delta(\Delta^*\phi \to \Delta\Delta^*\phi) \to (\neg\Delta^*\phi \to \Delta\neg\Delta^*\phi)$, is an instance of $\mathsf{B}^*$ in the infinitary language, so we can imediately infer (2), i.e. $\neg\Delta^*\phi \to \Delta\neg\Delta^*\phi$, by modus ponens, as required.

In terms of frame conditions, $\mathsf{B}^n$ characterises the the property that if $Rxy$ then there are $n$ points, $z_1, \ldots, z_n$ such that $Ryz_n$, $Rz_nz_{n-1}$, …, $Rz_2z_1$ and $Rxz_i$ for each $i$ - i.e. if $x$ sees $y$ then you can get back from $y$ to $x$ in $n$ steps which $x$ can see. $\mathsf{B}^*$ characterises the the property that if $Rxy$ then for *some* $n$, there are $z_1, \ldots, z_n$ such that $Ryz_n$, $Rz_nz_{n-1}$, …, $Rz_2z_1$ and $Rxz_i$ for each $i$ - i.e. if $x$ sees $y$ then you can get back from $y$ to $x$ in finitely many steps which $x$ can see. Call this latter property the 'backtrack' property. In the lattice of modal logics, $\mathsf{KTB}^*$ is the infimum of $\{\mathsf{KTB}^n \mid n \in \omega\}$.

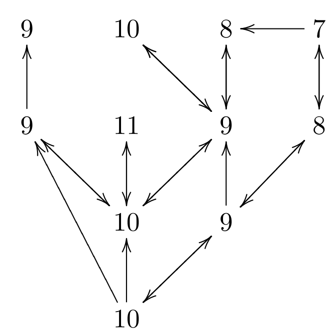
As stated, any one of these axioms is sufficient to close the gap between (1) and the existence sharp higher order cut-off points. I shall show that each frame validating $\mathsf{KTB}^n$ or $\mathsf{KTB}^*$ also validates $\neg\Delta^*p \to \Delta\neg\Delta^*p$ and hence $\Delta\Delta^*p \vee \Delta\neg\Delta^*p$. Suppose the frame $\mathcal{F} := \langle W, R\rangle$ validates $\mathsf{KTB}^n$ or $\mathsf{KTB}^*$. For any model based on $\mathcal{F}$, if $\neg\Delta^*p$ is true at $x$ and $Rxy$. Then (a) for some $n$ you can get from $x$ to a $\neg p$ world in $n$ steps. (b) for some $m$ you can get back to $x$ from $y$ in $m$ steps. Thus you may get from $y$ to a $\neg p$ world in $n + m$ steps, so $y \nVdash \Delta^{n+m}p$ and thus $y \Vdash \neg\Delta^*p$. But $y$ was an arbitrary point accessible from $x$, thus $x \Vdash \Delta\neg\Delta^*p$. So $x \Vdash \neg\Delta^*p \to \Delta\neg\Delta^*p$ for every $x$.

It should be clear by now that properly demonstrate that we can consistently deny higher order precision we will need a more principled way of generating counterexamples. This is what I shall attempt to do in the next section.

# 3 Prospects for a solution

To deny sharp cut-off points at all levels we must reject both $\mathsf{KTB}^n$ and $\mathsf{KTB}^*$. The rest of this paper is an evaluation of the prospects of this proposal.

Below is a model which demonstrates that it is at least possible to deny sharp cut-off points at every level. Each node represents an interpretation of English, with the number at each node representing the greatest number which satisfies 'small for a number less than 100' according to that interpretation. No point can see a point which differs from it by more than one. The converse fails, however, since interpretations may differ radically in the interpretation of other expressions. It is tacitly assumed that every point sees itself.



Remember that 'the last small number' according to a point is the number written beside it, so 'the last determinately$^*$ small number' at a point is the smallest number you can get to from that point by following the arrows. Note that the bottom node can see a node where the last determinately$^*$ small number is 7: follow the arrow to the right. But it can also see two nodes where the last determinately$^*$ small number is 8: follow the arrow up or left. Thus it is vague, at this point, whether the last determinately$^*$ small number is 7 or 8. Indeed, it's vague whether it's determinate$^*$ that 8 is small.

Of course we tried to make this model look realistic by having several points (we could have gotten away with two) and making sure that adjacent points didn't disagree substantially over the interpretation of 'small for a number less than 100'. However, it would be nice to have a general class of models that includes such models as a special case, but are also constrained by facts about vagueness in the same way Williamson's semantics was. In fact we can modify Williamson's fixed margin models in just the way Mahtani suggests to allow for variation in accessibility range. We must also make sure that close points don't interpret 'determinately' drastically differently, i.e. we must make sure close points have similar accessibility ranges. This motivates the following definition.

**Definition 3.0.1.** *A v-frame is a triple $\langle W, d(\cdot,\cdot), f(\cdot)\rangle$ where $\langle W, d\rangle$ is a metric space, and $f : W \to \mathbb{R}$ obeys the following:*

*(A)* $\forall w, v \in W, |f(w) - f(v)| \leq d(w, v)$

A formula of propositional modal logic is valid on a v-frame $\langle W, d, f\rangle$ iff it is valid on the Kripke frame $\langle W, R\rangle$ where $Rxy$ iff $d(x, y) \leq f(x)$. We shall talk about a v-frame and it's associated Kripke frame interchangeably from now on.

The elements of $W$ are to be thought of interpretations or precisifications of the language, with the metric $d$ representing how close they are to one another. A formula is determinately true at an interpretation if it is true at all nearby interpretations. What counts as nearby the interpretation $w$ is determined by $f(w)$: $v$ is nearby $w$ when the distance between them according to $d$ is less than $f(w)$. Note that what counts as 'nearby' depends on the precisification - the constraint (A) says, roughly, that the closer two interpretations are, the less they can differ over their interpretation of 'nearby'.

A useful fact is that there is a natural way to assign a metric over a (generated) Kripke frame. Simply assign a length to each arrow and define the distance between $x$ and $y$ to be the length, ignoring the direction of the arrows, of the shortest path between $x$ and $y$. With a bit of fiddling one can show that the model above is generated by a v-frame in this way. The fact that one can refute $\mathsf{B}^n$ and $\mathsf{B}^*$ over v-frames follows also from the more general fact that the logic of v-frames is $\mathsf{KT}$ (see the appendix.)

## 3.1 Realistic frames

Let us consider a toy propositional language whose only atomic sentences are English sentences of the form: '$a$ is red', where $a$ ranges over names for colours

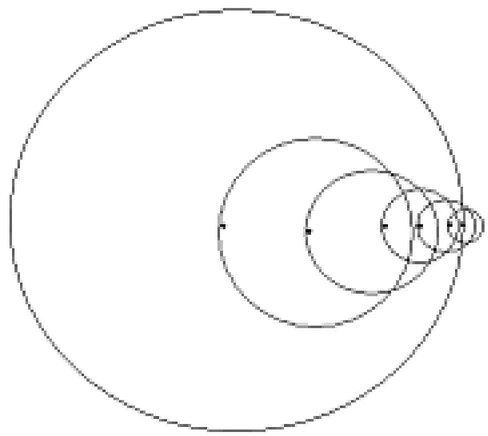


Figure 2: The validity of $\mathsf{B}^*$: $x$ sees a far out point on the edge, yet there is a finite path back to $x$.

in a fixed colour spectrum. It is natural to suppose the interpretations of this language are completely specified by the cutoff point for 'red' along this spectrum of colours. Each colour can be represented by a real number, and the distance between two interpretations can be modelled as the difference between the two numbers representing the cutoff points for those interpretations. Thus the metric of our v-frame is the standard notion of distance on $\mathbb{R}$.

This suggests very natural class of v-frames for modelling vague languages: those based on Euclidean space, $\mathbb{R}^n$. The good news about these v-frames is that each of the problematic principles $\mathsf{B}^n$ are refutable. To refute $\mathsf{B}^n$ we consider the standard metric over $\mathbb{R}$. Let $\epsilon = \frac{1}{2^{n+1}}$. Stipulate that $f(0) = 1$, that $f(x) = f(-x) = \epsilon$ for $x \in \mathbb{R}\backslash[-1+\epsilon, 1-\epsilon]$ and $f(x) = 1-x$ for $x \in (0, 1-\epsilon]$ and $x-1$ for $x \in [\epsilon - 1, 0)$. It is easy to check this satisfies condition (A) and is a v-frame. Now 0 can see 1, yet the shortest path back from 1 takes $n+1$ steps, thus $\mathsf{B}^n$ does not hold. Note that in this model there are no points which can only see themselves, i.e. no points, $x$, such that $f(x) = 0$.

The counterexamples traded on the idea that for any $n$ one can find a v-frame based on an $\epsilon$ small enough to ensure that the longest path from 1 to 0 is longer than $n$. There is, however, no single model which refutes all the $\mathsf{B}^n$ simultaneously. Indeed it is not hard to show that v-frames based on $\mathbb{R}^n$ where no points have a 0 accessibility range has the backtrack property: if $x$ sees $y$, then there is a finite path $z_0, \ldots, z_n$ such that $y$ sees $z_0$, $z_0$ sees $z_1$, …, $z_n$ sees $x$ and $x$ sees each $z_i$. Thus it follows that $\mathsf{B}^*$ holds in these frames (see figure 2.) Intuitively, the closer a point is to $x$, the closer in diameter its accessibility range must be to $x$'s thus one always can find a path leading back to $x$:

For our purposes this result would be devastating. It would entail, for example, that being determinately* red had completely precise boundaries, and this brings along with it all the problematic consequences already mentioned.

An obvious place to resist the result is to deny that the accessibility range of any point must be non-zero. Relaxing this constraint is independently motivated. Suppose we are working with the toy language described above, and the ordering of the relevant colour spectrum is not only dense but complete in the sense of containing a limit for any converging sequence of points (thus, for example, it's structure is like that of $\mathbb{R}$, but not of $\mathbb{Q}$.)[13] If one made the assumption that $f(x) > 0$ for any interpretation $x$, it follows that *no* colours in the spectrum are determinately* red. It is sufficient to show that any two interpretations, represented as real numbers, $x$ and $y$, can be connected by a path. Without loss of generality we may assume that $x < y$. Let $(a_n)_{n\in\omega}$ denote the sequence $x, x + f(x), x + f(x) + f(x + f(x)), \ldots$, i.e. $a_0 := x$ and $a_{n+1} = a_n + f(a_n)$. If $a_n < y$ for each $n$ then $(a_n)_{n\in\omega}$ must clearly be converge as it is a bounded monotonic sequence. Let $a_\infty$ be the point it converges to. I claim that $f(a_\infty) = 0$ contradicting the assumption that $f(x) > 0$ for any $x$. By condition (A) on v-frames we know that $|f(a_\infty) - f(a_n)| \leq a_\infty - a_n (= d(a_\infty, a_n))$ for each $n$. However, since the right hand side converges to 0 as $n$ increases, and $f(a_n)$ converges to 0, it follows that $f(a_\infty) = 0$.

Once one moves away from the simple toy example v-frames based on $\mathbb{R}^n$ become implausible for other reasons. For example, suppose now we are considering a language in which the only atomic sentences are of the form '$a$ is red' and '$a$ is orange' for $a$ a colour in a fixed spectrum. Modelling this language using $\mathbb{R}^2$ would be overly simplistic because the interpretation of 'red' and 'orange' are not independent. Any assignment of cutoff points that allowed 'red' and 'orange' to overlap should be intuitively very far away from the intended interpretation. Thus an interpretation that says that the red colours end at the colour representend by 10 and orange starts at the colour represented by 9 should be very far away from the sensible interpretation that says red ends at 9 and orange starts at 10. However their distance according to the standard metric on $\mathbb{R}^2$ is relatively small: $\sqrt{2} = \sqrt{(10-9)^2 + (9-10)^2}$.

A final worry in this ballpark is that even if $\mathbb{R}^n$ v-frames aren't suitable for modelling vagueness, the correct models might still have enough $\mathbb{R}^n$ like properties to guarantee that the backtrack properties hold. Let me finish by considering two such properties we might quite plausibly expect to hold in any realistic model:

- Density: for any $x$ and $y$ there is a $z$ such that $d(x, z), d(y, z) < d(x, y)$.
- Closeness: Whenever $d(x, y) \leq f(x)$ there is a $z$ such that $d(y, z) \leq f(y)$ and $d(x, z) < d(x, y)$.

However, neither of these principles, even in tandem, ensure that the relevant v-frame has the backtrack property. A simple example would be to let $W :=$

[13] If one were to object that some fact about colours prevents the existence of a such a spectrum, we could modify the example to be about the vague predicate 'small' as applied to real numbers.

$[0,1) \cup (2,3]$, $d(x,y) = |x-y|$, $f(x) = 1.5$ for $x \in [0,1)$ and $f(x) = 1$ for $x \in (2,3]$. Anything in the range $(\frac{1}{2}, 1)$ can see points in $(2,3]$, but there is no path from a point in $(2,3]$ to a point in $[0,1)$. Furthermore this v-frame is both dense and close. These example are *gappy*: for a given point $x$, there may be a range of real numbers $[\alpha, \beta]$, such no point is at a distance between $\alpha$ and $\beta$ away from $x$.

## 3.2 Precise ranges

When considering the simple model presented in this section one might object that it is no better to say that determinately 7 or 8 is the last determinately$^*$ small number (although it's indeterminate which) than to say that determinately 7 is the last determinately$^*$ small number. Indeed, it seems just as bad to say that the location of this cut-off point is vaguely located over a precise range as it is to say that it is precisely located somewhere.

One might question the intuition that this is just as bad. After all, if we consider the vague predicate 'small for a number less than 10' it seems fair enough to say that, determinately, either 3 or 4 is the last small number less that 10, although it's indeterminate which. 'small for a number less than 2' might even be precise even though it's constructed from vague vocabulary!

However there is no need to challenge the intuition. For according to that model there is no precise range in which the last determinately$^*$ small number vaguely falls. At the leftmost and middle point the bottom node can see it's vague whether the last determinately$^*$ small number is 8 or 9, and at the rightmost point it's vague whether it's 7 or 8. Indeed this is not just a peculiarity of our model. The contrapositive of (3) says $\neg\Delta\Delta^*p \to \neg\Delta^*p$ so can show fairly easily that $\nabla\Delta^*p \to \nabla\nabla\Delta^*p$.[14] In other words, if it's vague whether $p$ is determinate$^*$ then it's vaguely vague.

How much of a limitation does this place on our solution? Must we retract all our claims about there being no sharp cut-off points for 'determinately$^*$ small' - must we retract them not because they are false, but because they are vague? Evidently we must retract specific claims of the form 'it is vague whether $p$ is determinate$^*$' since they are all at best vague. I claim this is an advantage of the theory, even, since we can avoid the objection that we must have a precise range in which the last determinately$^*$ small number falls.

What about the crucial claim that it is vague where the last determinately$^*$ small number lies? This claim still stands, and is determinate. In our model not only is it vague, at the bottom node, where the last determinate$^*$ number lies, but it's determinately vague where the last determinate$^*$ small number lies. This should not be puzzling for the classicist - it is standard to allow it to be determinate that there are $F$'s whilst denying the existence of a determinate

[14] Proof: let $q$ be $\Delta^*p$, so that we have $\neg\Delta q \to \neg q$. So $\nabla q \to (\neg q \wedge \nabla q)$ by definitions and prositional logic. Since we can prove the consequent is not determinate, we can prove in $\mathsf{K}$ that the anteceded is not determinate, i.e. we can prove $\neg\Delta\nabla q$, so $\nabla q \to \neg\Delta\nabla q$. Since $\nabla q \to \neg\Delta\neg\nabla q$ we have $\nabla q \to \nabla\nabla q$. Indeed, given the equivalence between $q$ and $\Delta^n q$, it is possible to show that $\nabla q \to \nabla\Delta^n q$ for any $n$ whatsoever.

$F$. It is no different for the complex property of being vaguely determinately$^*$ small.

### 3.3 The Forced Sorites and Assertion

One of the more puzzling issues relating to vagueness and higher order vagueness is the so-called 'forced Sorites march.' One is to imagine that you are to be presented with the elements of a Sorites sequence for $F$ in succession, and in each case you are required to say to the best of your ability whether the element is $F$ on not. The puzzle is that there must surely be a first element at which you stop saying 'yes, it's $F$' and switch to doing something else. Perhaps that is saying 'no, it's not $F$', or saying 'I don't know' or perhaps it is not saying anything at all. The point is, whatever one does, it seems one is committed to a sharp boundary.[15]

Clearly the notion of commitment here is a pragmatic one which holds between an assertion and a proposition. It will be useful to first introduce a somewhat more technical notion of assertability. To say that $p$ is assertable, in this sense, is supposed to track what can be permissibly asserted in slightly idealised circumstances in which it is common knowledge that the audience wishes to know that $p$ and only that, and you do not have any relevant ignorance of the precise facts pertaining to the the truth of $p$. In such a circumstance I suggest you should assert that $p$ just in case you know that $p$, and given the description of your epistemic state above, you know $p$ just in case $p$ and it's not vague whether $p$.

Given the determinacy of these biconditionals it follows that if it's second order vague whether $p$, then it's vague whether it's permissible to assert that $p$, and it's vague whether you know that $p$.[16] It is true that I have built it into my definition of 'assertable' that one can assert $p$ whenever one is not ignorant about the precise proposition $p$, even if $p$ is not determinately precise. The assumption I am making is that the only barrier to knowledge in these idealised circumstances is vagueness. More specifically, the only barrier to knowing that $p$ is vagueness in $p$, and never vagueness in some proposition failing to be determinately equivalent to $p$. It follows that vagueness in the proposition that $p$ is determinate cannot be a barrier to my knowing $p$ unless $p$ is in fact not determinate. (Certainly this commits us to vagueness concerning what is permissibly assertable, but that is surely already independently plausible: consider a forced Sorites march in which the respondent always answers 'yes', and then ask which of those responses were permissible. It should be clear that this is a Sorites sequence for the notion of permissible assertion.)

The corresponding notion of commitment is one in which an assertion that $p$ commits one only to its being determinate that $p$. An alternative view, a view which I reject, is the view that assertion commits one to the determinacy$^*$

[15] The forced Sorites is supposed to also do this for the non-classical approaches which .

[16] I.e. given $\Delta(Ap \leftrightarrow \Delta p)$ it follows, using only the $\mathsf{K}$ principle for $\Delta$, that $\nabla\Delta p \leftrightarrow \nabla Ap$ (here $Ap$ is pronounced 'it is permissible to assert $p$'.)

of what is asserted.[17] A proponent of the latter view may argue against our view as follows. If in the proceedings of a forced march I was asked if 10 were small, I would answer 'yes, 10 is small'. How would I justify the stance just taken on 10's smallness? I would presumably reason as follows: it's permissible for me to assert that 10 is small because I have not asserted anything vague or false in doing so. But then one might want a justification of *this* assertion, concerning the justifiable assertibility of $p$. And again I would offer the same justification, this time about the assertibility and non-vagueness of $p$. If 10 were not determinate* small, however, then at some point I would no longer be able to justify my justifications in this way.

For the record I am not worried by this. I think that something like a knowledge norm on assertion and failures of the S4 principle for knowledge give rise to this phenomenon quite independently of the features of higher order vagueness. However it is worth noting that the alternative view fares no better, for given the vagueness of predicates like 'is determinately* small' there will be cases where $p$ is not determinate* but fails to be determinately not determinate* (and hence fail to be determinately* not determinate*.) In such a case $p$ will not be assertable, because $p$ is not determinate*, but one cannot justify the non-assertion in such an instance since it is not determinate* that it is not assertable.

To return to the forced Sorites march, we may consider the possible responses in turn. Certainly saying 'yes' up to a certain point, and then saying 'no' commits one to sharp boundaries, for that is to commit one to the elements being determinate cases up to a certain point, and determinate non-cases from there on. This is just what it is to say that $F$ is sharp. To respond by saying 'yes' up to a certain point, $a_n$, and then continue by saying 'I don't know', or 'it's indeterminate', is slightly more complicated. This commits one to the determinacy of $Fa_1, \ldots Fa_n$. Asserting 'I don't know' or 'it's indeterminate' to $a_{n+1} \ldots a_m$ commits one to the determinacy of $\nabla Fa_{n+1} \ldots \nabla Fa_m$. Thus we are committed to the following: $\Delta Fa_n$ and $\Delta\neg\Delta Fa_{n+1}$. This is not inconsistent, but it is uncomfortable since it rules out all vagueness in true statements of the form $\neg\Delta Fx$. On the other hand, saying 'yes' up to a certain point, and from then on saying nothing is a different matter altogether. This response pattern does not commit one to sharpness of any kind. Not asserting that $Fa_{n+1}$ is not the same as asserting that you don't know $Fa_{n+1}$, since the former is compatible with you not knowing that you don't know that $Fa_{n+1}$. It is completely compatible with this response pattern that every predicate of the form $\Delta^n Fx$ is vague. In other words, asserting that $Fa_i$ up to a certain point, remaining silent for a while, and then asserting $\neg Fa_j$ from then on, and then asserting that $\Delta^n Fx$ is vague for each $n$ does not commit you to a contradiction.

To demonstrate this, suppose that $Fa_1 \ldots Fa_n$ are determinate, and that $Fa_{n+1} \ldots Fa_m$ aren't. Of course, it is vague which number $n$ is in our example, but we may be certain there is *some* such $n$ by classical logic. If I happen to say 'yes' from cases $a_1$ to $a_n$ and remain silent for cases $a_{n+1} \ldots a_m$ I have (a)

[17] An explicit example of the latter view on commitment can be found in Fine [9].

asserted correctly, in the sense that I have asserted $p$ exactly when $p$ is determinate and (b) have committed my self to nothing incompatible with vagueness at all orders. Furthermore, despite the fact that $a_n$ is the last determinate $F$, it is presumably vague that $a_n$ is the last determinate $F$, so my assertions, despite being correct, fail to be determinately correct. If a perfect asserter is someone who asserts $p$ just in case it's determinate that $p$, it is not possible to be a determinately perfect asserter. (Given the assumption that it is always a determinate matter whether you have asserted $p$ or not.[18])

It is worth remarking that if you knew you were a perfect asserter, i.e. if you knew that you asserted just in case you knew, you would be able to infer from your having not asserted $Fa_{n+1}$ that you didn't know $Fa_{n+1}$ and that $Fa_{n+1}$ was not determinate. Since in a typical forced march Sorites, it is at best vague whether you are a perfect asserter, such knowledge would not be available to you.

## 3.4 Other paradoxes

There are a number of other paradoxes of higher order vagueness in the literature which do not rely on the principle $\mathsf{B}$ which I shall turn to now. They are both variations on an argument originally due to Wright [18].

Let me begin with an argument due to Delia Graff Fara [6]. Fara's argument shows that natural principles concerning higher order vagueness, so called 'gap principles' $\Delta\Delta^n Fa_k \rightarrow \neg\Delta\neg\Delta^n Fa_{k+1}$, with the rule of proof $\Delta$-intro:

If $\Gamma \vdash \phi$ then $\Gamma \vdash \Delta\phi$

lead to a contradiction. There is the preliminary question of whether the gap principles are stronger than what we need to avoid the problematic consequences of higher order precision. I won't pursue that issue here; I want to rather concentrate on the rule of proof of $\Delta$-introduction.

There has been some debate concerning whether a supervaluationist should accept classical logic, where 'classical logic' is construed broadly to include classical rules of inference. The rule Fara appeals to is incompatible with full classical rules of inference. For example one could not apply conditional proof to $p \vdash \Delta p$ (which can be obtained by $\Delta$ intro on $p \vdash p$) to obtain $\vdash p \rightarrow \Delta p$, since this would imply, as a matter of logic, that everything is precise.

There are some tricky questions in this area concerning the nature of logic, whether $\Delta$ is a logical constant, and on the normative impact both notions of consequence have. Since Fara's conclusion is essentially normative - that it is in some sense incoherent to accept vagueness at all orders - it would be nice to talk directly about the normative conclusions without the detour through 'consequence' talk which can be quite obscure in these contexts anyway. Let me introduce two notions of a 'good inference' from $p_1 \ldots p_n$ to $q$.

[18] This assumption may not be unassailable. For example, I might falter or hesitate as I say 'yes' in such a way as to make it vague whether I actually committed myself to the $F$ness of the case in question. This might be one way to be a determinate perfect asserter.

1. $Cr(q) = 1$ if $Cr(p_i) = 1$ for each $i \leq n$ and $Cr \in \mathsf{E}$.

2. $\Sigma_{i \leq n}(1 - Cr(p_i)) \leq 1 - Cr(q)$ for every $Cr \in \mathsf{E}$.

Here $\mathsf{E}$ is a set of probability functions representing possible epistemic situations one could find oneself in. One notion governs what can be inferred given what you already have, whereas the other constrains your beliefs when you are less than certain in the premisses. Something like these two notions are sometimes characterised in terms of global and local consequence, corresponding to the definiteness of the premisses strictly implying the definiteness of the conclusion (i.e. 'preservation of supertruth') and the premisses simply strictly implying their conclusion (i.e. 'preservation of disquotional truth'.) In formal terms that is $\Box(\Delta p_1 \wedge \ldots \wedge \Delta p_n \rightarrow \Delta q)$ and $\Box(p_1 \wedge \ldots \wedge p_n \rightarrow q)$ where $\Box$ represents some suitable notion of logical necessity.

Since one could never find oneself in an epistemic situation which *fully* supported $p \wedge \neg\Delta p$, but one could quite easily have evidence for $\neg p \vee \Delta p$ the former notion of good inference invalidates reductio: we have $p \wedge \neg\Delta p \vdash$ but not $\vdash \neg(p \wedge \neg\Delta p)$.

I am happy to engage in either talk provided it is clear what one means and one is careful which normative conclusions one draws. However, neither notion permits the inference form $p$ to $\Delta p$. Crucially the first notion, 1., does not permit this inference. Observe first that this rule does not preserve determinate truth, for example if $p$ is determinate but not determinately determinate, the premise of $p \vdash \Delta p$ is determinate and it's conclusion isn't. Similarly, since $p$ is precise (although not determinately so), one could in principle be certain of $p$, yet be uncertain in $\Delta p$ due to the vagueness in $\Delta p$. To be sure this counterexample relied of higher order vagueness, but this is clearly not the place to beg *that* question.

Note that the consequence relation that instead preserves determinacy$^*$ does appear to validate $\Delta$-intro. For example, according to this relation (3) guarantees that $p \vdash \Delta p$. As I have argued in the previous section, however, if your beliefs are 'inconsistent' according this consequence relation you are not necessarily being incoherent by committing yourself to a contradiction. The most any belief or assertion commits you to is the determinacy, not the determinacy$^*$, of what is believed or asserted.

In this vein Zardini presents an argument that does not rely on $\Delta$-intro [19]. His argument operates directly with the notion of determinacy$^*$. His argument shows that if we assume the determinacy$^*$ of (a) the vagueness of $\Delta^n Fx$ for each $n$, (b) the $F$ness of $a_0$ and (c) the non-$F$ness of $a_{1,000,000}$ we can derive a contradiction. More formally he assumes the following: $\Delta^*\exists x \nabla \Delta^n Fx$, $\Delta^* Fa_0$, $\Delta^*\neg Fa_{1,000,000}$.

What is surprising about Zardini's argument is that although I have throughout been arguing for the vagueness of predicates of the form $\Delta^n Fx$, i.e. for the claim $\exists x \nabla \Delta^n Fx$, I cannot maintain that this claim is determinate$^*$. This is interesting as it is a concrete example of something I would count as a permissible assertion which is not determinate$^*$.

To check that these assertions, including $\exists x \nabla \Delta^n Fx$ and $\exists x \nabla \Delta^* Fx$, are consistently permissible we need to show that not only are there models in which they are all true, but that there are models in which they are all determinately true. In fact, one can show for any $n \in \omega$ that there is a model in which these claims are determinate$^n$ true.

Let's start with an example in which $\exists x \nabla \Delta^* Fx$ is simply true. Note also that all these examples apply also to $\exists x \nabla \Delta^n Fx$.

$$\begin{tikzcd} 13 \arrow[loop left] \arrow[r] & 14 \arrow[loop right] \end{tikzcd}$$

Remember that a number is determinate$^*$ small at a point iff you can't get to a smaller number by following the arrows. So, for example, the left node sees a node (itself) in which 14 is not determinately$^*$ small, and can see a node in which it is (the right node.) Thus, at the left node it is vague whether 14 is determinately$^*$ small, so 'determinately$^*$ small' has a borderline case. However the vagueness of 'determinately$^*$ small' is not determinate because the left node sees a node in which 'determinately$^*$ small' is completely precise: the right node.

$$\begin{tikzcd} 12 \arrow[loop left] \arrow[r] & 13 \arrow[loop above] \arrow[r] & 14 \arrow[loop right] \end{tikzcd}$$

Here at the leftmost node it is vague whether 13 is determinately$^*$ small: it sees a world where it isn't (itself) and a world where it is (the middle node.) At the middle node it's vague whether 14 is determinately$^*$ small (see above). So at the leftmost node we have $\Delta(\nabla \Delta^* S(13) \vee \nabla \Delta^* S(14))$. Thus at every node the leftmost node sees 'determinately$^*$ small' has a borderline case, witnessed by 13 and 14 respectively. This gives a model for $\Delta \exists x \nabla \Delta^* Fx$ (and also $\Delta \exists x \nabla \Delta^n Fx$, for each $n$.)

$$\begin{tikzcd} 11 \arrow[loop left] \arrow[r] & 12 \arrow[loop above] \arrow[r] & 13 \arrow[loop above] \arrow[r] & 14 \arrow[loop right] \end{tikzcd}$$

Just as before, it is vague at the leftmost node whether 12 is determinately$^*$ small. At every world it sees either 12 or 13 is a borderline case of determinate$^*$ smallness (see above), and at every node seen by a node seen by the leftmost either 12, 13 or 14 is a borderline case of determinate$^*$ smallness. So we have $\Delta\Delta(\nabla \Delta^* S(12) \vee \nabla \Delta^* S(13) \vee \nabla \Delta^* S(14))$. Thus this is a model for $\Delta\Delta \exists x \nabla \Delta^* Fx$ (and also $\Delta\Delta \exists x \nabla \Delta^n Fx$, for each $n$.) It should be clear how to carry on this series.

### 3.5 Nihilism$^*$

One response to a Sorites paradox for a predicate of the form $\Delta^* Fx$ is simply to deny than *anything* satisfies $\Delta^* Fx$. I shall call this view 'nihilism$^*$' since it mimics the nihilist response to the ordinary Sorites paradox.

It should be noted that the necessitation principle for $\Delta$, a principle we have appealed to throughout this paper, already supplies counterexamples to nihilism$^*$ since, with a principle of infinitary conjunction introduction, we can easily show $\Delta^*(Fx \vee \neg Fx)$. Not only can we show that logical truths are determinate*, but we can also show things like $\Delta^*(\Delta Fx \to Fx)$, and indeed $\Delta^*(\Box Fx \to Fx)$ and $\Delta^*(KFx \to Fx)$ if we have necessity and knowledge operators in the language. Although certain logical and conceptual truths are among the determinate* propositions, some conceptual truths aren't. For example, if it is vague whether any foetus of a certain age is a person, then presumably it is is a conceptual truth one way or the other without being a determinate, and hence a determinate*, truth one way or the other.

Anyone who accepts necessitation for $\Delta$ has to admit a distinction between certain determinate* truths and others. I would be very skeptical that such a distinction would be a precise distinction.[19] And without a motivated precise distinction between the $\Delta^*$ truths and the rest a response to the paradoxes of higher order vagueness is needed.

A more radical approach is to deny necessitation altogether. This is the strategy that Dorr seems to endorse in [4]. A standard way to model failures of necessitation is to introduce a distinction between normal and non-normal nodes in the kind of Kripke frames we have been considering. Such a move invites variants of the kinds of paradoxes we have been considering. For example, we could define an operator $\Delta_N p$ saying that $p$ is true at all accessible *normal* worlds, and run the paradox for the operator $\Delta^*_N p$. Without appealing to this particular semantics for non-necessitatable operators, we must have some notion of normalcy which allows us to say things like '$\Delta p \to p$ is part of our logic and $\Delta p \to \Delta\Delta p$ isn't', i.e. something stronger than the mere truth or assertability of $p$, but weaker than the empty notion of determinate* truth.

# 4 Appendix

Here we demonstrate some relevant facts about the logic of higher order vagueness. Recall that:

**Definition 4.0.2.** *A v-frame is a triple $\langle W, d(\cdot,\cdot), f(\cdot)\rangle$ where $\langle W, d\rangle$ is a metric space, and $f : W \to \mathbb{R}^+$ obeys the following:*

*(A)* $\forall w, v \in W, |f(w) - f(v)| \leq d(w, v)$

A formula of propositional modal logic is valid on a v-frame $\langle W, d, f\rangle$ iff it is valid on the Kripke frame $\langle W, R\rangle$ where $Rxy$ iff $d(x, y) \leq f(x)$.

Dorr [3] shows, translating into the terminology of v-frames, that $\mathsf{B}$ is not valid over the v-frame $\langle (1, 2), |x - y|, \frac{x}{3}\rangle$ although the weaker principles $p \to \Delta\neg\Delta\Delta\neg p$ and $\mathsf{B}^2$ are valid in this frame. It is possible, however, to construct v-frames in which $p \to \Delta\neg\Delta^n\neg p$ is valid for no $n \in \mathbb{N}$. For example, let $W := \{0, 1\}$, $d(x, y) = |x - y|$, $f(0) = 1$ and $f(1) = \frac{1}{2}$.

What is the logic of v-frames? Clearly every v-frame generates a corresponding reflexive Kripke frame, so the logic of v-frames contains $\mathsf{KT}$. One might have hoped that every reflexive Kripke frame could be generated from a v-frame this

[19] At least, there isn't any obvious precise criteria for distinguishing the two such as 'is a tautology' and so on.

way ensuring a logic of exactly $\mathsf{KT}$. This reduces to the question of whether every reflexive digraph can be embedded into a metric space in such a way that there is a a closed ball around each node that contains all and only those nodes it can see. Unfortunately this does not hold:

**Fact:** Suppose $\mathcal{F}$ is a Kripke frame based on a v-frame. If $\mathcal{F}$ contains a cycle, it contains a 2-cycle.

To see this suppose that $\langle a_0, \ldots, a_n \rangle$ is a cycle in $\mathcal{F} = \langle W, R \rangle$ where $n > 2$. For convenience let $a_i = a_j$ where $j = i \bmod (n+1)$ for $i > n$. Now suppose that that $\neg R a_{i+1} a_i$ for every $i$. Since for each $i$ $Ra_i a_{i+1}$ we know that $d(a_i, a_{i+1}) \leq f(a_i)$ in the corresponding v-frame. We also know that $f(a_i) < d(a_{i-1}, a_i)$ since $\neg R a_i a_{i-1}$. Thus for each $i$, $d(a_i, a_{i+1}) \leq f(a_i) < d(a_{i-1}, a_i)$, so $f(a_n) < d(a_{n-1}, a_n) \leq f(a_{n-1}) < \ldots \leq f(a_{-1}) = f(a_n)$, i.e. $f(a_n) < f(a_n)$ which is a contradiction. So for some $i$, $Ra_i a_{i+1}$ and $Ra_{i+1} Ra_i$.

v-frames thus have more structure than reflexive frames. However, it turns out this does not make a difference to the logic:

**Theorem 4.1. Completeness**. *A set $\Sigma$ is valid on every v-frame iff it's members are theorem's of* $\mathsf{KT}$.

*Proof.* Suppose that $\Sigma$ is a $\mathsf{KT}$-consistent set of formulae. Then $\Sigma$ is satisfiable on the canonical frame $\mathcal{F}$. $\mathcal{F}$ may contain cycles without 2-cycles, so we cannot yet infer that $\Sigma$ is satisfiable on some v-frame. However we may construct a frame from $\mathcal{F}$, with all the cycles ironed out, that is equivalent to a v-frame.

Let $a_0$ be a maximal $\mathsf{KT}$-consistent set containing $\Sigma$. We may assume that $a_0$ is a root of $\mathcal{F}$ (if it isn't take the generated subframe around $a_0$ and work with that instead.) Define $\mathcal{F}^+ := \langle W^+, R^+ \rangle$ as follows

- $W^+ := \{s \mid s \text{ a path in } \mathcal{F} \text{ such that } s_0 = a_0\}$
- $R^+ := \{\langle s, t \rangle \mid |t| = |s| + 1 \text{ and } s_i = t_i \text{ for } i \leq |s| \text{ or } s = t\}$

**Claim:** $f(a_0, \ldots, a_n) = a_n$ is a bounded morphism from $\mathcal{F}^+$ to $\mathcal{F}$.

(1) Suppose $R^+ st$. If $s = t$ then $f(s) = f(t)$ so $Rf(s)f(t)$ since $R$ is reflexive. If $|t| = |s| + 1$ then $Rf(s)f(t)$ since $t$ and $s$ are paths.

(2) Suppose $Rf(s)f(t)$. We want to find a $u$ such that $R^+ su$ and $f(u) = f(t)$. If $f(t) = f(s)$ let $u = s$. Otherwise, let $u = \langle s, f(t) \rangle$.

Since anything valid on $\mathcal{F}^+$ is valid on every bounded morphic image of $\mathcal{F}^+$ (see for example [2]) it follows that $\Sigma$ is satisfiable on $\mathcal{F}^+$. Now we construct our v-frame as follows:

- We begin by defining distance between adjacent points. If $R^+ st$ then $e(s,t) = e(t,s) = \frac{1}{2^{|s|}}$. Always fix $e(s,s) = 0$
- $d(s,t) := inf\{\sum_{i=0}^{n} e(p_i, p_{i+1}) \mid p_0 = s, p_n = t, p \text{ a path in the symmetric closure of } \mathcal{F}^+\}$
- $f(s) := \frac{1}{2^{|s|}}$

It is now easy to check that $\langle W^+, d, f \rangle$ is a v-frame and that $R^+ st$ iff $d(s,t) \leq f(s)$. $\square$

Cian Dorr has pointed out to me that the constraint (A) on v-frames does not play much of a role in the proof of Theorem 4.1. This allows us to prove a slightly more general result:

**Definition 4.1.1.** *a* **difference measure** *is a function $g : \mathbb{R}^2 \to \mathbb{R}$ such that:*

- *$g$ is continuous in both arguments.*
- *$g(x, x) = 0$*
- *$g(x, y) = g(y, x)$ (this constraint is optional in what follows.)*

*For a given difference measure, $g$, a g-frame is a triple $\langle W, d(\cdot, \cdot), f(\cdot)\rangle$ where $\langle W, d\rangle$ is a metric space, and $f : W \to \mathbb{R}$ such that:*

*(A′)* $\forall w, v \in W, g(f(w), f(v)) \leq d(w, v)$

**Corollary 4.2.** *For any difference measure $g$, the logic of g-frames is* KT.

*Proof.* Note that for any positive $a$ there is a $b < a$ such that $g(a, b) \leq a$ since $g(a, a) = 0$ and $g$ is continuous in both arguments. For any $a$ pick a unique such $b$, $a_g$ (choice.)

Now modify the construction in Theorem 4.1 as follows.

- Fix $e(s, s) = 0$ for every $s$.
- Let $e(\langle a_0\rangle), t) = e(t, \langle a_0\rangle) := 1$ for $t \neq \langle a_0\rangle$ such that $R^+\langle a_0\rangle, t$.
- Suppose that $e(s, t) = e(t, s) = a$ has already been defined for $R^+st$, and suppose that $R^+tu$ $t \neq u$. Define $e(t, u) = e(u, t) = a_g$.
- $d(s, t) := inf\{\sum_{i=0}^{n} e(p_i, p_{i+1}) \mid p_0 = s, p_n = t, p$ a path in the symmetric closure of $\mathcal{F}^+\}$
- $f(s) := sup\{e(s, t) \mid R^+st\}$.

□

The interest in this generalization is that one might think that it becomes much harder for two points to differ on the interpretation of 'determinately' the closer together they are. Perhaps it is not the difference between $f(w)$ and $f(v)$ that must be less than $d(w, v)$ but the difference between their ratios, or some other such $g$.

## 4.1 Further restrictions

The proof of completeness and the counterexamples to the $\mathsf{B}^n$ principles relied heavily on our considering slightly artificial frames that were based on metric spaces that either aren't dense, or have points with zero accessibility range. A natural class of v-frames to consider are those based on metric spaces of the form $\mathbb{R}^n$ where $f(a) > 0$ for all $a \in \mathbb{R}^n$. In these frames whenever $x$ can see $y$, there is a path back from $y$ to $x$, even though there are frames invalidating

$\mathsf{B}^n$ for each $n \in \omega$ (i.e. there is no upperbound on how long these paths might be.) This is worrying since this means that $\Delta\Delta^* p \vee \Delta\neg\Delta^* p$ is valid over these frames.

We can express something like this principle in modal logic. I'll call it $\mathsf{B}^*$.

$$\mathsf{B}^*\colon\ \Delta(p \to \Delta p) \to (\neg p \to \Delta\neg p) \tag{16}$$

$\mathsf{B}^*$ is valid in the class of v-frames just described. In the presence of $\mathsf{KT}$, $\mathsf{B}^*$ defines what I shall call 'the backtrack principle'.

Whenever $Rxy$ there exists $z_1, \ldots, z_n$ such that (a) $z_1 = y$, $z_n = x$ and $Rz_iz_{i+1}$ for $1 \leq i < n$ and (b) $Rxz_i$ for $1 \leq i \leq n$. (17)

*Proof.* We shall show that $(\Delta(p \to \Delta p) \wedge \neg\Delta\neg p) \to p$ defines the requisite property. Suppose the reflexive frame $\mathcal{F} = \langle W, R\rangle$ has the backtrack property. Now suppose $x \Vdash (\Delta(p \to \Delta p) \wedge \neg\Delta\neg p)$. The second conjunct ensures that there is a $y$ such that $Rxy$ and $y \Vdash p$. Since $\mathcal{F}$ has the backtrack property there is a finite path back from $y$ to $x$, $z_1, \ldots, z_n$, which $x$ can see. Since $x \Vdash \Delta(p \to \Delta p)$ each $z_i \Vdash p \to \Delta p$. Since $z_1 = y$ and $y \Vdash p$, $y \Vdash \Delta p$ - by induction we can see that $z_i \Vdash p$ for each $i$ which means $z_n = x \Vdash p$ as required.

For the other direction suppose, for contradiction, that $\mathcal{F} \models (\Delta(p \to \Delta p) \wedge \neg\Delta\neg p) \to p$ but $\mathcal{F}$ lacks the backtrack property. This means that for some $x$ and $y$, $Rxy$ but there is no path back from $y$ to $x$ which $x$ can see. Define the following valuation on $\mathcal{F}$: $w \Vdash p$ iff there are $z_1, \ldots z_n$ such that (1) $z_1 = y$, $Rz_nw$ and $Rz_iz_{i+1}$ for $1 \leq i < n$ and (2) $Rxz_i$ for $1 \leq i \leq n$. Certainly if $x$ had this property then $z_1, \ldots, z_n, x$ would be a path back to $x$ which $x$ can see, so $x \nVdash p$. However $x \Vdash \Delta(p \to \Delta p)$ since if $Rxw$ and $w \Vdash p$ then there is a path from $y$ to $w$ satisfying (1) and (2): $z_1, \ldots z_n$. Furthermore, for any world that $w$ sees, $w'$, $z_1, \ldots, z_n, w$ will be a path from $y$ to $w'$ satisfying (1) and (2), since $Rxw$.

□

Is $\mathsf{KTB}^*$ the modal logic of these v-frames? We start with a negative result: $\mathsf{KTB}^*$ is not sound and *strongly* complete with respect to *any* class of frames. I.e. there is no class of frames, $\mathcal{C}$, such that a set is $\mathsf{KTB}^*$ consistent iff it's satisfiable on a frame in $\mathcal{C}$. The following also shows it is neither canonical nor compact.

*Proof.* To show this we shall show there is a $\mathsf{KTB}^*$-consistent set of sentences which is unsatisfiable on every frame validating $\mathsf{KTB}^*$.

Let $\Sigma := \{p, \neg\Delta\neg q\} \cup \{\Delta(q \to \Delta^n\neg p) \mid n \in \omega\}$. If $\Sigma$ were $\mathsf{KTB}^*$-inconsistent some finite subset would be $\mathsf{KTB}^*$-inconsistent (since proofs are finite.) We shall show that for every $m \in \omega$, $\Sigma_m := \{p, \neg\Delta\neg q\} \cup \{\Delta(q \to \Delta^n\neg p) \mid n \in m\}$ is $\mathsf{KTB}^*$-consistent. $\Sigma_m$ has a $\mathsf{KTB}^*$-model: $\langle m+1, R\rangle$ where $Rxy$ iff $x = 0$ or $x > 0$ and $|x - y| \leq 1$. 0 can see $m$ and there is a finite $m$ length path back from $m$ to 0 that 0 can see but no shorter path. Let $q$ be true only at $m$ and $p$ only at 0.

However, if $\mathcal{F}$ validates $\mathsf{KTB}^*$ then $\mathcal{F}$ has the backtrack property so at no point of $\mathcal{F}$ is every member of $\Sigma$ true: if $x \Vdash \neg\Delta\neg q$ then $x$ sees some $y \Vdash q$. By the backtrack property there is a path $z_1, \ldots, z_n$ back to $x$ which $x$ can see, so $\Delta(q \rightarrow \Delta^{n+1}\neg p)$ cannot be true at $x$ if $x \Vdash p$.

□

However there is a positive result, namely that $\mathsf{KTB}^*$ is sound and complete over the class of reflexive frames with the backtrack property. For this result I refer the reader to [1], who shows that $\mathsf{KTB}^*$ has the finite model property.

**Theorem 4.3.** *If $\phi$ is $\mathsf{KTB}^*$-consistent then it is satisfiable on a finite reflexive frame with the backtrack property.*

Let me end with an open question: is $\mathsf{KTB}^*$ the logic of v-frames over $\mathbb{R}^n$ in which $f(x) > 0$?

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