CLASS 7, SEPTEMBER 21: THE QUOTIENT TOPOLOGY

Next we study a topology that can be induced by any surjective map $f : X \to Y$ from a topological space X to a set Y. This adds a lot of depth to our knowledge of existent topological spaces and compares to the idea of a quotient group in abstract algebra.

Definition 7.1. Let X be a topological space and Y be a set. We define the **quotient topology** induced by a surjective map $p : X \to Y$ by the following property:

$$U \subseteq Y \text{ is open} \Leftrightarrow p^{-1}(U) \subseteq X \text{ is open}$$

In such a case, p is called a **quotient map**.

Note that this is stronger than continuity, which only implies the $\Rightarrow$ implication.

Proposition 7.2. *The quotient topology is a topology.*

Proof. Let $\tau = \{U \subseteq Y \mid p^{-1}(U) \subseteq X \text{is open}\}$.

1) $p^{-1}(Y) = X$ and $p^{-1}(\emptyset) = \emptyset$ are open subsets of X, so $Y, \emptyset \in \tau$.
2) If $U_\alpha \in \tau$, then
$$p^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha p^{-1}(U_\alpha).$$
Thus $\bigcup_\alpha U_\alpha \in \tau$.
3) Similarly,
$$p^{-1}(U_1 \cap U_2 \cap \ldots \cap U_n) = p^{-1}(U_1) \cap p^{-1}(U_2) \cap \ldots \cap p^{-1}(U_n)$$

Finally, since p is surjective, this satisfies the condition in the definition. That is to say, if $p^{-1}(U)$ is open, we know $U = p(p^{-1}(U))$, and thus $p^{-1}(U)$ being open implies U is. $\square$

Example 7.3. Consider $\mathbb{R}^2$ with the Euclidean topology. We can construct a map $f : \mathbb{R}^2 \to [0, 1)^2$

$$f(x, y) = (x \pmod 1, y \pmod 1)$$

Equivalently, take the decimal portion of a positive real number (or its complement if negative). This places a topology on the set $[0, 1)^2$ via the quotient topology. This is the Torus, the boundary of a doughnut, with its standard topology.

Example 7.4. Consider the subspace $X = [0, 1] \cup [2, 3] \subseteq \mathbb{R}$ and $Y = [0, 2] \subseteq \mathbb{R}$. Then we can construct the map $p : X \to Y$ with

$$p(x) = \begin{cases} x & 0 \leq x \leq 1 \\ x - 1 & 2 \leq x \leq 3 \end{cases}$$

This map does satisfy the axioms of a quotient map; it is surjective and open intervals pull back to open sets. However, this map is *not* open. In particular, $p([0, 1]) = [0, 1]$, which is not open even though $[0, 1] \subseteq X$ is.

Another particularly common example of a quotient mapping is found by 'collapsing' a subspace to a point.

Definition 7.5. Let $A \subseteq X$. Let Y be the set $Y = (X \setminus A) \cup \alpha$, where α is a distinguished element. We can define a map $p : X \to Y$ by the relation

$$p(x) = \begin{cases} x & x \notin A \\ \alpha & x \in A \end{cases}$$

Then Y with the quotient topology is called the quotient of X by A.

Example 7.6. Take $\mathbb{D}^2 = \bar{B}(0,1)$ the unit ball in $\mathbb{R}^2$ (or for the brave, $\mathbb{R}^n$). Take A to be the boundary subspace, defined by the points exactly of distance 1 from the origin. A can be naturally viewed as S^1 (resp. S^{n-1}). Then quotienting X by A produces a S^2 (resp. S^n) with its usual topology.

Theorem 7.7. *Let $p : X \to Y$ be a quotient map, and let $f : X \to Z$ be any map with the property that for a fixed $y \in Y$, and all $x, x' \in p^{-1}(y)$, $f(x) = f(x')$. Then there exists a map $g : Y \to Z$ with $g \circ p = f$. Furthermore, g is a continuous (resp. quotient) map if and only if f is continuous (resp. a quotient).*

Proof. If $x \in p^{-1}(y)$, we define $g(y) := f(x)$. This is well defined by the constancy condition of the theorem, and thus produces a map of sets. with $g \circ p = f$.

Now, to show the statement about continuity, suppose $U \subseteq Z$ is an open subset. Then by definition of p being a quotient map, we see

$$p^{-1}(g^{-1}(U)) = f^{-1}(U) \subseteq X \text{ is open} \Leftrightarrow g^{-1}(U) \subseteq Y \text{ is open}$$

Similarly, given p is surjective, it follows that g is surjective if and only if f is. Finally, again using the fact that p is a quotient map, we see that $U \subseteq Z$ is open would be equivalent to the following equivalent conditions:

$$f^{-1}(U) \subseteq X \text{ is open} \Leftrightarrow g^{-1}(U) \subseteq Y \text{ is open}$$

□

This allows us to realize that a quotient space is simply many quotients by individual subspaces.

Corollary 7.8. *Let $f : X \to Z$ be a continuous surjective map. Let $X^* = \{f^{-1}(z) \mid z \in Z\}$ be the set of fibers of the map f. Note $p : X \to X^* : x \mapsto f^{-1}(f(x))$ is a surjective map. Give X^* the quotient topology by p.*

Given this setup, the map $g : X^* \to Z$ from Theorem 7.7 is a continuous bijective map which is a homeomorphism if and only if f is a quotient map.

Proof. Indeed, if g is a homemorphism, then it is in particular a quotient map, thus f is as well by Theorem 7.7. On the flip side, if f is a quotient map, so is g. Therefore, g is a bijective map that is open: $g(U)$ is open since $f^{-1}(g(U)) = p^{-1}(U)$ is open. Thus g is a homeomorphism. □