CLASS 18, OCTOBER 25: THE ARGUMENT PRINCIPLE

Today we will discuss in more depth the idea of the logarithm in the complex world. We call log a ‘multivalued function’, since it can’t be defined unambiguously on all of $\mathbb{C} \setminus \{0\}$ unless we allow ourselves to work $\pmod{2\pi}$ in the imaginary axis. But if it does exist, then it would need to behave as we have mentioned previously:

$$\log(f(z)) = \log|f(z)| + i\arg(f(z))$$

for $f(z)$ living in some particular range. In such a case, we still have

$$\frac{\partial}{\partial z}\log(f(z)) = \frac{f'(z)}{f(z)}$$

As such, the integral of $\frac{f'(z)}{f(z)}$ represents the rate of change of the argument of $f(z)$.

Example 18.1. If $\gamma(t) = e^{it}$ for $t \in [a, b]$, then

$$\int_\gamma \frac{dz}{z} = i(b-a)$$

One can also observe that

$$\log(f_1 f_2) = \log(f_1) + \log(f_2)$$

where the equation makes sense due to the fact that

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

Suppose f has a zero of order m at z_0. Then $f(z) = (z-z_0)^m g(z)$ is such that $g(z)$ is non-vanishing near z_0. As a result,

$$\frac{f'(z)}{f(z)} = \frac{m}{z-z_0} + \frac{g'(z)}{g(z)}$$

As a result, if f has a zero of order m, then $\frac{f'(z)}{f(z)}$ also has a simple pole at z_0 with residue m. We can conclude the same but with a negative sign if f has a pole of order m, so we derive the formula

$$\mathrm{ord}_{z_0}(f) = \mathrm{res}_{z_0}\left(\frac{f'}{f}\right)$$

This reasoning yields the following result:

Theorem 18.2 (The Argument Principle)**.** *Suppose that f is a meromorphic function in Ω, and C is a simple positively oriented loop in its interior. If f has no poles nor zeroes on C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = (\#\text{ of zeroes in } C) \ - \ (\#\text{ of poles in } C)$$

where the # is counted with its order (sometimes called multiplicity).

This result is very impressive in the sense that it is very easy to compute integrals of this form. This yields even more impressive corollaries. The first we will tackle is Rouché's theorem, which gives a method to perturb a given holomorphic function without changing the number of zeroes inside a region.

Theorem 18.3 (Rouché's theorem). *Suppose f and g are holomorphic in an open set Ω, and C is a simple positively oriented curve with interior inside Ω. If*

$$|f(z)| > |g(z)| \qquad \forall z \in \mathbb{C}$$

then both f and $f+g$ have the same number of zeroes (counted with multiplicity) in C.

This may not be terribly surprising, but one should notice that we are only asking for the inequality **on** C, not within it!

Proof. We will consider this as a *deformation*. Let

$$f_t(z) = f(z) + tg(z)$$

for $t \in [0,1]$. Let n_t denote the number of zeroes of f_t in C. Our condition also yields that f_t has no zeroes on C, so Theorem 18.2 produces

$$n_t = \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}dz$$

To prove n_t is constant, it suffices to check that the RHS is a continuous function of t. Note that $\frac{f_t'(z)}{f_t(z)}$ is a jointly continuous function for t and $z \in C$, since the same is true for the numerator and denominator and the denominator is never 0.

As a result, the same is true for the integral since C is compact. Therefore, n_t is a continuous integer valued function of t, but the only way that is possible is if n_t is constant, which shows that $n_0 = n_1$. □

Example 18.4. Let's determine the number of zeroes of the polynomial $p(z) = z^7 - 2z^3 + 7$ inside $B(0,2)$. Notice that we can break up $p(z)$ as

$$p(z) = (z^7) + (-2z^3 + 7) = f(z) + g(z)$$

Now notice that $|f(z)| = |z|^7 = 2^7 = 128$, whereas $|g(z)| = |2z^3 - 7| \leq 2|z|^3 + 7 = 2^4 + 7 = 23$. So these two functions satisfy the conditions of Theorem 18.3, and therefore p shares the same number of zeroes as $f = z^7$. This is clearly just a single zero at 0 of order 7.

But by the fundamental theorem of algebra, we know that p has **exactly** 7 zeroes in $\mathbb{C}$, and thus we just showed that all of them are in $B(0,2)$. Further refinements can be made: for example, if we choose $r = 1.5$, then $r^7 > 17$ and $2r^3 + 7 = 13.75$, so really they exist within $B(0,1.5)$.

If you've ever studied Galois theory or more specifically generalizations of the quadratic formula, then you know that we have formulae to find the zeroes up degree 4 polynomials, and beyond that no formula *can* exist. As a result, the actual zeroes of $z^7 - 2z^3 + 7$ are difficult to find. Computer estimates are

$$z \approx -1.443, \quad -.71 - .98i, \quad -.71 + .98i, \quad .213 - 1.39i, \quad .213 + 1.39i$$

The fact that there are 5 distinct roots is not detectable by this method of course, as we count *with* multiplicity.