HOMEWORK 8: GEOMETRY & TOPOLOGY
DUE: WEDNESDAY APRIL 24TH

1) Let K be an algebraically closed field and let X_1, X_2 be 2 proper varieties in K^n. Let $U_i = X_i^c = K^n \setminus X_i$ be its open complement. Show that $U_1 \cap U_2 \neq \emptyset$. This shows that K^n is irreducible and also demonstrates that the Zariski Topology is not *Hausdorff*.

2) Given a field extension $K \subseteq L$, and a variety $X = V(J) \in K^n$, call $(a_1, \ldots, a_n) \in L^n$ an L**-valued point** of X if $f(a_1, \ldots, a_n) = 0$ for every $f \in J$. Prove an analogue of Hilbert-Nullstellensatz using all L-valued points where L/K is algebraic.

3) Suppose $R \subseteq S$ is an integral extension of Noetherian rings. Given $\mathfrak{p} \in \mathrm{Spec}(R)$, show that there are only finitely many prime ideals $\mathfrak{q} \in \mathrm{Spec}(S)$ lying over $\mathfrak{p}$.

4) We know that $\mathrm{Spec}(W^{-1}R)$ can be view as a subset of $\mathrm{Spec}(R)$. Show that $\mathrm{Spec}(R_f)$ is exactly the complement of $V(f)$ in $\mathrm{Spec}(R)$.

5) Consider the ring
$$R = K[x_{1,1}, x_{2,1}, x_{2,2}, x_{3,1}, \ldots] = K[x_{i,j}]_{i \geq j}$$
This is a non-Noetherian ring, since it has very natural ascending chains of ideals that never stabilize. Consider now the multiplicative set W which is defined as the complement of
$$W^c = \langle x_{1,1} \rangle \cup \langle x_{2,1}, x_{2,2} \rangle \cup \langle x_{3,1}, x_{3,2}, x_{3,3} \rangle \cup \cdots$$
Show that $W^{-1}R$ is a Noetherian ring. What does $\mathrm{Spec}(W^{-1}R)$ look like? [1]

6) If R is a Noetherian ring, we know it has finitely many minimal primes. Can you describe how to find them geometrically?
   What about algebraically? Show that minimal primes of R contain only zero divisors.

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[1] This yields an example of a Noetherian topological space of 'infinite dimension'.