CLASS 28, APRIL 26TH: PRIMARY IDEALS

Today we will approach the idea of decomposing any ideal in a Noetherian ring. So far, we've shown that radical ideals are exactly intersections of finitely many primes. Here we will attempt to broaden that horizon. Recall (from homework 1!) the definition of a primary ideal.

Definition 28.1. $q \subseteq R$ an ideal is said to be **primary** if when $x \cdot y \in \mathfrak{q}$, then either $x \in \mathfrak{q}$ or $y^m \in \mathfrak{q}$ for some $m \in \mathbb{N}$.

This is a slight weakening of the notion of prime. Slight is clarified here:

Proposition 28.2. *If $\mathfrak{q}$ is a primary ideal, then $\sqrt{q}$ is a prime ideal.*

Proof. Suppose $x \cdot y \in \sqrt{q}$. This is to say that $x^n \cdot y^n \in \mathfrak{q}$ for some n. As a result, either $x^n \in \mathfrak{q}$ or $y^{nm} \in \mathfrak{q}$. But naturally this implies either $x \in \sqrt{q}$ or $y \in \sqrt{q}$. $\square$

Thus it is also common to call a primary ideal **$\mathfrak{p}$-primary** to indicate directly the corresponding prime ideal $\mathfrak{p} = \sqrt{q}$.

Example 28.3. As checked in the first homework, $\langle p^n \rangle \subseteq \mathbb{Z}$ is a $\langle p \rangle$-primary ideal. Indeed, if $m \cdot l \in \langle p^n \rangle$, then either p^n divides m or it doesn't. If it doesn't, then p divides l which implies p^n divides l^n.

In fact you showed $\langle p^n \rangle$ is the only type of primary ideal in $\mathbb{Z}$.

It should be noted that if $\mathfrak{p}$ is a finitely generated ideal (e.g. R Noetherian), then since $\sqrt{\mathfrak{q}} = \mathfrak{p}$, we have that for some $N \gg 0$, we have that

$$\mathfrak{p}^N \subseteq \mathfrak{q} \subseteq \mathfrak{p}$$

Indeed, if $f_i^{n_i} \in \mathfrak{q}$ for the generators f_i, then $N = \sum_i (n_i - 1) + 1$ will suffice. One should however note that this is not enough to ensure that $\mathfrak{q}$ is a primary ideal.

Example 28.4. If $I = \langle x^2, xy \rangle$, then $\sqrt{I} = \langle x \rangle$. However, I is not primary, since $x \notin I$ and $y^n \notin I$ for any n.

This example shows that it is also not enough to ask either $x^n \in I$ or $y^n \in I$ for some I. This is a strictly weaker condition than 'primary'. There is however one example where this is not the case:

Proposition 28.5. *If $\sqrt{\mathfrak{q}} = \mathfrak{m}$ is a maximal ideal, then $\mathfrak{q}$ is $\mathfrak{m}$-primary.*

Proof. Let $f \notin \mathfrak{q}$. Then we can consider the ideal

$$I = \mathfrak{q} : f = \{g \in R \mid fg \in \mathfrak{q}\}$$

Note I is proper since $1 \notin I$. As a result, $I \subseteq \mathfrak{m}'$ for some maximal ideal $\mathfrak{m}'$. However, $\mathfrak{q} \subseteq \mathfrak{q} : f$, and $\mathfrak{m}$ is the only prime ideal containing $\mathfrak{q}$. As a result, $\mathfrak{m}' = \mathfrak{m}$. This shows that if $fg \in \mathfrak{q}$, and $f \notin \mathfrak{q}$, then $g \in \mathfrak{m}$ which implies $g^n \in \mathfrak{q}$. $\square$

$\mathfrak{m}$-primary ideals play a very important part in the study of Artinian rings. Recall that you proved the following result:

Proposition 1. *$\mathfrak{q}$ is a primary ideal if and only if $R/\mathfrak{q}$ contains only zero divisors which are non-reduced.*

By the analysis of Proposition 28.5, we can conclude that such a ring $R/\mathfrak{q}$ where $\mathfrak{q}$ is $\mathfrak{m}$-primary, is also local with a unique prime ideal $\mathfrak{m}$. If R was also Noetherian, we conclude by the above discussion that $\mathfrak{m}^n = 0$ in this ring. This yields a big class of Artinian rings.

Next, we will get into a slightly different classification of primary ideals;

Theorem 28.6. *If R is Noetherian, then*

$$\mathfrak{q} \text{ is } \mathfrak{p}\text{-primary} \iff \mathrm{Ass}(R/\mathfrak{q}) = \{\mathfrak{p}\}$$

Example 28.7. If R and I are as in Example 28.4, then we have that I is not $\langle x\rangle$-primary. Indeed, one can note that $\mathrm{Ass}(R/I) = \{\langle x\rangle, \langle x, y\rangle\}$, since $\mathrm{Ann}(y) = \langle x\rangle$ and $\mathrm{Ann}(x) = \langle x, y\rangle$. Therefore, we can conclude in a different way that I is not $\langle x\rangle$-primary.

The Noetherian assumption is absolutely paramount here.

Example 28.8. Consider the ring $R = K[x_1, x_2, \ldots]$ and $I = \langle x_1^2, x_2^2, \ldots\rangle$. By virtue of Proposition 28.5, we have that I is $\mathfrak{m}$-primary. However, if we consider R/I, then $\mathfrak{m}$ is NOT the annihilator of any single element. Indeed, to be annihilated by every variable, you need to be divisible by every variable. This can never happen. Since $\mathrm{Ann}(x) \supseteq I$ for every $x \in R/I$, we actually have $\mathrm{Ass}(R/I) = \emptyset$!

Proof. $\Rightarrow$: Suppose $\sqrt{q} = \mathfrak{p}$ is a $\mathfrak{p}$-primary ideal. By Proposition 1, we know that the zero divisors of $R/\mathfrak{q}$ are those elements that are non-reduced: $x^n = 0$. As a result, we have $x \in \mathfrak{p}$ and $\mathfrak{p} \supseteq \mathrm{Ann}(x) \supseteq \mathfrak{q}$. But every associated prime is $\mathrm{Ann}(x)$ for some x, so the only possibility for a prime is $\mathfrak{p}$.

$\Leftarrow$: The main claim is as follows: if $\mathrm{Ass}(R/\mathfrak{q}) = \{\mathfrak{p}\}$, and M is a non-zero submodule of $R/\mathfrak{q}$, then $\sqrt{\mathrm{Ann}(M)} = \mathfrak{p}$. This follows since $\sqrt{\mathrm{Ann}(M)}$ is the intersection of all prime ideals containing $\mathrm{Ann}(M)$. But this can be realized as the intersection of all minimal primes containing $\mathrm{Ann}(M)$, which are the minimal elements of $\mathrm{Supp}(M)$. Therefore we can conclude by Theorem 27.1 that these minimal elements are in fact associated primes. But $\mathrm{Ass}(R/\mathfrak{q}) = \{\mathfrak{p}\}$, and therefore $\mathrm{Ass}(M) = \{\mathfrak{p}\}$. This proves the assertion.

As a result we have that $\mathfrak{q} = \mathrm{Ann}(R/\mathfrak{q})$ has radical $\mathfrak{p}$. Suppose $fg \in \mathfrak{q}$ and $f \notin \mathfrak{q}$. Consider $\bar{f} \in R/\mathfrak{q}$, we have $g \in \mathrm{Ann}(f) \subseteq \sqrt{\mathrm{Ann}(f)} = \mathfrak{p}$. So $g^n \in \mathfrak{q}$, which implies $\mathfrak{q}$ is $\mathfrak{p}$-primary. □

Finally, we get to the most important reason to concern yourself with primary ideals; the primary decomposition.

Definition 28.9. If I is an ideal in a ring R, then I has a **primary decomposition** if

$$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$$

where $\mathfrak{q}_i$ is a primary ideal. It is **shortest** if $\mathfrak{q}_i \not\supseteq \mathfrak{q}_1 \cap \cdots \cap \hat{\mathfrak{q}}_i \cap \cdots \cap \mathfrak{q}_n$ and $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$ has the property that $\mathfrak{p}_i \neq \mathfrak{p}_j$ for any $i \neq j$.

Given any primary decomposition, you can easily make it shortest by using the following lemma:

Lemma 28.10. *If $\mathfrak{q}_1, \mathfrak{q}_2$ are $\mathfrak{p}$-primary ideals, so is $\mathfrak{q}_1 \cap \mathfrak{q}_2$.*

As a result we can combine away all primary ideals with the same prime. In addition, this also implies no $\mathfrak{q}_i$ is redundant since we could simply omit it.