Zermelian Extensibility

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> The two diametrically opposed tendencies of the thinking mind, the ideas of creative progress and summary completion, which form also the basis of Kant's "antinomies", find their symbolic representation as well as their symbolic reconciliation in the transfinite number series, which rests upon the notion of well-ordering and which, though lacking in true completion on account of its boundless progressing, possesses relative way stations, namely those "boundary numbers", which separate the higher from the lower model types. .
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> Zermelo (1930), p431.

In his *Grundlagen einer Allgemeinen Mannigfaltigkeitslehre*, Cantor introduces two principles of infinity. The first principle lets one create potential infinities—never ending series of larger transfinite numbers. The second tells us that to every potentially infinite series there corresponds a complete infinity. However, integral to Cantor's picture is the idea that these two processes of creation and completion can be continued forever. This line of thinking has been very influential in the philosophy of set theory, but has also remained fraught. In this article I explore, in the framework of higher-order logic, one particularly flatfooted formulation of the idea drawing inspiration from some remarks of Zermelo. Extant approaches to the topic of "indefinite extensibility" take the phenomenon to specifically concern the metaphysics of certain types of abstract objects, like sets and ordinals. The most popular approaches either deny the possibility of unrestricted quantification, or posit special mathematical modalities according to which the length of the set theoretic hierarchy

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is contingent. The present approach, by contrast, is compatible with nominalism, unrestricted quantification, and extensionalism, and articulates how the concept of a well-order (or a ZF relation) can be indefinitely extensible independently of the metaphysics of any particular sort of abstract object.

The paper is organized as follows. In section 1, I introduce Aristotle's and Cantor's theories of infinity, and the paradoxes associated with them. Higher-order (as opposed to a modal) analyses of the notion a potential infinity are emphasized and are shown to avoid certain puzzles implicit in Aristotle's notion. In section 2 I turn to the idea of indefinite extensibility, as it applies to the notion of well-order and a ZF relation. Following some remarks of Zermelo, I give a straightforward formulation of the indefinite extensibility of the notion of a well-order and ZF relation in higher-order language — every well-order or ZF relation can be extended to a larger one, and that any sequence of well-orders or ZF relations has a completion. Section 3 discusses the incompatibility between these higher-order indefinite extensibility principles and a higher-order well-ordering principle, and argues that certain weakenings, such as the idea that the sets of any ZF relation are well-orderable, are consistent. In the final section, I turn to the more general question of whether one needs to posit special first-order entities — like sets, transfinite numbers, and so on — to represent the structure of the higher-order. I explore the nominalist position that we do not need special purpose entities to represent the higher-order, and that the Zermelian logics developed are especially useful for developing this view for they remove dangling questions concerning the size of the universe that would have to be answered if the universe could be well-ordered. In appendix A.1 higher-order logics that contain these extensibility principles are defined, and are shown to be consistent using elementary methods (i.e. without forcing, along the lines of Fraenkel (1922)).

1 Higher-Order Formulations of the Potentially Infinite

Aristotle famously drew a distinction between potential infinities and completed infinities, rejecting the latter but not the former. The best way I can think of to spell this out employs second-order or plural resources.[1] An example of a potential infinity might include a series of stretches of time of increasing length, $t_1, t_2, t_3, ...$, each properly including its predecessors. Every initial segment of this series can be counted or listed and its members are

[1] Rosen (2021).

in good standing by Aristotle's lights, but according to Aristotle there is no single individual, an infinite stretch of time, that includes them all as parts; this would be a completed infinity. This situation is perfectly consistent provided we reject the mereological principle that any things whatsoever compose a whole. The stretches of time together are potentially infinite, but none of them alone, nor any finite number of them together are potentially infinite.

Mathematicians interested in the foundations of analysis understood the distinction as having to do with quantifier order (cf. Cantor's distinction between the *proper* and *improper* infinite[2]): an order relation, $<$, is a potentially infinite order when for every element there is a greater element, $\forall x \exists y. x < y$, and is a completed infinity when there is an element which is as great as every element, $\exists y \forall x. x \leq y$. So at a first pass, *being potentially infinite* is an irreducibly third-order predicate or, perhaps, a plural predicate. To posit a completed infinity is to posit a single individual that contains, or stands in some other similar relation, to all those things at once. In Aristotle the concept of potential infinity is usually applied to things that are ordered by some sort of part-whole relation, $<_t$, (representing proper parthood) and so is naturally represented by a higher-order predicate that can combine with a binary predicate to form a sentence: that every one of our times bears $<_t$ to another. In general this only guarantees infinitude when the relation in question is a *strict order*, i.e., a transitive irreflexive relation. Writing $\forall_e$ for the first-order quantifier, $\mathrm{Dom}\, Rx$ for $\exists_e y(Rxy \vee Ryx)$ ("x is in the domain of R") and SO for $\forall_e xyz(Rxy \wedge Ryz \rightarrow Rxz) \wedge \forall_e x \neg Rxx \wedge \exists_e xyRxy$ ("R is a non-empty strict order"), we may define the relevant higher-order predicate as follows:

$$\mathrm{PotInf}\, R := \forall_e x(\mathrm{Dom}\, Rx \rightarrow \exists_e yRxy) \wedge \mathrm{SO}\, R$$

Thus we can capture the potential infinity of the stretches of time $t_1, t_2, ...$ by ascribing this predicate to $<_t$, the parthood relation restricted to those times. In doing so we are not saying that there is a *series* $t_1, t_2, ...$ that is potentially infinite. For otherwise Aristotle's position seems to be incoherent: if one can have potentially infinite series, like $t_1, t_2, t_3, ...$, don't we also have an actually infinite individual, namely the infinite series itself? The higher-order nature of our formulation is thus essential here. It is hard to see how a first-order predicate, 'is a potential infinity', could take its place, for if the potentially infinite required, in addition to the times $t_1, t_2, ...$, a further individual—*a*

[2] See, for instance, Cantor (1883) §2.3-5, Bolzano (1996), D'Alembert (1996). In these discussions it is a *variable quantity* that is said to be potentially or actually infinite: potentially infinite when it only takes finite values, but with no upper bound, and actually infinite when it can take infinite values.

potential infinity—to be the logical subject of this predicate, whether it be a series or something else, it seems we are committed to something relevantly like a completed infinity.

We will take this idea as our starting point and not—as many have assumed—an irreducibly modal analysis of the potentially infinite.[3] Of course, whether this is a faithful interpretation of Aristotle is another question, and one I shall not pursue here. (Although it is not an entirely idiosyncratic place to start either: Rosen (2021), for instance, argues that Aristotle was entirely open to their being infinitely many things in actuality, unbounded in their size, provided each of those individuals is itself finite.) In order to theorize in a sufficiently general way, it is consequently necessary to introduce 'higher-order' quantifiers that can generalize into the position occupied by predicates as well as names. As we wrote $\exists_e$ for quantifiers binding into name position, we write $\exists_{(e)}$ and $\exists_{(ee)}$ for quantifiers binding into unary and binary predicate position, and so on (in general, $\exists_{(\sigma_1 \ldots \sigma_n)}$ for quantification into the position of an n-ary relation between things of types $\sigma_1, ..., \sigma_n$). Thus, writing Fa for 'Socrates is wise' we can generalize both into the position of the name 'Socrates' and into the position of the predicate 'is wise', so that not only $\exists_e x\, Fx$ but also $\exists_{(e)} X\, Xa$ follows from Fa by existential generalization. Following Prior (1971), I will understand $\exists_{(e)}$ as a device for forming generalizations in predicate position rather than as a covert first-order quantifier over properties, sets or classes, or as a device for quantifying plurally over individuals. A higher-order existential, on this interpretation, bears the same logical relationship to its instances as a first-order existential does: *Socrates is wise*, Fa, immediately entails that *something is wise*, $\exists_e x.Fx$, so the latter is logically weaker and cannot entail the existence of anything that this instance, *Socrates is wise*, doesn't already entail. By parallel reasoning, $\exists_{(e)} X\, Xa$ is also weaker and cannot entail the existence of anything that an instance, *Socrates is wise*, does not already entail (and *Socrates' being wise* does not seem to imply the existence of abstract objects like sets or properties or anything like that).[4] Nonetheless, I will follow the convention of *pronouncing* $\exists_{(e)}$ and $\exists_{(ee)}$ as 'some property' and 'some relation' respectively, to avoid overly formal prose. (Relatedly, quantification into predicate position is not plural quantification, for *Socrates is contingently wise* entails $\exists_{(e)} X.$*Socrates is contingently* X, even though there aren't any things that Socrates is contingently one of.)

Applying this to our previous remarks: the existence of the potentially

[3]See, for instance, Lear (1980) and Linnebo and Shapiro (2019).

[4]For more on this way of understanding higher-order generalizations see, for instance, Prior (1971), Williamson (2003), Trueman (2020), Bacon (forthcomingb).

infinite, expressed with a second-order existential, may commit us to infinitely many individuals, but not to an infinite individual. Given that we are understanding $\exists_{(ee)}R\,\mathrm{PotInf}\,R$ as expressing existential generality in the position of a binary predicate, it does not commit us to any completed infinite entities like sequences, infinite relations or infinite domains of such relations. $\exists_{(ee)}R\forall_e x\exists_e y Rxy$ is entailed by $\forall_e x\exists_e y.x <_t y$, the claim that *for every stretch of time there is a strictly longer stretch*, by existentially generalizing into the position that the binary predicate 'is strictly longer than' occupies. We have just argued that this claim does not entail that there are any individuals other than finite stretches of time, so neither do any weaker claims it entails.

In contrast to Aristotle, Cantor famously embraced completed infinities. In his *Grundlagen* (Cantor (1883)) he states two principles of generation for 'creating' infinities. The first principle of generation—which I will simply call *Successor*—ensures that one can always create a potential infinity by adding one to a sequence: "the principle of adding a unity to an already formed and existing number".[5] The second principle of generation—which I will call *Limit*—ensures that from any potential infinity one can always create a completed infinity: "if any definite succession of defined integers is put forward of which no greatest exists a new number is created by means of this second principle of generation, which is thought of as the limit of those numbers; that is, it is defined as the next number greater than all of them". Cantor's principles do not apply to stretches of time but to special mathematical objects—'transfinite numbers', or 'ordinals', ordered by a relation $<_\Omega$—which are governed by these principles. These transfinite numbers are totally ordered by $<_\Omega$—that is they are strict orders in the previously defined sense such that $\forall_e xy(Rxy \vee Ryx \vee x = y)$. We write this $\mathrm{Tot} <_\Omega$. Writing Ord for the property of being in the domain of $<_\Omega$ (i.e. the predicate $\mathrm{Dom} <_\Omega$) we might naïvely axiomatize Cantor's theory by adding to the principle that the ordinals are totally ordered:

Successor

$$\forall_e x(\mathrm{Ord}\,x \to \exists_e y(\mathrm{Ord}\,y \wedge \mathrm{Suc}\,xy))$$

Limit

$$\forall_{(e)}X((\forall_e y(Xy \to \mathrm{Ord}\,y) \to \exists_e y(\mathrm{Ord}\,y \wedge \mathrm{LUB}\,Xy)$$

UB, LUB and Suc stand for upper bound, least upper bound and successor

[5] Ewald (1996) p907.

and are defined in a footnote.[6] The ordinals are closely related to the notion of a well-order, a notion also introduced into mathematics by Cantor. A totally ordered relation, R, is well-ordered iff there is always an R *least* individual among any $\mathrm{Dom}\, R$ individuals, and can be straightforwardly defined in higher-order language.[7] It is easy to prove from Cantor's principles that the transfinite numbers are well-ordered.[8] Arguably for Cantor, the notion of a well-order is prior to that of a transfinite number: in later work, a transfinite number is an abstraction from the notion of a well-order—transfinite numbers represent the order-types of well-orders.[9]

Unfortunately Cantor's two principles of generation, if left unrestricted, lead to the Burali-Forti paradox. Cantor was aware of the paradox early on (prior to Burali-Forti).[10], and describes it quite clearly in a letter to Hilbert:[11]

> The totality of alephs is one that cannot be conceived as a determinate well-defined, *finished* set. If this were the case, then this totality would be *followed* in size by a *determinate aleph*, which would therefore both *belong* to this totality (as an element) and *not belong*, which would be a contradiction. (Letter from Cantor to Hilbert, 26 Sept 1897, translated in Ewald (1996).)

After Hilbert points out that the alephs are perfectly determinate and well-defined, Cantor insists that it is the notion of being *finished* that is of central

[6]

$$x \leq_\Omega y := x = y \vee x <_\Omega y$$

$$\mathrm{UB}\, Xy := \forall_e x(Xx \rightarrow x \leq_\Omega y)$$

$$\mathrm{LUB}\, Xy := \mathrm{UB}\, Xy \wedge \forall_e z(\mathrm{UB}\, Xz \rightarrow y \leq_\Omega z)$$

$$\mathrm{Suc}\, xy := \mathrm{LUB}\, \lambda z(z <_\Omega y)x$$

[7]

$$\mathrm{WO}\, R := \mathrm{Tot}\, R \wedge \forall_{(e)} X(\exists y Xy \wedge \forall_e y(Xy \rightarrow \mathrm{Dom}\, Ry) \rightarrow \exists_e x(Xx \wedge \forall_e y(Xy \rightarrow x = y \vee Rxy)))$$

[8]Given some ordinals, X, consider the property of not being strictly greater than any X: $\lambda z \neg \exists_e y(Xy \wedge y <_\Omega z)$. Its least upper-bound will be a minimal element of X.

[9]p86 of and pp111-112 Cantor (1915).

[10]The fact that Cantor is careful about how he introduces the second principle of generation and the principle of limitation in the *Grundlangen* strongly suggests that he was aware of a problem in with the unrestricted use of the second principle of generation in 1883 (see Menzel (1984)).

[11]Here he is applying his principle to the alephs rather than the transfinite ordinals, but the argument is essentially the same in any case.

importance:

> One must only understand the expression 'finished' correctly. I say of a set that it can be thought of as *finished* [...] if it is possible without contradiction [...] to think of *all its elements as existing together*, and so to think of the set itself as a *compounded thing for itself*, or[...] if it *possible* to imagine the set as *actually existing* with the totality of its elements.

This is not so different from Aristotle's notion of an actual infinity—the finite stretches of times cannot be brought together into a single infinite stretch. Cantor's position is that the ordering $<_\Omega$ is not a finished order. It is not only a potential infinity, in the higher-order sense defined earlier, but an order that cannot be completed (unlike the potential infinities discussed by Aristotle). $<_\Omega$, however, isn't the only incompletable order: there are many well-orders that properly extend $<_\Omega$. One can be made, for instance, removing the first element of $<_\Omega$, 0, adding it to the end of this ordering it above all the others — $x <_{\Omega+1} y$ iff $x <_\Omega y$ and $x \neq 0$ or $\mathrm{Ord}\, x$ and $y = 0$. $<_\Omega$ is merely the first well-order that is incompletable, and cannot be assigned an individual representing its order-type.

Cantor's remarks here are notoriously enigmatic. There is both an interpretive question of what Cantor actually means here by a set that is 'finished', and 'whose elements can be thought of as existing together'. And then, setting aside what Cantor had in mind, there is a question of whether *any* more precise notion can be substituted for these phrases in a way that would satisfactorily explain why some well-orders can be 'completed' ('finished', 'thought of as existing together all at once', etc.) and thus assigned an individual as an order type, and others cannot.

There are some promising answers to this second question in the literature already, but they are, to my mind still surrounded by some significant question marks. For example, much has been made of the modal language that Cantor uses in some of his formulations: the difference between the finite ordinals (say) and the totality of all ordinals is that while the former could have existed all together, the latter couldn't. But *what is it* that makes it possible for the finite ordinals, the countable ordinals, etc. to exist together, and not all the ordinals — modal facts also call out for explanation! Are we forced to just posit a "brute necessity" (in the sense of Dorr (2008))?[12] (Even friends of brute necessities—essentialists such as Fine (1994)—should

[12] I am here formulating the question of possible coexistence in terms of what is possible in the broadest sense of 'possible'. Of course, one can cook up restricted notions of possibility in which any collection of things you choose are incompossible by simply ruling out possibilities

feel pressure to explain why the ordinals specifically are incompossible with one another. After all, no ordinal is incompossible with any other since any pair—or indeed set—of ordinals can coexist. So the barrier to the coexistence of the totality of ordinals is fundamentally collective and can't be rooted in the essences of particular individuals as it can in the more familiar cases of incompossible objects.[13]) Some have focused instead on Cantor's use of words like 'determinate', 'definite' and 'well-defined': while the finite ordinals, the countable ordinals, and so on, each form a definite succession, the succession of all ordinals does not, and only definite successions can be completed. But here again there are explanatory demands to be met: what is the notion of definiteness being appealed to here, and why, as Hilbert asked, is the succession of finite ordinals but not the totality of ordinals definite.[14] (Notice how the Aristotelian, by contrast, has a rather principled answer to these questions. Because there are no actual infinities, no potentially infinite order can be completed: there is no difference between the completeability of $<_\omega$ and $<_\Omega$ to explain.[15])

The explanatory challenge — *Why are some well-orders, like $<_\Omega$, impossible to represent and not others?* — appears to be one faced by *any* theory of the transfinite that purports to represent infinite well-orders using special purpose abstract individuals. There is a *de dicto* way of understanding this challenge where it can be met by a straighforward explanation: a mathematical proof along the lines of the Burali-Forti paradox. The totality of ordinals, whatever they are, can't by represented by an ordinal otherwise it would have to be strictly greater than every ordinal and thus greater than itself. This reasoning is general and says nothing about the *specific* order type that the ordinals in fact possess. The total ordering of finite ordinals cannot be repre-

where they all exist by fiat. These cooked up notions clearly cannot meet the explanatory demands we are making, and it is does not seem to me that anyone has proposed a reasonably clear non cooked-up notion that meets these demands either.

[13]Two possible people originating from the same egg but different sperm are a standard example of incompossible objects. The incompossibility here cannot be identified with other possible sources of brute necessity discussed by Dorr, including those involving non-factual notions, like goodness, or semantically empty notions like phlogiston.

[14]Compare the notion of extensional definiteness in Florio and Linnebo (2021). They leave the notion as a primitive and some analyses of this notion are gestured at; but absent an analysis this notion seems no clearer than Cantor's notion of a 'finished' set.

[15]Another principle sometimes discussed in this context is the principle of limitation of size: that things equinumerous with the entire universe are to big to form a set. I cannot find it articulated like this in Cantor, but Hallett (1984) p176 describes this as a 'spiritual descendent' of Cantor's theory. This principle that one can assign something a transfinite number only if it is smaller than the universe is less obviously circular, but still smacks of something that is motivated only by the fact that it avoids inconsistency.

sented by a finite ordinal otherwise that finite ordinal would have to be greater than every finite ordinal; similarly the ordering of countable ordinals cannot be represented by a countable ordinal, the ordering of ordinals less than 7 can't be represented by a ordinal less then 7. However, if the ordinals did cut out at the number 7, say, we might ask for an explanation for they cut out at 7 and not 8?, and the Burali-Forti reasoning does nothing to help explain this arbitrary fact. When we think of the explanatory demand in this *de re* way, explanations are harder to come by. There is nothing *inconsistent* about a theory that assigns $<_\Omega$ a special individual as a representative: if the proper initial segments of $<_\Omega$ can consistently be assigned individuals as order-types, we can create a new way of representing order-types by individuals that does assign $<_\Omega$ an order-type by, as it were, 'making room in Hilbert's hotel'—i.e. shifting the individuals representing the finite order types up by one, and assign $<_\Omega$ what used to be playing the role of 0. So it's just not true that Ω can't be enumerated—we could assign it an individual representing its ordered type if we wanted—it's simply that Ω *isn't* enumerated by Cantor's particular way of doing it.[16] If we didn't have to do it this way, why is $<_\Omega$ special?

Perhaps the explanation comes from a feature of $<_\Omega$ that do not supervene on its order-theoretic properties but depends on the sorts of individuals in its domain. Perhaps an explanation appealing to the metaphysics of abstract objects like ordinals? But such explanations would be insufficiently general if we were of the mind that a well-order of concrete individuals isomorphic to $<_\Omega$ would be just as problematic — that it would also involve a inconsistent multiplicities, or things that cannot exist altogether. The inconsistency of certain well-orders of concrete things cannot obviously be explained by the metaphysics of abstract objects.[17] At any rate, this is the explanatory challenge.

[16] Here and throughout I am using 'enumerate' in the sense that Cantor means it: can be numbered by a (possibly) transfinite ordinal. In modern set theory this term is reserved for sets that can be numbered by the finite ordinals; i.e. whose members can be injectively mapped into ω.

[17] When we move from order-type to cardinality we find other authors adopting different representations: Frege, for instance, put forward a different (and consistent–see Boolos (1987)) theory of cardinality to Cantor based on Hume's principle: according to this any things whatsoever are be assigned an individual as number, including the totality of all things.

2 Higher-Order Formulations of Indefinite Extensibility

I have argued that standard Cantorian accounts of the transfinite seem to incur further explanatory demands, such as explicate the notion of possible co-existence, or definiteness. Perhaps these can be met, perhaps they cannot, but it is valuable, still, to investigate other accounts that do not incur these further demands.

The approach I will explore does away with special abstract individuals representing the order-types of well-orders. One reason for doing this is broadly speaking abductive. Why postulate special individuals to represent the different order-types of well-orders at all, if not all the well-orders can be represented? This seems like an unnecessary posit when we can formulate absolutely general principles, and reason with absolute generality, about the well-orders directly; we needn't restrict our reasoning to the class of well-orders that can be represented by special individuals. One might complain at this juncture that traditional approaches to indefinite extensibility tend to assume platonism and have identified the phenomena as having specifically to do with special sorts of abstract objects, like ordinals and sets, and so the present investigation will be of no interest to authors in this tradition.[18] Perhaps. But as we will see, even a nominalist who rejects an ontology of abstract sets and ordinals, can still recognize the indefinite extensibility of possible well-orders over concrete things; the phenomenon of indefinite extensibility may not be intrinsically tied to special kinds of abstract objects after all. And as we have seen, the platonist faces challenges in articulating the conditions under which properties define sets, which well-orders have ordinals, and so on, which the present approach deflates (we will return to this issue in section 4 where some nominalist friendly translations of platonic set theory into the language of pure higher-order logic are discussed).

Let us return to our general definition of a potential infinity. In some sense the completion of a potential infinity, R, is an individual, a, that can be placed 'above' each of the individuals in the domain of R. a is not itself in the domain of R, or else R would not be a potential infinity and a would not be above all the $\mathrm{Dom}\, R$ elements (nothing can be above itself). We must think of R being extended by a to make another well-order, R^{+a}, that includes a in its domain as lying above each of the elements of R. Although we have only talked of individuals as completed infinities—a in this case—there's also an

[18]The platonistic tradition I am aluding to includes Dummett (1991) p316, Parsons (1983), Linnebo (2013) among others. Thanks to an anonymous referee for pressing me on this point.

extended sense in which R^{+a} itself can be thought of a completion of R. Note, of course, that in this general sense R can be completed in multiple different ways. Cantor appears to get into trouble by assuming that there is a particular relation, $<_\Omega$, (defined on special individuals, the transfinite numbers), when it seems that any well-order, $<_\Omega$ included, can be extended.

I want to explore the idea that the notion of a well-order is 'indefinitely extensible' in a way that goes beyond what both Aristotle and Cantor have put forward. The only way I can think to formulate the indefinite extensibility of the notion of a well-order is not to formulate it in terms of some particular well-order, $<_\Omega$, which must already have some particular order-type, but by higher-order quantification over well-orders. The result is, in a loose sense, higher-order analogues of Cantor's two principles of generation. We have the principle that every well-order can be extended by one, and that every sequence of well-orders ordered by the initial segment relation has a well-order containing them as initial segments. To make these easier to state we introduce some abbreviations (formal definitions can be found in the footnotes). We will use $R \leq S$ to mean that the R is an initial segment of S, which may be defined.[19] R is a proper initial segment of S, written $R < S$, when $R \leq S$ but $S \not\leq R$. We will later apply these definitions to arbitrary relations, not just well-orders. When X is a higher-order property of relations, of type $((ee))$, we write $\mathrm{Lin}\, X$ to mean that the Xs are *linearly ordered* by $\leq$ and $\mathrm{UB}\, RX$ to mean that R is an upperbound for the Xs.[20] With these abbreviations in place we may formulate the two principles as follows.

Successor (Higher-Order)

$$\forall_{(ee)} R(\mathrm{WO}\, R \to \exists_{(ee)} S(\mathrm{WO}\, S \wedge R < S))$$

Limit (Higher-Order)

$$\forall_{((ee))} X((\mathrm{Lin}\, X \wedge \forall_{(ee)} R(XR \to \mathrm{WO}\, R)) \to \exists_{(ee)} T(\mathrm{WO}\, T \wedge \mathrm{UB}\, TX))$$

If these principles are consistent they imply that every well-order can be extended by one, even $<_\Omega$. One can repeat this indefinitely to obtain a potentially infinite well-order that can be completed by the second principle.[21] Observe

[19] $R \leq S := \forall x(\mathrm{Dom}\, Rx \to \forall y(Ryx \leftrightarrow Syx)$.

[20] $\mathrm{Lin}\, X := \forall_{(ee)} RS(XR \wedge XS \to R \leq S \vee S \leq R)$,
$\mathrm{UB}\, RX := \forall_{(ee)} S(XS \to S \leq R)$.

[21] In fact, the second principle is a theorem of a minimal higher-order order logic–if X is a collection of well-orders linearly ordered by $\leq$, $Rxy := \exists_{(ee)} S(XS \wedge Sxy)$ will complete them. However, it is useful to state in its own right because later we will consider variants of the principle.

that initial segment relation, $R \leq S$, is not the same as the relation of R being *isomorphic* to an initial segment of S, which I'll write $R \preceq S$[22] The latter is a linear order of the well-orders whereas the initial segment relation is not. This will turn out to be important later.

Indeed, we seem to find similar higher-order formulations of indefinite extensibility in Zermelo. In 1908 Zermelo introduced and axiomatized the *iterative* conception of sets, according to which they are built up in stages $V_0, V_1, ...$ in a well-ordered sequence much like Cantor's theory of transfinite ordinals. One can always add a stage, $V_{\alpha+1}$, by taking the powerset of the previous stage V_α (the set containing all its subsets), and given a sequence of stages V_α ordered by inclusion one can take their 'limit' by unioning them together.[23]

Zermelo was theorizing in a higher-order language with a single non-logical binary predicate $\in$.[24] In this language Zermelo axiomatized the iterative conception of set theory with a finite list of axioms, the conjunction of which we will call $\mathrm{ZF}^{\in}$. By replacing the membership predicate in this axiom with another binary predicate, R, we can formulate the claim that R satisfies Zermelo's conditions ZF^R. In this way we obtain a purely logical predicate, ZF, allowing us to talk about ZF relations in general.

In Zermelo (1930), Zermelo distances himself from the idea that there is a special ZF relation, $\in$, about which set theory is concerned. Just like $<_\Omega$, Zermelo maintains that any ZF relation—$\in$ included—can be extended to more inclusive ZF relations. Instead of fixating on one particular ZF relation he takes up the the investigation of ZF relations in general: a project that can be undertaken in the purely logical language of higher-order logic, without any set-theoretic primitives.

Zermelo's picture was that every ZF relation is properly included in a larger one, and any collection of ZF relations ordered by inclusion are included in

[22]This relation is defined as follows

$$\mathrm{Bij}\, RXY := \forall_e x(Xx \to \exists_e ! y(Yy \wedge Rxy) \wedge \forall_e y(Yy \to \exists_e ! x(Xx \wedge Rxy)$$

$$R \cong S := \exists T(\mathrm{Bij}\, T(\mathrm{Dom}\, R)(\mathrm{Dom}\, S) \wedge \forall xyx'y'(Txx' \wedge Tyy' \to (Rxy \leftrightarrow Sx'y'))$$

$$R \preceq S := \exists T(T \leq S \wedge R \cong S)$$

.

[23]Like with Cantor's principles this is inconsistent if applied unrestrictedly; in order to maintain consistency this principle is restricted to sequences of stages that can be indexed by a set that already exists; this latter idea is essentially due to Fraenkel.

[24]Zermelo follows the terminology of Whitehead and Russell (1910-1913), who are more explicit about the fact that second-order quantifiers bind into predicate position. Zermelo by contrast talks informally, using Russell's term 'propositional function' when higher-order quantification is intended.

some ZF relation—there is no special relation $\in$, which is itself indefinitely extensible. This is what Zermelo says: [25]

> Let us now put forth the general hypothesis that every categorically determined domain can also be conceived of as a "set" in one way or another; that is, that it can occur as an element of a (suitably chosen) normal domain. It then follows that there corresponds to any normal domain a higher one [...] Likewise, a categorically determined domain of sets arises through union and fusion from every infinite sequence of different normal domains [...] where one always contains the other as a canonical segment.

A 'normal domain', for Zermelo, is the domain of a ZF-relation R, and a 'domain' a collection contained in a normal domain. This allows Zermelo to avoid the problems associated with inconsistent multiplicities. As Geoffrey Hellman puts it, according to Zermelo "set theory should be seen, not as the theory of a unique, all-embracing structure, but instead as a theory of an endless infinity of intimately related structures."[26]

Of course, underlying this is the thought that well-orders themselves are indefinitely extensible. Of the transfinite numbers, Zermelo writes that they rest

> upon the notion of well-ordering and which, though lacking in true completion on account of its boundless progressing, possesses relative way stations, namely those "boundary numbers" [i.e. inaccessibles], which separate the higher from the lower model types.

If we flatfootedly formalize Zermelo's two remarks in higher-order logic we obtain the following pair of principles: every ZF relation is a proper initial segment of some other ZF relation, and (ii) whenever you have some ZF relations ordered under initiality there is a ZF-relation containing them all as initial segments.

Progress $\forall_{(ee)}R(\mathrm{ZF}\,R \to \exists_{(ee)}S(\mathrm{ZF}\,S \wedge R < S))$

Completion $\forall_{((ee))}X(\mathrm{Lin}\,X \wedge \forall_{(ee)}R(XR \to \mathrm{ZF}\,R) \to \exists_{(ee)}T(\mathrm{ZF}\,T \wedge \mathrm{UB}\,TX))$

[25]Below I suppress several qualifications Zermelo makes regarding differences between ZF relations that purely concern urelements, as they are not relevant to our present discussion of pure sets.

[26]Hellman (1989) p56. Hellman does not think Zermelo is successful in resolving the tension, essentially because Hellman thinks the only way to make sense of the relevant higher-order quantification is in terms of singular quantification over proper classes.

The idea that the ordinals are indefinitely extensible leads to a variant pair of principles about well-orders: every well-order of inaccessible order type is a proper initial segment of another such relation, and any collection of well-orders that are linearly ordered by $\leq$ are initial segments of some well-order of inaccessible order type.[27]

Progress$^{\mathrm{WO}}$ $\forall_{(ee)} R(\mathrm{Inaccessible}\, R \to \exists_{(ee)} S(\mathrm{Inaccessible}\, S \wedge R < S))$

Completion$^{\mathrm{WO}}$ $\forall_{((ee))} X(\mathrm{Lin}\, X \wedge \forall_{(ee)} R(XR \to \mathrm{WO}\, R)) \to \exists_{\rho} T(\mathrm{Inaccessible}\, T \wedge \mathrm{UB}\, TX))$

Of course, this harkens back to Cantor's principles of generation, that one can add one to any transfinite number, and given any sequence of transfinite numbers we can find the least number which is greater than them all.

Zermelo's remarks capture an attractive, but somewhat elusive idea. Many philosophers have been seduced by this picture of the set theoretic hierarchy as indefinitely extensible, but have had trouble articulating the idea precisely. Common to these formulations is the assumption that there is a distinguished relation, $\in$, or in the case of the ordinals $<_\Omega$, and it is *this* relation that is said to be indefinitely extensible, or not as the case may be. (This picture, it should be noted, is on its face importantly different from Zermelo's; for Zermelo it is the higher-order property of being a ZF relation or being a well-order that are indefinitely extensible.)

Let's consider (briefly) two major attempts to express the indefinite extensibility of *particular* relations, such as $\in$ and $<_\Omega$. According to some authors, one must give up on the idea that we can quantify unrestrictedly.[28] Each quantifier gives us a restricted view of the totality of ordinals, as it were, and from no viewpoint can we see them all at once. But to say that a given quantifier is restricted we do so by way of another quantificational claim: we mean there is *something* not in its range. This is only true if this new quantificational claim ranges more widely than the the original one, and if it too is restricted this can only be articulated by a yet wider quantifier. Zermelo was staunchly against this sort of relativism:

> In general, the concept of "allness", or "quantification", must lie at the foundation of any mathematical consideration as a basic logical

[27] The notion of *inaccessible* can be defined in pure higher-order logic. It can be obtained by essentially λing out $<_\Omega$ from a suitable version of Cantor's theory of ordinals: i.e. Inaccessible R means Tot R, and the two principles of generation (with a restriction on the principle Limit that applies only to collections of ordinals that can be indexed by an already existing ordinal.

[28] The literature on this is extensive; see for instance Rayo and Uzquiano (2006).

> category incapable of further analysis. If we were to restrict the allness in a particular case by means of special conditions, then we would have to do so using quantifications, which would lead us to a regressus in infinitum. Zermelo (1931).

Whether this regress is troublesome remains to be seen, but it does bring to salience a difficulty. How should the quantifier relativist state their positive view that every first-order quantifier is restricted? Presumably they should do this by quantifying into the position of a first-order quantifier—but if this higher-order quantifier is also restricted, it fails to have the required force. And if the higher-order 'quantifier quantifier' is unrestricted then one can define an unrestricted first-order quantifier: absolutely everything is F when F satisfies every first-order universal quantifier.[29] (It is worth mentioning, at this juncture, that some philosophers believe that the higher-order formalism faces a similar set of challenges. I do not find all versions of these challenges to be intelligible, and when they are they are not obviously analogous to the challenges for unrestricted quantification from the paradoxes of set theory.[30] But the literature on this is extensive and I will not attempt to defend higher-order language from such charges here.[31])

Other philosophers have suggested that the indefinite extensibility of the sets and the ordinals must be glossed in inherently modal terms.[32] Unlike generality relativism, this position has an exact statement; one which involves modal language. In this case, however, I believe that the approach is not able to preserve, as we have, the two essential components of Cantor's conception

[29] See Williamson (2003) for a discussion of related problems.

[30] They often involve somehow applying set-theoretic intuitions to "types", as though there were such things as types. But as Wittgenstein often pointed out, "type theory" is a misnomer: it is not a theory, it is just grammar. It is not *about* anything; thinking of it as though it were is bound to produce false analogies. In this way it is fundamentally unlike set theory, which is a theory in the usual sense—a set of sentences—stating facts about certain abstract objects; "the theory of types" does not state anything and so cannot state things about something either.

[31] Williamson (2003) outlines the classic version of the position that first-order quantification is absolutely general and defends the higher-order framework from challenges; Williamson's understanding of higher-order quantification is articulated Prior (1971) chapter 3 and is the prevalent one in the recent higher-order metaphysics literature. Various challenges to the higher-order formalism analogous to those faced by the generality relativist are articulated and discussed, for example, in Linnebo and Rayo (2012), Florio and Jones (2021), Florio and Jones (2023), Pickel (2024).

[32] An early modal articulation of indefinite extensibility can be found in Putnam (1967). This idea is developed in Parsons (1983), Linnebo (2013), and in several subsequent papers of his; a different formalization of the modal idea based on tense logic is given in Studd (2013).

of transfinite numbers in their unrestricted forms: that we can extend the transfinite by taking successors and by taking arbitrary limits.

In this tradition, much emphasis is given to the *potential* infinity of sets, as articulated in modal terms: necessarily, for any things it is always *possible* that they form a set. It is always possible to construct the next layer of sets. The analogue for ordinals is a modal version of Cantor's first principle of generation concerning successors: necessarily, every ordinal could have had a successor.

Successor (Modal)

$$\Box\forall_e x(\mathrm{Ord}\, x \to \Diamond\exists_e y(\mathrm{Ord}\, y \wedge x <_\Omega y))$$

writing Ord for Dom $<_\Omega$. But just as essential to the picture of the ordinals as extendible by adding one (or by adding a new layer of sets in the case of sets) is this idea of their being extendible by taking limits, captured in Cantor's second principle of generation, Limit. The successor thought on its own delivers arbitrarily high finite numbers, but never lets us push past the finite. The most naïve way to formulate Limit modally would say that whenever we have a property of ordinals picking out ordinals across worlds (not just a single world) it should be possible for there to be an ordinal that is necessarily at least as big as each of them.

Limit (Modal)

$$\Box\forall_{(e)} X(\Box\forall_e y(Xy \to \mathrm{Ord}\, y) \to \Diamond\exists_e x(\mathrm{Ord}\, x \wedge \Box\forall_e y(Xy \to y \leq_\Omega x))$$

The problem for the conjunction of these two principles is essentially the same problem that besets the original Successor and Limit principle: they are susceptible to a modal version of the Burali-Forti paradox. Assume that the ordinals are necessarily well-ordered. Plugging Ord into X in the limit principle we get the possibility of an ordinal, x, that is necessarily at least as big as every ordinal, but then the first principle implies the possibility of an ordinal strictly bigger than x.

Which principle is to blame? As it happens, modalists uniformly reject Limit. Notice, though, that this rejection cannot be motivated by appealing to the Burali-Forti paradox: from a purely logical perspective, both principles are individually consistent. More importantly, the standard kinds of model theory for modalism contains models of both principles, depending on the result of one choice. *The models of Limit are just as natural as the models of Successor*; a fact which I think vindicates the idea that there is a substantive choice to be made in our modal metaphysics between two attractive principles: a successor principle and an unrestricted limit principle. In a modalist model the possible worlds represent stages of the hierarchy, and so can be represented by ordinals.[33] The models can be describe roughly as follows. In a model of length α the worlds consist of the ordinals less than α. The extension $<_\Omega$ at the world $\beta < \alpha$ is just the given by restricting the relation $<_\Omega$ to the ordinals no greater than β.[34] Models of length α, when α is a *limit* ordinal, validate the modal *successor* principle. And when α is a *successor* ordinal, models of length α validate the modal *limit* principle.[35] Since they are together inconsistent, however, we are forced to choose between the two components of the Cantorian vision in their unrestricted form — that the ordinals are indefinitely extensible through the operations of taking successors, and of taking limits.

As a sociological fact, modalists keep Successor and restrict Limit. One cannot take limits of arbitrary persistent properties, but only of properties that are "eventually stable", in the sense that, possibly, their extension becomes a necessary matter (see Linnebo (2013) §7.2). Of course, Successor and the restricted limit principle, on their own, are far too weak. To see this it is instructive to see how the unrestricted principles easily generate the possibility of all the Cantorian ordinals. By repeatedly applying the modal successor principle, we can obtain the possibility of any finite ordinal. Then, by plugging the property of being a finite ordinal into the modal limit principle, we can obtain the possibility of the ordinal ω, that is as great as any finite ordinal. Repeating this reasoning we obtain the possibility of $\omega.2, \omega.3, ...$, so plugging the property of being a finite multiple of ω into the limit principle we get the possibility of ω^2. It is clear how to continue. But note that in this reasoning we needed to plug in properties that might have expanded forever: the property of being a finite ordinal is not eventually stable in the models of length ω described in the previous paragraph. Similarly, the property of being a finite multiple of ω is not eventually stable in models of length ω^2.

To overcome this weakness modalists have to make further assumptions and these assumptions face justificatory challenges of their own. For instance,

[33]Our model theory is closer then to that in Studd (2013) than Linnebo (2013), but essentially the same points can be made in the latter framework.

[34]The simplest way to make these into a model of higher-order logic is take relations of type $(\sigma_1, \ldots, \sigma_n)$ to be arbitrary functions from entities of types $\sigma_1, \ldots, \sigma_n$ to sets of worlds. (In the framework of Bacon (2018) this means that we end up with a universal accessibility relation and a constant domain semantics). The extension of $<_\Omega$ at β is the set of pairs (γ, δ) with $0 \leq \gamma < \delta \leq \beta$. More complicated constructions can capture the idea that later stages are accessible to earlier stages, but not conversely, and that which sets exist is contingent on the stage, but these are not essential to the consistency claim being made here.

[35]For instance, when $W = \omega$, for every world there's another world with one more ordinal, and when $W = \omega + 1$, for every property of ordinals ω is necessarily $\geq$ every ordinal with that property— for ω is the greatest possible ordinal.

strength is often achieved by essentially translating the replacement axiom into modal set theory, or appealing to a modal reflection principle. But as Berry (2022) §3.3.1, §5.1 notes, these further principles are hard to justify. We cannot, after all, justify the restricted limit principle (replacement) on the basis of the same intuitions that justify the unrestricted limit principle! In any case, the modalist must appeal to principles that seems to complicate the pure and simple idea of generating the infinite by taking successors and limits. Our higher-order principles, by contrast, capture these two principles in their unrestricted form directly.

3 The Well-Ordering Principle

The reader with an eye for paradoxes might wonder whether Zermelo has not committed himself to the Burali-Forti paradox. Zermelo seems to recognise the tension, and likens the principles to a Kantian antinomy of 'progress' and 'completion', from the epigraph. Zermelo does not, however, formalize or otherwise develop his remarks.

Let us see what happens if we attempt to naïvely apply the Burali-Forti reasoning. Suppose that the higher-order Progress and Completion principles are true: every ZF relation, or well-order, can be extended to another, and every totally ordered collection of ZF relations has a limit. The flatfooted way to reinstate the Burali-Forti paradox is to take the limit of all the ZF relations to make a mega ZF relation containing all others, and then paradoxically make it bigger than itself by applying the successor principle. However this argument does not work, because Completion only allows us to take limits of chains of ZF relations, but a single ZF relation can be extended to a greater one in multiple different ways without either of the extensions being contained in the other. The relation of extension, $\leq$, is not a total order, so we cannot apply the limit principle. A similar road block would be encountered if we attempted to run the Burali-Forti reasoning with the higher-order Successor and Limit principles: the well-orders are not totally ordered by $\leq$ and so we cannot apply Limit.

In a fresh bid to reinstate paradox, perhaps we shouldn't consider *all* ZF relations, but instead some chain of ZF relations which is as long as possible: a maximal chain. Then the limit principle (Completion) would tell us this chain had a limit, and the successor principle (Progress) would let us create a ZF relation properly extending ZF relation in the chain, contradicting the assumption that this was a maximal chain. However to run this argument, we needed some guarantee that there is a maximal chain of well-orders under the initial segment relation.

One could obtain a contradiction if we additionally assumed a higher-order version of the well-ordering principle (an equivalent of the higher-order axiom of choice, see Shapiro (1991)).

The Well-Ordering Principle$^{\sigma}$ $\exists_{(\sigma\sigma)} R(\mathrm{WO}\, R \wedge \forall_{\sigma} x\, \mathrm{Dom}\, Rx)$

Completion tells us that every chain of ZF relations ordered by $\leq$ has an upperbound that is also a ZF relation, and so by Zorn's lemma—a consequence of The Well-Ordering Principle$^{((ee))}$—the ordering of ZF relations under $\leq$ has a maximal element, contradicting Progress.

The inconsistency of our higher-order principles concerning well-orders, Successor and Limit, can also be derived from the well-ordering principle via Zorn's lemma, but in this case there is also a more direct proof from the higher-order well-ordering principle that is quite instructive. Suppose that R is a well-order of all the individuals. By Successor, there is a well-order strictly extending R.[36] But there cannot be a strict extension, because all of the individuals have been used up in the ordering of R—we cannot reuse an individual appearing in Rs domain, for a well-order cannot contain a cycle. This argument also illustrates a difference between the initial segment relation, $\leq$, and the more general relation between two well-orders where one is *isomorphic* to an initial segment of the other, which we will write $\preceq$. While no well-order has R as a proper initial segment, we can of course find a well-order that has as a proper initial segment something isomorphic to R: simply remove Rs least element—an operation that will leave Rs order type alone, provided R is infinite—and tag it to the end of R to make a strictly longer well-order under $\preceq$. So in the presence of the the well-ordering principle these two relations are governed by different principles.

Now, absent the higher-order well-ordering principle no contradiction can be derived from Progress and Completion, or from Successor and Limit. This is established by constructing a model, described in the appendix. We thus have a completely flatfooted articulation of the indefinite extensibility of the notion of ZF relation and well-order, that does not require one to deny the ability to quantify unrestrictedly, or to posit special mathematical modalities according to which the length of the set theoretic hierarchy is contingent. Indeed, our extensibility principles are consistent with the Fregean principle of extensionality, which rules out any sort of contingency whatsoever (this consequence of extensionalism does, however, render it implausible as a more general principle of higher-order logic).

[36]Zermelo's principles of Progress$^{\mathrm{WO}}$ and Completion$^{\mathrm{WO}}$ also imply every well-order can be extended.

Before we proceed, there is a certain kind of "bad company" objection to our principles that must be addressed. There are versions of Progress and Completion, and Successor and Limit, in which $\leq$, the relation of *being an initial segment of*, is replaced with the relation of *being isomorphic to an initial segment of*: every ZF relation/well-order can be extend by taking successors and limits *modulo isomorphism*. One might have thought, naïvely, that these principles should be on as good a footing as the original principles, since all we are doing is ignoring differences between isomorphic relations. Yet surprisingly they are not: the variants are in fact inconsistent. Let X be the property of being a well-order. It is easily shown that well-orders are linearly ordered (indeed well-ordered) by $\preceq$, and so by "Limit-up-to-isomorphism", there must be a well-order R, containing an initial segment isomorphic to any well-order.[37] But then by "Successor-up-to-isomorphism", there is a well-order R^+ that has a proper initial segment isomorphic to R, and thus any well-order relation is isomorphic to a proper initial segment of R^+. This includes R^+ itself, a contradiction! The inconsistency extends to variants of Zermelo's principles with $\preceq$ replacing $\leq$: this time we must appeal to a theorem proved by Zermelo himself, that ZF relations are linearly ordered by $\preceq$.

The cause of the problem here is the $\preceq$ variants of the limit principles (Completeness and Limit). *But*, the bad company objection goes, *shouldn't the limit principles and their variants stand or fall together?* Actually the answer is *demonstrably* no. Despite a superficial similarity in logical form, our original Limit principle for $\leq$ is demonstrably good (it is a theorem of a minimal higher-order logic), and its variant for $\preceq$ demonstrably bad (it implies, in that same logic, that there are finitely many things). This can all be shown using assumptions that everyone can agree upon so *nobody* should think these principles should stand or fall together. Let us call the two variant limit principles for well-orders, call them $\leq$-Limit and $\preceq$-Limit.

1. The $\leq$-Limit principle is a theorem of the minimal higher-order logic.[38]

2. The $\preceq$-Limit principle is inconsistent in this minimal logic with the assumption that there are infinitely many things.

For 1, note that if X are some well-orders totally ordered by $\leq$, their "union", i.e. the relation $Sxy := \exists_{(ee)}R(XR \wedge Rxy)$, is a well-order extending each

[37] Note that our definitions of linear order are suitable for preorders, and do not build in antisymmetry. $\preceq$ is not antisymmetric, since one has distinct but isomorphic relations, whereas $\leq$ is antisymmetric.

[38] The system called H in Bacon (2018).

relation in X. For 2, let R be a well-order that has an initial segment isomorphic to any well-order. But if Rs domain is infinite, we can construct a strictly larger well-order by removing the initial element of R and gluing it to the end, as we described earlier. This, I submit, addresses the bad company objection to Successor and Limit. (And once this is recognized I think the bad company objection to Progress and Completion can also be understood to be misguided).

What solace might Cantor or Zermelo draw from the consistency result of Successor and Limit, and Progress and Completion? Cantor calls the principle that 'it is always possible to bring any well-defined set into the form of a well-ordered set' a *law of thought*—'a law which seems to me fundamental and momentous and quite astonishing by reason of its general validity.' [39] Zermelo too accepted Cantor's principle, although justified it from what he took to be a more basic principle, the axiom of choice, saying that whenever you have some pairwise disjoint non-empty sets, there is a set which contains exactly one element for each of those sets. Both of these principles are principles about *sets*.

The higher-order well-ordering principle is stronger than this, since it implies that even properties whose extensions do not form a set can be enumerated. The foregoing remarks suggest to me a somewhat precise way to understand Cantor's notion of an inconsistent multiplicity—which he describes variously as some things which cannot be thought of as existing together all at once, cannot be counted, are not finished, are beyond enumeration, are absolutely infinite, and so on. We are unable to think of some things as all existing together when it is not possible to list all of those things, even by means of an infinite list. Consider, for instance how Cantor explains what sorts of things can be counted *Grundlagen*

> I believe however that I have proved above [...] that determinate countings can be carried out just as well for infinite sets as for finite ones, provided that one gives the sets a determinate law that turns them into well-ordered sets. That without such a law-like succession of the elements of a set it cannot be counted—this lies in concept of *counting*". Ewald (1996) p889

He then goes on to note that how a set is counted may depend on how it is ordered. It is implicit here that things that cannot be well-ordered by a law cannot be counted at all.[40] Consider also his *Grundlagen* definition of a set as an:

[39] *Grundlagen* §3.1, translated in Ewald (1996) p886.

[40] See Hallett (1984) p150, Lavine (1994) Chapter III§4, Newstead (2009) p546.

> aggregate of determinate elements which can be united into a whole by a law. Ewald (1996) p916.

Note again the emphasis on the existence of a law or rule ("gesetz") being a prerequisite for being combinable into a whole. This would, at least, vindicate the idea that the principle that any well-defined set can be well-ordered is a law of thought, rather than a substantive principle: because being well-orderable is a necessary condition for being a well-defined set and being non-well-orderable is sufficient for being an ill-defined, or unfinished, multiplicity.[41] Perhaps, this is what Cantor had in mind? We will see in the appendix, at any rate, that being well-orderable cannot also be a sufficient condition for set formation on pain of paradox.

Since the Zermelian approach to indefinite extensibility is purely logical, there is a question about whether we can make sense of this set-theoretic criteria for when properties can be well-ordered in this setting. The idea that only properties defined by sets need be well-orderable takes for granted a particular property of sethood and corresponding membership relation $\in$. From the present perspective there is nothing special about any particular ZF relation, and so any principle that relies on a particular membership relation could be deemed parochial. What we would like is a principle of pure higher-order logic that ensures choice holds for *any* ZF relation.

Note, firstly, there are principles of pure logic that imply set-theoretic choice. In the same way that we defined the higher-order predicate of relations, ZF, in terms of the Zermelo-Fraenkel axioms, we can introduce the notion of a ZFC relation, satisfying also the axiom of choice. Given the higher-order well-ordering principle it is easily seen that there is no difference between these relations:

Theorem 3.1. *Given the Higher-Order Well-Ordering Principle, every* ZF *relation is a* ZFC *relation.*

For suppose that S is a global well-order, and that R is a ZF relation. We will write 'element^R' for R 'set^R' for $dom(R)$. Suppose that x is a set^R of non-empty disjoint sets^R. Then I can define the relevant choice set^R by taking the set^R consisting of the S-least elements^R of each element^R of x.

The converse to this theorem need not hold. The claim that every ZF relation is a ZFC relation does not imply the higher-order choice principles. In fact, in the models described in appendix A.1 every ZF relation is a ZFC

[41] It should be noted, however, that Cantor in later writing sometimes does use the stronger form of choice—for instance in his proof that if a multiplicity does not have an $\aleph$ number, then it is a inconsistent multiplicity, and does not form a set.

relation but there is no global well-ordering of the universe.[42] Thus we might consider adding the following principle of higher-order logic to our existing Cantorian principles:

Local Choice $\forall_{(ee)}R(\mathrm{ZF}\,R \rightarrow \mathrm{ZFC}\,R)$

Whether the picture we have outlined would ultimately be acceptable to Cantor, or indeed Zermelo, is unclear. However the view does substantiate several distinctively Cantorian ideas. First, we straightforwardly obtain from our principles the thesis that every potential infinity can be completed — $\forall_{(ee)}R(\mathrm{WO}\,R \wedge \mathrm{PotInf}\,R \rightarrow \exists_{(ee)}S(\mathrm{WO}\,S \wedge R \leq S \wedge \neg\,\mathrm{PotInf}\,S)$. Cantor endorses principles like this when outlining his disagreements with Aristotle, although he later has to walk them back on account of the apparent incompletability of the potentially infinite series of ordinals.[43] Of inconsistent multiplicities, Cantor mysteriously writes in the *Grundlagen* that we can "never achieve even an approximate conception of the absolute".[44] Whatever this might mean, the present view vindicates something in the vicinity: absolutely infinite totalities, such as the totality of all individuals, cannot be approximated by a potentially infinite series, for no well-ordered list spans every individual.

4 Can the First-order Reflect the Higher-order?

The principles formulated and discussed in the previous sections are formulated in pure higher-order logic, and do not concern any sort of abstract mathematical objects. Cantor, Frege and many others since have posited special abstract individuals that, to some extent, reflect the structure of the higher-order and constitute the subject matter of mathematics. For Cantor these included transfinite numbers, cardinals and sets which are abstracted from higher-order entities like well-orders and properties. But the more general question is: to what extent can special purpose mathematical objects represent the structure of the higher-order?

In contemporary philosophy of mathematics this question has most commonly taken the form 'when do some things form a set?', but the general issue is of longstanding significance. According to our analysis, Aristotle's rejection

[42]This is related to the well-known fact that set-theoretic choice doesn't imply global choice in Morse-Kelley set theory.

[43]As noted in footnote 10, he was surely aware of the Burali-Forti paradox at the time of the *Grundlagen*.

[44]Ewald (1996) p916.

of completed infinities is an instance of this general issue—the higher-order claim $\exists_{(ee)} R\, \mathrm{PotInf}\, R$ entails the existence of an infinite series of individuals standing in the part-whole relation to one another, but not of any single infinite individual reflecting in its part-whole structure of the higher-order entity $<_p$. Cantor too was preoccupied with the question of when 'a many can be thought of as one'—when a higher-order property corresponds to a single individual, a set, when a higher-order well-order can be assigned an individual representing its order-type, and so on.

If we take the position explored in section 2 that no ZF relation is metaphysically distinguished, it's possible that the set formation question doesn't really have an absolute answer. Different ZF relations answer the set-formation question differently.[45] Different ZF relations may count more or fewer properties as defining a set. If there were a single 'maximal' ZF relation perhaps we could use it to provide a non-arbitrary condition for set-formation: it would count as set-making those properties defining a set according to some ZF relation or other. But we have no guarantee that there is a single 'maximal' ZF relation, even assuming higher-order choice.

Let us then consider a different idea: we do not need to posit special purpose individuals to represent the higher-order and to be the subject matter of mathematics, we can simply reason about the higher-order directly and then find some way to interpret mathematics in higher-order logic, by replacing each mathematical statement with a suitable sentence of pure higher-order logic. A flatfooted account would be to paraphrase a mathematical statement A—let's say, a sentence of ZF—with the purely logical sentence $\forall_{(ee)} R(\mathrm{ZF}\, R \to A^R)$, A^R replaces the membership relation $\in$ with the variable R, and replace all universals of the form "every set is such that ..." with "for every ZF relation S extending R every set of S is such that ...":

- $(x \in y)^R := Rxy$
- $(\neg A)^R := \neg A^R$
- $(A \wedge B)^R := A^R \wedge B^R$
- $(\forall x.A)^R := \forall_{(ee)} S \forall_e x.(\mathrm{ZF}\, S \wedge R \leq S \wedge \mathrm{Dom}\, Sx \to A^S)$

[45] In second-order ZF, if some pure sets, X, are well-ordered by R which is isomorphic to the ordinal $\leq \alpha$, then there is a one-to-one correspondence, S, between the ordinals less than or equal to α and X; since the former is a set, the axiom of replacement lets us infer that the range of S is too. One has to be a little careful here, since the standard version of ZFC doesn't posit any impure sets, so that even singletons of non-sets will fail to form sets. Here I simply interpret the question of when some pure sets form a single set.

In this way we obtain a logicist account of set theory, where mathematical primitives like $\in$ are eliminated in favour of logical ones.[46]

If this is to be viable, our higher-order logic must at least contain the principle that there's at least one ZF relation, for otherwise these paraphrases would be vacuously true (and thus, so would their negations!).[47] There's one sense in which this higher-order commitment is quarantined from the first-order realm. The resulting logic is conservative over the logical sentences of first-order logic that we already had (non-mathematical) reason to believe in: apart from the theorems of classical first-order logic, it implies, for each n, the claim that there are at least n things, $\exists_n x\, x = x$, which we already had reason to believe in—there are at least n space-time regions. But once we move beyond purely logical matters, and start asking questions involving non-logical predicates we confront many awkward questions. For instance, once we have renounced special purpose abstract objects to reflect the higher-order, shouldn't (or at least, couldn't) everything be concrete? But $\exists_{(ee)} R\, \mathrm{ZF}\, R$ implies that there are far more individuals than, for instance, regions of space-time, according standard theories of space-time.[48] Arguably there must be more individuals than there are concrete things more generally.[49] And so, at least in this respect, the higher-order perspective is in the same boat as the standard view about mathematical objects—they exist, and there are lots of them. There are, however, some important differences. One central difficulty for standard platonism concerns how we secure reference to particular mathematical objects, like $\emptyset$, when we do not have any sort of causal contact with them.[50] The present view posits lots of mathematical objects, but is compatible with the view that these objects are all indistinguishable from one another and cannot be referred to uniquely (except, perhaps, by radically indeterminate names). Any mathematical role that can be played by any one of these individuals can be played by any other.

Nonetheless, once we have granted that there is at least one ZF relation

[46]ZF $\in$ guarantees $\exists_{(ee)} R\, \mathrm{ZF}\, R$, and Zermelo's quasi-categoricity theorem ensures that when R and S are ZF relations, $R \leq S$ only if the sets of a given rank according to R are exactly the sets of that rank according to S.

[47]In order to validate the translation of the replacement axiom, we need to make further assumptions: one sufficient condition states that all maximal chains of ZF relations are inaccessibly large, and a form of higher-order choice telling us that every chain of ZF relations extends to a maximal chain.

[48]'Far more than' is a notion which can be spelled out in terms of higher-order quantification; see footnote 22.

[49]Note that certain plenitudinous views about material constitution do posit enough concrete individuals for the truth of $\exists_{(ee)} R\, \mathrm{ZF}\, R$. See for instance Dorr et al. (2021).

[50]Benacerraf (1965).

a host of further question seem to dangle. The mere existential is compatible with there being there being, say, exactly five ZF relations up to isomorphism, or with there being some other number. Questions like these arise because it seems there must be a brute fact about how many mathematical objects there are. And these questions seem awkward because they have a feeling of arbitrariness to them: if the number of individuals is the fifth inaccessible, one might wonder why it wasn't the fourth, or the sixth? Corresponding questions about physical objects feel less troublesome—there the physical sciences offer guidance, and the answers are at any rate contingent so there are no apparent brute necessities concerning how many things there are.

The Zermelian logics we have developed in this paper seem well placed to remove these dangling questions, not by answering them but rejecting their presuppositions. The ZF relations are indefinitely extendible, so there is no question of a biggest ZF relation. And there is no answer to the question 'how many things are there' when the universe is not well-orderable. We can still make comparisons of size between the universe and other things, but without assuming some form of the axiom of choice the 'size' of the universe does not occupy some arbitrary position on a linear scale, like the ordinals. Of course, in a choiceless setting, one can still talk about "cardinalities" in a Fregean sense, identifying them with equivalence classes of equinumerous properties, but it strikes me that such entities cannot be informative answers to questions about size. Without higher-order choice we cannot prove that for any two properties, X and Y, one is at least as big as the other, so these "cardinalities" are not linearly ordered. But more importantly, it is a scale defined in terms of the thing you are trying to measure. "He is six foot" is an informative answer to the question of how tall John is, because it is defined with respect to an independent, pre-existing scale (like the ordinals). "He is John's height" is not a helpful answer, nor is "his height is represented by the equivalence class of people the same height as John".[51]

Another option would be to adopt a modal paraphrase of mathematical statements. The simplest approach is to paraphrase a ZF sentence A as

[51] There is another sense in which the question 'how many things are there' might not have an answer: the cardinality of the universe may not be expressible in pure higher-order logic (after all, only countably many cardinalities can be expressed in a countable language). But this does not mean that the question has no answer—it just means that one cannot express it in a certain language. If the universe was well-orderable, then the question could be answered in more expressive languages containing primitive cardinality quantifiers for the size of the universe (and if the universe wasn't well-orderable then such a cardinality quantifier couldn't be introduced semantically, and such languages would presumably not exist).

$\Box\forall_{(ee)}R(\mathrm{ZF}\,R \to A^R)$, where A^R now replaces $\in$ with R and universals of the form "every set ..." with modal statements "necessarily, for every individual in the domain of a rigid ZF relation S extending R ...".[52] Here we no longer have to posit *any* distinctively mathematical objects, or brute facts about how many mathematical individuals there are. We only need to posit the possibility of a sufficient number of things, concrete or otherwise. Understanding the possibility in this claim in terms of Kripke's notion of 'metaphysical' possibility introduces a number of distracting questions that I'd rather circumvent—needless to say, the matter is more fraught. Hellman (1989) suggests, instead, that the $\Box$ here should be interpreted as a *logical* modality. Taking the linguistic notion of logical consistency as our guide to logical possibility, the assumption that there could have been a ZF relation is modest. The consistency of first-order ZF is certainly still an assumption here, but it is completely uncontentious among set theorists. More contentious is the idea that there is a propositional notion of logical possibility at all. While we have several reasonable accounts of the notion of a logically consistent sentence, some might argue that there is nothing that stands to reality as logical consistency stands to language. While I myself agree with the general concern that one must tread carefully when introducing a propositional notion in the vicinity of a linguistic one, I believe that the notion of logical necessity *can* be put in good standing in higher-order logic, with suitable axioms ensuring that the operator notion behaves like the logical one. It would take me too far afield to develop a proper defence of this here.[53] It is worth noting, however, that in certain logics for logical necessity, the claim $\Diamond\exists_{(ee)}R\,\mathrm{ZF}\,R$ is in fact a theorem, and so the non-vacuity of mathematical claims fall out of the logic

[52]A version of this following translation, inspired by some remarks in Putnam (1967), can also be found in Hellman (1989). Let Rig stand for property of being a rigid relation, $\lambda R.\Box(\forall_{(ee)}S(\Box\forall_e xy(Rxy \to Sxy) \leftrightarrow \forall_e xy(Rxy \to \Box Sxy)))$, and let R and S be relation variables.

- $(x \in y)^S = Sxy$
- $(\forall x.A)^S := \Box\forall R(\mathrm{Rig}\,R \wedge \mathrm{ZF}\,R \wedge S \leq R \to \forall x.(A)^R)$
- $(A \wedge B)^S := A^S \wedge B^S$
- $(\neg A)^S := \neg A^S$

A sentence of first-order ZF, A, may then be translated as $\Box\forall_{(ee)}S(\mathrm{Rig}\,S \wedge \mathrm{ZF}\,S \to A^S)$. Here the logical hypotheses needed to prove that this translation secures the vacuous quantifier axiom and the replacement axiom are a bit more subtle.

[53]I have undertaken this elsewhere. See, for instance, Bacon (forthcominga) chapter 7, Bacon and Zeng (2022).

of logical necessity rather than as a special mathematical posit.[54]

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[54] In Bacon and Dorr (forthcoming) the authors consider adding to a certain higher-order logic, Classicism, a schema that would have $\Diamond\exists_{(ee)} R\, \mathrm{ZF}\, R$ as an instance if first-order ZFC were consistent (as widely assumed). At present my favoured theory of logical necessity is a schema containing all sentences of the form $\Diamond A$ where A is a closed sentences of higher-order logic that has a certain sort of set theoretic model. There is a concern here that to specify the instances of the schema one has to already assume a platonistic set-theory, or at least the non-vacuity of a suitable higher-order paraphrase of a platonistic set theory. Perhaps that is the order of understanding needed for a contemporary logician already familiar with ZFC to arrive at the statement of the schema, but it need not be the order of things in any deeper sense. Zermelo once objected to Skolem's reformulation of his single higher-order separation axiom as a first-order schema on the grounds that to know what a wff, and thus an instance, is, one has to already have the notion of a finite number in order to know which sentences are formable from the constants by a finite number of applications of the formation rules, and this was not available prior to understanding set-theory itself. These days, few would agree with Zermelo that first-order ZFC (or ever first-order Peano arithmetic) is an inadequate foundation for the theory of finite numbers on these grounds.

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A Appendices

A.1 Logics of Zermelian Extensibility

In this appendix I outline some higher-order logics that formalize, in purely logical terms, certain Zermelian theses about indefinite extensibility, and prove they are consistent relative to a more standard mathematical theory. The style of proof is very reminiscent of Fraenkle's proof of the independence of the axiom of choice from ZFU (set theory with urelemente Fraenkel (1922)). Because the urelemente are essentially indistinguishable in the language of set-theory the sorts of forcing techniques involved in Cohen proof are not necessary.

Let's begin by being a bit more precise about the language we have been working in. As previously described, there is a type of singular terms, e, and whenever $\sigma_1, ..., \sigma_n$ are types we also have a type, $(\sigma_1 ... \sigma_n)$ of n-ary relations between entities of these respective types. Terms are formed inductively: we have logical constants $\rightarrow$, $\perp$ and $\forall_\sigma$ of types $(()())$, $()$ and $((\sigma))$ respectively, and an infinite stock of variables of each type. Given a term R of type $(\sigma_1 ... \sigma_n ... \sigma_{n+m})$ and terms $a_1, ..., a_n$ of types $\sigma_1, ..., \sigma_n$ we can a complex term of type $(\sigma_{n+1} ... \sigma_{n+m})$ by application: $Ra_1 ... a_n$. And given a term A of type $(\sigma_1 ... \sigma_n)$ we and variable x_0 of type σ_1 we can form a complex predicate of type $(\sigma_0 ... \sigma_n)$ by $\lambda x_0.A$. Common symbols, like $\wedge$ and $\exists_\sigma$, are introduced by abbreviation—of particular note is the higher-order identity relation $=_\sigma$ of type $(\sigma\sigma)$, which is defined as $\lambda xy.\forall_{(\sigma)} X(Xx \rightarrow Xy)$.

A *logic* is just a set of sentences in this logical language that contains the axioms PC, UI and $\beta\eta$ below, and is closed under the rules MP and Gen.[55] Following Bacon (2018) I'll call the smallest such theory H.

PC All instances of propositional tautologies.

MP From A and $A \rightarrow B$ infer B

Gen From $A \rightarrow B$ infer $A \rightarrow \forall_\sigma \lambda x.B$ when x does not occur free in A.

UI $\forall_\sigma F \rightarrow Ft$ (where t is a term of type σ)

$\beta\eta$ $A \rightarrow B$ whenever A and B are $\beta\eta$ equivalent terms of type t.[56]

[55] If we wanted we could consider languages with non-logical constants, and distinguish between a logic and a theory, but in the present language every theory is a logic.

[56] For the notion of $\beta\eta$-equivalence see, for instance, Bacon (forthcominga) chapter 3.

We can now consider logics obtained by adding to H principles discussed in the previous sections. They are not all independent of one another. For instance the higher-order Limit principle in fact is already a theorem of H, as noted in footnote 21. The higher-order Successor principle follows from Progress$^{\mathrm{WO}}$. We will thus focus on the following logics.

- HZ is the higher-order logic axiomatized by Progress and Completion.
- $\mathsf{HZ}^{\mathrm{WO}}$ is axiomatized by Progress$^{\mathrm{WO}}$ and Completion$^{\mathrm{WO}}$.
- We will use (LC) and (Ext) respectively to denote Local Choice and the principle of Extensionalism, so that, e.g., $\mathsf{HZ(LC)}$ is HZ plus Local Choice, and $\mathsf{HZ(Ext)}$ is HZ plus Extensionalism.

Extensionalism is the principle stating that proposition, properties and relations are individuated by coextensiveness.

Extensionalism $\forall_{(\sigma_1...\sigma_n)}RS(\forall_{\sigma_1}x_1...\forall_{\sigma_n}x_n(Rx_1...x_n \leftrightarrow Sx_1...x_n) \rightarrow R =_{(\sigma_1...\sigma_n)} S)$

It is not a plausible principle: it implies that all operators are truth-functional, yet there appears to be plenty of genuine contingency. However, it is quite strong and implies many principles of higher-order logic that do appear to be desirable.[57] So the consistency of any of the above with Extensionalism implies the consistency with these principles.

We now proceed to establish the following theorem:

Theorem A.1. $\mathsf{HZ(LC)(Ext)}$ *and* $\mathsf{HZ}^{\mathrm{WO}}\mathsf{(LC)(Ext)}$ *are consistent relative to the consistency of ZFC + "there are infinitely many inaccessibles".*

Our strategy is to develop a fairly standard 'Henkin model' for these logics.[58]. A Henkin model is a type of set-theoretic entity, so the metalanguage with which we will reason about these structures will be the language of first-order set theory (i.e. first-order logic with a single non-logical predicate $\in$) with the axioms and axiom schemas of first-order ZFC and the assumption that there are infinitely many inaccessibles. It's worth here pausing on the fact that in this section we do appear to operate with a distinguished relation $\in$, contrary to the philosophical vision being pursued. But this appearance can ultimately be dispensed with: once we have proven the consistency of,

[57] See the principles discussed in section 2 of Bacon and Dorr (forthcoming).

[58] Henkin (1950)

say, $\mathsf{HZ(LC)(Ext)}$ in this theory, we may obtain a finitary proof of the conditional 'if ZFC+"there are infinitely many inaccessibles" is consistent, then so is $\mathsf{HZ(LC)(Ext)}$'.

We have argued above that HZ (and by extension any stronger logic) implies the negation of The Well-Ordering Principle, and of the higher-order axiom of choice. Thus the models we construct here must invalidate these higher-order choice principles. Nonetheless, our construction of a Henkin model invalidating these higher-order choice principles will be elementary. This is in stark contrast to the situation in standard set theory, where one must undertake much more involved model theoretic constructions in order to invalidate choice, such as Cohen's method of forcing.[59]

Let's start with the promised notion of a Henkin model.

Definition A.1 (Henkin Structure)**.** *A Henkin structure $D^{\cdot}$ is a type indexed collection of sets, D^σ for each type σ, subject to the constraint:*

$$D^{(\sigma_1 \ldots \sigma_n)} \subseteq P(D^{\sigma_1} \times \ldots \times D^{\sigma_n})$$

A structure is full iff this inclusion is always an identity.

Define $\mathrm{ff} = \emptyset$ and $\mathrm{tt} := \{()\}$.

Note we could adopt a convention of identifying a unary product of D with itself, or else we $D^{(\sigma)}$ is a set of sets of singleton sequences from D^σ have to continually distinguish $\{a\}$ and $\{(a)\}$.

A variable assignment for a Henkin structure is a function, g, defined on variables mapping variables of type σ to elements of D^σ. We write $g[x \mapsto a]$ for the variable assignment like g except in assigning a to x

Definition A.2 (Henkin Model)**.** *A model is a pair $(D^{\cdot}, [\![\cdot]\!]^{\cdot})$ where $D^{\cdot}$ is a Henkin structure and $[\![\cdot]\!]^{\cdot}$ is a function taking a term of higher-order logic and a variable assignment as arguments such that*

- $[\![M]\!]^g \in D^\sigma$ *for every term M of type σ.*
- $[\![x]\!]^g = g(x)$ *for every variable x.*
- $[\![MN]\!]^g = \{(a_1, \ldots, a_n) \mid ([\![N]\!]^g, a_1, \ldots, a_n) \in [\![M]\!]^g\}$
- $[\![\lambda x.M]\!]^g = \{(a_1, \ldots, a_n) \mid (a_2, \ldots, a_n) \in [\![M]\!]^{g[x \mapsto a_1]}\}$

[59] If it seems surprising that we can invalidate choice without forcing, it's worth remembering that in the context of ZFCU—where we allow impure sets—we can construct models in which choice fails straightforwardly for sets containin urelements. See Fraenkel (1922).

- $[\![\forall_\sigma]\!]^g = \{D^\sigma\}$
- $[\![\wedge]\!]^g = \{(\mathrm{tt}, \mathrm{tt})\}$
- $[\![\neg]\!]^g = \{(\mathrm{ff})\}$

A sentence, A, is true in a model relative to g iff $[\![A]\!]^g = tt$

Henkin (1950) shows in essence that every sentence of $\mathsf{H(Ext)}$ is true in every Henkin model.

Note that not every Henkin structure can be extended to a Henkin model. The structure may fail to contain the interpretations of the logical constants (the final three conditions) or be closed under the operations that correspond to application and λ-abstraction (the second and third). If the structure is closed under these operations, then constraints in definition A.2 can be reinterpreted as an inductive *definition* of $[\![\cdot]\!]$. If a Henkin structure D can be extended to a model $(D, [\![\cdot]\!])$ then that model is unique, so we will often simply refer to these Henkin structures as models.

Below is a method for constructing Henkin models.

Definition A.3 (Permutations)**.** *Let* $\pi : D^e \to D^e$ *be a permutation.* π *may be extended to arbitrary elements of the full Henkin structure based on on* D^e *as follows.*

- $\pi^e = \pi$
- $\pi^t = \mathrm{id}$
- $\pi^{(\sigma_1 \ldots \sigma_n)} R = \{(\pi a_1, ..., \pi a_n) \mid (a_1, ..., a_n) \in R\}$

An element $a \in D^\sigma$ *is fixed by* π *iff* $\pi a = a$

By a straightforward induction, one can show that $(\pi^{-1})^\sigma$ is an inverse of π^σ, so that π^σ is a permutation of D^σ for each type σ. Since there is no difference between $(\pi^{-1})^\sigma$ and $(\pi^\sigma)^{-1}$, I will henceforth omit superscripts from permutations, letting them be determined by context. It follows that a is fixed by π iff it is fixed by π^{-1}, since if $\pi a = a$ then $\pi^{-1}\pi a = \pi^{-1} a$ and so $a = \pi^{-1}$.

Here are some useful notions:

Definition A.4 (Metaphysical definability)**.** *Let D be a full Henkin structure. Let* $X \subseteq \bigcup_\sigma D^\sigma$ *be some collection of relations. Say that* $a \in D^\sigma$ *is m-defined from X, or 'fixed by X', iff every permutation that fixes every element of X also fixes a.*

(A full Henkin model equipped with the full set of permutations is a substitution structure, in the sense of Bacon (2019), which is why I am adopting the terminology of metaphysical definition).

Definition A.5 (Directedness)**.** *Let D be a full Henkin structure. Say that $X \subseteq \bigcup_\sigma D^\sigma$ is* directed *iff, for any R and S in X, there exists at $T \in X$ such that T fixes R and S.*

The next result tells us that from any full Henkin structure, D, and any directed collection of its elements, X, we can form another Henkin structure D/X that can be extended to a model. In fact, the application for which we need this theorem, the Henkin model D/X can be described a bit more simply. However, the general technique is very useful for generating models with various properties where second-order choice fails, so I state the more general theorem.

Definition A.6. *Let D be a full Henkin structure, and X a directed collection of elements from D. Then the structure D/X is defined by setting $(D/X)^\sigma := \{a \in D^\sigma_0 \mid a \text{ is fixed by some element of } X\}$.*

Theorem A.2. *D/X is a Henkin model.*

Proof. We will show by induction that for every term M, $[\![M]\!]^g$ is defined for every assigment g, and that there exist a $R \in X$ such that for every π fixing R and every assignment g, $[\![M]\!]^{\pi\circ g} = \pi([\![M]\!]^g)$.[60]

It is easily checked that $[\![\forall_\sigma]\!]$, $[\![\wedge]\!]$ and $[\![\neg]\!]$ are all fixed by every permutation and are independent of the assignment. Clearly for variables, $\pi[\![x]\!]^g = \pi(g(x)) = [\![x]\!]^{\pi\circ g}$. It remains to show that $[\![\cdot]\!]$ can be extended inductively to application terms and λ-terms.

Let us suppose that M and N satisfy the inductive hypothesis, witnessed respectively by R and S in X. Suppose that T m-defines both R and S, so that for any π fixing T, $[\![M]\!]^{\pi\circ g} = \pi([\![M]\!]^g)$ and $[\![N]\!]^{\pi\circ g} = \pi([\![N]\!]^g)$. We will show that if π fixes T, then for every assignment g, $[\![MN]\!]^{\pi\circ g} = \pi([\![MN]\!]^g)$. That is, we must show $\{(a_1, ..., a_n) \mid ([\![N]\!]^{\pi\circ g}, a_1, ..., a_n) \in [\![M]\!]^{\pi\circ g}\} = \{(\pi a_1, ..., \pi a_n) \mid ([\![N]\!]^g, a_1, ..., a_n) \in [\![M]\!]^g\}$. We begin with the right to left inclusion. Any tuple in the right-hand-side is of the form $(\pi a_1, ..., \pi a_n)$ where $([\![N]\!]^g, a_1, ..., a_n) \in [\![M]\!]^g$. Then by the way $\pi[\![M]\!]^g$ is defined, $(\pi[\![N]\!]^g, \pi a_1, ..., \pi a_n) \in \pi[\![M]\!]^g$. Since $\pi[\![M]\!]^g = [\![M]\!]^{\pi\circ g}$ and $\pi[\![M]\!]^g = [\![M]\!]^{\pi\circ g}$, we have that $([\![N]\!]^{\pi\circ g}, \pi a_1, ..., \pi a_n) \in$

[60] A little more precisely, we are showing by induction on complexity that there is a partial function satisfying the clauses of definition A.2 for all expressions of that complexity. The union of these partial functions clearly satisfies the conditions for all expressions.

$[\![M]\!]^{\pi\circ g}$ giving us the right-to-left inclusion. For the other inclusion, we may use the previously noted fact that π^{-1} also fixes T, so that we may apply the inductive hypothesis, using π^{-1} as the permutation, and $\pi \circ g$ as the assignment, to obtain the identities $[\![M]\!]^g = \pi^{-1}[\![M]\!]^{\pi\circ g}$ and $[\![N]\!]^g = \pi^{-1}[\![N]\!]^{\pi\circ g}$. Now we reason as before: if $([\![N]\!]^{\pi\circ g}, a_1, ..., a_n) \in [\![M]\!]^{\pi\circ g}$, then $(\pi^{-1}[\![N]\!]^{\pi\circ g}, \pi^{-1}a_1, ..., \pi^{-1}a_n) \in \pi^{-1}[\![M]\!]^{\pi\circ g}$, and using the two identities we obtained from the inductive hypothesis, $([\![N]\!]^g, \pi^{-1}a_1, ..., \pi^{-1}a_n) \in [\![M]\!]^g$. So $(\pi\pi^{-1}a_1, ..., \pi\pi^{-1}a_n)$, that is $(a_1, ..., a_n)$, belongs to the right-hand-side.

We may now show that $[\![MN]\!]^g$ is defined for every assignment g — i.e. that the third clause from definition A.2 defines an element of D. Let g be an arbitrary assignment. Using directedness find an $R \in X$ that m-defines T and $g(x)$ for every x appearing in MN. Now by the above $[\![MN]\!]^{\pi\circ g} = \pi[\![MN]\!]^g$. But $[\![MN]\!]^{\pi\circ g} = [\![MN]\!]^g$ since $\pi g(x) = g(x)$ for every x appearing in MN.

Now suppose the inductive hypothesis holds for M. So there is some $T \in X$ such that that for every permutation π fixing T and every assignmente g, $[\![M]\!]^{\pi\circ g} = \pi[\![M]\!]^g$. We will show that for every π fixing T and assignment g, $[\![\lambda x.M]\!]^{\pi\circ g} = \pi[\![\lambda x.M]\!]^g$. If a tuple is in the right-hand-side it is of the form $(\pi a_1, ..., \pi a_n)$ where $(a_2, ..., a_n) \in [\![M]\!]^{g[x\mapsto a_1]}$. So as before $(\pi a_2, ..., \pi a_n) \in \pi[\![M]\!]^{g[x\mapsto a_1]}$ which $= [\![M]\!]^{\pi\circ(g[x\mapsto a_1]}$ by the inductive hypothesis, which $= [\![M]\!]^{(\pi\circ g)[x\mapsto \pi a_1]}$. So $(\pi a_1, ..., \pi a_n) \in [\![\lambda x.M]\!]^{\pi\circ g}$ as required. As before we may also reverse this reasoning, by using the fact that π^{-1} also fixes T.

The argument that $[\![\lambda x.M]\!]^g$ is well-defined is identical to the argument for MN.

□

We now describe two examples that can be used to generate models of HZ and $\mathsf{HZ}^{\mathsf{WO}}$. Let κ be a limit of inaccessibles. Let D be the full Henkin structure obtained by setting $D = \kappa$.

Example A.1. *We let $<_\alpha$ be the ordering of the ordinals restricted to the ordinal $\alpha < \kappa$. The set $X_1 = \{<_\alpha \mid \alpha < \kappa\}$ forms a directed set, since $<_\alpha$ m-defines $<_\beta$ whenever $\alpha \geq \beta$.*

For the second example we let $D = V_\kappa$ (in fact this same domain could be used in the first example).

Example A.2. *Let $\in_\alpha$ be the membership relation restricted to the sets of rank α (i.e. $\in \cap V_\alpha$). $X_2 = \{\in_\alpha \mid \alpha < \kappa\}$ is directed, since $\in_\alpha$ m-defines $\in_\beta$ whenever $\alpha \geq \beta$.*

Theorem A.3. *For $i = 1, 2$, a relation $R \subseteq D^{\sigma_1} \times \ldots \times D^{\sigma_n}$ is in $(D/X_i)^{\sigma_1 \times \ldots \times \sigma_n}$ iff, for some $\alpha < \kappa$, every permutation that is the identity restricted to α (resp. V_α) fixes R.*

Proof. A permutation π fixes $<_\alpha$ iff $\pi \restriction_\alpha$ is the identity. This is because there are no non-trivial automorphisms of well-orders. Similarly, π fixes $\in_\alpha$ iff $\pi \restriction_\alpha$ is the identity, because there are no non-trivial automorphisms of V_α.

We prove the latter by $\in$-induction. Assume that $\pi y = y$ for all $y \in x$. The members of πx are of the form πy for $y \in x$, so $\pi x = x$ by extensionality. The former can be proved similarly by transifinite induction. $\square$

It follows that the models obtained from X_1 and X_2 are essentially the same. In fact, given choice in the metalanguage κ and V_κ have the same size.

It will be convenient in what follows to say that an element of D/X_1 (or D/X_2) is 'pinned down' by λ iff every permutation that is identity on λ (V_λ respectively) fixes that element.

Theorem A.4. *$M = (D/X_1, [\![\cdot]\!])$ is a model of $\mathsf{HZ}^{\mathrm{WO}}(\mathsf{LC})(\mathsf{Ext})$. $M' = (D/X_2, [\![\cdot]\!])$ is a model of $\mathsf{HZ}(\mathsf{LC})(\mathsf{Ext})$.*

Proof. As noted $\mathsf{H}(\mathsf{Ext})$ is validated in any Henkin model. It remains to show Progress, Completion and Local Choice are true in M. We treat these in order.

Progress: Suppose that $R \in (D/X_1)^{(e,e)}$ is a (well-order of inaccessible order type)^M. Then R must in fact be a well-order (indeed an inaccessible well-order), for there must exist some $\alpha < \kappa$ such that R is fixed by every permutation that is the identity on α. Thus $Dom(R) \subseteq \alpha$. Moreover, every subset of α is in $(D/X_1)^{(e)}$ for a similar reason, so that the second-order quantifiers in the claim that R is a (well-order of inaccessible order type)^M are essentially unrestricted, so R is in fact a well-order of inaccessible order type. Since κ is a limit of inaccessibles, there is an inaccessible, $\lambda < \kappa$, of greater order type than R, and using choice we may pick an $R' \subseteq \kappa \times \kappa$ containing R with $Dom(R') \subseteq \lambda$ such that R has order type λ. R' is pinned down by λ because it is a subset of λ, so R' is in $(D/X_1)^{(e,e)}$. Moreover, as with R, we can see that R' (well-order of inaccessible order type)^M iff it is a well-order of inaccessible order type (which it is). Thus we have shown that any inaccessible well-order of M is a proper initial segment of another inaccessible well-order of M. So M is a model of Progress$^{\mathrm{WO}}$.

Completion: Now suppose that $X \in (D/X_1)^{((e,e)}$ is a set of (well-orders)^M that are (linearly ordered by the initial segment relation)^M, and that X is pinned down by λ. We show that for every $R \in X$, $dom(R) \subseteq \lambda$. Suppose not, and $a \in dom(R) \setminus \lambda$. Let π be a transposition that fixes λ and swaps a

with some element b also outside of λ. Since π fixes X and $R \in X$, $\pi R \in X$, and since X is linearly ordered then either πR is an initial segment of R or R is an initial segment of πR. Thus $dom(R) \subseteq dom(\pi R)$ or $dom(\pi R) \subseteq dom(R)$. Of course, $a \in dom(R)$ and $\pi a = b \in dom(\pi R)$, so that either a and b both belong to $dom(R)$ or to $dom(\pi R)$. Without loss of generality suppose the former. Since R is linear either $(a, b) \in R$ or $(b, a) \in R$, in which case $(b, a) \in \pi R$ or $(a, b) \in \pi R$ respectively, and either case is impossible given than one is an initial segment of the other (and both are asymmetric orders).

So $\bigcup X \subseteq \lambda$, and is a well-order. Since $\lambda < \kappa$ there is an inaccessible, γ, between λ and κ and we may extend $\bigcup X$ to a well-order, S, of inaccessible order-type whose domain is γ and is thus pinned down by γ.

Local Choice: Suppose that $R \in (D/X_1)^{(e,e,)}$ is a ZF^M relation, and is pinned down by $\lambda < \kappa$. By the same sort of reasoning the domain of R is contained in λ, and thus R is a ZF(C) relation iff it is a $\mathrm{ZF(C)}^M$ relation, by the fact that the second-order quantifiers in M range over all subsets of the domain of R. Since R is thus a ZF relation, it is isomorphic to V_γ for some inaccessible γ by Zermelo's theorem (Zermelo (1930)), and since V_γ is a ZFC relation (by the axiom of choice), R is a ZFC relation too, and finally a ZFC^M relation.

The proof $(D/X_2, [\![\cdot]\!])$ is a model of $\mathsf{HZ(LC)(Ext)}$ is essentially the same so I do repeat it here. □

A.2 Appendix: The Inconsistency of a Cantorian Criteria of Set Formation

In this appendix I show that the theory one gets by formalizing a broadly Cantorian account of set formation—according to which some things form a set when they can be *listed*—is subject to the Burali-Forti paradox. However here the argument is somewhat less obvious, so it should be presented it in detail. The theory inspired by the *Grundlagen* may be axiomatized as follows

Extensionality $\forall_e xy(\forall_e z(z \in x \leftrightarrow z \in y) \to x =_e y)$.

Well-Ordered Comprehension $\forall_{(ee)} R(\mathrm{WO}\, R \to \exists_e x \forall_e y(y \in x \leftrightarrow \mathrm{Dom}\, Ry)$

Let us define a *von Neumann* ordinal as a transitive set that is well-ordered by $\in$:[61]

$$\mathrm{Ord}\, \alpha := (\forall x(x \in \alpha \to x \subseteq \alpha) \wedge \mathrm{WO}\, \lambda xy(x \in y \wedge y \in \alpha)$$

[61] von Neumann (1923).

We can then show that the von Neumann ordinals are well-ordered by $\in$, and thus form a set by Well-Ordered Comprehension. The argument that the von Neumann ordinals are well-ordered is not at all new, but it needs to be checked that it can be carried out in the present set theory.

We begin by showing that von Neumann ordinals are linearly ordered. Let Lemma (a) be the claim that if α and β are ordinals and α is a proper subset of β then $\alpha \in \beta$.[62] let Lemma (b) be the claim that if α and β are ordinals then the set of things belonging to both, $\alpha \cap \beta$, exists and is an ordinal.[63]

We now see that von Neumman ordinals linearly ordered, for suppose that α and β are ordinals, and $\alpha \neq \beta$. So $\alpha \cap \beta$ is an ordinal by Lemma (b), and is a proper subset of α or of β. Without loss of generality, we assume the former. Then by Lemma (a) $\alpha \cap \beta \in \alpha$. Now $\alpha \cap \beta$ cannot also be a proper subset of β. For otherwise, by Lemma (a) it is an element of β, and it is already an element of α, in which case $\alpha \cap \beta \in \alpha \cap \beta$ contradicting the fact that the elements of α (which includes $\alpha \cap \beta$) are well-ordered by membership. So $\beta \subseteq \alpha \cap \beta$ — the other inclusion is clear, so $\beta = \alpha \cap \beta \in \alpha$. In the case that $\alpha \cap \beta$ is a proper subset of β we reason analogously, and conclude $\alpha \in \beta$. So Ord is linearly ordered.

Von Neumann ordinals are also well-ordered by $\in$. Suppose that all Xs are ordinals and α is X. If α is not already the $\in$-least X, then there is at least one $\beta \in \alpha$ that is X, and so a $\in$-least $\beta \in \alpha$ that is X. If γ is also an X ordinal, then $\gamma \notin \beta$, for otherwise $\gamma \in \alpha$ by the transitivity of α, contradicting the assumption that β was the $\in$-least element of α that was X. So either $\gamma = \beta$ or $\beta \in \gamma$, by the fact that $\in$ linearly orders the Xs.

[62] It is proved by noting that by Extensionality there is at least one member of β not in α, and so there must be a least such element, x, under membership since β is well-ordered. If $y \in x$ then $y \in \beta$, since β is transitive, and so $y \in \alpha$ or else x would not be the $\in$-least element of β not in α. Conversely if $y \in \alpha$ then $y \in \beta$, since $\alpha \subseteq \beta$. Since β is linearly ordered by $\in$, either $y = x$, $x \in y$ or $y \in x$. y can't be the same as x, since $x \notin \alpha$. Nor can x belong to y, because otherwise x would again belong to α by the transitivity of α and the fact that y belongs to α. Thus $y \in x$. So x and α have the same elements, and are identical by Extensionality. Since $x \in \beta$, $\alpha \in \beta$.

[63] If all Xs belong to both α and β, then there is an $\in$-least X in α, since α is well-ordered by $\in$. Similarly, if x and y belong to both α and β, then either $x = y$, $x \in y$ or $y \in x$ by the fact that α is well-ordered. So by Well-Ordered Comprehension, there is a set of things belonging to both α and β. It is of course well-ordered by $\in$, as we have just seen. And it is transitive, by the transitivity of both α and β, so $\alpha \cap \beta$ is an ordinal.