CLASS 10, SEPTEMBER 28: A GENERALIZED MVT

Today I will introduce a new topology that can be put on any totally ordered set. This, together with connectedness can drastically improve the applicability of the Intermediate Value Theorem from calculus to more geometric contexts.

Definition 10.1. A **totally ordered set** is a pair $(X, \leq)$ where X is a set $\leq$ is a binary operation on X such that

1) $a \leq b$ and $b \leq a$ implies $a = b$.
2) $a \leq b$ and $b \leq c$ implies $a \leq c$.
3) $a \leq b$ or $b \leq a$.

Let X be a totally ordered set. Then the order topology τ is the topology generated by the basis

$$\{(a, b), (-\infty, b), (a, \infty)\}$$

where $(a, b) = \{c \mid a < c < b\}$, and the ∞ omit one of the inequalities.

Canonical examples of this are already known to us:

Example 10.2.

1) $\mathbb{R}$ with the Euclidean topology is exactly $\mathbb{R}$ with the order topology given by inequalities of real numbers.
2) $\mathbb{R}^2$ can be endowed with the **dictionary topology**; $(x_1, x_2) \leq (y_1, y_2)$ if either $x_1 < y_1$ or $x_1 = y_1$ and $x_2 < y_2$.
3) By our usual method of induction, the previous example can be bootstrapped to $\mathbb{R}^n$. This is fine than (thanks Ben) the Euclidean topology for $n \geq 2$.
4) Any set of cardinal or ordinal numbers has a natural ordering by size.

Now, we can recall a classical result from Calculus and see what we can tweak;

Theorem 10.3 (The Intermediate Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < f(b)$ (or flipped). Then for any $c' \in R$ such that $f(a) < c' < f(b)$, there exists $c \in (a, b)$ such that $f(c) = c'$.*

Theorem 10.4 (The *Improved* Intermediate Value Theorem)**.** *Let $f : X \to Y$ be a continuous map between X a connected topological space and Y an ordered set with the order topology. Assume $a, b \in X$ are such that $f(a) < r < f(b)$. Then there exists $x \in X$ such that $f(x) = r$.*

Notice that here X has just 1 property of $[a, b]$, and Y 1 property of $\mathbb{R}$.

Proof. We have two open sets in the order topology of interest: $U_< = (-\infty, r)$ and $U_> = (r, \infty)$. These are disjoint open subsets of Y, and thus their preimages are as well. But X is connected, so $X \neq f^{-1}(U_<) \cup f^{-1}(U_<)$. This implies there exists $x \in X \setminus (f^{-1}(U_<) \cup f^{-1}(U_>))$, which of course imples $f(x) = r$. $\square$

Example 10.5. The n-sphere S^n is a connected space, and we can endow $\mathbb{R}$ with the order/Euclidean topology. For any given continuous map $t : S^n \to \mathbb{R}$ a corresponding map $T : S^n \to \mathbb{R}$ such that $T(\mathbf{x}) = t(\mathbf{x}) - t(-\mathbf{x})$. This map is either 0 constantly or has some value $(y_1, y_2) > (0, 0)$. Moreover, this function is odd: $T(\mathbf{x}) = -T(-\mathbf{x})$. The Improve

Intermediate Value Theorem thus implies (in either case) that there exists $\mathbf{x} \in S^n$ such that $-y < T(\mathbf{x}) < y$. This demonstrates for example the following fact:

There exists antipodal points on Earth that share exactly the same temperature.[1]

To finish up, I want to give a slightly stronger notion of connected and compare it with the original notion.

Definition 10.6. A space X is called **path connected** if for any 2 points $x, y \in X$, there exists a continuous function (path) $\gamma : [0,1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Rephrased, a space is path connected if all of its points can be connected by paths.

Proposition 10.7. *A path connected space X is also connected.*

Proof. Suppose that X is separated by U and V. Let $x \in U$ and $y \in V$. Consider a path γ connecting these points. We can consider the open disjoint sets $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$. Their union is $[0,1]$, which is connected. Therefore, we may assume $\gamma^{-1}(V) = \emptyset$. But $1 \in \gamma^{-1}(V)$, a contradiction.

Alternatively: The image of γ is connected. Since $Im(\gamma) = \gamma([0,1])$ is a connected subset of $X = U \cup V$, it must belong to one or the other. $\square$

However, a connected space need not be path connected.

Example 10.8. Consider the following set with the subspace topology:

$$X = \{(x, \sin(\frac{1}{x})) \mid 0 < x \leq 1\} \cup \{0\} \times (-1,1) \subseteq \mathbb{R}^2$$

I claim this set is connected. Indeed, if U, V separate X, then the 2 components (which are images of intervals, thus connected) must belong to one or the other. Call the graph Γ and the interval I. WLOG, suppose $\Gamma \subseteq U$ and $I \subseteq V$. But V is open in the subspace topology, therefore for $x \in V$, $B(x,r) \subseteq V$ for some $r > 0$. However, this would imply that

$$\emptyset \neq \Gamma \cap V \subseteq U \cap V = \emptyset$$

an immediate contradiction. So X is connected.

To show it is not path connected, suppose we have a path from $(0,0$ to $(1, \sin(1))$. By connectedness and closedness, there exists some largest $b < 1$ for which $\gamma([0,b]) \subseteq I$. By continuity, we have that $\gamma(b) = \lim_{x \to b} \gamma(x)$. However, there exist infinitely many x near b for which $\sin(x) = 1$ and $\sin(x) = -1$, so the limit can't exist. Therefore, X is not path connected.

[1]With some stronger algebraic topological methods, we can show any map $S^n \to \mathbb{R}^n$ also has this property. Thus in particular, we have antipodal points of earth with the same temperature and pressure!