CLASS 26, WEDNESDAY, MAY 2: F-REGULARITY

So F-splitting is a wonderful property that is relatively easy to check by Fedder. However, being weakly normal and reduced are often unsatisfactory for a 'nice' ring. Therefore, we introduce a property between being regular and being F-split.

We can think of being F-split as having the inclusion (because of F-split) $R \to F_*R$ being a split inclusion. The next notion perturbs this by an ϵ.

Definition 0.1. A ring R is called F-regular if for every non-zero divisor $c \in R$, we have that the following inclusion splits for some $e \gg 0$:

$$R \xrightarrow{F} F_*^e R \xhookrightarrow{\cdot F_*^e c} F_*^e R$$

I think of this as an ϵ perturbation since thinking of $F_*^e c$ as $c^{\frac{1}{p^e}}$ makes c quite small. On homework 6, you are asked to show that this property is also a local property. I will first show that regular rings (such as polynomial rings) are F-regular.

Proposition 0.2. *If R is a regular, then R is F-regular. If R is F-regular, then it is F-split.*

Proof. The case of units is handled by being F-split. Otherwise, by Krull's intersection theorem, we can take $n > 0$ such that $c \notin \mathfrak{m}^n$. Choose $e > 0$ such that $p^e > n$. If $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$, then

$$c = \sum_{\substack{\beta \\ \alpha_i < p^e}} c_\alpha F_*^e k_\beta x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

where $c_\alpha, k_\alpha \in K$ and $F_*^e k_\beta$ is a basis for $F_*^e K$ over K. Choose a $c_\alpha \neq 0$, and take $\varphi \in \mathrm{Hom}_R(F_*R, R)$ to be the projection from the $F_*^e k_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ free summand. This map will be nonzero, and map c to c_α. Post-composing by $\cdot c_\alpha^{-1}$ completes the proof.

The second statement follows by taking $c = 1$. $\square$

Example 0.3. We have shown that $R = K[x_1, \ldots, x_n]/\langle x_1 \cdots x_n \rangle$ is F-split. It is not F-regular, since if we take $c = x_1$, then

$$\Phi_S^e(F_* x_1^{p^e-1} \cdots x_n^{p^e-1} \cdot r \cdot x_1) = x_1 \Phi_S(F_* x_2^{p^e-1} \cdots x_n^{p^e-1} \cdot r) \in \langle x_1 \rangle$$

In fact, $\phi(\mathfrak{m}) \subseteq \mathfrak{m}$ for any $\phi \in \mathrm{Hom}_R(F_*^e R, R)$.

Example 0.4. If $R = K[x, y]/\langle f = x^2 + y^2 \rangle$, then similarly R is not F-regular. Last time we showed that R is F-split except when $p = 2$. In the case of $p > 2$, we can consider the condition $f^{p^e-1} \cdot c \notin \mathfrak{m}^{[p^e]}$. Let $c = xy$. We notice that

$$f^{p^e-1} = \sum_{i+j=p^e-1} c_{ij} x^{2i} y^{2j}$$

In this setting, it is clear that either $2i \geq p^e - 1$ or $2j \geq p^e - 1$ for integers i, j. Therefore $c \cdot f^{p^e-1} \in \mathfrak{m}^{[p^e]}$.

Now lets look at some positive examples:

Example 0.5. Consider $R = K[x, y, z]/\langle x^2 + y^2 + z^2\rangle$, where K is perfect of characteristic > 2. Again, by Fedder's Criterion it goes to show that for any given c, there is a sufficiently large $e \gg 0$ such that $f^{p^e-1}c \notin \mathfrak{m}^{[p^e]}$. That is to say that there exists a monomial of the left hand side of x, y, z-degree less than p^e; $cx^iy^jz^k$ with $i, j, k < p^e$.

$$f^{p^e-1} = \sum_{i+j+k=p^e-1} \binom{p^e-1}{i, j, k} x^iy^jz^k$$

Let m be the maximum of the x, y, z degree of c. It now suffices to show by the above observation that $\binom{p^e-1}{i,j,k} \neq 0$ and $i + m, j + m, k + m < p^e$.

Lemma 0.6 (Lucas's Theorem). *$\binom{m}{n}$ is divisible by $p > 0$ if and only if expressing $n = \sum_{i=1}^k n_ip^i$ and $m = \sum_{i=1}^l m_ip^i$, for some i, $n_i > m_i$.*

Now, noting that $\binom{p^e-1}{i,j,k} = \binom{p^e-1}{i}\binom{p^e-1-i}{j}$, and that

$$p^e - 1 = (p - 1) + (p - 1)p + \ldots + (p - 1)p^{e-1}$$

Lucas's Theorem allows us to conclude that $\binom{p^e-1}{i,j,k} \neq 0$ for $i = (p-1)p^{e-1}$, $j = p^{e-1} - 1$, and $k = 0$. We can choose $e \gg 0$ so that $p^{e-1} - 1 > m$, and this shows that R is F-regular.

Many other examples can be computed in a similar fashion. So the question becomes why are F-regular rings so great? The following 2 results demonstrate it's importance as a singularity class:

Theorem 0.7. *If R is an F-regular domain, then R is Cohen-Macaulay and Normal.*

Cohen-Macaulay was mentioned with regard to its correspondence with depth. Normalcy is another fantastic condition, which in particular isolates your singularities (or irregularities) to height 2 and above prime ideals. In this case we say R is regular in codimension 1.

Definition 0.8. A domain R is called **normal** if R is integrally closed in $K(R) = R_{\langle 0\rangle}$. That is to say, $x \in K(R) \setminus R$ is not the zero of a monic polynomial with coefficients in R. If R is not normal, we call its integral closure in $K(R)$ by R^N (the **normalization** of R).

Theorem 0.7. To show that R is CM requires the techniques of local cohomology. This will be omitted for now.

To show R is normal, we consider the **conductor** of R; $\mathfrak{c} := \mathrm{Ann}_R(R^N/R)$. A ring R is normal if and only if $\mathfrak{c} = R$.

Lemma 0.9. *If $\varphi \in \mathrm{Hom}_R(F^e_*R, R)$, the $\varphi(F_*\mathfrak{c}) \subseteq \mathfrak{c}$.*

Proof. We can consider $\varphi \otimes 1_{K(R)} : F_*K(R) \to K(R)$. If $x \in \mathfrak{c}$ and $r \in R^N$, then $r\varphi(F_*x) = \varphi(F_*r^{p^e}x)$. But $r^{p^e} \in R^N$, and $x \in \mathfrak{c}$, and therefore $F_*r^{p^e}x \in R$. Therefore, $\varphi(F^e_*\mathfrak{c}) \cdot R^N \subseteq R$. □

To complete the proof, notice that if $\mathfrak{c} \neq R$, then we can take $c \neq 0$ in $\mathfrak{c}$, and find φ such that $\varphi(F_*c) = 1 \in \mathfrak{c}$ by the lemma. This is a contradiction! □