$$\begin{tikzcd} & A \arrow[dl, hook] \arrow[dr, hook] & \\ X & & Y \end{tikzcd}$$

$\boxed{\text{Oct 2}}$ Prop .19: If (X,A) and (Y,A) have the HEP, and $f: X \to Y$ is a homotopy equivalence, with $f|_A = \mathbb{1}_A$. Then f is a homotopy equiv rel A.

Note: What does this say? ~~I~~ ~~say~~

$\exists g: \cancel{X}\, Y \to X$ w/ $g \circ f: X \to X \simeq Id_X$ rel A

and $f \circ g \simeq Id_Y$ rel A.

Cor .20 If (X,A) has HEP, and $A \hookrightarrow X$ is a homotopy equiv, then A is a def retract of X.

Pf: $A \hookrightarrow \cancel{X}$ has the properties, and being a homotopy equiv rel A keeps A fixed (required for a def ret.

Cor. 21 $f: X \to Y$ is a homotopy eq $\iff$ M_f deform ret to X. Thus $X \simeq Y$ $\iff$ $\exists Z \simeq X, Y$ def ret to both.

Pf:
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, hook, "i"'] & Y \arrow[d, shift right, "j"'] \\ & M_f \arrow[u, shift right, "r"'] \end{tikzcd}$$

$f = r \circ i$ $\quad$ $i \simeq j \circ f$

Since ~~j&r~~ j & r are homotopy eq, f is $\iff$ i is a ~~Hom eq~~ eq.

So $X \hookrightarrow M_f$ is a hom. eq. ~~By~~ $(\cancel{X}, \cancel{M_f})$ $(M_f, \cancel{X})$ is a HEP pair (Ex .15) in book. So

f is a hom. eq. $\iff$ M_f def rets to X

Ex. 15: (X,A) has HEP if

- $\exists A \subseteq N$ w/ N closed, $B \subseteq N$ s.t. $A \subseteq N \setminus B$ is open
- $\exists f: B \to A$
- $\exists h: M_f \xrightarrow{\text{homeo}} N$ w/ $h|_{A\cup B} = \mathrm{Id}$

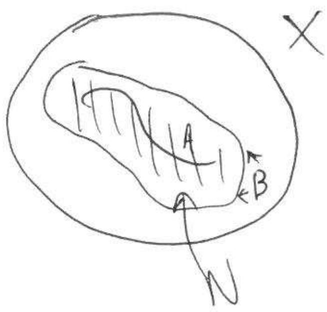
Note: $\exists\ I\times I \xrightarrow{\text{ret}} I\times\{0\} \cup \partial I \times I$

$\Longrightarrow B\times I\times I \xrightarrow{\text{ret}} B\times I\times\{0\} \cup B \cup \partial I \cup I$

$\Longrightarrow M_f \times I \xrightarrow{\text{ret}} M_f \times \{0\} \cup (A\cup B)\times I$

$\Rightarrow (M_f, A\cup B)$ has HEP $\Rightarrow (N, A\cup B)$ has HEP

~~Take~~ $\cancel{F: X\times I \to Y}$ $f_0: X \to Y$ and $F: A\times I \to Y$
Take Const Homotopy on $X \setminus (N\setminus B)$, and extend to N via HEP on $(N, A\cup B)$.



Pf of .19: Let $g: Y \to X$ be the h.e. pair for f.

1) $g \simeq g_1$, w/ $g|_A = \mathbb{1}_A$

Let $h_t: X \to X$ be s.t $f \circ g \simeq_{h_t} \mathbb{1}_X$.

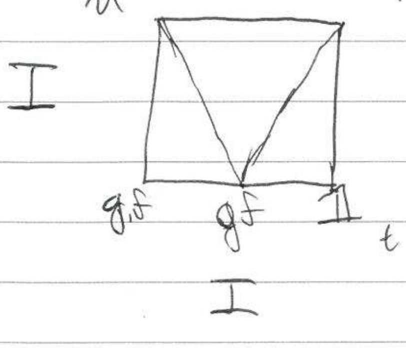
Since $f|_A = \mathbb{1}_A$, $h_t|_A \simeq f|_A \circ g|_A \simeq g|A \simeq \mathbb{1}_A$

Since (Y, A) have HEP, we can extend $h_t|_A$ to $g_t: Y \to X$. This yields
$g \simeq_{g_t} g_1$

2) $g_1 \circ f \simeq \mathbb{1}_X$ rel A. The idea here is to send g_1 back to g, then use the original homotopy $g \circ f \simeq_{h_t} \mathbb{1}$

$$K_t(x) = \begin{cases} g_{1-2t} \circ f & t \leq \frac{1}{2} \\ h_{2t-1} & t \geq \frac{1}{2} \end{cases}$$
$: A \to X$

This is a continuous map $(t = \frac{1}{2})$. The issue here is that K_t does not fix A. Thus we add another parameter $u \in I$:



Below V, define
$K_{t,u} = K_t$
Above V,
$K_{t,u} = K_u$
$K_t = K_{1-t}$ on A

(X, A) HEP $\Rightarrow (X \times I, A \times I)$ HEP

$\Rightarrow K_{tu}$ extends to $K_{tu}: X \to X$ w/ $K_{t0} = K_t$

$\Rightarrow$ Left $\Rightarrow$ ~~Right~~ Top $\Rightarrow$ Right yields
$g_1 f \simeq \mathbb{1}$ rel A.

3) $fg_1 \simeq \mathbb{1}$ rel A :

$$g_1 \simeq g \Rightarrow fg_1 \simeq fg \simeq \mathbb{1}_Y$$

Same pf as 2 $\Rightarrow$ can be made rel A

$\boxed{\text{Oct } 4}$ Refer to notes 1,

Bottom of pg 7 — Top of pg 10

$\boxed{\text{Oct } 6/11}$ or Mountain Day : Paths, Homotopy

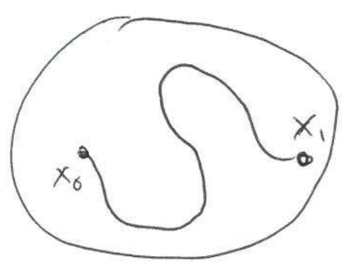
or 11/13

We will study a space X by study all of the loops / paths in X mod homotopy

<u>Defn</u> a path in X from x_0 to x_1 is a continuous map

$$p: [0,1] = I \to X$$

s.t. $p(0) = x_0$, $p(1) = x_1$.



Ex / Let S^1 be parameterized by $\theta \in [0,1]$, w/ $0=1$. Equivalently, $S^1 = \mathbb{R}/\mathbb{Z}$.

Consider $p^\alpha: [0,1] \to S^1$
$p^\alpha: x \longmapsto \alpha \cdot x$ for $\alpha \in \mathbb{R}$.

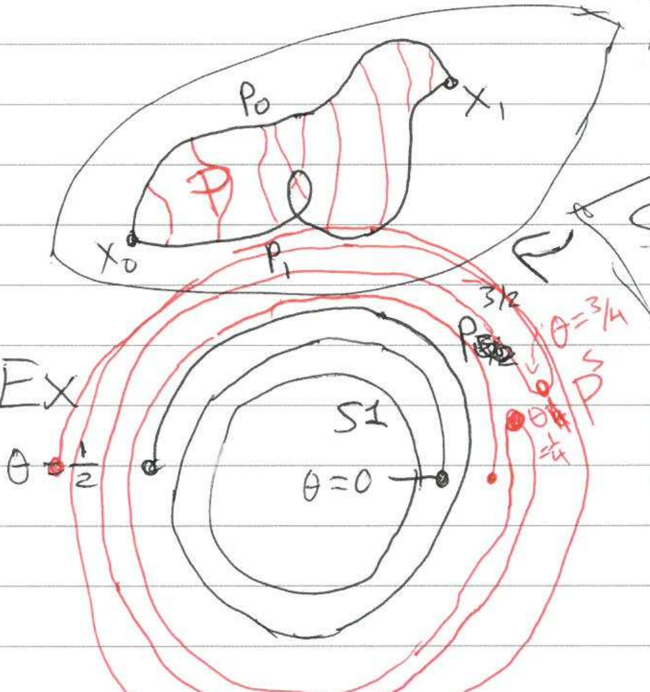
p^{α} is a path from $x_0=0$ to $x_1=\{\alpha\}$.
It goes around S^1 $\lfloor\alpha\rfloor$-many times (if $\alpha>0$)

Defn: A homotopy of paths is a homotopy P of p_0 to p_1 for two paths, w/

$$p_0 \simeq_P p_1 \ \text{rel}\ \{0,1\}$$

Unwinding, $P: I\times I \to X$ w/
(s,t)

$$P(s,0)=p_0(s),\quad P(s,1)=p_1(s)$$
$$P(0,t)=x_0,\quad P(1,s)=x_1$$



Sorry pictures mixed

Ex

$$\boxed{\begin{aligned} p^s: [0,1] &\to S^1=\mathbb{R}/\mathbb{Z} \\ x &\longmapsto \tfrac{3}{2}x+\sin(2\pi x)\end{aligned}}$$

$p(0)=0$, $p(1)=\frac{3}{2}=\frac{1}{2}$

$p(\frac{1}{2})=\frac{1}{2}$ $\quad p(\frac{1}{4})=1+\frac{3}{8}=\frac{11}{8}$ $\quad p(\frac{3}{4})=\frac{9}{8}-1=\frac{1}{8}$

$p^{3/2}\simeq p^s$ by

$$P(x,s,t)=\frac{3}{2}s+t\sin(2\pi x)$$

Prop 1.2 $p_1 \simeq p_2 \simeq p_3 \Rightarrow p_1 \simeq p_3$. Thus homotopy of paths is an eq. rel.

Pf: Standard $2t$ and $2t-1$ trick.

Composition: $f \cdot g\,(t) = \begin{cases} f(2t) & t \leq \frac{1}{2} \\ g(2t-1) & t \geq \frac{1}{2} \end{cases}$

for f a path from x_0 to x_1 and g a path from x_1 to x_2

Consider the set of paths from x_0 to x_0, ~~$L(x_0)$~~ $L(X,x_0)$. These are called loops at x_0 in X.

~~Prop: $L(X,x_0)$ is a group under composition~~

Defn The Fundamental Group of X @ x_0 is $\boxed{\pi_1(X,x_0) = L(X,x_0)/\sim}$

where $p_0 \simeq p_1 \Leftrightarrow p_0 \sim p_1$

Prop: $\pi_1(X, x_0)$ is a group under Composition.

Pf: • Identity: $p = e_{x_0} = t \mapsto x_0$

• Composition of paths is a path:

$p_1 \cdot p_2$ is a continuous map $I \to X$

• Inverses exist: Let p be a path.
Let (suggestively)

$$p^{-1} = \bar{p} : [0,1] \to X : t \mapsto p(1-t)$$

Note $\bar{p} \in \pi_1(X, x_0)$, as $t \mapsto 1-t$ is continuous and $\bar{p}(0) = \bar{p}(1) = x_0$.

Consider $$\bar{p} \cdot p(t) = \begin{cases} \bar{p}(2t) & t \le \frac{1}{2} \\ p(2t-1) & t \ge \frac{1}{2} \end{cases}$$

Consider

$$\cancel{P : I \times I \to X :}$$

$$\cancel{P(t,s) = \begin{cases} \bar{p}(2t) & t \le \frac{1}{2}(1-s) \\ p(1-s) & \frac{1}{2}(1-s) \le t \le 1 - \frac{1}{2}(1-s) \end{cases}}$$

$$P : I \times I \to X$$

$$P(t,s) = \begin{cases} \bar{p}(2t) & t \le \frac{1}{2}(1-s) \\ \bar{p}(1-s) = p(s) & \frac{1}{2}(1-s) \le t \le 1 - \frac{1}{2}(1-s) \\ p(2t-1) & t \ge 1 - \frac{1}{2}(1-s) \end{cases}$$