CLASS 2, SEPTEMBER 10: TOPOLOGICAL SPACES

We now have the required properties to define a topological space, following the principals from last class and in particular Proposition 1.4:

Definition 2.1. A **Topological Space** is a pair (X, τ), where X is a set and τ is a collection of subsets (concisely, $\tau \subseteq \mathcal{P}(X)$) with the following properties:

1) $\emptyset, X \in \tau$.
2) If $U_\alpha \in \tau$, with $\alpha \in \Lambda$ any indexing set, then
$$U = \bigcup_{\alpha \in \Lambda} U_\alpha \in \tau.$$
3) If $U_1, \ldots, U_n \in \tau$, then $V = U_1 \cap \ldots \cap U_n \in \tau$.

Oftentimes people refer to X as a topological space, suppressing τ. The elements of τ are called **open sets**. Their complements are called **closed sets**. τ itself is called a **topology**.

Therefore, a topological space is simply a set with a selection of open sets, where we want the open sets to satisfy axioms like the ones we know best. This gives us a canonical example of topological spaces: metric spaces! The induced topology is often referred to as the **metric** topology. In fact, many common objects in mathematics can be realized topologically.

Example 2.2. Let X be any set.

a) If $\tau = \{X, \emptyset\}$, then τ satisfies the axioms of a topology and therefore gives X a topological structure. This is called the **indiscrete** or **trivial topology**.
b) If $\tau = \mathcal{P}(X)$, then τ is also a topology, called the **discrete topology**.
c) If $\tau = \{Y \subseteq X \mid Y^c \text{ is finite}\}$, then τ is a topology. It is (cleverly) named the **finite complement topology**.
d) If $\tau = \{Y \subseteq X \mid Y^c \text{ is at most countable}\}$, then τ is a topology.

Example 2.3 (2 points)**.** If $X = \{a, b\}$, there there are 4 possible topologies; the discrete topology, the indiscrete topology, $\tau = \{\emptyset, \{a\}, X\}$, and $\tau = \{\emptyset, \{b\}, X\}$. Up to renaming a and b, there are thus 3 topologies.

The 3 point case, for which there are 29 total topologies or 9 up to reordering of points, is illustrated on page 76 on Munkres (or on wikipedia, c.f. finite topological spaces).

Topologies are often compared to one another, and we have words to make there discussions more seamless:

Definition 2.4. Let X be a set, and let τ and σ be 2 topologies on X. Suppose $\tau \supseteq \sigma$ (so that every set open in the σ-topology is open in the τ-topology as well). Then we say τ is **finer** than σ, or σ is **coarser** than τ. In either of these case, we say the topologies are **comparable**. Otherwise, we call them **incomparable**.

Therefore, the indiscrete topology is coarser than every other possible topology. Similarly, the discrete topology is the finest topology.

We can also produce a notion of neighborhoods to add to our favorable comparison with metric spaces:

Definition 2.5. If x is a point in X, and V is any subset containing x, then V is called a **neighborhood of** x if there exists an open subset U such that $x \in U \subseteq V$.

If $S \subset X$ is any subset, then we say V is a **neighborhood of** S if V is a neighborhood of any point of S.

Similar to the case of metric spaces, neighborhoods can be refined:

Proposition 2.6. *If X is a topological space, $x \in X$, and V, V' are 2 neighborhoods of x, then so is $V \cap V'$.*

The proof is left to the reader. Next, we discuss some of the operations that can be performed on sets in a topological space.

Definition 2.7. Let $S \subseteq X$ be a subset of a topological space. We define the **interior** of S, denoted S°, to be the largest open set contained within S. Similarly, we define the **closure** of S, denoted $\bar{S}$, to be the smallest closed set containing S.

Proposition 2.8. *S° and $\bar{S}$ are well defined objects.*

Proof. Note that there is always an open set contained within S, namely the empty set. Therefore, we can simply define

$$S^\circ = \bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U$$

Because this is a union of open sets, it is itself open by the second axiom of a topological space. Of course, any open subset contained within S is inside this union, so this is necessarily the largest such set.

For $\bar{S}$, we can use this operation along with the complement to produce the result:

$$\bar{S} = ((S^c)^\circ)^c$$

That is to say $\bar{S}$ is the complement of the interior of the complement of S. Note that this is closed since the interior is open by definition and we take its complement. It goes to show this is the smallest closed set containing S. Suppose there is a closed Z such that $S \subseteq Z \subseteq \bar{S}$. Taking complements, we retrieve

$$S^c \supseteq Z^c \supseteq \bar{S}^c = (S^c)^\circ$$

Since $(S^c)^\circ$ is the largest open set inside S^c, we see that $Z^c = (S^c)^\circ$ and thus $Z = \bar{S}$. $\square$

Proposition 2.9. *If A, B are 2 subsets of X, then $(A \cap B)^\circ = A^\circ \cap B^\circ$. However, generally $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.*

Proof. By the third axiom, $A^\circ \cap B^\circ$ is an open set, contained within $A \cap B$. Therefore, $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$. Additionally, $(A \cap B)^\circ \subseteq A, B$. Therefore, since it is open, we have $(A \cap B)^\circ \subseteq A^\circ, B^\circ$. Therefore, $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$ and thus equality is shown.

For the second claim, consider $A = [0, 1]$ and $B = [1, 2]$ inside $\mathbb{R}$ with the (standard) metric topology. It is easy to check $A^\circ = (0, 1)$ and $B^\circ = (1, 2)$, but $(A \cup B)^\circ = (0, 2)$. $\square$