CLASS 4, FEBRUARY 11TH: EXISTENCE OF MAXIMAL IDEALS

Today we will study Zorn's Lemma, a result from logic, and show that every ring has a maximal ideal (and potentially many).

Lemma 4.1 (Zorn's Lemma)**.** *Let $\mathcal{S}$ be a non-empty partially ordered set, with the property that every ascending chain has an upper bound. Then there exists a maximal element.*

There is a little bit of information to unravel here. I state the definitions formally for your convenience.

Definition 4.2. A **partially ordered set**, or **poset**, is a collection $\mathcal{S}$ with an binary relation $\leq$ (not applicable to all elements, thus *partially*) on $\mathcal{S} \times \mathcal{S}$ satsfying the following properties for every $a, b, c \in \mathcal{S}$:

1) $a \leq a$
2) If $a \leq b$ and $b \leq a$, then $a = b$.
3) If $a \leq b$ and $b \leq c$, then $a \leq c$.

An **ascending chain** in a poset $\mathcal{S}$ are elements $a_i \ldots \in \mathcal{S}$ satisfying

$$a_1 \leq a_2 \leq a_3 \leq \ldots$$

An **upper bound** for such a chain is an element $b \in \mathcal{S}$ such that $a_i \leq b \; \forall i \in \mathbb{N}$. Finally, a **maximal element** of $\mathcal{S}$ is an element $M \in \mathcal{S}$ such that the only $N \in \mathcal{S}$ such that $M \leq N$ is $N = M$.

The result of Zorn's lemma is quite general and has applications throughout mathematics. However, instead of doing a deep dive into transfinite induction, we simply use this result to show the following result:

Theorem 4.3. *If $R \neq 0$ is a ring, then there exists $\mathfrak{m} \subsetneq R$ a maximal ideal.*

Setup 4.4. Let $\mathcal{S} = \{I \subsetneq R \text{ an ideal}\}$, and let $\leq$ denote inclusion of ideals; i.e. $I \leq J$ if and only if $I \subseteq J$. For simplicity I will refer in the following proof only to the latter operation $\subseteq$.

Proof. Note that for any ring $R \neq 0$, the set $\mathcal{S}$ is non-empty. Given $I_1 \subseteq I_2 \subseteq \ldots$ an ascending chain of ideals in $\mathcal{S}$, consider the set

$$I = \bigcup_{i=1}^{\infty} I_i$$

I claim that this is an ideal. Suppose $a, b \in I$. Then $a \in I_i$ and $b \in I_j$ for some $i, j \in \mathbb{N}$, and therefore $a, b \in I_{\max(i,j)}$. But this is an ideal, and therefore, $a + b \in I_{\max(i,j)} \subseteq I$.

Similarly, if $r \in R$ and $a \in I_i \subseteq I$, then $r \cdot a \in I_i \subseteq I$ since I_i is an ideal.

Finally, note that a subset $I \subseteq R$ satisfying these properties is *equal* to R if and only if it contains 1. Since I_i are ideals, none of them contain 1, and therefore I also doesn't contain 1. Therefore I is an upper bound for the above chain.

As a result of Lemma 4.1, we see that the set $\mathcal{S}$ contains a maximal element, which is exactly the definition of a maximal ideal. $\square$

Note that with a subtle manipulation to our setup, we can prove a greater result.

Theorem 4.5. *Given $I \subsetneq R$ an ideal, there exists $\mathfrak{m} \subsetneq R$ a maximal ideal containing I.*

Proof. The same proof goes through replacing $\mathcal{S} = \mathcal{S}_I = \{J \subsetneq R \text{ an ideal} \mid I \subset J\}$. ☐

This gives us a very nice way to break up the structure of a ring:

Corollary 4.6. *Given $R \neq 0$ a ring, we have that*

$$R = R^\times \cup \bigcup_{\mathfrak{m} \in m\text{-}\operatorname{Spec}(R)} \mathfrak{m} = R^\times \cup \bigcup_{\mathfrak{p} \in \operatorname{Spec}(R)} \mathfrak{p}$$

The $\cup$ are in fact disjoint unions.

Here I am denoting by $R^\times$ the set of (multiplicative) units in R.

Proof. Given $f \in R$, f is a unit if and only if $f \cdot g = 1$ for some $g \in R$, which occurs if and only if $\langle f \rangle = R$.

As a result, if $f \notin R^\times$, then $\langle f \rangle \neq R$ is an ideal. Thus there exists some maximal ideal $\mathfrak{m}$ containing $\langle f \rangle$. So $f \in \bigcup_{\mathfrak{m} \in \mathrm{m}--\operatorname{Spec}(R)} \mathfrak{m}$.

The second equality is trivial, since each $\mathfrak{p} \subseteq \mathfrak{m}$ for some $\mathfrak{m} \in \mathrm{m}\text{-} - \operatorname{Spec}(R)$. ☐

As one final application of Zorn's Lemma, we have a powerful result that will tell us the structure of certain *localized* rings later on.

Theorem 4.7. *Let $R \neq 0$ be a ring, and S a multiplicative subset (c.f. Homework 1 #3). Then if I is an ideal of R disjoint from S (meaning $I \cap S = \emptyset$), then there exists a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p}$ is disjoint from S and $I \subseteq \mathfrak{p}$.*

Proof. First, consider the set $\mathcal{S} = \{I \subsetneq R \text{ an ideal} \mid S \cap I = \emptyset\}$. It is clear that this set is non-empty, since it contains I. Similar to the previous proof, given $I_1 \subseteq I_2 \subseteq \ldots$ an ascending chain in $\mathcal{S}$, we have that $I = \cup_{i=1}^\infty I_i$ is again an ideal in $\mathcal{S}$. Therefore, there exists a maximal element $\mathfrak{p} \in \mathcal{S}$.

I assert that $\mathfrak{p}$ is a prime ideal. Suppose $f, g \notin \mathfrak{p}$ but $f \cdot g \in \mathfrak{p}$. Then we can consider

$$\mathfrak{p} + \langle f \rangle = \{a + r \cdot f \mid r \in R, a \in \mathfrak{p}\}$$

and similarly $\mathfrak{p} + \langle g \rangle$. These are again ideals, and they contain $\mathfrak{p}$. Therefore, by assumption of maximality, we have that $\exists a + rf \in S \cap \mathfrak{p} + \langle f \rangle$ and $\exists b + r'g \in S \cap \mathfrak{p} + \langle g \rangle$. But S is a multiplicative set, so

$$(a + rf)(b + r'g) = ab + ar'g + brf + rr'fg \in S$$

Note that $ab, ar'g, brf \in \mathfrak{p}$ since $a, b \in \mathfrak{p}$. Similarly, $rr'fg \in \mathfrak{p}$ since $fg \in \mathfrak{p}$ by assumption. So $(a + rf)(b + r'g) \in \mathfrak{p} \cap S$, contradicting our assumptions. This proves the result. ☐

Example 4.8. $R = K[x, y]$ and $S = \{1, x, x^2, x^3, x^4, \ldots\}$. Then an example of an ideal disjoint from S not contained in a larger ideal is $\langle y - \alpha \rangle$ and $\langle x - \beta \rangle$ for $\beta \neq 0$, which are indeed prime. Other examples are also possible, and in fact one can detect that

$$\{\mathfrak{p} \in \operatorname{Spec}(R) \mid S \cap \mathfrak{p} = \emptyset\} = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid x \notin \mathfrak{p}\}$$