COMPLEX ANALYSIS: FINAL EXAM

Instructions

This exam is to be completed within 24 hours. If you need additional pages, staple them to the exam and point to them within the problem.

To attain full points on a given problem, be sure to write clear concise solutions which properly reference the results used (either by name, e.g. "Goursat's Theorem", or by number within the notes, or by statement).

I recommend that you draft your solutions first (on a private board or scratch paper) before transferring them onto the exam.

Things that are accessible for this exam:

- Notes, be they mine or yours.
- Homework assignments.
- The midterm.
- Stein and Shakarchi.

Things that are NOT accessible:

- Other people.
- The internet.
- Other books, math or otherwise.

By signing your name below you assert that you have taken this exam under the above stated rules. Violations are treated as violations of the honor code!

NAME: ____________________

Score by page:

Page 1: ________ /20

Page 2: ________ /20

Page 3: ________ /25

Page 4: ________ /30

Page 5: ________ /25

Page 6: ________ /30

TOTAL: ________ /150

1) **(10 points)** Consider the power series expansion about the origin of $f(z) = \frac{1}{(1-z)^m}$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Find an explicit formula for a_n, and show that

$$a_n \sim \frac{1}{(m-1)!} n^{m-1}$$

as $n \to \infty$. Using Hadamard's formula show the radius of convergence is 1.

2) **(10 points)** Show explicitly that if $\gamma : [a, b] \to \mathbb{C}$ is a curve, and

$$\bar{\gamma} : [a, b] \to \mathbb{C} : t \mapsto \gamma(b + a - t)$$

is the curve traversed backwards, then

$$\int_{\gamma} f(z)\ dz = -\int_{\bar{\gamma}} f(z)\ dz$$

3) **(20 points)** Given $a > 0$ and $b \in \mathbb{R}$, evaluate the integral

$$\int_0^\infty e^{-ax} \cos(bx) dx \qquad \int_0^\infty e^{-ax} \sin(bx) dx$$

by integrating e^{-Az}, where $A = \sqrt{a^2 + b^2}$ over a sector with angle ω such that $\cos(\omega) = \frac{a}{A}$.

4) **(15 points)** Suppose Ω is an open bounded region and L is a line in $\mathbb{C}$ that intersects Ω in an interval. Write Ω_1 and Ω_2 be the sections of Ω on either side of L, so that $\Omega = \Omega_1 \cup (\Omega \cap L) \cup \Omega_2$ are all disjoint. Show that if Ω_1 and Ω_2 are simply connected, then so is Ω. Pictures with color may be helpful but are not complete solutions!

5) **(10 points)** Show that

$$f(z) = z^{10} + 3z^8 + 3z^6 + 2z^4 + 5z - 2$$

has all of its roots in $B(0, 2)$.

6) **(30 points)** Here we will formalize how Fourier Transforms are useful for solving differential equations of the form

$$(*) \qquad a_n \frac{\partial^n}{\partial t^n} u(t) + a_{n-1} \frac{\partial^{n-1}}{\partial t^{n-1}} u(t) + \ldots + a_0 u(t) = f(t)$$

for u given f an analytic function.

i. Use induction and integration by parts to deduce that if $f \in \mathfrak{F}_a$, then if $g = f^{(n)}$ is the n^{th} derivative of f, then

$$\hat{g}(\xi) = (2\pi i \xi)^n \hat{f}(\xi)$$

ii. As a result, describe how to find a solution to the differential equation (*) above.

7) **(10 points)** Suppose f is entire and non-vanishing, and that no derivatives of f vanish. Assuming Hadamard's theorem, prove that if f is of finite order, then $f(z) = e^{az+b}$ for some $a, b \in \mathbb{C}$. As a result, deduce that f has order 1.

8) **(15 points)** Using Picard's big theorem, the stronger version of Riemann's Theorem on essential singularities, show that $f(z) = e^z - z$ has infinitely many zeroes in $\mathbb{C}$.

9) **(15 points)** Show that $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$ is a conformal map from the upper half disc $\mathbb{D} \cap \mathbb{H}$ to the upper half plane $\mathbb{H}$.

10) **(15 points)** A point z is a fixed point for $f : \mathbb{D} \to \mathbb{D}$ if $f(z) = z$. Show that if f is holomorphic with 2 fixed points, then f is the identity map. Does every holomorphic map $f : \mathbb{D} \to \mathbb{D}$ have a fixed point? (**hint:** consider $\mathbb{H}$)