HOMEWORK 2: MODULE THEORY
DUE: FRIDAY, MARCH 2ND

1) Show that for a prime ideal $\mathfrak{p}$, $R \setminus \mathfrak{p}$ is a multiplicatively closed set.

2) Compute the localization of the ring $R = \mathbb{Z}/\langle 10 \rangle$ at the multiplicative set $W = \langle 1, 2, 4, 8, 6 \rangle$. In particular, write down all the elements.

3) Show that a map of modules $\varphi : M \to N$ is injective and surjective if and only if there is a 2-sided inverse to φ: $\exists\ \varphi^{-1} : N \to M$ with $\varphi^{-1} \circ \varphi = Id_M$ and $\varphi \circ \varphi^{-1} = Id_N$.

4) Prove the following Proposition from class:

   Proposition 0.1. *There is a natural map* $\mathrm{Hom}_R(N, P) \times \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, P)$ *given by composition.*

   In addition, $End_R(M) := \mathrm{Hom}_R(M, M)$ *has a natural structure as an* R*-algebra.*

5) We define the tensor product of two R-modules M, N to be

$$M \otimes_R N = \{\sum_{i=1}^{l} m_i \otimes n_i \mid m_i \in M, n_i \in N\} / \sim$$

   where $\sim$ is defined by
   i. $rm \otimes n \sim m \otimes rn$
   ii. $m \otimes n + m' \otimes n \sim (m + m') \otimes n$
   iii. $m \otimes n + m \otimes n' \sim m \otimes (n + n')$
   Give $M \otimes_R N$ the structure of an R-module.

6) Recall that if $\varphi : R \to S$ is a ring homomorphism, then S is an R-module. If M is another R-module, show that $M \otimes_R S$ has the structure of an S module.

   Additionally, note that if M is an S-module, M is also an R-module with action $r \cdot m = \varphi(r)m$. Therefore, we can pass modules between rings that are connected by a ring homomorphism.

7) Show that if $N \subseteq M$, and that M/N and N are finitely generated modules, then so is M.

8) Show that if F, F' are free modules, then so is $F \otimes_R F'$. Calculate its rank.

9) Show that two free R-modules are isomorphic, $F \cong F'$, if and only if F and F' have the same rank.

   Hint: You may want to quotient out by the maximal ideal. This is the same as tensoring by $R/\mathfrak{m}$ over R by Class 9, Prop 0.4.