GENERAL TOPOLOGY: MIDTERM 2

1) [**8 pts**] What does it mean for a topological space X to be first-countable?

Solution: Given $x \in X$, there exists $U_1, U_2, \ldots$ open neighborhoods of x such that for any open neighborhood U of x, $U_n \subseteq U$ for some $n \in \mathbb{N}$.

2) [**7 pts**] What does it mean for a space to be T3/regular?

Solution: Given a point $x \in X$ and $A \subseteq X$ a closed subset such that $x \notin A$, there exists open disjoint sets U, V separating them: $x \in U$ and $A \subseteq V$.

3) [**10 pts**] Recall the statement of Urysohn's Lemma.

Solution: Given X a normal topological space, and A, B two disjoint closed subsets of X, there exists $f : X \to [0, 1]$ continuous such that $f(A) = 0$ and $f(B) = 1$.

4) [**10 pts**] Consider $X = \mathbb{R}$ with two topologies; let τ be the Euclidean topology on $\mathbb{R}$ and let τ' be the topology generated by τ and $\{\mathbb{Q}\}$. That is to say that τ' is the smallest topology containing τ and $\mathbb{Q}$. Show that $(\mathbb{R}, \tau')$ is Hausdorff, but not T4/normal.

Solution: Consider the closed sets $A = \{x\}$, where $x \in \mathbb{Q}$, and $A = \mathbb{R} \setminus \mathbb{Q}$. For each $y \in \mathbb{R} \setminus \mathbb{Q}$, a neighborhood base is given by $\{y\} \cap ((y - \epsilon, y + \epsilon) \cap \mathbb{Q})$. Therefore, if U is an open neighborhood of A, $x \in \mathbb{Q} \subseteq U$. This is a contradiction, showing that in fact X is not even T3.

5) [**15 pts**] State Urysohn's Metrization Theorem. Now use it to show the following: Suppose X is a compact Hausdorff space which is **locally metrizable**, i.e. every point has a neighborhood which is metrizable. Show that X is metrizable.

Solution: UMT: If X is a T3 + T1 second-countable space, then X is metrizable.

Let U_x be an open neighborhood of x for which is $\bar{U_x}$ is metizable. Note we may assume this since X is $T3$; if V is a neighborhood of x which is metrizable, then there exists U_x such that

$$x \in U_x \subseteq \overline{U_x} \subseteq V_x^\circ \subseteq V_x$$

and a subset of a metric space is a metric space. Therefore,

$$X = \bigcup_x U_x$$

By compactness, we may assume that

$$X = U_{x_1} \cup \cdots \cup U_{x_n}$$

Now, since X is compact and Hausdorff, we know that $\overline{U_{x_i}}$ is compact and Hausdorff. But Compact, Hausdorff, and Metrizable imply that they are second-countable.

Therefore, X is second countable, since it has a basis for which the elements come from the countable bases of U_{x_i}.

6) [**15 pts**] A space X is said to be **locally compact** if for every $x \in X$, and U a neighborhood of X, there exists a compact neighborhood K of x contained within U; e.g. $x \in K^\circ \subseteq K \subseteq U$.

Given a *locally* compact Hausdorff space X, a compact subspace K, and an open neighborhood $U \supseteq K$, show that there exists $f : X \to [0, 1]$ such that $f(K) = 1$ and $f(U^c) = 0$.

Solution: I want to apply Urysohn's Lemma, but don't know that all of X is normal. Therefore, I restrict. Given $x \in K$, let K_x be a compact neighborhood of x inside U:

$$x \in K_x^\circ \subseteq K_x \subseteq U$$

Then $K \subseteq \bigcup K_x^\circ$ is an open cover, so we may assume $K \subseteq K_{x_1}^\circ \cup \cdots \cup K_{x_n}^\circ$ by compactness. By Urysohn's Lemma, there exists a function $f' : K_{x_1} \cup \cdots \cup K_{x_n} \to [0, 1]$ such that $f'(K) = 1$ and $f'((K_{x_1}^\circ \cup \cdots \cup K_{x_n}^\circ)^c) = 0$. Let

$$f(x) = \begin{cases} f'(x) & x \in K_{x_1} \cup \ldots \cup K_{x_n} \\ 0 & x \notin K_{x_1}^\circ \cup \cdots \cup K_{x_n}^\circ \end{cases}$$

This is continuous by the pasting lemma and satisfies the desired properties.

7) [**10 pts**] Show that a product of an m-manifold X with an n-manifold Y is an $m + n$-manifold.

Solution: A finite product of second countable spaces is second-countable.
A product of T2 spaces is T2.
If $x \in U \cong U' \subseteq \mathbb{R}^m$ and $y \in V \cong V' \subseteq \mathbb{R}^n$, then

$$(x, y) \in U \times V \cong U' \times V' \subseteq \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{n+m}$$

So $X \times Y$ is a $(n + m)$-manifold.

8) [**20 pts**] The **Axiom of Choice** is assumed throughout almost every branch of mathematics. It is stated as saying if S is a set of non-empty sets, then there is a way of choosing an element of each $C \in S$ (called a choice function). It can be restated as follows:

Axiom. *If X_α is a collection of non-empty sets, then $\prod_\alpha X_\alpha \neq \emptyset$.*

We used this in the proof of Tychnoff's Theorem. Here, we show that they are equivalent by demonstrating that Tychonoff implies the axiom of choice.

- State Tychonoff's Theorem.

  Solution: If X_α are compact sets, then $\prod_\alpha X_\alpha$ with the product topology is compact.

- Let X_α be sets as in the axiom. Give each X_α the indiscrete topology. Show that $X_\alpha \cup \{\infty\}$, obtained by adding 1 extra point and making X_α a closed subset. Show the following space is compact:

$$Y = \prod_\alpha (X_\alpha \cup \{\infty\})$$

  Solution: The topology of $X_\alpha \cup \{\infty\}$ is exactly $\{\emptyset, \{\infty\}, X_\alpha \cup \{\infty\}\}$. Therefore any open cover must contain $X_\alpha \cup \{\infty\}$! As a result, Tychonoff implies Y is compact.

- Recall the **finite intersection property**.

  Solution: A collection $\mathcal{C}$ of closed subsets has the finite intersection property if every $C_1, \ldots, C_n \in \mathcal{C}$ has the property that $C_1 \cap \ldots \cap C_n \neq \emptyset$.

- Let $\pi_\alpha : Y \to X_\alpha \cup \{\infty\}$ be the projection map. Show that $\pi_\alpha^{-1}(X_\alpha)$ is a closed subset of Y. Show the collection has the finite intersection property.

  Solution: π_α is continuous, so $\pi_\alpha^{-1}(X_\alpha) = X_\alpha \times \prod_{\alpha' \neq \alpha} X_{\alpha'} \cup \{\infty\}$ is closed. The collection of $\pi_\alpha^{-1}(X_\alpha)$ has the finite intersection property, since

$$\pi_{\alpha_1}^{-1}(X_{\alpha_1}) \cap \ldots \cap \pi_{\alpha_n}^{-1}(X_{\alpha_n}) = X_{\alpha_1} \times \ldots \times X_{\alpha_n} \times \prod_{\alpha' \neq \alpha_1, \ldots, \alpha_n} X_{\alpha'} \times \{\infty\}$$

  contains $(x_{\alpha_1}, \ldots, x_{\alpha_n}, (\infty)_{\alpha'})$.

- What is $\bigcap_\alpha \pi_\alpha^{-1}(X_\alpha)$? Conclude that the axiom is true.

  Solution: By virtue of the fact that X is compact, we know

$$\bigcap_\alpha \pi_\alpha^{-1}(X_\alpha) \neq \emptyset.$$

  On the other hand,

$$\bigcap_\alpha \pi_\alpha^{-1}(X_\alpha) = \prod_\alpha X_\alpha$$

  Therefore, the axiom of choice is true (in a universe in which Tychonoff holds, such as topology :).

9) [**10 pts**] Let X, Y, Z be two T3.5 spaces. Let $\beta(X)$ be the the Stone-Cech Compactification of X, and $\beta(f) : \beta(X) \to \beta(Y)$ be the map associated to a continuous map $f : X \to Y \subseteq \beta(Y)$. Verify the following two facts:

- $\beta(Id_X) = Id_{\beta(X)} : \beta(X) \to \beta(X)$.
- If $f : X \to Y$ and $g : Y \to Z$ are two continuous maps, then

$$\beta(g \circ f) = \beta(g) \circ \beta(f) : \beta(X) \to \beta(Z)$$

Solution: Recall that $\beta(f)$ is the *unique* extension of $f : X \to Y \subseteq \beta(Y)$ to $\beta(X)$.

Note that $Id_{\beta(X)}(x) = x$ for every $x \in X$, so it is an extension of Id_X. By uniqueness, $\beta(Id_X) = Id_{\beta(X)}$.

Similarly, I claim $\beta(g) \circ \beta(f) : \beta(X) \to \beta(Z)$ is an extension of $g \circ f : X \to Z$. Indeed, for $x \in X$, $\beta(f)(x) = f(x) \in Y$ and for $y \in Y$, $\beta(g)(y) = g(y)$. Therefore, by uniqueness,

$$\beta(g \circ f) = \beta(g) \circ \beta(f) : \beta(X) \to \beta(Z).$$