Nov 6: Introduction to Homology.

$\pi_1(X)$, though a beautiful tool, is not all powerful. For example, we showed

$$\pi_1(S^n) = \begin{cases} \mathbb{Z} & n=1 \\ 0 & n=2,3,4,\ldots \end{cases}$$

However, S^n is not homotically equiv to S^m for any $m \neq n$. We can distinguish them by considering higher homotopy groups:

$$\pi_n(X, x_0) = \{ \gamma : I^n \to X \mid \gamma(\partial I^n) = x_0 \} / \simeq$$

This, like π_1, depends only on the n+1 cell structure for CW complexes. Also

$$\pi_n(S^m) = 0 \quad \text{for} \quad n < m,$$
$$\pi_m(S^m) = \mathbb{Z}$$

So it does distinguish spheres. However, the groups are notoriously difficult to compute:

$$\pi_3(S^2) = \mathbb{Z}$$

Open Question: Compute $\pi_n(S^m)$ $\forall n, m \in \mathbb{N}$

$\pi_n(S^m)$

$n \backslash m$	1	2	3	4	5
1	$\mathbb{Z}$	0	0	0	0
2	0	$\mathbb{Z}$	0	0	0
3	0	$\mathbb{Z}$	$\mathbb{Z}$	0	0
4	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	0
5	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$
6	0	$\mathbb{Z}/12$	$\mathbb{Z}/12$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
7	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}\times\mathbb{Z}/12$	$\mathbb{Z}/2$
8	0	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/24$

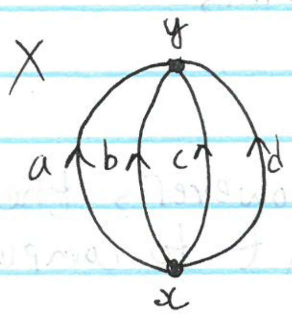
...

Some nice (hard) results show some patterns:

- $\pi_i(S^2) = \pi_i(S^3)$ $i \geq 3$
-

Thus Homology was introduced:

- $H_n(X)$ no basepoint, is Abelian!
- $H_n(X^{n+1}) = H_n(X)$ for CW complexes
- $H_i(X^n) = 0 \quad \forall i > n$
- H_i are easier to compute usually.



With fundamental groups, this is $\pi_1(X) = \mathbb{Z}^{*3}$ (exam)

If we abelianize, $\pi_1(X)^{ab} = \mathbb{Z}^{\oplus 3}$

$$(ab^{-1})\cdot(cd^{-1}) = a - b + c - d$$
$$= (ad^{-1})\cdot(c\cdot b^{-1})$$

1- Chains

Edges ②

Cycles: $C_1 = \{$ Free abelian Group gen by ~~[illegible]~~ $\}$

$= a\mathbb{Z} \oplus b\mathbb{Z} \oplus c\mathbb{Z} \oplus d\mathbb{Z}$

"Counts # times traveled on each edge"

$C_0 = \{ \quad$ '' '' vertices $\}$

0-Chains $= x\mathbb{Z} \oplus y\mathbb{Z}$

Boundary $\partial : (a,b,c,d) \longmapsto y - x$

extend by linearity:

$ka + lb + mc + nd \mapsto (k+l+m+\cancel{m})(y-x)$

$Z_1(X) =$

1-cycles: $\ker(\partial) = \{ c \in C_1 : \partial c = 0 \}$

$n = -(k+l+m)$

$(k+l+\cancel{m}+\cancel{m}) = 0$ or $\cancel{k = -(l+n+m)}$

$\rightsquigarrow (ka - kd) + (lb - ld) + (mc - md)$

$\rightsquigarrow Z_1(X) = (a-d)\mathbb{Z} \oplus (b-d)\mathbb{Z} \oplus (c-d)\mathbb{Z}$

$= H_1(X)$

Attach 2-cells A, B along $a-b$

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

$C_2 = A\mathbb{Z} \oplus B\mathbb{Z}$

$\partial_2 : A, B \mapsto a - b$

$\ker(\partial_2) = (A-B)\mathbb{Z} \quad = H_2(X)$

$\mathrm{Im}(\partial_2) = (a-b)\mathbb{Z}$

$$\ker(\partial_1) = (a-b)\mathbb{Z} \oplus (a-c)\mathbb{Z} \oplus (a-d)\mathbb{Z}$$
$$\operatorname{Im}(\partial_1) = (y-x)\mathbb{Z}$$

$$\ker(\partial_0) = x\mathbb{Z} \oplus y\mathbb{Z}$$

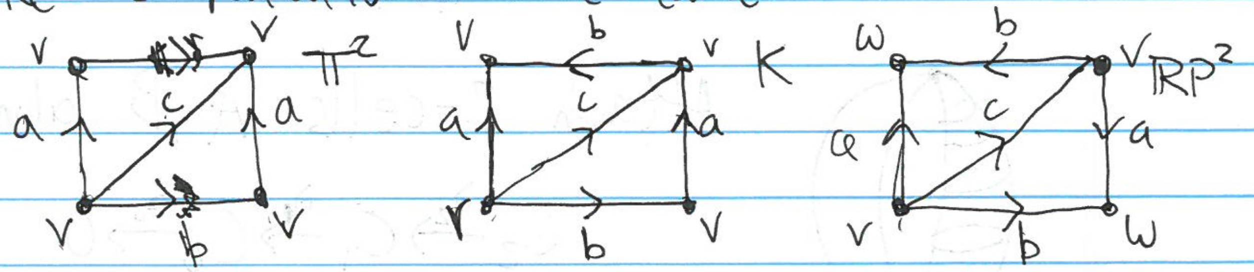
$$\Rightarrow H_2(X) \cong \mathbb{Z}$$
$$H_1(X) = \ker(\partial_2)/\operatorname{Im}(\partial_2) \cong (a-c)\mathbb{Z} \oplus (a-d)\mathbb{Z} \cong \mathbb{Z}^2$$
$$H_0(X) = \mathbb{Z}$$

$$Z_n(X) = \ker(d_n), \quad B_n(X) = \operatorname{Im}(d_{n+1})$$
$$H_n(X) = Z_n(X)/B_n(X)$$

Oct 8: Simplices / Δ-Complexes

The idea here is to "triangulate" spaces to make computations more tame.

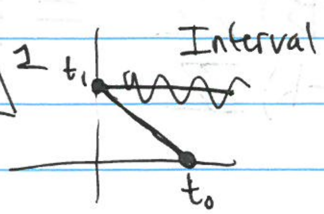
Ex



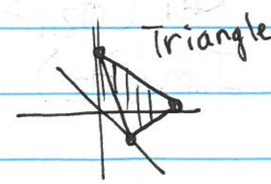
We genalize this to a notion of a simplex:

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} : \sum t_i = 1,\ t_i \geq 0\ \forall i\}$$

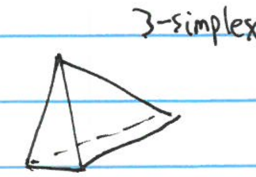
Δ^1



Δ^2



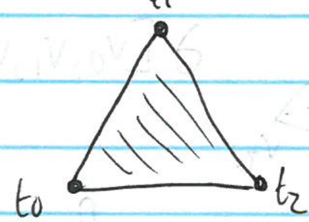
Δ^3



Order matters : $t_0 \to t_1$ vs $t_1 \leftarrow t_0$

Boundary map: If Δ^n is an n-simplex, $(t_0, \dots, t_n)$, setting $t_i = 0$ for some i yields $(t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)$ which reps an n-1 simplex by "forgetting zero", F_i. This is a <u>face</u> of Δ^n.

$$\partial \Delta^n = \bigcup_{i=0}^{n} F_i$$ is called it's boundary.



$$\partial \Delta^2 = [t_0, t_1] \cup [t_0, t_2] \cup [t_1, t_2]$$

$$\mathring{\Delta} = \Delta \setminus \partial \Delta$$

Defn: A Δ-complex structure on a space X, is a collection of maps $\Delta^n_\alpha \xrightarrow{\sigma_\alpha} X$ s.t. $n = n(\alpha)$

1) $\sigma_\alpha|_{\mathring{\Delta}_n}$ is injective, and each $x \in X$ is in exactly one.
2) $\sigma_\alpha|_{F_i^\alpha} = \sigma_\beta : \Delta^{n-1} \to X$ for some $\beta = \beta(\alpha, i)$
3) $A \subseteq X$ is open $\Leftrightarrow$ $\sigma_\alpha^{-1}(A)$ is open

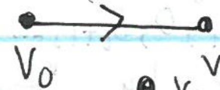
(3) Rules out goofy things like $\sigma_x : e^0 \to X$, $\{pt\} \mapsto x$. Moreover

$$X = \coprod_\alpha \Delta^n / \sim \qquad \underset{\Delta^n_\alpha}{\overset{\cap}{x}} \sim \underset{\Delta^n_\beta}{\overset{\cap}{y}} \Leftrightarrow \sigma_\alpha(x) = \sigma_\beta(y)$$

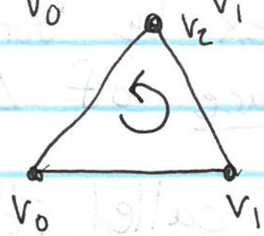
Oct 10: Simplicial Homology.

Let $\Delta_n(X) = \mathbb{Z}\cdot\{e^n_\alpha\} = \bigoplus_\alpha e^n_\alpha \cdot \mathbb{Z}$

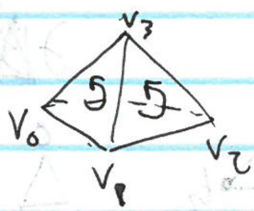
Boundary:



Boundary: $v_1 - v_0 = \partial[v_0, v_1]$



$$\partial[v_0,v_1,v_2] = [v_1, v_0] + [v_1,v_2] + [v_2,v_0]$$
$$= [v_0,v_1] + [v_1,v_2] - [v_0,v_2]$$



$$\partial[v_0,v_1,v_2,v_3] = [v_1,v_2,v_3] - [v_0,v_2,v_3] + [v_0,v_1,v_3] - [v_0,v_1,v_2]$$

extend by linearity

$$\partial : \Delta_n(X) \to \Delta_{n-1}(X) : \sigma_\alpha \mapsto \sum_{i=0}^{n} (-1)^i \sigma_\alpha|[v_0,\dots,\hat{v}_i,\dots,v_n]$$

removed

Lemma: $\partial_{n-1}\circ\partial_n = 0$.

Pf: $\partial_{n-1}\circ\partial_n(\sigma_n) = \partial_{n-1}\left[\sum_{i=0}^{n}(-1)^i \sigma_\alpha|_{[v_0,\dots,\hat{v}_i,\dots,v_n]}\right]$

$$= \sum_{i=0}^{n}\sum_{j<i}(-1)^i(-1)^j \sigma_\alpha|_{[v_0,\dots,\hat{v}_j,\dots,\hat{v}_i,\dots,v_n]}$$
$$+ \sum_{i=0}^{n}\sum_{j>i}(-1)^i(-1)^{j-1}\sigma_\alpha|_{[v_0,\dots,\hat{v}_i,\dots,\hat{v}_j,\dots,v_n]}$$

Comparing $i<j$ with $j<i$ $\Rightarrow 0$.

Looking Closer:

$$\cdots \to \Delta_n \xrightarrow{\partial_n} \Delta_{n-1} \xrightarrow{\partial_{n-1}} \Delta_{n-2} \to \cdots \to \Delta_0 \to 0 \qquad (*)$$

$$\partial_{n-1}\circ\partial_n = 0 \iff \mathrm{Ker}(\partial_{n-1}) \supseteq \mathrm{Im}(\partial_n)$$
$$\iff (*) \text{ is a "Chain Complex"}$$

So we can consider their difference:

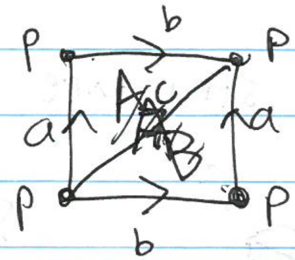
$$H_n^{\Delta}(X) = \mathrm{Ker}(\partial_{n-1})/\mathrm{Im}(\partial_n)$$

Ex/ $X = S^1$

$$0 \xrightarrow{\partial_2} \Delta_1(S^1) \xrightarrow{\partial_1} \Delta_0(S^2) \xrightarrow{\partial_0} 0$$
$$\Delta_1(S^1) = a\mathbb{Z}, \qquad \Delta_0(S^2) = p\mathbb{Z}$$
$$a \longmapsto p - p = 0$$

$$H_1^{\Delta}(S^1) \cong \mathbb{Z}, \quad H_0^{\Delta}(S^1) \cong \mathbb{Z}$$

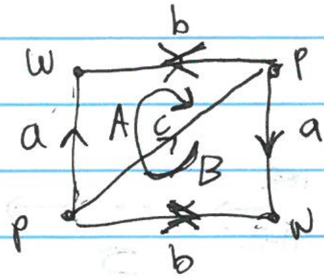
Ex/ $X = \mathbb{T}^2$



$$0 \to \Delta_2 \to \Delta_1 \to \Delta_0 \to 0$$
$$A\mathbb{Z}\oplus B\mathbb{Z} \qquad a\mathbb{Z}\oplus b\mathbb{Z}\oplus c\mathbb{Z} \qquad p\mathbb{Z}$$
$$A \mapsto a + b - c$$
$$B \mapsto c - a - b \qquad a \longmapsto p - p = 0$$
$$b \longmapsto p - p = 0$$
$$c \longmapsto p - p = 0$$

$$H_2^{\Delta}(\mathbb{T}^2) \cong \mathbb{Z}$$
$$H_1^{\Delta}(\mathbb{T}^2) = \mathbb{Z}\oplus\mathbb{Z}$$
$$H_0^{\Delta}(\mathbb{T}^2) = \mathbb{Z}$$

Ex/ $\mathbb{RP}^2$



$$(A, B) \longmapsto (a+b-c, a+b+c)$$

$$(a, b, c) \longmapsto (w-p, p-w, p-p)$$

$\mathrm{Ker}(\partial_2) = 0$

$\mathrm{Im}(\partial_2) = (a+b-c)\mathbb{Z} \oplus (a+b+c)\mathbb{Z}$

$\mathrm{Ker}(\partial_1) = \langle a-b, c\rangle$

$\mathrm{Im}(\partial_1) = \langle w-p\rangle\mathbb{Z}$

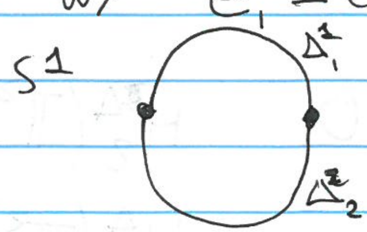
$$H_i^{\Delta}(\mathbb{RP}^2) = \begin{cases} 0 & i=2 \\ \mathbb{Z}/2\mathbb{Z} & i=1 \\ \mathbb{Z}^2/\mathbb{Z} \cong \mathbb{Z} & i=0 \end{cases}$$

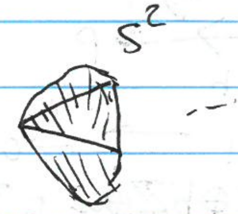
$\cancel{a \neq b} \Rightarrow \langle 2a+c, c\rangle = \langle 2a, c\rangle$

$\Rightarrow \langle a-$

$\langle a-b, c\rangle / \langle a-b, c^2\rangle \cong \mathbb{Z}/2\mathbb{Z}$

S^n has n-simplex structure of two Δ^n_1, Δ^n_2 w/ $e^n_1 = e^n_2$:





So $H_n^{\Delta}(S^n) = (\Delta^n_1 - \Delta^n_2)\mathbb{Z} \cong \mathbb{Z}$