CLASS 15, MONDAY APRIL 2ND: NAKAYAMA'S LEMMA & APPLICATIONS

Nakayama's lemma is one of the most useful tools in commutative algebra. It gives a strong passage of local properties and global properties of a ring. Today, I will state the result, and use it to prove several slight extensions of it as well as some applications. We will prove the result (starting) next time.

Theorem 0.1 (Nakayama's Lemma). *If R is a local ring with unique maximal ideal $\mathfrak{m}$, and M is a finitely generated R-module, then $M = \mathfrak{m}M$ implies $M = 0$.*

Some equivalent formulations are as follows:

Theorem 0.2 (Nakayama's Lemma+). *If R is a local ring with unique maximal ideal $\mathfrak{m}$, $N \subseteq N$ are finitely generated R-modules, then $M = \mathfrak{m}M + N$ implies $M = N$.*

Proof. Given the setup, we know that $M = \mathfrak{m}M + N$. If we mod out by N, we see that

$$M/N = (\mathfrak{m}M + N)/N = \mathfrak{m}M/(N \cap \mathfrak{m}M) = \mathfrak{m}M/N$$

The second equality is by the 3rd module isomorphism theorem. By Nakayama, we have $M/N = 0$, or equivalently, $M = N$. □

Theorem 0.3 (Nakayama's Lemma++). *If R is a local ring with unique maximal ideal $\mathfrak{m}$ and M a finitely generated R-module. If $m_1, \dots, m_n \in M$ are such that $\langle \bar{m}_1, \dots, \bar{m}_n \rangle = M/\mathfrak{m}M$, then $\langle m_1, \dots, m_n \rangle = M$.*

Proof. Given the setup, we note that $M = \langle m_1, \dots, m_n \rangle + \mathfrak{m}M$. By Nakayama+, the result is implied directly. □

There is another formulation which gives a little more than Nakayama's Lemma.

Theorem 0.4 (Nakayama's Lemma+++). *If I is an ideal of R and M is a finitely generated module such that $IM = M$, then $\exists r \equiv 1 \pmod I$ such that $rM = 0$.*

Note that there is no local assumption here. If we take $I = \mathfrak{m}$, then $r = 1 + m$ for $m \in \mathfrak{m}$. So $r \cdot M/\mathfrak{m}M = 1 \cdot M/\mathfrak{m}M = 0$ which implies $M = \mathfrak{m}M$. We will prove this variant on Wednesday.

We can get around the local assumptions by replacing $\mathfrak{m}$ by the following object

Definition 0.5. The **Jacobson radical** of a ring R is

$$Jac(R) = \bigcap_{\mathfrak{m}\ max'l} \mathfrak{m}$$

We note that we can pass from a non-local ring to a local one via localization at $\mathfrak{p}$, sending M to $M_\mathfrak{p}$. Moreover, we can provide a partial inverse to this procedure as follows:

Proposition 0.6 (Locally zero modules are zero). *Let R be any ring, and M be any module. $M_\mathfrak{m} = 0$ for each maximal ideal $\mathfrak{m}$ if and only if $M = 0$.*

Proof. The localization of the zero module is certainly 0, since $M \otimes W^{-1}R = 0 \otimes W^{-1}R \cong 0$. Therefore it only goes to prove the $\Rightarrow$ direction. I will prove this statements contrapositive.

Suppose $M \neq 0$. Consider the set $Ann_R(m) = \{r \in R \mid rm = 0\}$. This is an ideal of R, and if we assume $m \neq 0$, this is a proper ideal since in particular it doesn't contain 1 (M is unital). Therefore, there exists a maximal ideal $\mathfrak{m}$ containing $Ann_R(m)$. I claim $(1, m) \neq 0$ in $M_\mathfrak{m}$. Indeed, otherwise

$$i(m - 1 \cdot 0) = i \cdot m = 0$$

for some $i \in R \setminus \mathfrak{m}$. This is impossible, since $i \notin Ann_R(m)$ by assumption. So $M_\mathfrak{m} \neq 0$. □

Therefore, if we are faced with a situation where $M = Jac(R)M$, we can localize at each maximal ideal and see

$$M_\mathfrak{m} \supseteq \mathfrak{m}M_\mathfrak{m} \supseteq Jac(R)M_\mathfrak{m} \supseteq M_\mathfrak{m}$$

Therefore $M_\mathfrak{m} = \mathfrak{m}M_\mathfrak{m}$, which implies $M_\mathfrak{m} = 0$ by Nakayama's lemma, and therefore $M = 0$ by Proposition 0.6.

One other neat application is the following, which is known in general due to Vasconcelos.

Proposition 0.7. *If $\varphi : M \to M$ is a surjective R-module homomorphism, then it is also injective.*

This is very similar to the case of finite dimensional vector spaces.

Proof. We can give M the structure of an $R[x]$-module by allowing x to act by φ:

$$(r_n x^n + \ldots + r_1 x + r_0)m := r_n\varphi^n(m) + \ldots + r_1\varphi(m) + r_0$$

The surjectivity assumption is stating that $I = \langle x \rangle$ has the property that $M = IM$. Nakayama+++ now implies that $\exists p(x) \in R[x]$ such that $1 - p(x) = x \cdot q(x)$. Since $p(x) \cdot m = 0$ for every $m \in M$, we note that $x \cdot q(x)m = m$. Therefore, $x \cdot m = \varphi(m) \neq 0$ for every $m \in M$. (MAGIC!) □

As a final remark, I want to add a nice note about Projective modules:

Theorem 0.8. *Let R be Noetherian and P be a finitely generated R-module. Then P is a projective module if and only if $P_\mathfrak{m}$ is a free $R_\mathfrak{m}$-module for every maximal ideal $\mathfrak{m}$. In this case, P is called* **locally free***.*

Proof. P is projective if and only if $P \oplus P' \cong R^n$. Localizing at $\mathfrak{m}$, we can then quotient by $\mathfrak{m}$:

$$P_\mathfrak{m}/\mathfrak{m}P_\mathfrak{m} \oplus P'_\mathfrak{m}/\mathfrak{m}P'_\mathfrak{m} \cong (R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m})^n \cong (R/\mathfrak{m})^n$$

Therefore, the RHS is a vector space, and we can produce a basis with $n - m$ elements $\bar{p}_1, \ldots, \bar{p}_{n-m}$ of $P_\mathfrak{m}/\mathfrak{m}P_\mathfrak{m}$ and m elements of $P'_\mathfrak{m}/\mathfrak{m}P'_\mathfrak{m}$. As a result of Nakayama++, we see lifts $p_1, \ldots, p_{n-m}$ generate $P_\mathfrak{m}$ (similarly for $P'_\mathfrak{m}$). This shows $P'_\mathfrak{m} \cong R_\mathfrak{m}^{n-m}$.

On the other hand, let P be locally free. If $M \to N$ is a surjection, consider

$$\mathrm{Hom}_R(P, M) \xrightarrow{\psi} \mathrm{Hom}_R(P, N) \to \mathrm{coker}(\psi)$$

Localizing at each maximal ideal $\mathfrak{m}$, we see

$$\mathrm{Hom}_{R_\mathfrak{m}}(R_\mathfrak{m}^n, M_\mathfrak{m}) \xrightarrow{\psi} \mathrm{Hom}_R(R_\mathfrak{m}^n, N_\mathfrak{m}) \to \mathrm{coker}(\psi)_\mathfrak{m} = 0$$

Since $R_\mathfrak{m}^n$ is free (projective) as an $R_\mathfrak{m}$-module. By Proposition 0.6, we see $\mathrm{coker}(\psi) = 0$, and thus the desired surjectivity holds! □