CLASS 29, APRIL 29TH: PRIMARY DECOMPOSITIONS EXIST!

Today we ask the question of when a primary decomposition exists for all ideals of a given ring. The first result in this direction is for Noetherian rings.

Theorem 29.1. *If R is a Noetherian ring, and $I \subsetneq R$ is a proper ideal, then I has a primary decomposition*

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$$

To prove this result, we will use an idea similar to irreducibility but for ideals:

Definition 29.2. An ideal I is called **indecomposable** if there exists no strictly larger ideals J, K such that $I = J \cap K$.

Prime ideals are examples of indecomposable ideals, and the following shows a direct comparison with irreducible decompositions of varieties. The proof in fact has many similarities.

Lemma 29.3. *If R is a Noetherian ring, then every ideal is an intersection of a finite number of indecomposable ideals.*

Proof. Let $\mathcal{S}$ be the set of ideals that can't be written as a finite intersection of indecomposables. If $\mathcal{S} \neq \emptyset$, then $\mathcal{S}$ contains a maximal element I by the Noetherian property and Zorn's Lemma. Clearly I cannot be indecomposable, so $I = J \cap K$ for two larger ideals. But these each can be expressed as a finite intersection of indecomposables by maximality of I in $\mathcal{S}$. As a result, we contradict the fact that $I \in \mathcal{S}$. □

Lemma 29.4. *If R is a Noetherian ring, every indecomposable ideal is primary.*

Proof. Note that $\mathfrak{q} \subseteq R$ is indecomposable if and only if $0 \subseteq R/\mathfrak{q}$ is indecomposable (by the ideal correspondence). Therefore we reduce to the case $\mathfrak{q} = 0$. Suppose $xy = 0$, i.e. $y \in \mathrm{Ann}(x)$. We can consider the chain of ideals

$$\mathrm{Ann}(x) \subseteq \mathrm{Ann}(x^2) \subseteq \ldots \subseteq \mathrm{Ann}(x^n) = \mathrm{Ann}(x^{n+1}) = \ldots$$

I claim $\langle x^n \rangle \cap \langle y \rangle = 0$. Suppose $a \in \langle x^n \rangle \cap \langle y \rangle$. Then $ax = 0$ since $y|a$. This implies $ax = (bx^n) \cdot x = 0$. This is to say $b \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$. This is only possible if $a = bx^n = 0$, proving the result.

Therefore, if 0 is indecomposable, we have that $xy = 0$ implies either $x^n = 0$ or $y = 0$, which demonstrates 0 is primary. □

This yields Theorem 29.1 immediately, because a decomposition into indecomposables is already an intersection of primary ideal! Now we move onto the question of uniqueness of a decomposition.

Theorem 29.5. *If R is Noetherian, and $I \subsetneq R$ is an ideal with shortest primary decomposition*

$$I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$$

then $\mathrm{Ass}(R/I) = \{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{q_n}\}$.

Proof. Given $I = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_n$, we can use our standard inclusion

$$R/I \hookrightarrow \oplus_{i=1}^n R/\mathfrak{q}_i$$

As a result, $\mathrm{Ass}(R/I) \subseteq \cup \mathrm{Ass}(R/\mathfrak{q}_i) = \{\sqrt{\mathfrak{q}_1}, \ldots, \sqrt{\mathfrak{q}_n}\}$. Since we chose our decomposition to be shortest possible, we have that $M = (\cap_{i\neq j}\mathfrak{q}_i)/I \neq 0$. Therefore M has an associated prime. But $M \hookrightarrow R/\mathfrak{q}_j \subseteq \oplus_{i=1}^n R/\mathfrak{q}_i$, since all other factors map to 0. As a result, $\mathrm{Ass}(M) \subseteq \mathrm{Ass}(R/\mathfrak{q}_j) = \{\sqrt{\mathfrak{q}_j}\}$, which shows every ideal in the list is necessary. □

As a small note, this does NOT show that the choices of $\mathfrak{q}_i$ are uniquely determined in a shortest decomposition.

Example 29.6. Consider again our famous example of $I = \langle x^2, xy\rangle \subseteq K[x,y]$. We found that the associated primes of I are $\langle x\rangle$ and $\langle x, y\rangle$. Now, note that $I = \langle x^2, xy, y^n\rangle$ is $\langle x, y\rangle$-primary for any choice of n. Indeed, the radical is clearly $\langle x, y\rangle$. Furthermore, if we consider $K[x,y]/\langle x^2, xy, y^n\rangle$, then we should note that the zero divisors of this ring are any element of $\langle x, y\rangle$. Considering

$$f = ax + b_1 y + \ldots + b_{n-1}y^{n-1}$$

where $a, b \in K$, we see that $f^n = 0$. Indeed, every xy term is 0, $x^n = 0$, and $y^m = 0$ for $m \geq n$. This shows $\langle x^2, xy, y^n\rangle$ is primary. Finally,

$$\langle x^2, xy\rangle = \langle x\rangle \cap \langle x^2, xy, y^n\rangle \qquad \forall n \geq 1.$$

To finish up with primary decompositions, I would like to mention how they behave under localization. Note that if $\mathfrak{q}$ is a $\mathfrak{p}$-primary ideal, and $\mathfrak{p} \cap W = \emptyset$, then $\cdot W^{-1}\mathfrak{q}$ is a $\cdot W^{-1}\mathfrak{p}$-primary ideal in the localization, and even $\varphi^{\#}(\cdot W^{-1}\mathfrak{q}) = \mathfrak{q}$, where $\varphi : R \to W^{-1}R$ is the localization map.

Corollary 29.7. *If I is an ideal with shortest primary decomposition*

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$$

and let $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Reorder $\mathfrak{q}_i$ so that $\mathfrak{p}_i \cap W = \emptyset$ for $i \leq m$ and $\mathfrak{p}_i \cap W \neq \emptyset$ for $i > m$. Then

$$W^{-1}I = W^{-1}\mathfrak{q}_1 \cap \cdots \cap W^{-1}\mathfrak{q}_m$$

$$\varphi^{-1}(W^{-1}I) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_m$$

In particular, one should note that this yields an example of extension and contraction not being inverse to one another:

$$I \subseteq \varphi^{-1}(W^{-1}I)$$

Proof. Recall that localization and intersections commute (can be interchanged). Therefore we get

$$W^{-1}I = W^{-1}\mathfrak{q}_1 \cap \cdots \cap W^{-1}\mathfrak{q}_m \cap W^{-1}\mathfrak{q}_{m+1} \cap \cdots \cap W^{-1}\mathfrak{q}_n$$

but $W \cap \mathfrak{q}_i \neq \emptyset$ implies $W^{-1}\mathfrak{q}_i = W^{-1}R$. This completes the proof. □

Corollary 29.8. *If I is as in Corollary 29.7, and $\mathfrak{p}_i$ is minimal among $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$, then localizing at $W = R \setminus \mathfrak{p}_i$ yields*

$$\mathfrak{q}_i = \varphi^{-1}(W^{-1}I)$$

Therefore, such a $\mathfrak{q}_i$ primary to a minimal prime is unique!

Corollary 29.8 demonstrates that $\langle x\rangle$ cannot be modified in Example 29.6.