CLASS 2, SEPTEMBER 11TH: COMPLEX FUNCTIONS

Last time we talk about the complex number system in various frames of reference. Today, we will study complex functions and how we can verify differentiability. First, we need to review a few notions from real analysis to produce a domain for our functions.

Definition 2.1. We define the **open ball** of radius r and centered at $z \in \mathbb{C}$ to be

$$B(z,r) = \{w \in \mathbb{C} \mid |w-z| < r\}$$

Similarly, we can define the **closed ball** with the same parameters:

$$\bar{B}(z,r) = \{w \in \mathbb{C} \mid |w-z| \leq r\}$$

Note that the book uses the notation $D_r(z)$ instead of $B(z,r)$ for a 'disc'. You may use either but my notation is used more commonly in analysis. Worthy of its own name and notation is the **unit disc** $\mathbb{D} = B(0,1)$.

This allows us to define the notion of an open and closed set in the complex plane:

Definition 2.2. If $\Omega \subseteq \mathbb{C}$ is a subset, $z \in \Omega$ is called an **interior point** if there exists $r > 0$ such that $B(z,r) \subseteq \Omega$.

Ω is **open** if every point is an interior point.

Ω is called **closed** if its complement $\Omega^c = \mathbb{C} \setminus \Omega$ is open.

An alternative characterization (agreeing with the situation of real analysis) of Ω being closed is as follows:

Proposition 2.3. *Ω is closed if and only if every every convergent sequence $z_n \to z$ in $\mathbb{C}$ with $z_n \in \Omega$ implies that $z \in \Omega$.*

Proof. $\Rightarrow$: If Ω is closed, Ω^c is open. If $z \in \Omega^c$, then $B(z,r) \subseteq \Omega^c$ for some $r > 0$. But then z_n cannot converge to z, since $z_n \notin B(z,r)$ (which implies $|z - z_n| \geq r > 0$).

$\Leftarrow$: Suppose Ω is not closed. Then there exists $z \in \Omega^c$ such that $B(z, \frac{1}{n}) \bigcap \Omega \neq \emptyset$ for each $n \in \mathbb{N}$. Choose $z_n \in B(z, \frac{1}{n}) \bigcap \Omega$. Then $z_n \to z$ but $z \notin \Omega$. $\square$

Finally, recall the definition of **bounded**:

Definition 2.4. $\Omega \subseteq \mathbb{C}$ is called **bounded** if there exists $z \in \mathbb{C}$ and $r > 0$ such that

$$\Omega \subseteq B(z,r).$$

As a result, we have the following equivalence from real analysis:

Theorem 2.5. *Let $\Omega \subseteq \mathbb{C}$. Then TFAE (The Following Are Equivalent):*

1) *If U_α is any collection of open sets* **covering** *Ω, i.e. $\Omega \subseteq \bigcup_\alpha U_\alpha$, then there exists $\alpha_1, \ldots, \alpha_n$ such that $\Omega \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ is a finite subcover.*
2) *If $z_n \subseteq \Omega$, then there exists a subsequence z_{n_i} which converges.*
3) *Ω is complete and* **totally bounded**.
4) *Ω is closed and bounded.*

Definition 2.6. Ω satisfying any of the equivalent conditions of Theorem 2.5 is **compact**.

Note condition 2) is often called **sequential compactness**. The first 3 conditions are equivalent for any metric space. 4) is strictly weaker in the infinite dimensional cases.

We can now pivot to the notion of a continuous function. This is exactly as it was in real analysis:

Definition 2.7. If $\Omega \subseteq \mathbb{C}$, and $f : \Omega \to \mathbb{C}$ is a function, then f is said to be **continuous at** $z \in \Omega$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $w \in \Omega$ with $|w - z| < \delta$, necessarily $|f(w) - f(z)| < \epsilon$.

f is **continuous** if f is continuous at z for every $z \in \Omega$.

There are some immediate and convenient rephrasings of this statement in terms of preimages. Recall that if $\Gamma \subseteq \mathbb{C}$, then

$$f^{-1}(\Gamma) = \{z \in \Omega \mid f(z) \in \Gamma\}$$

This should NOT be confused as an inverse function!

Theorem 2.8. *Let $f : \Omega \to \mathbb{C}$ be a function. TFAE:*

0) If $x_n \to x$ in Ω, then $f(x_n) \to f(x)$.
1) f is continuous.
2) For every $z \in \Omega$ and $\epsilon > 0$, there exists $\delta > 0$ such that $f(B(z, \delta)) \subseteq B(f(z), \epsilon)$.
3) For every $z \in \Omega$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B(z, \delta) \subseteq f^{-1}(B(f(z), \epsilon))$.

If Ω is itself open, these conditions are equivalent to the following:

5) If $U \subseteq \mathbb{C}$ is an open set, then $f^{-1}(U)$ is an open set.

Proof. Sketch. For 0) implies 1), it is convenient to prove the contrapositive. If f is not continuous at x, there exists ϵ such that no $\delta > 0$ has the property that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Choose x_n to be y violating this property with $\delta = \frac{1}{n}$. 2) is exactly the statement of 1) but with set notation. Applying f^{-1} to the inclusion in 2) yields 3) once you notice that

$$\Gamma \subseteq f^{-1}(f(\Gamma))$$

is true for any set Γ. 3) implies 4) is a result of the fact that preimages and unions can be interchanged:

$$\bigcup_\alpha f^{-1}(\Gamma_\alpha) = f^{-1}(\bigcup_\alpha \Gamma_\alpha)$$

(here let $U = \bigcup_\alpha \Gamma_\alpha$ where each Γ_α is an open ball). 4) implies 0) is simply established by considering $f^{-1}(B(f(x), \epsilon)$ for any given ϵ! □

For context, 4) is the definition of continuity given outside of the context of metric spaces. A useful thing to recover from real analysis is the **extreme value theorem**:

Theorem 2.9. *If Ω is a compact set and $f : \Omega \to \mathbb{C}$ is a continuous function, then Ω obtains its maximum and minimum with respect to absolute value.*

Proof. We can note that the image of a compact set is compact (this is easiest to see with the open cover definition of compactness). But this implies that $f(\Omega)$ is a closed and bounded set. Boundedness then yields an $R > 0$ such that $f(\Omega) \subseteq \bar{B}(0, R)$. Choose R minimally satisfying this condition. If $z_n \in \Omega$ is a convergent sequence such that $|z_n| \to R$, then closedness of Ω implies its limit z is in Ω. Thus maxima are achieved.

If $0 \in \Omega$, we are done. Otherwise, $\frac{1}{f}$ is a continuous function on Ω. Applying the previous result shows $\frac{1}{f}$ has a maximum, which in turn implies f has a minimum. □

Next time we will get into the idea of when a complex function is differentiable.