HOMEWORK 9: SUPPORT & ASSOCIATED PRIMES
DUE: WEDNESDAY, MAY 1ST

1) (From the discussion of 7.2) Given M an R-module, construct $\mathcal{M} = \coprod_{\mathfrak{p}\in\mathrm{Spec}(R)} M_{\mathfrak{p}}$. This can be thought of as a copy of M lying over each $\mathfrak{p} \in \mathrm{Spec}(R)$. $\mathrm{Supp}(M)$ now marks the closed subset, by Proposition 25.2 (d), of points that matter in this construction: $\mathcal{M} = \coprod_{\mathfrak{p}\in\mathrm{Supp}(M)} M_{\mathfrak{p}}$.

   We have a natural map of spaces $p : \mathcal{M} \to \mathrm{Spec}(R)$ sending $\frac{m}{p} \in M_{\mathfrak{p}}$ to $\mathfrak{p} \in \mathrm{Spec}(R)$. Show that M can be identified as a subset of the *sections* of p; that is to say $s_m : \mathrm{Spec}(R) \to \mathcal{M}$ such that $p \circ s_m = Id_{\mathrm{Spec}(R)}$.

   Additionally, show $W^{-1}M$ represents partially defined sections, for those $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathfrak{p} \cap W = \emptyset$.

   Solution: Given m, we can design a section

$$s_m : \mathrm{Spec}(R) \to \mathcal{M} : \mathfrak{p} \mapsto \frac{m}{1} \in M_{\mathfrak{p}}$$

   This is very naturally a section.

   If $m \in W^{-1}M$, for each $\mathfrak{p} \in \mathrm{Spec}(W^{-1}R) \subseteq \mathrm{Spec}(R)$, we can define a partial section

$$s_m : \mathrm{Spec}(W^{-1}R) \to \mathcal{M} : \mathfrak{p} \mapsto \frac{m}{1} \in M_{\mathfrak{p}}$$

   Trying to extend this to other points would yield 0, since $W^{-1}M_{\mathfrak{p}} = 0$. The domain can vary quite a lot, from open in the case of R_f to very small in the case of $R_{\mathfrak{p}}$ (in particular when $\mathfrak{p}$ is a minimal prime).

   For those students interested, this is an example of a fiber bundle over a variety in algebraic geometry.

2) Verify the claim of Example 25.3; If we consider $M = \oplus_{n\in\mathbb{N}}\mathbb{Z}/n\mathbb{Z}$, we can conclude that $\mathrm{Supp}(M) \neq V(\mathrm{Ann}(M))$. Show all of your assertions.

   Solution: First, I claim $\mathrm{Ann}(M) = 0$. Indeed, if $a \in \mathbb{Z}$ annihilates every element of M, then it must annihilate $1 \in \mathbb{Z}/n\mathbb{Z}$ for each $n \in \mathbb{N}$. But this is saying every integer divides a. Therefore $a = 0$. As a result, $V(\mathrm{Ann}(M)) = \mathrm{Spec}(\mathbb{Z})$.

   On the other hand, I claim that $0 \notin \mathrm{Supp}(M)$. Let $(a_1, \ldots, a_k) \in \mathbb{Z}/n_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/n_k\mathbb{Z}$. Then we have inverted $n_1 \cdots n_k$. Therefore, $(1, (a_1, \ldots, a_k)) \sim 0 = (n_1 \cdots n_k, 0)$. This proves the assertion.

   As a result, $\mathrm{Supp}(M) \subsetneq V(\mathrm{Ann}(M))$.

3) Consider $M = \mathbb{Z} \oplus \mathbb{Z}/\langle 2\rangle$ as a $\mathbb{Z}$-module. Find the associated primes of M. Find 2 modules M_1, M_2, both isomorphic to $\mathbb{Z}$, such that $M_1 + M_2 = M$. What does this tell you about $\mathrm{Ass}(M)$ vs. $\mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2)$?

Solution: It is clear that $\operatorname{Ass}(M) = \{0, \langle 2\rangle\}$, by use of split exact sequences. We can consider M_1 generated by $(1,0)$ and M_2 generated by $(1,1)$. In either case, projection onto the first factor yields an isomorphism with $\mathbb{Z}$. Moreover,

$$(a,b) = (a-b)(1,0) + b(1,1)$$

so $M_1 + M_2 = M$. Of course, since $\operatorname{Ass}(M_i) = \operatorname{Ass}(\mathbb{Z}) = \{0\}$, we have that $\operatorname{Ass}(M) \neq \operatorname{Ass}(M_1) \cup \operatorname{Ass}(M_2)$.

4) Consider the ring $R = K[x,y,z]/\langle xz - y^2\rangle$ and the prime ideal $\mathfrak{p} = \langle x, y\rangle$. Let $M = R/\mathfrak{p}^2$. Compute $\operatorname{Ass}(M)$, and find all $m \in M$ for which $\operatorname{Ann}(m) = \mathfrak{p}$ for each $\mathfrak{p} \in \operatorname{Ass}(M)$. Finally, find an ascending chain $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Ass}(M)$.

Solution: One should note that

$$M = R/\mathfrak{p}^2 = K[x,y,z]/\langle xz - y^2, x^2, xy, y^2\rangle = K[x,y,z]/\langle x^2, xy, xz, y^2\rangle$$

Therefore, since the 0 divisors of M are anything in $\mathfrak{m} = \langle x,y,z\rangle$ (since $x\mathfrak{m} = 0$), we have that

$$\operatorname{Ann}(x) = \mathfrak{m} \qquad \operatorname{Ann}(y) = \langle x, y\rangle \qquad \operatorname{Ann}(z) = \langle x\rangle$$

$\operatorname{Ann}(z)$ is not a prime ideal of R, modding out by it yields $K[y,z]/\langle y^2\rangle$, which clearly is not a domain. The others are however. Therefore $\operatorname{Ass}(M) \supseteq \{\langle x,y\rangle, \mathfrak{m}\}$.

I claim this is all of the assassins. Indeed, we have primary decomposition

$$\mathfrak{p}^2 = \langle x, y^2\rangle \cap \langle z, xy, x^2\rangle$$

This shortest (since $\mathfrak{p}^2$ is non-primary) decomposition shows the claim.

Note that every element of M can be presented as $f(z) + c \cdot x + yg(z)$ for some $c \in K$. Therefore, we have $\operatorname{Ann}(m) = \langle x, y\rangle$ if and only if $f(z) = 0$. Additionally, $\operatorname{Ann}(m) = \mathfrak{m}$ if and only if $f(z) = 0$.

Considering $M_1 = R/\langle x,y,z\rangle \cong x \cdot K \subseteq M$, we see the only things that remain upon modding out are $f(z) + yg(z)$. Therefore, we can consider $M_2 = M_1 + yg(z)$ and $M_3 = M$. Noting $M_2/M_1 \cong K[z] = R/\mathfrak{p} \cong M_3/M_2$, we produce our ascending chain:

$$0 \subsetneq xK \subsetneq yK[z] + xK \subsetneq K[z] + yK[z] + xK = M$$

Note that each is itself an R-submodule.

5) If $N, N' \subseteq M$, show that

$$\operatorname{Ass}(M/N \cap N') \subseteq \operatorname{Ass}(M/N) \cup \operatorname{Ass}(M/N')$$

Solution: We have that $\varphi : R/\mathfrak{p} \hookrightarrow M/N \cap N'$ and $M/N \cap N' \to M/N$ and $M/N \cap N' \to M/N'$ are surjections. Composing these maps, if either is injective we are done. So suppose not, i.e. $q(\varphi(r)) = 0 \in M/N$ and $q'(\varphi(r')) = 0 \in M/N'$ for $r, r' \neq 0$. Thus we can consider the image of $r \cdot r'$ in $M/N \cap N'$. As a result, $q(\varphi(rr')) = r'q(\varphi(r)) = 0$ in M/N (i.e. it is in N). Similarly for N'. Thus $\varphi(rr') = 0 \in M/N \cap N'$. This is only possible if $rr' = 0 \in R/\mathfrak{p}$, which is an integral domain. This contradicts our choices of r, r', proving the claim.

6) If $\varphi : R \to S$ is a ring homomorphism, and $\mathfrak{q}$ is $\mathfrak{p}$-primary in S, is it true that $\varphi^{-1}(\mathfrak{q})$ is $\varphi^{-1}(\mathfrak{p})$-primary? In the reverse direction? I.e. is $\varphi(\mathfrak{q}) \cdot S$ primary?

Solution: First, checking $\varphi^{-1}(\mathfrak{q})$ is primary. Let $x \cdot y \in \varphi^{-1}(\mathfrak{q})$. Then $\varphi(x)\varphi(y) \in \mathfrak{q}$. As a result, either $\varphi(x)$ or $\varphi(y)^n = \varphi(y^n)$ are in $\mathfrak{q}$. But this implies either $x \in \varphi^{-1}(\varphi(x)) \subseteq \varphi^{-1}(\mathfrak{q})$ or $y^n \in \varphi^{-1}(\varphi(y^n)) \subseteq \varphi^{-1}(\mathfrak{q})$.

Now it goes to check $\sqrt{\varphi^{-1}(\mathfrak{q})} = \varphi^{-1}(\mathfrak{p})$. Since $\varphi^{-1}(\mathfrak{q}) \subseteq \varphi^{-1}(\mathfrak{p})$, it is clear that $\subseteq$ holds. Now let $x \in \varphi^{-1}(\mathfrak{p})$. This implies $\varphi(x) \in \varphi(\varphi^{-1}(\mathfrak{p})) \subseteq \mathfrak{p}$. Therefore, $\varphi(x)^n = \varphi(x^n) \in \mathfrak{q}$, which again shows $x^n \in \varphi^{-1}(\varphi(x^n)) \subseteq \varphi^{-1}(\mathfrak{q})$, i.e. $x \in \sqrt{\varphi^{-1}(\mathfrak{q})}$.