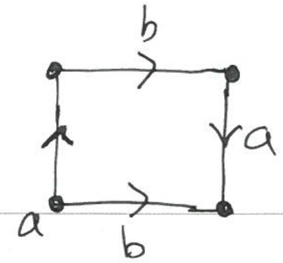
Start

Ex : Klein Bottle 

$X^0 = \cdot$

$X^1 =$ 

$\pi_1(X^1) = \mathbb{Z} * \mathbb{Z}$

$\mathbb{RP}^2 = X^2 : e^2 = a\,b\,a\,b^{-1} \quad \pi_1(X^2) = \pi_1(X^1)/N$

$= \mathbb{Z} * \mathbb{Z} / N = \langle a, b \mid aba = b \rangle$

A note about General Groups G.

Oct 25: Covering Spaces: We want a way to find data of a space X by consitering (topologically) ~~...~~ simpler spaces.

Defn: A covering space of a space X is a map $p : \tilde{X} \to X$ with the following prop:

For each $x \in X$, $\exists\, \mathcal{U} \ni x$ open w/ $p^{-1}(\mathcal{U}) = \coprod_\alpha \mathcal{U} \subseteq \tilde{X}$. and

$\coprod_\alpha \mathcal{U} \longrightarrow \mathcal{U}$ is "$u \mapsto u$"

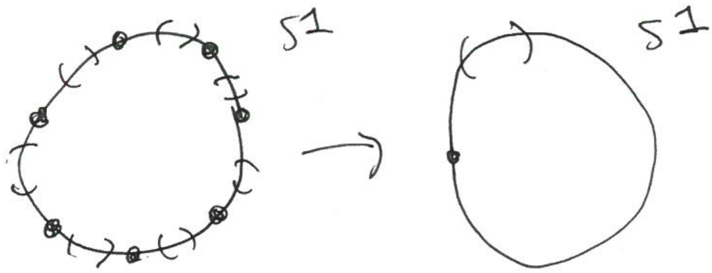
Such a $\mathcal{U}$ is called evenly covered, and each $\mathcal{U} \in \coprod_\alpha \mathcal{U}$ is called a sheet of the covering space.

The number of sheets is locally a constant on X (it is on $\mathcal{U}$), so it is constant on a connected comp of X.

Ex/ S^1: There are a lot of covering spaces. For each n,

$$p_n : S^1 \to S^1 : \theta \mapsto n\theta$$

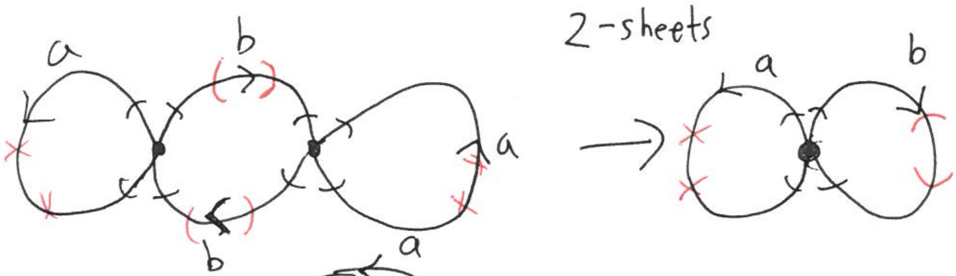
$$p : \mathbb{R} \to S^1$$

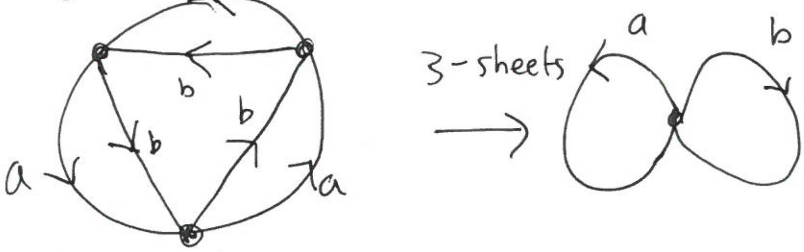


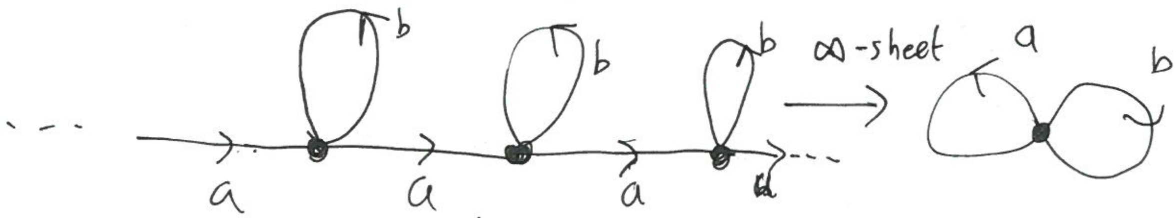
p_n has n-sheets.

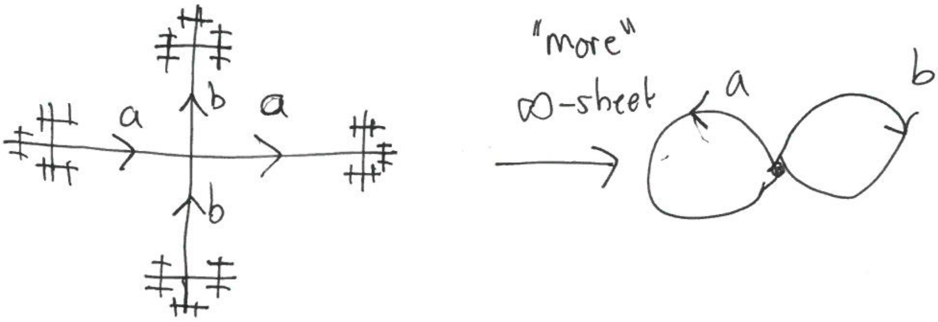
$\mathbb{R}$?

Ex/ $S^1 \vee S^1$:









$\mathbb{R}P^n = S^n/\sim$

Some nice things: • Open covers have cov sp realization
Locally finite

• If $p:\tilde{X}\to X$ and $q:\tilde{Y}\to Y$ are covers so is $p\times q$

• If $p:\tilde{X}\to\hat{X}$ and $q:\hat{X}\to X$ are CS, then so is $q\circ p$.

• p_* is injective (To be shown later)

• Much like

$$\begin{tikzcd} Y\times\{0\} \arrow[r] \arrow[d, hook] & \mathbb{R} \arrow[d] \\ Y\times I \arrow[r, "\varphi"] \arrow[ur, dashed, "\tilde{\varphi}"] & S^1 \end{tikzcd}$$

$$\begin{tikzcd} 0\times Y \arrow[r] \arrow[d, hook] & \tilde{X} \arrow[d, "p"] \\ I\times Y \arrow[r] \arrow[ur, dashed] & X \end{tikzcd}$$

• Good correspondence with $\pi_1(X)$

Oct 27: Homotopy lifting property:

Prop: Given a covering space $p:\tilde{X}\to X$, and a homotopy $f_t:Y\to X$ w/ $\tilde{f}_0:Y\to\tilde{X}$, then $\exists!\ \tilde{f}_t:Y\to\tilde{X}$ w/
$$p\circ\tilde{f}_t = f_t$$

$$\begin{tikzcd} Y\times\{0\} \arrow[r] \arrow[d, hook] & \tilde{X} \arrow[d, "p"] \\ Y\times I \arrow[r] \arrow[ur, dashed, "\exists!"] & X \end{tikzcd}$$

Pf: Same as the proof that homotopies lift from $X=S^1$ to $\tilde{X}=\mathbb{R}$.

If Y is a point, then we get the path lifting property. Thus, constant loops lift to constant loops.

If $Y = I$, we see that homotopies of loops lift. So if γ_0, γ_1 are homotopic loops in X, $\tilde{\gamma}_0, \tilde{\gamma}_1$ lifts to $\tilde{X}$ w/ same basepoint, then $\tilde{\gamma}_0 \simeq \tilde{\gamma}_1$.

Here is an application: <u>Prop</u>: If $p: \tilde{X} \to X$ is a covering space, then $\pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{p_*} \pi_1(X, x_0)$ is injective. The image of p_* is homotopy classes of loops in X @ x_0 lifting to loops in $\tilde{X}$ at $\tilde{x}_0$.

Pf: Suppose $p_*: \tilde{\gamma} \mapsto c_{x_0}$. Then

$$p \circ \tilde{\gamma} \simeq c_{x_0}$$

but homotopies of loops lift, so

$$\tilde{\gamma} \simeq c_{\tilde{x}_0} \Rightarrow [\tilde{\gamma}] = c_{\tilde{x}_0}.$$

Let's consider the image. $\mathrm{Im}(p_*) \subseteq$ loops $\subseteq X$ lifting to loops in $\tilde{X}$. Similarly if γ lifts to a loop $\tilde{\gamma}$, $p_* \tilde{\gamma} = \gamma$.

Prop: If X is connected, and $p: \tilde{X} \to X$ is a cov. sp, then

$$|p^{-1}(x_0)| = \#\text{ Sheets of } p = [\pi_1(X) : p_*\pi_1(\tilde{X})]$$

Exs/ $p_n: S^1 \to S^1$ & $p: \mathbb{R} \to S^1$

Pf: For $g: I \to X$ a loop based @ x_0, let $\tilde{g}$ be its lift to $\tilde{X}$ (a path) @ $\tilde{x}_0$. If

$h \in p_*(\pi_1(\tilde{X}, x_0))$, then $h \cdot g$ lifts to $\tilde{h} \cdot \tilde{g}$
(to avoid composing paths @ different points.
Thus, we can define

$$\Phi : \underbrace{\pi_1(X) : p_*\pi_1(X)}_{\text{cosets}} \longrightarrow \text{Sheets}, \qquad p_*\pi_1(X) = H$$

$$\underbrace{p_*\pi_1(X) g}_{H \cdot g} \longmapsto \tilde{g}(1)$$

X: Path connected $\Rightarrow \Phi$ is surjective

Φ injective: If $\Phi(H \cdot g) = \tilde{x}_0$, then $\tilde{g}(1) = x_0$
$\Rightarrow \tilde{g}$ is a loop @ $\tilde{x}_0 \Rightarrow g \in H$.

More Generally, it is interesting to see when maps lift at all (indep of the map $\tilde{f}_0$).

Lifting criterion: Suppose $p: (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a CS and $f: (Y, y_0) \to (X, x_0)$. Then

$$\exists \tilde{f}: (Y, y_0) \to (\tilde{X}, x_0) \Leftrightarrow f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$$
w/ $p \circ \tilde{f} = f$

$$\begin{tikzcd} & \tilde{X} \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed, "\tilde{f}"] & X \end{tikzcd} \iff \begin{tikzcd} & \pi_1(\tilde{X}, x_0) \arrow[d, "p_*"] \\ \pi_1(Y, y_0) \arrow[r, "f_*"] & \subseteq \pi_1(X, x_0) \end{tikzcd}$$

Pf: "⇒" $p_* \circ \tilde{f}_* = f_*$

$\Rightarrow \subseteq$

"⇐" Let γ be a path from y_0 to y.

Then $f_*(\gamma)$ at x_0 lifts to $\widetilde{f_*\gamma} \in \pi_1(\tilde{X}, x_0)$

Let $\tilde{f}(y) = \widetilde{f_*\gamma}(y)$.

Well-defined: Suppose γ' is another path.

Then $(f_*\gamma')\cdot(\overline{f_*\gamma})$ is a loop h_0 at x_0. Thus $\exists\, h_t$ a homotopy of h_0 to h_1 lifting to $\tilde{h}_1$ in $\tilde{X}$ based at $\tilde{x}_0$. $\Rightarrow \tilde{h}_t$ is a lift. $\Rightarrow \tilde{h}_0$ is a loop at x_0, since $\tilde{h}_1$ is. Thus

$$\tilde{h}_0 = (\widetilde{f_*\gamma'})\cdot(\overline{\widetilde{f_*\gamma}}) \underset{\tilde{X}}{\Rightarrow} \tilde{\gamma}'(1) = \overline{\tilde{\gamma}(0)} = \tilde{x}_0$$

$\tilde{f}$ is also continuous because $\tilde{X}$ is a local homeo to X.

$$\tilde{f}^{-1}(\tilde{\mathcal{U}}) = \tilde{f}^{-1}\Big(\bigcup_\alpha \tilde{\mathcal{U}}_\alpha\Big)$$

$$= \bigcup_\alpha \tilde{f}^{-1}(\tilde{\mathcal{U}}_\alpha)$$

$$= \bigcup_\alpha f^{-1}(\mathcal{U}_\alpha)$$

$$\begin{array}{c} \tilde{X} \supseteq \mathcal{U}_\alpha \\ \downarrow \cong \\ X \supseteq \mathcal{U}_\alpha \end{array}$$

Prop: Additionally, if two lifts agree at 1 point, they agree everywhere.