CLASS 11, OCTOBER 1: COMPACTNESS

In metric spaces, a notion of being a small space makes sense. For general topological spaces not so much. However, we have a basis for the topology of X, dividing not only X but every open subset of X into simple components. As we've seen on the homework, when there are finitely many such components, the local to global behavior is often well understood. Today, we formalize this with the notion of compactness.

Definition 11.1. A topological space X is said to be **covered** by a collection of open sets $U_\alpha \in \tau$ if

$$X = \bigcup_\alpha X_\alpha$$

X is called **compact** if any open cover can be made finite; there exists $\alpha_1, \ldots, \alpha_n$ such that

$$X = X_{\alpha_1} \cup \ldots \cup X_{\alpha_n}$$

Example 11.2.

1) $\mathbb{R}$ is not compact. Take $B(x, 1)$ for every $x \in \mathbb{R}$. If finitely many cover it, their length (measure) is bounded above by $2n$ where n is as in the definition of compactness. This is a contradiction, because the length of $\mathbb{R}$ is ∞.
2) The same argument extends to $\mathbb{R}^m$.
3) Any finite space is compact, As is any space with the finite complement topology.
4) The subspace $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ is compact! Indeed, if U_α cover X, then there exists U_{α_0} containing 0. But then $B(0, \epsilon) \subseteq U_{\alpha_0}$ for some ϵ. So U contains $\frac{1}{n}$ for any $n > \frac{1}{\epsilon}$. This leaves only finitely many points outside of U_{α_0}, which at worst can be covered individually by the open set containing them.

   A good exercise is to prove that removing $\{0\}$ from X makes it non-compact.
5) The set $[a, b) \subseteq \mathbb{R}$ (or any interval not of the form $[a, b]$) is not compact. It can be covered, for example, by $[a, b - \frac{1}{n})$. However, every finite refinement will miss some points close to b.

A nice thing to note is that if $\tau \subseteq \tau'$, and X is τ'-compact, then it is also τ-compact. Furthermore, we can create many nice examples from *closed* subspaces:

Theorem 11.3. *If $Y \subseteq X$ is a closed subset, and X is compact, then Y is compact.*

Proof. Suppose U_α covers Y. Then there exists V_α open in X such that $V_\alpha \cap Y = U_\alpha$ (by definition of the subspace topology). Then we can consider the open covering of X given by

$$X = (\bigcup_\alpha V_\alpha) \cup (X \setminus Y)$$

Because X is compact, finitely many will do (for which we adjoin $(X \setminus Y)$ to avoid cases):

$$Y \subseteq X = V_{\alpha_1} \cup \ldots \cup V_{\alpha_n} \cup (X \setminus Y)$$

But this implies $Y \subseteq V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$, or equivalently

$$Y = U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$$

□

Corollary 11.4 (Of the proof)**.** *A subspace $Y \subseteq X$ is compact if and only if any open covering of Y in X can be refined to a finite collection.*

Next, I would like to provide a partial converse to Theorem 11.3. Note that the converse is not true in general;

Example 11.5. A finite proper subset of X with the trivial topology is compact but not closed.

Therefore, we need an extra condition that we will study in some detail later; Hausdorff.

Definition 11.6. A topological space X is **Hausdorff** if for any 2 points $x, y \in X$, there exists open disjoint set U, V containing x, y respectively.

Examples, similar to that of homework 1, are metric spaces: Let $U = B(x, \frac{d(x,y)}{2})$ and $V = B(y, \frac{d(x,y)}{2})$. This allows us to find the desired converse statement.

Theorem 11.7. *If X is Hausdorff, and $Y \subseteq X$ is compact, then Y is closed.*

Proof. We will prove Y^c is open. Assume it is non-empty (or we are done). Choose $x \in Y^c$ and $y \in Y$. Since X is Hausdorff, there exist U, V as in the definition. Fix $x \in X$ and label these opens U_y and V_y. This produces a covering of Y; $Y = \bigcup_{y \in Y} V_y$. But by compactness we see that $Y = V_{y_1} \cup \ldots \cup V_{y_n}$. But this set is thus disjoint from the open neighborhood of x:

$$U_x = U_{y_1} \cap \ldots \cap U_{y_n}$$

Therefore, $Y^c = \bigcup_{x \in Y^c} U_x$ is an open set! □

Finally, similar to the notion of connected, we see that images of compact sets remain compact:

Proposition 11.8. *If $f : X \to Y$ is a continuous map, and $A \subseteq X$ is a compact set, so is $f(A)$.*

Proof. Let $f(A) = \bigcup_\alpha U_\alpha$ be an open cover. Then

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f^{-1}(U_\alpha)$$

and thus a finite collection of these will do:

$$A \subseteq f^{-1}(U_{\alpha_1}) \cup \ldots \cup f^{-1}(U_{\alpha_n})$$

This of course implies

$$f(A) \subseteq f\left(f^{-1}(U_{\alpha_1}) \cup \ldots \cup f^{-1}(U_{\alpha_n})\right) = f\left(f^{-1}(U_{\alpha_1})\right) \cup \ldots \cup f\left(f^{-1}(U_{\alpha_n})\right) \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}.$$

□

The same again does not hold for preimages: consider any compact set inside $\mathbb{R}$ and consider the projection map $\pi : \mathbb{R}^2 \to \mathbb{R}$. Its preimage will not be compact.