A Philosophical Introduction to Higher-order Logics

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Contents

Nomenclature

Some common symbols:

H, C, etc.	Names for particular higher-order logics, the minimal system H, Classicism, etc.
$\mathbf{L}, \mathbf{L}', ...$	Name for an arbitrary logic
M, N, P, Q, a, b, c	Terms of arbitrary type, lower case reserved for terms in argument position
A, B, C	Terms of type t
X, Y, Z, x, y, z, w	Variables, lower case reserved for variables appearing exclusively in argument position
$\sigma, \tau, \rho, ...$	Types
Σ	A name for a signature
Λ	The logical signature of higher-order logic
$\mathcal{L}(\Sigma)$	The full λ-language in the signature Σ
$\mathcal{J}(\Sigma)$	An arbitrary λ-language in the signature Σ
$M \sim_{\beta\eta} N$	M and N are $\beta\eta$ equivalent
$CL[\Sigma, \{B, C, K\}]$, etc.	The combinatory language in signature Σ with combinators B, C and K
$ND[PCW]$, etc.	The natural deduction Curry system containing the rules P, C, and W
$\mathcal{A}, \mathcal{B}, \mathcal{C}...,\ \mathcal{A}^\sigma, \mathcal{B}^\sigma, \mathcal{C}^\sigma...$	Names for applicative structures, names for domains of type σ
$\tilde{f}$	The counterpart of the function f in an applicative structure
$\mathbf{M}, \mathbf{N}, ...$	Models of higher-order languages
$g, h, ...$	Variables assignments
$[\![M]\!]^g$	The denotation of a term M relative to an assignment in a given model
$\mathcal{R}, \mathcal{S}, \mathcal{T}, ...,\ \mathcal{R}^\sigma, \mathcal{S}^\sigma, \mathcal{T}^\sigma...$	Names for logical relations or Kripke logical relations, names for logical or Kripke logical relation at a given type

Some common abbreviations:

$M : \sigma$	'M is a term of type σ', 'M, of type σ,'
$MN_1N_2...N_k$	$(...((MN_1)N_2)...N_k)$
$\lambda x_1x_2...x_n.M$	$\lambda x_1.(\lambda x_2.(...\lambda x_n.M))...)$
$\sigma_1 \to \sigma_2 \to ... \to \sigma_n \to \tau$	$(\sigma_1 \to (\sigma_2 \to (... \to (\sigma_n \to \tau)...)))$
$\forall_\sigma x.A$	$\forall_\sigma(\lambda x.A)$
$A \wedge B$, $a =_\sigma b$, etc	$((\wedge A)B)$, $((=_\sigma a)b)$
$\bar{x}$, $\bar{a}$, $\bar{\sigma}$	$x_1, ..., x_n$, $a_1, ..., a_n$, $\sigma_1, ..., \sigma_n$
$(\bar{\sigma} \to \tau)$	$\sigma_1 \to \sigma_2 \to ... \to \sigma_n \to \tau$
$\lambda\bar{x}.M$, $M\bar{x}$	$\lambda x_1...\lambda x_n.M$, $Mx_1...x_n$
$\bar{x} : \bar{\sigma}$:	$x_1 : \sigma$, $x_2 : \sigma_2...$ $x_n : \sigma_n$
$\forall\bar{x}.A$	$\forall_{\sigma_1}x_1...\forall_{\sigma_n}x_n.A$ where $\bar{x} : \bar{\sigma}$
$(MN)^n_m$	$\lambda\bar{x}\bar{y}.M\bar{x}(N\bar{y})$ where $\bar{x} = x_1...x_m$, $\bar{y} = y_1...y_n$, $\bar{x} : \bar{\sigma}, \bar{y} : \bar{\rho}, N : \bar{\rho} \to \tau, M : \bar{\sigma} \to \tau \to \theta$
$\wedge_{\bar{\sigma}\to t}, \neg_{\bar{\sigma}\to t}$, etc.	$\lambda XY\bar{x}\bar{y}(X\bar{x}\wedge Y\bar{y})$, $\lambda X\bar{y}.\neg(X\bar{y})$, etc. where $\bar{x}, \bar{y} : \bar{\sigma}$
I^σ	$\lambda x.x$ where $x : \sigma$
$S^{\sigma\tau\rho}$	$\lambda XYz.Xz(Yz)$ where $X : \sigma \to \tau \to \rho$, $Y : \sigma \to \tau$, $z : \sigma$
$K^{\sigma\tau}$	$\lambda xy.x$ where $x : \sigma, y : \tau$
$B^{\sigma\tau\rho}, B'^{\sigma\tau\rho}$	$\lambda XYz.X(Yz)$, $\lambda YXz.X(Yz)$ where $Y : \sigma \to \tau, X : \tau \to \rho, z : \sigma$
$C^{\sigma\tau\rho}$	$\lambda Xyz.Xzy$ where $X : \sigma \to \tau \to \rho, z : \sigma, y : \tau$
$W^{\sigma\tau}$	$\lambda Xy.Xyy$ where $X : \sigma \to \sigma \to \tau, y : \sigma$
$\Box_\top$	$\lambda p.p =_t \top$

Preface

There are some ways in which this book resembles a textbook, and other ways in which it resembles a monograph. Like a textbook it aims to impart to the reader certain logical tools that I believe to have a great number of applications in philosophy, specifically metaphysics. This represents a majority of the book. On the other hand, I have made no effort to be comprehensive in my coverage of those applications: I have simply taken two topics—modal metaphysics (chapter 9), and metaphysical structure (chapter 10)—that have personally captivated my interest, and to which the tools presented in this book have helped me get things straight in my own head. These cases, I hope, illustrate the power of higher-order logic both as a language for formulating important claims in metaphysics, and as a framework for investigating those questions. But I have followed my own inclinations in deciding what to explore, and this part of the book is in no way representative of the full variety of possible applications, or indeed, existing applications of these tools in the burgeoning literature on higher-order metaphysics. Some of these omissions are discussed below.[1] My hope is that the reader will be able to take the apparatus in this book and apply them to whatever questions captivate their interest.

I was motivated to write this book after struggling myself to find the right tools for my purposes in existing philosophical texts. I found, increasingly, that I was reinventing wheels that had previously been invented by computer scientists with completely different applications in mind. These include several logical tools described in this book, for instance, substructural type theory, the concept of a logical relation, and various ideas from category theory. However, because these wheels have been designed for different cars (as it were), and the expository texts written for a different audience, there is no resource which a philosopher can simply consult in order to learn about them in a way that transparently relates them to philosophical concerns. My hope is this book will fill that lacuna.

A couple of brief remarks on the title of the book are in order. Perhaps a more accurate (but less catchy) name would have been *A Philosophical Introduction to Higher-order Logics and λ-Calculi*. About a quarter of the book is devoted to the simply typed λ-languages: a very general class of languages that

[1]The reader interested in the broader field of higher-order metaphysics might wish to consult the recent anthology Fritz and Jones (forthcoming).

includes many logical languages, including propositional, first-order and higher-order logic, and even non-logical languages such as programming languages. Pedagogically, the relationship between the λ-calculus and higher-order logic is a bit like the relationship between propositional logic and first-order logic: very few philosophically interesting theories can be formulated in propositional logic alone, but one needs to become reasonably fluent in it before learning first-order logic. I believe the λ-calculus stands as much in the same relationship to higher-order logic as propositional and first-order logic do.

The other remark regarding the title concerns the use of a plural noun. There is an important difference between propositional and first-order logic on the one hand, and higher-order logics on the other. One cannot consistently extend classical propositional logic with further logical axioms, and the only logical principles one can consistently add to first-order logic with identity make fairly uninteresting statements about how many different things there are. By contrast there are many different ways to consistently extend classical higher-order logic with further purely logical axioms.[2] Although higher-order logics are neutral on questions of mathematical *ontology*—they are formulated without reference to primitive mathematical notions, like number and set membership—some of these logical axioms seem mathematical in nature. There are purely logical statements you can make that settle the continuum hypothesis in the sense that they would imply the continuum hypothesis if there *were* any sets.[3] Similarly, while higher-order logics are neutral about the ontology of propositions, conceived of as certain kinds of abstract individuals, one can formulate purely logical statements that make structural claims about reality: claims that imply that propositions would be structured, if there were any abstract entities that bore an appropriately close relationship to reality (statements like these are explored in chapter 10). One can similarly make purely logical statements that correspond in this way to the existence or non-existence of maximally specific propositions, 'possible worlds', or statements that imply that modal reality admits a lot of different possibilities, or not very many at all, or statements that contradict the $\mathsf{S5}$ axiom for a suitably broad kind of necessity (statements like these are explored in chapter 9). The line between logic, mathematics and metaphysics becomes somewhat blurry: logic can constrain mathematical and metaphysical theorizing in new and non-obvious ways. The wider point here is that there are a great many interesting higher-order logics, and so the subject should be studied with this in mind: it is not the study of a single system, like first-order logic, but rather of a class of interestingly different systems. In this sense the study of higher-order logic is much more like the study of modal logic.

[2] 'Purely logical' here means stateable using only logical words (or logical words with the addition of schematic constants) – in this case the truth-functional connectives and the first and higher-order quantifiers.

[3] In higher-order logic one can formulate a general claim that says that any relation satisfying the principles of Zermelo-Fraenkle set theory when taking the place of the membership relation is one in which the continuum hypothesis holds. Given the non-logical assumption that $\in$ — i.e. *set membership* — satisfies the axioms of Zermelo-Fraenkle set theory, we can infer that the continuum hypothesis is true. See Shapiro (1991) p105 for another formulation of a CH-like principle of higher-order logic.

Part I of the book concerns typed languages. Most languages we are familiar with, such as the language of propositional logic, first-order logic, propositional modal logic, and so on, are implicitly typed languages. There are rules about which expressions can be combined with which — one can apply a predicate to an individual constant, but not to a sentence or another predicate, whereas one can apply operators to sentences, but not to predicates, individual constants, other operators, and so on. We also know how to introduce new devices that behave in grammatically novel ways: for instance we could easily add to first-order logic predicate modifiers, that combine with predicates to make other predicates; then we could introduce predicate modifier modifiers that combine with predicate modifiers to make new predicate modifiers, and so on. Type theory systematizes these rules and a typed language is simply a language which fits this schema (chapter 2). There is a particularly important device, λ, that guarantees that there is a predicate corresponding any open formula parametrized by a variable x, and makes similar guarantees for expressions of other types. Chapter 3 introduces the simply typed λ-calculus: a theory concerning typed languages containing the λ device, governed by two central principles β and η. The chapter also outlines relations between the λ-calculus and a certain variable free alternative to it called combinatory logic. The full λ-calculus is ontologically committal: it contains binary predicate expressions that are converses of other predicates, committing us to (or at least biasing us in favour of) the existence of converse relations. This is not a first-order ontological commitment, but in a higher-order one; λ-languages contain many other types of expressions that are ontologically contentious. Chapters 4 and 5 explore general λ-calculi that do not have these expressions.

Part II of the book concerns a certain kind of typed language that contains, for each grammatical type, devices that stand to that type as the first-order order quantifiers stand to individual terms. They can bind variables of that type in the same way that the first-order quantifiers bind individual variables, and they are subject to logical laws that are completely parallel to the first-order universal and existential quantifiers. We thus call them 'higher-order quantifiers'. Higher-order quantifiers let us express generality in sentence position, predicate position, operator position, and so on, in the same way that the first-order quantifiers express generality into name position. Chapter 6 outlines these quantificational devices in some detail, and chapter 7 explores some of their philosophical consequences. In chapter 8, the key concept of a higher-order logic is introduced.

Part III of the book consists of two extended applications of the machinery of the book to two areas of metaphysics. Chapter 9 is an extended investigation of the higher-order logic Classicism, introduced in chapter 8, and its extensions, in application to questions of in modal metaphysics. In higher-order logic it is possible to define the analogue in reality of an operator expression with a normal modal logic, and many other notions from modal metaphysics—'necessity in the highest degree', entailment, possible world—have a claim to being reducible to pure logic. In chapter 10 the general λ-languages introduced in chapters 4 and 5 are applied to questions about the structure of reality. Various ideas about the structure of reality discussed in the philosophical literature are formalized in higher-order languages, and some limitative results are presented.

Part IV concerns the model theory of higher-order logics. Chapters 11 and 12 introduce the notion of a model for an arbitrary λ-language and the logical language of higher-order logic respectively. Chapter 13 introduces the key notion of a *logical relation*, an important model theoretic tool which can be used to construct partial quotients of models and partial homomorphisms between models, as well as to establish definability and undefinability results. Chapter 14 introduces concepts from category theory that are useful in the study of type theory, and explores three concrete categories – the category of sets, the category of modalized domains and the category of M-sets – which have particular applications to the study of higher-order logic. Chapter 15 investigates the model theory of Classicism in particular and outlines some model theoretic constructions that can be wielded to settle the consistency of several higher-order logics discussed in chapter 9.

This book should certainly not be your introduction to logic. Familiarity with the syntax and model theory of propositional and first-order logic will be assumed. Apart from this, however, there are no hard prerequisites for reading the book. There are parts of the book where the reader would benefit from knowing some elementary modal logic, especially chapter 9, and to a lesser extent chapters 14 and 15. While the book is self-contained in this respect, and defines basic notions like that of a *normal modal logic* and a *Kripke frame*, the book does not attempt to teach these concepts. Chapters 1-4 and 6 of Cresswell and Hughes (1996) will be more than sufficient; there are also routes through the book that do not require any modal logic.

The book is intended to take the reader progressively through some logical and meta-theoretical concepts in higher-order logic. However, chapters do not always depend on all the chapters prior to them, and there are consequently a couple of routes through the book that one can focus on depending on your goals. One route focuses on the higher-order logic *Classicism*, its applications to modal metaphysics and its model theory, and the other on general λ-languages and their applications to structured theories of propositions.

- Classicism and modal metaphysics: Chapters 1 - 3 for the simply typed λ-calculus, 6- 8 for higher-order logic and the system Classicism, chapter 9 investigates what modal notions can be defined in Classicism, the chapters on model theory (chapters 11-15), but especially chapter 15 for the model theory of Classicism.

- Structure: Chapters 1 - 3 for the simply typed λ-calculus, chapters 4-5 for general lambda calculi, 6- 8 for higher-order logic, 10 for applications to structured theories of reality, and the chapters on model theory (chapters 11-14).

Higher-order logic and type theory is a vast subject with practitioners belonging to variety of disciplines: mathematical logic[4], intuitionistic mathemat-

[4]See the references in the next paragraph.

ics[5] and category theory[6], theorem proving[7], the foundations of programming languages[8], linguistics[9], and in philosophy, the philosophy of mathematics[10], semantics and metaphysics[11]. As a result any book on higher-order logic will by necessity contain many omissions. In the case of this book, these omissions are for the most part well-covered elsewhere. The reader may consult the references provided here for further details.

Higher-order logic originated with mathematicians, and for a while higher-order logic, not set theory, was the preferred language for foundational mathematics. Actual higher-order mathematics has a long and prestigious lineage, tracing back to Frege (1879), and continuing with Peano (1889)[12], Whitehead and Russell (1963), Zermelo (1908)[13], Hilbert and Ackermann (1928)[14], Bernays and Schönfinkel (1928), Tarski (1931)[15], Carnap (1947), Church (1940), Kreisel (1967), Henkin (1950), Montague (1965), Montague (2014), Friedman (1975c), Simpson (2009).[16] Opposing the higher-order mathematicians were Skolem and Gödel, who propounded the use of first-order logic in mathematics, an attitude that Quine advocated for more generally in philosophy.[17] I will end these prefatory remarks by briefly drawing attention to some topics in higher-order philosophy that have not been treated as directly in this book as I would have

[5]See especially intuitionistic type theories, originating with Martin-Löf (1972), and continuing recently as homotopy type theory Univalent Foundations Program (2013).

[6]Bell (1982), Lambek and Scott (1988), MacLane and Moerdijk (2012).

[7]See for instance Gordon and Melham (1993), Brown (2007).

[8]Mitchell (1996).

[9]Carpenter (1997), Jacobson (1999), Heim and Kratzer (1998), Barker and Shan (2014).

[10]Linnebo (2013), Fine (2002a), Studd (2013), Scambler (2021), Goodsell (2022) for combine modal and higher-order resources in the philosophy of mathematics. Other papers in the philosophy of logic and mathematics involving higher-order logic include: Burgess (2005), Wright and Hale (2001), Boolos (1984), McGee (1997), Shapiro (1987), Rayo and Williamson (2003), Uzquiano (1999).

[11]See, for instance, Dorr (2016), Bacon (2020), Jones (2018),Trueman (2020), and the citations below.

[12]In Peano's original axiomatization of arithmetic the induction principle was formalized as a single second-order generalization (in modern presentations it is presented in a first-order context by an axiom schema with infinitely many instances). There is more discussion of Frege and Peano in section 2.5.

[13]This paper contained the first axiomatization of modern set theory. Modern mathematicians often use 'Zermelo Fraenkel set theory', or 'ZF', to refer to a first-order theory, however Zermelo's original system was second order.

[14]While Hilbert and Ackermann (1928) is often cited as the first modern treatment of first-order logic, the book covers in its four chapters propositional logic, the monadic predicate logic (a version of propositional logic where the letters are interpreted as monadic predicates), predicate logic, and higher-order logic.

[15]Tarski's original definition of truth, and the original foundational papers in the discipline that would later become model theory were formulated in higher-order languages, as opposed to the first-order language of set theory.

[16]The reader should note that there is a very fuzzy distinction (which I haven' tried to draw) between 'first-order' higher-order mathematics, as it were, and mathematical work on the meta-theory of higher-order logic, exemplified in Church (1940), Henkin (1950). A selection of themes in higher-order mathematics can be found in Bell (2022), Väänänen (2012).

[17]For a good overview of how the default framework in mathematics moved from higher-order logic to first-order logic see Moore (1988).

liked: plural logic[18], ramified type theory[19], the metaphysics of grounding[20], higher-order contingentism[21], and applications of higher-order logic to propositional attitudes[22]. While discussions of these topics are limited, my hope is that this book will equip the reader with the necessary tools explore these topics on their own.

[There will be a paragraph with thanks here.]

[18]Plural logic was an important stepping stone to the rehabilitation of higher-order notions in post-Quinean philosophy; see, e.g., Boolos (1984), Schein (1993).

[19]A venerable list of philosophers have expressed sympathy towards the ramified theory of types, including Bertrand Russell, Alonzo Church, David Kaplan, Saul Kripke, and Harold Hodes. There is a brief discussion of ramified type theories in section 10.1. See Church (1976), Hodes (2015), Hatcher (1982) for a modern presentation of the ramified theory. Some critical discussion can be found in Bacon et al. (2016), Uzquiano (forthcoming).

[20]Higher-order logic is especially pertinent here in relation to the Russell-Myhill paradox, and other puzzles of ground. The reader should consult Fine (2010), Krämer (2013), Fritz (forthcominga), Fritz (forthcomingb), Fritz (2020), Litland (2020), Goodman (2022), Zeng [REF], for a start into this literature.

[21]See for instance Fine (1977), Williamson (2013), Stalnaker (2012), Fritz and Goodman (2016). Further references are given in section 9.10.

[22]These papers include Bacon and Russell (2019), Caie et al. (2020), Yli-Vakkuri and Hawthorne (2021). The literature on Prior's paradox is also relevant here: see Prior (1961), Priest (1991), Rapaport et al. (1988), Tucker and Thomason (2011), Bacon et al. (2016), Bacon (2021), Bacon and Uzquiano (2018), Uzquiano (2021).