HOMEWORK 10: ASCOLI & HOMOTOPY
DUE: DECEMBER 7

1) Suppose X_n is metrizable with metric d_n. Show that $X = \prod_{i=1}^{\infty} X_n$ is metrizable:

$$D(x, y) = \sup_i \left(\frac{\min\{1, d_i(x_i, y_i)\}}{i} \right)$$

Also, show that if each X_n is totally bounded, then so is X. As a result, without' Tychonoff, we have a countable product of compact metric spaces is a compact metric space.

Solution: It was already shown that X is metrizable by the claimed metric (at least in the case of $\mathbb{R}^n$. It is reiterated here for convenience.

Given $x \in X$ and $\epsilon > 0$, then

$$B(x, \epsilon) = B(x_1, \epsilon) \times \ldots \times B(x_n, n\epsilon) \times \prod_{i=n+1}^{\infty} X_i$$

where n is chosen minimally such that $(n+1)\epsilon \geq 1$. This is open in the product topology.

On the other hand, for $x \in U_1 \times \ldots \times U_n \times \prod_{i=n+1}^{\infty} X_i$, choose $0 < \epsilon \ll 1$ such that $B(x_i, i\epsilon) \subseteq U_i$. Then

$$x \in B(x, \epsilon) \subseteq U_1 \times \ldots \times U_n \times \prod_{i=n+1}^{\infty} X_i$$

This shows they are topologically equivalent.

Suppose each X_i is totally bounded. Then for fixed $\epsilon > 0$, choosing $x_j \in X_i$ such that $B(x_j, i \cdot \epsilon)$ cover X_i for $j \in \Lambda_i$ a finite set. Then for N such that $N\epsilon > 1$, then

$$\{B(x, \epsilon) \mid x_i \in \Lambda_i,\ i = 1, \ldots, N\}$$

is a finite covering. This completes the proof.

2) Show *Arzela's Theorem*: If X is compact, and $f_n \in C(X, \mathbb{R}^m)$ is a sequence of equicontinuous and pointwise bounded functions, then f_n has a uniformly convergent subsequence.

Solution: The collection $\{f_n\}$ is equicontinuous and pointwise bounded by assumption. Therefore, by Ascoli, we can conclude that $\overline{\{f_n\}}$ is a compact set in $C(X, Y)$. Therefore, f_n is a sequence in a (sequentially) compact space, and therefore has a convergent subsequence f_{n_i} to let's say f.

The question then remains what does it mean to converge in the uniform topology? For any $\epsilon > 0$, there exists $N \gg 0$ such that for $n \geq N$, we have

$$\rho(f, f_n) = \sup\{d(f_n(x), f(x))\} < \epsilon$$

This implies it is uniformly convergent to f, and is why it is called the uniform topology.

3) Show that if $f : X \to Y$ is a continuous function, and there exist $g, h : Y \to X$ such that $f \circ g \simeq Id_Y$ and $h \circ f \simeq Id_X$, then $X \simeq Y$.

Solution: This follows from from the following chain:

$$g = g \circ Id_Y \simeq g \circ (f \circ h) = (g \circ f) \circ h \simeq (Id_X) \circ h = h$$

The result now follows by realizing that

$$g \circ f \simeq h \circ f \simeq Id_X$$

4) X is contractible if $X \simeq x$, where x is representative of a point with its unique topology. Show that if Y is contractible, then every map $X \to Y$ is homotopic to one another. If X is path connected, show the same is true for functions $Y \to X$.

Solution: Suppose $f : X \to Y$ is a a map. Let $F : Y \times I \to Y$ be the contraction map, where $F(y, 0) = y$ and $F(y, 0) = y_0$ for some fixed $y_0 \in Y$. Then the homotopy

$$G : X \times I \to Y : (x, t) \mapsto F(f(x), t)$$

Then $G(x, 0) = f(x)$ and $G(x, 1) = y_0$. This shows every map is equivalent to the constant map, and $\simeq$ is an equivalence relation, completing the proof.

Next, assume X is path connected. Suppose $f, g : Y \to X$ are such that $f(y_0) = x$ and $g(y_0) = x'$. Let $\gamma : I \to X$ be the path connecting x to x'. Then the homotopy

$$G : Y \times I \to X : (y, t) \mapsto \begin{cases} f(F(y, 3t)) & t \leq \frac{1}{3} \\ \gamma(3t - 1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ g(F(y, 3 - 3t)) & t \geq \frac{2}{3} \end{cases}$$

This is continuous by the pasting lemma:

$$f(F(y, 1)) = f(y_0) = x = \gamma(0) \qquad g(F(y, 1)) = g(y_0) = x' = \gamma(1)$$

and $G(y, 0) = f(y)$, and $G(y, 1) = g(y)$, as desired.

5) Show that if $r : Y \to Z$ is a retraction (cf homework 7), then the induced map

$$r : \pi_1(Y, y) \to \pi_1(Z, r(y)) : \gamma \mapsto r \circ \gamma$$

is surjective.

Solution: Note that $(r \circ i)_* = r_* \circ i_*$ (simply by definition). But $r \circ i = Id_Z$, and $(Id_Z)_* = Id_{\pi_1(Z, r(y))}$. Therefore, when $f = g \circ h$, then f surjective implies g is as well. Thus r_* is surjective.

6) Suppose $Y \subseteq \mathbb{R}^n$, and $f : Y \to Z$ is a continuous map. Show that if f extends to a map from $\tilde{f} : \mathbb{R}^n \to Z$, then the induced map $f_* : \pi_1(Y, y) \to \pi_1(Z, z)$ is 0.

Solution: Similarly to the previous problem, if $f = \tilde{f} \circ \iota$, then $f_* = \tilde{f}_* \circ \iota_*$. But note that

$$\iota_* : \pi_1(Y, y) \to \pi_1(\mathbb{R}^n) = 0$$

Therefore $f_*(\gamma) = \tilde{f}_*(e) = e$.

7) Show that $\pi_1(X, x_0)$ is abelian if and only if for every 2 paths γ_0, γ_1 connecting x_0 to x_1, the change of base point maps

$$\pi_1(X, x_0) \to \pi_1(X, x_1) : \sigma \mapsto \bar{\gamma}_i * \sigma * \gamma_i$$

are equal as group homomorphisms.

Solution: $(\Rightarrow)$: To be equal as group homomorphisms is to say

$$\bar{\gamma}_0 * \sigma * \gamma_0 \simeq \bar{\gamma}_1 * \sigma * \gamma_1$$

This is equivalent to saying that (using group inversion)

$$\gamma_1 * \bar{\gamma}_0 * \sigma * \gamma_0 * \bar{\gamma}_1 \simeq \sigma$$

Applying associativity and Abelianicity, we see that after noting $\gamma_1 * \bar{\gamma}_0$ and $\gamma_0 * \bar{\gamma}_1$ are loops based at x_1,

$$\gamma_1 * \bar{\gamma}_0 * \sigma * \gamma_0 * \bar{\gamma}_1 \simeq \sigma * \gamma_1 * \bar{\gamma}_0 * \gamma_0 * \bar{\gamma}_1 \simeq \sigma$$

as asserted.

$(\Leftarrow)$: Suppose $\gamma * \gamma' \not\simeq \gamma' * \gamma$. This is equivalent to

$$\gamma \not\simeq \gamma' * \gamma * \bar{\gamma}'$$

But this is literally the change of base by $\bar{\gamma}'$, a contradiction.