~~All~~ Both of these notions can be formalized as "Homotopy" operations.

Note: Haydee's Talk

Defn: A **homotopy** is a family of continuous maps

$$F : X \times I \longrightarrow Y \qquad F(x,t) = f_t(x)$$

Thursday Office Hours

Two maps $f,g : X \to Y$ are **Homotopic** if there is a homotopy $F : X \times I \to Y$ w/

$$F(x,0) = f_0(x) = f(x)$$
$$F(x,1) = f_1(x) = g(x)$$

So we can continuously deform one function to the other.

Sept 22 Recall the definition of a homotopy of maps

Defn A retraction of X to A is a map $r : X \to X$ s.t. $r(X) = A$ and $r|_A = Id_A$.

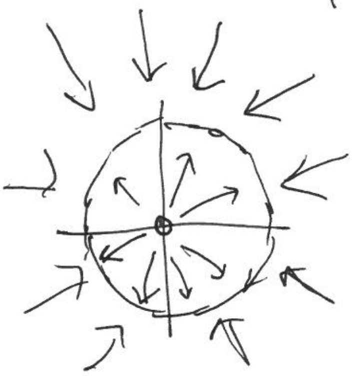
Note retractions are much weaker than deformation retractions. Every space retracts to a point via the constant map.

We can extend the notion of homotopy to relative homotopy:

Defn A Homotopy $F : X \times I \to X$ is a homotopy rel. $A \subseteq X$ if $f_t(a)$ is constant w.r.t. t.

Ex/ $X = \mathbb{R}^2 \setminus \{0\}$ and S^1

$$F : X \times I \to X : (\vec{x}, t) \mapsto t \cdot \frac{\vec{x}}{\|\vec{x}\|} + (1-t)\vec{x}$$



Note: $f_0 \equiv Id_X$

$$f_1(\vec{x}) = \frac{\vec{x}}{\|x\|}$$

and for $\vec{x} \in S^1$, $\|\vec{x}\| = 1$, so

$$F(x,t) = t \cdot \vec{x} + (1-t)\vec{x} = \vec{x}$$

So F is a (linear) homotopy rel S^1.

<u>Homotopy Equivalence</u>: A much weaker notion of equivalence than a homeomorphism:

<u>Defn</u>: $f : X \to Y$ is a homotopy equivalence if $\exists\, g : Y \to X$ s.t. $g \circ f \simeq Id_X$ and $f \circ g \simeq Id_Y$. If such a map f exists, X & Y are Homotopy equivalent.

<u>Compare w/ homeo.</u>

Contractibility: If X is homotopy equivalent to a point, then X is called Contractible.

Let's compare this to a deformation retract to a point:

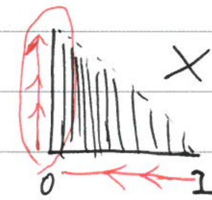
If $\exists\, F: X \times I \to X$ a deformation retract to a point, then $f_0 \equiv Id_X$ and $f_1(x) = p$, with $f_t(p) = p$ always.

So $f: X \to p$ and $g: p \hookrightarrow X$, $f \circ g = Id_p$ and $g \circ f = f_1 \simeq f_0$.

Deformation ret $\to$ pt $\Rightarrow$ contractible.

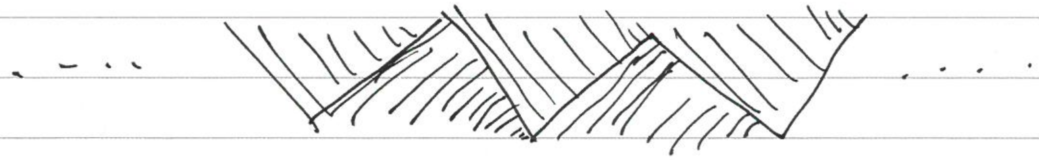
What about the other way?

Consider $X = [0,1] \times \{0\} \cup \bigcup_{r \in \mathbb{Q} \cap [0,1]} \{r\} \times [0, 1-r]$



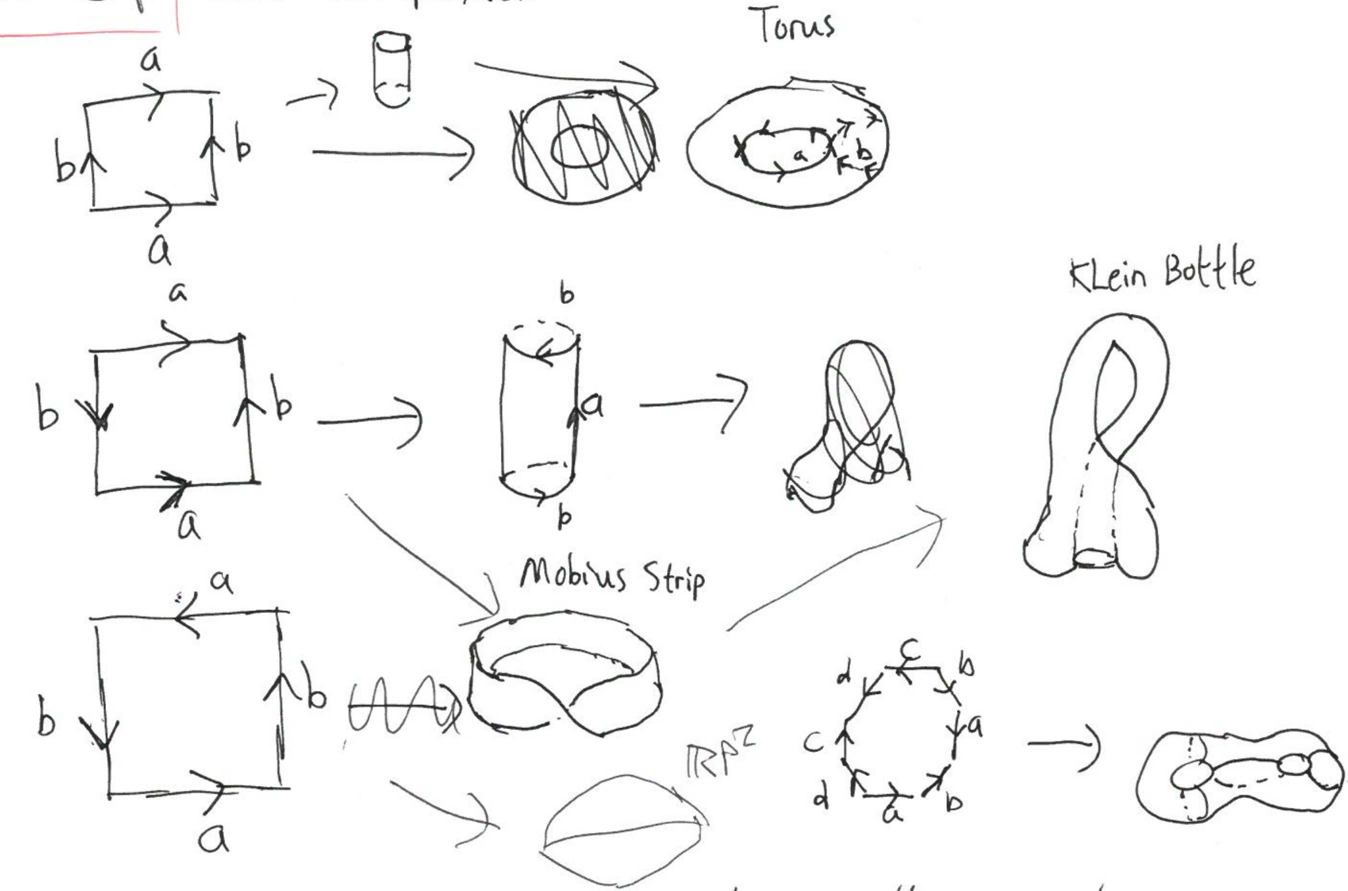
X def retracts to $p \in [0,1] \times \{0\}$, but nowhere else and contracts to any point.

Construct Y by gluing ∞-many of these together as shown



This space is contractible, but not a def retract.

Sept 25: Cell Complexes



Q: How can we generalize this notion to higher dimensions?

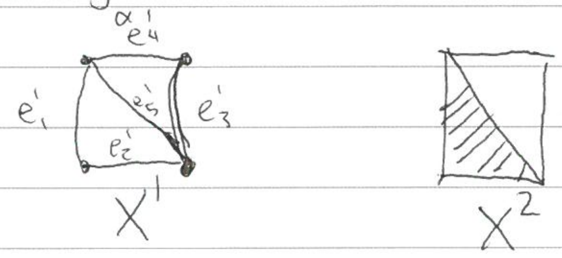
Cell Complex: ~~Idea~~: We want to build a topological space by adjoining disks of higher dimensions to a space obtained previously.

Formally: 1) Start w/ a discrete set X^0 (a collection of points). These are called 0-cells.

2) ~~Inductively~~ we form the "n-skeleton" X^n from X^{n-1} by attaching n-cells e_α^n to X^{n-1}. S^{n-1}

Explicitly $e_\alpha^n : \longrightarrow X^{n-1}$, and Let

$Y^n = X^{n-1} \amalg \coprod_\alpha \mathbb{D}^n_\alpha$. Then X^n is the quotient space $X^n = Y^n/\sim$, where $x \in X^n$ and $y \in \mathbb{D}^n_\alpha$ are equivalent iff $e^n_\alpha(y) = x$.
$\in S^{n-1}_\alpha$



3) You can either iterate this process for finite n and Let $X = X^n$, or continue indefinitely, and let $X = X^n$. In the infinite case, $A \subseteq X$ is open if $A \cap X^n$ is open for each n

- Building S^n: Take a point $P = X^0$

Let $X^0 = X^1 = X^2 = \cdots = X^{n-1}$

$e^n_\alpha : \partial D^n = S^{n-1} \longrightarrow P$

This gives S^n!

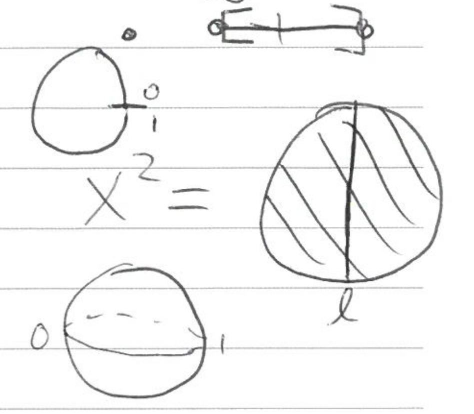
$Y_1 \quad [-1,1] \cup .$

Ex/ $S^1 = X_1$: $X^0 = P$ $\quad X^1 = $

$e^1_1(-1) = e^1_1(1) = P$

$S^2 = X$: $X^0 = P = X^1$ $\quad X^2 =$

$e^2_1 : \partial \mathbb{D}^2 = S^1 \longrightarrow P$

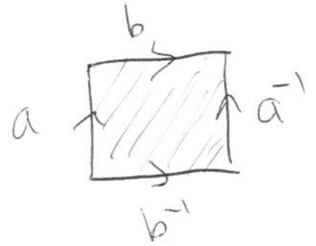


2 ways $S^{n-1} \rightsquigarrow S^n$

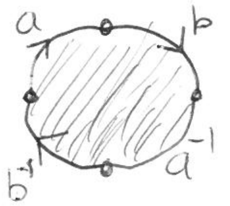
- Building a Torus: $X^0 = P$, $X^1 = [-1,1] \sqcup [-1,1] \cup P$
$= \mathbb{D}^1_1 \cup \mathbb{D}^1_2 \cup P$

- $e^1_a : \partial[-1,1] = \{-1,1\} \to P$
$= e^1_b$

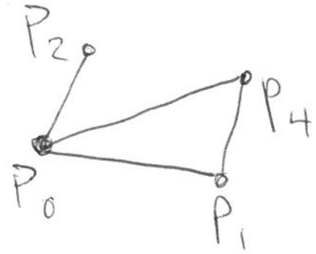
- $X^2 = \mathbb{D}^2 \cup X^1 / \sim$



$e^2_* : \partial \mathbb{D}^2 = S^1 \to X^1$

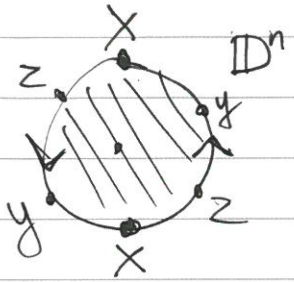


- Building a graph: $X^0 = \{P_i \mid P_i \text{ a vertices}\}$
$X^1 = X^0 \cup \{[0,1]_{i,j} \mid P_i - P_j \text{ an edge}\}$



- $\mathbb{R}P^n$: Space of lines in $\mathbb{R}^{n+1}$
$= \mathbb{R}^{n+1} \setminus \{0\} / \sim \qquad \vec{x} \sim \vec{y} \iff \lambda \cdot \vec{x} = y$, $\lambda \in \mathbb{R}$
$= S^{n} / \sim \qquad \vec{x} \sim -\vec{x}$

We can view it as follows:



$\delta \mathbb{D}^n$ identified to itself by antipodal map.

$\mathbb{R}P^{n-1} \rightsquigarrow \mathbb{R}P^n$ by attaching on n-cell

$$e^n_\alpha : \delta\mathbb{D}^n \longrightarrow \mathbb{R}P^{n-1}$$
$$\overset{\shortparallel}{S^{n-1}}$$
$$\vec{x} \longrightarrow \pm\vec{x}$$

Notes
$\mathbb{R}P^n = \sim\sim =$ attaching maps
Cell Complex = Cw
subcomplex

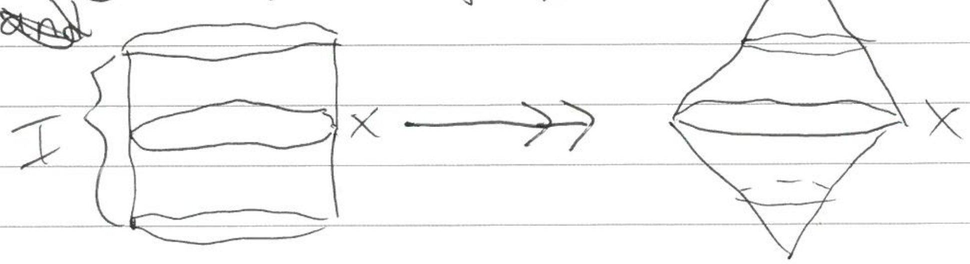
$\boxed{\text{Sept 27}}$ Operations on Spaces

So far we ~~H~~ave Subspace, Product, Quotient Topologies for any space. Let's add a few more!

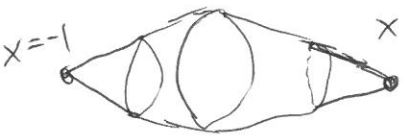
I Suspension of a space X:

$$S(X) \text{ or } \cancel{SX}\ SX = X \times I / \sim$$

where $(x,1) \sim (y,1)$ $\leftarrow \forall x, y \in X$
~~and~~ & $(x,0) \sim (y,0)$ $\leftarrow$



Example: $S^2 = S(S^1)$



$\cong S^2$

$$S^2 = \{x^2+y^2+z^2=1\} \subseteq \mathbb{R}^3$$

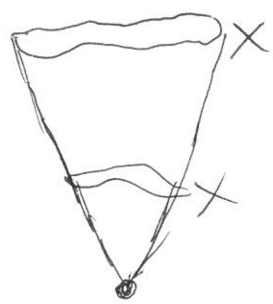
If $x \neq \pm 1$, then $1-x^2=r^2$ for some $r>0$

$$= \{(\pm 1,0,0)\} \cup \{(x,y,z) \mid y^2+z^2=r^2=1-x^2\}$$

In general, $S(S^n) = S^{n+1}$.

One can also form the cone over a space X by

$$CX = X \times I/\sim$$
$$(x,1)\sim(y,1) \quad \forall x,y \in X$$



$$SX = CX \cup CX$$

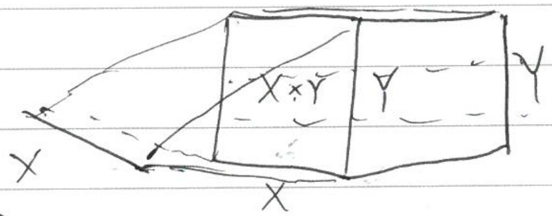
Join of 2 Spaces: One can do better.

CX and SX can be thought of as all the line segments from X to an exterior point (or 2).

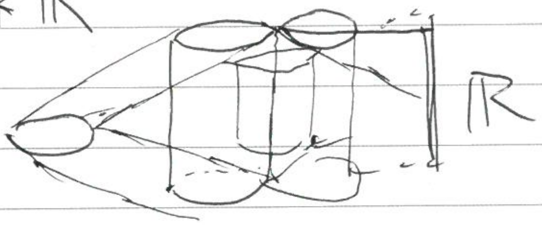
Take 2 Spaces X, Y. We can form the space

$$J(X,Y) = X * Y = X \times Y \times I / \sim$$

$$(x, y_1, 0) \sim (x, y_2, 0) \quad \forall y_1, y_2 \in Y$$
$$(x_1, y, 0) \sim (x_2, y, 0) \quad \forall x_1, x_2 \in X$$



Ex: $S^1 * \mathbb{R}^1$

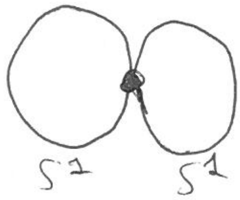


All lines from $X \times Y$ to X and Y preserving id_X and id_Y.

Wedge Sum: Take $x_\alpha \in X_\alpha$. We can form

$$\bigvee_\alpha X_\alpha = \coprod_\alpha X_\alpha / \sim \qquad X_\alpha \sim X_\beta \ \forall \alpha, \beta$$

$S^1 \vee S^1$



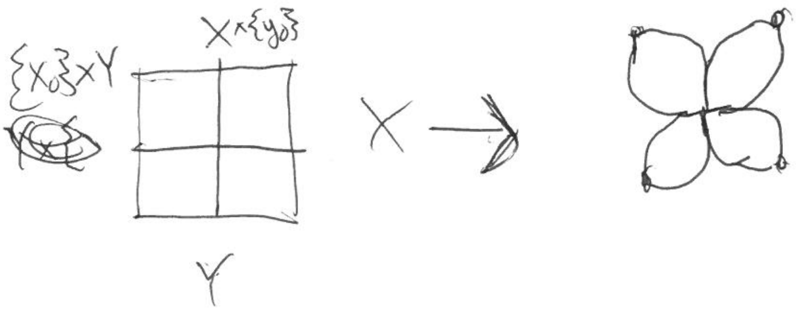
= figure 8

From Cell Complexes, ~~Every~~ X^n / X^{n-1} has the structure of

$$\bigvee_\alpha S^n_\alpha$$

α runs through the n-cells of a complex

(smash product? $X \times Y / X \vee Y$)



$\boxed{\text{Sept 29}}$ Homotopy Extension Property [HEP]

Defn: A pair (X, A) with $A \subseteq X$ a subspace is said to have the homotopy extension property if for every $f_0 : X \to Y$ and ~~a homotopy w/~~ $F_A : A \times I \to Y$ w/

$$\boxed{f_0|_A(a) = F_A(a, 0)}$$

Then $\exists\, F : X \times I \to Y$ extending F_A:

$$F(a,t) = F_A(a,t):$$

$$\begin{tikzcd}
 & X \arrow[r, hook, "\times\{0\}"] \arrow[dr, "f_0"'] & X \times I \arrow[d, dashed, "F"] \\
A \cong A \times \{0\} \arrow[ur] \arrow[d, hook] & & \\
A \times I \arrow[rr, "F_A"] & & Y
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times I \arrow[dr, dashed, "F"] & \\
X \times \{0\} \arrow[u, hook] \arrow[r, "f"] & Y \\
\cup \; A \times I \arrow[ur, "F_A"'] &
\end{tikzcd}$$

where this makes sense

A different way to phrase this:

$$X \times \{0\} \cup A \times I \circlearrowleft \mathrm{Id}$$

Extends to a retraction

$$X \times I \to X \times \{0\} \cup A \times I$$

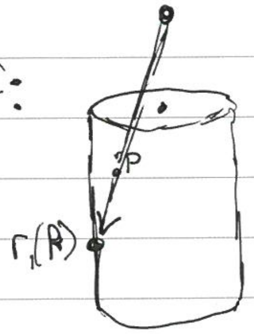
Thus, if (X, A) has the homotopy extn prop so does $(X \times Z, A \times Z)$.

Prop .16: If (X,A) is a CW pair, then (X,A) has the Homotopy extension property

Pf: First let's show our building blocks behave as desired:

Lemma: $\exists\ r_t : \mathbb{D}^n \times I \to \mathbb{D}^n \times I$ a deformation retraction of $\mathbb{D}^n \times I$ to $\mathbb{D}^n \times \{0\} \cup \partial\mathbb{D}^n \times I$

Pf:



$r_1 : \mathbb{D}^n \times I \to \mathbb{D}^n \times \{0\} \cup \partial \mathbb{D}^n \times I$ is a retraction. We can let

$$r_t = (1-t)\,\mathrm{Id} + t \cdot r_1$$

So, it suffice to show a more general result:

Prop: $X^n \times \{0\} \cup (X^n \cap A) \times I$ is a def. ret. of $X^n \times I$

Pf: Note $X^n \times I$ is obtained from $X^n \times 0 \cup (X^{n-1} \cup A) \times I$ by attaching $\mathbb{D}^n \times I$ to it along $\partial \mathbb{D}^n \times I$.

Lemma shows we can ~~contra~~ def retract each of these so we do each in a subinterval of I. ✓

Prop .17: If (X,A) has HEP, and A is contractible, then $X \to X/A$ is a homotopy eq.

Pf

$$\begin{tikzcd} X \arrow[dr, "f_0 = Id_X"] & \\ A \arrow[u, hook] \arrow[r, "f_t"] & X \end{tikzcd} \Rightarrow \quad X \xrightarrow{f_t} X \xrightarrow{q} X/A \quad \text{sends } A \text{ to a point}$$

$$= \quad X \xrightarrow{q} X/A \xrightarrow{\bar{f}_t} X/A$$

$$q f_t = \bar{f}_t q$$

$$\begin{tikzcd} X \arrow[r, "f_t"] \arrow[d, "q"] & X \arrow[d, "q"] \\ X/A \arrow[r, "\bar{f}_t"] \arrow[ur, dashed] & X/A \end{tikzcd}$$

Since $f_1(A) = p$, $\bar{f}_1 : X/A \to X$. Call it g.

$$\begin{tikzcd} X \arrow[r, bend left, "q"] & X/A \arrow[l, bend left, "g"] \end{tikzcd} \quad \checkmark$$

$$q \circ g = \bar{f}_1 \simeq \bar{f}_0 = Id_{X/A}$$
$$g \circ q = f_1 \simeq f_0 = Id_X$$

Prop .18: If (X, A) is a CW pair and $f \simeq g : A \to X_0$, then

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \quad \text{rel } X_0$$

First,

$$X_0 \sqcup_f X_1 = (X_0 \sqcup X_1)/\sim$$

$x_0 \sim x_1$ if $x_0 \in A$ and $f(x_0) = x_1$

Pf: Consider $X_0 \sqcup_F (X_1 \times I) \supseteq \begin{array}{l} X_0 \sqcup_f X_1 \\ X_0 \sqcup_g X_1 \end{array}$

By .16 $\Rightarrow$ They def retract to each subspace so they are hom equiv.