CLASS 25, NOVEMBER 11TH: POISSON SUMMATION FORMULA

Last time we proved that the Fourier inversion holds for functions in the class $\mathcal{F}$. This was done through the method of contour integration, which as we know is a powerful technique that we've built up for the entire semester. Today we will again institute these methods to prove the wonderful Poisson summation formula.

Theorem 25.1 (Poisson Summation Formula)**.** *If $f \in \mathcal{F}$, then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

Proof. First note that by our previous estimates and assumptions, both sums converge! Suppose $f \in \mathcal{F}_a$. We can consider the function $g(z) = \frac{f(z)}{e^{2\pi i z}-1}$, which has simple poles at the integers with residues $\frac{f(n)}{2\pi i}$. If $0 < b < a$, we can consider the rectangle R_N of height $2b$ and of length $2N+1$ centered at the origin. Note that this encompasses the poles $-N, -N+1, \dots, N$. Thus we have

$$\sum_{n=-N}^{N} f(n) = \int_{R_N} g(z)dz$$

Sending N off to infinity, we get

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{-\infty-ib}^{\infty-ib} g(z)dz - \int_{-\infty+ib}^{\infty+ib} g(z)dz$$

Now we will use the identity that if $|w| > 1$, we have

$$\frac{1}{w-1} = \frac{\frac{1}{w}}{1-\frac{1}{w}} = w^{-1}\sum_{n=0}^{\infty} w^{-n}$$

Applying this to $\frac{1}{e^{2\pi i z}-1}$ in the first integral, we produce

$$\int_{-\infty-ib}^{\infty-ib} g(z)dz = \int_{-\infty-ib}^{\infty-ib} f(z)e^{-2\pi i z}\sum_{n=0}^{\infty} e^{-2\pi i n z}dz$$

Similarly, using the more standard $\frac{1}{w-1} = -\sum_{j=0}^{\infty} w^n$ produces:

$$\int_{-\infty+ib}^{\infty+ib} g(z)dz = -\int_{-\infty+ib}^{\infty+ib} f(z)\sum_{n=0}^{\infty} e^{2\pi i n z}dz$$

So in total, we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} f(n) &= \int_{-\infty-ib}^{\infty-ib} f(z)e^{-2\pi iz}\sum_{n=0}^{\infty} e^{-2\pi inz}dz + \int_{-\infty+ib}^{\infty+ib} f(z)\sum_{n=0}^{\infty} e^{2\pi inz}dz \\ &= \sum_{n=0}^{\infty}\int_{-\infty-ib}^{\infty-ib} f(z)e^{-2\pi iz}e^{-2\pi inz}dz + \sum_{n=0}^{\infty}\int_{-\infty+ib}^{\infty+ib} f(z)e^{2\pi inz}dz \\ &= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f(z)e^{-2\pi i(n+1)z}dz + \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f(z)e^{2\pi inz}dz \\ &= \sum_{n=-\infty}^{-1}\int_{-\infty}^{\infty} f(z)e^{2\pi inz}dz + \sum_{n=0}^{\infty}\int_{-\infty}^{\infty} f(z)e^{2\pi inz}dz\end{aligned}$$

The last term is the desired $\sum_{n=-\infty}^{\infty}\hat{f}(n)$. Note that the switching of $\sum$ and $\int$ again requires absolute convergence (or Fubini-Tonelli). And we can move back to the real line by the equality in the proof of Theorem 23.4. $\square$

Example 25.2. Recall we have proven that if $f(z) = e^{-\pi z^2}$, then $\hat{f}(\xi) = e^{-\pi\xi^2}$. If we do the change of variables $x \mapsto t^{\frac{1}{2}}(x+a)$ (where $t > 0$ and $a \in \mathbb{R}$), then the result its that the Fourier transform of $f(z) = e^{-\pi t^{\frac{1}{2}}(z+a)^2}$ is $\hat{f}(\xi) = t^{-\frac{1}{2}}e^{-\frac{\pi\xi^2}{t}}e^{2\pi ia\xi}$.

Now, looking at Theorem 25.1, we get the following result:

$$\sum_{n=-\infty}^{\infty} e^{-\pi t(n+a)^2} = \sum_{n=-\infty}^{\infty} t^{-\frac{1}{2}}e^{-\frac{\pi n^2}{t}}e^{2\pi ina}$$

This has some noteworthy consequences in number theory and algebraic geometry. The θ-function is defined by

$$\theta(t) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 t}$$

for $t > 0$. This is exactly the case of $a = 0$ in our previous relation. Thus we achieve a totally new expression

$$\theta(t) = \sum_{n=-\infty}^{\infty} t^{-\frac{1}{2}}e^{-\frac{\pi n^2}{t}} = t^{-\frac{1}{2}}\theta\left(\frac{1}{t}\right) \qquad t > 0$$

Example 25.3. A similar story can be done for $\frac{1}{\cosh(\pi x)}$. The above transformation can be used to produce the equality

$$\sum_{n=-\infty}^{\infty}\frac{e^{-2\pi ian}}{\cosh(\frac{\pi n}{t})} = \sum_{n=-\infty}^{\infty}\frac{t}{\cosh(\pi(n+a)t)}$$

This is relevant to the two squares theorem, which says that every positive integer (without an odd prime in its factorization to an odd power) can be written as a sum of 2 squares. See chapter 10 if interested.

Chapter 4 concludes with the Pailey-Wiener theorem, which gives a very nice condition to prove analyticity of a function based on the decay of its Fourier Transform. I recommend reading this section if you are interested. Otherwise, we will move to entire functions next time.