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$\boxed{\text{Dec 4}}$ Euler Characteristic

$X$ a $\Delta$-complex. Then

$$\chi(X) := \sum (-1)^n \# C_n(X)$$

This definition makes sense (independent of $\Delta$-cx structure

$$\chi(X) = \sum_i (-1)^n B_i$$

$$= \sum_i (-1)^n \, rk\,(H_n(X))$$

$rk(H_n(X)) = n \quad : \quad H_n(X) = \mathbb{Z}^n \oplus T$

Pf: $0 \to C_n(X) \to \cdots \to C_0(X) \to 0$

$0 \to Z_n \to C_n \to B_{n-1} \to 0 \quad , \quad 0 \to B_n \to Z_n \to H_n \to 0$

$rk(C_n) = rk(B_{n-1}) + rk(Z_n)$, $rk(Z_n) = rk(H_n) + rk(B_n)$

$= rk(B_{n-1}) + rk(B_n) + rk(H_n)$

$\sum (-1)^n (rk(B_{n-1}) + rk(B_n) + rk(H_n))$

$$= \sum (-1)^n \, rk(H_n)$$

These numbers ~~determ~~ are clearly homotopy invariants

Ex $\chi(M_g) = 2-2g \qquad \chi(N_g) = 2-g$

Split exact sequences

A short exact sequence being split is optimal for understanding the components

<u>Defn:</u> $0 \to \cancel{H} \xrightarrow{i} \cancel{G} \xrightarrow{q} G/H \to 0$ is split exact iff $\exists j: G \to H$ s.t.

$$j \circ i = Id_H$$

$j$ "splits" the sequence.

Ex/ Let $r: X \to A$ be a retraction. We have shown $r_*$ is surjective. Also, $i_*$ is injective:

$$0 \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{q_*} \underset{\underset{H_n(X)/H_n(A)}{\text{IIS}}}{H_n(X,A)} \to 0$$

$r_*$ is a splitting.

Thm: TFAE for $0 \to \cancel{H} \xrightarrow{i} G \xrightarrow{q} G/H \to 0$:

1) The sequence is split exact
2) $\exists q': G/H \to G$ s.t. $q \circ q' = Id$
3) $G \cong G \oplus G/H$

Pf: We will show $1 \Leftrightarrow 3 \Leftrightarrow 2$

$3 \Rightarrow 1,2$: is trivial

$1 \Rightarrow 3$: Let $i' : G \to H$ be the splitting

$$\begin{tikzcd} & & G \arrow[dr, "q"] \arrow[d] & & \\ 0 \arrow[r] & H \arrow[ur, "i"] \arrow[r] & H \oplus G/H \arrow[r] & G/H \arrow[r] & 0 \end{tikzcd}$$

$\mathrm{Ker}(q) = \mathrm{Im}(i)$, so $G \xrightarrow{\varphi} H \oplus G/H$
$g \longmapsto (i'(g), q(g))$

$\varphi$ is injective: Suppose $\varphi(g) = 0$. Then
$q(g) = 0 \Rightarrow g \in \mathrm{Ker}(q) = \mathrm{Im}(i)$

$\Rightarrow g = i(a)$. but $i'(g) = i'(i(a)) = a = 0$
$\Rightarrow g = i(0) = 0$.

$2 \Rightarrow 3$: Similar.

So if $\exists r$ as above $H_n(X) \cong H_n(A) \oplus H_n(X,A)$

We will develop this w/ Meyer Vietoris next time.

## Dec 6: Meyer-Vietoris Sequence (Homological Vankampen)

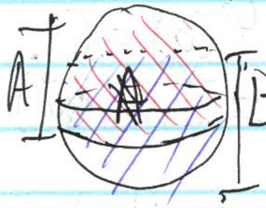
$$\begin{tikzcd} & A \arrow[dr, "i'"] & \\ A\cap B \arrow[ur, "i"] \arrow[dr, "j"'] & & X = \mathring{A}\cup\mathring{B} \\ & B \arrow[ur, "j'"'] & \end{tikzcd}$$

$$0 \to C_n(A\cap B) \xrightarrow{i_*\oplus j_*} C_n(A)\oplus C_n(B) \xrightarrow{i'_*-j'_*} C_n(\cancel{X}\ A+B) \to 0$$

⇝ LES

$$\cdots \to H_{n+1}(X) \xrightarrow{\delta} H_n(A\cap B) \to H_n(A)\oplus H_n(B) \to H_n(X) \to H_{n-1}(A\cap B) \to \cdots$$

Ex/ $S^n$: A = Northern Hem, B = Southern

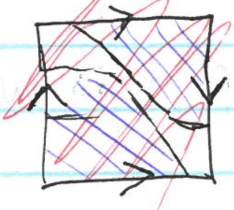


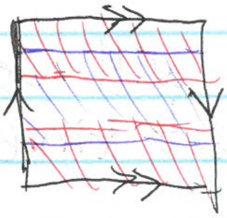

A, B cont, $A\cap B \simeq S^{n-1}$

$$H_i(A)\oplus H_i(B) = 0 \to H_i(S^n) \to H_{i-1}(S^{n-1}) \to 0$$

$$H_i(S^n) \cong H_{i-1}(S^{n-1})$$

Ex/ K = klein Bottle:





$X = K$; $A, B, A\cap B \simeq S^1$

$0 \to$

LES: $H_2(K) \xrightarrow{\delta} H_1(S^1) \to H_1(S^1)^2 \to H_1(K) \xrightarrow{0} H_0(S^1)$

$0 \to H_2(K) \xrightarrow[0]{} \mathbb{Z} \to \mathbb{Z}^2 \xrightarrow[i_* - j_*]{} H_1(K) \to 0$

$\begin{array}{c} \| \\ 0 \end{array}$ $\Leftarrow$ inj $\quad 1 \mapsto (2,-2) \quad \Rightarrow H_1(K) \cong \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

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Relating notions: Recall

$$\pi_1(X,x_0) = \{[\gamma : I \to X] \mid \gamma(0) = \gamma(1) = x_0\}$$

We can regard $\gamma$ as a singular 1-cycle. This gives a map

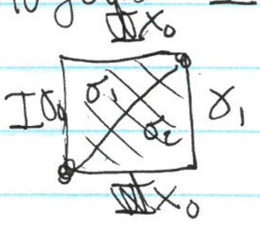
$L(x,x_0) \mapsto$

$$\pi_1(X,x_0) \xrightarrow{\rho} H_1(X)$$

Thm: If $X$ is path connected, then $\rho$ exists and is surjective and has kernel $[\pi_1(X), \pi_1(X)]$. Thus

$$\pi_1(X)^{ab} \cong H_1(X)$$

It takes some work to show $\rho$ is well defined: eg homotopic maps yield homologous 1-cycles




$\gamma_0 \gamma_1^{-1} = \partial_2(\cancel{\sigma}$

$\gamma_0 - \gamma_1 = \partial_2(\sigma_1 + \sigma_2)$

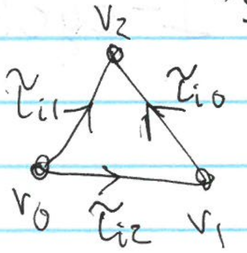
Similarly, surjectivity follows w/ usual trick:

For $\sigma = \sum_i n_i \sigma_i$

Let $\gamma_i$ be paths $x_0 \rightsquigarrow \sigma_i(0)$, then

$$\gamma_0 \sigma_0^{n_0} \overline{\gamma_0}\, \gamma_1 \sigma_1^{n_1} \overline{\gamma_1} \cdots \gamma_m \sigma_m^{n_m} \overline{\gamma_m} \mapsto \sigma$$

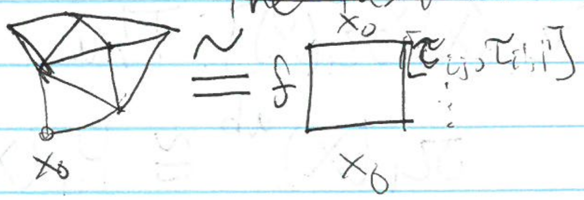
Finally, $\ker(p) \supseteq [\pi_1(X), \pi_1(X)]$ since $H_1(X)$ is Abelian.




"$\subseteq$" $f \in \ker(p) \Rightarrow \partial_2(\Delta^2) = f$

$$\partial_2(\sigma) = \partial_2\Big(\sum_i n_i \sigma_i\Big) = \sum_i \sum_{j=0}^{2} (-1)^j n_i \tau_{ij} = f$$

We can adjoin these $\Delta$ together so that Each ~~cance~~ adjoining is one $+1$, one $-1$. ~~except one.~~ ~~The last is equal to f~~




$\Rightarrow f \in$ Commutator

Dec 8 Topological Graduation

From here, we have found techniques to say whether or not two spaces are homotopically equivalent. There are many directions to go from here.

I) Develop further exciting / refined invariants to detect.

0) $\pi_n(X)$

1) Cohomology: $0 \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0$

We can apply $\mathrm{Hom}_{\mathbb{Z}}(-,\mathbb{Z})$ to this sequence, effectively "dualizing" it:

$$0 \leftarrow C^n(X) \xleftarrow{\delta_n} \cdots \xleftarrow{\delta_2} C^1(X) \xleftarrow{\delta_1} C^0(X) \xleftarrow{\delta_0} 0$$

Where $C^i(X) = \mathrm{Hom}_{\mathbb{Z}}(C_i(X), \mathbb{Z})$

$$\cancel{\delta_2(f) = \delta} \quad (\delta_2(f))(\sigma) = f(\partial_2(\sigma))$$

Taking $H^i(X) = \ker(\delta_{i+1}) / \mathrm{Im}(\delta_i)$

This is called Cohomology. We can give cohomology the structure of a ring

2) Homology w/ coefficients: Why only allow $\mathbb{Z}$-coefficients in our chains? Because it's "More geometric"!? Pew--- Let $R$ be a ring:

$$C_n(X,R) = \{\textstyle\sum c_i\sigma^n \mid c\in R\}.$$

Then $C_\bullet$ is still a complex and we can form

$$H_n(X,R) = \ker(\partial_n^R)/\mathrm{Im}(\partial_{n+1}^R)$$

as a homotopy invariant.

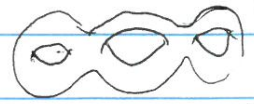
$R=\mathbb{R},\mathbb{C},\mathbb{Z}/2\mathbb{Z}$ are of particular interest

3) Cobordism / K-Theory / DeRham-Cohomology / Ricci Flow

II) Try to classify a smaller collection of spaces:

Manifolds: Locally like $\mathbb{R}^n$

Surfaces: $M_g$, $N_g$ $\quad N_1=\mathbb{RP}^2$, $N_2=K$



Thm [Whitehead]

CW complexes: If $f:X\to Y$, then $f_*:\pi_n(X)\xrightarrow{\sim}\pi_n(Y) \iff X\simeq Y$.

## III Relate concepts

<u>Hurewicz theorem:</u> $\exists\ \pi_K(X) \xrightarrow{h_*} H_K(X)$ Group Homs, & if $\pi_1(X) = \cdots = \pi_{n-1}(X)$, then $h_*$ is an isomorphism for $2 \leq K \leq n$, and $h_*$ is the abelianization for $\pi_1(X)$.

## IV Moduli of topological spaces / homotopy.