HOMEWORK 4: HAUSDORFF & COMPACTNESS
DUE: FRIDAY, OCTOBER 5TH, 2018

1) Show that if X is a compact Hausdorff space under 2 topologies τ, τ', then either $\tau = \tau'$ or they are incomparable.

   Solution: Suppose $\tau \supseteq \tau'$. Then $Id : X_\tau \to X_{\tau'}$ is a continuous map between a compact space and a Hausdorff space. Therefore, applying Corollary 12.1 from class, we see Id is a homeomorphism. However, this implies that $\tau = \tau'$, proving the claim.

2) Show that every compact subspace $Y \subseteq X$ of a metric space X is bounded in that metric space (e.g. there exists $x \in Y$ and $r > 0$ such that $Y \subseteq B(x, r)$. Find an example of a metric space X, and $Y \subseteq X$ which is closed and bounded but NOT compact.

   Solution: $U_r = B(x, r)$ forms a cover of Y in X as $r > 0$ varies. Therefore, since Y is compact we see that
$$Y = B(x, r_1) \cup B(x, r_2) \cup \ldots \cup B(x, r_n).$$
   Letting $r = \max\{r_i\}$, we see that $Y \subseteq B(x, r)$.

   To find the example, consider $\mathbb{R}^{\mathbb{N}}$ with the uniform metric topology of homework 3. We then note that $X = [0, 1]^{\mathbb{N}}$ is a closed subset of this space, since in particular it is $\bar{B}((\frac{1}{2}, \frac{1}{2}, \ldots), \frac{1}{2})$. X is in particular bounded by $B(\mathbf{\frac{1}{2}}, r)$ for any $r > \frac{1}{2}$ (in fact, in this space EVERY subspace is bounded, since $\mathbb{R}^{\mathbb{N}} = B(\mathbf{0}, 2)$. However, given the cover $\{B(x, r) \mid x \in [0, 1]^{\mathbb{N}}\}$ for any $r \leq \frac{1}{2}$, we see that no finite refinement will cover $[0, 1]^{\mathbb{N}}$. Indeed, if some selection of $\mathbf{x}_i$ did, we could write
$$\mathbf{x}_i = (x_{i1}, x_{i2}, \ldots)$$
   and let $y_i = 1$ if $x_{ii} \leq \frac{1}{2}$ and $y_i = 0$ if $x_{ii} > \frac{1}{2}$. For $i > n$ where n is as in the definition of compactness, we can let $y_i = 0$. Then we note
$$|\mathbf{x}_{ii} - y_i| \geq \frac{1}{2}$$
   For any i, so letting $\mathbf{y} = (y_1, y_2, \ldots) \in X$, we see
$$y \notin \bigcup_{i=1}^{n} (B(\mathbf{x}_i, \frac{1}{2}).$$

3) Show that if $f : X \to Y$ is a continuous map from a compact space to a Hausdorff space, then f is a closed map.

   Solution: Let $Z \subseteq X$ be a closed set. Then Z is also compact by Theorem 11.3. It goes to show that $f(Z)$ is closed. But $f(Z)$ is compact by Proposition 11.8. Therefore, since Y is Hausdorff, Theorem 11.7 implies $f(Z)$ is closed.

4) Show that if Y is compact, then the projection map $\pi : X \times Y \to X$ is a closed map.

Solution: Let $Z \subsetneq X \times Y$ be a closed subset. As usual, I will demonstrate that $\pi(Z)^c$ is open. Suppose $x \notin \pi(Z)^c$. This implies no element of Z maps to x. Moreover, since Z is assumed to be closed, we know that there exists an open neighborhood $U_y \times V_y$ of (x, y) such that $Z \cap U_y = \emptyset$. But this operation produces an open cover:

$$\{x\} \times Y \subseteq \bigcup_{y \in Y} U_y \times V_y.$$

Since the LHS is homeomorphic to Y, we can apply compactness:

$$\{x\} \times Y \subseteq (U_{y_1} \times V_{y_1}) \cup \ldots \cup (U_{y_n} \times V_{y_n})$$

Since this covers the previous set, we see

$$\{x\} \times Y \subseteq (U_{y_1} \cap \ldots \cap U_{y_n}) \times Y$$

and therefore, $y \in U_{y_1} \cap \ldots \cap U_{y_n} \subseteq \pi(Z)^c$. This completes the proof.

5) Let $f : X \to Y$ be a map with Y compact and Hausdorff. Show that f is continuous if and only if the graph

$$\Gamma = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$$

is closed.

Solution:
($\Rightarrow$): Suppose f is continuous. I will show that Γ^c is open. Given $(x, y) \subseteq \Gamma^c$, we know that $f(x) \neq y$. Therefore, since Y is Hausdorff, we note that there exists U, V open disjoint subsets of Y such that $f(x) \in U$ and $y \in V$. Since f is assumed continuous, we know that there exists an open neighborhood U' of x such that $f(U') \subseteq U$. This shows that $U' \times V$ is an open subset disjoint for $(x, f(x)) \in \Gamma$. Taking the the union over all such sets, we see Γ^c is open as desired.
($\Leftarrow$): Suppose the graph is closed. Then we have a sequence of maps

$$X \to \Gamma \subseteq X \times Y \to Y$$

The composition is the original map f. Let C be a closed subset of Y. Since the second map is a projection, it is certainly continuous. Therefore, $\pi_Y^{-1}(C) = X \times C$ is closed, so by assumption, $(X \times C) \cap \Gamma$ is closed in $X \times Y$. By the previous problem, we have that $\pi_X((X \times C) \cap \Gamma)$ is closed in X. But this is exactly the preimage of C under f! If $f(x) \in C$, then $(x, f(x)) \in (X \times C) \cap \Gamma$.

6) Let $p : X \to Y$ be a closed, continuous, and surjective map with compact fibers $p^{-1}(y)$. Show that if Y is compact, so is X.[1]

Solution: I'll begin by proving the footnote. Consider $U^c \subseteq X$ a closed set. Taking its image, we get a closed subset of Y. Furthermore, $U^c \subseteq f^{-1}(f(U^c))$ always. Therefore, we get that

$$U \supseteq f^{-1}(f(U^c)^c)$$

[1]You may assume the following: If $p : X \to Y$ is a closed map, then if $p^{-1}(y) \subseteq U$ is some open neighborhood of the fiber, there exists an open neighborhood V of y such that $f^{-1}(V) \subseteq U$.

demonstrating that $V = f(U^c)^c$ is the desired set. Note that this is not the same as $f(U)$ necessarily.

Let $X = \bigcup_\alpha X_\alpha$ be an open cover. For a given fiber $X_y = p^{-1}(y) \subseteq X$, since X_y is compact there exists a finite subcovering

$$X_y \subseteq U_{\alpha_1^y} \cup \ldots \cup U_{\alpha_{n_y}^y} = U^y$$

Moreover, U^y is an open neighborhood of X_y, so we can take $V^y \subseteq Y$ open such that $f^{-1}(V^y) \subseteq U^y$. Now, we can use the compactness of Y to refine the cover by the V^y:

$$Y = V^{y_1} \cup \ldots \cup V^{y_m}$$

But then we note

$$X = f^{-1}(Y) = \bigcup_{i=1}^{m} f^{-1}(V^{y_i}) = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n_{y_i}} U_{\alpha_j^{y_i}}$$

Therefore any cover has a finite refinement, and thus X is compact.