CLASS 15, OCTOBER 16: APPLICATIONS OF RESIDUES

Today we will consider a few other important examples where the residue theorem can be directly applied.

Example 15.1. Consider the following integral with $a \in (0,1)$:

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$$

To evaluate this, we will use the rectangle with sides the real axis and $Im(z) = 2\pi$. Now, notice that the only singular point occurs when $e^z = 1$, which occurs exactly at πi. Therefore, to compute this integral, we only need to compute its residue here. Consider $(z - \pi i)f(z)$. Given the fact that $e^{i\pi} = -1$, we can rewrite this as

$$(z-\pi i)f(z) = \frac{z-\pi i}{1+e^z} \cdot e^{az} = \frac{z-\pi i}{e^z - e^{i\pi}} \cdot e^{az} = \frac{1}{\frac{e^z - e^{i\pi}}{z-\pi i}} \cdot e^{az}$$

Now, as we send $z \to \pi i$, we acquire that this is the inverted derivative of e^z:

$$\text{res}_{\pi i}((z-\pi i)f(z)) = \lim_{z\to\pi i}(z-\pi i)f(z) = \frac{e^{a\pi}}{e^{\pi i}}$$

As a result, the residue formula yields

$$\int_\gamma \frac{e^{az}}{1+e^z} dz = 2\pi i \frac{e^{a\pi i}}{e^{\pi i}} = -2\pi i e^{a\pi i}$$

Now, it goes to consider the parts individually. For the right side of the integral, we achieve

$$\int_0^{2\pi} f(R+it) i dt = \int_0^{2\pi} \frac{e^{a(R+it})}{1+e^{R+it}} i dt$$

In absolute value, the integrand is bounded by

$$2\pi \frac{e^{aR}}{e^R - 1} \to 0$$

as $R \to \infty$. As a result, the integral is bounded by 2π times something going to 0. The same is true for the left hand side of the rectangle with slight modification of the bound. Therefore, it only goes to compute the remaining portions. The bottom is the quantity we are interested in. The top is

$$-\int_{-R}^{R} f(t+2\pi i)dt \to -\int_{-\infty}^{\infty} \frac{e^{at}e^{2\pi i a}}{1+e^t} dt = -e^{2\pi i a} \int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t} dt$$

So in total, we get

$$(1-e^{2\pi i a}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = -2\pi i e^{a\pi i}$$

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = -\frac{2\pi i e^{a\pi i}}{1-e^{2\pi i a}} = -2\pi i \frac{1}{e^{-\pi i a} - e^{\pi i a}} = \frac{\pi}{\sin(\pi a)}$$

We have already seen that $e^{-\pi x^2}$ is its own Fourier transform. Here we will find another function with this property.

Example 15.2. I will demonstrate that

$$\frac{1}{\cosh(\pi\xi)} = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)} dx$$

We will again use the rectangle, but this time let the height be $2i$. Since $\cosh(z) = \frac{e^z+e^{-z}}{2}$, we have $\cosh(\pi z) = 0$ precisely when $e^{\pi z} = -e^{-\pi z}$, or when $e^{2\pi z} = -1$. This occurs exactly when $z = \alpha := i\frac{1+2n}{2}$ for $n \in \mathbb{Z}$. Again, one can check that these are simple poles:

$$(z-\alpha)f(z) = 2e^{-2\pi i z\xi}e^{\pi z}\frac{z-\alpha}{e^{2\pi z} - e^{2\pi\alpha}} \to 2e^{-2\pi i \frac{i}{2}\xi}e^{2\pi\alpha}\frac{1}{2\pi e^{2\pi\alpha}} = \frac{e^{\pi\xi}}{\pi i}$$

as $z \to \alpha$. Again, the vertical sides go to 0 as $R \to \infty$, since

$$|e^{2\pi i z\xi}| \leq e^{4\pi|\xi|}$$

and ξ is fixed, where as

$$\cosh(z)| \geq \frac{1}{2}\left|\left(|e^{\pi z}| - |e^{-\pi z}|\right)\right| = \frac{1}{2}(e^{\pi R} - e^{-\pi R}) \to \infty$$

Thus again we have

$$(1-e^{4\pi\xi})\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)} dx = 2\pi i\left(\frac{e^{\pi\xi}}{\pi i} + \frac{-e^{3\pi\xi}}{\pi i}\right) = 2(e^{\pi\xi} - e^{3\pi\xi})$$

Finally, rearranging cosh yields the desired result:

$$\frac{1}{\cosh(\pi\xi)} = \frac{2}{e^{\pi\xi}+e^{-\pi\xi}} = \frac{2}{e^{\pi\xi}+e^{-\pi\xi}}\frac{e^{\pi\xi}-e^{-\pi\xi}}{e^{\pi\xi}-e^{-\pi\xi}} = 2\frac{e^{\pi\xi}-e^{-\pi\xi}}{e^{2\pi\xi}-e^{-2\pi\xi}}$$

The last quantity is exactly the desired integral.

The importance of this integral is again to do with the Fourier transform. What we have just proven is that the Fourier transform of $\frac{1}{\cosh(\pi x)}$ is itself. We will go into more depth in chapter 4 regarding these transforms.

Next time we will talk singularities more in general. We will see that not all of them are simply poles. There is an idea of an **essential singularity**, which is something like an infinite order pole!

Example 15.3. Consider the function $e^{\frac{1}{z}}$. At 0, it is easy to check that the following limit never exists:

$$\lim_{z\to 0} z^n e^{\frac{1}{z}}$$

One can think about this as when we plug $\frac{1}{z}$ into the taylor expansion for e^z, we will never remove all of the powers of $\frac{1}{z}$.

So we will attempt to tackle functions like this and create a notion of a function that avoids such difficult properties.