HOMEWORK 4: NAKAYAMA'S LEMMA AND REGULARITY
DUE: WEDNESDAY, APRIL 11

1) Let M and N be finitely generated modules over a ring R. Show that $M \otimes_R N = 0$ if and only if $ann_R(M) + ann_R(N) = R$. In addition, if we assume R is local, show that $M \otimes_R N = 0$ implies either $M = 0$ or $N = 0$.

2) Show that $Jac(R)$, the Jacobson radical of R, can be characterized as
$$Jac(R) = \{r \in R \mid 1 + rr' \text{ is a unit for every } r' \in R\}$$

3) Let $(R, \mathfrak{m})$ be a local ring, and $I \subseteq R$ an ideal. Suppose that x is an element of $\mathfrak{m}$ is such that its image in R/I is a non-zero divisor. Show that a minimal generating set of I is also a minimal generating set of $I \cdot R/\langle x \rangle$. Give an example to show this is not true when x is allowed to be a zero divisor of R/I.[1]

4) Let R be a ring, and S an R-algebra. Finally, let M be a finitely generated S module. Show that if S is finite as an R-module, then M is finitely generated as an R-module.

5) Suppose that I is a nilpotent ideal (e.g. $I^n = 0$ for some $n \gg 0$. Show that if $M = IM$ for some (not necessarily finitely generated module M), then $M = 0$.[2]

6) Consider the ring
$$R = K[x_{ij}]_{i \leq j} = K[x_{11}, x_{12}, x_{22}, x_{13}, \ldots]$$
and $W = R \setminus \langle x_{11} \rangle \cup \langle x_{12}, x_{22} \rangle \cup \langle x_{13}, x_{23}, x_{33} \rangle \cup \ldots$ Show that $W^{-1}R$ is a regular Noetherian ring, but is not finite dimensional.[3]

7) Show that if R is a local ring, and I is a proper ideal such that $gr_I(R)$ is an integral domain, then R is as well.

8) Let R be a Noetherian ring, and let I be an ideal and M a finitely generated module. Show that there is a largest submodule $N \subseteq M$ where $(1 - r)N = 0$ for some $r \in I$. Then show that $\cap_{n \geq 0} I^n M = N$.[4]

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[1] Recall a minimal generating set is one in which removing any element stops generation.

[2] This gives a nice extension of Nakayama to the non-finitely generated case.

[3] This is also an example of a non-equidimensional ring.

[4] This provides a partial converse to Krull's Intersection Theorem.