HOMEWORK 6: HILBERT-NULLSTELLENSATZ
DUE: APRIL 8TH

1) Given 2 R-submodules $M_1, M_2 \subseteq N$, is it true that
$$W^{-1}(M_1 + M_2) \cong W^{-1}M_1 + W^{-1}M_2?$$

Solution: Indeed it is true. Recall the SES
$$0 \to M_1 \cap M_2 \to M_1 \oplus M_2 \to M_1 + M_2 \to 0$$
Localizing at W produces
$$0 \to W^{-1}(M_1 \cap M_2) \to W^{-1}(M_1 \oplus M_2) \to W^{-1}(M_1 + M_2) \to 0$$
We have shown that $W^{-1}(M_1 \cap M_2) \cong W^{-1}M_1 \cap W^{-1}M_2$. Additionally, localizing the standard split exact sequence yields $W^{-1}(M_1 \oplus M_2) = W^{-1}M_1 \oplus W^{-1}M_2$. As a result, this short exact sequence can be replaced by
$$0 \to W^{-1}M_1 \cap W^{-1}M_2 \to W^{-1}M_1 \oplus W^{-1}M_2 \to W^{-1}(M_1 + M_2) \to 0$$
Comparing this with the original sequence, we can conclude $W^{-1}(M_1 + M_2) \cong W^{-1}M_1 + W^{-1}M_2$.

2) Give an example of 2 multiplicative sets W_1 and W_2 which are distinct but $W_1^{-1}R \cong W_2^{-1}R$. Show that there exists a unique largest such multiplicative set $W \supseteq W_0$ for a given localization $W_0^{-1}R$.

Solution: For the first part, the easiest example is $W_1 = \{1, f, f^2, f^3, \ldots\}$ and $W_2 = \{1, f^2, f^4, \ldots\}$. The isomorphism is then given by
$$W_1^{-1}R \to W_2^{-1}R : (f^n, r) \mapsto (f^{2n}, rf^n)$$
The inverse is obtained by inclusion.

For the second part, we use Zorn's Lemma. Let $\mathcal{S}$ be the collection of all multiplicative sets whose localization is $W_0^{-1}R$. $\mathcal{S} \neq \emptyset$ since $W_0 \in \mathcal{S}$. Given an ascending chain
$$W_1 \subseteq W_2 \subseteq \ldots$$
in $\mathcal{S}$, we have $W_\infty = \cup_{i=1}^{\infty} W_i$ a multiplicative set. Clearly $W_\infty^{-1}R = W_0^{-1}R$, since every fraction on the right has denominator in some W_i, and we can choose our isomorphisms $W_i^{-1}R \cong W_0^{-1}$ compatibly with the localization maps $W_i^{-1} \to W_{i+1}^{-1}$. Therefore a maximal element exists in $\mathcal{S}$. Call it W. If W' is a multiplicative set in $\mathcal{S}$ not contained within W, then we can form
$$W \cdot W' = \{w \cdot w' \mid w \in W,\ w' \in W'\}$$
This set contains W properly and localizes to $W_0^{-1}R$:
$$(ww', r) = (w, 1) \cdot (w', r) \sim (w_0, 1) \cdot (w_0', r_0) = (w_0 w_0', r_0)$$
for some $w_0, w_0' \in W_0$, $r_0 \in R$. This contradicts maximality of W.

3) If $A = A' \times A''$ is decomposable as a product of 2 rings, show that A' and A'' are localizations of A.

Solution: For the first part, it suffices to check this for A'. We can localize at the very simple multiplicative set $W = \{(1,1),(1,0)\}$. In this case, we note that by Example 17.3 in the Notes,

$$\ker(R \to W^{-1}A) = \{a \in A \mid a \cdot (1,0) = 0\} = 0 \times A''$$

Additionally, the map is surjective as to acquire $((1,0),(a',a''))$ we have the natural choice of $a' \in A' \subseteq A$. This shows $W^{-1}A = A'$.

4) Assume $2 \neq 0$ in K (i.e. *char*$(K) \neq 2$). Given the ideal $J = \langle x+y, (x-y)^2 \rangle \subseteq K[x,y]$, show that $x \in I(V(J)) = \sqrt{J}$ but $x \notin J$.

Solution: $x - y \in \sqrt{J}$, so $x + y + x - y = 2x \in \sqrt{J}$. Since $2 \neq 0$, it is a unit, so $x \in \sqrt{J}$. To show $x \notin J$, suppose to the contrary that it is. Then

$$x = f(x+y) + g(x-y)^2 = fx + fy + gx^2 + 2gxy + gy^2$$

Comparing x-degrees, we need f to have constant coefficient 1. But this forces a y to appear on the right hand side which can't be canceled by any other term.

5) Recall the result of Hilbert-Nullstellensatz:

Theorem 0.1 (Hilbert-Nullstellensatz)**.** *Assume K is an algebraically closed field. If $J \subsetneq K[x_1,\dots,x_n]$ is an ideal, then $V(J) \neq \emptyset$. Furthermore, $I(V(J)) = \sqrt{J}$.*

Show that this is true if and **only if** K is algebraically closed.

Solution: If K is not algebraically closed, we can consider

$$K[x_1,\dots,x_n] \subseteq \bar{K}[x_1,\dots,x_n]$$

We can pick $\mathfrak{m}$ to correspond to a point of $\bar{K}^n \setminus K^n$. Then we know $I(V(\mathfrak{m})) = \sqrt{\mathfrak{m}} = \mathfrak{m}$. Therefore

$$\mathfrak{m} \cap K[x_1,\dots,x_n]$$

is a maximal ideal for which $V(\mathfrak{m} \cap K[x_1,\dots,x_n]) = \emptyset$. As a result, $I(V(\mathfrak{m} \cap K[x_1,\dots,x_n]))$ is the set of functions vanishing at the emptyset, which is an empty condition, so you get $I(V(\mathfrak{m} \cap K[x_1,\dots,x_n])) = K[x_1,\dots,x_n]$.

6) Explain why Hilbert-Nullstellensatz is sometimes referred to as the 'Generalized Fundamental Theorem of Algebra'.

Solution: The first part of the statement says that any proper ideal has non-empty vanishing set for an algebraically closed field. In the case where $n = 1$, this is the statement that if $I = \langle f \rangle$, then f has a root α. This is equivalent to being divisible by $x - \alpha$. To see this, proceed by induction on degree. Degree 1 is obvious. For the degree n case, the division algorithm yields

$$f = s(x - \alpha) + r$$

but r has degree less than n, thus must itself be divisible by $x - \alpha$, implying f is.

7) Show that if f is irreducible in $K[x_1, \ldots, x_n]$, and f doesn't divide g, then $V(f) \not\subseteq V(g)$. As a result, if $X = V(g)$ and $g = u \cdot f_1^{e_1} \cdots f_m^{e_m}$ where f_i are irreducible and u is a unit, then

$$V(g) = V(f_1) \cup \cdots \cup V(f_m)$$

This is a decomposition of a hypersurface into **irreducible components**.

Solution: Assume to the contrary. Given f is irreducible, it generates a prime ideal since $K[x_1, \ldots, x_n]$ is a UFD. Thus we have that

$$\langle f \rangle = I(V(f)) \supseteq I(V(g)) = \sqrt{\langle g \rangle} \supseteq \langle g \rangle$$

This is only possible if $g = f \cdot h$ for some h, and therefore means f divides g, contradicting our hypothesis.

The *as a result* statement follows directly from the fact that

$$V(I \cdot J) = V(I) \cup V(J)$$
$$\langle g \rangle = \langle f_1^{e_1} \rangle \cdots \langle f_m^{e_m} \rangle.$$