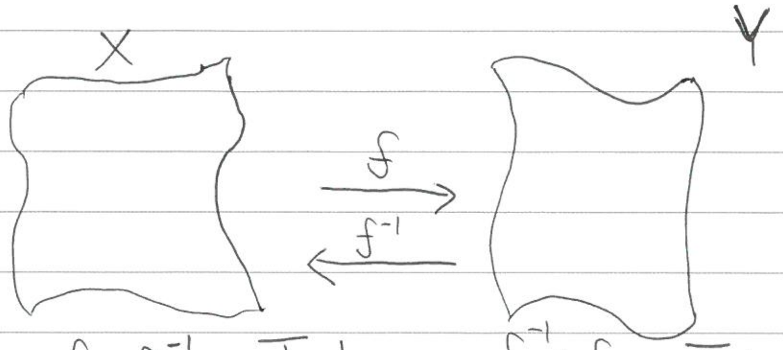
Intro Day: What does Algebraic Topology do for us?

$\boxed{\text{SEP 8}}$

The primary question in AT is are spaces "the same"?

The most stringent definition of the same: A homeomorphism:

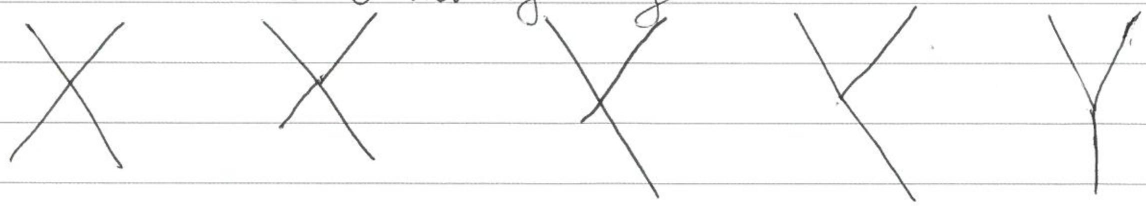


$f \circ f^{-1} = Id_X \quad f^{-1} \circ f = Id_Y$

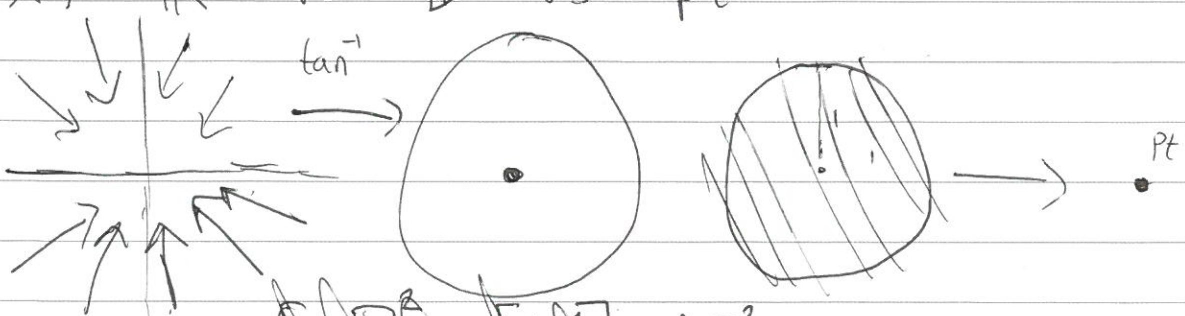
w/ f and f^{-1} both continuous functions.
Allows for streches, flips, etc.

This in practice is too strong of an equivalence. What we are interested in is called Homotopy equivalence. This allows us to deform one space to another.

Example/ take the letters X and Y. We can deform X to Y in the following way



Ex/ $\mathbb{R}^2$ vs $\mathbb{D}^2$ vs pt



~~$f: \mathbb{R}^2 \times [0,1] \to \mathbb{R}^2$~~

~~$(x,y) \times t \longmapsto (x \cdot \tan(\pi t), y \cdot \tan(t))$~~

$(\tan^{-1}(x), \tan^{-1}(y))$

Defn: ~~[Topological Space]~~ A Topological Space is a pair of a set X and a ~~topology~~ subset $\tau \subseteq \mathcal{P}(X) = \{$subsets $S \subseteq X$ subject to the following conditions:

1) $X, \phi \in \tau$
2) If $X_\alpha \in \tau$, then $\bigcup_\alpha X_\alpha \in \tau$
3) If $X_1, \dots, X_n \in \tau$, then $X_1 \cap X_2 \cap \dots \cap X_n \in \tau$

τ is called a Topology on X. $\mathcal{U} \in \tau$ is called an open set.

A topological space is a very general notion

Ex/ Let S be ANY set.

1) Let $\tau = \{S, \phi\}$. Then (S, τ) is a topological space. τ is called the indiscrete topology.
2) Let $\tau = \mathcal{P}(S)$. Then (S, τ) is a topological space. τ is called the discrete topology.

Note: These are the smallest and biggest topology

3) Let $\tau = \{T \in \mathcal{P}(S) \mid S \setminus T \text{ is a finite set}\} \cup \{\phi\}$

[Prove it is a Topological Space]. This is called the finite complement topology.

~~Defn: We say $\mathcal{B} \subseteq \tau$ is a basis for the topology τ if every element $\mathcal{U} \in \tau$ can be represented as~~

$$\cancel{\bigcup_\alpha X_\alpha = \mathcal{U}}$$

<u>Defn</u>: ~~Basis~~ If X is a set, $\mathcal{B} \subseteq P(X)$ is called a <u>basis</u> for a topology on X if

1) For each $x \in X$, there is $X_\alpha \in \mathcal{B}$ s.t. $x \in X_\alpha$
2) If $x \in B_1, B_2 \in \mathcal{B}$, then $\exists B_3 \in \mathcal{B}$ s.t. $x \in$

The topology generated by $\mathcal{B}$ is the smallest top τ containing $\mathcal{B}$.

<u>Ex</u>/ Metric spaces (V,d) are topological spaces: $\mathcal{B} = \{B_\varepsilon(x) \mid x \in V, \varepsilon > 0\}$. Elements of τ generated by $\mathcal{B}$ are the usual open sets from real analysis.

Set $\rightsquigarrow$ Topological Spaces $\leftsquigarrow$ Metric Spaces

<u>Ex</u>/ $\cdot_A \quad \cdot_B \qquad \{\phi\}, \{A\}, \{A,B\}$

~~Suppose X is a set w/ a total order.~~

$\boxed{\text{Sept 13}}$: Building Topological Spaces from others I

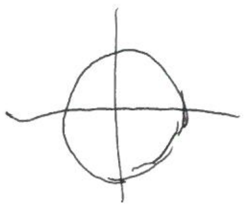
~~Product~~ Subspace topology: Let (X,τ) be a topological space, and $Y \subseteq X$. We create the ~~subspace~~ sub topological space as follows. Let

$$\tau_Y = \{ Y \cap U \mid U \in \tau \}$$

Then (Y, τ_Y) is a topological space.

Ex/ Give $\mathbb{R}$ the standard topology (i.e. the metric topology, or the one with basis (a,b) for $a<b$)
Consider $[0,1] \subseteq \mathbb{R}$ w/ subspace topology. What are all the open sets?

~~Product Topology~~ Ex/ Give $\mathbb{R}^2$ the metric topology (a basis given by $B(x,\varepsilon)$, or $\{(a,b)\times(c,d) \mid a<b, c<d\}$)
Consider $S^1 \subseteq \mathbb{R}^2$. Endow S^1 w/ the subspace topology.



This is the usual topology on S^1 (as a manifold for example)

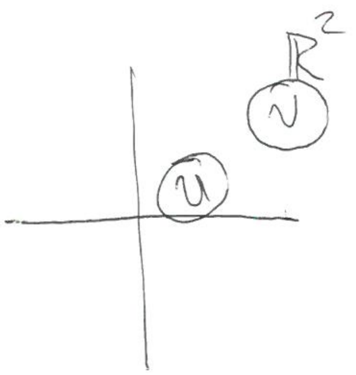
Product Topology: Given (X,τ) and (Y,σ) two topological spaces, one can create a third topo space "$X \times Y$":

$$X \times Y = \{(x,y) \mid x \in X, y \in Y\}$$

Topology generated by $\mathcal{U} \times \mathcal{V}$ w/ $\mathcal{U} \in \tau$, $\mathcal{V} \in \sigma$

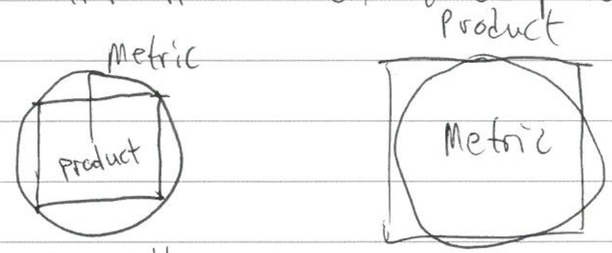
Note: This is NOT exclusively given by products

Ex/



$\mathcal{U} \cup \mathcal{V}$ is open but Not a product

<u>Ex</u>/ $\mathbb{R}^2$ w/ it's usual metric topology is the same as $\mathbb{R}\times\mathbb{R}$ with the product topology:



(Real analysis Fact: all ~~metrics~~ Norms are equivalent)

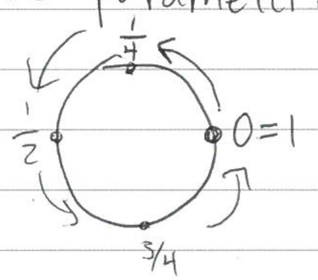
Quotient Topology: Equivalence Relations $\sim$ on a topological space

$X/\sim$ is the set of equivalence classes

We give $X/\sim$ the quotient topology defined as follows:

$$\mathcal{U} \subseteq X/\sim \text{ is open } \quad (\text{i.e. } \mathcal{U} \in \mathcal{T})$$
$$\Updownarrow$$
$$p^{-1}(\mathcal{U}) \subseteq X \text{ is open}, \quad p: X \longrightarrow X/\sim \text{ is the quotient map}$$

<u>Ex</u>/ Let S^1 be parameterized by $[0,1]$:



There is a natural map $\mathbb{R} \to S^1 : t \mapsto t \bmod 1$ i.e. give your t a decimal expansion and forget the integer part. This is an equivalence relation.

$$\mathbb{R} \to \mathbb{R}/\sim = \mathbb{R}/\mathbb{Z} = S^1$$

Endowing S^1 w/ the quotient topology is equivalent to the subspace topo from $\mathbb{R}^2$.

Sept 15: Continuous Functions

The main way of transferring info from one topo space to another is a continuous function:

Defn: $f: X \to Y$ is said to be continuous if for every $\mathcal{U} \in \tau_Y$, $f^{-1}(\mathcal{U}) = \{x \in X \mid f(x) \in \mathcal{U}\} \in \tau_X$.

Note: this is a natural generalization of continuity of metric spaces

$f: X \to Y$ is continuous at x if for every $\varepsilon > 0, \exists \delta > 0$ such that $\mu_X(x, x_0) < \delta \Rightarrow \mu_Y(f(x), f(x_0)) < \varepsilon$

Char of open sets in ~~topology~~ metric: $\forall x \in \mathcal{U}$ $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq \mathcal{U}$

$x \in f^{-1}(\mathcal{U}) \iff f(x) \in \mathcal{U}$
$\exists \varepsilon > 0$ s.t. $B_\varepsilon(f(x)) \in \mathcal{U}$
$\Rightarrow B_\delta(x) \subseteq f^{-1}(\mathcal{U})$ for some $\delta > 0$

Easy Continuous \ Discontinuous functions

$Id_X : \underset{\text{any } \tau}{X} \to \underset{\text{indiscrete}}{X}$ Continuous

$Id_X : \underset{\text{discrete}}{X} \longrightarrow \underset{\text{any } \tau}{X}$ Continuous

What about flip side?

Topological spaces with continuous maps form a $\boxed{\text{category}}$

Composition of continuous functions is continuous

Associative, Id. Algebraic Topology: ...

$\boxed{\text{Maps into products}}$: If $f: X \to Y \times Z$, then

$f = f_Y \times f_Z$ and

f is continuous $\iff$ f_Y and f_Z are continuous

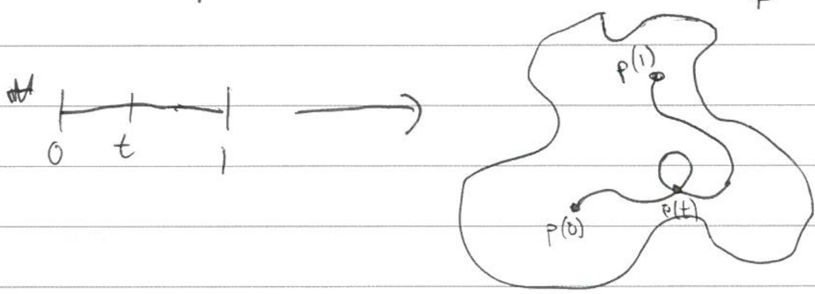
"$\Rightarrow$" easy direction

"$\Leftarrow$" It goes to show $f^{-1}(U \times V)$ is open in X (since $U \times V$ form a basis for the product topology)

but $f^{-1}(U \times V) = f_Y^{-1}(U) \cap f_Z^{-1}(U)$ which is open by definition of a topology

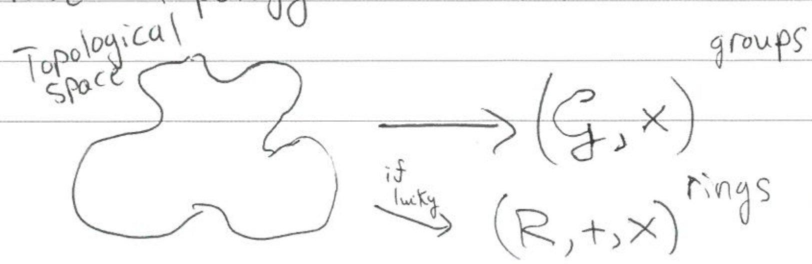
One Continuous map of interest later on is "paths"

<u>Defn</u>: A continuous map $[0,1] \xrightarrow{P} X$ is called a path in X.



$\boxed{\text{Sept 18}}$ Day of Groups

Algebraic Topology in a nutshell



Defn: A group is a set G with an operation $\cdot$ such that

1) $\exists\, e \in G$ s.t. $g \cdot e = e \cdot g = g \quad \forall g \in G$
2) $\forall g \in G, \exists\, g^{-1} \in G$ s.t. $g \cdot g^{-1} = g^{-1} \cdot g = e$
3) $\forall g, h, f \in G, \ (g \cdot h) \cdot f = g \cdot (h \cdot f)$
4) $\forall g, h \in G, \ g \cdot h \in G$

If in addition, 5) $g \cdot h = h \cdot g$, G is called abelian

Examples: $(\mathbb{Z}, +)$, $(\mathbb{Q}^{\times}, \cdot)$, $(S_n, \circ)$, (continuous functions, $\circ$)

(Rigid motions of a regular n-gon, $\circ$), $\{\pm 1, \pm i, \pm j, \pm k\}$, etc...

Defn: $H \subseteq G$ w/ $(H, \cdot)$ itself a group, is called a subgroup of G. H is called <u>normal</u> if for every $g \in G$, $g \cdot H \cdot g^{-1} \subseteq H$

> Normal Subgroups form an important class of subgroups, as one can use them to form Quotient Groups $G/H = \{g \cdot H \mid g \in G\}$.

Examples
- $A_4 \subseteq S_4$ is normal (even permutations)
- $\{e, (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3)\} = K_4 \subseteq S_4$ is normal
- All other proper subgroups of S_4 are not! (there are 30 in total)

Products: If $(G, \cdot)$ and $(H, \times)$ are groups, one can form $(G \times H, *)$ where $(g, h) * (g', h') = (g \cdot g', h \times h')$. This is the standard product

One can also form a "non-abelian" version of this, the free product $*$:

Defn: If G and H are groups, $G * H$ is the group composed of expressions (words)

$$\overset{\text{and/or}}{\boxed{h_0}} \cdot g_1 \cdot h_1 \cdot g_2 \cdot h_2 \cdots \cdot g_n \cdot h_n \cdot \overset{\text{and/or}}{\boxed{g_{n+1}}}$$

Multiplication is performed by concatenating the 2 expressions

$$g_1 \cdot \cancel{g_2} \cdot g_2 \cdots g_n \cdot h_1' \cdot g_2' \cdot h_2' \cdots g_m'$$

If 2 elements of the same group meet, multiply them together

$$\text{eg.} \quad g_1 h_1 g_2 * g_1' h_1' = g_1 h_1 (g_2 \cdot g_1') h_1'$$

and any time e_G or e_H appear, remove them.

Ex/ $\mathbb{Z}/\langle 2\rangle * \mathbb{Z}/\langle 2\rangle$

$\{0,1\} * \{e,f\}$

elements: $E, e, f, ef, fe, efe, fef, \ldots$

As sets, this has the same cardinality as the integers.

Abelianization: Let G be a group. Consider the subgroup $[G,G] = \{g \cdot h \cdot g^{-1} \cdot h^{-1} \mid g, h \in G\}$

~~This is a normal subgroup: $f \cdot g \cdot h \cdot g^{-1} \cdot h^{-1} \cdot f^{-1}$~~

$u \in [G,G]$, then $g^{-1} \cdot u \cdot g =$

$u \cdot u^{-1} g^{-1} u g = u \cdot (u^{-1} g^{-1} u g) \in [G,G]$

$G/[G,G] = G^{ab}$ is the largest Quotient of G which is abelian. $G \to G^{ab}$ is called the abelianization of G.

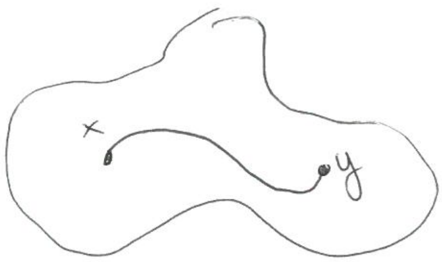
Relation: If G and H are ~~free~~ abelian groups how is
$$(G * H)^{ab} \text{ related to } G \times H?$$

Sept 20 Connectedness & Path Connectedness (and local variants)

Defn: A topological space (X,τ) is called connected if $U, V \subseteq X$ are open, and $U \cup V = X$, then $U \cap V \neq \phi$. Otherwise, X is called disconnected.

- Examples and Non-examples: $\mathbb{R}$, $\mathbb{Z}$,
- Image of connected set is connected.

Defn: (X,τ) is called path-connected if for every $x, y \in X$, there exists $\gamma : [0,1] \to X$ continuous such that ~~A~~ $\gamma(0) = x$ and $\gamma(1) = y$.



How do these relate?

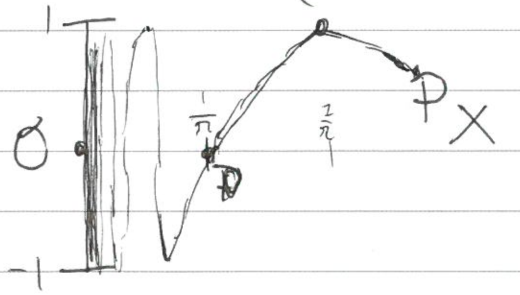
Thm: If X is path connected, then X is connected

Q: Proof Suppose not. Let $X = U \cup V$, w/ $U \cap V = \phi$
Choose $x \in U$, $y \in V$, and $\gamma : [0,1] \to X$ as above.
$\gamma^{-1}(U)$ is open and $\gamma^{-1}(V)$ is open, but
$\gamma^{-1}(U \cap V) = \gamma^{-1}(U) \cap \gamma^{-1}(V) = \phi$. But $[0,1]$
is connected ↯

What about Connected $\not\Rightarrow$ Path connected?

Consider $\Gamma_f = \{(x, \underset{f(x)}{\sin(\tfrac{1}{x})}) \in \mathbb{R}^2 \mid 0 < x \leq 1\}$

and let $X = (\{0\} \times [-1,1]) \cup \Gamma_f \subseteq \mathbb{R}^2$



This set is connected, because any open set containing $(0,x)$ must contain some (ε, y), and since $0 \times [0,1]$ and Γ_f are both ~~path~~ connected, must be all of X

On the other hand, there is no path from

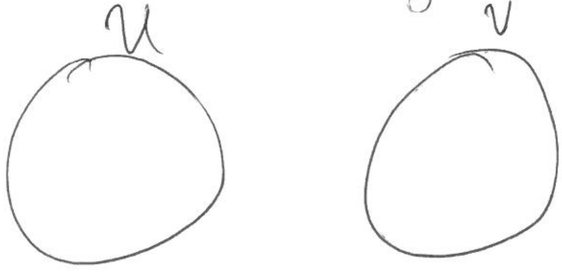
P to $(0,0) = O$. Reason: Image of a compact set is compact. and Γ_f is not

There are also local versions of these notions.

If $\mathcal{P}$ is a property of topological and spaces, we say (X,τ) is "locally $\mathcal{P}$" if $\forall x \in X$, $\forall U \in \tau$ w/ $x \in U$, ~~such that U is locally P~~ $\exists V \subseteq U$, $V \in \tau$ w/ V satisfying $\mathcal{P}$.

So ... Locally-Connected and Locally Path Connected

Ex/ Disconnected but locally connected



disjoint union

$$X = U \sqcup V = U \cup V$$

Same for path connected.

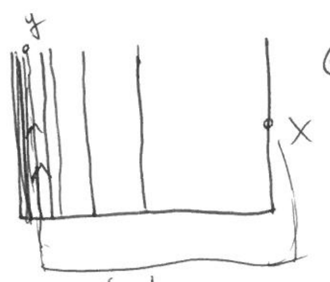
Locally Path connected $\Rightarrow$ Locally Connected

Ex/ Path Connected but not Locally (Path)-Connected

Consider

$$X \subseteq \mathbb{R}^2$$

$$X = \left([0,1]\times\{0\}\right) \cup \left(\{0\}\times[0,1]\right) \cup \bigcup_{\substack{n \geq 1 \\ n \in \mathbb{Z}}} \left\{\tfrac{1}{n}\right\}\times[0,1]$$



Comb space

Path connected

$x = (0,1)$

Not path connected and not connected

Start of Hatcher

$\checkmark / \checkmark - /$
$\times / \times - / \times$

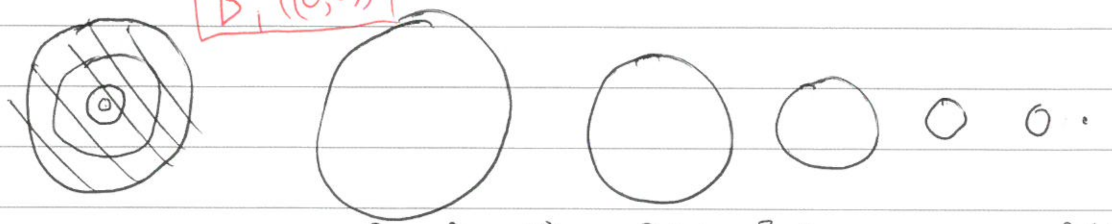
7

$\boxed{\text{Sept 22}}$ Homotopy and Homotopy Type

Concise, Learn, A couple bad homeworks vs. Great final

$\boxed{\text{Idea of Homotopy}}$: We want to classify spaces up to continuous deformation. Can we say objects have the same shape?

$\boxed{B_1((0,0))}$



$f(\vec{x}) = \vec{x}$ $f(\vec{x}) = \frac{2}{3}\vec{x}$ - - - - $f(\vec{x}) = 0$

Explanation of Homotopy

What we do is adjoin an extra parameter $t \in [0,1] = I$, which allows us to ~~slowly~~ deform a space over time.

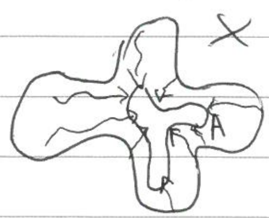
Formally: If X is a topological space, then a deformation retract of X onto a subspace $A \subseteq X$ is a continuous map

$$F : X \times I \to X \qquad (F(x,t) = f_t(x))$$

with

Defn: Def ret

1) $F(x,0) = x$ $(f_0 = Id_X)$
2) $F(a,t) = a$ $(f_t|_A = Id_A)$
3) $F(X,1) \subseteq A$ $(f_1(X) = A)$



Ex/ Closed Disc Def retract to a point

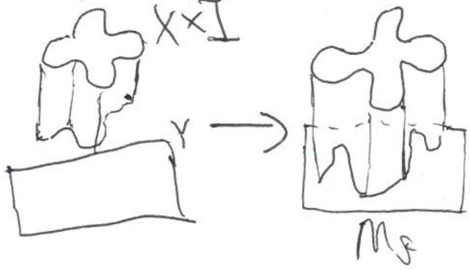
Ex/ Closed Disc \ pt Def retracts to S^1

An easy way to create such def retracts is the mapping cylinder.

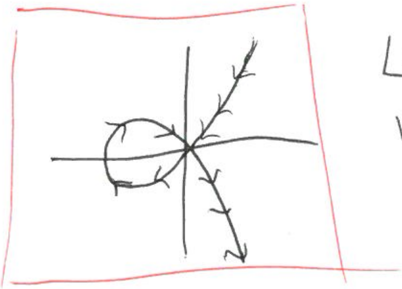
Defn Given $f: X \to Y$ a continuous map, the mapping cylinder is the topological space

$$M_f = (X \times I) \amalg Y / \sim$$

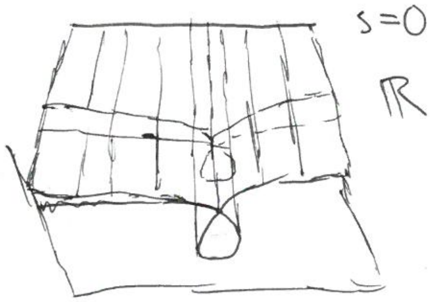
$$(x,1) \sim y \iff f(x) = y$$



Ex/ Consider ~~f(x)=~~ ~~$f: \mathbb{R} \to \mathbb{R}^2$~~ the real roots of the equation $y^2 = x^2(x+1)$



Let $t \in \mathbb{R}$ parameterize this curve via $f: \mathbb{R} \to \mathbb{R}^2 : t \mapsto (x(t), y(t))$. Form M_f:



The advantage of a mapping cone: M_f always def. retracts onto Y:

$$F: M_f \times I \to M_f$$

$$F(m,s) = \begin{cases} (m', t\cdot(1-s)) & \text{if } m = (m',t) \in X \times I \\ m & \text{if } m \in Y \end{cases}$$

"We can squeeze X onto Y by collapsing I to $\{1\}$"

~~If, in addition, $f(X)$ is a deformation retraction of Y, then we can def. retract~~