HOMEWORK 1: RING THEORY
DUE: MONDAY, FEBRUARY 19TH

1) Determine which of the following are ring homomorphisms from $M_2(\mathbb{Z})$ to $\mathbb{Z}$:
   i.
$$\phi : M_2(\mathbb{Z}) \to \mathbb{Z} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a$$
   ii.
$$\Phi : M_2(\mathbb{Z}) \to \mathbb{Z} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a + d$$
   iii.
$$\det : M_2(\mathbb{Z}) \to \mathbb{Z} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto ad - bc$$

2) Prove Proposition 0.2 from Notes 2:

   Proposition 1. *The following conditions are equivalent:*
   - *R is a Noetherian ring, e.g. every ideal is finitely generated.*
   - *Every ascending chain of ideals eventually* **stabilizes***: if*
   $$I_1 \subseteq I_2 \subseteq \dots$$
   the $\exists n > 0$ such that $I_n = I_{n+1} = I_{n+2} = \dots$.
   - *Every collection of Ideals $\{I_\alpha\}_{\alpha \in \Lambda}$ contains a maximal element. That is to say that there exists $\beta \in \Lambda$ such that there are no $\alpha \in \Lambda$ such that $I_\beta \subsetneq I_\alpha$.*

3) Show that if $\varphi : R \to S$ is a ring homomorphism, and $\mathfrak{p} \subseteq S$ is a prime ideal, then $\varphi^{-1}(\mathfrak{p})$ is also a prime ideal. As a result, show that if φ is a surjective map, preimages gives an injective map from prime ideals of S to prime ideals of R.

   Is the same true in the opposite direction? That is to say, if $\mathfrak{p} \subseteq R$ is a prime ideal, then is $\varphi(\mathfrak{p}) \subseteq S$ a prime ideal? Or perhaps $\langle \varphi(\mathfrak{p}) \rangle \subseteq S$?

4) Let R be commutative. The **radical** of an ideal I, denoted by $\sqrt{I}$, is the set of elements $r \in R$ such that $r^n \in I$ for some $n \gg 0$. Show that this is an ideal. For every prime ideal $\mathfrak{p}$, show that $\mathfrak{p} = \sqrt{\mathfrak{p}}$.

   In addition, show that the nil-radical (radical of the zero ideal) $\mathcal{N} = \sqrt{0}$ is contained in every other prime ideal.[1]

5) Let R be a commutative ring. An ideal I is called **primary** if whenever $a \cdot b \in I$, we have that $a \in I$ or $b^n \in I$. This is a slight generalization of being prime.
   i. What are the primary ideals of $\mathbb{Z}$?
   ii. I is primary if and only if every element of R/I is either a non-zero-divior or in $\mathcal{N} \subseteq R/I$.

[1]Later on we will show that $\mathcal{N}$ is precisely the intersection of all prime ideals.

iii. If $\mathfrak{q}$ is a primary ideal, then $\sqrt{\mathfrak{q}}$ is a prime ideal.

6) We will now show that not all powers of prime ideals are primary ideals. Consider the ring $R = K[x, y, z]/\langle xy - z^2\rangle$. Then show that the ideal $\mathfrak{p} = \langle x, z\rangle$ is prime. However, $\mathfrak{p}^2 = \langle x^2, xz, z^2 = xy\rangle$. Show that $xy \in \mathfrak{p}^2$ implies that $\mathfrak{p}^2$ can NOT be primary.