CLASS 6, WEDNESDAY FEBRUARY 21ST: MODULE THEORY

Assumption: From now on we will assume R is a commutative ring with unity.

As with many fields of the mathematics, many times the objects of interest are really the structures you can put on top of another more common object. This immediately makes modules an intellectually profitable realm of study.

Definition 0.1. A **module** over a commutative ring R is an abelian group $(M, +)$ with a multiplication by elements of R, that respects the additive structure of R. Let $r, s \in R$ and $m, m' \in M$:

1) **R-Distributive:** $(r + s) \cdot m = rm + sm$.
2) **M-Distributive:** $r(m + m') = rm + rm'$.
3) **Associative:** $(rs)m = r(sm)$.
4) **Unital:** If R is assumed to be a unital ring, then we assume $1 \cdot m = m$.

M is often referred to as an R-module.

Technically, this is the notion of a 2-sided module. You can guess what left/right modules are. The next few examples show the prevalence of R-modules:

Example 0.2 (**Vector Spaces).** If V is a vector space over a field K, then V is also a module over K. So you can view $\mathbb{R}^n$ as a $\mathbb{R}$-module. In fact, every K-module is a vector space!

More generally, every vector space is a free K-module:

Definition 0.3. A module M of R is called **free** if

$$M = R^{\oplus\Lambda} = R^{\Lambda} = \{(r_\lambda)_{\lambda\in\Lambda} \mid r_\lambda \in R\}$$

Modules also generalize the notions of this class so far!

Example 0.4 (**Ideals).** If I is an ideal of R, then I is naturally an R-module. In fact, it inherits all of the above properties from R! In particular, R is an R-module. We can say I is a submodule of R if we want to keep track of where it lives.

Example 0.5 (**Ring Homomorphisms).** Let $\varphi : R \to S$ be a unital ($\varphi(1_R) = 1_S$) ring homomorphism. Then S can be viewed as an R-module via the following action:

$$r \cdot s = \varphi(r)s$$

where the second multiplication is simply multiplication in S. One checks respectively:

1) $(r + r') \cdot s = \varphi(r + r')s = (\varphi(r) + \varphi(r'))s = \varphi(r)s + \varphi(r')s = r \cdot s + r' \cdot s$.
2) $r \cdot (s + s') = \varphi(r)(s + s') = \varphi(r)s + \varphi(r)s' = rs + rs'$
3) It is associative since S-multiplication is.
4) We assume unital.

We can also manipulate modules over a ring R to be more tame:

Definition 0.6. The **annihilator** of a module M is the following set:

$$Ann_R(M) = \{r \in R \mid r \cdot m = 0\ \forall m \in M\}$$

Proposition 0.7. *$Ann_R(M)$ forms an ideal of R. If M is an R-module, then R can naturally be viewed as an $R/Ann_R(M)$ module. Therefore, if $Ann_R(M)$ is a maximal ideal, we can view M as an $R/Ann_R(M)$ vector space.*

Proof. Let $i \in Ann_R(M)$ and $r \in R$. Then $i \cdot m = 0$ implies $(ri)m = r(im) = r0 = 0$. Similarly, if $i_1, i_2 \in Ann_R(M)$, then $i_1 m = i_2 m = 0 = (i_1 + i_2)m$. So $Ann_R(M)$ is an ideal.

Consider the action of $R/Ann_R(M)$ on M given by $(r + Ann_R(M))m = rm$. This is well defined since $a \in Ann_R(M)$ implies $rm = (r + a)m$.

The final statement follows from previous observations, such as $R/Ann_R(M)$ being a field if $Ann_R(M)$ is maximal and Example 0.2. □

An additional example is in fact a major theorem (cf class 7).

Example 0.8 (Abelian groups)**.** There is a natural bijection between the set of Abelian groups and the set of $\mathbb{Z}$-modules. Given an Abelian group G, we have a $\mathbb{Z}$-action given by $n \cdot g = ng \in G$, given by applying the G group operation n times to itself: $g + g + \ldots + g$. In addition, the conditions of being a $\mathbb{Z}$-modules are precisely those required to form a group.

Because for any unital ring R we have a natural map $\mathbb{Z} \to R : 1 \mapsto 1_R$, every R-module is a $\mathbb{Z}$-module by Example 0.5.

Moreover, if for every element $x \in M$, $nx = 0$, M is naturally a $\mathbb{Z}/n\mathbb{Z}$-module by Proposition 0.7.

Example 0.9 (An alternative to linear algebra)**.** One way to study linear algebra is to consider $K[x]$-modules! In particular, given a K vector space V and a linear transformation $M : V \to V$, we can produce a $K[x]$-module structure by

$$(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0) \cdot v = a_n M^n(v) + a_{n-1}M^{n-1}(v) + \ldots + a_1 M(v) + a_0 v$$

Moreover, we can view any modules as a K-module by considering the action of K induced by the inclusion $K \to K[x]$ under Example 0.5. This yields the desired bijection

$$\{K[x] - modules\} \leftrightarrow \{\text{V a } K\text{-Vector spaces and a linear map } T : V \to V\}$$

One can study many maps simultaneously by consideration of $K\{x_1, x_2, \ldots, x_n\}$, or even $K\{A\}$ for A a set of linear maps $V \to V$. Here the notation $\{A\}$ is to denote the fact that $x_1 \cdot x_2 \neq x_2 \cdot x_1$. In effect, we are adjoining a free group on n (or $|A|$) generators.

Example 0.10 (An alternative to calculus)**.** One can also study the ring $K[\frac{\partial}{\partial x}]$ and the module $C^\infty(K)$ of infinitely differentiable function from K to K.

As a final example of a module (for today), we can look at R-Algebras.

Definition 0.11. If R is a commutative unital ring, an **R-algebra** A is an R-module with a notion of multiplication. That is there is a ring homomorphism $R \to A$ with image in the center of A. We do not assume commutativity.

Example 0.12 (Group Rings)**.** Much like we adjoin variables to a ring, we can adjoin a (potentially non-Abelian) group G. It's objects are of the form

$$r_1 g_1 + r_2 g_2 + \ldots r_n g_n$$

for $r_i \in R$ and $g_i \in G$. We multiply $rg \cdot r'g' = rr'gg'$, noting the importance of order in gg'. This is an example of an R-algebra.

As a sub-example, $K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}]$ can be viewed as the group ring $K[\mathbb{Z}^n]$, where $r \cdot x_1^{m_1} \cdots x_n^{m_n}$ corresponds to $r \cdot (m_1, \ldots, m_n)$. This is the degree!