CLASS 31, NOVEMBER 25TH: EXAMPLES IN CONFORMAL MAPPINGS

Today we will study further known examples of conformal mappings and discuss the Dirichlet problem on the strip.

Example 31.1. First note that affine transformations of the form $z \mapsto az + b$ with $a \neq 0$ are always conformal mappings of $\mathbb{C}$ to $\mathbb{C}$. Restricting to smaller sets gives that every disc is conformal to every other disc.

Specific cases are dilations ($a > 0$ real, $b = 0$), translations ($a = 1$), and rotations $a = e^{i\theta}$. The general affine transform is a composition of these things: if $a = re^{i\theta}$, then it is given by dilate by r, rotate by θ, then translate by b.

Example 31.2. If $n > 0$, then $z \mapsto z^n$ is a conformal mapping between the sector

$$S = \left\{ z \in \mathbb{C} \mid 0 < Arg(z) < \frac{\pi}{n} \right\}$$

and $\mathbb{H}$. It's inverse it the n^{th} root function.

This generalizes naturally to $z \mapsto z^\alpha$ for any $\alpha \in (0, 2)$, which takes $\mathbb{H}$ to $S = \{z \in \mathbb{C} \mid 0 < Arg(z) < \pi\alpha\}$.

Example 31.3. The map $z \mapsto \frac{1+z}{1-z}$ is a conformal map from the upper half disc to the first quadrant: $\mathbb{H} \cap \mathbb{D} \to Q$. Indeed, notice that

$$\frac{1+z}{1-z} = \frac{(1+z)(1-\bar{z})}{(1+z)(1-\bar{z})} = c(1 + z - \bar{z} - z\bar{z}) = c(1 - |z|^2) + 2cIm(z)$$

Note that $c > 0$ and thus both the real and imaginary parts are positive. It's inverse is given by $w \mapsto \frac{w-1}{w+1}$.

Example 31.4. One can think of the exponential as a conformal map. Indeed, if we consider

$$\exp : \mathbb{R} \times (-\pi, \pi) \to \mathbb{C} \setminus \mathbb{R}_{\leq 0}$$

Then this is a holomorphic bijection. Its inverse is the principle branch of the log.

Further restrictions on the domain are possible: if we restrict to the negative strip, we get

$$\exp : \mathbb{R}_{<0} \times (0, \pi) \to \mathbb{D} \cap \mathbb{H}$$

is a conformal equivalence. Similarly,

$$\exp : \mathbb{R}_{>0} \times (0, \pi) \to \mathbb{H} \setminus \bar{\mathbb{D}}$$

Example 31.5. The sine function is also conformal:

$$\sin : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}_{>0} \to \mathbb{H}$$

This shows many things that we know (and love) are conformally equivalent. We'll end the course with a proof of a theorem which says this in a rigorous way. We'll now transfer to a questions about the class of self conformal mappings of our most popular players: the disc and the upper half plane.

Lemma 31.6 (Schwarz Lemma)**.** *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map with $f(0) = 0$. Then*
- $|f(z)| \leq |z|$ *for all* $z \in \mathbb{D}$.
- *If* $z_0 \neq 0$ *and* $|f(z_0)| = |z_0|$, *then* f *is a rotation.*
- $|f'(0)| \leq 1$, *and equality holds iff* f *is a rotation.*

An instant corollary of the second fact is that if $f(z_0) = z_0$ for some z_0, then f is the identity map!

Proof. Let $f(z) = a_0 + a_1 z + \dots$ be a power series expansion. We know $a_0 = 0$ by our condition.
- If $|z| = r < 1$, then since $|f(z)| \leq 1$, we always have
$$\left|\frac{f(z)}{z}\right| \leq \frac{1}{r}$$
By the maximum modulus principle, this is true for all $z \in \mathbb{D}_r$. Letting $r \to 1$ proves the result.
- Again by the MMP, we have that $\frac{f(z)}{z}$ attains its maximum inside $\mathbb{D}$. This is only possible if it is constant. This implies $f(z) = cz$ for some $|c| = 1$.
- Again, by the first bullet, $g(z) = \frac{f(z)}{z}$ is bounded above by 1. Sending $z \to 0$ preserves this. If $|f'(0)| = 1$, we are in the situation of the previous bullet, and again $g(z)$ is constant.

□

Now we turn to the idea of an automorphism:

Definition 31.7. A conformal mapping $\Omega \to \Omega$ is said to be an **automorphism**. The set of automorphisms is denoted $\mathrm{Aut}(\Omega)$, and forms a group under composition.

Examples of automorphisms include the identity map, rotations of a disc, dilations of the upper half plane, etc. A good question is what are *all* of them? An interesting non-trivial class of automorphisms is given as follows:

Example 31.8. For each $\alpha \in \mathbb{D}$, define
$$\psi_\alpha : \mathbb{D} \to \mathbb{D} : z \mapsto \frac{\alpha - z}{1 - \bar{\alpha} z}$$
Each ψ_α is an automorphism.

First, note that if $z = 1$, then
$$\psi_\alpha(z) = \frac{\alpha - e^{i\theta}}{e^{i\theta}(e^{-i\theta} - \bar{\alpha})} = e^{i\theta}\frac{w}{\bar{w}}$$
which has absolute value 1. As a result, the range is well define. Next, I claim it is actually its own inverse!
$$\psi_\alpha^2(z) = \frac{\alpha - \frac{\alpha - z}{1 - \bar{\alpha} z}}{1 - \bar{\alpha}\frac{\alpha - z}{1 - \bar{\alpha} z}} = \frac{\alpha(1 - \bar{\alpha} z) - \alpha + z}{1 - \bar{\alpha} z - \bar{\alpha}(\alpha - z)} = \frac{(1 - |\alpha|^2) z}{(1 - |\alpha|^2)} = z$$

Next time we will check that this is nearly all of them!