CLASS 3, SEPTEMBER 12: BASES AND SUBSPACES

As with many things in mathematics, it's often nice to think of an object based on some significantly smaller defining objects. For example, in the study of finite dimensional linear algebra, we refine an uncountable collection of vectors to only finitely many basis vectors without reducing information. A similar notion exists for topology; instead of listing every open set in a topology, why not only list some building blocks?

Definition 3.1. Let X be a set. A **basis for a topology on X** is a subset $\mathcal{B} \subseteq \mathcal{P}(X)$ such that

1) For each $x \in X$, there exists $U \in \mathcal{B}$ such that $x \in U$.
2) If $x \in B_1, B_2$ for $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Notice that we can also generate (or span) a topology given a basis:

Proposition 3.2. *Given any collection $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfying properties 1) and 2), there exists a unique smallest topology τ containing $\mathcal{B}$. In this case, $\mathcal{B}$* **generates** *τ.*

Proof. Let

$$\tau = \{\bigcup_{\alpha \in \Lambda} U_\alpha \mid U_\alpha \in \mathcal{B}\} \cup \{\emptyset\}.$$

Of course, this contains $\mathcal{B}$ and is contained within any topology containing $\mathcal{B}$ by axiom 2) of a topology. Additionally, it contains X by property 1) above.

Finally, if $V_1, \ldots, V_n \in \tau$, then for each $x \in V_1 \cap \ldots, \cap V_n$, we can (inductively using property 2)) find a set $V_x \in \mathcal{B}$ such that $V_x \subseteq V_1 \cap \ldots \cap V_n$. This results in

$$V = V_1 \cap \ldots \cap V_n = \bigcup_{x \in V} V_x$$

and thus $V \in \tau$. □

Example 3.3. Consider $\mathbb{R}^n$ with the metric topology τ. Letting $\mathcal{B}$ be any of the following yields a basis for τ:

- τ itself.
- $\{B_x(d) \mid x \in \mathbb{R}^n,\ d > 0\}$.
- $\{(a_1, b_1) \times \ldots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}$.
- $\{B_x(d) \mid x \in \mathbb{Q}^n,\ d \in \mathbb{Q}_+\}$. That is to say we consider open balls of rational radius centered at points with rational coordinates.
- $\{B_x(d) \mid x \in \mathbb{Q}^n,\ d \in \mathbb{Q} \text{ with } 0 < d < 1\}$

The point here is that unlike in the case of (finite-dimensional) vector spaces, bases can have completely different sizes! Note that the first 3 bases above are uncountable, where as the last 2 are countable (since they are in particular finite products of copies of $\mathbb{Q}$).

Next, we check that a basis does determine enough information to make statements about the topology itself.

Lemma 3.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for τ and τ' respectively. Then TFAE:*

- *τ' is finer than τ.*
- *For each $x \in X$ and $B \in \mathcal{B}$ containing x, there exists $B' \in \mathcal{B}'$ containing x such that $B' \subseteq B$.*

Proof. $(\Rightarrow)$: Suppose τ' is finer than τ. Then B is an open set in the τ' topology. Therefore, by definition of generation, there exists $B'_\alpha \in \mathcal{B}'$ such that $B = \cup_{\alpha \in \Lambda} B'_\alpha$. Since $x \in B$, there exists some $\alpha_0 \in \Lambda$ for which $x \in B'_{\alpha_0}$. Letting $B' = B'_{\alpha_0}$ completes the proof.

$(\Rightarrow)$: Suppose U is open in the τ-topology. Then $U = \cup_{\alpha \in \Lambda} B_\alpha$ for some $B_\alpha \in \mathcal{B}$. Now, the second property implies that for each $x \in B_\alpha$, there exists $B'_{\alpha,x} \in \mathcal{B}'$ such that $x \in B'_{\alpha,x} \subseteq B_\alpha$. This then implies that

$$U = \bigcup_{\alpha \in \Lambda} \bigcup_{x \in B_\alpha} B_{x,\alpha}$$

and that U is open in the $\mathcal{B}'$-topology. This completes the proof. □

It is left as a homework exercise to use this to show that some of the topologies listed above are equivalent.

Next up, we will start a trend of producing new topological spaces from old. The first method of doing so is by taking the subspace topology:

Definition 3.5. Let (X, τ) be a topological space, and $Y \subseteq X$ be any subset. Define

$$\tau_Y = \tau|_Y = \{V \subseteq Y \mid V = Y \cap U \text{ for some } U \in \tau\}$$

This is called the **subspace topology** on Y.

It is a quick check (left for the reader) that this is in fact a topology on Y. Many of the common objects, such as a space of interest sitting inside of $\mathbb{R}^n$, can thus naturally receive a topology. These spaces include spheres, tori, Mobius Strips, Klein Bottles, space filling curves, etc.

Example 3.6. Let $\mathbb{R}$ be given with the Euclidean topology.

- The integers $\mathbb{Z} \subseteq \mathbb{R}$ with the subspace topology yields the discrete topology.
- Giving $[0, 1]$ the subspace topology of $\mathbb{R}$ produces open sets of the form (a, b), $[0, a)$, $(b, 1]$, and $[0, 1]$ (as well as unions of such things) where $0 < a < b < 1$. Note in particular these sets are not all open in $\mathbb{R}$ itself!
- If Y is an open (respectively closed) subset of X, and $U \subset Y$ is open (resp. closed) in the subspace topology, then $U \subseteq X$ is also open (resp. closed). Compare this with the previous bullet.

Relating back to bases, we have the following result:

Lemma 3.7. *If $\mathcal{B}$ is a basis for a topology on X, and we let*

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

Then $\mathcal{B}_Y$ forms a basis for the subspace topology on Y.

This proof is again left for the curious reader, though both axioms follow very naturally. Next time we will talk about the product topology.