CLASS 18, OCTOBER 22: THE URYSOHN LEMMA

Today, we will begin a study of some very important theorems for normal topological spaces. The first of which is called Urysohn's Lemma. The proof is substantial and brings in an idea that we haven't seen before. Today I will state the theorem, provide one corollary/application, and finally prove the theorem.

Theorem 18.1 (Urysohn Lemma)**.** *Let X be a normal subspace, and A and B be two disjoint closed subspaces. Then there exists a continuous function $f : X \to [a, b] \subseteq \mathbb{R}$ such that $f(a') = a$ and $f(b') = b$ for all $a' \in A$ and $b' \in B$.*

Therefore, we can separate any two closed sets, not only with open sets, but with a function! This seems more general ($A \subseteq f^{-1}((a, \frac{a+b}{2})) = U$ and $B \subseteq f^{-1}((\frac{a+b}{2}, b)) = U$), but Urysohn shows that they are equivalent notions. The same is not true for T_3, which brings about the notion of $T_{3\frac{1}{2}}$:

Definition 18.2. A space X is $T_{3\frac{1}{2}}$/**completely regular** if for every point $x \in X$ and closed subset A, there exists $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$.

So again, if X has closed points, we see

$$\text{T4} \Rightarrow T_{3\frac{1}{2}} \Rightarrow \text{T3}$$

Here is a beautiful application of this result, allowing us to produce something called a *partition of unity* on a compact Hausdorff space (or Manifold).

Corollary 18.3. *If X is a compact Hausdorff space, and $U_1, \ldots, U_n$ is an open cover, then there exists functions $f_i : X \to [0, 1]$ such that $f = f_1 + \ldots + f_n$ has the property that $f(x) = 1$ for all $x \in X$, and $f_i^{-1}((0, 1]) \subseteq U_i$.*[1]

Note that any open cover can be refined as such, so this implies something for general covers as well.

Proof. By virtue of Theorem 17.2 from last class, we see that X is a normal space. I claim we can choose V_i with $\bar{V_i} \subseteq U_i$ still covering X. Indeed, consider $A = (U_2 \cup \ldots \cup U_n)^c \subseteq U_1$. Now, as a result of our equivalent definition of T4 (Theorem 16.4), we note that there exists $V_1 \subseteq U_1$ with

$$A \subseteq V_1 \subseteq \bar{V_1} \subseteq U_1$$

So $V_1, U_2, \ldots, U_n$ still covers X. Repeat this process finitely many times.

Given this tiny lemma, we now have $V_i \subseteq \bar{V_i} \subseteq U_i$, with the V_i covering X. Repeat this procedure with the V_i to produce $W_i \subseteq \bar{W_i} \subseteq V_i$ covering X. Now we can apply Urysohn's Lemma to the pair of closed sets $(\bar{W_i}, V_i^c)$: $\exists g_i : X \to [0, 1]$ with $g_i(W_i) = 1$ and $g_i(V_i^c) = 0$. Note that since the W_i cover X, the function $g_1 + g_2 + \ldots + g_n$ is never 0 (in fact always ≥ 1). Therefore, the desired function is

$$f_i(x) = \frac{g_i(x)}{g_1(x) + \ldots + g_n(x)}$$

[1] We say f_i is **supported** in U_i. *Supp*$(f) = \overline{\{x \in X \mid f(x) \neq 0\}} \subseteq U_i$.

Note that $f_i(x) = 0$ for $x \in U_i^c \subseteq \bar{V}_i^c$. □

Proof. (of Theorem 18.1) It suffices to prove the statement when $a = 0$ and $b = 1$, since $[0, 1] \cong [a, b]$ by a linear homeomorphism. We proceed with this proof in several steps:

Step 1) We will construct $U_r \subseteq X$ an open subset for each $r \in \mathbb{Q} \cap [0, 1]$ such that for $p < q$, we have

$$p < q \quad \Longrightarrow \quad \bar{U}_p \subseteq U_q$$

This goes as follows: Choose $U_1 = X \setminus B$. Now choose U_0 satisfying this property and containing A by Theorem 16.4. Now, enumerate $\mathbb{Q} \cap (0, 1)$, say by $p_1, p_2, \ldots$. We construct U_{p_i} inductively by taking $p_j < p_i < p_k$, where $j, k < i$, and p_j, p_k are the largest and smallest numbers with this property so far constructed[2]. Then let U_{p_i} again be the open set as in Theorem 16.4:

$$Z = \overline{U_{p_j}} \subseteq U_{p_i} \subseteq \overline{U_{p_i}} \subseteq U_{p_k}$$

This produces the desired sets U_r.

Step 2) Next, we construct the desired function. Let $f : X \to [0, 1]$ be the function

$$f(x) = \begin{cases} 1 & x \in B \\ \inf\{r \in \mathbb{Q} \cap [0, 1] \mid x \in U_r\} & x \notin B \end{cases}$$

Note first that this is a well defined function since if $x \notin B$, $x \in U_1$. Therefore, the infimum set is non-empty and bounded below by 0. Furthermore, it is easy to check that $f(b) = 1$ and $f(a) = 0$ for $b \in B$ and $a \in A$.

Step 3) Finally, it goes to check the continuity of f. This can be verified by showing that for any neighborhood $y \in N_y \subseteq [0, 1] \cap \mathbb{Q}$, and for any $x \in X$ with $f(x) = y$, there exists a neighborhood $x \in N_x \subseteq X$ with $f(N_x) \subseteq N_y$. This follows by noting the following two facts:

- If $x \in U_r$, then $f(x) \leq r$.
- If $x \notin \bar{U}_r$, then $f(x) \geq r$.

The first claim is obvious, since the set $S_x = \{r \in \mathbb{Q} \cap [0, 1] \mid x \in U_r\}$ has the property that $r \in S_x$. The infimum is the largest lower bound, so we have $f(x) \leq r$.

For the second claim, we note that if $x \notin \bar{U}_r$, then

$$S_x = \{r \in \mathbb{Q} \cap [0, 1] \mid x \in U_r\} = \{r \in \mathbb{Q} \cap (r, 1] \mid x \in U_r\}$$

and thus r is a (potentially not largest) lower bound for $f(x)$.

- Given $y = 1$, and a neighborhood $N_y = (a, 1]$, we can take $N_x = X \setminus \overline{U_{\frac{a+1}{2}}}$. The notes above imply $f(N_x) \subseteq [\frac{a+1}{2}, 1] \subseteq N_y$.
- Given $y = 0$, and a neighborhood $N_y = [0, b)$, we can take $N_x = U_{\frac{b}{2}}$. The notes above imply $f(N_x) \subseteq [0, \frac{b}{2}] \subseteq N_y$.
- Finally, if $0 < y < 1$ and $N_y = (a, b)$, set $N_x = U_{\frac{b+y}{2}} \setminus \overline{U_{\frac{a+y}{2}}}$, and note

$$f(N_x) \subseteq [\frac{a+y}{2}, \frac{b+y}{2}] \subseteq N_y$$

This completes the proof. □

[2] $S = \{0, 1, p_1, p_2, \ldots, p_{i-1}\}$, then $p_j = \max(S \cap [0, p_i))$ and $p_k = \min(S \cap (p_i, 1])$.