HOMEWORK 9: METRIZATION AND COMPLETENESS
DUE: NOVEMBER 30

1) Show that if X is a T3+T1 space, and $X = \bigcup_{i \in \mathbb{N}} K_i$ where K_i are compact subspaces, then X is paracompact. Use this to show that
$$\mathbb{R}^{\oplus \mathbb{N}} = \{x \in \mathbb{R}^{\mathbb{N}} \mid x_i = 0 \ \forall i \gg 0\}$$
with the box topology is paracompact.

2) Show that if X is a T3 space, then X is paracompact if either
   - $X = X_1 \cup \ldots \cup X_n$, where X_i are paracompact closed subspaces.
   - $X = \bigcup_{i \in \mathbb{N}} X_i$, where X_i are paracompact closed subspaces with X_i° still covering X.

3) Show that if X is a complete metric space, and $A_1 \supseteq A_2 \supseteq \ldots$ is a nested sequence of closed subsets for which $\mathrm{diam}(A_n) \to 0$, then $\bigcap_i A_i \neq \emptyset$. Note that here the **diameter** is given by
$$\mathrm{diam}(A) = \sup\{d(x, y) \mid x, y \in A\}$$

4) Given X and Y spaces, consider $\mathcal{C}$ the space of continuous functions $X \to Y$ and the evaluation map
$$ev : X \times \mathcal{C} \to Y : (x, f) \mapsto f(x)$$
Show that if Y is a metric space, and $\mathcal{C}$ has the uniform topology, then ev is continuous.

5) Show that the completion of a metric space is unique[1]. That is to say if there exist Y, Y' completions of X, then there exist distance preserving continuous maps (**isometries**) $f : Y \to Y'$ and $g : Y' \to Y$ which preserve X.

6) A map $p : Y \to X$ is said to be **perfect** if it is continuous, surjective, closed, and for each $x \in X$, $p^{-1}(x) \subseteq Y$ is compact. You have encountered perfect maps in Homework 4.

   Let X be a Hausdorff space. If $\gamma : I \to X$ is a space filling curve, show γ is a perfect map.

   Perfect maps preserve many properties of a space, e.g. if X is second-countable, so is Y. Use this to show X with a space filling curve is metrizable.

7) The converse of the previous problem is the **Hahn-Mazurkiewicz Theorem**: If X is compact, connected, locally connected, and metrizable, then there exists a space filling curve in X. Use it to show there exists a space filling curve in $I^{\mathbb{N}}$ with the product topology.

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[1]Similar to compactifications, Y is a completion of X if it is a complete metric space, and $\bar{X} = Y$.