CLASS 29, NOVEMBER 20TH: WEIRSTRASS INFINITE PRODUCTS

Today we will return to the question posed in Class 27; given a non-accumulating sequence a_n, can we find an entire function vanishing precisely at these points? We pointed out that the naive guess is

$$f(z) = (z - a_1) \cdot (z - a_2) \cdots$$

but we would need to deal with convergence issues. This issue was tackled by Weirstrass:

Theorem 29.1 (Weirstrass Infinite Products)**.** *Suppose a_n is a sequence with $|a_n| \to \infty$. There exists f entire such that $f(a_n) = 0$ and $f(z) \neq 0$ for $z \neq a_n$. Any other function with these properties has the form $f(z)e^{g(z)}$ for some entire function g.*

Note that since a_n and a_m can agree for various n, m, we can achieve zeroes of any order as well!

Proof. We begin with the last statement. Suppose f_1 and f_2 have the properties in Theorem 29.1. Consider $h(z) = \frac{f_1(z)}{f_2(z)}$. By our previous results, h has removable singularities at a_n and is no other zeroes. So by our analysis of the logarithm, since h is entire and non-vanishing, we have that $h(z) = e^{g(z)}$ for some entire function $g(z)$ (Theorem 21.5). Of course, this implies the desired result exactly.

So it goes to show the existence. For each $k \geq 0$, consider the **canonical factors**

$$E_0(z) = 1 - z \qquad\qquad E_k(z) = (1 - z)e^{z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}} \quad \forall k > 0$$

k is the **degree** of the canonical factor.

Lemma 29.2. *If $|z| \leq \frac{1}{2}$, then $|1 - E_k(z)| \leq c|z|^{k+1}$ for some constant c.*

Proof. Note that the logarithm $\log(1 - z)$ has a power series expansion

$$\log(1 - z) = -\sum_{k \geq 1} \frac{z^k}{k} = -(z + \frac{z^2}{2} + \cdots + \frac{z^k}{k} + \cdots)$$

As a result,

$$E_k(z) = e^{\log(1-z) + z + \frac{z^2}{2} + \cdots + \frac{z^k}{k}} =: e^{-\sum_{l \geq k+1} \frac{z^l}{l}}$$

Now, we note that

$$\left| \sum_{l \geq k+1} \frac{z^l}{l} \right| \leq |z|^{k+1} \sum_{l \geq k+1} \left| \frac{z}{l} \right| \leq |z|^{k+1} \sum_{l} 2^{-l} \leq 2|z|^{k+1}$$

So term being exponenitated is bounded above by $2|z|^{k+1} < 1$. Finally, noting that $e^x - 1 < e \cdot x$ when $0 < x < 1$, we can conclude

$$|1 - E_k(z)| \leq e \left| \sum_{l \geq k+1} \frac{z}{l} \right| \leq 2e|z|^{k+1}$$

So $c = 2e$ will do. ∎

Now returning to the proof, suppose there are m 0s among the a_n. Reordering so that the zeroes are removed and the a_n are all non-zero, we claim

$$f(z) = z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right)$$

is the desired function. Note this avoids the convergence issues of the naive approach. We call this the **Weirstrass product**.

Let $R > 0$ and consider $\mathbb{D}_R$. We can consider the factors separately for cases $|a_n| \leq 2R$ and $|a_n| > 2R$. The finite products vanish for a_n of the first kind. If a_n is of the second kind, then $|\frac{z}{a_n}| \leq \frac{1}{2}$. So by Lemma 29.2, we have that

$$\left|1 - E_n\left(\frac{z}{a_n}\right)\right| \leq c\left|\frac{z}{a_n}\right|^{n+1} \leq c2^{-n-1}$$

Writing our product as

$$\prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right) = \prod_{n=1}^{\infty} 1 - \left(1 - E_n\left(\frac{z}{a_n}\right)\right)$$

Then Proposition 28.2 allows us to ensure the convergence of f on $\mathbb{D}_R$, and vanishes precisely at $|a_n| \leq 2R$. But R is arbitrary, so this holds in $\mathbb{C}$. □

We can bootstrap this result to meromorphic functions as well:

Corollary 29.3. *If $a_n \to \infty$ and $b_n \to \infty$, then there exists f a meromorphic function with zeroes at a_n and poles at b_n (precisely).*

Proof. Create g for a_n and h for b_n by Theorem 29.1. Then divide! □

We will now state a result of Hadamard which improved upon Weirstrass's work using all of the techniques of chapter 5. The statement is as follows:

Theorem 29.4 (Hadamard)**.** *Suppose f is entire and has growth order ρ_0. Set $k = \lfloor \rho_0 \rfloor$. If $0 \neq a_1, a_2, \ldots$ are the zeroes of f, then*

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right)$$

where P is a polynomial of degree $\leq k$, and some m.

Hadamard proved this by showing that the degree of the canonical factors can be taken to be constant. The proof is illustrated in chapter 5, section 5 of the book (pgs 147-153) if you are interested. But since only 5 classes remain, we will move instead to conformal mappings.