HOMEWORK 6: NORMALITY AND URYSOHN THEOREMS
DUE: OCTOBER 26

1) Show a closed subspace of a normal space is normal.

2) Show that if X_α are non-empty topological spaces, and $\prod_\alpha X_\alpha$ is T2 or T3 or T4, then so is each X_α.

3) Show that the following 2 conditions are equivalent:
   1) Every subspace of X is normal.
   2) For all A, B subsets of X such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$, there exists U, V open disjoint sets separating A and B; $A \subseteq U$ and $B \subseteq V$.

   In such a case, X is said to be **T5**, or **completely normal**.

4) Show that any connected normal space X containing 2 disjoint non-empty closed sets A,B is uncountable.

5) We say $Y \subseteq X$ is a $\mathbf{G}_\delta$ set if Y is an intersection of countably many open sets. Similarly, Y is a $\mathbf{F}_\sigma$ set if it is a countable union of closed sets. Use the techniques of the proof of Urysohn's Lemma to show the following result:

   Theorem. *If X is normal, then there exists $f : X \to [0, 1]$ a continuous function such that $f^{-1}(0) = A$ iff A is a closed G_δ set.*[1]

6) X is **T6** or **perfectly normal** if it is T4 and every closed set is a G_δ-set. Show every metric space is T6 and that T6 implies T5.[2]

7) Show that if X is a compact Hausdorff space, then X is metrizable if and only if X is second-countable.

[1] Urysohn's Lemma holds in a exact sense when A and B are closed G_δ-sets.

[2] Apply the previous problem.