CLASS 15, MARCH 11TH: INTEGRAL CLOSURES

Recall that last time we proved the following result:

Proposition 15.1. *The subset $\tilde{R} \subseteq A$ given by*

$$\tilde{R} = \{a \in A \mid a \text{ is integral over } R\}$$

forms a subring of A. If $a \in A$ is integral over $\tilde{R}$, then it is integral over R, thus in $\tilde{R}$.

This is an extremely excellent result, as it tells us that $\tilde{\cdot}$ is a **closure-operation**; applying it twice gives back the result of applying it once! Thus we give it a special name:

Definition 15.2. If $R \subseteq A$, then we call $\tilde{R}$ obtained as in Proposition 15.1 the **integral closure** of R in A. If $\tilde{R} = R$, then R is said to be **integrally closed**. If R an integral domain is integrally closed inside of $\mathrm{Frac}(R)$, then R is said to be **normal**.

Example 15.3.
- If $\mathbb{Q} \subseteq K$ is a finite extension of fields, then we can consider the integral closure of $\mathbb{Z}$ inside K. This is how one obtains the *ring of integers* of K, named $\mathcal{O}_K = \tilde{Z}$.
- In line with the previous example, if we consider $K = \mathbb{Q}(\sqrt{n})$, then

$$\mathcal{O}_K = \begin{cases} \mathbb{Z}[\sqrt{n}] & n \equiv 2, 3 \pmod 4 \\ \mathbb{Z}[\frac{\sqrt{n}+1}{2}] & n \equiv 1 \pmod 4 \end{cases}$$

  For a fairly accessible write up of this result I invite you to check out `https://math.stackexchange.com/questions/654202/determining-ring-of-integers-for-mathbbq-sqrt17`.
- Last time we showed that if R is a UFD, then R is integrally closed in its field of fractions. Thus we acquire the result all UDFs are normal!
- If R is an integral domain, another important closure is the *absolute integral closure*, often denoted by R^+. It is precisely the integral closure inside $\overline{\mathrm{Frac}(R)}$.

Example 15.4. $R = K[x, y]/\langle y^2 - x^3\rangle$: You showed that this is an integral domain on Homework 3. Therefore we can consider its integral closure inside $\mathrm{Frac}(R)$.

First, let's check that $Frac(R) = K(\frac{y}{x})$. Note that

$$x = \frac{x^3}{x^2} = \frac{y^2}{x^2} = \left(\frac{y}{x}\right)^2$$

$$y = \frac{y \cdot x^3}{x^3} = \frac{y^3}{x^3} = \left(\frac{y}{x}\right)^3$$

Set $t = \frac{y}{x}$, and consider the map $\iota : R \hookrightarrow K(t)$ with $\iota(x) = t^2$ and $\iota(y) = t^3$. Note t is integral in R, since it satisfies $t^2 - x = 0$. Additionally, $K[t]$ is itself normal (since it is a PID, thus a UFD). Therefore the **normalization** of R (i.e. the integral closure of R in $Frac(R)$) is $K[t]$. Thinking about this on Spec yields an interesting interpretation of normalizations of 'curves'.

Next, we move toward the Noether Normalization Theorem. This allows us to think of K-algebras as integral extensions of polynomial rings! Let A be a K-algebra throughout.

Definition 15.5. Elements $z_1, \dots, z_n \in A$ are **algebraically independent** if the surjection

$$K[x_1, \dots, x_n] \to K[z_1, \dots, z_n] : x_i \mapsto z_i$$

is an isomorphism.

The kernel being 0 is simply saying there exist no polynomial relations on the z_i; i.e. if $F \in K[x_1, \dots, x_n]$, then

$$F(z_1, \dots, z_n) = 0 \quad \implies \quad F = 0$$

Theorem 15.6 (Noether Normalization)**.** *If $A \cong K[x_1, \dots, x_N]/I$ is a finitely generated K-algebra, then there exists $z_1, \dots, z_n \in A$ algebraically independent over K such that A is a finite $B = K[z_1, \dots, z_n]$-module.*

Example 15.7. In the case of Example 15.4, we can let $z_1 = x$ (or y). Then we can view

$$R = (K[x])[y]/\langle y^2 - x^3 \rangle$$

But y is integral over $K[x]$ by the relation defining the ideal. Thus $K[x]$ is *a* Noether Normalization of R.

We prove this theorem by a sort of descending induction argument, stating that if there is an algebraic relation on a finite set of generators, then we can cleverly reduce to a smaller collection:

Lemma 15.8. *Given the set up of Theorem 15.6, if $z_1, \dots, z_n \in A = K[z_1, \dots, z_n]$ are not algebraically independent, then there exists $z_1^*, \dots, z_{n-1}^*$ such that z_n is integral over $A^* = K[z_1^*, \dots, z_{n-1}^*]$. Moreover, $A = A^*[z_n]$.*

I will now prove Theorem 15.6 assuming Lemma 15.8. We will return to the proof of Lemma 15.8 next time.

Proof. (of Theorem 15.6): We proceed by induction on N. If $N = 0$, then there is nothing to do. Suppose the result is true for up to $N - 1$ generated algebras. If there does not exist any polynomial relation on the x_i, i.e. $I = 0$, then we are also done; let $z_i = x_i$ as A is already a polynomial ring. Let F be a non-zero algebraic relation on the generators of A:

$$F(x_1, \dots, x_N) = 0$$

Lemma 15.8 implies that there exist $x_1^*, \dots, x_{N-1}^* \in A$ such that $A = A^*[x_N] = K[x_1^*, \dots, x_{N-1}^*][x_N]$ and x_N is integral over A^*. By the inductive hypothesis, we can conclude the existence of elements $z_1, \dots, z_n$ such that A^* is a finite extension of $K[z_1, \dots, z_n]$. But by our tower laws, this further implies that

$$K[z_1, \dots, z_n] \subseteq A^* \subseteq A$$

are 2 finite extensions, thus so is their composition. This proves the result.

□