HOMEWORK 9: METRIZATION AND COMPLETENESS
DUE: NOVEMBER 30

1) Show that if X is a T3 space, and $X = \bigcup_{i \in \mathbb{N}} K_i$ where K_i are compact subspaces, then X is paracompact. Use this to show that

$$\mathbb{R}^{\oplus \mathbb{N}} = \{x \in \mathbb{R}^{\mathbb{N}} \mid x_i = 0 \ \forall i \gg 0\}$$

with the box topology is paracompact.

Solution: Let $\{U_\alpha\}$ be an open covering of X. By Lemma 27.6, note that it suffices to check that there is a countably locally finite open covering refinement of $\{U_\alpha\}$. However, this is easy! Since U_α is also a covering of K_i for each i, choose $U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$ a finite subcover and call the collection of these $\mathfrak{B}_n$. Then $\mathfrak{B}_n$ is finite and in particular locally finite. Therefore, $\mathfrak{B} = \bigcup_i \mathfrak{B}_i$ is a countably locally finite open refinement of U_α.

As a result, we can conclude that

$$\mathbb{R}^{\mathbb{N}} = \bigcup_{i=1}^{\infty} \left([-i, i]^i \times \{0\}^{\mathbb{N} \setminus [i]}\right) = \bigcup_i K_i$$

is paracompact. Note that this union is the whole space since every element $(x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ has the property that it is in K_N where $N = \max\{|x_1|, \ldots, |x_n|, n\}$.

2) Show that if X is a T3 space, then X is paracompact if either
   - $X = X_1 \cup \ldots \cup X_n$, where X_i are paracompact closed subspaces.
   - $X = \bigcup_{i \in \mathbb{N}} X_i$, where X_i are paracompact closed subspaces with X_i° still covering X.

Solution:

- Note that since X is T3, so is each X_i. Let $\{U_\alpha\}$ be an open cover. By Lemma 27.6, it suffices to show that we can find a locally finite closed refinement of $\{U_\alpha\}$. We can do this on each X_i since they are paracompact. Call $\mathfrak{B}_i$ the resulting refinement. The note since we have a collection of closed subsets of a closed set, $\mathfrak{B}_i$ is composed of closed subsets of X!

  Finally, I claim that $\mathfrak{B} = \mathfrak{B}_1 \cup \ldots \cup \mathfrak{B}_n$ is the desired refinement. Given $x \in X$, let $X_{i_1}, \ldots, X_{i_m}$ be the subsets which contain x. For each, there exists an open neighborhood U_{i_j} of x in X intersecting finitely many elements of $\mathfrak{B}_{i_j}$. Then for

  $$x \in U = U_{i_1} \cap \ldots \cap U_{i_m} \cap \bigcap_{i \neq i_j} X_i^C$$

  we find that $\mathfrak{B}$ intersects U at most finitely many times.
- Let $\{U_\alpha\}$ be a cover of X. This is of course still a cover when restricted to X_i, which is paracompact. Therefore, there exists a locally finite refinement $\mathfrak{B}_i = \{V_\beta\}$ of U_α of U_i. Note V_β is only open as a subset of X_i, not X. So let V'_β be an open set of X such that $V'_\beta \cap X_i = V_\beta$. Then note that $V'_\beta \cap X_i^\circ$ is an open

subset of X contained within U_α. Therefore, $\{V'_\beta\}$ is a locally finite open cover of X_i°. Call it $\mathfrak{B}_i$.
As a result $\mathfrak{B} = \bigcup_i \mathfrak{B}_i$ is a countably locally finite refinement of $\{U_\alpha\}$. Note that it is a cover since each $\mathfrak{B}_i$ covers X_i°, and thus the union of all of them covers $X = \bigcup_{i\in\mathbb{N}} X_i^\circ$. Since X is assumed T3, Lemma 27.6 shows that X is paracompact.

3) Show that if X is a complete metric space, and $A_1 \supseteq A_2 \supseteq \dots$ is a nested sequence of closed subsets for which $\operatorname{diam}(A_n) \to 0$, then $\bigcap_i A_i \neq \emptyset$. Note that here the **diameter** is given by
$$\operatorname{diam}(A) = \sup\{d(x,y) \mid x, y \in A\}$$

Solution: Choose $x_n \in A_n$. Since $\operatorname{diam}(A_n) \to 0$, we can see that for $m > n$,
$$d(x_n, x_m) \leq \operatorname{diam}(A_n)$$
Therefore for $\epsilon > 0$, choose $n \gg 0$ such that $\operatorname{diam}(A_m) < \epsilon$ for every $m > n$. As a result, x_n is a Cauchy sequence. Therefore it converges to some $x \in X$. Therefore, it only goes to show that $x \in \bigcap_{n\in\mathbb{N}} A_n$. The right hand side is a closed set, so given $B(x,\epsilon)$, by the previous step we know that
$$x_n \in B(x,\epsilon) \cap A_n \subseteq B(x,\epsilon) \cap A_m \neq \emptyset$$
For all $n \geq m$. But this implies every neighborhood of x intersects A_n for all n, and thus $x \in A_n$ for all n, or equivalently $x \in \bigcap_{n\in\mathbb{N}} A_n$.

4) Given X and Y spaces, consider $\mathcal{C}$ the space of continuous functions $X \to Y$ and the evaluation map
$$ev : X \times \mathcal{C} \to Y : (x, f) \mapsto f(x)$$
Show that if Y is a metric space, and $\mathcal{C}$ has the uniform topology, then ev is continuous.

Solution: Given $\epsilon > 0$, and $y \in Y$, suppose $ev(x,f) = y$. It goes to find a neighborhood U of (x,f) such that $ev(U) \subseteq B(y,\epsilon)$. Consider
$$U_1 = f^{-1}\left(B(y, \frac{\epsilon}{2})\right) \subseteq X$$
$$U_2 = B\left(f, \frac{\epsilon}{2}\right) = \{g \mid d(g(x), f(x)) < \frac{\epsilon}{2} \quad \forall x \in X\} \subseteq \mathcal{C}$$
Note U_1 is open since f is continuous. I claim that $U = U_1 \times U_2$ works. Let $(x', g) \in U$. Then the triangle inequality implies
$$d(f(x), g(x')) \leq d(f(x), f(x')) + d(f(x'), g(x')) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$
This proves the claim.

5) Show that the completion of a metric space is unique. That is to say if there exist Y, Y' completions of X, then there exist distance preserving continuous maps (**isometries**) $f : Y \to Y'$ and $g : Y' \to Y$ which preserve X.

Solution: If x_n is a Cauchy sequence in X, it converges to point $y \in Y$ and $y \in Y'$. Define $f : Y \to Y'$ by sending y to y' and $g : Y' \to Y$ sending y' to y. Note that this gives a definition to every point of Y and Y', since the closure of X in these spaces is Y or Y' respectively.

Now I show this is well defined. Suppose x_n and x'_n are two Cauchy sequences converging to $y \in Y$. For a given $\epsilon > 0$, $\exists n \gg 0$ such that $d(x_m, x'_{m'}) < \epsilon$ for all $m, m' \geq n$:

$$d(x_m, x'_{m'}) \leq d(x_m, y) + d(y, x'_{m'}) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

As a result, x_n and x'_n converge to the same point of Y'.

Next, I show that distance is well defined. For every $y, y' \in Y$, we can find x, x' such that $d_Y(y, x), d_Y(y', x') < \frac{\epsilon}{2}$. As a result

$$d(x, x') = d_Y(x, x') \leq d_Y(x, y) + d_Y(y, y') + d_Y(y', x') < d(y, y') + \epsilon$$

$$d_Y(y, y') \leq d_Y(y, x) + d_Y(x, x') + d_Y(x', y') < d(x, x') + \epsilon$$

So symmetrically $d_Y(y, y'), d_{Y'}(f(y), f(y')) \in B(d(x, x'), \epsilon)$. But this is true for any ϵ, so they agree.

Finally, we can take the constant Cauchy sequence x to show x is preserved. This completes the proof.

6) A map $p : Y \to X$ is said to be **perfect** if it is continuous, surjective, closed, and for each $x \in X$, $p^{-1}(x)$ is compact. You have encountered perfect maps in Homework 4.

Let X be a Hausdorff space. If $\gamma : I \to X$ is a space filling curve, show γ is a perfect map.

Perfect maps preserve many properties of a space, e.g. if X is second-countable, so is Y. Use this to show X with a space filling curve is metrizable.

Solution: γ is continuous and surjective by assumption. Moreover, if $x \in X$, then $\{x\}$ is a closed set since X is Hausdorff. Therefore, $\gamma^{-1}(x)$ is a closed subset of a compact set and therefore compact.

It only goes to show γ is closed. Note that $\gamma(I) = X$, so X is necessarily compact. If $C \subseteq I$ is a closed set, then it is compact and thus $\gamma(C)$ is a compact set by continuity. But X is Hausdorff, so it is necessarily closed. Therefore γ is perfect.

Since X is a compact Hausdorff space, X is metrizable if and only if it is second-countbale. Since I is second-countable, X is as well.

7) The converse of the previous problem is the **Hahn-Mazurkiewicz Theorem**: If X is compact, connected, locally connected, and metrizable, then there exists a space filling curve in X. Use it to show there exists a space filling curve in $I^{\mathbb{N}}$ with the product topology.

Solution: Note $I^{\mathbb{N}}$ is connected and compact since it is a product of such sets.

Given $x \in I^{\mathbb{N}}$ and an open neighborhood U of x, we can find

$$x \in U_1 \times \cdots \times U_n \times I^{\mathbb{N} \setminus [n]} \subseteq U$$

But we can simply take a connected neighborhood of x_i in U_i, say $C_i \subseteq U_i$, and consider $x \in C_1 \times \cdots \times C_n \times I^{\mathbb{N} \setminus [n]}$. This in fact shows that $I^{\mathbb{N}}$ is locally connected.

Finally, $I^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$ has the subspace topology, and therefore is metrizable.

Applying Hahn-Mazurkiewicz implies the desired result.