HOMEWORK 7: HILBERT-NULLSTELLENSATZ
DUE: WEDNESDAY, APRIL 17TH

1) Given 2 R-submodules $M_1, M_2 \subseteq N$, is it true that
$$W^{-1}(M_1 + M_2) \cong W^{-1}M_1 + W^{-1}M_2?$$

2) Give an example of 2 multiplicative sets W_1 and W_2 which are distinct but $W_1^{-1}R \cong W_2^{-1}R$. Show that there exists a unique largest such multiplicative set $W \supseteq W_0$ for a given localization $W_0^{-1}R$.

3) If $A = A' \times A''$ is decomposable as a product of 2 rings, show that A' and A'' are localizations of A.

4) Assume $2 \neq 0$ in K (i.e. $char(K) \neq 2$ or $\mathbb{Z}/2\mathbb{Z} \not\subseteq K$). Given the ideal $J = \langle x + y, (x - y)^2 \rangle$, show that $x \in I(V(J)) = \sqrt{J}$ but $x \notin J$.

5) Recall the result of Hilbert-Nullstellensatz:

   Theorem 0.1 (Hilbert-Nullstellensatz)**.** *Assume K is an algebraically closed field. If $J \subsetneq K[x_1, \ldots, x_n]$ is an ideal, then $V(J) \neq \emptyset$. Furthermore, $I(V(J)) = \sqrt{J}$.*

   Show that this is true if and **only if** K is algebraically closed.

6) Explain why Hilbert-Nullstellensatz is sometimes referred to as the 'Generalized Fundamental Theorem of Algebra'.

7) Show that if f is irreducible in $K[x_1, \ldots, x_n]$, and f doesn't divide g, then $V(f) \not\subseteq V(g)$. As a result, if $X = V(g)$ and $g = u \cdot f_1^{e_1} \cdots f_m^{e_m}$ where f_i are irreducible and u is a unit, then
$$V(g) = V(f_1) \cup \cdots \cup V(f_m)$$
This is a decomposition of a hypersurface into **irreducible components**.