CLASS 20, OCTOBER 30: HOMOTOPY

Today we will study the idea of a homotopy, a feature that permeates geometry, topology, and even abstract algebra at the highest level. We already saw an example of the idea of homotopy in our proof of Riemann's theorem on removable singularities (Lemma 16.3), when we study how an integral *deforms*.

Definition 20.1. Let $\gamma_0, \gamma_1 : [a, b] \to \Omega$ be two curves in Ω an open set that share common endpoints α, β (representing the start and finish). Then $\gamma_0 \simeq \gamma_1$ are said to be **homotopic** in Ω if there exists a continuous function $F : [a, b] \times [0, 1] \to \Omega$ such that

$$F(t, 0) = \gamma_1(t) \qquad\qquad F(t, 1) = \gamma_1(t)$$

$$F(a, s) = \alpha \qquad\qquad F(b, s) = \beta$$

F is called a **homotopy** connecting γ_0 to γ_1.

It is convenient to think of this geometrically: for a fixed value of s, $\gamma_s(t) = F(t, s)$ is a curve with endpoints α, β. Therefore, as s varies from 0 to 1, we deform γ_0 to γ_1. A nice feature of homotopic curves is that their integrals of holomorphic functions agree!

Theorem 20.2. *If f is holomorphic in Ω, and $\gamma_1 \simeq \gamma_2$, then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

Proof. Our proof will rely on the fact that the statement is true locally, i.e. if the curves are sufficiently close to one another then the theorem holds.

Let K be the image of F. Since the image of a compact set is compact, K is compact. Therefore, there exists $\epsilon > 0$ such that $B(z, 3\epsilon) \subseteq \Omega$ for each $z \in K$. Now, since F is continuous on a compact set, it is uniformly continuous. Thus we can select δ with

$$\sup_{t\in[a,b]} |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \epsilon \qquad\qquad \text{if } |s_1 - s_2| < \delta$$

Fixing such a choice of s_1, s_2, we can choose discs $D_0, \ldots, D_n$ of radius 2ϵ and consecutive points z_i, w_i on $\gamma_{s_1}, \gamma_{s_2}$ respectively for $i = 0, \ldots, n + 1$ with

$$z_i, z_{i+1}, w_i, w_{i+1} \in D_i$$

Further assume $z_0 = w_0 = \alpha$ and $z_{n+1}, w_{n+1} = \beta$ so our points go from start to finish. By Cauchy's theorem, we have that on each disc D_i we have a primative for f, say F_i. On adjacent overlaps, we get that $F_{i+1} - F_i = C_i$ is a constant (since both are primatives of the same function). This implies centrally that

$$F_{i+1}(z_{i+1}) - F_i(z_{i+1}) = F_{i+1}(w_{i+1}) - F_i(w_{i+1})$$

$$F_{i+1}(z_{i+1}) - F_{i+1}(w_{i+1}) = F_i(z_{i+1}) - F_i(w_{i+1})$$

As a result, we have that

$$\begin{aligned}\int_{\gamma_{s_1}} f - \int_{\gamma_{s_2}} f &= \sum_{i=0}^{n} [F_{i+1}(z_{i+1}) - F_i(z_i)] - \sum_{i=0}^{n} [F_{i+1}(w_{i+1}) - F_i(w_i))] \\ &= \sum_{i=0}^{n} [F_i(z_{i+1}) - F_i(w_{i+1}) - (F_i(z_i) - F_i(w_i)))]\end{aligned}$$

Canceling nearby signed terms leaves us with

$$\int_{\gamma_{s_1}} f - \int_{\gamma_{s_2}} f = F_{n+1}(z_{n+1}) - F_{n+1}(w_{n+1}) - (F_0(z_0) - F_i(w_0)) = 0$$

Since $z_{n+1} = w_{n+1}$ and $z_0 = w_0$. This shows the equality. Subdividing the whole interval $[0,1]$ into $\frac{\delta}{2}$-sized pieces, we can prove that

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

□

An important type of domain is one in which all curves starting and ending at the same point are homotopic. This notion is called a **simply connected** domain, and should be thought of as the next order of (path) connectedness. In a path connected space, all points can be connected by a path. In a simply connected domain, all paths (with the same starting and ending points) can be connected by a path of paths.

Example 20.3. A disc $\bar{B}(z_0, r)$ is an example of a simply connected domain. Indeed, we can transform one curve to another by the straight line homotopy:

$$F : [a,b] \times [0,1] \to \bar{B}(z_0, r) : (s,t) \mapsto t\gamma_0(s) + (1-t)\gamma_1(s)$$

This in fact works for any convex domain!

Example 20.4. A non-convex domain that is still simply connected is $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Remember that this is where we were for example able to define $\log(z) = \log(re^{i\theta}) = \log(r) + i\theta$.

This can be seen by writing the curves in polar coordinates: $\gamma_i(s) = r_i(s)e^{i\theta_i(s)}$. Now again, we can apply the straight line contours to r_i and θ_i again:

$$F : [a,b] \times [0,1] \to \bar{B}(z_0, r) : (s,t) \mapsto (tr_0(s) + (1-t)r_1(s))\, e^{i(t\theta_0(s) + (1-t)\theta_1(s))}$$

Noticing that since $r_i > 0$ and $\theta_i \in (-\pi, \pi)$, so are they for each $t \in [0,1]$.

Example 20.5. For an example of a non-simply connected domain, consider $\bar{B}(0,1) \setminus \{0\}$. If we consider, for example, the constant path at 1, and the path $\gamma(s) = e^{2\pi i s}$ for $s \in (0,1)$. Intuitively, there is certainly no way to deform one loop to the other, as looping is always undone. Formally, this is provable by virtue of the fact that

$$\int_1 \frac{1}{z} dz = 0 \neq \int_C \frac{1}{z} dz = 2\pi i$$

Corollary 20.6. *Any holomorphic function $f : \Omega \to \mathbb{C}$ in a simply connected domain Ω has a primitive given by*

$$F(w) = \int_{\gamma_w} f(z) dz$$

where γ_w is (any) path from some fixed point z_0 to w.