HOMEWORK 3: QUOTIENTS AND CONNECTEDNESS
DUE: FRIDAY, SEPTEMBER 28

1) Suppose $f : X \to Y$ is a continuous map. If there exists $g : Y \to X$ a continuous map such that $f \circ g$ is the identity map on Y, show that f is a quotient map.
2) If $A \subseteq X$, $r : X \to A$ a continuous map is said to be a **retraction** if $r(a) = a$ for every $a \in A$. Show that r is a quotient map.
3) Consider the map $\pi : \mathbb{R}^2 \to \mathbb{R} : (x, y) \mapsto x$. Let $A \subseteq \mathbb{R}^2$ be the set of points such that $x \geq 0$ or (inclusive) $y = 0$. Show that the restricted map $f : A \to \mathbb{R}$ is a quotient map that is neither open nor closed.
4) If $X_1, X_2, \ldots \subseteq X$ are a sequence of connected subspaces, such that $X_i \cap X_{i+1} \neq \emptyset$, show that $\bigcup_{i=1}^{\infty} X_i$ is connected.
5) If $p : X \to Y$ is a quotient map, with Y connected and $p^{-1}(y) \subseteq X$ connected for each $y \in Y$, then X is connected.
6) Let $Y \subseteq X$ be two connected topological spaces. Suppose that $X \setminus Y$ is disconnected with a separation by A, B. Show that $Y \cup A$ and $Y \cup B$ are connected.[1]
7) Show any infinite cardinality space X with the finite complement topology is connected.[2]
8) We have shown that $X = \mathbb{R}^{\mathbb{N}}$ with the product topology is connected, whereas X with the box topology is not. Consider the **uniform topology**; define $d(\mathbf{x}, \mathbf{y}) = \sup\{d_i(x_i, y_i)\}$, where $d_i(a, b) = \min\{|x_i - y_i|, 1\}$ is the *standard bounded metric.* The induced metric topology is the uniform topology, which by 20.4 of Munkres is finer than the product yet coarser than the box topology.

   Prove or disprove that X with the uniform topology is connected.[3]

[1]This result is very helpful for identifying connected subsets of $\mathbb{R}^n$. As a **hint**, suppose $Y \cup A = U \cup V$. Apply Prop 8.4 from class to Y and U WLOG. Then show that V and $U \cup B$ separate X.

[2]This mildly generalizes 'every space is connected with the trivial/indiscrete topology'.

[3]**Hint:** Are the sets we used for the box topology open in the uniform topology?