The Euler characteristic $\chi(X)$ for a finite Δ-complex X is defined to be

$$\chi(X) := \sum_{i=0}^{n} (-1)^i \operatorname{rank}(\Delta_i(X))$$

Here, rank is defined to be the largest n such that $\mathbb{Z}^n \hookrightarrow \Delta_i(X)$. We showed that $\chi(X)$ is well defined (independent of Δ-complex structure) by showing that

$$\chi(X) = \sum_{i=0}^{n} \operatorname{rank}(H_i(X))$$

This was done using two short exact sequences and a lemma:

$$0 \to Z_i(X) \to \Delta_i(X) \xrightarrow{\partial_n} B_{i-1}(X) \to 0$$
$$0 \to B_i(X) \to Z_i(X) \to H_i(X) \to 0$$

Here $Z_n(X)$ are the n-cycles, or elements of $\ker(\partial_n)$. Similarly, $B_n(X)$ is the n-boundaries of X, or $\operatorname{im}(\partial_{n+1})$. The lemma is as follows:

Lemma 0.1. *If*

$$0 \to H \to G \to G/H \to 0$$

is a short exact sequence of finitely generated Abelian groups, then

$$\operatorname{rank}(G) = \operatorname{rank}(H) + \operatorname{rank}(G/H).$$

Proof. What I said in class was as follows: If you apply $- \otimes_{\mathbb{Z}} \mathbb{Q}$, what you are left with is an exact sequence

$$0 \to \mathbb{Q}^{\operatorname{rank}(H)} \to \mathbb{Q}^{\operatorname{rank}(G)} \to \mathbb{Q}^{\operatorname{rank}(G/H)} \to 0$$

It is exact since $\mathbb{Q}$ is a flat $\mathbb{Z}$ module, which follows from the fact that $\mathbb{Q}$ is torsion-free. In particular, if you consider the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

then tensoring by $\mathbb{Q}$ yields

$$0 \to \mathbb{Q} \xrightarrow{\cdot n} \mathbb{Q} \to 0 \to 0$$

Here $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ since by the definition of tensor product

$$a \otimes \alpha = an \otimes \frac{\alpha}{n} = 0 \otimes \frac{\alpha}{n} = 0 \otimes 0 = 0.$$

Note that $\mathbb{Q} \xrightarrow{\cdot n} \mathbb{Q}$ is now an isomorphism, since n is invertible. $\square$

As a result, the sums agree because all of the ker and im terms cancel or are 0 to begin with.