CLASS 10, SEPTEMBER 30: CAUCHY'S COROLLARIES

Last time to great effect we used Cauchy's integral theorem to prove that if f is holomorphic, then it has infinitely many complex derivatives. Today we will improve this to showing it is in fact analytic! But first, we can show the following inequality.

Corollary 10.1 (Cauchy's inequality)**.** *If f is holomorphic in an open set containing a $\bar{B}(z_0, r)$, and $C = \partial\bar{B}(z_0, r)$, then*

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{r^n}$$

where $\|f\|_C = \sup_{z\in C} |f(z)|$.

Proof. This follows by our usual inequality for an integral via its sup:

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}dw\right| = \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}}re^{i\theta}d\theta\right| \leq \frac{n!}{2\pi}\frac{\|f\|_C}{r^n}2\pi$$

□

This yields an easy numeric method to test the size of a given integral. Now we can get to the amazing result about holomorphic functions being analytic (thus it is an if and only if statement). This is especially surprising since we are assuming a single derivative exists. We have seen an example of a real valued function having infinitely many derivatives but not being analytic on homework 2.

Theorem 10.2. *If f is holomorphic in Ω an open set, then if $\bar{B}(z_0, R) \subseteq \Omega$, then f is analytic at z_0. Furthermore,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

Proof. The last equality follows from our previous analysis of derivatives of power series. Fix $z \in \bar{B}(z_0, r)$. Cauchy then yields

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z_0}dw$$

We can write

$$\frac{1}{w-z} = \frac{1}{w-z_0-(z-z_0)} = \frac{1}{w-z_0}\cdot\frac{1}{1-\frac{z-z_0}{w-z_0}}$$

There exists $0 < r < 1$ such that

$$\left|\frac{z-z_0}{w-z_0}\right| < r$$

Thus the geometric series yields

$$\frac{1}{1-\frac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$$

Here the series converges uniformly for $w \in C$. As a result, we can interchange the sum and the integral in the previous equations:

$$f(z) = \sum_{n=0}^{\infty}\frac{1}{2\pi i}\left(\int_C \frac{f(w)}{(w-z_0)^{n+1}}dw\right)\cdot(z-z_0)^n$$

This proves the result given Cauchy's integral theorem. ☐

These major results also yields quite a great deal about the structure of holomorphic functions. For example, in the complex world, everywhere holomorphic functions which are bounded are necessarily constant. This differs drastically with the real case, where functions like $\tan^{-1}(x)$ exist.

Corollary 10.3 (Louiville's Theorem). *If $f : \mathbb{C} \to B(0, R)$ is an entire function, then f is constant.*

Proof. Given *Corollary* 10.1, we have that

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{r^n}$$

for any $r > 0$. Furthermore, since f is bounded, $\|f\|_C < R$ by our assumption. Therefore, sending r to infinity produces $|f^{(n)}(z_0)| = 0$. Now, given Theorem 10.2, we can conclude f is necessarily constant (equal to $f(z_0)$). ☐

As a final major corollary, we have something that you likely know quite well; the fundamental theorem of algebra. It states that every polynomial factors into linear polynomials over the complex numbers.

Theorem 10.4 (FTOA). *If $p(z) = a_n z^n + \ldots + a_1 z + a_0$ with $n > 0$ and $a_n \neq 0$, then $p(z)$ has a root: there exists z_0 such that $p(z_0) = 0$.*

As a result, $p(z)$ factors as

$$p(z) = a_n(z - z_1) \cdots (z - z_n)$$

for some $z_i \in \mathbb{C}$ potentially repeating.

Proof. Suppose p has no roots. I assert that $\frac{1}{p(z)}$ is a bounded function. To see this, consider

$$\frac{p(z)}{z^n} = a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}$$

As $|z|$ becomes large, say bigger than R, this function from below by $c = |\frac{a_n}{2}|$, thus

$$|p(z)| \geq c|z|^n$$

or equivalent $\frac{1}{|p(z)|} \leq C|z|^{-n} \leq CR^{-n}$. Now, for $|z| < R$, we have a closed disc for which $p(z)$ is never 0. Therefore, $p(z)$ is bounded away from 0, say by r. Immediately, we can conclude $\frac{1}{p(z)}$ is bounded above by $\max\{\frac{1}{r}, CR^{-n}\}$. But by Corollary 10.3, we have that $\frac{1}{p(z)}$ is constant, and therefore so is p. This contradicts our assumption that $n > 0$.

To show the final statement, we could use the division algorithm. But a natural way to realize this is the following: start by factoring out a_n. If z_0 is a root of p, then using the binomial theorem, we can write

$$p(z) = (z - z_0)^n + b_{n-1}(z - z_0)^{n-1} + \ldots + b_1(z - z_0) + b_0$$

Plugging in z_0 yields 0 on the left, so $b_0 = 0$. Therefore, we can write

$$p(z) = p_1(z)(z - z_0) = (z - z_0)^{n-1} + b_{n-1}(z - z_0)^{n-2} + \ldots + b_1$$

Now induction will yield the desired result.

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