HOMEWORK 1: TOPOLOGICAL SPACES
DUE: FRIDAY, SEPTEMBER 14

1) Write down the axioms for a topological space in terms of its collection of closed sets.

Solution: The axioms can be realized by taking complements of the conditions for open sets; namely $\sigma = \{Z \subseteq X \text{ closed}\}$ should have the following properties:
1) $\emptyset, X \in \sigma$.
2) If Z_α is a collection of closed subsets, $\alpha \in \Lambda$, then so is $Z = \bigcap_{\alpha\in\Lambda} Z_\alpha$.
3) If $Z_1, \ldots, Z_n$ is a finite collection of closed subsets, then so is $Z = Z_1 \cup \ldots \cup Z_n$.

2) Let τ_α be a collection of topologies. Is it true that $\bigcap_\alpha \tau_\alpha$ is a topology? What about $\bigcup_\alpha \tau_\alpha$?

Solution: I claim that the intersection of topologies is in fact a topology, but the union is not necessarily.
i. Given $X, \emptyset \in \tau_\alpha$ for each $\alpha \in \Lambda$, $X, \emptyset \in \bigcap_\alpha \tau_\alpha$.
ii. If $Z_\beta \in \bigcap_\alpha \tau_\alpha$, then $Z_\beta \in \tau_\alpha$ for each β, α, and therefore by definition of a topology, we have $\bigcup_\beta Z_\beta \in \tau_\alpha$ for each α, thus $\bigcup_\beta Z_\beta \in \bigcap_\alpha \tau_\alpha$.
iii. Similarly, if $Z_1, \ldots, Z_n \in \bigcap_\alpha \tau_\alpha$, then $Z_i \in \tau_\alpha$ for $i = 1, \ldots, n$ and each α, and therefore by definition of a topology, we have $Z_1 \cap \ldots \cap Z_n \in \tau_\alpha$ for each α, thus $Z_1 \cap \ldots \cap Z_n \in \bigcap_\alpha \tau_\alpha$.

The problem that arises for unions of topologies is that if $U_1 \in \tau$ but not σ, and $U_2 \in \sigma$ but not τ, then there is no reason to expect $U_1 \cap U_2$ or $U_1 \cup U_2$ is in τ or σ. An example of this is as follows:

Take $X = \{a, b, c\}$ a 3-point set, and let $\tau_1 = \{\emptyset, \{a\}, X\}$ and $\tau_2 = \{\emptyset, \{b\}, X\}$. Then $\tau_1 \cup \tau_2 = \{\emptyset, \{a\}, \{b\}, X\}$ which would need to contain $\{a\} \cup \{b\} = \{a, b\}$ to form a topology but does not.

3) Show that on $\mathbb{R}^n$ the following bases generate the same (metric) topology:
- $\mathcal{B}_1 = \{B(x, r) \mid x \in \mathbb{R}^n,\ r > 0\}$.
- $\mathcal{B}_2 = \{(a_1, b_1) \times \ldots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}$.
- $\mathcal{B}_3 = \{B(x, r) \mid x \in \mathbb{Q}^n,\ r \in \mathbb{Q}_+\}$. That is to say we consider open balls of rational radius centered at points with rational coordinates.

Solution: Using Lemma 3.4 from class, we have only to show that for each $x \in X$ and $B \in \mathcal{B}_i$, there is a $B' \in \mathcal{B}_j$ such that $B' \subseteq B$ (for each i, j). Let τ_i be their respective topologies for $i = 1, 2, 3$.
- $\tau_1 \supseteq \tau_3$: Since $\mathcal{B}_1 \supseteq \mathcal{B}_3$, this follows immediately.
- $\tau_3 \supseteq \tau_1$: Fix $x \in \mathbb{R}^n$ and $r > 0$. The for $y \in B(x, r)$, we have $B(y, r') \subseteq B(x, r)$ for $r' = r - d(x, y)$. Furthermore, there exists $z \in \mathbb{Q}^n$ such that $d(y, z) < \frac{r'}{2}$. Therefore, by the triangle inequality $B(z, \frac{r'}{2}) \in \mathcal{B}_3$ and $B(z, \frac{r'}{2}) \subseteq B(y, r') \subseteq B(x, r)$. By the (TFAE) proposition from class, we know $\tau_3 \supseteq \tau_1$.

We may assume the point of interest is the center, possibly by shrinking the basis element.

- $\tau_1 \supseteq \tau_2$: It suffices to show that given $(a_1, b_1) \times \ldots (a_n, b_n)$, we can find x, r such that $B(x, r) \subseteq (a_1, b_1) \times \ldots (a_n, b_n)$. Let $x_i = \frac{b_i + a_i}{2}$ be the midpoint of the interval and $r_i = \frac{b_i - a_i}{2}$. Let $x = (x_1, \ldots, x_n)$. Then if $r = \min\{r_1, \ldots, r_n\}$, and $z \in B(x, r)$, we have

$$|z_i - c_i| \leq \sqrt{|z_1 - c_2|^2 + \ldots + |z_n - c_n|^2} < r \leq \frac{b_i - a_i}{2}$$

implying $z \in (a_1, b_1) \times \ldots \times (a_n, b_n)$.

- $\tau_2 \supseteq \tau_1$: Similarly, given $B(x, r)$, let $r' = \frac{r}{\sqrt{n}}$ and consider the set

$$S = (x_1 - r', x_1 + r') \times (x_n - r', x_n + r')$$

Then if $z \in S$, notice that

$$d(x, z) = \sqrt{(x_1 - z_1)^2 + \ldots + (x_n - z_n)^2} < \sqrt{n \cdot r'^2} = \sqrt{r^2} = r$$

so $z \in B(x, r)$, and $S \subseteq B(x, r)$.

4) Consider the following topologies on $\mathbb{R}$:
- $\mathcal{T}_1$ = the standard Euclidean/metric topology.
- $\mathcal{T}_2$ = the finite complement topology.
- $\mathcal{T}_3$ = the topology with basis $(a, b]$, where $a, b \in \mathbb{R}$.
- $\mathcal{T}_4$ = the topology with basis $(-\infty, b)$, where $b \in \mathbb{R}$.
- $\mathcal{T}_5$ = the topology with basis (a, b) and $(a, b) \setminus K$, where $K = \cup_{n \in \mathbb{Z}} \frac{1}{n}$.

Order them in terms of comparability, i.e. finer, coarser, or incomparable. You can do this with as few as 8 pairwise comparisons.

Solution:I claim that the data is filtered as follows: $\mathcal{T}_i \supset \mathcal{T}_1 \supset \mathcal{T}_j$, for $i = 3, 5$ and $j = 2, 4$, $\mathcal{T}_3 \neq \mathcal{T}_5$ and $\mathcal{T}_2 \neq \mathcal{T}_4$.

- $\mathcal{T}_5 \supset \mathcal{T}_1$ is tautilogical, given $(-1, 1) \setminus K \notin \mathcal{T}_1$.
- Next, from the above computation one can check $\mathcal{T}_5$ has as a basis (a, b) and $(a, b) \cap (-1, 1) \setminus K$. Therefore, it contains

$$(-1, -\frac{1}{2}) \cup (-\frac{1}{2}, -\frac{1}{3}) \cup \ldots \cup \{0\} \cup \ldots (\frac{1}{3}, \frac{1}{2}) \cup (\frac{1}{2}, 1).$$

This is not an element of $\mathcal{T}_3$, and $(2, 3] \notin \mathcal{T}_5$, showing they are incomparable.

- $\mathcal{T}_3 \supset \mathcal{T}_1$, since $(a, b) = \bigcup_{c<b} (a, c]$.
- $\mathcal{T}_2 \subset \mathcal{T}_1$: since every open set of $\mathcal{T}_2$ is of the form

$$(-\infty, a_1) \cup (a_1, a_2) \cup \ldots \cup (a_{n-1}, a_n) \cup (a_n, \infty)$$

- $\mathcal{T}_4 \subset \mathcal{T}_1$ is tautilogical as well (bases include).
- The second incomparability statement: Note that $\mathbb{R} \setminus \{0\} \notin \mathcal{T}_4$, since all open sets of $\mathcal{T}_4$ look like $(-\infty, a)$ for some a. Additionally, $(-\infty, 0) \notin \mathcal{T}_2$ since its complement is infinite.

5) If τ and σ are 2 topologies on X with τ *strictly* finer than σ (i.e. $\tau \supsetneq \sigma$), what can you say about the subspace topology on $Y \subseteq X$?

Solution: You can only say that the subspace topology is finer, no longer necessarily strict. A perfect example of this comes from the previous problem; $\mathcal{T}_5$ is strictly finer than $\mathcal{T}_1$. However, if we restrict our attention to $Y = (2, \infty)$, we see that the topologies are identical.

6) Verify that the following are topologies on a 3-point set $X = \{a, b, c\}$:
   - $\tau_1 = \{\emptyset, \{a, b, c\}\}$
   - $\tau_2 = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$
   - $\tau_3 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$
   - $\tau_4 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

   Additionally, which can be realized as metric topologies? Can you notice a pattern?

Solution: Addressing each individually:
- This is the indiscrete topology on 3 points. Therefore, it is a topology by general principles.
- Intersecting and taking unions with $\emptyset$ or X are meaningless (yielding only the original set or those sets themselves). Therefore, noting that
$$\{c\} \cap \{a, b\} = \emptyset \in \tau_2$$
$$\{c\} \cup \{a, b\} = X \in \tau_2$$
We see that τ_3 is a topology.
- Similarly to the previous problem, the notable considerations is
$$\{a, b\} \cap \{a, c\} = \{a\} \in \tau_3$$
$$\{a, b\} \cup \{a, c\} = X \in \tau_3$$
- This is the discrete topology on 3 points. Therefore, it is a topology by general principles.

Now, I claim that the only possible metric topology on any finite set is the discrete topology. The pattern to realize this is as follows: Label the points $a_1, \ldots, a_n$. For a given $i \in \{1, \ldots, n\}$, consider $d_i = \min_{j \neq i}\{d(a_i, a_j)\}$. Note that this is positive, since we took the minimum of n positive (by the separability axiom) numbers. Then we note $B(a_i, d_i) = \{a_i\}$. Therefore, by taking unions of these sets, we realize that every set is open in a finite metric space, and is therefore discrete!