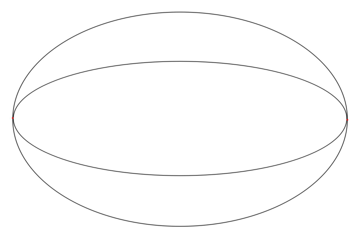
ALGEBRAIC TOPOLOGY MIDTERM

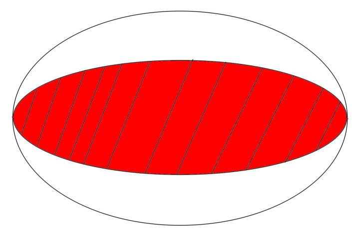
1) [10 pts] Define the fundamental group of a space X at basepoint x_0. Be precise about any equivalence relations involved.

2) [10 pts] What is the fundamental group of the space X obtained by taking two circles and identifying 2 distinct points on one circle with 2 distinct points on the other?



3) [7 pts] If X is a topological space, and γ is a loop based at x_0 in X, what is the effect of adjoining a 2-cell by $e^2 = \gamma$ to $\pi_1(X, x_0)$?

4) [10 pts] What is the fundamental group of the space obtained from part 2 by filling in one region?



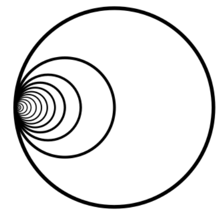
5) [10 pts] State the (simplified) Van Kampen Theorem.

6) [12 pts] Let X be a path connected space. Recall that the suspension of X, called $S(X) = SX$, is defined by

$$S(X) = X \times I / \sim$$

where $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$. That is, we pinch the sides of the interval to a point. Find $\pi_1(SX)$. (**Hint:** Since we know nothing about X, it may be wise to divide the space along I using Question 5.)

7) [20 pts] **Shrinking Wedge of Circles:** Let X be the subspace of $\mathbb{R}^2$ formed by taking a wedge sum at the origin of all the circles C_n of radius $\frac{1}{n}$ centered at $(0, \frac{1}{n})$. We will show that this easily obtained space X has an uncountable fundamental group:



i. [5 pts] The group $G = \prod_{i=1}^{\infty} \mathbb{Z}$ is the ordered set of infinitely many integers. Show that it is uncountable (for example, by injecting $\mathbb{R}$ into it).

ii. [8 pts] Show that for $\boldsymbol{a} = (a_1, a_2, \ldots) \in G$, there is $\gamma_{\boldsymbol{a}} \in \pi_1(X)$ such that $\gamma_{\boldsymbol{a}}$ loops a_1-times around C_1, then a_2-times around C_2, and so on (say on timescale $[\frac{1}{n+1}, \frac{1}{n}]$). In particular, show continuity at $t = 0$ (ϵ-δ may be useful).

iii. [7 pts] Show that the $\gamma_{\boldsymbol{a}}$ are non-homotopic, by considering retractions $r_n : X \to C_n$ sending all other circles to the origin.[1]

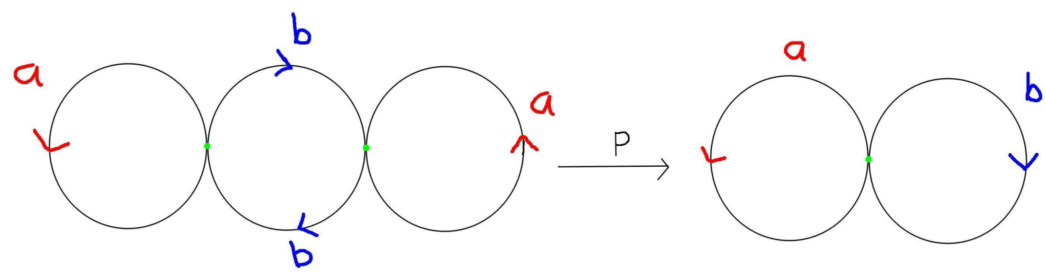
[1]Note this also distinguishes the space X from $\vee_{i=1}^{\infty} S^1$, which has countable fundamental group $\mathbb{Z}^{*\infty}$.

8) [5 pts] Define a covering space.

9) [10 pts] Let $X = \cup_\alpha X_\alpha$ be a locally finite open cover of X. If $\tilde{X} = \coprod_\alpha X_\alpha$ is the disjoint union of the open sets in the cover, show that $p : \tilde{X} \to X$ is a covering space.[2]

[2]Therefore, nice open covers can be viewed as covering spaces. Recall locally finite means every point is in only finitely many X_α.

10) [10 pts] Consider the cover of $X = S^1 \vee S^1$ given by the following picture. Present and describe in words $G(\tilde{X})$, the group of deck transformations of $\tilde{X}$ over X.



11) [Extra Credit 10] Let X and Y be path connected, locally connected spaces, and let $\tilde{X}$ and $\tilde{Y}$ be their respective universal covering spaces (so that $\tilde{X}$ and $\tilde{Y}$ are simply connected). Show that if $X \simeq Y$, then $\tilde{X} \simeq \tilde{Y}$. (**Hint:** Lifting properties!)

$$\begin{tikzcd}
\tilde{X} \arrow[d, "p"'] & \tilde{Y} \arrow[d, "q"] \\
X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"]
\end{tikzcd}$$

You may use the following without proof: $X \xrightarrow{f} Y$ is a homotopy equivalence if there exists $g, h : Y \to X$ with $g \circ f \simeq Id_X$ and $f \circ h \simeq Id_Y$.