CLASS 19, APRIL 5TH: LOCALIZING MODULES

Today we will naturally extend the notion of localization at a multiplicative set to its modules. This has several advantages, reducing aspects of our study to modules over local rings. Begin by recalling the result of Homework 3, #2:

Proposition 1. $\mathrm{Spec}(W^{-1}R) \longleftrightarrow \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \cap W = \emptyset\}$

We can immediately extend this to modules.

Proposition 19.1. *Let Mod_R be the collection of R-modules for a ring R. Then*

$$Mod_{W^{-1}R} = \{M \in Mod_R \mid M \xrightarrow{\cdot w} M \text{ is bijective } \forall w \in W\}$$

Proof. If M is a $W^{-1}R$-module, then it gets the structure of an R-module via the localization map $R \to W^{-1}R : r \mapsto (1, r)$. Also, clearly $\cdot w$ is bijective with inverse $\cdot(w, 1)$. This yields $\subseteq$.

For the reverse, we can give M a $W^{-1}R$-module structure by multiplication $(w, r) \cdot m = r \cdot m'$, where m' is the unique element in the preimage of m under $\cdot w$. $\square$

If M is any R-module, then we can still produce a $W^{-1}R$-module via localization. It is defined analogously to the procedure for rings:

Definition 19.2. The **localization of M at W** is the $W^{-1}R$-module given as

$$W^{-1}M = W \times M / \sim$$

where $(w, m) \sim (w', m')$ if and only if there exists $s \in W$ such that $s(wm' - w'm) = 0$ in M. The multiplicative and additive structure are identical to the case of rings.

I leave it to you to check that this yields a well defined $W^{-1}R$-module, though it is identical to the case of rings. As usual, in the special cases of $W = R \setminus \mathfrak{p}$ and $W = \{1, f, f^2, \ldots\}$, it is common to write $M_{\mathfrak{p}}$ and M_f. We can also localize homomorphisms:

Definition 19.3. If $f : M \to N$ is an R-module homomorphism, its **localization** is the $W^{-1}R$-module map

$$W^{-1}f : W^{-1}M \to W^{-1}N : (w, m) \mapsto (w, f(m))$$

Again, it is natural to check that this is well defined, but simple to do so. This gives us a way to relate localization of modules and exact sequences in a natural way:

Proposition 19.4. *If $M' \xrightarrow{\alpha} M \xrightarrow{\beta} M''$ is an exact sequence of R-modules, then $W^{-1}M' \xrightarrow{W^{-1}\alpha} W^{-1}M \xrightarrow{W^{-1}\beta} W^{-1}M''$is an exact sequence of $W^{-1}R$-modules.*

Proof. For the $\ker(W^{-1}\beta) \supseteq \mathrm{im}(W^{-1}\alpha)$ direction, note

$$W^{-1}\beta(W^{-1}\alpha(w, m')) = (w, \beta(\alpha(m')) = (w, 0) = 0$$

Now suppose $(w, m) \in \ker(W^{-1}\beta)$. This is to say there exists $s \in W$ such that $0 = s\beta(m) = \beta(sm)$ in M''. Thus $sm \in \ker(\beta) = \mathrm{im}(\alpha)$. Take $m' \in M'$ mapping to sm (by exactness of the original sequence). Then if we consider $(sw, m') \in W^{-1}M'$, we have

$$W^{-1}\alpha(sw, m') = (sw, \alpha(m')) = (sw, sm') = (w, m')$$

This demonstrates the $\subseteq$ direction and proves the claim. $\square$

This result is often stated as **localization is an exact functor** and is central to many corollaries regarding localization.

Corollary 19.5. (a) *$W^{-1}(M/N) \cong W^{-1}M/W^{-1}N$ as $W^{-1}R$-modules. In particular, $W^{-1}(R/I) \cong W^{-1}R/W^{-1}I$ as rings!*
(b) *If $M, M' \subseteq N$, then $W^{-1}(M \cap M') = W^{-1}M \cap W^{-1}M'$.*
(c) *Given a module homomorphism $f : M \to N$, then $\ker(W^{-1}f) = W^{-1}\ker(f)$ and $\operatorname{coker}(W^{-1}f) = W^{-1}\operatorname{coker}(f)$. In particular, surjectivity and injectivity are preserved under localization.*

Proof. Most of these results are acquired by applying Proposition 19.4 appropriately:
(a) Localize the sequence $0 \to N \to M \to M/N \to 0$.
(b) The exact sequence of interest is

$$0 \to M \cap M' \to M \to N/M'$$

which yields the localized sequence

$$0 \to W^{-1}(M \cap M') \to W^{-1}(M) \to W^{-1}(N/M') \cong W^{-1}N/W^{-1}M'$$

We can replace $W^{-1}(M \cap M')$ by $W^{-1}M \cap W^{-1}M'$ without changing exactness, so they are isomorphic and thus equal.
(c) Localize the sequences $0 \to \ker(\varphi) \to M \to N$ and $M \to N \to \operatorname{coker}(\varphi) \to 0$.

$\square$

Finally, a neat result which shows that if a module is *locally* zero, then it in fact is zero. One might even say that being 0 is a **local property**.

Proposition 19.6. *If $f : M \to N$ is a map of R-modules such that $f_{\mathfrak{m}}$ is the zero map for every maximal ideal $\mathfrak{m}$, then f was 0 to begin with. In particular, if $M_{\mathfrak{m}} = 0$ for every maximal ideal, then $M = 0$.*

Proof. The first result follows from the second when combined with part (c) of Corollary 19.5. Suppose $m \neq 0$ in M. Then since $1 \cdot m = m \neq 0$, we have that $\operatorname{Ann}_R(m)$ is a proper ideal of R. Let $\mathfrak{m}$ be a maximal ideal containing it. Then $(1, m) \neq 0$ in $M_{\mathfrak{m}}$, since there exists no $s \notin \operatorname{Ann}_R(m) \subseteq \mathfrak{m}$ such that $sm = 0$. $\square$

Corollary 19.7 (Non-local Nakayama II). *If I is an ideal such that*

$$I \subseteq Jac(R) = \bigcap_{\mathfrak{m} \text{ maximal}} \mathfrak{m}$$

and M is a finitely generated module with $M = IM$, then $M = 0$.

$Jac(R)$ is called the **Jacobson Radical** of R.

Proof. Since $I \subseteq \mathfrak{m}$ for each $\mathfrak{m}$, $I_{\mathfrak{m}}$ is a proper ideal of $R_{\mathfrak{m}}$ contained within $\mathfrak{m}R_{\mathfrak{m}}$. Then

$$M_{\mathfrak{m}} \supseteq \mathfrak{m}M_{\mathfrak{m}} \supseteq IM_{\mathfrak{m}} \supseteq M_{\mathfrak{m}}$$

Thus everything is equal. By NL2, we see $M_{\mathfrak{m}} = 0$ for each maximal ideal $\mathfrak{m}$, so Proposition 19.6 yields the desired result. $\square$