CLASS 27, NOVEMBER 15TH: FUNCTIONS OF FINITE ORDER

Today we will measure the asymptotic behavior of a function and study how this relates to the number of zeroes $\mathfrak{n}_f(r)$. This is rate of growth is measured as the order:

Definition 27.1. If there exists a positive number $\rho > 0$ and constants $A, B > 0$ such that

$$|f(z)| \leq Ae^{B|z|^\rho} \qquad \forall z \in \mathbb{C}$$

then we say f has **order of growth** $\leq \rho$. The order of f is the infimal such ρ.

Thus the order of growth of e^{z^2} is 2 and the order of growth of a polynomial is 0. We stick with the assumption that $f \neq 0$ to simplify statements.

Theorem 27.2. *If f is an entire function that has order of growth $\leq \rho$, then*

- *$\mathfrak{n}_f(r) \leq Cr^\rho$ for some C and r sufficiently large.*
- *If $z_1, z_2, \ldots$ denotes the zeroes of f with $z_k \neq 0$, then for $s > \rho$, we have*

$$\sum_{j=1}^{\infty} \frac{1}{|z_k|^s} < \infty$$

Proof. It is enough to prove the first bullet in the case $f(0) \neq 0$, since we can divide by z^l and only modify the result by a constant. Thus we can replace C with $C + l$ for $r > 1$ and not lose any generality.

Using the corollary of Jensen's formula, namely

$$\int_0^R \frac{\mathfrak{n}_f(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)|$$

we can choose $R = 2r$ and derive:

$$\int_r^{2r} \frac{\mathfrak{n}_f(x)}{x} dx \leq \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)|$$

Note that $\mathfrak{n}_f(r)$ is an increasing function, so we have

$$\int_r^{2r} \frac{\mathfrak{n}_f(x)}{x} dx \geq \mathfrak{n}_f(r) \int_r^{2r} \frac{dx}{x} = \mathfrak{n}_f(r) \log(2)$$

But the growth condition on f yields us

$$\frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta \leq \int_0^{2\pi} \log|Ae^{BR^\rho}| d\theta \leq C' r^\rho$$

This proves

$$\mathfrak{n}_f(r) \leq \frac{1}{2\pi \log(2)} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)| \leq Cr^\rho$$

for some $C > 0$ and $r \gg 0$.

For the second bullet point, we may break up the complex plane into discs of radius 2^j:

$$\begin{aligned}\sum_{j=1}^{\infty}\frac{1}{|z_k|^s} &= \sum_{j=0}^{\infty}\sum_{z_k\in\mathbb{D}_{2^{j+1}}\setminus\mathbb{D}_{2^j}}|z_k|^{-s}\\ &\leq \sum_{j=0}^{\infty}2^{-js}\mathfrak{n}_f(2^{j+1})\\ &\leq c\sum_{j=0}^{\infty}2^{-js}2^{(j+1)\rho}\\ &\leq 2^{\rho}c\sum_{j=0}^{\infty}(2^{\rho-s})^j = \frac{2^{\rho}c}{1-2^{\rho-s}} < \infty\end{aligned}$$

□

The following examples show that the $s > \rho$ is a sharp bound on this result.

Example 27.3. If we consider $f(z) = \sin(\pi z)$, then we know this function has zeroes precisely at the integers. So we can calculate $\mathfrak{n}_f(r)$ explicitly:

$$\mathfrak{n}_f(r) = 2\lfloor r\rfloor + 1.$$

Let's now consider the rate of growth. Since $\sin(z) = \frac{e^{i\pi z}-e^{-i\pi z}}{2i}$, it naturally has a rate of growth of 1. So if we look at the sum in bullet 2 of Theorem 27.2, we get

$$\sum_{n\neq 0}\frac{1}{|n|^s} = 2\cdot\zeta(s)$$

which converges precisely when $s > 1$.

Example 27.4. Let

$$f(z) = \cos(\sqrt{z}) = \sum_{n=0}^{\infty}(-1)^n\frac{z^n}{(2n)!}$$

One can quickly check by comparing Taylor series that $|f(z)| \leq e^{|z|^{\frac{1}{2}}}$, and thus the order of growth is 1/2. Moreover, f has zeroes exactly at $z_n = ((n+\frac{1}{2})\pi)^2$, and thus

$$\sum_{j=1}^{\infty}\frac{1}{|z_k|^s} = \sum_{j=1}^{\infty}\frac{1}{((n+\frac{1}{2})\pi)^{2s}} < \infty$$

Next time we will try to tackle the following question: given any sequence $z_1, z_2, \ldots$ such that $|z_i| \to \infty$, can we construct an entire function vanishing precisely at z_i. The naive guess would be $\prod_{i=1}^{\infty}(z - z_i)$. But there is an issue of convergence. We'll look at infinite products next week.