HOMEWORK 4: HAUSDORFF & COMPACTNESS
DUE: FRIDAY, OCTOBER 5TH, 2018

1) Show that if X is a compact Hausdorff space under 2 topologies τ, τ', then either $\tau = \tau'$ or they are incomparable.

2) Show that every compact subspace $Y \subseteq X$ of a metric space X is bounded in that metric space (e.g. there exists $x \in Y$ and $r > 0$ such that $Y \subseteq B(x, r)$. Find an example of a metric space X, and $Y \subseteq X$ which is closed and bounded but NOT compact.

3) Show that if $f : X \to Y$ is a continuous map from a compact space to a Hausdorff space, then f is a closed map.

4) Show that if Y is compact, then the projection map $X \times Y \to X$ is a closed map.

5) Let $f : X \to Y$ be a map with Y compact and Hausdorff. Show that f is continuous if and only if the graph
$$\Gamma = \{(x, f(x)) \mid x \in X\} \subseteq X \times Y$$
is closed.[1]

6) Let $p : X \to Y$ be a closed, continuous, and surjective map with compact fibers $p^{-1}(y)$. Show that if Y is compact, so is X.[2,3]

[1] **Hint:** The previous problem may help! Also, equivalent formulations of continuity could be useful.

[2] This is one of the rare times where a preimage of a compact set is compact.

[3] **Hint:** If $p : X \to Y$ is a closed map, then if $p^{-1}(y) \subseteq U$ is some open neighborhood of the fiber, there exists an open neighborhood V of y such that $f^{-1}(V) \subseteq U$. You needn't prove this but it is a nice exercise.