# HOMEWORK 10: CONFORMAL MAPPINGS
# DUE: FRIDAY DECEMBER 6TH

(1) Prove that the following product converges and the result is $\frac{\sin(z)}{z}$:

$$\cos\left(\frac{z}{2}\right)\cos\left(\frac{z}{4}\right)\cdots = \prod_{n=1}^{\infty}\cos\left(\frac{z}{2^n}\right)$$

As a hint, recall the double angle identity for sin.

**Solution:** The double angle identity is stated as

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

Applying this to $\sin(z)$, we can produce

$$\sin(z) = 2\sin\left(\frac{z}{2}\right)\cos\left(\frac{z}{2}\right) = 4\sin\left(\frac{z}{4}\right)\cos\left(\frac{z}{4}\right)\cos\left(\frac{z}{4}\right)$$

And in general,

$$\sin(z) = 2^n\sin\left(\frac{z}{2^n}\right)\cos\left(\frac{z}{2}\right)\cos\left(\frac{z}{4}\right)\cdots\cos\left(\frac{z}{2^n}\right) = 2^n\sin\left(\frac{z}{2^n}\right)P_n(z)$$

So in general, we have that

$$P_n(z) = \frac{\sin(z)}{2^n\sin\left(\frac{z}{2^n}\right)}$$

Sending $n \to \infty$ yields

$$\lim_{n\to\infty} P_n(z) = \sin(z)\lim_{n\to\infty}\frac{\frac{1}{2^n}}{\sin\left(\frac{z}{2^n}\right)} = \frac{\sin(z)}{z}$$

(2) If $|z| < 1$, show that

$$(1+z)(1+z^2)(1+z^4)\cdots = \prod_{n=1}^{\infty}(1+z^{2^n}) = \frac{1}{1-z}$$

**Solution:** Notice that

$$(1+z)(1+z^2)(1+z^4)\cdots(1+z^{2^n}) = 1+z+z^2+\ldots+z^{2^n+2^{n-1}+\ldots+1}$$

This is realized by consider a number $n$ expressed base 2, and presenting a given exponent $\leq z^{2^n+2^{n-1}+\ldots+1}$ exactly by multiplying $z^{2^j}$ where the $j^{th}$ coefficient is non-zero. As a result, we get that

$$\lim_n P_n(z) = \sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

(3) Assuming the result of Hadamard, stated as Theorem 29.4 in the notes, show Picard's Little Theorem:

**Theorem 0.1.** *If $f$ is an entire function of finite order that omits 2 values, then $f$ is constant.*

Picard's 'big theorem' is the one about essential singularities having infinite sheeted coverings nearby missing perhaps 1 point.

**Solution:** Suppose $a, b \in \mathbb{C}$ are missed. Then we can consider $f(z)-a$. Hadamard's result states that we can write

$$f(z) - a = e^{P(z)} z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right)$$

where $P$ is a polynomial of degree at most $\rho = \operatorname{ord}(f)$, 0 has order $m$, and $a_n$ are the other zeroes of $f(z) - a$. But $f(z) - a$ has no zeroes by assumption! As a result,

$$f(z) - a = e^{P(z)}$$

But if $P(z)$ has positive degree, the $P(z)$ surjects onto $\mathbb{C}$. As a result, $e^{P(z)} = b - a \neq 0$ has a solution. On the other hand, this is impossible since $f(z) - a = b - a$. This concludes the proof.

(4) Show that if $f : U \to V$ is a conformal map, then if $U$ is connected or simply connected, then $V$ is also. Therefore these properties are preserved by conformal equivalence.

**Solution:** First, the connected case. Given $f$ is continuous, if there are two disjoint open sets $A, B$ such that $V = A \cup B$, then

$$U = f^{-1}(V) = f^{-1}(A) \cup f^{-1}(B)$$

But since $U$ is connected, either $f^{-1}(A) = U$ or $f^{-1}(B) = U$. But by symmetry, in the first case

$$V = f(U) = A = f(f^{-1}(A))$$

implying $B = \emptyset$. On could further note that even if $f$ is only surjective, we would have

$$V = f(U) f(f^{-1}(A)) \subseteq A$$

So the same holds for continuous surjections in general.

Now suppose $U$ is simply connected. Let $\gamma : [0, 1] \to V$ be a loop. We can then consider

$$f^{-1} \circ \gamma : [0, 1] \to U$$

is a loop in $U$. Therefore it is contractible to $f^{-1}(\gamma(0))$. Let $F$ be this homotopy. Then if we consider

$$f \circ F : [0, 1] \times [0, 1] \to V$$

is a map with $F(t, 0) = f \circ f^{-1} \circ \gamma(t) = \gamma(t)$ and $F(t, 1) = f(\gamma(0)$. So $f^{-1} \circ \gamma$ is constant.

(5) Is there a holomorphic surjection from the disc onto $\mathbb{C}$?

**Solution:** Yes! Consider the conformal map

$$G : B(0, 1) \to \mathbb{H} : z \mapsto i\frac{1 - z}{1 + z}$$

We can then consider the translation of the upper half plane to the upper $-1$ halfplane given by $t : \mathbb{C} \to \mathbb{C} : z \mapsto z - i$. It is very bijective and holomorphic.

Finally, we can consider the map $p : \mathbb{C} \to \mathbb{C} : z \mapsto z^2$. This is holomorphic and surjective. As a result,

$$p \circ t \circ G : B(0,1) \to \mathbb{C}$$

is a surjective and holomorphic map.

(6) Suppose $F$ is holomorphic at 0, and $F(0) = F'(0) = 0$, but $F''(0) \neq 0$. Show that there exist two curves $\gamma_1, \gamma_2 : [-1,1] \to \mathbb{C}$ with $\gamma_i(0) = 0$ and such that $F \circ \gamma_1$ is real valued with a minimum at 0 and $F \circ \gamma_2$ is real valued with a maximum at 0. (**hint:** Write $F(z) = (g(z))^2$ for some $g$, and consider $g$ and its inverse)[1]

**Solution:** Given $F$ has an order 2 zero at $z = 0$, we can write $F$ as

$$F(z) = z^2 g(z)$$

where $g(0) \neq 0$ for some holomorphic function $g$ in a neighborhood of 0. But since $g$ is non-vanishing in a (perhaps smaller) neighborhood, $g(z) = e^{h(z)}$. So, we have

$$F(z) = ze^{h(z)}$$

Therefore, we can notice $F(z) = (ze^{\frac{h(z)}{2}})^2$. Call the squared function $G(z)$. Then noticing that

$$G'(0) = 0 \cdot \frac{h'(0)}{2} e^{\frac{h(z)}{2}} + e^{\frac{h(0)}{2}} = e^{\frac{h(0)}{2}} \neq 0$$

we have that $G$ is invertible by Proposition 30.2. This yields

$$F(G^{-1}(z)) = G^2(G^{-1}(z)) = z^2$$

Now, we can consider the fact that $z^2$ has the desired property. If we consider the path along the real axis (positively), we would get $x^2$, which experiences its minimum there. Call this $\gamma_1$. Similarly, along the imaginary axis $\gamma_2$, the function reads $-y^2$, which experiences its maximum. As a result, the curves $G^{-1}(\gamma_1(t))$ and $G^{-1}(\gamma_2(t)$ would have the desired properties.

(7) If $F : \mathbb{H} \to \mathbb{C}$ is holomorphic satisfying

$$|F(z)| \leq 1 \qquad\qquad F(i) = 0$$

Prove that

$$|F(z)| \leq \left| \frac{z - i}{z + i} \right| \qquad\qquad \forall z \in \mathbb{H}$$

**Solution:** Consider again the holomorphic map $G : B(0,1) \to \mathbb{H}$ from the previous exercise. If we couple this with the map $F$, then we achieve

$$H = F \circ G : B(0,1) \to B(0,1)$$

[1]This is an analog of a saddle point in calculus and real analysis.

where the range is demonstrated via the maximum modulus principle. Note further that $H(0) = F(G(0)) = F(i) = 0$. Now, by virtue of the Schwarz Lemma, we have $|H(z)| < |z|$. Finally, we have that $G^{-1}(z) = \frac{z-i}{z+i}$. So replacing $z$ with $G^{-1}(z)$, we get

$$|H(G^{-1}(z))| = |F(z)| \leq |G^{-1}(z)| = \left|\frac{z-i}{z+i}\right|$$