Similar to the list of things for the fundamental group and covering spaces, here is one for homology. I will update this from time to time with new information.

1) A Δ-complex structure on a space X is a collection of maps $\{\sigma_\alpha^n : \Delta^n \to X\}$ (where n depends on α) with each map meeting the following criteria:
   - σ_α^n restricted to the interior $\overset{\circ}{\Delta}{}^n$ is an injective map.
   - $\sigma_\alpha^n|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$ is representable as some other σ_β^{n-1}.
   - A set $U \subseteq X$ is open if and only if $(\sigma_\alpha^n)^{-1}(U)$ is open.

   The intuition for such requirements is as follows: 1) makes it so that Δ^n can be 'seen' within the space. 2) makes it so that the boundary map makes sense. 3) gives X the structure of a quotient of $\coprod_\alpha \Delta^n$.

2) $\Delta_n(X)$ is the free abelian group generated by σ_α^n. Elements look like $a_1\sigma_1^n + \ldots + a_m\sigma_m^n$ for $a_i \in \mathbb{Z}$.

3) We define the boundary map $\partial_n : C_n(X) \to C_{n-1}(X)$ by

$$\partial_n(\sigma^n) = \sum_{i=0}^{n} (-1)^i \sigma^n|_{[v_0,\ldots,\hat{v}_i,\ldots,v_n]}$$

4) This map implies that $\partial_n \circ \partial_{n+1} = 0$. Equivalently, $\ker(\partial_n) \supseteq \operatorname{im}(\partial_{n+1})$. Since both of these groups are subgroups of an abelian group, they are abelian, so every subgroup is normal. Therefore, we can define

$$H_n^\Delta(X) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$$

   We have computed the Homology groups of some low dimensional spaces using this property.

5) Not all spaces X are Δ-complexes. Therefore, we can extend this notion to the singular case. Instead of restricting our attention to σ in a Δ-complex structure, let $C_n(X)$ be the free abelian group generated by ALL maps $\sigma : \Delta^n \to X$. The same boundary map makes sense, so we can define

$$H_n(X) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$$

   where $\partial_n : C_n(X) \to C_{n-1}(X)$.

6) We showed that Homology breaks up as a direct sum over path connected components. Thus $H_0(X) = \mathbb{Z}^{\oplus(\#\text{ Path Components of X})}$, and that for a point, $H_i(pt) = 0$ for all $i > 0$.

7) We defined $\tilde{H}_i(X)$ to be exactly $H_i(X)$ when $i > 0$, but $H_0(X)/\mathbb{Z}$ when $i = 0$. Note that this quotient is via the identification with the image of a ϵ^{-1}.

8) A primary reason for liking Homology is the following:

   Theorem 0.1 (Homotopy Invariance)**.** *If $f \simeq g : X \to Y$, then $f_* = g_* : H_i(X) \to H_i(Y)$. Thus homotopy equivalent spaces have isomorphic homology.*

   An immediate consequence is that $\tilde{H}_i(X) = 0$ for all i if X is contractible.

9) A pair (X, A) is called a good pair if $A \subseteq X$ is a closed subset and $\exists U \supseteq A$ open deformation retracting to A. In this case we can naturally define maps $C_n(A) \to C_n(X) \to C_n(X/A)$ for every n. These induce a long exact sequence of homology:

$$\ldots \xrightarrow{q_*} \tilde{H}_{n+1}(X/A) \xrightarrow{\delta} \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} \tilde{H}_n(X/A) \xrightarrow{\delta} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \ldots$$

Note that $H_{-1}(X) = 0$ for any space X, so this terminates with

$$\ldots \xrightarrow{\delta} \tilde{H}_0(A) \xrightarrow{i_*} \tilde{H}_0(X) \xrightarrow{q_*} \tilde{H}_0(X/A) \xrightarrow{\delta} 0$$

Often times we study finite dimensional spaces as well, so we can find the left most end to look like

$$0 \to \tilde{H}_m(A) \xrightarrow{i_*} \tilde{H}_m(X) \xrightarrow{q_*} \tilde{H}_m(X/A) \xrightarrow{\delta} \tilde{H}_{m-1}(A) \xrightarrow{i_*} \ldots$$

where m is the dimension of X.

10) As with most things in topology, not all pairs are good (However, it can be shown CW pair $\Rightarrow$ HEP pair $\Rightarrow$ Good pair). Therefore we define $C_n(X, A)$ to be $C_n(X)/C_n(A)$. One can equivalently present this as the free abelian group generated by n-simplices in X with image outside of A (not entirely in A). This allows us to produce an exact sequence of complexes:

$$\begin{tikzcd}
0 \arrow[r] & C_n(A) \arrow[r, "\iota_\#"] \arrow[d, "\partial_n^A"] & C_n(X) \arrow[r, "q"] \arrow[d, "\partial_n^X"] & C_n(X, A) \arrow[r] \arrow[d, "\partial_n^{X,A}"] & 0 \\
0 \arrow[r] & C_{n-1}(A) \arrow[r, "\iota_\#"] & C_{n-1}(X) \arrow[r, "q"] & C_{n-1}(X, A) \arrow[r] & 0
\end{tikzcd}$$

Where all of the squares above commute, and the horizontal arrows form exact sequences. $\partial^{X,A}$ is just the restriction of ∂^X to the smaller generating set.

11) With this, next time we will construct a map going from $H_n(X, A) = \ker(\partial_n^{X,A})/\operatorname{im}(\partial_{n+1}^{X,A})$ to $H_{n-1}(A)$ and form a LES

$$\ldots \xrightarrow{q_*} H_{n+1}(X, A) \xrightarrow{\delta} \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \xrightarrow{q_*} H_n(X, A) \xrightarrow{\delta} \tilde{H}_{n-1}(A) \xrightarrow{i_*} \ldots$$

We will the relate this back to the case of good pairs.

12) Next we can produce Excision:

Theorem 0.2. *Let $Z \subseteq A \subseteq X$ with the property that the closure of Z is still contained in the interior of A: $\bar{Z} \subseteq \overset{\circ}{A}$. Then for every $n \geq 0$, the inclusion induces*

$$i_* : H_n(X \setminus Z, A \setminus Z) \to H_n(X, A)$$

is an isomorphism of groups.

An equivalent formulation is as follows: If A, B are subsets of X such that $X = \overset{\circ}{A} \cup \overset{\circ}{B}$, then

$$i_* : H_n(B, A \cap B) \to H_n(X, A)$$

is an isomorphism.

You can go between the formulations by setting $B = X \setminus Z$ or $Z = X \setminus B$.

13) This allows to define a notion of local homology about a (closed) point (or more generally any closed subset A). Note that $C_n(X, X \setminus A)$ is generated by simplices $\sigma^n : \Delta^n \to X$ whose image is not entirely in A. If U is any open neighborhood of A, then excision implies

$$H_n(X, X \setminus A) = H_n(U, U \setminus A)$$

This is given by taking $B = U$ and $A = A$ in the second version of excision. Therefore, this object only depends on the structure of X near A. Sometimes it is denoted by

$$H_n^A(X) := H_n(X, X \setminus A)$$

It is useful for checking if $f : X \to Y$ is a local homeomorphism near a set A even though it may not be globally.

14) Utilizing excision again, we can show that for good pairs (X, A),

$$H_n(X, A) \cong H_n(X/A, A/A) \cong H_n(X/A, pt) \cong \tilde{H}_n(X/A)$$

Therefore, the LES above listed in 9 is exactly that listed in 11.

15) A corollary of the preceding statement is the case of the wedge sum: If (X_α, x_α) are good pairs, where $x_\alpha \in X$, then

$$\tilde{H}_n(\vee_\alpha X_\alpha) \cong H_n(\vee_\alpha X_\alpha, x) \cong H_n(\coprod_\alpha X, \coprod_\alpha x_\alpha) \cong \oplus_\alpha H_n(X_\alpha, x_\alpha) \cong \oplus_\alpha \tilde{H}_n(X_\alpha)$$

Note that here the wedge sum is being taken with $x_\alpha \in X_\alpha$ being identified with $x_{\alpha'} \in X_{\alpha'}$.

16) **Strong Invariance of Dimension:** If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open sets in the Euclidean topology, and $U \cong V$ are homeomorphic, then $n = m$.

17) The 5-lemma is stated as follows, and used to easily prove things in a LES are isomorphic.

Lemma 0.3. *Consider the following diagram of groups and group homomorphisms (or more generally, objects and arrows in any Abelian category):*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "l"] \arrow[d, "\delta"] & E \arrow[d, "\epsilon"] \\
A' \arrow[r, "f'"] & B' \arrow[r, "g'"] & C' \arrow[r, "h'"] & D' \arrow[r, "l'"] & E'
\end{tikzcd}$$

If this diagram is commutative, with exact rows, α is surjective, ϵ is injective, and β, δ are isomorphisms, then γ is an isomorphism.

18) A beautiful induction on dimension allows us to show that our two notions of Homology are equivalent:

Theorem 0.4. *If (X, A) is a Δ-complex, then for all $n \geq 0$,*

$$H_n^{\Delta}(X, A) \cong H_n(X, A)$$

Thus all the singular cycles are representable as standard cycles of simplices up to boundary.

19) Split Exact Sequences: A short exact sequence

$$0 \to H \xrightarrow{\iota} G \xrightarrow{q} G/H \to 0$$

is said to be **split** if one of the following equivalent conditions is met:

i. $\exists \varphi : G \to H$ such that $\varphi \circ \iota = Id_H$.

ii. $\exists \varphi : G/H \to G$ such that $q \circ \varphi = Id_{G/H}$.

iii. $G \cong G/H \oplus H$.

A non-split sequence is $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$.

20) If $\exists r : X \to A$ a retraction, then the following sequence is split exact:

$$0 \to H_n(A) \to H_n(X) \to H_n(X, A) \to 0$$

In particular the connecting map $\delta : H_n(X, A) \to H_{n-1}(X, A)$ is 0.

21) **Meyer-Vietoris Sequence:** Similar to Van Kampen's Theorem, sometimes it's easier to work with smaller components of a space instead of the whole.

Theorem 0.5. *If $A, B \subseteq X$ are sets whose interiors cover X, then the following sequence is exact:*

$$\ldots \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(X) \to H_{n-1}(A \cap B) \to \ldots$$

The first map is (i_*, j_*), where i and j are the inclusions of $A \cap B$ into A, B respectively. Then the second map is the difference of the inclusion maps from A, B into X. The last is the connecting map δ.

22) If X is a path connected space, then

$$\pi_1(X)^{ab} \cong H_1(X)$$