HOMEWORK 1: TOPOLOGICAL SPACES
DUE: FRIDAY, SEPTEMBER 14

1) Write down the axioms for a topological space in terms of its collection of closed sets.

2) Let $\mathcal{T}_\alpha$ be a collection of topologies. Is it true that $\bigcap_\alpha \mathcal{T}_\alpha$ is a topology? What about $\bigcup_\alpha \mathcal{T}_\alpha$?

3) Show that on $\mathbb{R}^n$ the following bases generate the same (metric) topology:
   - $\mathcal{B}_1 = \{B(x,r) \mid x \in \mathbb{R}^n,\ r > 0\}$.
   - $\mathcal{B}_2 = \{(a_1, b_1) \times \ldots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}$.
   - $\mathcal{B}_3 = \{B(x,r) \mid x \in \mathbb{Q}^n,\ r \in \mathbb{Q}_+\}$. That is to say we consider open balls of rational radius centered at points with rational coordinates.

4) Consider the following topologies on $\mathbb{R}$:
   - $\mathcal{T}_1 =$ the standard Euclidean/metric topology.
   - $\mathcal{T}_2 =$ the finite complement topology.
   - $\mathcal{T}_3 =$ the topology with basis $(a, b]$, where $a, b \in \mathbb{R}$.
   - $\mathcal{T}_4 =$ the topology with basis $(-\infty, b)$, where $b \in \mathbb{R}$.
   - $\mathcal{T}_5 =$ the topology with basis (a, b) and $(a, b) \setminus K$, where $K = \cup_{n \in \mathbb{Z}} \frac{1}{n}$.

   Order them in terms of comparability, i.e. finer, coarser, or incomparable. You can do this with as few as 8 pairwise comparisons.

5) If τ and σ are 2 topologies on X with τ *strictly* finer than σ (i.e. $\tau \supsetneq \sigma$), what can you say about the subspace topology on $Y \subseteq X$?

6) Verify that the following are topologies on a 3-point set $X = \{a, b, c\}$:
   - $\tau_1 = \{\emptyset, \{a, b, c\}\}$
   - $\tau_2 = \{\emptyset, \{c\}, \{a, b\}, \{a, b, c\}\}$
   - $\tau_3 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$
   - $\tau_4 = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

   Additionally, which can be realized as metric topologies? Can you notice a pattern?