CLASS 15, OCTOBER 15: COUNTABILITY AXIOMS

Recall from last time the notion of first countable, which played an essential role in relating Hausdorffness to convergence of sequences. Today we will discuss first and second countability in greater detail.

Definition 15.1. A topological space X is said to be **second-countable** if there exists a countable basis for the topology.

Therefore, we note immediately that second-countable implies first-countable, as we can take the subset of the basis illustrated in the definition of open sets containing a given point $x \in X$. The following example illustrates that they are not equivalent in general.

Example 15.2.
- $X = \mathbb{R}^n$ with the Euclidean topology is a second countable space. As previously discussed (in ancient times, Class 3), we can take
$$\mathcal{B} = \{B(\mathbf{x}, d) \mid \mathbf{x} = (x_1, \ldots, x_n),\ x_i \in \mathbb{Q},\ d \in \mathbb{Q}_+\}$$
This is countable, since we have a finite $n+1$-fold product of countable sets, thus countable.
- Let $X = \mathbb{R}^{\mathbb{N}}$ with the uniform topology. Last time, we demonstrated that every metric space is first-countable by taking rational-radii. I claim X is not second-countable. Using the lemma that follows, Lemma 15.3, we note that $A = \mathbb{Z}^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$ has the discrete topology, taking $U_a = B(a, \frac{1}{2})$ for example. On the other hand, note that we have a surjection
$$A \to \mathbb{R} : (a_1, a_2, \ldots) \mapsto \ldots \bar{a}_5 \bar{a}_3 \bar{a}_1 . \bar{a}_2 \bar{a}_4 \ldots$$
where we are taking a decimal expansion of a number, and letting $\bar{a}_i$ is the singles digit (remainder $\pmod{10}$) of a_i. Since $\mathbb{R}$ is uncountable, so is A. Therefore, we note X is not second-countable by the contrapositive of Lemma 15.3.

Lemma 15.3. *If X is a second-countable space, then if $A \subseteq X$ has the discrete topology (as a subspace), then A is countable.*

Proof. Since A has the discrete topology, for any point $a \in A$, there exists U_a an open basis element such that $U_a \cap A = \{a\}$. However, this induces an injection
$$\iota : A \to \mathcal{B} : a \mapsto U_a.$$
Therefore, since $\mathcal{B}$ is countable, so is A. □

We now note that our standard operations on spaces maintain the countability axioms:

Theorem 15.4.
1) If X is first (or second) countable, then so is $Y \subseteq X$ with the subspace topology.
2) If $X_1, \ldots, X_n$ is first (or second) countable, then so is $X_1 \times \ldots \times X_n$.

Just in case you haven't seen the typical logical tricks, note that a countable union of countable sets is countable. Also, a finite product of countable sets is countable.

Proof. 1) The subspace topology has a basis given (by Lemma 3.7) as
$$\mathcal{B}_Y = \{U \cap Y \mid U \in \mathcal{B}\}$$
So if X is first countable, then the countable neighborhood base of $y \in Y \subseteq X$ will do. Similarly, if $\mathcal{B}$ is countable, then $\mathcal{B} \to \mathcal{B}_Y : U \mapsto U \cap Y$ is a surjection.

2) The product topology has a basis given (by Proposition 4.3) as
$$\mathcal{B} = \{U_1 \times \ldots \times U_n \mid U_i \in \mathcal{B}_i\}$$
If X_i is first countable for all i, then the product of the countable neighborhood bases will do. Similarly, if $\mathcal{B}$ is countable, then
$$\mathcal{B}_1 \times \ldots \times \mathcal{B}_n \to \mathcal{B} : (U_1, \ldots, U_n) \mapsto U_1 \times \ldots \times U_n$$
is a surjection.

□

Finally, we proceed to some statements on density of subsets. This plays an essential role in many of our countability axioms in practice; namely that $\mathbb{Q} \subseteq \mathbb{R}$ is dense.

Definition 15.5. $A \subseteq X$ is said to be **dense** if $\bar{A} = X$.

Theorem 15.6. *Suppose X is second-countable.*

1) Every open cover of $Y \subseteq X$ has a countable refinement.
2) There exists a countable dense subset of X.

In this theorem, we are saying that every subset of a second countable space is something weaker than compact, but still has controllable covers. The second statement generalizes the idea $\mathbb{Q} \subseteq \mathbb{R}$ to any second countable space!

Proof. Let $X_1, X_2, \ldots$ be a countable base for X.

1) It suffices to prove the statement for X by Theorem 15.4, part 1. Let $X = \bigcup_\alpha U_\alpha$. Choose $U_{\alpha_i} \supset X_i$ (which exists by definition of basis). Then
$$X = \bigcup_\alpha U_\alpha \supseteq \bigcup_{i=1}^\infty U_{\alpha_i} \supseteq \bigcup_{i=1}^\infty X_i = X$$
where the last equality again follows by definition of a basis noting X is open. Therefore, everything above is equal. Thus $X = \bigcup_{i=1}^\infty U_{\alpha_i}$ is a countable refinement.

2) Choose $x_i \in X_i$ arbitrarily. I claim the set $D = \{x_1, x_2, \ldots\}$ is the desired dense subset. Indeed, if $x \in \bar{D}^c$, then there exists some X_i such that $x \in X_i \in \bar{D}^c \subseteq D^c$. But $x_i \in D \cap X_i$. This is a contradiction.

□

This leads us to a pretty startling realization:

Example 15.7. I claim $\mathbb{R}^\mathbb{N}$ is second-countable with the product topology.[1] Recall that the product topology is given by
$$\mathcal{B} = \{\left(\prod_{i=1}^n U_i\right) \times \mathbb{R}^{\mathbb{N} \setminus \{1,\ldots,n\}} \mid U_i \subseteq \mathbb{R} \text{ is open.}, n > 0\}$$
If we fix a specific n, the elements of the form $(\prod_{i=1}^n U_i) \times \mathbb{R}^{\mathbb{N}\setminus\{1,\ldots,n\}}$ are in bijection with the countable basis of the topology $\mathbb{R}^n$. Call this set $\mathcal{B}_n$. Then
$$\mathcal{B} = \bigcup_{n>0} \mathcal{B}_n$$
Since a countable union of countable sets is countable, $\mathcal{B}$ is countable!

Therefore, there exists a countable dense subset of $\mathbb{R}^\mathbb{N}$ with the Euclidean topology.

[1]This follows directly from the following more general statement: A countable product of second-countable spaces is second countable. The proof is identical.