CLASS 7, SEPTEMBER 23: GOURSAT'S THEOREM

Chapter 2 of the book is filled with many theorems which are at the heart of complex analysis. We will begin with one of the assertions made previously; Goursat's Theorem. This provides a similar result to the case of primatives from last class. It should be noted that this result derives many of the other important upcoming theorems.

Theorem 7.1 (Goursat's' Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $T \subseteq \Omega$ is a triangle, then*

$$\int_T f(z)dz = 0$$

The theorem seems very restrictive, i.e. why study triangles? They are able to closely represent other shapes as well through use of a triangulation of a given region. We will formalize this idea later.

Proof. The proof is divided into some steps due to its length.

1) **Barycentric Subdivision:** Write $T = T^{(0)}$. Here we represent a natural way to subdivide a triangle into 4 triangular pieces. Given a triangle with sides ABC, choose the midpoints a, b, c of each side. Connecting them with lines naturally subdivides ABC into 4 pieces; $T_1^{(1)}, \dots, T_4^{(1)}$.

   The important aspect with regards to integrals is the orientation of the triangle. We already know that if we reverse the orientation, we negate the corresponding integral.

   To do this, we maintain the natural positive/counterclockwise orientation of the volume. That is to say, we give each of the new triangles the same counterclockwise orientation. This allows us to cancel equal sides and produce the following subdivision of the integral:

$$\int_{T^{(0)}} f(z)dz = \int_{T_1^{(1)}} f(z)dz + \int_{T_2^{(1)}} f(z)dz + \int_{T_3^{(1)}} f(z)dz + \int_{T_4^{(1)}} f(z)dz$$

   As a result, we can obtain that for some j (maximal say), we have

$$\int_{T^{(0)}} f(z)dz \leq 4 \left| \int_{T_j^{(1)}} f(z)dz \right|$$

2) **Proceed by induction:** We can continue the barycentric subdivision of the chosen triangle $T_j^{(1)}$ satisfying the previous inequality. This would yield the inequality

$$\int_{T^{(0)}} f(z)dz \leq 4 \left| \int_{T^{(1)}} f(z)dz \right| \leq 16 \left| \int_{T_j^{(2)}} f(z)dz \right|$$

   for some j. Inductively, we can produce the smaller and smaller triangles $T^{(n)}$ satisfying

$$\int_{T^{(0)}} f(z)dz \leq 4 \left| \int_{T^{(1)}} f(z)dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z)dz \right|$$

3) **Notes about the triangles:** If we let d_n and p_n denote the diameter and perimeter respectively of $T^{(n)}$, then
$$d_n = 2^{-n} d_0 \qquad\qquad p_n = 2^{-n} p_0$$
If we denote by $\mathcal{T}^{(n)}$ the solid closed triangle enclosed by $T^{(n)}$, then we have constructed
$$\mathcal{T}^{(0)} \supseteq \mathcal{T}^{(1)} \supseteq \mathcal{T}^{(2)} \supseteq \dots$$
a nested sequence of triangles whose diameter goes to 0. Since each triangle was compact, this implies that there is a unique point z_0 belonging to $\bigcap_{n=0}^{\infty} \mathcal{T}^{(n)}$.

4) **Holomorphicity:** Given f is holomorphic at z_0, we can write
$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$
where $\psi(z) \to 0$ as $z \to z_0$. Now notice that $F(z) = f(z_0)z + \frac{f'(z_0)}{2}(z - z_0)^2$ is a primative for $f(z_0) + f'(z_0)(z - z_0)$, so their integrals are 0. As a result,
$$\int_{T^{(n)}} f(z)dz = \int_{T^{(n)}} \psi(z)(z - z_0)dz$$
Now, since $z_0 \in \mathcal{T}^{(n)}$ and $z \in T^{(n)}$, we have $|z - z_0| < d_n = 2^{-n} d_0$. As a result, we can estimate
$$\left| \int_{T^{(n)}} \psi(z)(z - z_0)dz \right| \leq \int_{T^{(n)}} |\psi(z)| \cdot |(z - z_0)|dz \leq \sup_{z \in T^{(n)}} \{\psi(z)\} \cdot p_n \cdot d_n$$
If we let $\epsilon_n = \sup_{z \in T^{(n)}}\{\psi(z)\}$, we know $\lim_{n \to \infty}(\epsilon_n) = 0$.

5) **Combining previous estimates:** We have
$$\left| \int_{T^{(n)}} f(z)dz \right| \leq \epsilon_n 4^{-n} d_0 p_0$$
Furthermore,
$$\left| \int_{T^{(0)}} f(z)dz \right| \leq 4^n \left| \int_{T^{(0)}} f(z)dz \right| \leq \epsilon_n d_0 p_0$$
Since the right-hand side goes to 0 as $n \to \infty$, we conclude that the independent of n left hand side must also be 0. This proves the theorem.

□

Corollary 7.2. *Let $\Omega \subseteq \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $P \subseteq \Omega$ is a n-gon, then*
$$\int_P f(z)dz = 0$$

Proof. There is a very natural way to triangulate an n-gon. Each triangle satisfies the previous theorem and is therefore 0 in integral. □