CLASS 3, FEBRUARY 8TH: $\mathrm{Spec}(K[x, y])$ AND $\mathrm{Spec}(\mathbb{Z}[x])$

Today I intend to compute the spectra of two 2-dimensional (to be interpreted) rings. We will cover this in bigger generality in section 4, but it is very useful to go through the details in a few cases.

Theorem 3.1. $\mathrm{Spec}(K[x, y])$ *is exactly the set of prime ideals of the following form:*

0) 0*, the zero ideal.*

1) $\langle f(x, y)\rangle$*, where* $f = f(x, y)$[1] *is an irreducible polynomial.*

2) The maximal ideals $\mathfrak{m}$*, for which* $\mathfrak{m} = \langle f(x), g(x, y)\rangle$*, and* $f = f(x)$ *is an irreducible polynomial, and* $g = g(x, y)$ *is a polynomial whose reduction* $(\mathrm{mod}\, f)$ *is irreducible in* $(K[x]/\langle f\rangle)[y]$*. This implies* $K[x, y]/\mathfrak{m}$ *is a finite field extension of* K*.*

Definition 3.2. A polynomial $f \in R[x]$, where R is a unique factorization domain (**U.F.D**), is said to be **primative** if $f = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ and a_i share no common factor.

Lemma 3.3 (Gauss's Lemma)**.** *If* R *is a U.F.D,* $f, g \in R[x]$ *are primative polynomials, then so is* $f \cdot g$*.*

Proof. Suppose $f \cdot g$ is not primative, i.e. there exists $p \in R$ such that p divides all the coefficients of $f \cdot g$. Note by assumption, p does not divide all the coefficients of f or g. Let

$$f = a_0 + a_1 x + \ldots + a_n x^n$$
$$g = b_0 + b_1 x + \ldots + b_m x^m$$

and let a_r and b_s be the first coefficients of f and g respectively not divisible by p. Then the x^{r+s} term of $f \cdot g$ reads

$$\alpha = \sum_{i+j=r+s} a_i b_j = a_r b_s + a_{r+1}b_{s-1} + \ldots + a_{r-1}b_{s+1} + \ldots$$

For $i \neq r$ and $j \neq s$, $a_i b_j$ is divisible by p by assumption. However, $a_r b_s$ is not. This implies p does not divide α, a contradiction. $\square$

Proof. Let $R = K[x]$ and $L = K(x)$ be the ring (field) of rational functions in x. Then R is a P.I.D. and L is its field of fractions. The ring of interest is $S = K[x, y] = R[y]$. If $\mathfrak{p} \in \mathrm{Spec}(S)$, then we may assume $f_1, f_2 \in \mathfrak{p}$ are elements with no common factor in S (otherwise, they fall into case 0) or case 1).

1) f_1, f_2 also share no factors in $L[y]$.

Suppose $f_1 = h \cdot g_1$ and $f_2 = h \cdot g_2$, with $h, g_1, g_2 \in L[y]$, $\deg(h) \geq 1$. We can factor an element of L so that the coefficients of the h, g_1, and g_2 share no common factors: $h = ah_0$, $g_1 = b_1\gamma_1$, and $g_2 = b_2\gamma_2$, with $a, b_1, b_2 \in L$. Now as a result of Lemma 3.3, we get that $h_0\gamma_1$ and $h_0\gamma_2$ are also primative. Therefore $f_1 = hg_1 = (ab_1)(h_0\gamma_1) \in K[x, y]$ implies $ab_1 \in R$. Symmetrically, the same is true of ab_2. Therefore h_0 divides f_1 and f_2, a contradiction.

[1] calling the polynomial without variables is notationally convenient

2) If $I = \langle f_1, f_2\rangle$, then $I \cap R \neq 0$.
   Since $L[y]$ is a P.I.D., and $gcd(f_1, f_2) = 1$, there exists $a, b \in L[y]$ such that $af_1 + bf_2 = 1$. Therefore, clearing denominators by multiplying by $c \in K[x]$,
$$0 \neq c = caf_1 + cbf_2 \in R$$
3) If $\mathfrak{p}$ is a prime ideal of $K[x, y]$, then $R \cap \mathfrak{p}$ is a prime ideal of R. This follows by Homework 1 #2, given the inclusion $R = K[x] \hookrightarrow K[x, y] = S$. By 2), we have that if $\mathfrak{p}$ is not principal, $\mathfrak{p} \cap R \neq 0$. But since R is a P.I.D., we see that $\mathfrak{p} \cap R = \langle f\rangle$ is a maximal ideal. As a result, $K[x]/\langle f\rangle$ is a field, so again
$$(K[x]/\langle f\rangle)[y] = K[x, y]/\langle f\rangle$$
is a P.I.D. Therefore, there is a $g \in K[x, y]$ irreducible so that $\mathfrak{p} = \langle f, g\rangle$.

□

Corollary 3.4. *The prime ideals of $\mathbb{Z}[y]$ are exactly*

0) 0, the zero ideal.
1) $\langle f(y)\rangle$, where $f = f(y)$ is an irreducible polynomial.
2) The maximal ideals $\mathfrak{m}$, for which $\mathfrak{m} = \langle p, f(y)\rangle$, and p is a prime number, and $f(y)$ is a polynomial whose reduction $\pmod p$ is irreducible in $\mathbb{F}_p[y] = (\mathbb{Z}/p\mathbb{Z})[y]$. This implies $\mathbb{Z}[y]/\mathfrak{m}$ is a finite field extension of $\mathbb{F}_p$, and thus of the form $\mathbb{F}_{p^e}$.

Proof. The assertion follows precisely by replacing $R = K[x]$ with $R = \mathbb{Z}$, and $L = K(x)$ by $L = \mathbb{Q}$.[2] □

As a direct application of this result, we can see that geometric interpretation of these 2 rings:

Example 3.5 (The Affine K-Plane)**.** Suppose K is an algebraically closed field. Then we have that every maximal ideal $\mathfrak{m} = \langle f, g\rangle$, with $f \in K[x]$ irreducible, and $g(x, y)$ is irreducible $\pmod f$. Since K is algebraically closed, we have that $f = x - \alpha$ for some $\alpha \in K$ (as in the case of $\mathbb{A}^1_K$). But then $x = \alpha \in K$ inside $(K[x]/\langle f\rangle)[y]$, so $K[x, y]/\langle f\rangle \cong K[y]$. As a result, $g = y - \beta$ (up to subtracting some multiple of f, which is fine in terms of generation of an ideal)!

Therefore, the maximal ideals are exactly given as
$$\text{m-}\operatorname{Spec}(K[x, y]) = \{\langle x - \alpha, y - \beta\rangle \mid \alpha, \beta \in K\} \longleftrightarrow K^2.$$
This is canonically a 2-dimensional vector space over K, thus the terminology K-plane! We call
$$\mathbb{A}^2_K = \operatorname{Spec}(K[x, y])$$
the affine K-plane. The more general considerations of
$$\mathbb{A}^n_K = \operatorname{Spec}(K[x_1, x_2, \ldots, x_n])$$
behave similarly (c.f. Hilbert-Nullstellensatz), though the statement of a proposition like Theorem 3.1 is much harder.

[2]The proof can in fact be generalized to *any* P.I.D. R in its field of fractions L!