HOMEWORK 3: MODULES, TENSORS, AND LOCALIZATION
DUE: FRIDAY, MARCH 16TH

1) Show that $\mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

2) If $\mathfrak{q}$ is a minimal prime ideal of a reduced ($\mathcal{N} = 0$) Noetherian ring R show that $R_{\mathfrak{q}}$ is a field. (**Hint:** A minimal prime ideal is a prime ideal not containing any other prime ideal. In a Noetherian ring, a minimal prime is composed entirely of zero divisors. You may assume this. What does the localization set to 0?)

3) Show that if I, J are ideals of R, then $R/I \otimes_R R/J \cong R/(I + J)$.

4) Show that the following conditions are equivalent for a short exact sequence

$$0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$$

   i. $M \cong M' \oplus M''$.
   ii. There is a homomorphism $\varphi' : M \to M'$ such that $\varphi' \circ \varphi = Id_{M'}$.
   iii. There is a homomorphism $\psi' : M'' \to M$ such that $\psi \circ \psi' = Id_{M''}$.

5) Show that if P and P' are projective modules, so is $P \otimes_R P'$.

6) Prove the following lemma from class 12:

   Lemma 0.1. *For modules M_λ, N, $\lambda \in \Lambda$, we have isomorphisms*

$$\mathrm{Hom}_R\Big(\bigoplus_{\lambda\in\Lambda} M_\lambda, N\Big) \cong \prod_{\lambda} \mathrm{Hom}_R(M_\lambda, N)$$

$$\mathrm{Hom}_R\Big(N, \bigoplus_{\lambda\in\Lambda} M_\lambda\Big) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_R(N, M_\lambda)$$

   Recall here that $\prod$ (the direct product) differs from $\oplus$ in the sense that $(m_\lambda) \in \oplus_\lambda M_\lambda$ requires all but finitely many m_λ be zero. The direct product has no such restriction.

7) Prove the following universal property of the tensor product:

   Theorem 0.2. *The pair $(M \otimes_R N, \otimes : M \times N \to M \otimes_R N)$ satisfies the following: Given a bilinear map $\varphi : M \times N \to P$, then $\exists! \varphi' : M \otimes_R N \to P$ a homomorphism factoring φ. That is to say $\varphi = \varphi' \circ q$. If T is any other module with this factorization property, then T is uniquely isomorphic to $M \otimes_R N$.*

8) Show that for a given ring R, every R-module is projective if and only if every R-module is injective.