HOMEWORK 10: GRAND FINALE
DUE: WEDNESDAY MAY 8TH

1) Recall every Noetherian ring R has only finitely many associated primes (Theorem 27.3). Let $W = \{r \in R \mid r \text{ is not a zero-divisor}\}$. Show that $W^{-1}R$ has only finitely many maximal ideals.

2) Prove the following Lemma using ideas of Lemma 29.4:

   Lemma 0.1 (Fitting Lemma). *If $\varphi : M \to M$ is a homomorphism, with M a Noetherian Module, show that there exists $n \gg 0$ such that $\operatorname{im}(\varphi^n) \cap \ker(\varphi^n) = 0$.*

3) Show that if R is a Noetherian local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is principal if and only if $\mathfrak{m}/\mathfrak{m}^2$ is a 1-dimensional $\mathbb{R}/\mathfrak{m}$-vector space.[1] This allows us to say R is a DVR if and only if R is local Noetherian with $\operatorname{Spec}(R) = \{0, \mathfrak{m}\}$ and $\mathfrak{m}/\mathfrak{m}^2$ a 1-dimensional vector space.

4) If $R = K[x, y]$ with K algebraically closed, let f is an irreducible polynomial of the form
$$f = f' + f''$$
where $f' = ax + by$ and $f'' \in \langle x, y\rangle^2$. Consider $\mathfrak{m} = \langle x, y\rangle \subseteq A = R/\langle f\rangle$. Show that $A_{\mathfrak{m}}$ is a DVR if and only if $f' \neq 0$.

   This shows f is smooth (in the case of $\mathbb{C}$) at $P = (0, 0) \in K^2$ if and only if $A_{\mathfrak{m}}$ is a DVR.

5) If R is an intermediate ring $K[x] \subseteq R \subseteq K[\![x]\!]$ which is local, maximal ideal $\langle x\rangle$, show R is a DVR and thus in particular Noetherian.

6) Given R a DVR with maximal ideal $\mathfrak{m} = \langle t\rangle$, consider the sequence
$$\ldots \xrightarrow{\pi_4} R/\mathfrak{m}^3 \xrightarrow{\pi_3} R/\mathfrak{m}^2 \xrightarrow{\pi_2} R/\mathfrak{m}$$
of ring homomorphisms. Given such an arrangement, we can take the inverse limit:
$$\hat{R} = \varprojlim(R/\mathfrak{m}^n) = \{(a_1, a_2, \ldots) \in \prod_{n=1}^{\infty} R/\mathfrak{m}^n \mid \pi_i(a_i) = a_{i-1}\ \forall i \geq 2\}$$
This is called the **completion** of R with respect to $\mathfrak{m}$, which is a ring with coordinate-wise addition and multiplication. Show $\hat{R}$ is also a DVR with maximal ideal $\mathfrak{m} = \langle \hat{t}\rangle$, where $\hat{t} = (t, t, t, t, \ldots)$.

[1] This is what it means to be **regular** for dimension 1 rings.