CLASS 23, NOVEMBER 2: STONE-ČECH COMPACTIFICATION

Today we will add to our repertoire of methods to compactify a space. Since we didn't cover the 1-point compacfication in class (it is in Class 13 notes), I recall the procedure here.

Definition 23.1. A space Y is called a **compactification** of another space X if $\exists \iota : X \hookrightarrow Y$ an embedding such that Y is compact and Hausdorff and $\bar{X} = Y$. Two such compactications are **equivalent** if there is a homeomorphism $f : Y \to Y'$ such that $f(\iota(x)) = \iota'(x)$ for every $x \in X$:

Example 23.2 (1-point compactification)**.** Assume X is a locally Hausdorff space. Let $Y = X \cup \{\infty\}$, where ∞ is just a name for a new distinguished point. It goes to define a topology. A subset $U \subseteq Y$ is open if either

- $\infty \notin U$ (or equivalently $U \subseteq X$) and U is open in the topology of X.
- $\infty \in U$ and $U^c \subseteq X$ is a compact subset.

Note that this is in fact a topology. Y has the second property and $\emptyset$ has the first. The other 2 facts follow from the fact that arbitrary intersections of closed subsets are closed and finite unions of compact sets are compact. Y is called the **one-point compactification** of X.

1) If $X = \mathbb{R}$, then $Y = \mathbb{R} \cup \{\infty\} \cong S^1$.
2) If $X = \mathbb{C}$, then Y is the Riemann Sphere.
3) If $X = \mathbb{R}^n$, then $Y \cong S^n$.

Lemma 23.3. *Let X be a space, and $f : X \to Z$ be an embedding where Z is a compact Hausdorff space. Then there exists Y a compactification of X such that $\exists \iota : Y \to Z$ an embedding such that $f(x) = \iota(x)$ for each $x \in X$. Y is unique up to equivalence.*

Proof. Let $Y = \overline{f(X)} \subseteq Z$, and ι represent the inclusion as a map. Since Y is a closed subset of a compact Hausdorff space, Y with the subspace topology is compact and Hausdorff. Moreover, ι is still an embedding, since all we did was take a subspace of the range of f. Therefore, Y is a compactification of X.

Suppose Y' is another compactification with the desired properties, and let $\iota' : Y' \to Z$ be its embedding into Z. Note that we have $\iota(x) = f(x) = \iota'(x)$ for all $x \in X$. It suffices to show that Y and Y' are homeomorphic.

Note that since $f(X) \subseteq \iota'(Y')$, and $f(\bar{X}) = Y \subseteq Z$, we must have that $\iota'(Y') \subseteq \iota(Y)$. On the other hand, $\iota(Y')$ is the image of a compact set, and therefore is itself compact in a Hausdorff space. Therefore, it is closed. But $\iota(Y) = f(\bar{X})$. Therefore, $\iota'(Y') = \iota(Y)$.

Finally, since $\iota : Y \to \iota(Y) = \iota'(Y')$ and $\iota' : Y' \to \iota(Y')$ are homeomorphisms, we see that

$$\iota^{-1} \circ \iota' : Y' \to Y$$

is also a homeomorphism, with $\iota^{-1}(\iota'(x)) = \iota^{-1}(x) = x$. □

There are many ways non-homeomorphic ways to compactify a space in general:

Example 23.4. Above we noted that the 1-point compactification of $\mathbb{R}$ is homeomorphic to S^1. Another compactification is by adding 2-points; $-\infty, \infty$. This yields the extended real line which is sometimes written $\bar{\mathbb{R}}$. Consider the corresponding embedding: $\iota : \mathbb{R} \to$

$[-\infty, \infty]$. Note $[-\infty, \infty] \cong [0,1]$ by use of a piecewise defined $\tan^{-1}$-function. Of course, $\iota(\bar{\mathbb{R}}) = [-\infty, \infty]$. This should be viewed as the 2-point compactification of a space.

On the other hand, we can embed $\mathbb{R} \cong (0,1)$ into $[0,1] \times [0,1]^2 \subseteq \mathbb{R}^2$ via the topologist's sin curve: $x \mapsto (x, \sin\left(\frac{1}{x}\right))$. The resulting compactification adds an entire line segment (uncountable set) to $\mathbb{R}$!

The 1-point compactification is somehow the smallest possible compactification of a space X; indeed, at least one point must be added to make the space compact when X itself isn't compact. The main theorem for today introduces a new way to compactify a space, which should be thought of as the *largest* compactification.

Theorem 23.5 (Stone-Čech Compactification Theorem)**.** *Let X be a T3.5 space*[1]*. Then there exists Y a compactification of X such that every bounded continuous map $f : X \to \mathbb{R}$ extends uniquely to a continuous map $f' : Y \to \mathbb{R}$.*

Example 23.6. Continuing with the previous example, consider the extended real line $\bar{\mathbb{R}}$. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function, then we can consider $a = \lim_{x\to-\infty} f(x)$ and $b = \lim_{x\to\infty} f(x)$. f extends to $\bar{\mathbb{R}}$ if and only if both limits exist. Therefore, a function like $f(x) = x\sin(x)$ tells us that $\bar{B}$ is not the compactification Y in Theorem 23.5.

With our topologists sin curve example, the set of functions which can be extended increases. If both limits exist, we define $f(x) = \lim_{y\to\infty} f(y)$ for each y in the newly adjoined line segment. On the other hand, we can extend the function $f(x)\sin(\frac{1}{x})$ (viewed as $(0,1) \to [0,1]$) as well! We would take $f((0,y)) = y$.

Proof. (of Theorem 23.5). Let $C_0(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous, bounded}\}$. Additionally, for each $f \in C_0(X)$, let $I_f = [\inf f, \sup f]$. Then we have a (ultimate) function

$$F : X \to \prod_{f\in C_0(X)} I_f : x \mapsto (f(x))$$

By the Tychonoff Theorem, $\prod_{f\in C_0(X)} I_f$ is a compact space. Moreover, since X is T3.5, we have that functions separates points from closed sets. Therefore, by The Embedding Theorem (Corollary 19.5), we know that F is an embedding.

Therefore, by Lemma 23.3 we have that there exists a subspace $\iota : Y \hookrightarrow \prod_{f\in C_0(X)} I_f$ such that Y is a compactification of X. It suffices to show that $f \in C_0(X)$ extends to Y. Note that $f : X \to I_f \subseteq \mathbb{R}$ is the composition $\pi_f \circ F$. Therefore, I claim that $\pi_f \circ \iota : Y \to I_f$ is the desired extension of f. It suffices to check that it is unique, which follows by the following lemma: □

Lemma 23.7. *If $A \subseteq X$, and $f : A \to Z$ is a continuous map to a Hausdorff space Z, then if f extends to $\bar{A}$, it extends uniquely.*

Proof. This follows from our standard trick; suppose $f', f'' : \bar{A} \to Z$ are two extensions. Then if $f'(x) \neq f''(x)$, we can separate them by open sets U, V respectively. Choose $x \in U', V' \subseteq \bar{A}$ such that $f'(U') \subseteq U$ and $f''(V') \subseteq V$. Then $\exists a \in A$ such that $a \in U$. Therefore, $f'(a) = f''(a)$. But this implies

$$f'(a) \in f(U') \cap f'(V') \subseteq U \cap V = \emptyset$$

a contradiction. □

[1]To maximize generality. You may assume T4+T1 for comforts sake.