CLASS 11, WEDNESDAY MARCH 7TH: EXACTNESS

$\otimes$ and Hom_R are two of the most important operations in commutative algebra. When applied to a given module M, they can be used to measure the complexity of M through failure of **exactness**. This is a measure of how well 2 modules approximate another.

Just a quick recall and expansion of some previous definitions:

Definition 0.1. Let $\varphi : M \to N$.

$$\ker(\varphi) = \{m \in M \mid \varphi(m) = 0\}$$

$$\mathrm{im}(\varphi) = \{n \in N\}$$

$\ker(\varphi) \subseteq M$ and $\mathrm{im}(\varphi) \subseteq N$ are submodules, so we can also quotient:

$$\mathrm{coim}(\varphi) = M/\ker(\varphi)$$

$$\mathrm{coker}(\varphi) = N/\mathrm{im}(\varphi)$$

Now for the definition of exactness:

Definition 0.2. If $\varphi : M' \to M$ and $\psi : M \to M''$ are 2 homomorphisms, we say that the **sequence**

$$M' \xrightarrow{\varphi} M \xrightarrow{\psi} M''$$

is **exact** if $\ker(\psi) = \mathrm{im}(\varphi) \subseteq M$. We can do this at infinitum:

$$\ldots \xrightarrow{\varphi_{-2}} M_{-2} \xrightarrow{\varphi_{-1}} M_{-1} \xrightarrow{\varphi_0} M_0 \xrightarrow{\varphi_1} M_1 \xrightarrow{\varphi_2} M_2 \xrightarrow{\varphi_3} \ldots$$

is an **exact sequence** if $\ker(\varphi_i) = \mathrm{im}(\varphi_{i-1})$ for every $i \in \mathbb{Z}$.

This notion gives a proper generalization of several notions we have already spoken about:

Proposition 0.3 (Exactness vs other properties of maps)**.**

1) *A sequence $0 \to M \xrightarrow{\varphi} N$ is exact if and only if φ is injective.*
2) *A sequence $M \xrightarrow{\varphi} N \to 0$ is exact if and only if φ is surjective.*
3) *A sequence $0 \to M \xrightarrow{\varphi} N \to 0$ is exact if and only if φ is an isomorphism.*
4) *A sequence $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ is exact if and only if φ is injective, ψ is surjective, and $M' = \ker(\psi)$ (or equivalently $M'' = \mathrm{coker}(\varphi) = M/M'$). This is special enough to give it's own name, a* **short exact sequence***. We also call M an* **extension** *of M'' by M'.*

Proof. 1) $0 \to M \xrightarrow{\varphi} N$ is exact if and only if $\ker(\varphi) = \mathrm{im}(0 \to M) = 0$ if and only if φ is injective.

2) $M \xrightarrow{\varphi} N \to 0$ is exact if and only if $N = \ker(N \to 0) = \mathrm{im}(\varphi)$ if and only if φ is surjective.

3) This follows directly from the previous 2 parts.

4) The only new piece of information here is that $M'' = M/M'$. Since $M \xrightarrow{\psi} M''$ is a surjective map, we know that
$$M'' \cong M/\ker(\psi) \cong M/\operatorname{im}(\varphi) \cong M/M'.$$

$\square$

Example 0.4.
- Given ANY R-modules M, N, we can form the exact sequence
$$0 \to M \to M \oplus N \to N \to 0$$
where we send m to $(m, 0)$ and (m, n) to n.
- The following is an exact sequence of $\mathbb{Z}$-modules:
$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$
- As $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ modules, we can form the SES (by the 2nd isomorphism theorem)
$$0 \to \mathbb{Z}/m\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/(n/m)\mathbb{Z} \to 0$$
where $m|n$, $\psi(1) = \frac{n}{m}$, and $\varphi(1) = \bar{1}$.
- More generally, given any ideal $I \subseteq R$, we can form the SES
$$0 \to I \to R \to R/I \to 0$$
- By the 1st isomorphism theorem, given any R-module homomorphism $M \xrightarrow{\psi} N$, we have a SES
$$0 \to \ker(\psi) \to M \to \operatorname{im}(\psi) \to 0$$

Example 0.5 (Free Resolution)**.** Recall that for any R-module M, we can find a generating set m_λ for $\lambda \in \Lambda_0$ and form a surjection from a free module:
$$R^{\Lambda_0} \to M \to 0$$
We can then look at the kernel of this map, which is a submodule of R^{Λ_0}. Thus we can repeat the process finding a generating set of the kernel, and surjecting onto it via a free module:
$$R^{\Lambda_1} \to R^{\Lambda_0} \to M \to 0$$
This is an exact sequence by design. Iterating this procedure indefinitely produces a **long exact sequence**
$$\ldots \to R^{\Lambda_2} \to R^{\Lambda_1} \to R^{\Lambda_0} \to M \to 0$$
which is called a **free resolution of** M.

Finally, we give the definition of a split exact sequence:

Definition 0.6. A SES $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ is said to be **split exact** if one of the following equivalent conditions is met:

1º $M \cong M' \oplus M''$.
2º There is a homomorphism $\varphi' : M \to M'$ such that $\varphi' \circ \varphi = Id_{M'}$.
3º There is a homomorphism $\psi' : M'' \to M$ such that $\psi \circ \psi' = Id_{M''}$.

Proof. See homework 3. $\square$