CLASS 26, NOVEMBER 12: NAGATA-SMIRNOFF METRIZATION

Today we will explore the relationship between the T3/regularity of a space and its metrizability. It turns out that they are fairly close if we are willing to assume our new condition of being countably locally finite.

Lemma 26.1. *Let X be a topological space which is T3 and has a countably locally finite basis. Then X is T6.*

Proof. It goes to show X is normal and every closed set $Z \subseteq X$ is a G_δ-set.

Let W be an open set. I claim that W can be written as

$$W = \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} \bar{U_i}$$

for some open subsets $U_i \subseteq W$. Divide a basis $\mathcal{B}$ into a countable collection of locally finite subsets $\mathcal{B}_n$. Let $\mathcal{C}_n \subseteq \mathcal{B}_n$ be the subset of elements U which have the property that $\bar{U} \subseteq W$. Define $U_n = \bigcup_{U \in \mathcal{C}_n} U$. By Lemma 25.3 part 3), we have that

$$\bar{U_n} = \overline{\bigcup_{U \in \mathcal{C}_n} U} = \bigcup_{U \in \mathcal{C}_n} \bar{U}$$

So in particular $\bar{U_n} \subseteq W$. On the other hand, $\mathcal{B}$ is a basis of a regular space, so there always exists $x \in U \subseteq \bar{U} \subseteq W$. So equality holds as asserted.

We now show that this implies every closed set $C \subseteq X$ is a G_δ set. Consider $C^c = U$. By the previous part, we know that

$$C = \left(\bigcup_{i=1}^{\infty} \bar{U_n} \right)^c = \bigcap_{i=1}^{\infty} \bar{U_n}^c$$

Finally, it goes to show that X is T4. Let Z, Z' be closed disjoint subsets. By the previous step, we know $Z = \bigcap_{i=1}^{\infty} U_n = \bigcap_{i=1}^{\infty} \bar{U_n}$ and $Z' = \bigcap_{i=1}^{\infty} U_n' = \bigcap_{i=1}^{\infty} \bar{U_n}'$ for some open subsets U_n, U_n'. As a result, we can repeat the proof that a second-countable regular space is normal (Theorem 17.1). That is to say, create a new collection $V_n = U_n \setminus (\overline{U_1'} \cup \ldots \cup \overline{U_n'})$ (similarly V_n') which cover Z, Z' and are disjoint. □

Also recall Homework 6, #5.

Lemma 26.2. *In a normal space, a closed subset $A \subseteq X$ is a G_δ if and only if there is a continuous function $f : X \to [0, 1]$ with $f^{-1}(0) = A$.*

These 2 results combine to give our main theorem of today:

Theorem 26.3 (Nagata-Smirnoff Metrization Theorem)**.** *A topological space X is metrizable if and only if it is regular and has a countably locally finite basis.*

Compare this with homework 6, #7. It generalizes it! NST gives an equivalent condition to metrizable spaces in general, whereas the exercise only gives the compact metric spaces. Furthermore, compact + countably locally finite basis implies second countable.

Proof. ($\Rightarrow$): We know that metric spaces are T0-6, so it only goes to show they are countably locally finite. Consider the basis resulting from our analysis in the exercise; namely

$$\mathcal{B}_n = \{B(x, \frac{1}{n}) \mid x \in X\}$$

By Lemma 25.6 from last class, we have that there exists a $\mathcal{B}'_n$ a refinement which is countably locally finite. Let $\mathcal{B} = \bigcup_{i=1}^{n} \mathcal{B}'_n$ be their union. Since each a countable union of countable sets is countable, we note that this set is also countably locally finite. Furthermore, in the exercise you have shown that this is a basis for the topology.

($\Leftarrow$) : Similar to the proof of Urysohn's Metrization Theorem, we show that the space is metrizable by embedding it into a metric space; namely $(\mathbb{R}^J, \rho)$, where ρ is the uniform metric.

Let $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_n$ be the countably locally finite basis, with each $\mathcal{B}_n$ being locally finite. Lemma 26.1 allows us to conclude X is normal. Therefore, we can apply Urysohn to construct for each $B \in \mathcal{B}_n$ a continuous function

$$f_{n,B} : X \to \left[0, \frac{1}{n}\right]$$

with $f_{n,B}^{-1}(0) = B^c$. Note that this collection of functions separates points from closed sets. Indeed, being a basis ensures there is an open set containing every point x whose complement contains a neighborhood of Z.

Consider $J = \{(n, B) \mid B \in \mathcal{B}_n\} \subseteq \mathbb{N} \times \mathcal{B}$. Then we can define a function

$$F : X \to [0, 1]^J : x \mapsto (f_{n,B}(x))$$

Note that by the the embedding theorem, Corollary 19.5, we know that if we give the right side the product topology this is already an embedding! However, the product topology on an uncountable product of copies of $\mathbb{R}$ is not-metrizable. Therefore, we need to show it is still an embedding under the finer uniform topology. Since F is an open mapping to its image in the product topology, it also true in the uniform topology.

Therefore, it only goes to show that F is continuous. Note that on $[0, 1]$, we may ignore the min; $\rho(x, y) = \sup_{j \in J}(|x_j - y_j|)$. Consider for $f(x) = y$ the open set $B(y, \epsilon) \subseteq [0, 1]^J$. It goes to show that there exists some neighborhood V of x in X such that $F(V) \subseteq B(y, \epsilon)$.

Fix n and choose U_n a neighborhood of x with the locally finite condition of $\mathcal{B}_n$. This implies that there are only finitely many functions with non-zero values on U_n among $f_{n,B}$. Additionally, we can choose $V_n \subseteq U_n$ a neighborhood of x such that the finitely many remaining functions vary by $\leq \frac{\epsilon}{2}$.

Choose $N \gg 0$ such that $\frac{1}{N} \leq \frac{\epsilon}{2}$ and choose $W = V_1 \cap V_2 \cap \cdots \cap V_N$. If $w \in W$, and $n < N$, we have

$$|f_{n,B}(x) - f_{n,B}(w)| \leq \frac{\epsilon}{2}$$

But this inquality is also true if $n > N$, since then $f(w) \leq \frac{1}{n} \leq \frac{\epsilon}{2}$. But this implies $\rho(F(x), F(w)) \leq \frac{\epsilon}{2} < \epsilon$, and therefore the supremum is also bounded above by $\frac{\epsilon}{2}$. $\square$