So I wanted to pose a more formal solution to Micheal's question from today's class, that is 'How do we know that the topologies are the same?' in a variety of settings.

First and foremost, this returns us to a question about continuity of a homotopy built up from 2 other homotopies $X \times I \to Y$:

$$H(x,t) = \begin{cases} F(x,2t) & 0 \leq t \leq \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

If $U \subset Y$, it goes to show that $H^{-1}(U)$ is open. Note that $F^{-1}(U), G^{-1}(U) \subseteq X \times I$ are open sets by assumption. Therefore, since a basis for the product topology is given by squares $V_1 \times V_2$, we can write $F^{-1}(U) = \bigcup_\alpha V_1^\alpha \times V_2^\alpha$, and similarly for $G^{-1}(U) = \bigcup_\beta W_1^\beta \times W_2^\beta$.

Since $F(x,2t) = G(x,2t-1)$ at $t = \frac{1}{2}$, and $H^{-1}(U) = S_F(F^{-1}(U)) \cup S_G(G^{-1}(U))$ where $S_F : I \to [0, \frac{1}{2}]$ and $S_G : I \to [\frac{1}{2}, 1]$ in the second coordinate, there is only one type of point to check: $(x, \frac{1}{2}) \in X \times I$. But in this case, in basis form, $x \in V_1^\alpha \times (\frac{1}{2} - \epsilon_\alpha, \frac{1}{2}]$ and $x \in W_1^\alpha \times [\frac{1}{2}, \frac{1}{2} + \epsilon_\beta)$. This implies $x \in V_1^\alpha \cap W_1^\beta \times (\frac{1}{2} - \epsilon_\alpha, \epsilon_\beta) \subseteq H^{-1}(U)$. Since we can do this for any point $x \in H^{-1}(U)$, we see that H is continuous.

Now that we have this tool, we can be adult mathematicians and freely combine homotopies together! (and never prove it again :)

The same type of thing applies when one wants to compare $S(X) \cong C(X) \cup C(X)$ as we did in class today, but with respect to topologies.

Another consideration was to compare the suspension of S^n with that of S^{n+1}. I believe it was reasonably convincing in class that they are in bijection, but can we further prove that they are homeomorphic?

First, the bijection part. For $\boldsymbol{x} = (x,t) = (x_1, \ldots, x_{n+1}, t) \in S(S^n) = S^n \times I/\sim$, consider the map $(x,t) \mapsto (c_t \cdot x_1, \ldots, c_t \cdot x_n, 2t-1)$, where $c_t = \sqrt{1-(2t-1)^2}$. This does the trick and should be thought of as a way to may the S^n part of the domain to a circle of radius r with $r^2 = 1^2 - (2t-1)^2$ by scaling.

The inverse of this map is given by $(x_1, \ldots, x_{n+2}) \mapsto (c_x x_1, \ldots, c_x x_{n+1}, \frac{x_{n+2}+1}{2})$, where $c_x = \sqrt{1 - x_{n+2}^2}$ (again, correcting the radius to 1).

Note that the composition $S^n \times I \to S(S^n) \to S^{n+1}$ is a continuous map (as written above) due to the way maps from a product are determined to be continuous (from class). This implies, by definition of the quotient topology, that the map constructed above is continuous.

The inverse map is continuous as follows: $U \subseteq S(S^n)$ is open if and only if $q^{-1}(U)$ is open in $S^n \times I$. These both have the Euclidean topology, and are isomorphisms away from $t = 0, 1$. So the inverse image of an open set away from $(x, 0)$ and $(x, 1)$ is clearly open. An open set containing $(x, 0)$ is a set containing $S^n \times \{0\}$ at these points. This is exactly what happens near these points in the Euclidean topology. Intersect an open ball with the sphere! So the inverse map is also continuous as all of the maps are continuous in the Euclidean setting.