CLASS 4, MONDAY FEBRUARY 12TH: NOETHERIAN PROPERTY & IDEALS

We already discussed how to generate an ideal by select elements, stating that it is the smallest two-sided ideal generated by the elements $A = \{f_1, f_2, \ldots, f_n\}$ (or potentially even an infinite set). The notation was

$$\langle A \rangle = \langle f_1, f_2 \ldots, f_n \rangle$$

We can also introduce notation for the left and right ideal; RA and AR respectively. How do we know that such a smallest ideal exists?

$$\langle A \rangle = \bigcap_{\substack{I \supseteq A \\ \text{I is an ideal}}} I$$

If there are finitely many f_i, we call the ideal **finitely generated**. If there is only a single generator, the ideal is called **principal**.

Definition 0.1. A ring R is called **Noetherian** if every ideal is finitely generated. This naturally also yields left and right Noetherian rings by putting the adjective on ideal.

This is a fantastic property, named after the great mathematician Emmy Noether. Here some equivalent ways to specify this property:

Proposition 0.2. *The following conditions are equivalent:*

- *R is a Noetherian ring.*
- *Every ascending chain of ideals eventually* **stabilizes***: if*
  $$I_1 \subseteq I_2 \subseteq \ldots$$
  the $\exists n > 0$ such that $I_n = I_{n+1} = I_{n+2} = \ldots$.
- *Every collection of Ideals $\{I_\alpha\}_{\alpha \in \Lambda}$ contains a maximal element. That is to say that there exists $\beta \in \Lambda$ such that there are no $\alpha \in \Lambda$ such that $I_\beta \subsetneq I_\alpha$.*

Proof. See homework. □

This condition will become extremely important later when we study **modules** and the **spectrum** of a ring, since it puts a measure on the size of a ring. Examples include $\mathbb{Z}$, $K[x_1, \ldots, x_n]$, or even $R[x_1, \ldots, x_n]$ and $R[\![x_1, \ldots, x_n]\!]$ where R is a Noetherian ring (a theorem of Hilbert that we will return to later). In fact, most rings you will study in practice are Noetherian. A simple non-example is a polynomial ring in infinitely many variables: $K[x_1, x_2, x_3, \ldots]$.

Definition 0.3. An ideal $\mathfrak{m} \neq R$ is called **maximal** if the only ideal properly containing $\mathfrak{m}$ is R itself.

An ideal $\mathfrak{p}$ is called prime if for every $r, s \in R$, if $r \cdot s \in \mathfrak{p}$, then either $r \in \mathfrak{p}$ or $s \in \mathfrak{p}$.

Proposition 0.4. *Every proper ideal ($\neq R, 0$) I in a unital ring R is contained in some maximal ideal.*

We require a Lemma from set theory; Zorn's Lemma:

Lemma 0.5 (Zorn's Lemma). *Let S be a partially ordered set, with the property that every ascending chain has an upper bound. Then there exists a maximal element.*

Proof. Let $\mathcal{C}$ be the set of all proper ideals containing I. Note in particular that $\mathcal{C} \neq \emptyset$, since it contains I. If $I_1 \subseteq I_2 \subseteq \dots$ is an ascending chain of ideals in $\mathcal{C}$, then

$$J = \bigcup_{i \geq 1} I_i$$

is a proper ideal containing I which is an upper bound for the chain. Therefore, Zorn's Lemma applies, and there is a maximal element $\mathfrak{m}$ of the set $\mathcal{C}$. This is necessarily a maximal ideal since if it were contained in another proper ideal, it would contain I and therefore make $\mathfrak{m}$ non-maximal in $\mathcal{C}$. This completes the proof. □

Next up, we see an equivalent way to detect whether an ideal is maximal or prime. In addition, it demonstrates that all maximal ideals are in fact prime.

Proposition 0.6. *Let R be a commutative ring.*

- $\mathfrak{p}$ *is a prime ideal if and only if* $R/\mathfrak{p}$ *is an integral domain.*
- $\mathfrak{m}$ *is a maximal ideal if and only if* $R/\mathfrak{m}$ *is a field.*

Since fields are in particular integral domains;

Corollary 0.7. *All maximal ideals are prime!*

Proof. of Proposition 0.6: I will prove the statements in order. Let $\mathfrak{p}$ be a prime ideal. Then there exist no elements $a, b \in R$ not in $\mathfrak{p}$ with the property that $a \cdot b \in \mathfrak{p}$. Suppose that $\bar{a}, \bar{b} \in R/\mathfrak{p}$ are such that $\bar{a} \cdot \bar{b} = 0$. Then since $R \to R/\mathfrak{p}$ is surjective, there exist a, b mapping to $\bar{a}, \bar{b}$. This implies that $a \cdot b \in \mathfrak{p}$, a contradiction.

The converse follows by an identical argument.

Now I consider the second statement. Let $\mathfrak{m}$ be a maximal ideal. Then there exists no proper ideals containing $\mathfrak{m}$. By the fourth isomorphism theorem, we know that the ideals of $R/\mathfrak{m}$ are exactly those which contain $\mathfrak{m}$, which is exactly $\mathfrak{m}$. Therefore, the only ideal of $R/\mathfrak{m}$ is the zero ideal. This is precisely the condition of a field:

Lemma 0.8. *A commutative ring is a field if and only if it's only ideal is the* 0 *ideal.*

Proof. If R is a field, then every element is a unit. Therefore, any non-zero ideal contains 1 and thus everything. Since R is commutative, it is necessarily a field.

On the other hand, if R is a commutative ring which is not a field, then R necessarily contains a non-unit r. This implies $\langle r \rangle$ is a proper, non-zero ideal. □

This completes the proof of the forward direction of the theorem. The other direction also uses the fourth isomorphism theorem naturally. □