CLASS 24, APRIL 17TH: SCHEME THEORETIC PERSPECTIVE

As we know, there are many rings which aren't (quotients, or localizations, or completions of) polynomial rings over a field. The easiest examples to reconcile are $\mathbb{Z}$ or $\mathbb{Z}_p$, the p-adic integers (power series in p). This gives us a good reason to study a topological space which is very similar to a variety, but with some extra points which represent subvarieties. Recall Proposition 21.5:

Proposition 1. *X is irreducible if and only if $I(X)$ is prime:*

$$Spec(K[x_1,\dots,x_n]) = \{J \subseteq K[x_1,\dots,x_n] \text{ prime}\} \longleftrightarrow \{V = V(J) \subseteq K^n \text{ irreducible}\}$$

Definition 24.1. Given a ring R, we can endow the set $X = \mathrm{Spec}(R)$ with the Zariski Topology generated by closed sets

$$V(J) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid J \subseteq \mathfrak{p}\}$$

We call the resulting topological space an **affine scheme**.

Sometimes the two notions are conflated and people refer to this more modern approach as an affine variety. There are a few advantages to this change.

(a) There exists a map from $V(J)$ to $\mathrm{Spec}(K[x_1,\dots,x_n]/J)$ sending $(a_1,\dots,a_n)$ to $\mathfrak{m} = \langle x_1 - a_1,\dots,x_n - a_n\rangle \in \mathrm{Spec}(K[x_1,\dots,x_n])$. For those of you who have dealt with topological spaces before, this is an injective continuous map inducing a homeomorphism onto its image. Therefore, it is a topological embedding!

(b) To get any irreducible subvariety of K^n, we simply need to take a *point* (i.e. a prime ideal) of $\mathrm{Spec}(K[x_1,\dots,x_n])$ and take its closure. Therefore $\mathrm{Spec}(K[x_1,\dots,x_n])$ contains much more information than K^n itself.

(c) We can produce a natural way of defining a map between varieties: It should look like $\varphi^\# : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ for some ring homomorphism $\varphi : R \to S$. It turns out that this is actually a continuous map when we put the Zariski topology on each Spec.

(d) One doesn't need to view a variety inside K^n. This is a general theme in geometry; you care about the variety itself, not where it lives.

(e) It generalizes naturally to broader rings.

This style of reasoning yields more general results then for varieties without too much hard work.

Proposition 24.2. *Let R be a Noetherian ring with J a proper ideal.*

(a) $V(J) \subseteq \mathrm{Spec}(R)$ *contains only finitely many minimal primes.*

(b) $\sqrt{J} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$.

(c) *Considering $J = 0$ in the previous case, if R is a ring with zero divisors, then either R has nilpotents or R has a finite number $n \geq 2$ of minimal primes.*

Proof. (a) $V(J)$ has an irreducible decomposition $X_1 \cup \cdots \cup X_n$ where each X_i is irreducible and $X_i \not\subseteq X_j$ for all $i \neq j$. Proposition 1 extended to general Noetherian rings (with the same proof) allows us to conclude $I(X_i) = \mathfrak{p}_i$ is prime. Thus $\mathfrak{p}_i$

contains J by the inclusion reversing property of I. The maximal such X_i are minimal among primes containing J, as desired.

(b) This follows from part (a) and the fact that

$$\sqrt{J} = \bigcap_{\mathfrak{p} \supseteq I} \mathfrak{p}$$

which was proved as Corollary 5.5 in our notes.

(c) $\sqrt{0} = 0$ if and only if R is reduced ring. Since R contains zero divisors, 0 is itself not prime. Thus the result of part (b) implies there are at least 2 primes represented in the intersection.

□

Example 24.3. If we consider $V(n) \subseteq \operatorname{Spec}(\mathbb{Z})$, the statement shows there are only finitely many primes (minimally) containing. Let $n = p_1^{e_1} \cdots p_n^{e_n}$ be its prime decomposition. Then

$$\sqrt{\langle n \rangle} = \langle p_1 \rangle \cap \cdots \cap \langle p_n \rangle$$

This recovers a result from an early homework.

Example 24.4. Sometimes the decomposition is less clear. We know $\mathbb{Z}[x, y, z]/I$ is a Noetherian ring by virtue of the Hilbert Basis Theorem. As a result, any radical ideal J has such a decomposition. For example, considering

$$J = \langle -5\,y^5 - 5\,z^3 + 5\,x^2, -5\,y^6 - 5\,y\,z^3 + 5\,x^2y, -y^8 + x^2y^5 - y^3z^3 + x^2y^3 + x^2z^3 - x^4 \rangle$$

It can be checked through hard computation (or by use of a computer) that J is a radical ideal, and it has a decomposition into primes

$$J = \langle x^2 - y^3 - z^5 \rangle \cap \langle 5, x^2 - y^3 \rangle$$

As a corollary, we have the following cool result:

Corollary 24.5. *If R is a reduced Noetherian ring, then R injects into a finite product of integral domains.*

Proof. Given R is Noetherian and reduced, we get

$$0 = \sqrt{0} = \cap_{\mathfrak{p} \text{ prime}} \mathfrak{p} = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n$$

Where $\mathfrak{p}_i$ are the finitely many minimal primes of R. Therefore, we see that the kernel of the map $R \to R/\mathfrak{p}_1 \times \cdots \times R/\mathfrak{p}_n$ is the intersection of the primes, which is 0. □

It should be noted that non-reduced elements can't map to non-zero elements of a reduced ring (or domain in particular). So in some sense this is best possible in that regard.

Example 24.6. $R = K[x_1, \ldots, x_n]/\langle x_1 x_2 \cdots x_n \rangle$ is often called the **simple normal crossing** ring, because geometrically it looks like n hyperplanes intersecting with 'perpendicular' crossings. The previous result shows that

$$R \hookrightarrow R/\langle x_1 \rangle \times \cdots \times R/\langle x_n \rangle \cong K[x_2, \ldots, x_n] \times K[x_1, x_2, \ldots, x_n] \times \cdots \times K[x_1, \ldots, x_{n-1}]$$

So we can view $\operatorname{Spec}(R) \leftarrow \mathbb{A}_K^{n-1} \cup \overset{n-\text{times}}{\cdots\cdots} \cup \mathbb{A}_K^{n-1}$ as a nice surjective map. This map is in fact always surjective, since every prime ideal or R contains some minimal prime.