HOMEWORK 1: THE COMPLEX PLANE
DUE: WEDNESDAY, SEPTEMBER 18TH

1) Write down a piecewise function to determine the argument of any given complex number $z = a + ib$. Be sure to justify your assertions.

Solution: The desired function is as follows:

$$\theta = \begin{cases} \tan^{-1}\left(\frac{b}{a}\right) & a > 0 \\ \pi + \tan^{-1}\left(\frac{b}{a}\right) & a < 0, b > 0 \\ -\pi + \tan^{-1}\left(\frac{b}{a}\right) & a < 0, b < 0 \\ \frac{\pi}{2} & a = 0, b > 0 \\ -\frac{\pi}{2} & a < 0, b = 0 \\ \pi & a = 0, b < 0 \end{cases}$$

This is easily justified using high school geometry.

2) Verify the assertion that $re^{i\theta} \cdot se^{i\phi} = rse^{i(\theta+\phi)}$ by using the Cartesian representation of a complex number.

Solution: Let $z = re^{i\theta}$ and $w = se^{i\phi}$. Then converting to cartesian coordinates, we have

$$z = r\cos(\theta) + ir\sin(\theta)$$
$$w = s\cos(\theta) + is\sin(\theta)$$

Therefore

$$z \cdot w = rs\left(\cos(\theta)\cos(\phi) - sin(\theta)\sin(\phi)\right) + irs(\cos(\theta)\sin(\phi) + \cos(\phi)\sin(\theta))$$

Using our rules from trig, this can be identified as

$$z \cdot w = rs\left(\cos(\theta + \phi)\right) + irs(\sin(\theta + \phi))$$

Thus we naturally conclude that $|z \cdot w| = rs$ and $Arg(z \cdot w) = \theta + \phi$.

3) Given $w = re^{i\theta}$, solve the equation $z^n = w$ explicitly. How many solutions are there? To simplify matters, you may give your solutions with $Arg(z) \in [0, 2\pi)$ instead of our usual $(-\pi, \pi)$.

Solution: Using the previous result, we can conclude that $|z^n| = |z|^n$ and $Arg(z^n) = nArg(z)$. Therefore, there is a unique choice for $|z| = \sqrt[n]{r}$, and for $Arg(z) = \phi$, we must solve $n\phi = \theta \pmod{2\pi}$. The possibilities are then $\phi = \frac{\theta}{n} + \frac{2\pi j}{n}$ for $j = 0, \ldots, n-1$ if $\theta > 0$, or $j = 1, \ldots, n$ if $\theta < 0$. Thus there are n-many solutions.

However, these values of ϕ do not fit into our $(-\pi, \pi]$ paradigm. To fix this, we would need to convert: Any integer j such that

$$-\pi n < \theta + 2\pi j \leq \pi n$$

will do. Thus the appropriate range is

$$-\frac{\theta}{2\pi} - \frac{n}{2} < j \leq -\frac{\theta}{2\pi} + \frac{n}{2}$$

Some case checking ($\theta > 0$ and $\theta \leq 0$) will allow you to conclude that there are indeed n solutions again.

4) Show that it is impossible to define a total ordering $<$ on $\mathbb{C}$ such that
   1) For any $z, w \in \mathbb{C}$, either $z = w$, $z < w$, or $w < z$.
   2) If $a, b, c \in \mathbb{C}$ and $a < b$, then $a + c < b + c$.
   3) If $a, b, c \in \mathbb{C}$ and $0 < a$, then $b < c$ implies $ab < ac$

   (**hint:** What happens when you consider $0 < i$ and $i < 0$?)

Solution: (Case 1: $0 < i$) By property 3), we may multiply by i and i^3 and conclude that $0 < i^2 = -1$ and $0 < i^4 = 1$. But since $0 < 1$, by property 2) we can add 1 to both sides of $0 < -1$ and conclude that $1 < 0$. This is a contradiction to our assumptions, so Case 1 cannot hold.

(Case 2: $i < 0$) Proceed exactly as in the previous case, but utilizing the fact that $i - i = 0 < -i = 0 - i$.

Since $0 \neq i$, we can conclude that no total ordering can exist.

5) Show that in polar coordinates, the Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad\qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

Therefore, if we define $\log(z) = \log(r) + i\theta$, where $z = re^{i\theta}$, then log is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

Solution: Applying the change of variable formula, we have $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Thus

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial (r\cos(\theta))}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial (r\sin(\theta))}{\partial r} = \frac{\partial u}{\partial x}\cos(\theta) + \frac{\partial u}{\partial y}\sin(\theta)$$

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial v}{\partial x}\frac{\partial (r\cos(\theta))}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial (r\sin(\theta))}{\partial \theta} = -\frac{\partial v}{\partial x}r\sin(\theta) + \frac{\partial v}{\partial y}r\cos(\theta)$$

But the original CR equations yield $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. Therefore, we get the first equation in polar form.

Similarly,

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial (r\cos(\theta))}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial (r\sin(\theta))}{\partial \theta} = -\frac{\partial u}{\partial x}r\sin(\theta) + \frac{\partial u}{\partial y}r\cos(\theta)$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial (r\cos(\theta))}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial (r\sin(\theta))}{\partial r} = \frac{\partial v}{\partial x}\cos(\theta) + \frac{\partial v}{\partial y}\sin(\theta)$$

So again we can conclude identically the second equation holds.

As a final step, utilizing Theorem 4.1 in the notes (or 2.4 in the book), we can easily check that $u = \log(r)$ and $v = i\theta$ satisfy

$$\frac{\partial \log(r)}{\partial r} = \frac{1}{r} = \frac{1}{r}\frac{\partial \theta}{\partial \theta}$$

$$\frac{\partial \log(r)}{\partial \theta} = 0 = \frac{1}{r}\frac{\partial \theta}{\partial r}$$

Moreover, each of these functions is continuously differentiable, so Theorem 4.1 implies holomorphicity.

6) Show that the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

while acting on twice continuously differentiable functions satisfies the following equality:

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$$

Why is this assumption necessary? Conclude that if f is holomorphic (with this property), then the real and imaginary parts are **harmonic**. That is to say $\Delta f = 0$.

Solution: Plugging in our definition of the differentials yields

$$4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + i\left(-\frac{\partial}{\partial y}\frac{\partial}{\partial x} + \frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)$$

$$4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + i\left(\frac{\partial}{\partial y}\frac{\partial}{\partial x} - \frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)$$

The twice continuously differentiable assumption ensures that $\frac{\partial}{\partial x}\frac{\partial}{\partial y} = \frac{\partial}{\partial y}\frac{\partial}{\partial x}$ (Clairaut's Theorem)[1].

If $f = u + iv$ is holomorphic, then it satisfies the CR equations. As a result,

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}u = \frac{\partial}{\partial x}\frac{\partial}{\partial y}v = \frac{\partial}{\partial y}\frac{\partial}{\partial x}v = -\frac{\partial}{\partial y}\frac{\partial}{\partial y}u$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial x}v = -\frac{\partial}{\partial x}\frac{\partial}{\partial y}u = -\frac{\partial}{\partial y}\frac{\partial}{\partial x}u = -\frac{\partial}{\partial y}\frac{\partial}{\partial y}v$$

7) Define a function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = f(x + iy) = \sqrt{|x||y|}$$

Show that although f satisfies the Cauchy-Riemann equations, f is not holomorphic at 0.

Solution: f has only a real part, so it suffices to check that $\frac{\partial}{\partial x}\sqrt{|x||y|}$ and $\frac{\partial}{\partial y}\sqrt{|x||y|}$ are both 0 at the origin. However, this is clear, since $\sqrt{|x||y|}$ is the zero function on the x and y axes!

However, it is clear that this function cannot be holomorphic. Approaching along the line $y = x$, $f(x, y) = |x|$, which is not differentiable.

[1]We will later show that this assumption is unnecessary.