CLASS 19, OCTOBER 28: ARGUMENTATIVE COROLLARIES

Last time we proved the argument principal using the idea of the logarithm. It is stated as follows:

Theorem. *Suppose that f is a meromorphic function in Ω, and C is a simple positively oriented loop in its interior. If f has no poles nor zeroes on C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = (\#\text{ of zeroes in } C) \ - \ (\#\text{ of poles in } C)$$

where the # is counted with its order (sometimes called multiplicity).

This allowed us to prove Rouche's theorem, which allows one to perturb holomorphic functions without changing the number of zeroes in a given region. Today we will examine some other very important corollaries of this result.

The first is the open mapping theorem. Thus it would be helpful to know what an open mapping is:

Definition 19.1. Let $\Omega \subseteq \mathbb{C}$ be an open set. Then $f : \Omega \to \mathbb{C}$ is said to be an **open mapping** if whenever $\Omega' \subseteq \Omega$ is an open set, then so is $f(\Omega')$ open.

Notice that this is very different from continuity!

Example 19.2. The constant map $f : \mathbb{C} \to \mathbb{C} : z \to z_0$ is continuous but not open.

Example 19.3. If $f : \Omega \to \mathbb{C}$ is a bijective function, then in fact f is continuous if and only if f^{-1} is an open map.

Constructing maps which are open but not continuous in the complex world (as opposed to the topological world) is a bit more difficult, but a non-constructive example can be constructed as follows:

Example 19.4. Consider an equivalence relation $\sim$ on $\mathbb{C}$ defined by $z \sim z'$ if and only if $Re(z-z')$, $Im(z-z') \in \mathbb{Q}$. Then there are uncountably many equivalences classes in $\mathbb{C}/\sim$. So there exists a bijection p between $\mathbb{C}/\sim$ and $\mathbb{C}$. Now consider the map which sends z to its equivalence class $[z]$, then to $p([z])$. Call it f:

$$f : \mathbb{C} \to \mathbb{C} : z \mapsto p([z])$$

This map has an astounding property:

$$f(B(z,r)) = \mathbb{C}$$

for any $z \in \mathbb{C}$ and $r > 0$. This is because $\mathbb{Q}^2 \subseteq \mathbb{C}$ is dense! As a result it is necessarily open. However, for the same reason it can't be continuous; the preimage of $B(f(z),\epsilon)$ can't contain any neighborhood of z.

This example is pathological, but does demonstrate the assertion that open and continuous do not imply one another. Now we can move onto the open mapping theorem:

Theorem 19.5 (Open mapping theorem)**.** *If f is non-constant and holomorphic on Ω, then f is an open map on Ω.*

Proof. Consider $\Lambda = f(B(z_0, r))$ and suppose $f(z_0) = w_0$. It goes to demonstrate the existence of $\epsilon > 0$ such that $B(w_0, \epsilon) \subseteq \Lambda$.

Consider $g(z) = g_w(z) = f(z) - w$ and write

$$g(z) = (f(z) - w_0) + (w_0 - w) = F(z) + G(z)$$

Choose $0 < \delta < r$ such that $f(z) \neq w_0$ for $|z - z_0| = \delta$. Choose $\epsilon > 0$ such that $|f(z) - w_0| > \epsilon$ on this circle (by compactness). If $|w - w_0| < \epsilon$, we get that $|F(z)| > |G(z)|$ on this circle. Rouche's theorem ensures that $F(z)$ has the same number of zeroes as $g(z)$. But this implies $g(z)$ has a zero, since we assumed $F(z)$ has one. This shows the assertion. □

Another wonderful corollary is the maximum modulus principle, which states that maxima for holomorphic functions can only exist on the boundary.

Theorem 19.6 (Maximum Modulus Principle). *If f is non-constant and holomorphic in Ω an open set, then f cannot attain its maximum in Ω.*

Proof. Suppose $z_0 \in \Omega$ was a maximum for f. Then let $B(z_0, r) \subseteq \Omega$. In this case, we know $f(B(z_0, r))$ is an open set containing $f(z_0)$. But this implies that larger values for f exist! A direct contradiction to our assumptions on z_0. □

Rephrasing this realization a bit, we get the following:

Corollary 19.7. *If Ω is an open set with compact closure, and if f is holomorphic on Ω and continuous on $\bar{\Omega}$, then*

$$\sup_{z \in \Omega}(f(z)) \leq \sup_{z \in \bar{\Omega} \setminus \Omega}(f(z))$$

Proof. Since $\bar{\Omega}$ is compact and f is continuous, f attains its maximum on $\bar{\Omega}$. But Theorem 19.6 informs us that no point in the interior can be maximal. □

Example 19.8. It should be noted here that the compactness assumption is necessary to our claim. Otherwise, you could consider something like the upper half plane and the function $\sin(z)$. On the boundary, the real line, the function is bounded. However, in the upper half plane the values of $\sin(z)$ can be chosen arbitrarily large.

Example 19.9. A special case of Theorem 19.6 is the story for analytic functions at the origin with radius of convergence $R > 1$. If we consider

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

we know that on $\bar{B}(0, 1)$, f must attain its maximum for some $|z| = 1$. This can be interpreted as saying we can solve a system of linear equations to make the first N terms have near the same argument (or be 0) and thus to make f as large as possible, the absolute value of z with such argument should also be maximized.

To specialize even further, we can consider $f(z) = \cos(\frac{\pi}{2} z)$. If we consider only real values, we know f is maximized at $z = 0$ where it have value 1. at ± 1, the function is 0. So Theorem 19.6 implies there is $|z| = 1$ such that $|f(z)| > 1$. It is easy to check that $z = \pm i$ works:

$$|\cos(\pm \frac{\pi}{2} i)| = \frac{1}{2}\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right) > 2.3$$