CLASS 4, SEPTEMBER 14: THE PRODUCT TOPOLOGY

When one has 2 different topological spaces X and Y, it may be typical (and productive) to ask if there is a natural topological space encapsulating the information of both simultaneously. This is where the product topology enters.

Definition 4.1. If (X, τ) and (Y, σ) are 2 topological spaces, we define the **product topology** on the cartesian product

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$$

to be defined by the basis $\mathcal{B}$ consisting of $U \times V$, where $U \in \tau$ and $V \in \sigma$. For simplicity, we denote this by $\tau \times \sigma$.

This should really be a definition-proposition because it isn't immediately clear that this is a basis. Axiom 1) of a basis is immediate, since $(x, y) \in X \times Y$ (which is in $\mathcal{B}$. Furthermore, if $x \in U_1 \times V_1$ and $(u, v) \in U_2 \times V_2$ are 2 elements of $\mathcal{B}$, we know that $(u, v) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$ is in $\mathcal{B}$ and contained within the intersection. This verifies axiom 2).

Example 4.2. We can give $\mathbb{R}^n$ the product topology of n-copies of $\mathbb{R}$ inductively. In homework 1 question 3, you have demonstrated that in fact this topology is equivalent to the standard metric topology.

Note. *An important consideration, and common mistake, is that not all elements of the product topology are 'squares'; that is of the form $U \times V$. In particular, if we consider $(1, 2) \times (1, 2) \cup (3, 4) \times (3, 4)$ in $\mathbb{R}^2$, we get a set not expressible in this form.*

Though this is a fairly common misconception, the idea does work on bases:

Proposition 4.3. *If $\mathcal{B}_\tau$ is a basis for τ on X, and $\mathcal{B}_\sigma$ is a basis for σ on Y, then*

$$\mathcal{B} = \mathcal{B}_\tau \times \mathcal{B}_\sigma = \{U \times V \mid U \in \mathcal{B}_\tau,\ V \in \mathcal{B}_\sigma\}$$

is a basis for the product topology $\tau \times \sigma$.

So in particular we can make the basis much smaller than initially demonstrated.

Proof. By definition of the product topology, we can express any open set $A \in \tau \times \sigma$ by

$$A = \bigcup_\alpha U_\alpha \times V_\alpha$$

where each of U_α and V_α are open in τ and σ respectively. By definition of a bases, each U_α and V_α can be expressed as

$$U_\alpha = \bigcup_\beta U_{\alpha,\beta}$$

$$V_\alpha = \bigcup_\gamma V_{\alpha,\gamma}$$

where $U_{\alpha,\beta} \in \mathcal{B}_\tau$ and $V_{\alpha,\gamma} \in \mathcal{B}_\sigma$. Noting that

$$U_\alpha \times V_\alpha = \bigcup_{\beta,\gamma} U_{\alpha,\beta} \times V_{\alpha,\gamma}$$

we can conclude that

$$A = \bigcup_{\alpha,\beta,\gamma} U_{\alpha,\beta} \times V_{\alpha,\gamma}.$$

□

Next, just for context, I introduce the projection maps:

Definition 4.4. Given $X \times Y$ with the product topology, we define $\pi_X : X \times Y \to X$ where $\pi_X(x, y) = x$. A similar definition is given to $\pi_Y : X \times Y \to Y$. These are called the **projection maps**.

These maps allow us to later formalize the statement in the introduction; namely that $X \times Y$ encodes all of the topological data of X and Y simultaneously. We will return to this once we develop the notion of a continuous map.

Finally, our definition allows us to construct a topology on a finite Cartesian product by induction. However, how to handle the infinite case? This arises typically when studying function spaces, such as $C(\mathbb{R})$ (continuous functions $\mathbb{R} \to \mathbb{R}$) or $C^\infty(\mathbb{R})$ (infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$).

Definition 4.5. Given (X_α, τ_α) a collection of topological spaces, the **product topology** on $\prod_\alpha X_\alpha$ is defined by a basis

$$\mathcal{B} = \{\prod_\alpha U_\alpha \mid \emptyset \neq U_\alpha \in \tau_\alpha, \text{all but finitely many } U_\alpha = X_\alpha\}$$

The clear ambiguous feature is in the finiteness requirement, which lives implicitly in finite products. If we drop this requirement, we are left with the (much finer) **box topology**. Though this may make more intuitive sense, there is good reason to define the product topology in this way.

Example 4.6. Consider $X = \mathbb{R}^\mathbb{N}$, which is a countable product of copies of $\mathbb{R}$ (one for each natural number). If we assign the box topology to X, an example of an open set is

$$U = \prod_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \dots$$

Even if we pick the most simple function, $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x$, then took the product of these functions,

$$F : \mathbb{R} \to \mathbb{R}^\mathbb{N} : x \mapsto (x, x, x, x, \dots)$$

the only element which can map into U is 0. Since $\{0\}$ is a closed (non-open) set, F doesn't behave well with respect to the topologies of the respective spaces.

However, if we instead consider the product topology, we couldn't have this shrinking interval problem, so the elements which can map to a given open set will be the intersection of *finitely many* open intervals, thus open by axiom 3) of a topological space!