HOMEWORK 2: MODULE THEORY
DUE: FRIDAY, MARCH 2ND

1) Show that for a prime ideal $\mathfrak{p}$, $R \setminus \mathfrak{p}$ is a multiplicatively closed set.

Solution: Suppose $a, b \in R \setminus \mathfrak{p}$. Then $a \cdot b \notin R \setminus \mathfrak{p}$ since $\mathfrak{p}$ is prime.

2) Compute the localization of the ring $R = \mathbb{Z}/\langle 10 \rangle$ at the multiplicative set $W = \langle 1, 2, 4, 8, 6 \rangle$. In particular, write down all the elements.

Solution: By inverting 2, 5 is set equal to 0 by the bad/ugly example from class. More explicitly, we note that

$$(w, 5) = (1, 0) = (w', 0)$$
$$2(5w' - 0w) = 10w' = 0$$

In fact, this is all of the only elements set to 0, since to divide 10 by an element of W, you must be divisible by 5. As a result, we see that

$$(w, n) \sim (w, n + 5)$$

For $n = 0, 1, 2, 3, 4$. In addition, we can conclude

$$(2, 1) \sim (1, 3)$$

since $5(3 \cdot 2 - 1 \cdot 1) = 0$. This gives us the desired result: $W^{-1}\mathbb{Z}/10\mathbb{Z} \cong \mathbb{Z}/5\mathbb{Z}$ given by the isomorphism $(2^n, a) \mapsto a \cdot 3^n$. This is injective since $(2^n, a) \mapsto 0$ if and only if $5|a$ if and only if $(2^n, a) \sim (1, 0)$. It is also surjective, since $(1, a) \mapsto a$.

3) Show that a map of modules $\varphi : M \to N$ is injective and surjective if and only if there is a 2-sided inverse to φ: $\exists\ \varphi^{-1} : N \to M$ with $\varphi^{-1} \circ \varphi = Id_M$ and $\varphi \circ \varphi^{-1} = Id_N$.

Solution: If φ is as described, then if $n \in N$, then $\exists! m \in M$ such that $\varphi(m) = n$. This allows us to define $\varphi^{-1}(n) = m$, where m is the unique m with $\varphi(m) = n$. This is a well defined map by uniqueness, and is a homomorphism since φ is: $\varphi^{-1}(n + n') = m + m'$ since $\varphi(m) = n$ and $\varphi(m') = n'$, thus $\varphi(m + m') = \varphi(m) + \varphi(m') = n + n'$.

Now suppose that there exists a map φ^{-1} as above. Then $\varphi^{-1} \circ \varphi = Id_M$ and $\varphi \circ \varphi^{-1} = Id_N$. Since Id_M is surjective, we see φ is surjective. Similarly, since Id_N is injective, we see φ is injective.

4) Prove the following Proposition from class:

Proposition 0.1. *There is a natural map* $\mathrm{Hom}_R(N, P) \times \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, P)$ *given by composition.*

In addition, $End_R(M) := \mathrm{Hom}_R(M, M)$ has a natural structure as an R-algebra.

Solution: Define $(\varphi, \psi) \mapsto \varphi \circ \psi$. This is a well defined map (since $\varphi \circ \psi : M \to P$. It only goes to show that $End(M)$ is an R-algebra. Define the action of R by $r \cdot \psi(m) = r\psi(m) = \psi(rm)$. This forces R to be in the center since M is a 2-sided module. Finally, the multiplicative structure is given by composition. This is indeed distributive since it always is for functions.

5) We define the tensor product of two R-modules M, N to be

$$M \otimes_R N = \{\sum_{i=1}^{l} m_i \otimes n_i \mid m_i \in M, n_i \in N\}/\sim$$

where $\sim$ is defined by
i. $rm \otimes n \sim m \otimes rn$
ii. $m \otimes n + m' \otimes n \sim (m + m') \otimes n$
iii. $m \otimes n + m \otimes n' \sim m \otimes (n + n')$
Give $M \otimes_R N$ the structure of an R-module.

Solution: We can think of $M \otimes_R N$ as the free module generated by symbols $m \otimes n$ and quotient by the submodule

$$\langle rm \otimes n - m \otimes rn, m \otimes n + m' \otimes n - (m + m') \otimes n, m \otimes n + m \otimes n' - m \otimes (n + n')\rangle$$

This makes the tensor product of two modules well defined. One can see that $M \otimes_R N$ is an R-module by multiplication:

$$r \cdot (m \otimes n) = (rm) \otimes n = m \otimes (rn)$$

It is additive by conditions ii. and iii.

6) Recall that if $\varphi : R \to S$ is a ring homomorphism, then S is an R-module. If M is another R-module, show that $M \otimes_R S$ has the structure of an S module.
Additionally, note that if M is an S-module, M is also an R-module with action $r \cdot m = \varphi(r)m$. Therefore, we can pass modules between rings that are connected by a ring homomorphism.

Solution: We can create a natural S-module structure on $M \otimes_R S$ as follows:

$$s \cdot (m \otimes s') = m \otimes ss'$$

and extend it over sums by distribution.

7) Show that if $N \subseteq M$, and that M/N and N are finitely generated modules, then so is M.

Solution: Suppose that N is generated by $n_1, \ldots, n_l$ and M/N is generated by $\bar{q}_1, \ldots, \bar{q}_k$. Let $q_1, \ldots, q_k$ be elements of M such that $q_i + N = \bar{q}_i$. Then I claim

$$M = \langle n_1, \ldots, n_l, q_1, \ldots, q_k \rangle$$

Let $m \in M$. Then $\bar{m} = m + N \subseteq M/N$ implies that $\exists r_i \in R$ with

$$\bar{m} = r_1\bar{q}_1 + \ldots + r_k\bar{q}_k$$

Therefore, if we consider

$$Q(m - r_1q_1 - \ldots - r_kq_k) = \bar{m} - (r_1\bar{q}_1 + \ldots + r_k\bar{q}_k) = 0$$

where $Q : M \to M/N$ is the quotient map. Therefore $m - r_1q_1 - \ldots r_kq_k \in N$ to begin with, and therefore $\exists r_i' \in R$ with

$$m - r_1q_1 - \ldots - r_kq_k = r_1'n_1 + \ldots + r_l'n'$$

or equivalently,

$$m = r_1q_1 + \ldots + r_kq_k + r_1'n_1 + \ldots + r_l'n'$$

This shows the desired generation statement for M!

8) Show that if F, F' are free modules, then so is $F \otimes_R F'$. Calculate its rank.

Solution: If $\text{rank}(F) = n$ and $\text{rank}(F) = m$, I claim that $\text{rank}(F \otimes_R F') = mn$. Let 1_i denote 1 in the i^{th}-entry of F, and 0 otherwise. Similarly for 1_j in F'. Then I claim the desired basis of $F \otimes_R F'$ is

$$F \otimes_R F' = \langle 1_i \otimes 1_j \mid i = 1, \ldots, n,\ j = 1, \ldots, m \rangle$$

Note that

$$(r_1, \ldots, r_n) \otimes (r_1', \ldots, r_m') = \sum_i \sum_j r_ir_j' \cdot 1_i \otimes 1_j$$

Therefore, the above basis spans $F \otimes_R F'$. Moreover, they are clearly linearly independent, as in the previous equation there is no other way to represent the given element.

9) Show that two free R-modules are isomorphic, $F \cong F'$, if and only if F and F' have the same rank. (**Hint:** It may be useful to quotient out by a maximal ideal).

Solution: It is clear that if $F \cong R^n$ and $F' \cong R^n$, then $F \cong R^n \cong F'$.

For the opposing direction, following the hint we can consider $\varphi : F \cong R^n \to F' \cong R^m$ an isomorphism. Tensoring this isomorphism by $R/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$, we get an isomorphism

$$\varphi \otimes 1 : F \otimes_R R/\mathfrak{m} \to F' \otimes_R R/\mathfrak{m} : \alpha \otimes r \mapsto \varphi(\alpha) \otimes r$$

extended by linearity. This is well defined since

$$r\alpha \otimes 1 \mapsto \varphi(r\alpha) \otimes 1 = r\varphi(\alpha) \otimes 1 = \varphi(\alpha) \otimes r$$

It is also an isomorphism by consideration of $\varphi^{-1} \otimes 1$.