CLASS 27, APRIL 24TH: $\mathrm{Supp}(R)$ VS $\mathrm{Ass}(R)$

Today we will discuss the relationship between associated primes and the support of a given module M. The primary result is as follows:

Theorem 27.1. *If M is an R-module, then $\bigcup_{\mathfrak{p}\in\mathrm{Ass}(M)} V(\mathfrak{p}) \subseteq \mathrm{Supp}(M)$. In particular, associated primes are in the support. Additionally, if R is Noetherian, and $\mathfrak{p} \in \mathrm{Supp}(M)$ is a minimal element, then $\mathfrak{p} \in \mathrm{Ass}(M)$.*

Proof. By Proposition 25.2 (a), it suffices to check that $\mathrm{Ass}(R) \subseteq \mathrm{Supp}(M)$. Let $\mathfrak{p} = \mathrm{Ann}(m)$ be an associated prime and consider the residue field $k(\mathfrak{p}) := (R/\mathfrak{p})_{\mathfrak{p}}$. Since $R/\mathfrak{p} \subseteq M$, we have $k(\mathfrak{p}) \subseteq M_{\mathfrak{p}}$. Therefore, $\mathfrak{p} \in \mathrm{Supp}(M)$.

Now on to the additional statement. Suppose $\mathfrak{p} \in \mathrm{Supp}(M)$ is minimal. This is to say for any $\mathfrak{q} \subsetneq \mathfrak{p}$, we have $M_{\mathfrak{q}} = 0$ but $M_{\mathfrak{p}} \neq 0$.

Step 1: (Prove the result in the local case). Consider $M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module. Since $M_{\mathfrak{p}}$ is a non-zero module over a Noetherian ring, $\mathrm{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq \emptyset$. I claim that the only possibility is $\mathfrak{p}R_{\mathfrak{p}}$ itself. Suppose $\mathfrak{q} \subseteq \mathfrak{p}$. Then

$$(M_{\mathfrak{p}})_{\mathfrak{q}R_{\mathfrak{p}}} = M_{\mathfrak{q}} = 0$$

by our minimality assumption. As a result, $\mathrm{Supp}(M) = \{\mathfrak{p}R_{\mathfrak{p}}\}$. This necessarily implies $\mathrm{Ass}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \{\mathfrak{p}R_{\mathfrak{p}}\}$, since $\mathrm{Ass}(R_{\mathfrak{p}}) \subseteq \mathrm{Supp}(R_{\mathfrak{p}})$.

Step 2: (Trace back to R itself). We note that step 1 yields that there exists $(s, m) \in M_{\mathfrak{p}}$ whose annihilator is precisely $\mathfrak{p}R_{\mathfrak{p}}$. If $t, u \notin \mathfrak{p}$, then t, u map to units in $R_{\mathfrak{p}}$. Therefore $u \notin \mathrm{Ann}(tm)$. This shows that $\mathrm{Ann}(tm) \subseteq \mathfrak{p}$. The assertion that remains to show is that there is a t to make this an equality.

Since $(1, p) \cdot (s, m) \sim (1, 0)$ for all $p \in \mathfrak{p}$, we have that $t \cdot pm = 0$ for some $t \notin \mathfrak{p}$ depending on p. $\mathfrak{p}$ is a finitely generated ideal by the Noetherian condition. As a result, we can choose t_i for $i = 1, \dots, m$ for each of the generators, then notice $\mathrm{Ann}(t_1 \cdots t_n \cdot m) = \mathfrak{p}$. This completes the proof. □

This allows us to somewhat reverse the containment of Theorem 27.1:

Corollary 27.2. *If M is a finitely generated R-module, and R is Noetherian, then*

$$\mathrm{Supp}(M) = \cup_{i=1}^{n} V(\mathfrak{p}_i)$$

where $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ are the minimal primes containing $\mathrm{Ann}(M)$. Each $\mathfrak{p}_i \in \mathrm{Ass}(M)$.

Proof. We know that $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$, since M is finitely generated (by Proposition 25.2 (d)). Additionally, by Corollary 23.4, there are only finitely minimal primes $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ containing $\mathrm{Ann}(M)$, and $V(\mathrm{Ann}(M)) = \cup_{i=1}^{n} V(\mathfrak{p}_i)$. Since they are minimal in the support, Theorem 27.1 shows they are associated primes. □

This style of reasoning allows us to truly get at the structure of finitely generated modules over Noetherian rings:

Theorem 27.3. *If R is a Noetherian ring, and M is a finitely generated R-module, then there exists a chain of submodules*

$$0 = M_0 \subsetneq M_1 \subsetneq \dots \subsetneq M_n = M$$

such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ *for various* $\mathfrak{p}_i \in \operatorname{Spec}(R)$. *In addition,*

$$\operatorname{Ass}(M) \subseteq \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.$$

Proof. If $M \neq 0$, then there exists some associate prime $\mathfrak{p}_1 \in \operatorname{Ass}(M)$. This means precisely that $R/\mathfrak{p}_1 \subseteq M$. Call this module M_1. Then we can consider M/M_1. Either this is 0 or there exists an associated prime $\mathfrak{p}_2$. By the same procedure, we can construct $M_2' \cong R/\mathfrak{p}_2 \subseteq M/M_1$.This corresponds to modules

$$0 \subseteq M_1 \subseteq M_2 = M_2' + M_1 \subseteq M$$

Iterating this procedure yields an ascending chain

$$0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots \subseteq M$$

But M is Noetherian, since it is finitely generated over a Noetherian ring. Therefore the chain eventually must stabilize, at M.

For the latter statement, we can consider $0 \to M_i \to M_{i+1} \to M_{i+1}/M_i \cong R/\mathfrak{p}_{i+1} \to 0$. Induction shows $\operatorname{Ass}(M_{i+1}) \subseteq \operatorname{Ass}(M_i) \cup \operatorname{Ass}(R/\mathfrak{p}_{i+1}) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_{i+1}\}$. □

Definition 27.4. A series associated to M as in Theorem 27.3 is called a **prime filtration** for M.

Prime filtrations are non-unique, this is easily detectable for things such as direct sums: $R/\mathfrak{p} \oplus R/\mathfrak{q}$. Here is another few examples:

Example 27.5. The $\mathbb{Z}$-module $\mathbb{Z}/n\mathbb{Z}$ has a filtration of length $e_1 + \ldots + e_m$, where $n = p_1^{e_1} \cdots p_n^{e_n}$;

$$0 \subseteq Z/p_1\mathbb{Z} \subseteq \mathbb{Z}/p_1^2\mathbb{Z} \subseteq \mathbb{Z}/p_1^n\mathbb{Z} \subseteq \mathbb{Z}/p_1^n p_2\mathbb{Z} \subseteq \ldots \subseteq \mathbb{Z}/n\mathbb{Z}$$

Of course, there are many distinct possibilities for a series. Up to isomorphism, there are (at least)

$$\binom{e_1 + \ldots + e_n}{e_1, \ldots, e_n} = \binom{e_1 + \ldots + e_n}{e_1}\binom{e_2 + \ldots + e_n}{e_2} \cdots \binom{e_{n-1} + e_n}{e_{n-1}}\text{-many}$$

It should be noted that in this case the associated primes are exactly those which appear in the composition series.

Example 27.6. Consider the ring $R = K[x^4, x^3y, xy^3, y^4]$ and the ideal $I = \langle x^4 \rangle$. It can be checked that

$$\operatorname{Ass}(R/I) = \{\langle x^4, x^3y, xy^3\rangle, \mathfrak{m} = \langle x^4, x^3y, xy^3, y^4\rangle\}$$

The first is the annihilator of $(xy^3)^3 = x^3y^9$. Notice

$$x^3y \cdot x^3y^9 = x^4y^{10} = x^6y^{10} = x^4 \cdot (xy^3)^2 \cdot y^4 = 0$$
$$xy^3 \cdot x^3y^9 = x^4y^{12} = x^4(y^4)^3 = 0$$

Of $xy^3 \cdot y^{4i} \neq 0$ in R/I. I leave it to you to check $\operatorname{Ann}((x^3y)^2) = \mathfrak{m}$. This is already weird, as we have a non-minimal element in the associated primes. Such a thing is sometimes called an embedded component. Modding out by the first ideal, we get

$$0 \subseteq M_1 = R/\langle x^4, x^3y, xy^3\rangle \cong K[y^4] \subseteq R/I$$

The cokernel of this inclusion is nothing but a K-vector space generated by $(x^3y)^i$ and $(xy^3)^i$. For $i = 1, 2, 3$. This accounted for with 6 iterations of $R/\mathfrak{m}$ (from the highest degree to lowest).