HOMEWORK 2: CONNECTEDNESS AND HOMOTOPY TYPE
DUE: SEPTEMBER 24

1) Suppose that $A \subset X$ is a connected subspace of X. Let $\bar{A}$ be the closure of A inside X, the smallest closed subset of X containing $A^{\dagger}$. Suppose that $A \subseteq B \subseteq \bar{A}$. Show that B is connected.

2) Suppose $\tau_1 \subseteq \tau_2$ are two topologies on a set X. How does connectedness in one topology differ from connectedness in the other? Give an example to demonstrate your claim(s).

3) Let $p : X \to Y$ be a quotient map. If $p^{-1}(y)$ is connected for each $y \in Y$, and Y is connected, show that X is itself connected. As a result, "connectedness can be checked fiber-wise".

4) (Chapter 0, #3)
   - ∘ Suppose $f : X \to Y$ and $g : Y \to Z$ are homotopy equivalences. Show that $g \circ f : X \to Z$ is a homotopy equivalence. Show as a result that homotopy equivalence is an equivalence relation.
   - ∘ Show that the relation of 'homotopic maps' $f, g : X \to Y$ is an equivalence relation.
   - ∘ show that a map homotopic to a homotopy equivalence is itself a homotopy equivalence.

5) (Chapter 0, #13) Show that two deformation retractions r_t^0 and r_t^1 of a space X onto a subspace A can be joined by a **continuous** family of deformation retractions $r_t^s : X \times I \times I \to X$ from X to A. That is to say that r_t^s is a deformation retraction for each s.

   Appended Note: Check out the function

$$r(x,t,s) = r_t^s(x) = \begin{cases} r_t^0 \circ r_{2st}^1(x) & s \leq \frac{1}{2} \\ r_{2t(1-s)}^0 \circ r_t^1(x) & s \geq \frac{1}{2} \end{cases}$$

   Show that this map has the desired properties:
   - ∘ $r(x,t,0) = r_t^0(x)$
   - ∘ $r(x,t,1) = r_t^1(x)$
   - ∘ At $s = \frac{1}{2}$, both definitions of the function can be used.
   - ∘ For fixed $s \in I$, $r(x,t,s)$ is a deformation retraction.

6) (Chapter 0, #3) A **weak deformation retraction** of X to A is a homotopy $f_t : X \to X$ such that
   - ∘ $f_0 = Id_X$
   - ∘ $f_1(X) \subseteq A$
   - ∘ $f_t(A) \subseteq A$

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†**Note:** The closure exists because infinite intersections of closed sets are closed.

Note that this is a weakening of a deformation retraction in the sense that we allow the points in A to move about as $t \in [0, 1]$ varies.

Show that if there is a weak deformation retraction from X to A, then the inclusion $A \overset{\iota}{\hookrightarrow} X$ is a homotopy equivalence.

Thought Experiment*: Can you produce a weak deformation retraction which is not a deformation retraction?

***A.K.A.** An extra question that has no effect on your grade.