HOMEWORK 4: NAKAYAMA'S LEMMA AND REGULARITY DUE: WEDNESDAY, APRIL 11

1) Let M and N be finitely generated modules over a ring R. Show that $M \otimes_R N = 0$ if and only if $ann_R(M) + ann_R(N) = R$. In addition, if we assume R is local, show that $M \otimes_R N = 0$ implies either $M = 0$ or $N = 0$.

Solution: The $\Leftarrow$ is easy to see: If $ann_R(M)+ann_R(N) = R$, then $r_M+r_N = 1$. Therefore,

$$1 \cdot m \otimes n = (r_M + r_N)m \otimes n = r_M m \otimes n + m \otimes r_N n = 0$$

For $\Rightarrow$, it suffices to prove the local version of the statement. This is because if $ann_R(M) + ann_R(N) \neq R$, we can take a maximal ideal containing the left side, localize it, and this would contradict the assumption that $M = 0$ or $N = 0$.

So we can assume $(R, \mathfrak{m})$ is local. If $M \neq 0$, then by Nakayama's lemma, we have $M/\mathfrak{m}M \neq 0$. But this is a $k = R/\mathfrak{m}$ vector space, so we have a (many) surjection $M/\mathfrak{m}M \to k$. Therefore, since $\otimes$ is right exact, we have a surjection

$$0 = M \otimes_R N \to M/\mathfrak{m}M \otimes_R N \to k \otimes_R N$$

Therefore, $k \otimes_R N = 0$. But this is exactly $N/\mathfrak{m}N$. By Nakayama's lemma, we note $N = 0$. This completes the proof.

2) Show that $Jac(R)$, the Jacobson radical of R, can be characterized as

$$Jac(R) = \{r \in R \mid 1 + rr' \text{ is a unit for every } r' \in R\}$$

Solution: Suppose that $1 - rr'$ is not a unit for some $r' \in R$. Then $1 - rr' \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Therefore, if $r \in Jac(R)$, we know that $rr' \in Jac(R)$ and thus $1 \in Jac(R)$. This is clearly impossible. So LHS $\subseteq$ RHS.

For the other direction, suppose $x \notin Jac(R)$. Then $x \notin \mathfrak{m}$ for some $\mathfrak{m}$ maximal. Therefore, $\mathfrak{m}+\langle x \rangle = R$ and $1 = m+rx$ for some $m \in \mathfrak{m}$ and $r \in R$. But $m = 1-rx$ cannot be a unit (since otherwise it would generate R).

3) Let $(R, \mathfrak{m})$ be a local ring, and $I \subseteq R$ an ideal. Suppose that x is an element of $\mathfrak{m}$ is such that its image in R/I is a non-zero divisor. Show that a minimal generating set of I is also a minimal generating set of $I \cdot R/\langle x \rangle$. Give an example to show this is not true when x is allowed to be a zero divisor of R/I.[1]

Solution: It goes to show that if $\langle i_1, \ldots, i_n \rangle = I$ is minimal, then $\langle \bar{i}_1, \ldots, \bar{i}_n \rangle$ is minimal for $I \cdot R/\langle x \rangle$. Suppose WLOG that

$$\bar{i}_1 = r_2\bar{i}_2 + \ldots + r_n\bar{i}_n + \langle x \rangle$$

Then

$$i_1 = r_2 i_2 + \ldots + r_n i_n + rx$$
$$rx = i_1 - (r_2 i_2 + \ldots + r_n i_n)$$

[1]Recall a minimal generating set is one in which removing any element stops generation.

But this implies that $\bar{r}\bar{x} = 0 \in R/I$. Therefore, $\bar{r} = 0$ or equivalently, $r \in I$ (since $\bar{x}$ is a non-zero divisor). Therefore, we can express $r = r_1' i_1 + \ldots + r_n' i_n$. Rewriting the previous equation, we see that

$$(1 - xr_1)i_1 = (r_2 + r_2' x)i_2 + \ldots + (r_n + r_n')i_n$$

But since $x \in \mathfrak{m}$, by problem 2, $1 - xr_1$ is a unit. Therefore, $i_1 \in \langle i_2 \ldots i_n \rangle$. This contradicts minimality and proves the claim.

For the example, suppose we consider $R = K[x, y]$ and $I = \langle xy \rangle$. Choosing x, we see that $I \cdot R/\langle x \rangle$ is simply 0 and thus 0-generated.

4) Let R be a ring, and S an R-algebra. Finally, let M be a finitely generated S module. Show that if S is finite as an R-module, then M is finitely generated as an R-module.

Solution: Suppose that $S = Rs_1 + \ldots + Rs_n$ and $M = \langle m_1, \ldots, m_l \rangle_S$. Given $m \in M$, we know

$$m = s_1' m_1 + \ldots + s_l' m_l = \sum_{i=1}^{l} s_i' m_i$$

But each s_i' lives in the R-span of s_j:

$$s_i' = r_{1i}s_1 + \ldots + r_{ni}s_n = \sum_{j=1}^{n} r_{ji}s_j$$

Therefore, $m = \sum_j \sum_i r_{ji}s_j m_i$. Therefore, our generators are $s_j m_i$ for all i, j.

5) Suppose that I is a nilpotent ideal (e.g. $I^n = 0$ for some $n \gg 0$. Show that if $M = IM$ for some (not necessarily finitely generated module M), then $M = 0$.[2]

Solution: Suppose $M = IM$. Then by induction, we see

$$I^n M = I^{n-1}(IM) = I^{n-1}M = M$$

But $I^n = 0$, so $I^n M = 0 = M$.

6) Consider the ring

$$R = K[x_{ij}]_{i \leq j} = K[x_{11}, x_{12}, x_{22}, x_{13}, \ldots]$$

and $W = R \setminus \langle x_{11} \rangle \cup \langle x_{12}, x_{22} \rangle \cup \langle x_{13}, x_{23}, x_{33} \rangle \cup \ldots$ Show that $W^{-1}R$ is a regular Noetherian ring, but is not finite dimensional.[3]

Solution: The prime ideals of $W^{-1}R$ are exactly the prime ideals of R which do not intersect W. Therefore, we see the prime ideals are exactly those contained in some $\langle x_{1j}, \ldots, x_{jj} \rangle$. Therefore, the maximal ideals of this ring are exactly those $\mathfrak{m}_j := \langle x_{1j}, \ldots, x_{jj} \rangle$. Localizing at one of these maximal ideals yields

$$(W^{-1}R)_{\mathfrak{m}_j} = R_{\mathfrak{m}_j} = L[x_{1j}, \ldots, x_{jj}]_{\langle x_{1j}, \ldots, x_{jj} \rangle}$$

where L is a field extension of K. The unique maximal ideal of this is $\langle x_{1j}, \ldots, x_{jj} \rangle$, and the dimension is j, thus regular. It is Noetherian since every chain lives in one

[2]This gives a nice extension of Nakayama to the non-finitely generated case.

[3]This is also an example of a non-equidimensional ring.

of these polynomial rings, which are themselves Noetherian. On the other hand, the dimension j of these localizations can be taken arbitrarily large. Therefore, it is not finite dimensional.

7) Show that if R is a local ring, and I is a proper ideal such that $gr_I(R)$ is an integral domain, then R is as well.

Solution: Let $f \cdot g = 0$ in R. Not that by the Krull intersection theorem, we know that $f, g \notin \mathfrak{m}^n$ for all $n \gg 0$, so we can send $R \to gr_I(R) : f \mapsto \bar{f}$. Take their image $\bar{f}, \bar{g}$ in $gr_I(R)$. Then $\bar{f} \cdot \bar{g} = 0$, because it is the image of 0 under a ring homomorphism. Therefore, $\bar{f} = 0$ or $\bar{g} = 0$. Now, applying Krull intersection theorem again, we know that if $f \in I^n$ for all n, $f = 0$. Therefore, either f or g was 0 to begin with.

8) Let R be a Noetherian ring, and let I be an ideal and M a finitely generated module. Show that there is a largest submodule $N \subseteq M$ where $(1-r)N = 0$ for some $r \in I$. Then show that $\cap_{n \geq 0} I^n M = N$.[4]

Solution: Note that this property is equivalent to saying $N = I \cdot N$. Let $\mathcal{S}$ be the set of all modules with this property. It is non-empty, because clearly the 0 submodule is in $\mathcal{S}$. If $N_1 \subseteq N_2 \subseteq \ldots$ is an ascending chain of submodules in $\mathcal{S}$, then if $N = \cup_{i \geq 0} N_i$, we have that for any $n \in N$, $n \in N_i$ for some i, so $n \in IN_i$. Therefore, $N = IN$ is in $\mathcal{S}$. Therefore, by Zorn's lemma, we know there is a maximal element of this set. Furthermore, since if $N, N' \in \mathcal{S}$ are maximal, $N + N' \in \mathcal{S}$ since

$$(1-r)(1-r')(N+N') = (1-(r+r'-rr'))(N+N') = 0$$

so the maximum is unique.

For the second portion, I will demonstrate the equality by showing 2 containments. Certainly $\cap_{n \geq 0} I^n M$ is in $\mathcal{S}$, since

$$I \cdot \cap_{n \geq 0} I^n M = \cap_{n \geq 1} I^n M = \cap_{n \geq 0} I^n M$$

by Artin-Rees, and N is the largest such module. On the other hand, if $n \in N$, we know that $n = rn$ for some $r \in I$, so $n \in IN$. But we can iterate this procedure and say $n = r^l n$ for any $l \gg 0$. Therefore $N \subseteq \cap_{n \geq 0} I^n M$. This completes the proof.

[4]This provides a partial converse to Krull's Intersection Theorem.