CLASS 25, NOVEMBER 9: LOCAL FINITENESS

We have already demonstrated that there exist partitions of unity subordinate to a finite cover $X = U_1 \cup \ldots \cup U_n$. I defined a partition of unity in a broader setting of locally finite covers. For a given cover, we can refine it to a finite cover (in general) only in a compact space. For locally finite, we only need need the space to be **locally compact**. This includes things like $\mathbb{R}^n$, broadening our generality quite a lot. Today we will explore this notion further which will lead to great results.

Definition 25.1. Let X be any space and $\mathcal{A}$ be a collection of subsets of X. $\mathcal{A}$ is said to be **locally finite** if for any given point $x \in X$, there exists a neighborhood U of x such that only finitely many $A_1, \ldots, A_n$ intersect U: $A_i \cap U \neq \emptyset$.

We have seen the power of such a condition in Homework 1. I note here the importance of the neighborhood condition:

Example 25.2. Consider the collection $\mathcal{A} = \{(0, \frac{1}{n})\}$. This is a locally finite collection in $(0,1)$, as for any α we can choose n large enough such that $\frac{1}{n} \leq \alpha$. On the other hand, if we consider this collection in $\mathbb{R}$, we realize it is no longer locally finite. Indeed, a neighborhood of 0 has the form $(-\epsilon, \epsilon)$. And this intersects every element of $\mathcal{A}$. The same is true if we restrict even more to $\mathcal{A}' = \{(\frac{1}{n+1}, \frac{1}{n})\}$.

Of course, no element of $\mathcal{A}$ contains 0.

Here I list some nice properties of locally finite collections.

Lemma 25.3. *Let $\mathcal{A}$ be a locally finite collection of subsets of X.*

1) *If $\mathcal{A}' \subseteq \mathcal{A}$, then $\mathcal{A}'$ is locally finite.*
2) *If $\mathcal{B} = \{\bar{A} \mid A \in \mathcal{A}\}$, then $\mathcal{B}$ is locally finite.*
3) $\bigcup_{A \in \mathcal{A}} \bar{A} = \overline{\bigcup_{A \in \mathcal{A}} A}$.

This last property is pretty strange generally speaking, and shows how nice the locally finite property is. In general, we only have that $\bigcup_\alpha \bar{U_\alpha} \subseteq \overline{\bigcup_\alpha U_\alpha}$. This follows since $V \subset U$ implies $\bar{V} \subseteq \bar{U}$. In general however the left side isn't even closed!

Proof.

1) Trivial.
2) To see this, note that every open set which intersects $\bar{A}$ also intersects A (contrapositively, $U \subseteq A^c$ implies $U \subseteq (A^c)^\circ$, whose complement is $\bar{A}$). Therefore, we can take the same neighborhood and elements of $\mathcal{A}$ that show it is locally finite.
3) It suffices to check that $\bigcup_{A \in \mathcal{A}} \bar{A}$ is a closed subset. Let $x \in \overline{\bigcup_{A \in \mathcal{A}} A}$. Let $U \subseteq X$ be an open neighborhood of x intersecting only finitely many sets in $\mathcal{A}$, say $A_1, \ldots, A_n$. We claim $x \in \bar{A}_i$ for some i. If not, then $x \in U' = U \setminus (\bar{A}_1 \cup \ldots \cup \bar{A}_n)$ which is an open set. But this implies U' doesn't intersect *any* element of $\mathcal{A}$, and thus is in the complement's interior. This contradicts our choice of x.

$\square$

Definition 25.4. A collection $\mathcal{A}$ is said to be **countably locally finite** if $\mathcal{A}$ can be decomposed into a countable collection $\mathcal{A}_1, \mathcal{A}_2, \ldots$ such that each $\mathcal{A}_i$ is locally finite.

Definition 25.5. Given $\mathcal{A}$ and $\mathcal{B}$ two collections of subsets of X, we say $\mathcal{B}$ **refines** (or is a **refinement** of) $\mathcal{A}$ if every element $B \in \mathcal{B}$ has a $A \in \mathcal{A}$ such that $B \subseteq A$. If $\mathcal{B}$ is composed of open (or closed) sets, it is called an **open** (or **closed**) **refinement**.

To relate the previous notions and a key lemma to a future metrization theorem, we have the following:

Lemma 25.6. *If X is a metric space, and $\mathcal{A}$ is an open cover, then there exists $\mathcal{B}$ an open covering refinement which is countably locally finite.*

Proof. We will using the well ordering theorem, which states that given any set we can well order the elements. This is actually easy to check, since we can inject any *set* into the space of ordinals. Applying the natural ordering there to its image produces such an ordering.

Let $U \in \mathcal{A}$. We can define $U_n \subseteq U$ to be the $\frac{1}{n}$-shrinking of U;

$$U_n = \{x \in X \mid B\left(x, \frac{1}{n}\right) \subseteq U\}$$

Note eventually U_n is non-empty, since U is open. We can define a further refinement of U:

$$U_n' = U_n \setminus \bigcup_{V<U} V$$

Again, one of these must be non-empty by the previous part. This collection of subsets is disjoint, since either $V < U$ or $U < V$. Even stronger, for a fixed n they are separated by distance at least $\frac{1}{n}$. Indeed, if $V < U$, then V_n has the property that $B(x, \frac{1}{n}) \subseteq V \subseteq U_n^c$ for each $x \in V_n$.

The only issue left to resolve is that these sets are not open (they are even closed!). Indeed, we subtracted a bunch of open sets from an arbitrary (closed) set, so they likely are not open. However, we may consider

$$U_n'' = \bigcup_{x \in U_n'} B\left(x, \frac{1}{3n}\right)$$

This is a union of open sets and therefore is itself open. Furthermore, if $x' \in B\left(x, \frac{1}{3n}\right) \subseteq U_n''$ and $y' \in B\left(y, \frac{1}{3n}\right) \subseteq V_n''$, then the triangle inequality implies

$$d(x', y') \geq d(x, y) - d(x', x) - d(y, y') \geq \frac{1}{n} - \frac{2}{3n} = \frac{1}{3n} > 0$$

so $x' \neq y'$ and therefore $U_n'' \cap V_n'' = \emptyset$.

Finally, I claim that if we let $\mathfrak{B}_n = \{U_n'' \mid U \in \mathcal{A}\}$, and let $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathfrak{B}_n$ we are done. Indeed, $U_n'' \subseteq U$ for every n by design, and

$$\bigcup_{U \in \mathcal{A}} \bigcup_{n \in \mathbb{N}} U_n'' \supseteq \bigcup_{U \in \mathcal{A}} \bigcup_{n \in \mathbb{N}} U_n' = \bigcup_{U \in \mathcal{A}} \bigcup_{n \in \mathbb{N}} \left(U_n \setminus \bigcup_{V<U} V\right) = \bigcup_{U \in \mathcal{A}} \left(U \setminus \bigcup_{V<U} V\right) = \bigcup_{U \in \mathcal{A}} U = X$$

So $\mathcal{B}$ covers X. Finally, each $\mathfrak{B}_n$ is composed of pairwise disjoint sets! So they are automatically locally finite. This completes the proof. □

Corollary 25.7. *Metric spaces have a countably locally finite basis.*