CLASS 34, DECEMBER 6TH: FINALE! (RMT)

We are only left to prove the Riemann Mapping Theorem:

Theorem (Riemann Mapping Theorem)**.** *If $\Omega \subseteq \mathbb{C}$ is a proper, open, and simply-connected subset, then Ω is conformally equivalent to $\mathbb{D}$.*

Proof. **1)** Choose $\alpha \notin \Omega$. Then $f(z) = \log(z - \alpha)$ is a non-vanishing holomorphic function on Ω. As a result, $e^{f(z)} = z - \alpha$ is injective, which implies f is as well. For $w \in \Omega$,

$$f(z) \neq f(w) + 2\pi i \forall z \in \Omega$$

Otherwise, exponentiating would for $z = w$. Using sequences, we can even find a disc about $f(w) + 2\pi i$ for which no values of f reach it. Consider

$$F(z) = \frac{1}{f(z) - (f(w) + 2\pi i)}$$

Since f is conformal, so is F. $F(\Omega)$ is also bounded, thus by translation and dilation, we may assume $0 \in \Omega \subseteq \mathbb{D}$ up to conformal equivalence.

2) Consider the uniformly bounded class

$$\mathcal{F} = \{f : \Omega \to \mathbb{D} \mid f(0) = 0, f \text{ is holomorphic and injective}\}$$

We want to find $f \in \mathcal{F}$ maximizing $f'(0)$. Note they are bounded by Cauchy's inequality. By Montel's theorem, we can choose a sequence f_n with $|f_n'(0)|$ approaching this non-zero supremum. Thus by our lemma from last time, the limit f is injective. Furthermore, by the MMP $|f(z)| < 1$. All in all, $f \in \mathcal{F}$.

3) We need to show that f is surjective. Suppose $\alpha \notin f(\Omega)$. Then $\psi_\alpha : \mathbb{D} \to \mathbb{D}$ swaps α with the origin. If we consider $U = \psi_\alpha(f(\Omega))$, then $0 \notin U$ is simply connected. Therefore we can define a log and thus a square root: $g(w) = e^{\frac{1}{2}\log(w)}$. Then I claim the composition

$$F : \Omega \xrightarrow{f} \mathbb{D} \setminus \{\alpha\} \xrightarrow{\psi_\alpha} \mathbb{D} \setminus \{0\} \xrightarrow{g} \mathbb{D} \setminus \{0\} \xrightarrow{\psi_{g(\alpha)}} \mathbb{D} \setminus \{g(\alpha)\}$$

is in $\mathcal{F}$. Indeed, it is holomorphic, injective, and each maps 0 to 0. Additionally, if $h(w) = w^2$, then

$$f = \psi_\alpha^{-1} \circ h \circ \psi_{g(\alpha)}^{-1} \circ F = \Phi \circ F$$

$\Phi(0) = 0$ is not injective since h isn't. As a result, Schwarz Lemma (bullet 3) implies that $|\Phi'(0)| < 0$, since it can't possibly be a rotation. Finally, the chain rule yields

$$f'(0) = \Phi'(0) \cdot F'(0)$$

but this implies $f'(0)$ was not maximal! A contradiction. $\square$