CLASS 35, DECEMBER 7: COVERING SPACES

One final example of a fundamental group should be noted:

Example 35.1. Consider the space $X = S^1 \vee S^1$, obtained by identifying 1 point on 2 distinct circles (looks like an 8). This space has a non-abelian fundamental group. Indeed, the group can be described explicity as $\mathbb{Z} * \mathbb{Z}$, the free product on 2 copies of $\mathbb{Z}$.

Without going into too much detail, the non-abelianess of this group can be realized as follows: let γ_1 be once around the left circle, and let γ_2 be once around the right circle. Then

$$\gamma_1 * \gamma_2 \not\simeq \gamma_2 * \gamma_1 \text{ rel } \{0, 1\}$$

Proposition 35.2 (Functoriality of π_1)**.** *Given a continuous map $f : X \to Y$ of topological spaces, we can induce a group homomorphism*

$$\pi_1(f) = f_* : \pi_1(X, x) \to \pi_1(Y, f(x)) : \gamma \to f \circ \gamma$$

Moreover, if $g : Y \to Z$ is another, $(g \circ f)_* = g_* \circ f_* : \pi_1(X, x) \to \pi_1(Z, g(f(z)))$.

Proof. It only goes to show that $f_*(\gamma_1 * \gamma_2) \simeq f_*(\gamma_1) * f_*(\gamma_2)$. They are in fact equal!

$$f_*(\gamma_1 * \gamma_2) \simeq f_*(\gamma_1) * f_*(\gamma_2) = \begin{cases} f(\gamma_1(2t)) & t \leq \frac{1}{2} \\ f(\gamma_2(2t-1)) & t \geq \frac{1}{2} \end{cases}$$

The second statement follows immediately from the definition of f_*. $\square$

Now we come to a geometric analogy with Galois Theory:

Definition 35.3. A covering space of X is a surjective map $p : X' \to X$ such that for each $x \in X$, there exists U a neighborhood of x such that

$$p^{-1}(U) \cong \coprod_{\alpha} U$$

that is to say that the preimage of U is many copies of U in X'.

Some examples of this phenomenon we have already seen are as follows:

Example 35.4. The map $\theta_n : S^1 \to S^1 : t \mapsto nt$ is a covering space. Indeed, for a given x, we can choose a sufficiently small $\epsilon < \frac{1}{2n}$ and then for $t \in S^1$, consider $U = (t - \epsilon, t + \epsilon)$. In this case,

$$\theta_n^{-1}(U) = \bigcup_{i=0}^{n-1} \left(\frac{t + i - \epsilon}{n}, \frac{t + i + \epsilon}{n} \right) \cong \coprod_{i=0}^{n-1} U$$

such a thing is called an n-sheeted covering space. Note that this induces the (injective!) map

$$(\theta_n)_* : \pi_1(S^1) \to \pi_1(S^1) : m \mapsto n \cdot m$$

Another example of a covering space of the circle is given by $p : \mathbb{R} \to S^1 : t \to \lfloor t \rfloor$. Of course, given $\epsilon < \frac{1}{2}$, we see that

$$p^{-1}((t - \epsilon, t + \epsilon)) = \bigcup_{i \in \mathbb{Z}} (t + i - \epsilon, t + i + \epsilon) \cong \coprod_{\mathbb{Z}} (t - \epsilon, t + \epsilon)$$

Lemma 35.5. *If X satisfies some mild conditions*[1]*, then there exists a simply connected covering space $\tilde{X} \to X$.*

$\tilde{X}$ is called the **Universal Cover** of X. $\mathbb{R}$ is the universal cover of S^1. It is constructed by taking the space of all equivalence classes of paths in X which start at a selected basepoint x_0. The difficultly in the proof of this lemma is showing that it is simply connected. This gets us to the classification of covering spaces:

Theorem 35.6 (Classification of Covering Spaces)**.** *If X satisfies the same mild conditions, then there is a bijection between basepoint preserving path connected covering spaces and*

$$\{p : X' \to X \mid p \text{ is a covering space, } p(x_0') = x_0\}/\cong \;\longleftrightarrow\; \{H \subseteq \pi_1(X, x_0) \mid H \text{ a subgroup}\}$$
$$p \mapsto p_*\pi_1(X', x_0')$$

If we forget about the choice of basepoint, then we get

$$\{p : X' \to X \mid p \text{ is a covering space}\}/\cong \;\longleftrightarrow\; \{H \subseteq \pi_1(X, x_0) \mid H \text{ a subgroup}\}/\sim$$

where $\sim$ denotes conjugacy equivalence.

Finally, if $H' \subseteq H \subseteq \pi_1(X)$, and $X_{H'}$ and X_H are their associated path connected covering spaces, then

$$\exists p'' : X_{H'} \to X_H$$

a covering space which factors the covering space $p' : X_{H'} \to X$ and $p : X_H \to X$; $p \circ p'' = p'$.

There is also a notion of a *normal* cover, which corresponds exactly to the notion of a normal subgroup, and a notion of **deck transformations**, which plays a nearly identical role to the automorphisms fixing the base field in Galois Theory:

Theorem 35.7. *If L/K is a Galois extension of fields, then*

$$\{K' \mid K \subseteq K' \subseteq L\}/ \longleftrightarrow \{H \subseteq Gal(L/K) \mid H \text{ a subgroup}\}$$

[1] X is path connected, locally path connected, and semilocally simply-connected.