CLASS 9, WEDNESDAY FEBRUARY 28TH: TENSOR PRODUCTS I

In Homework 2, I defined the tensor product of 2 modules M, N. This is one of the most powerful tools in the study of module theory, and today we will study some of the basic properties. Recall the definition:

Definition 0.1. The **tensor product** of two R-modules M, N is

$$M \otimes_R N = \{\sum_{i=1}^{l} m_i \otimes n_i \mid m_i \in M, n_i \in N\}/\sim$$

where $\sim$ is defined by

1) $mr \otimes n \sim m \otimes rn$
2) $m \otimes n + m' \otimes n \sim (m + m') \otimes n$
3) $m \otimes n + m \otimes n' \sim m \otimes (n + n')$

In the homework, you are asked to prove the following 2 facts: $M \otimes_R N$ has a natural R-module structure, and tensoring by a ring S which is also an R-module can upgrade M to an S-module (cf Homework 2, 5/6). These facts are invaluable. I now state some further basic properties:

Proposition 0.2. *1) If $\phi : R \to S$ is a ring homomorphism, then there is a natural R-module homomorphism $M \to M \otimes_R S$ given by $m \mapsto m \otimes 1$.*

2) There is a natural map $\otimes : M \oplus N \to M \otimes_R N$ given by $(m, n) \mapsto m \otimes n$. This is not a homomorphism!

3) If $\varphi : M \oplus N \to P$ is a bilinear map, then $\exists!\ \Phi : M \otimes_R N \to P$ such that $\varphi = \Phi \circ \otimes$.

Proof. I prove the statements in order:

1) If $r \in R$, then $r\phi(m) = r(m \otimes 1) = (rm) \otimes 1 = \varphi(rm)$. Similarly, relation 2) above implies ϕ is additive. Therefore it is a homomorphism of R-modules.
2) There is almost nothing to prove. This is what is called a **balanced product**, and is key to the definition of the tensor product. Note that this is not a homomorphism of R-modules because
$$\otimes((m, n) + (m', n')) = \otimes(m + m', n + n') = (m + m') \otimes (n + n') =$$
$$m \otimes n + m \otimes n' + m' \otimes n + m' \otimes n' \neq m \otimes n + m' \otimes n'$$
3) First note that a map is called **bilinear** if restricting to a specific $m \in M$ or $n \in N$ makes φ into a homomorphism:
$$\varphi(rm + m', n) = r\varphi(m, n) + \varphi(m', n)$$
$$\varphi(m, rn + n') = r\varphi(m, n) + \varphi(m, n')$$
Therefore, in part 2), $\otimes$ is a natural bilinear map. As a result, we define
$$\Phi : M \oplus N \to P : \sum_i m_i \otimes n_i \mapsto \sum_i \varphi(m_i, n_i)$$

This forces $\varphi = \Phi \circ \otimes$, since

$$(\Phi \circ \otimes)(m, n) = \Phi(m \otimes n) = \varphi(m, n)$$

Moreover, this is a homomorphism as a result of the conditions of the definition of $M \otimes_R N$:

$$\Phi(rm \otimes n + m' \otimes n') = \Phi(rm \otimes n) + \Phi(m' \otimes n') = r\varphi(m, n) + \varphi(m', n')$$

$\square$

Note: This gives an important way to convert a pair of homomorphism $M \otimes P$ and $N \otimes P$ into either a bilinear map $M \oplus N \to P$ OR a single homomorphism $M \otimes_R N \to P$.

Example 0.3 (Size of the Tensor Product)**.** It is tempting to believe that the tensor product of two modules M, N is always bigger than M or N individually, much like the direct sum. However, the following nice proposition shows this is not the case:

Proposition 0.4. *Let R be a ring, and I be an ideal. Then*

$$M \otimes_R R/I \cong M/IM$$

Proof. I claim that the map is given by

$$\varphi : M \otimes_R R/I \to M/IM : m \otimes \bar{r} \mapsto \bar{r} \cdot m$$

This map is surjective, since if $m + IM \in M/IM$, we can consider $m \otimes \bar{1} \in M \otimes_R R/I$, whose image is the desired element. In addition, if $\varphi(m \otimes \bar{r}) = \bar{0}$, then $rm \in IM$. Therefore, $rm = im'$ for $i \in I$. Therefore, the following equalities show that φ is injective:

$$m \otimes \bar{r} = rm \otimes \bar{1} = m' \otimes \bar{i} = m' \otimes 0 = 0$$

$\square$

Another typical example is as follows: Consider $R = \mathbb{Z}$;

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \cong \mathbb{Z}/\gcd(m, n)\mathbb{Z} : a \otimes b \mapsto a \cdot b$$

$$\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0$$

Another natural example we will use extensively is the localization of a module:

Example 0.5 (Localization)**.** We have already described $W^{-1}R$, the localization of R at W. This can be extended to modules in the following way: There is a natural localization map $R \to W^{-1}R : r \mapsto (1, r)$. Thus we can form the localization of an R-module:

$$W^{-1}M := M \otimes_R W^{-1}R$$

This naturally has the structure of a $W^{-1}R$ module by the Homework 2 Exercise 6. It is also an R-module by virtue of the localization map.

It is useful to check that the localization of a module, $W^{-1}M$, is equivalent to that given by copying the conditions for a ring:

$$W^{-1}M = \{(w, m) \ : \ w \in W, \ m \in R\}/\sim$$

$(w, m) \sim (w', m')$ if there exists $s \in W$ such that

$$s(wm' - w'm) = 0$$

We will see soon that a module locally satisfying some properties gives us 'global' information as well!