HOMEWORK 3: QUOTIENTS AND CONNECTEDNESS
DUE: FRIDAY, SEPTEMBER 28

1) Suppose $f : X \to Y$ is a continuous map. If there exists $g : Y \to X$ such that $f \circ g$ is the identity map on Y, show that f is a quotient map.

   Solution: Since the identity is surjective, this implies f is surjective (as $f \circ g = Id_Y$). If $U \subseteq Y$ is open, then so is $f^{-1}(U)$ since f is continuous. Furthermore, if $U \subseteq Y$ is such that $f^{-1}(U)$ is open, we can consider

   $$g^{-1}(f^{-1}(U)) = (f \circ g)^{-1}(U) = Id_Y^{-1}(U) = U$$

   is also open. Therefore, f is a quotient map.

2) If $A \subseteq X$, $r : X \to A$ a continuous map is said to be a **retraction** if $r(a) = a$ for every $a \in A$. Show that r is a quotient map.

   Solution: This follows directly from the previous exercise, since $(r \circ i) = Id_A$.
   Alternatively, r is surjective, since it is surjective even on the restricted domain of A. Since r is continuous, it only goes to show that if $U \subseteq A$ is such that $r^{-1}(U) \subseteq X$ is open, we can similarly consider

   $$i^{-1}(r^{-1}(U)) = (r \circ i)^{-1}(U) = (Id_A)^{-1}(U) = U$$

   is open, where $i : A \to X$ is the (continuous!) inclusion.

3) Consider the map $\pi : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$. Let $A \subseteq \mathbb{R}^2$ be the set of points such that $x \geq 0$ or (inclusive) $y = 0$. Show that the restricted map $f : A \to \mathbb{R}$ is a quotient map that is neither open nor closed.

   Solution:
   - **Quotient Map:** f is surjective, since A contains in particular the x-axis. Furthermore, f is continuous (composition of 2 continuous maps). Suppose U is such that $f^{-1}(U)$ is open. Since A has the subspace topology, we know

     $$f^{-1}(U) = \bigcup_{\alpha} (B((x_\alpha, y_\alpha), r_\alpha) \cap A)$$

     In such a case, $U = \cup_\alpha (x_\alpha - r, x_\alpha + r)$, which is open in $\mathbb{R}$. Therefore, f is a quotient map.
   - **Not Open:** Consider the open subset $U = B((0, 2), 1) \cap A \subseteq A$. This is open by virtue of the subspace topology. It is easy to see that $f(U) = [0, 1)$, which is not open.
   - **Not Closed:** Consider the graph

     $$\Gamma = \{(x, \tan(x)) \mid x \in (\frac{\pi}{2}, \frac{3\pi}{2})\}$$

     This is closed in $\mathbb{R}^2$ and contained in A, and thus closed in A. However, $f(\Gamma) = (\frac{\pi}{2}, \frac{3\pi}{2})$ which is even open.

Of course many other examples work as well...

4) If $X_1, X_2, \ldots \subseteq X$ are a sequence of connected subspaces, such that $X_i \cap X_{i+1} \neq \emptyset$, show that $\bigcup_{i=1}^{\infty} X_i$ is connected.

Solution: Suppose that $\bigcup_{i=1}^{\infty} X_i$ is separated by non-empty open subsets U, V. Since each X_i is connected, it falls into either U or V. Since $U, V \neq \emptyset$, there exists a n such that $X_n \subseteq U$ and $X_{n+1} \subseteq V$. But they share a point; $x \in X_n \cap X_{n+1} \subseteq U \cap V = \emptyset$, a contradiction. So $\bigcup_{i=1}^{\infty} X_i$ is connected.

5) If $p : X \to Y$ is a quotient map, with Y connected and $p^{-1}(y) \subseteq X$ connected for each $y \in Y$, then X is connected.

Solution: Suppose that X is separated by U, V. This implies that every fiber $p^{-1}(y)$ is entirely contained within either U or V. This yields a partition of the base space Y: $Y = U' \cup V'$ where $U' = \{y \mid p^{-1}(y) \subseteq U\}$ and $V' = \{y \mid p^{-1}(y) \subseteq V\}$. Even better, $p^{-1}(U') = U$ and $p^{-1}(V') = V$, so they are open non-empty disjoint sets covering X (by surjectivity of a quotient map). But this contradicts the connectedness of Y, proving the claim.

6) Let $Y \subseteq X$ be two connected topological spaces. Suppose that $X \setminus Y$ is disconnected with a separation by A, B. Show that $Y \cup A$ and $Y \cup B$ are connected.[1]

Solution: Suppose $Y \cup A$ is separated by U, V non-empty open subsets. Since Y is connected, we may assume $Y \subseteq U$ WLOG. Therefore $A \supseteq V$. But this implies

$$X = Y \cup A \cup B = U \cup V \cup B.$$

Now, note that $\bar{V} \subseteq \bar{A} \subseteq X \setminus B$ (similarly for $\bar{B} \subseteq X \setminus A$, $\bar{U} \subseteq X \setminus V$, $\bar{V} \subseteq X \setminus U$). This follows by virtue of being a separation in the subspace topology, and therefore there exists closed sets covering the space A in X that don't intersect within A. Thus we can conclude

$$\overline{B \cup U} = \bar{B} \cup \bar{U} \subseteq (X \setminus A) \cup (X \setminus V) = (B \cup Y) \cup (B \cup U) \subset B \cup U$$

$$\bar{V} \subseteq (X \setminus B) \cap (X \setminus U) = V$$

So $V \subseteq X$ is a closed subset with closed complement, thus open. This contradicts the connectness of X.

7) Show any infinite cardinality space X with the finite complement topology is connected.[2]

Solution: If U, V are 2 non-empty open subsets X, then $(U \cap V)^c = U^c \cup V^c$, which is a finite set, thus not all of X. Therefore, $U \cap V \neq \emptyset$, and therefore no separation of X can exist.

8) We have shown that $X = \mathbb{R}^{\mathbb{N}}$ with the product topology is connected, whereas X with the box topology is not. Consider the **uniform topology**; define $d(\mathbf{x}, \mathbf{y}) = \sup\{d_i(x_i, y_i)\}$, where $d_i(a, b) = \min\{|x_i - y_i|, 1\}$ is the *standard bounded metric.*

[1]This result is very helpful for identifying connected subsets of $\mathbb{R}^n$.

[2]This mildly generalizes 'every space is connected with the trivial/indiscrete topology'.

The induced metric topology is the uniform topology, which by 20.4 of Munkres is finer than the product yet coarser than the box topology.

Prove or disprove that X with the uniform topology is connected.

Solution: I claim that the same collection of subsets forming a separation for the box topology are open in the uniform topology, therefore it is disconnected. Namely $U = \{\text{bounded sequences}\}$ and $V = \{\text{unbounded sequences}\}$.

Fortunately, all that needs to be shown is that U and V are both open in the uniform topology. For a given $\mathbf{x} \in U$, consider $B = B(\mathbf{x}, r)$ for any $0 < r \leq 1$. This implies that for $\mathbf{y} \in B$, every coordinate y_i has the property that $|x_i - y_i| < r - \epsilon \leq 1$ for some $\epsilon > 0$, since being of distance less than 1 away from x negates the minimum statement and we need to bound the supremum away from r. This actually demonstrates that

$$B(\mathbf{x}, r) = \bigcup_{\epsilon > 0} \prod_{i \in \mathbb{N}} (x_i - r + \epsilon, x_i + r - \epsilon)$$

which is somewhat spectacular, since none of the boxes are open themselves! Indeed, we can find no open ball around the point $\mathbf{x} + (\frac{r}{2}, \frac{2r}{3}, \frac{3r}{4}, \dots)$ inside $B(\mathbf{x}, r)$.

Therefore, $\mathbf{y}$ is also bounded (by at worst r more than $\mathbf{x}$), implying $B \subseteq U$, thus open. The same logic and B apply to unbounded sequences.