CLASS 24, NOVEMBER 8TH: FOURIER INVERSION

Last time we proved that the Fourier transform of a function of moderate descent has quite steep descent. This allows us to apply an inversion formula to calculate the inversion formula:

Theorem 24.1 (Fourier Inversion Formula)**.** *If* $f \in \mathcal{F}$*, then*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi$$

The proof will use Contour integration, much like Theorem 23.4 from last class. In addition, we need the following Lemma:

Lemma 24.2. *If* $a, b \in \mathbb{R}$ *and* $a > 0$*, then*

$$\int_{0}^{\infty} e^{-(a+ib)\xi} d\xi = \frac{1}{a+ib}$$

Proof. Indeed, this is calculus!

$$\int_{0}^{\infty} e^{-(a+ib)\xi} d\xi = \frac{1}{a+ib} \int_{-\infty}^{b} e^{u} du = \frac{1}{a+ib} \left[e^{u}\right]_{u=-\infty}^{0} = \frac{1}{a+ib}$$

□

Now we can proceed to the proof of Theorem 24.1.

Proof. Let $f \in \mathcal{F}_a$ and write

$$\int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi = \int_{-\infty}^{0} \hat{f}(\xi) e^{2\pi i x\xi} d\xi + \int_{0}^{\infty} \hat{f}(\xi) e^{2\pi i x\xi} d\xi =: (-) + (+)$$

to separate into cases based on sign. For $(+)$, let $0 < b < a$. As in the proof from last time, we may conclude that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(u - ib) e^{-2\pi i (u-ib)\xi} du$$

This yields a computation:

$$\begin{aligned}
(+) &= \int_{0}^{\infty} \int_{-\infty}^{\infty} f(u-ib) e^{-2\pi i (u-ib)\xi} e^{2\pi i x\xi} du d\xi \\
&= \int_{-\infty}^{\infty} f(u-ib) \int_{0}^{\infty} e^{-2\pi i (u-ib)\xi} e^{2\pi i x\xi} d\xi du \\
&= \int_{-\infty}^{\infty} f(u-ib) \frac{1}{2\pi b + 2\pi i (u-x)} du \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(u-ib)}{u - ib - x} du \\
&= \frac{1}{2\pi i} \int_{-\infty-ib}^{\infty-ib} \frac{f(\zeta)}{\zeta - x} d\zeta
\end{aligned}$$

Going through this set of equations requires some explanation. The first line is the definition. The second equality is through a process of flipping the 2 integrals. In general, this is a delicate process. But since in this case we have that the integrals of the absolute values converge, it is easy to ensure using only the finite case. One should see Fubini's theorem for any further clarification.

The 3rd equality is Lemma 24.2. The forth is simply rearranging terms. The final is using the substitution $\zeta = u - ib$, $d\zeta = du$. The integral is stated more precisely as the integral along the straight line path with imaginary part b.

A similar computation works for $(-)$ and shows

$$(-) = -\frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{f(\zeta)}{\zeta - x}d\zeta$$

Now we can consider the rectangle $\mathcal{R}$ of height 2b centered at the origin. Cauchy's integral theorem tells us that

$$\frac{1}{2\pi i}\int_{\mathcal{R}}\frac{f(\zeta)}{\zeta - x}d\zeta = f(x)$$

Just like in the proof of the Theorem 23.4, the integrals over the vertical sides of this rectangle approach 0. Thus we are left with

$$\begin{aligned} f(x) &= -\frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{f(\zeta)}{\zeta - x}d\zeta + \frac{1}{2\pi i}\int_{-\infty-ib}^{\infty-ib}\frac{f(\zeta)}{\zeta - x}d\zeta \\ &= \int_{-\infty}^{0}\hat{f}(\xi)e^{2\pi i x\xi}d\xi + \int_{0}^{\infty}\hat{f}(\xi)e^{2\pi i x\xi}d\xi = \int_{-\infty}^{\infty}\hat{f}(\xi)e^{2\pi i x\xi}d\xi \end{aligned}$$

As an application of this analysis, we get some neat formulas involving all of the Fourier $\square$ transforms we have computed so far. An additional application which is very common in signal processing is the following example:

Example 24.3. Define the square function by

$$sq(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$

This function looks like a flat line seperated by 2 big jumps. Let's compute its Fourier transform:

$$\begin{aligned} \hat{sq}(\xi) &= \int_{-\infty}^{\infty} sq(x)e^{-2\pi i\xi x}dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i\xi x}dx \\ &= -\frac{1}{2\pi i\xi}[e^{-\pi i\xi} - e^{\pi i\xi}] = \frac{1}{\pi\xi}\frac{e^{-\pi i\xi} - e^{\pi i\xi}}{2i} = \frac{\sin(\pi\xi)}{\pi\xi} \end{aligned}$$

So the Fourier transform of the square function is a sin-wave.

An interesting application used for the past century is then the fact that a sin-wave has an inverse Fourier transform of one of these square functions. If we went through this analysis with a larger magnitude, that would pull out from the integral. Similarly, if we change the periodicity, that would be an adjusted square function (a rectangle). In particular, if $\hat{f} = M\frac{\sin(f\xi)}{x}$, then

$$f(x) = \begin{cases} \pi M & |x| \leq \frac{f}{2\pi} \\ 0 & |x| > \frac{f}{2\pi} \end{cases}$$