# CLASS 21, WEDNESDAY APRIL 18TH: $\mathrm{Tor}_i^R$

We have already noted the importance of flat modules; they allow us to conclude all modules inject into an injective $R$-module. Today we will study how close a module is to being flat. We will use some homological methods.

**Definition 0.1.** If $M$ is an $R$-module, we have already talked about how we can take a free resolution of $M$ producing an exact sequence

$$\cdots \xrightarrow{\psi_3} R^{\lambda_2} \xrightarrow{\psi_2} R^{\lambda_1} \xrightarrow{\psi_1} R^{\lambda_0} \longrightarrow M \to 0$$

We can tensor this sequence with $N$ and produce

$$\cdots \xrightarrow{\psi_3 \otimes 1_N} R^{\lambda_2} \otimes_R N \xrightarrow{\psi_2 \otimes 1_N} R^{\lambda_1} \otimes_R N \xrightarrow{\psi_1 \otimes 1_N} R^{\lambda_0} \otimes_R N \xrightarrow{\psi_0 \otimes 1_N} 0$$

This sequence is no longer exact (unless $N$ was flat to begin with). However, we do maintain the inclusion $\ker(\psi_i \otimes 1_N) \supseteq \mathrm{im}(\psi_{i+1} \otimes 1_N)$. Therefore, we measure

$$\mathrm{Tor}_i^R(M, N) = \ker(\psi_i \otimes 1)/\mathrm{im}(\psi_{i+1} \otimes 1)$$

An important note is that this is independent of the chosen free resolution. The primary advantage of Tor is that it defines a LES for the tensor product:

**Theorem 0.2.** *If $0 \to M' \to M \to M'' \to 0$ is a SES, then the following is exact:*

$$\begin{array}{rccc}
\cdots \longrightarrow & \mathrm{Tor}_2^R(M,N) \longrightarrow & \mathrm{Tor}_2^R(M'',N) \longrightarrow & \\
\mathrm{Tor}_1(M',N) \longrightarrow & \mathrm{Tor}_1^R(M,N) \longrightarrow & \mathrm{Tor}_1^R(M'',N) \longrightarrow & \\
M' \otimes_R N \longrightarrow & M \otimes_R N \longrightarrow & M'' \otimes_R N \longrightarrow & 0
\end{array}$$

This theorem allows us complete the SES corresponding to a tensor, since it is only in general right exact. The proof of this theorem requires us to use the snake lemma and complete a free resolution of $M'$ and $M''$ to one for $M$. I recommend reading about this proof independently.

To make this more succinct, I make the following note:

**Proposition 0.3.** *Consider $R$-modules $M, N$. Then*

$$M \otimes_R N \cong \mathrm{Tor}_0^R(M, N)$$

*Proof.* Note that we have a right exact sequence

$$R^{\lambda_1} \xrightarrow{\psi_1} R^{\lambda_0} \longrightarrow M \to 0$$

Therefore, tensoring with $N$ maintains this:

$$R^{\lambda_1} \otimes_R N \xrightarrow{\psi_1 \otimes 1_N} R^{\lambda_0} \otimes_R N \to M \otimes N \to 0$$

This implies

$$M \otimes N \cong R^{\lambda_0}/\mathrm{im}(\psi_1 \otimes 1_N) \cong \ker(\psi_0 \otimes 1_N)/\mathrm{im}(\psi_1 \otimes 1_N) = \mathrm{Tor}_0^R(M, N)$$

$\square$

Here is the theorem that represents the importance of Tor explicitly.

**Theorem 0.4.** *The following are equivalent:*

*1) $N$ is a flat $R$-module.*
*2) $\mathrm{Tor}_1^R(M,N) = 0$ for all $R$-modules $M$.*
*3) $\mathrm{Tor}_i^R(M,N) = 0$ for all $R$-modules $M$ and $i \geq 1$.*

*Proof.* 1) $\Leftrightarrow$ 2): Given Theorem 0.2, we have an exact sequence

$$\mathrm{Tor}_1(M'', N) \to M' \otimes_R N \to M \otimes_R N \to M'' \otimes_R N \to 0$$

Therefore, the desired SES holds if and only if $\mathrm{Tor}_1(M'', N) = 0$. Additionally, every $M''$ can be surjected onto by a free module, so $M''$ can always appear in such a SES sequence.

3) $\Rightarrow$ 2) Obvious.

1) $\Rightarrow$ 3) Given $M$ is flat, we see that

$$\cdots \xrightarrow{\psi_3 \otimes 1_N} R^{\lambda_2} \otimes_R N \xrightarrow{\psi_2 \otimes 1_N} R^{\lambda_1} \otimes_R N \xrightarrow{\psi_1 \otimes 1_N} R^{\lambda_0} \otimes_R N \to M \otimes N \to 0$$

is an exact sequence. Therefore, $\ker(\psi_i \otimes 1_N) = \mathrm{im}(\psi_{i+1} \otimes 1_N)$ for all $i \geq 1$, or equivalently, $\mathrm{Tor}_i^R(M,N) = \ker(\psi_i \otimes 1_N)/\mathrm{im}(\psi_{i+1} \otimes 1_N) = 0$. $\square$

Another important remark is that $\mathrm{Tor}_i$ is symmetric:

**Proposition 0.5.**
$$\mathrm{Tor}_i^R(M,N) \cong \mathrm{Tor}_i^R(N,M)$$

This requires tensoring together a free resolution of $M$ with one for $N$, then taking the total complex. This shares homology with both complexes simultaneously, and therefore allows one to conclude the desired isomorphism. Now onto some examples:

**Example 0.6.** $\mathbb{Q}$ is a flat $\mathbb{Z}$-module which is not projective (not locally-free). However, $\mathbb{Q}$ is a flat $\mathbb{Z}$-module since it is torsion-free over a PID. So we can look at the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

which induces an exact sequence

$$0 = \mathrm{Tor}_1^R(\mathbb{Q}, D) \to \mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, D) \to \mathbb{Z} \otimes D \to \mathbb{Q} \otimes D \to \mathbb{Q}/\mathbb{Z} \otimes D \to 0$$

for a given module $D$. Therefore, $\mathrm{Tor}_1(\mathbb{Q}/\mathbb{Z}, D) = \ker(\mathbb{Z} \otimes D \to \mathbb{Q} \otimes D)$. This is precisely the torsion subgroup of $D$.