CLASS 22, OCTOBER 31: THE TYCHONOFF THEOREM

Now we have the desired tools to return to the question of arbitrary products of compact sets. The statement of the Tychonoff Theorem is as follows:

Theorem 22.1 (Tychonoff Theorem). *If X_α is a collection of compact topological spaces, then so is $X = \prod_\alpha X_\alpha$ endowed with the product topology.*

As we've already seen, the box topology has far too many open sets to be viable for such a theorem even in the countable product case. This should further illustrate the desirability of the product topology. Recall the following equivalent definition of compactness for closed sets (Theorem 12.5 from our notes).

Theorem. *X is compact if and only if every collection $\mathcal{C}$ of closed sets such that $\emptyset \neq C_1 \cap C_2 \cap \cdots \cap C_n$ for any choice of $C_i \in \mathcal{C}$,[1] then*

$$\emptyset \neq \bigcap_{C \in \mathcal{C}} C.$$

Just to reiterate, this is proved by taking the complements the sets in an open cover, and vice-versa. It is in fact the contrapositive statement. To prove Theorem 22.1, we will show instead that X has the finite intersection property. To do this, we require a few lemmas. The first is actually a very common tool which is equivalent to the axiom of choice (which is always assumed in topology).

Lemma 22.2 (Zorn's Lemma). *If $(\mathcal{S}, <)$ is a non-empty partially ordered set in which every chain of elements of $\mathcal{S}$ has an upper bound $B \in \mathcal{S}$:*

$$B_1 < B_2 < B_3 < \ldots < B$$

Then $\mathcal{S}$ contains a maximal element $M \in S$, e.g. $\nexists M' \in \mathcal{S}$ such that $M < M'$.

The proof is a deep dive into the language of set theory. A nice accessible proof of this fact is given here: `https://arxiv.org/pdf/1207.6698.pdf`. We will instead choose to apply it in our specific scenario:

Lemma 22.3. *Let X be any set, and let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a collection of subsets with the finite intersection property. Then there exists a maximal collection $\mathcal{M} \subseteq \mathcal{P}(X)$ containing $\mathcal{A}$ with the property that $\mathcal{M}$ has the finite intersection property.*

Proof.

$$\mathcal{S} := \{\mathcal{B} \subseteq \mathcal{P}(X) \mid \mathcal{A} \subseteq \mathcal{B}, \text{ and } \mathcal{B} \text{ has the finite intersection property}\}.$$

Make it into a poset by inclusion. It suffices to check the condition of Zorn's Lemma for $\mathcal{S}$. Suppose $\mathcal{B}_i \in \mathcal{S}$ is such that

$$\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots$$

It suffices to check that $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i$. I claim $\mathcal{B} \in \mathcal{S}$. Suppose $C_1, \ldots, C_n \in \mathcal{B}$. Since $\mathcal{B}$ is a union of sets, we have that $C_i \in \mathcal{B}_{j_i}$ for some $j_i \in \mathbb{N}$ depending on i. Therefore, letting

[1] This is referred to as the finite intersection property.

$N = \max\{j_1, \dots, j_n\}$, we have that $C_1, \dots, C_n \in \mathcal{B}_N$. But $\mathcal{B}_N$ has the finite intersection property, so

$$C_1 \cap \ldots \cap C_n \neq \emptyset.$$

Zorn's lemma immediately implies the existence of the desired $\mathcal{M}$. □

Next up, we prove some nice properties of the resulting set $\mathcal{M}$.

Lemma 22.4. *Let $\mathcal{M}$ be the set obtained in Lemma 22.3. Then*

- *If $C_1, \dots, C_n \in \mathcal{M}$, then $C_1 \cap \dots \cap C_n \in \mathcal{M}$*
- *If $A \subseteq X$ has the property that $C \cap A \neq \emptyset$ for all $C \in \mathcal{M}$, then $A \in \mathcal{M}$.*

Proof.

- Suppose the statement is false, and consider $\mathcal{M}' = \mathcal{M} \cup \{C_1 \cap \ldots \cap C_n\}$. I claim $\mathcal{M}'$ has the finite intersection property. Indeed, if $C_i' \in \mathcal{M}$, then

$$(C_1 \cap \ldots \cap C_n) \cap C_1' \cap \ldots \cap C_m' = C_1 \cap \dots \cap C_n \cap C_1' \cap \dots \cap C_m' \neq \emptyset$$

  where the last equality follows by virtue of the fact that we intersected finitely many sets in $\mathcal{M}$. But $\mathcal{M} \subsetneq \mathcal{M}'$, which contradicts the maximality of $\mathcal{M}$.
- Similarly, suppose the statement is false, and consider $\mathcal{M}' = \mathcal{M} \cup \{A\}$. Note again that $\mathcal{M}'$ has the finite intersection property:

$$A \cap C_1 \cap \dots C_n = A \cap (C_1 \cap \dots C_n) \neq \emptyset$$

  where the last equality follows by virtue of the fact that $C_1 \cap \dots \cap C_n \in \mathcal{M}$ by the first statement. Again, this contradicts the maximality of $\mathcal{M}$.

□

Proof. (of Theorem 22.1): Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a collection of subsets with the finite intersection property. It suffices to check that

$$\bigcap_{A \in \mathcal{A}} \bar{A} \neq \emptyset$$

as applying Theorem shows X is compact. Choose $\mathcal{M}$ according to Lemma 22.3. Note that by adding more sets, we have made the intersection only smaller. Therefore, it suffices to check

$$\bigcap_{A \in \mathcal{M}} \bar{A} \neq \emptyset.$$

For a given α, consider the projection map $\pi_\alpha : X \to X_\alpha$. Now, consider the collection $\mathcal{M}_\alpha = \{\pi_\alpha(A) \mid A \in \mathcal{M}\}$. By the compactness of X_α, we note that $\mathcal{M}_\alpha$ having the finite intersection property implies $\exists\, x_\alpha \in \bigcap_{A_\alpha \in \mathcal{M}_\alpha} \bar{A}_\alpha$. I claim $\mathbf{x} = (x_\alpha) \in \bar{A}$ for every $A \in \mathcal{M}$.

First I claim that if $x_\alpha \in U_\alpha \subseteq X_\alpha$ is an open set, then $\pi^{-1}(U_\alpha) \cap \bar{A} \neq \emptyset$. Since $x \in U_\alpha$ is an open neighborhood, and $x_\alpha \in \pi_\alpha(\bar{A}) \subseteq \overline{\pi_\alpha(A)}$ by assumption[2], $\exists y_\alpha = \pi_\alpha(\mathbf{y}) \in U_\alpha \cap \pi_\alpha(A)$. Therefore, $\mathbf{y} \in \pi_\alpha^{-1}(U_\alpha) \cap A$. This demonstrates the claim.

Finally, by the second part of Lemma 22.4, we see that $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{M}$ for all U_α open neighborhoods of x_α in X_α. Therefore, by part 1 of Lemma 22.4, taking the intersection of finitely many such sets is in $\mathcal{M}$. But these are the basis elements of the product topology. But $\mathcal{M}$ has the finite intersection property, and every basis element containing $\mathbf{x}$ intersects every $A \in \mathcal{M}$. Therefore, $\mathbf{x} \in \bar{A}$ (it is a limit point) for all $A \in \mathcal{M}$, and thus $\mathcal{M}$ has the infinite intersection property. □

[2]They are in fact equal by HWK 4, #4.