CLASS 23, WEDNESDAY APRIL 25: KUNZ THEOREM II

It remains to show that the other direction of Kunz Theorem holds (aka the hard direction). Just to recall:

Theorem 0.1 (Kunz's Theorem)**.** *Let R be an F-finite ring. Then R is regular if and only if F_*R is a flat module.*

We have proved the only if direction of this Theorem already. What remains to prove is if F_*R is a flat module, then R is regular. The original proof due to Kunz was lengthy and chased elements around. The proof I will exhibit here uses more advanced techniques with the result of a shortened proof.

Definition 0.2. If R is a ring of characteristic $p > 0$, then we can consider the sequence

$$R \to F_*R \to F_*^2R \to \dots$$

The **perfection** of a ring, denoted R^∞ or $F_*^\infty R$, is the direct limit of this sequence. A ring is called **perfect** if $R = R^\infty$.

If R is reduced, we have that each map above is injective and $F_*^\infty R = \cup_{e\geq 0} F_*^\infty R$. Otherwise, non-reduced elements are set to zero in R^∞, and therefore if $R^\infty = (R/\mathcal{N})^\infty$ satisfies the previous statement. Note that perfections are often Non-Noetherian.

Example 0.3. If $R = K[x]$ with K perfect, then $R^\infty = K[x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, x^{\frac{1}{p^3}}, \dots]$. We have the naturally ascending chain of ideals which never stabilizes:

$$0 \subsetneq \langle x\rangle \subsetneq \langle x^{\frac{1}{p}}\rangle \subsetneq \langle x^{\frac{1}{p^2}}\rangle \subsetneq \dots$$

A very similar statement holds for $R = K[x_1, \dots, x_n]$.

We can use the following (recent) theorem of Bhatt-Scholze to prove the desired theorem:

Lemma 0.4. *If $S \to R$ and $S \to R'$ are surjections of rings of characteristic $p > 0$, then there are induced surjections $S^\infty \to R^\infty$ and $S \to R'^\infty$. For all $i > 0$, $\mathrm{Tor}_i^{S^\infty}(R^\infty, R'^\infty) = 0$. As a result,* $\mathrm{Hom}_{R^\infty}(M, N) \cong \mathrm{Hom}_{S^\infty}(M, N)$.

They actually prove this result for arbitrary perfect rings, not perfections of rings. This Lemma allows us to prove the desired result. For a proof, consult the professor privately.

Proposition 0.5. *If R is a complete Noetherian local ring, and R^∞ is its perfection, then R^∞ has finite global dimension.*

Proof. Given the assumptions, $R \cong K[\![x_1, \dots, x_n]\!]/I$. Let M be an R^∞-module. Then the proposition implies

$$M \cong M \otimes_{R^\infty} R^\infty \cong M \otimes_{R^\infty} (R^\infty \otimes_{S^\infty} R^\infty) \cong (M \otimes_{R^\infty} R^\infty) \otimes_{S^\infty} R^\infty \cong M \otimes_{S^\infty} R^\infty$$

Now, M is an S^∞-module. Furthermore, we may consider M as an F_*^eS-module, since $F_*^eS \to S^\infty$. Furthermore, if we let $M_e = M \otimes_{S^{\frac{1}{p^e}}} S^\infty$, we have maps given by increasing the amount of linearity:

$$M \otimes_S S^\infty \to \dots \to M \otimes_{S^{\frac{1}{p^e}}} S^\infty \to M \otimes_{S^{\frac{1}{p^{e+1}}}} S^\infty \to \dots \to M$$

and that M is the direct limit of M_e. This is because every element of S^∞ is in some $F^e_* S$.

Now, since M is an $S^{\frac{1}{p^e}}$-module, and $S^{\frac{1}{p^e}}$ is a regular ring of dimension n (thus has global dimension n), we have $\operatorname{pdim}_{S^{\frac{1}{p^e}}}(M) \leq n$. However, tensoring the sequence with S^∞ implies that $\operatorname{pdim}_{S^\infty} M_e \leq n$.

Finally, consider the module $M_+ = \oplus_{e \geq 0} M_e$. We can create an endomorphism φ of M_+ by sending $a_e \in M_e$ to $(a_e, -a_e) \in M_e \oplus M_{e+1}$ extended by linearity. The cokernel of φ is isomorphic to M. Therefore, the projective dimension of M is bounded by $n+1$. But M was arbitrary, so gl-dim$(S^\infty) \leq n+1$. This completes the proof. □

Given the technique of the previous lecture, the assumption of Kunz Theorem is that $F_* R$ is a flat R-module. But this naturally implies $F_*(F_* R) = F^2_* R$ is a flat $F_* R$-module, and thus a flat R-module. Therefore, $F^e_* R$ is a flat R-module for any $e \geq 0$. Now, the direct limit is an exact functor, so we can conclude that since $F^e_* R$ is a flat R-module for all e, so too is R^∞. This allows us to conclude the proof of Kunz.

Theorem 0.6. *If R is a complete Noetherian local ring of characteristic $p > 0$ and R^∞ is a flat R-module, then R is a regular ring.*

Note that this is enough to conclude the proof by Cohen's Structure Theorem.

Proof. Note that by Proposition 0.5, we have that every R^∞-module has finite projective dimension. Our assumption implies that $R \to R^\infty$ is faithfully flat, which implies $M \otimes_R R^\infty \neq 0$ if $M \neq 0$.

Let d be the global dimension of R^∞. Suppose that M is a module with projective resolution

$$\ldots \to P_{n+1} \to P_n \to \ldots \to P_0 \to M \to 0$$

Applying $\operatorname{Hom}_R(-, N)$,

$$\ldots \xleftarrow{\psi_{n+1}} \operatorname{Hom}_R(P_{n+1}, N) \xleftarrow{\psi_n} \operatorname{Hom}_R(P_n, N) \xleftarrow{\psi_{n-1}} \ldots \xleftarrow{\psi_0} \operatorname{Hom}_R(P_0, N) \leftarrow 0$$

Then gl-dim(R) is the largest (or ∞ if no finite one works) i such that $\ker(\psi_i)/\operatorname{im}(\psi_{i-1}) \neq 0$. This is most commonly called $\operatorname{Ext}^i_R(M, N)$. So assume the desired statement is false, and there is $i > d$ such that $\operatorname{Ext}^i_R(M, N) \neq 0$. Then by flatness,

$$\operatorname{Ext}^i_R(M, N) \otimes_R R^\infty = \operatorname{Ext}^i_{R^\infty}(M \otimes_R R^\infty, N \otimes_R R^\infty)$$

But the extension is faithful, so this is no zero, contradicting the global dimension of R^∞. This completes the proof. □