CLASS 33, DECEMBER 4TH: RIEMANN MAPPING THEOREM

Last time we finished our examining automorphisms of $\mathbb{D}$ and $\mathbb{H}$. We used the neat method of realizing one in terms of the other via conjugating by their biholomorphic mappings. So conformally equivalent spaces have a nice isomorphism:

$$\Gamma : \mathrm{Aut}(\mathbb{D}) \to \mathrm{Aut}(\mathbb{H}) : \psi \mapsto F^{-1} \circ \psi \circ F$$

Today I state the Riemann mapping theorem. Instead of focusing on one individual space, we will focus on a property of spaces.

Theorem 33.1 (Riemann Mapping Theorem)**.** *If $\Omega \subseteq \mathbb{C}$ is a proper, open, and simply-connected subset, then Ω is conformally equivalent to $\mathbb{D}$.*

By virtue of this being an equivalence relation, this also says that all proper simply connected domains are conformally equivalent.

Note that both of these conditions are necessary. Indeed, if $\Omega = \mathbb{C}$, then $F : \mathbb{C} \to \mathbb{D}$ would be entire and bounded. Additionally, if Ω weren't simply connected, we could consider γ a non-trivial loop. Then $F^{-1} \circ \gamma \circ F$ would be a loop in $\mathbb{D}$, thus trivial. But this would imply that γ was trivial to begin with.

We will prove Theorem 33.1 over the remaining 2 classes. We need to build up some machinery first and pass through Montel's Theorem.

Definition 33.2. A family of functions $\mathcal{F} \subseteq \{f : \Omega \to \mathbb{C} \mid f \text{ is holomorphic}\}$ is **normal** if every sequence in $\mathcal{F}$ has a uniformly convergent on compact sets subsequence.

$\mathcal{F}$ is normal is usually proven using the following two properties and our knowledge of compact sets:

Definition 33.3. $\mathcal{F}$ is **uniformly bounded on compact sets** if for each K compact, there exists R such that

$$|f(z)| \leq R \qquad\qquad z \in K,\ f \in \mathcal{F}$$

$\mathcal{F}$ is **equicontinuous on compact sets** if for each $\epsilon > 0$ and $K \subseteq \Omega$ compact, there exists a $\delta > 0$ such that for all $f \in \mathcal{F}$, $|f(z) - f(w)| < \epsilon$ whenever $|z - w| < \delta$.

This brings about Montel's theorem:

Theorem 33.4 (Montel's Theorem)**.** *If $\mathcal{F}$ is a family which is uniformly bounded on compact sets, then $\mathcal{F}$ is equicontinuous on compact sets. Additionally, $\mathcal{F}$ is a normal family.*

This is a simpler version of a theorem from topology called Ascoli's Theorem.

Sketch of the proof. The first part follows from Cauchy's Integral Theorem: if K is compact, we can choose $r > 0$ small enough so that $B(K, 3r) \subseteq \Omega$. If $|z - w| < r$, and γ is the boundary of $B(w, 2r)$, then

$$f(z) - f(w) = \frac{1}{2\pi i} \int_\gamma f(\zeta) \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - w} \right) d\zeta$$

It is easy to check that

$$\left|\frac{1}{\zeta - z} - \frac{1}{\zeta - w}\right| \leq \frac{|z-w|}{r^2}$$

As a result,

$$|f(z) - f(w)| \leq \frac{1}{2\pi}\frac{2\pi r}{r^2}R|z-w| = C|z-w|$$

which is to say $\mathcal{F}$ is equicontinuous on compact sets.

Let $f_i \in \mathcal{F}$ be a sequence and K a compact set. Choose $z_i \in K$ a sequence which is dense in K (by total boundedness of a compact set). Since f_i is uniformly bounded on K, there exists a subsequence $f_{1,i}$ for which $f_{1,i}(z_1)$ converges (by sequential compactness). But we can take a subsequence $f_{2,i}$ for which $f_{2,i}(z_i)$ converges. Inductively, we can produce a subsequence $f_{n,i}$ converging for all z_i with $i \leq n$.

Let $g_n = f_{n,n}$. I claim this is the desired sequence. Since it converges for each z_i, and the z_i are dense in Ω, equicontinuity and the triangle inequality yields

$$|g_n(z) - g_m(z)| \leq |g_n(z) - g_n(z_j)| + |g_n(z_j) - g_m(z_j)| + |g_m(z_j) - g_m(z)|$$

We can find z_j arbitrarily close to z, making the outer quantities smaller than $\frac{\epsilon}{3}$, and choose n, m sufficiently large to ensure the middle term is as well.

It only suffices to show that the convergence can be chosen uniform on any compact set. We can find compact sets K_i, $i \in \mathbb{N}$, which are nested (in each others interiors) and which build up to Ω. Such a thing is called an **exhaustion** of Ω. This ensures that we can find $z_i \in \Omega$ a sequence which is dense in Ω (since we can do it for each K_i).

If K_i is an exhaustion of Ω, and we chose $g_{i,n}$ as g_n for each K_i, then again the diagonalization $g_{n,n}$ converges uniformly on all K_i, and since the K_i exhaust Ω, eventually the engulf any compact set K by the interior property above. This completes the proof of Montell. ☐

One final thing we need is the following result which states that uniform limits of injective functions are either injective or as non-injective as possible.

Lemma 33.5. *If Ω is open and connected, and f_n is a sequence of holomorphic injective functions converging uniformly on compact sets to f, then f is either injective or constant.*

Proof. We know f is holomorphic by Theorem 12.1. Suppose f is non-injective; $f(z_1) = f(z_2)$. Consider $g_n(z) = f_n(z) - f_n(z_1)$. Then g_n is zero exactly at z_1. If $g(z) = f(z) - f(z_1)$ is not identically 0, then 12.1 again shows z_2 is an isolated zero. As a result,

$$\frac{1}{2\pi i}\int_\gamma \frac{g'(\zeta)}{g(\zeta)}d\zeta = 1$$

by the argument principle for small γ about z_2. But by uniform convergence,

$$\frac{1}{2\pi i}\int_\gamma \frac{g_n'(\zeta)}{g_n(\zeta)}d\zeta \to \frac{1}{2\pi i}\int_\gamma \frac{g'(\zeta)}{g(\zeta)}d\zeta$$

But $0 \not\to 1$, so this is a contradiction. ☐