CLASS 6, SEPTEMBER 19: CONTINUITY II

Now that we have defined continuous functions, some properties and related notions can be demonstrated.

Theorem 6.1. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following are equivalent (forevermore, TFAE):*

1) f is continuous
2) If $Z \subseteq Y$ is a closed set, $f^{-1}(Z)$ is closed.
3) For any subset $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.
4) For each $x \in X$ and neighborhood $f(x) \subseteq V \subseteq Y$, we can find a neighborhood $U \subseteq X$ with $x \in U$ and $f(U) \subseteq V$.

Condition 4) is markedly similar to the case of metric spaces.

Proof. 1) $\Rightarrow$ 2): Note that $(f^{-1}(Z^c))^c = f^{-1}(Z)$. This follows since every point of X must map to either Z or Z^c. Therefore, since Z^c is open, continuity of f implies $f^{-1}(Z^c)$ is open, and thus $f^{-1}(Z)$ is closed.

2) $\Rightarrow$ 3): By 2), we realize that $f^{-1}(\overline{f(A)})$ is a closed set. Moreover, as previously mentioned, for *any* set

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)}).$$

Taking closures, we see

$$\overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)}).$$

or

$$f(\bar{A}) \subseteq f(f^{-1}(\overline{f(A)})) \subseteq \overline{f(A)}.$$

3) $\Rightarrow$ 4): Given V is a neighborhood of $f(x)$, we know that it contains an open set V' containing $f(x)$. Therefore, $f(x) \notin \overline{V^c}$. Applying 3) to $f^{-1}(V^c)$, we see that

$$f(\overline{f^{-1}(V^c)}) \subseteq \overline{f(f^{-1}(V^c))} \subseteq \overline{V^c} \subseteq V'^c$$

Finally, since $\overline{f^{-1}(V^c)}$ contains all points (and potentially more) mapping to V^c, letting $U = (\overline{f^{-1}(V^c)})^c$ produces the desired neighborhood.

4) $\Rightarrow$ 1): Take $V \subseteq Y$ to be open. If $x \in X$ is such that $f(x) \in V$, we can find a neighborhood $U_x \subseteq X$ containing x such that $f(U_x) \subseteq V$. Since it is a neighborhood, its interior also contains x, allowing us to write

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x^\circ$$

a union of open sets, which is thus open.

$\square$

Recall that a function $f : X \to Y$ is said to be **bijective** if for every $y \in Y$, there exists (surjective) a unique (injective) point $x \in X$ such that $f(x) = y$. We give a special name to spaces to a collection of such special continuous maps which plays a similar role as isomorphisms in group theory (or algebra generally).

Definition 6.2. A continuous bijective map $f : X \to Y$ is said to be a **homeomorphism** if f^{-1} is also continuous.

Using some of the properties of functions demonstrated so far, we can go further with more assumptions:

Proposition 6.3.
- *If $f : X \to Y$ is a surjective map, then $f(f^{-1}(S)) = S$ for any subset $S \subseteq Y$.*
- *If $f : X \to Y$ is an injective map, then $f^{-1}(f(S)) = S$ for any subset $S \subseteq X$.*
- *A bijective continuous map $f : X \to Y$ is a homeomorphism if and only if f is open if and only if f is closed.*

Proof.
- We always have $f(f^{-1}(S)) \subseteq S$. If $s \in S$, then there exists $x \in X$ such that $f(x) = s$. Of course, since $x \in f^{-1}(s) \subseteq f^{-1}(S)$, we see $s \in f(f^{-1}(S))$.
- We always have $f^{-1}(f(S)) \supseteq S$. If $s \in f^{-1}(f(S))$, then $f(s) \in f(S)$. But there is only one $x \in X$ mapping to any give $y \in Y$, which implies $s \in S$.
- This is a combination of the two previous statements, which together imply $(f^{-1})^{-1}(U) = f(U)$. If U was open (resp closed) then so is one of the above equal sets by one of the given assumptions on f^{-1}.

Example 6.4. Consider the mapping $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ with the metric topology. This is a homeomorphism because it is continuous and $\tan^{-1}$ is its continuous inverse. Equivalently, $\tan((a, b)) = (\tan(a), \tan(b))$, which is open.

This yields a nice more general result: A continuous function $f : \mathbb{R} \to \mathbb{R}$ which is strictly increasing or decreasing is a homeomorphism onto its image.

$\square$

Example 6.5. Examples of continuous bijective functions which are not homeomorphisms are easy to construct. Take $Id : (X, \tau) \to (X, \tau')$ where τ is strictly finer than τ'. Then the map is continuous but not open.

Next is a list of easy to show properties of continuous functions:

Proposition 6.6.
1) *If $f : X \to Y$ and $g : Y \to Z$ are two continuous functions, then so is $g \circ f$.*
2) *If $f : X \to Y$ is continuous and $A \subseteq X$ has the subspace topology, so is $f|_A : A \to Y$.*
3) *If $X = \bigcup_\alpha U_\alpha$ where U_α are open sets, then $f : X \to Y$ is continuous if and only if $f|_{U_\alpha} : U_\alpha \to Y$ is for all α.*
4) **Pasting Lemma:** *If $X = A \cup B$ for A and B open (or closed), and $f : A \to Y$ and $g : B \to Y$ are two continuous functions which agree on $A \cap B$, then so is the piecewise function*

$$h = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$