CLASS 13, MONDAY MARCH 12TH: INJECTIVE MODULES

Now that we have studied projective modules, we will also study their dual notion: **injective modules**. These play a very important role in homological algebra, since many (left-exact) functors behave well with an injective resolution.

Definition 0.1. A module I is said to be **injective** if whenever $\varphi : M \to N$ is an injective homomorphism, and $\psi : M \to I$ is any homomorphism, there exists $\psi' : N \to I$ such that $\psi = \psi' \circ \varphi$.

Note the similarity to projectives, except the arrows are facing the opposing way. Similar to how a module P is projective if and only if $\mathrm{Hom}_R(P, -)$ is an exact functor, a module I is injective if and only if $\mathrm{Hom}_R(-, I)$ is an exact functor (note that arrows are flipped by this functor. This notion is called **contravariance**).

Checking something is injective seems like quite a chore; you need to check that every injection of R-modules satisfies a given property. However, a theorem of Baer allows this condition to be relaxed:

Theorem 0.2 (Baer's Criterion). *A module I is injective if and only if for every ideal J of R, and every $\psi : J \to I$, there is a $\psi' : R \to I$ such that $\psi = \psi' \circ \iota$ where $\iota : J \hookrightarrow R$ is the inclusion.*

Proof. The $\Rightarrow$ direction of this theorem is obvious (if it holds for all module inclusions, it certainly holds for a subset of them!)

So it goes to prove the $\Leftarrow$ direction. Suppose $\varphi : M \hookrightarrow N$ and $\psi : M \to I$. Let $\mathcal{S}$ be the set of submodules N' of N together with $\psi' : N' \to I$ such that $\psi = \psi' \circ \varphi$. This is a non-empty set, since it certainly contains $\varphi(M) \subseteq N$. We can put a partial ordering on this set by taking $(N', \psi') \leq (N'', \psi'')$ if $N' \subseteq N''$ and $\psi' = \psi''|_{N'}$. We can take the union of an ascending chain to produce a module and map in $\mathcal{S}$, so Zorn's Lemma applies and therefore there is a maximal element of $\mathcal{S}$, call it (N_0, ψ_0). If $N_0 = N$, we are done. If not, take $x \in N \setminus N_0$. Let

$$J = \{r \in R \ : \ rx \in N_0\} \subseteq R$$

J is an ideal of R, and we can make $g : J \to I : r \to \psi_0(rx)$. So we can apply the assumption: there is a map $g' : R \to I$ factoring ψ_0. But this produces a map

$$\psi_1 : N_0 + xR \to I : n + rx \to \psi_0(n) + g'(r)$$

from a strictly larger module of $\mathcal{S}$, contradicting maximality and proving the result. □

For a general ring, the proof of this theorem shows that if I is an injective R-module, then I is **divisible**: $r \cdot I = I$ for every $r \in R$ a NZD. A nice converse can be realized in the case of principal ideal domains:

Corollary 0.3. *If R is a PID, then I is an injective module if and only if I is divisible.*

Proof. We only need to prove the $\Leftarrow$ direction. Using Baer's criterion, we know I is injective if and only if the condition holds for ideals $J \subseteq R$. But R is a PID, so we know

$J = \langle x \rangle$, and $\psi : J \to I$ is completely determined by where it sends x. But I is divisible, so if $\psi(x) = \alpha$, then there is α' such that $x\alpha' = \alpha$. Therefore, we can define

$$\psi' : R \to I : 1 \mapsto \alpha'$$

This satisfies the desired condition and proves the corollary. □

This gives us a way to generate a lot of examples quickly:

Example 0.4.

1) R is an injective module over itself implies that R is divisible as an R-module. Therefore most rings do not satisfy this property.
2) As a special case of the previous item, $\mathbb{Z}$ is not an injective $\mathbb{Z}$-module. This is because $1 \notin 2\mathbb{Z}$.
3) $\mathbb{Q}$ is an injective $\mathbb{Z}$-module. In particular, every integer is invertible in $\mathbb{Q}$, so **??** applies.
4) $\mathbb{Q}/\mathbb{Z}$ is also injective. Not that we can still divide by n as a valid isomorphism.
5) A field is a PID, and every module is a vector space over a field which is divisible. Therefore, every modules over a field is injective (and projective if fg)!

The final portion of this class is devoted to the following claim: every module M is a subset of an injective module I.

Lemma 0.5. *Every $\mathbb{Z}$-module M is a subset of an injective module I.*

Proof. We have already shown that there exists a surjection $\mathbb{Z}^\Lambda \to M$. Let K be the kernel of this map, so that $M \cong \mathbb{Z}^\Lambda/K$. We see that $\mathbb{Q}^\Lambda$ is an injective $\mathbb{Z}$-module, since it is a direct sum of injectives, containing $\mathbb{Z}^\Lambda$ and thus K. We therefore conclude that $\mathbb{Q}^\Lambda/K$ is an injective module identically to the case of $\mathbb{Q}/\mathbb{Z}$. Finally, we see that $M = \mathbb{Z}^\Lambda/K \hookrightarrow \mathbb{Q}^\Lambda/K$. This completes the proof. □

This can be upgraded to any ring using the following adjointness theorem (as well as some information from next time):

Theorem 0.6 (Hom_R-$\otimes_R$ adjointness)**.** *If L, M, N are R-modules, then there is a natural isomorphism of R-modules*

$$\mathrm{Hom}_R(M \otimes_R N, L) \cong \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, L))$$

Proof. The strategy will be to construct mutually inverse homomorphisms. Given $\psi \in \mathrm{Hom}_R(M \otimes_R N, L)$, we construct $F(\psi) \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, L))$ as follows:

$$(F(\psi)(m))(n) = \psi(m \otimes n)$$

Note the notation $(F(\psi)(m))(n)$ is because we want to construct an element of $\mathrm{Hom}_R(N, L)$ given an element of M. This is easily checked to be a well-defined homomorphism. Finally, it goes to construct its inverse.

Given $\varphi \in \mathrm{Hom}_R(M, \mathrm{Hom}_R(N, L))$, define

$$G(\varphi)(m \otimes n) = (\varphi(m))(n)$$

This is well defined, since

$$(\varphi(rm))(n) = (r\varphi(m))(n) = (\varphi(m))(rn)$$

Finally, $G \circ F = Id$ and $F \circ G = Id$. This completes the proof. □