HOMEWORK 10: ASCOLI & HOMOTOPY
DUE: DECEMBER 7

1) Suppose X_n is metrizable with metric d_n. Show that $X = \prod_{i=1}^{\infty} X_n$ is metrizable:
$$D(x, y) = \sup_i \left(\frac{\min\{1, d_i(x_i, y_i)\}}{i} \right)$$
Also, show that if each X_n is totally bounded, then so is X. As a result, without Tychonoff, we have a countable product of compact metric spaces is a compact metric space.

2) Show *Arzela's Theorem*: If X is compact, and $f_n \in C(X, \mathbb{R}^m)$ is a sequence of equicontinuous and pointwise bounded functions, then f_n has a uniformly convergent subsequence.

3) Show that if $f : X \to Y$ is a continuous function, and there exist $g, h : Y \to X$ such that $f \circ g \simeq Id_Y$ and $h \circ f \simeq Id_X$, then $X \simeq Y$.

4) X is contractible if $X \simeq x$, where x is representative of a point with its unique topology. Show that if Y is contractible, then every map $X \to Y$ is homotopic to one another. If X is path connected, show the same is true for functions $Y \to X$.

5) Show that if $r : Y \to Z$ is a retraction (cf homework 7), and $z \in Z$, then the induced map
$$r : \pi_1(Y, z) \to \pi_1(Z, z) : \gamma \mapsto r \circ \gamma$$
is surjective.

6) Suppose $Y \subseteq \mathbb{R}^n$, and $f : Y \to Z$ is a continuous map. Show that if f extends to a map from $\tilde{f} : \mathbb{R}^n \to Z$, then the induced map $f_* : \pi_1(Y, y) \to \pi_1(Z, z)$ is 0.

7) Show that $\pi_1(X, x_0)$ is abelian if and only if for every 2 paths γ_0, γ_1 connecting x_0 to x_1, the change of base point maps
$$\pi_1(X, x_0) \to \pi_1(X, x_1) : \sigma \mapsto \bar{\gamma}_i * \sigma * \gamma_i$$
are equal as group homomorphisms.