HOMEWORK 2: THE COMPLEX PLANE
DUE: WEDNESDAY, SEPTEMBER 25TH

1) Prove the complex version of the chain rule: if $f : U \to V$ and $g : V \to \mathbb{C}$ are two differentiable functions, and $h = g \circ f$

$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}$$

$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}$$

Solution: We can write g and f as functions of z and $\bar{z}$ to simplify matters. Doing so

$$h(z, \bar{z}) = g(f(z, \bar{z}), \bar{f}(z, \bar{z}))$$

Now, we can compute using the standard chain-rule:

$$\frac{\partial h(z)}{\partial z} = \frac{\partial g(f(z,\bar{z}), \bar{f}(z,\bar{z}))}{\partial z} = \left[\frac{\partial g(w,\bar{w})}{\partial w}\right]_{w=f(z,\bar{z})} \cdot \frac{\partial f(z,\bar{z})}{\partial z} + \left[\frac{\partial g(w,\bar{w})}{\partial \bar{w}}\right]_{\bar{w}=\bar{f}(z,\bar{z})} \cdot \frac{\partial \bar{f}(z,\bar{z})}{\partial z}$$

This shows the first equality. The second is similar:

$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g(f(z,\bar{z}), \bar{f}(z,\bar{z}))}{\partial \bar{z}} = \left[\frac{\partial g(w,\bar{w})}{\partial w}\right]_{w=f(z,\bar{z})} \cdot \frac{\partial f(z,\bar{z})}{\partial \bar{z}} + \left[\frac{\partial g(w,\bar{w})}{\partial \bar{w}}\right]_{\bar{w}=\bar{f}(z,\bar{z})} \cdot \frac{\partial \bar{f}(z,\bar{z})}{\partial \bar{z}}$$

2) If $f : \Omega \to \mathbb{C}$ is holomorphic, then assuming any of the following conditions one can conclude f is constant:

   i. $Re(f)$ is constant.
   ii. $Im(f)$ is constant.
   iii. $|f|$ is constant.

Solution:

i. Let $Re(f) = u$ and $Im(f) = v$. Then $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial y} = 0$. But the CR equations ensure that $\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$. Therefore f is constant.

ii. The same argument yields this result.

iii. If $|f|$ is constant, then $|f|^2 = u^2 + v^2$ is also constant. Taking x and y derivatives yields

$$0 = uu_x + vv_x = uu_y + vv_y$$

CR then gives us

$$0 = uv_y + vv_x = -uv_x + vv_y$$

Multiplying through by u and v each of the sides then yields

$$0 = (u^2 + v^2)v_y = (u^2 + v^2)v_x$$

If $u^2 + v^2 = 0$, we're done. Otherwise, $v_y = v_x = 0$. Again, CR yields the same is true for u!. So f must be constant.

3) Verify the Euler relations for sin and cos:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2} \qquad\qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

Solution: We have that $\sin(z) := \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{(2n+1)!}$. Now considering the right hand side of the equation yields

$$\frac{e^{iz} - e^{-iz}}{2} = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} - \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right] = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^{2n}}{(2n)!} - \sum_{n=0}^{\infty}\frac{(iz)^{2n}}{(2n)!}\right] + \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^{2n+1}}{(2n+1)!} + \sum_{n=0}^{\infty}\frac{(iz)^{2n+1}}{(2n+1)!}\right]$$

The second equality is obtained by separating the sum into even and odd terms and viewing $(-1)^n$ in each case. The first sum cancels, and the second are identical, canceling the $\frac{1}{2}$. Thus equality is achieved.

Similarly, $\cos(z) := \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n}}{(2n)!}$ and the right hand side is

$$\frac{e^{iz} + e^{-iz}}{2} = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^n}{n!} + \sum_{n=0}^{\infty}\frac{(-iz)^n}{n!}\right] = \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^{2n}}{(2n)!} + \sum_{n=0}^{\infty}\frac{(iz)^{2n}}{(2n)!}\right] + \frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(iz)^{2n+1}}{(2n+1)!} - \sum_{n=0}^{\infty}\frac{(iz)^{2n+1}}{(2n+1)!}\right]$$

Again, the desired equality follows immediately.

4) Determine (and prove) the radii of convergence for the following power series:
   i. $\sum_{n=1}^{\infty}(\log(n))^2 z^n$
   ii. $\sum_{n=0}^{\infty}(n!)z^n$
   iii. $\sum_{n=0}^{\infty}\left(\frac{n^2}{4^n+3n}\right)z^n$
   iv. $\sum_{n=0}^{\infty}\left(\frac{(n!)^3}{(3n)!}\right)z^n$

For iv. it may be helpful to use Sterling's Formula: $n! \sim cn^{n+\frac{1}{2}}e^{-n}$ for some constant $c > 0$.

Solution: The key here is to use Hadamard's equality:

i. $\frac{1}{R} = \limsup(|a_n|^{\frac{1}{n}}) = \limsup|\log(n)|^{\frac{2}{n}} = \lim|\log(n)|^{\frac{2}{n}} = 1$. The last equality follows from the fact that $\log(n)^2 < n$ for n large enough and $\lim n^{\frac{1}{n}} = 1$ (by, for example, the binomial theorem).

ii. $\frac{1}{R} = \limsup|n!|^{\frac{1}{n}}$. Now, one can notice that

$$(n!)^{\frac{1}{n}} = n^{\frac{1}{n}} \cdot (n-1)^{\frac{1}{n}} \cdots 2^{\frac{1}{n}} \geq m^{\frac{1}{n}} \cdots m^{\frac{1}{n}} \cdot (m-1)^{\frac{1}{n}} \cdots 2^{\frac{1}{n}} \geq m^{\frac{n-m}{n}}$$

for any $1 \leq m \leq n$. As a result, taking the limit as $n \to \infty$, we see that $\frac{1}{R} \geq m$ for any $m \geq 0$. Thus $\frac{1}{R} = \infty$, which is to say $R = 0$.

iii. $\frac{1}{R} = \limsup\left(\frac{n^2}{4^n+3n}\right)^{\frac{1}{n}} = \limsup\left(\frac{n^{\frac{2}{n}}}{4}\right) = \frac{1}{4}$. The middle equality follows since the 2 fractions are asymptotically equivalent. So $R = 4$.

iv. Following Sterling, we can write

$$\frac{1}{R} = \limsup\left(\frac{(cn^{n+\frac{1}{2}}e^{-n})^3}{c(3n)^{3n+\frac{1}{2}}e^{-3n}}\right)^{\frac{1}{n}} = \limsup\left(\frac{c^3n^{3n+\frac{3}{2}}e^{-3n}}{c(3n)^{3n+\frac{1}{2}}e^{-3n}}\right)^{\frac{1}{n}}$$

$$= \limsup\left(\frac{c^2n^{3n+\frac{3}{2}}}{3^{\frac{1}{2}}27^n n^{3n+\frac{1}{2}}}\right)^{\frac{1}{n}} = \limsup\left(\frac{c^2n}{3^{\frac{1}{2}}27^n}\right)^{\frac{1}{n}} = \frac{1}{27}$$

So $R = 27$. I.e. the power series converges on $\mathbb{C}$.

5) Verify that our notion of 2 parameterized curves being equivalent forms an **equivalence relation**. There are 2 statements here: if $\gamma_1 \simeq \gamma_2$, then $\gamma_2 \simeq \gamma_1$. Additionally, show that if $\gamma_2 \simeq \gamma_3$, then $\gamma_1 \simeq \gamma_3$.

Solution: Let $s : [a, b] \to [c, d]$ and $r : [c, d] \to [e, f]$ be the transition functions for $\gamma_1 \simeq \gamma_2$ and $\gamma_2 \simeq \gamma_3$ respectively. Then we have

$$\gamma_1(s(r(t))) = \gamma_2(r(t)) = \gamma_3(t)$$

So using the transition function $s \circ r$ shows $\gamma_1 \simeq \gamma_3$. Now, the key is the following realization: if $s : [a, b] \to [c, d]$ is a bijective function with $s'(t) \neq 0$, then $s^{-1} : [c, d] \to [a, b]$ is also differentiable and in fact

$$(s^{-1})'(t) = \frac{1}{s'(s^{-1}(t))}$$

Therefore, we have that

$$\gamma_2(t) = \gamma_2(s(s^{-1}(t))) = \gamma_1(s^{-1}(t))$$

which shows $\gamma_2 \simeq \gamma_1$.

6) Suppose $|a| < r < |b|$, and let C be the circle of radius r. Show that

$$\int_C \frac{1}{(z-a)(z-b)}dz = \frac{2\pi i}{a-b}$$

Solution: We can parameterize C by $\gamma(t) = re^{it}$. As a result, using partial fractions we can derive

$$\int_C \frac{1}{(z-a)(z-b)}dz = \frac{1}{a-b}\int_C \frac{1}{z-a} - \frac{1}{z-b}dz$$

So we can focus on the question of $\int_C \frac{1}{z-c}dz$ for various $r \neq c \geq 0$. If $|c| > r$, then we have that $\frac{1}{z-c}$ is a holomorphic function in C. As a result, the integral is 0. On the other hand, if $|c| < r$, we can do the substitution $w = z - c$ to produce

$$\int_C \frac{1}{z-c}dz = \int_C \frac{1}{w}dw = 2\pi$$

As in our in class example. This yields the desired result

$$\int_C \frac{1}{(z-a)(z-b)}dz = \frac{2\pi i}{b-a}$$

7) Consider the real valued function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x^2}} & x > 0 \end{cases}$$

Show that this function is infinitely differentiable, but the n^{th}-derivative $f^{(n))}(0) = 0$ for every n. Conclude there is no power series for f at $x = 0$.[1]

Solution: Taking the first derivative of $e^{-\frac{1}{x^2}}$ yields the function

$$\frac{2}{x^3} \cdot e^{-\frac{1}{x^2}}$$

Taking the limit as $x \to 0^+$, we acquire

$$\lim_{x \to 0^+} \frac{2}{x^3} \cdot e^{-\frac{1}{x^2}} = \lim_{y \to \infty} 2y^3 e^{-y^2} = 0$$

by L'Hopital's rule. Similarly, we can compute the n^{th} derivative to be of the form

$$g_n(\frac{1}{x})e^{-\frac{1}{x^2}}$$

where g_n is a polynomial of degree $3n$ (by induction). Again, the right-handed limit of this function is 0 by L'Hopital.

As a result, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then taking derivatives would product $f^{(n)}(x) = 0 = n! a_n$, implying $a_n = 0$ for every n. But $f(x) > 0$ for every $x > 0$, so the power series representation holds in no neighborhood of $x = 0$.

[1]This is related to the complex case of the involved function not being holomorphic.