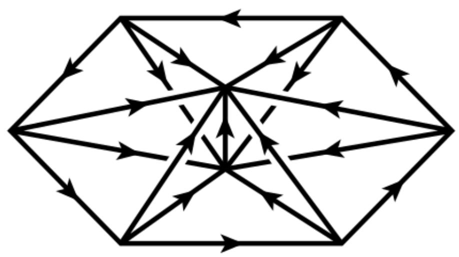
HOMEWORK 8: HOMOLOGY
DUE: MONDAY, NOVEMBER 27

1) Compute the simplicial homology groups of the Δ-complex of the space X obtained by identifying all faces of the same dimension of an n-simplex Δ^n via the identity map. The resulting space has one k-simplex in each dimension $k = 0, 1, \ldots, n$.

2) Analyze the simplicial structure of a 3-dimensional space L_n obtained by n-many Δ^3, say $\Delta^3_1, \ldots, \Delta^3_n$, with Δ^3_i and Δ^3_{i+1} sharing a face for $i = 1, \ldots, n$. Then identify the bottom 2-simplex of Δ^3_i with the top 2-simplex of Δ^3_{i+1}. Note that $i = n$ yields $\Delta_{i+1} = \Delta_i$. Compute the homology group $H^\Delta_3(X)$.



You should find $H^\Delta_3(L_n) = \mathbb{Z}$. One can also find with some clever book keeping that $H^\Delta_0(L_n) = \mathbb{Z}$, $H^\Delta_1(L_n) = \mathbb{Z}/n\mathbb{Z}$, and $H^\Delta_2(L_n) = 0$.

3) Determine whether there exists a short exact sequence

$$0 \to \mathbb{Z}/4\mathbb{Z} \to \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/4\mathbb{Z} \to 0$$

More generally (and optionally), conjecture which Abelian groups A fit into short exact sequences

$$0 \to \mathbb{Z}/p^m\mathbb{Z} \to A \to \mathbb{Z}/p^n\mathbb{Z} \to 0$$
$$0 \to \mathbb{Z} \to A \to \mathbb{Z}/n\mathbb{Z} \to 0$$

4) Show that $H_0(X, A) = 0$ if and only if for every path connected component $X_\alpha \subset X$, $X_\alpha \cap A \neq \emptyset$.

In addition, show that $H_1(X, A) = 0$ if and only if $H_1(A)$ surjects onto $H_1(X)$ and $A \cap X_\alpha$ is either empty or path-connected.

5) Compute the homology groups $H^\Delta_n(X)$ of the subspace

$$X = \{(x, y) \in I^2 \ : \ x \in \mathbb{Q}\} \cup [0, 1] \times \{0, 1\} \subset I^2 = [0, 1]^2$$

That is to say, the boundary of the square union all vertical lines at rational values.

6) Compute $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus x)$. You may assume excision and homotopy invariance. Use this to show that

$$\mathbb{R}^n \cong \mathbb{R}^m \ \Leftrightarrow \ n = m$$