CLASS 4, SEPTEMBER 16: CAUCHY-RIEMANN EQUATIONS

Last time we finished up with the following result:

Theorem. *If f is holomorphic at $z_0 \in \mathbb{C}$, then f satisfies the Cauchy-Riemann equations. Additionally,*

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \qquad\qquad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$$

Finally, f is differentiable in the sense of real analysis, and

$$|f'(z)|^2 = \det(J(x_0, y_0))$$

where J is the Jacobian as above.

I'll begin with a proof sketch:

Proof. f satisfies the CR equations was demonstrated last time. For the second equality;

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

The CR equations allow us to write

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\right)\right] = 2\frac{\partial u}{\partial z}(z_0)$$

The case of $\frac{\partial f}{\partial \bar{z}}(z_0)$ is similar but easier. A final computation shows the following:

$$\det(J(x_0, y_0)) = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(2\frac{\partial u}{\partial z}\right)^2 = |f'(z_0)|^2$$

□

We have put a lot of effort into the Cauchy-Riemann equations and it has been stated that they are essential in complex analysis. The following partial converse is one of the main motivations to study these equations:

Theorem 4.1. *Let $\Omega \subseteq \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$ a function. Write $f = u + iv$ to denote its real and imaginary parts. If u and v are* **continuously** *differentiable and satisfy the Cauchy-Riemann equations in Ω, then f is holomorphic in Ω and $f'(z) = \frac{\partial f}{\partial z}$.*

This is a beautiful result yielding not only a path from real to complex analysis, but a simple way to check holomorphicity of functions!

Proof. Given u and v are differentiable, write

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\psi_1(h)$$

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\psi_2(h)$$

where $h = (h_1, h_2)$ and $\lim_{h\to 0}\psi_i(h) = 0$. Using CR, we have that

$$f(z + h) - f(z) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + |h|\psi(h)$$

where $\psi = \psi_1 + \psi_2$. This implies f is differentiable by our equivalent characterizations. $\square$

Next up we will talk about power series. Many of the functions with power series from calculus are equivalently defined for complex analysis! They also provide canonical examples of holomorphic functions. Recall

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

For real values of z, this series converges absolutely! It turns out that this condition is enough to ensure convergence for any choice of complex number[1] and thus yields a definition for the exponential as a complex function.

Theorem 4.2. *Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \leq R \leq \infty$ such that the series converges absolutely for $|z| < R$ and diverges for $|z| > R$.*

Additionally, we can calculate R explicitly using **Hadamard's formula***:*

$$1/R = \limsup |a_n|^{\frac{1}{n}}$$

Definition 4.3. R as in Theorem 4.2 is called the **radius of convergence**, whereas $B(0, R)$ is called the **disc of convergence**.

Proof. Let $L = \frac{1}{R}$ be as in Hadamard's formula. Suppose $L \neq 0, \infty$ (which are easy special cases). If $|z| < R$, choose $\epsilon > 0$ (by openness) such that

$$(L + \epsilon)|z| < 1$$

Since $|a_n|^{\frac{1}{n}} \leq L + \epsilon$ for $n \gg 0$,

$$|a_n||z|^n \leq ((L + \epsilon)|z|)^n = r^n$$

By what we know about geometric series, the series converges absolutely. If $|z| > R$, the same argument shows the existence of a sequence of terms going to infinity. Since a series can only converge in any sense if the terms go to 0, we conclude that the sequence diverges. $\square$

The same ideas allow us to define our trig functions:

Definition 4.4. Define the complex functions

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

It is now easy to check $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ for real numbers θ by comparing power series. Additionally, we get some new **Euler relations**:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

Next time, we'll begin by proving the following result:

Theorem 4.5. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then f is holomorphic in its disc of convergence. Furthermore,*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

has the same radius of convergence R.

[1] This can be seen, for example, by considering real and imaginary parts.