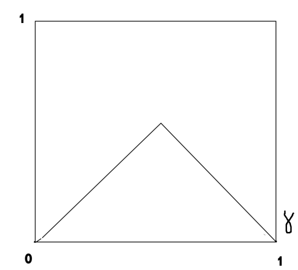
CLASS 31, NOVEMBER 28: SPACE FILLING CURVES

We can employ our results about complete metric spaces to realize some very interesting facts about the so-called 'Peano Curve'.

Theorem 31.1 (Peano)**.** *There exists a surjective continuous map* $\Gamma : [0,1] \to [0,1]^2$.

Note that this doesn't violate invariance of dimension, as the map we will construct is highly non-injective. I will make this precise after producing the construction.

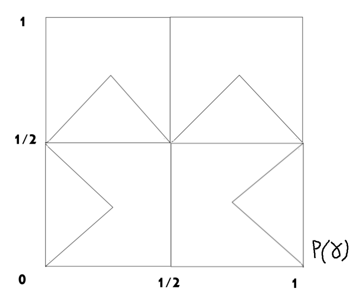
Proof. The proof uses 2 critical notions; a fractal construction (employing the fact that $[0, \frac{1}{2^n}] \cong [0,1]$) and the completeness of continuous functions to $[0,1]$.



Step 1: Let $I = [0,1]$. Given a path $\gamma : I \to I^2$ with $\gamma(0) = (0,0)$ and $\gamma(1) = (1,0)$, we can construct a new path $P(\gamma)$ with the same properties as follows: Let

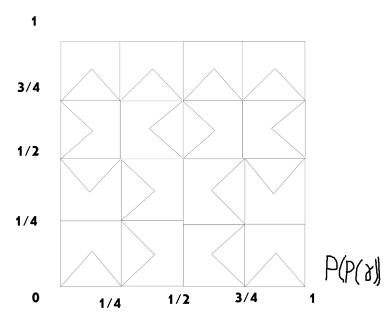
$$T : I^2 \to I^2 : (x,y) \mapsto (y,x)$$

this is a homeomorphism. Now we can write



$$P(\gamma)(t) := \begin{cases} T(\frac{1}{2} \cdot \gamma(4t)) & t \in [0, \frac{1}{4}] \\ (\frac{1}{2}, 0) + \frac{1}{2} \cdot \gamma(4t-1) & t \in [\frac{1}{4}, \frac{1}{2}] \\ (\frac{1}{2}, \frac{1}{2}) + \frac{1}{2} \cdot \gamma(4t-2) & t \in [\frac{1}{2}, \frac{3}{4}] \\ (1, \frac{1}{2}) - T(\frac{1}{2} \cdot \gamma(4t-3)) & t \in [\frac{3}{4}, 1] \end{cases}$$

As illustrated on the right, the effect of this is to iterate the same curve 4 times with different orientations. $\gamma^{(2)}$ is continuous by the pasting lemma: It is defined by continuous functions defined on 4 closed sets which agree on their overlap.



Step 2: We can apply this construction iteratively. Let $f_0 : I \to I^2$ be the function[1] defined by

$$f_0(t) = \begin{cases} (t,t) & t \leq \frac{1}{2} \\ (t, 1-t) & t \geq \frac{1}{2} \end{cases}$$

Now let f_n be obtained by applying either P (if the curve ends in the lower right corner) or $T \circ P \circ T$ (if it ends in the upper left) to each of the piecewise functions making up f_{n-1} inductively.

[1]This choice of function is NOT important in the grand scheme of the proof. You may in fact replace it with any choice of γ as above.

Step 3: Consider d_∞ on I^2 defined by $d_\infty((x,y),(x',y')) = \max\{|x-x'|, |y-y'|\}$. Let ρ be the uniform metric on $(I^2)^I$. By Theorem 30.7 this is a complete metric space, and therefore we can conclude that the subset of continuous functions $I \to I^2$ is as well. Therefore, to produce a limit of the f_n, it suffices to demonstrate that they form a Cauchy sequence.

The idea is that for a given $t \in [0,1]$, we can specify which 2^{-n}-square the function will be in and note that this is preserved when n is increased.

Given $\epsilon > 0$, choose N such that $2^{-N} < \epsilon$. By the previous observation, we note that for $n > N$ and a given t, we have that $f_N(t)$ and $f_n(t)$ lie in the same 2^{-N}-square. As a result

$$d_\infty(f_n(t), f_N(t)) \leq 2^{-N}$$

Since ρ is exactly the supremum of these numbers, $\rho(f_n, f_N) \leq 2^{-N} < \epsilon$, as desired. Let Γ be its limit.

Step 4: Surjectivity of Gamma! Since f_n has values in each square of size 2^{-n} which tile $[0,1]$, we note that for any given $x \in I^2$, there exists t such that $d(x, f_n(t)) \leq 2^{-n}$.

Let $\epsilon > 0$. I claim $B(x,\epsilon) \cap \Gamma(I) \neq \emptyset$. Choose $N \gg 0$ such that $\rho(\Gamma, f_N) < \frac{\epsilon}{2}$ and $\frac{1}{2^N} < \frac{\epsilon}{2}$. Fix such a t as in the previous paragraph. This implies

$$d(\Gamma(t), x) \leq d(\Gamma(t), f_N(t)) + d(f_N(t), x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

So the asserted claim is proved.

Since every neighborhood of x intersects $\Gamma(I)$, we know that $x \in \overline{\Gamma(I)}$. But $\Gamma(I)$ is the image of a compact set, therefore itself compact, in a Hausdorff space. Therefore it is closed, proving surjectivity. $\square$

Note that the 2 dimensional cube was not important at all.

Corollary 31.2. *There exists a surjective continuous map $\Gamma' : [0,1] \to [0,1]^n$ for any $n \in \mathbb{N}$.*

Proof. Choose $2^m > n$. Then we can define a surjective function

$$I \xrightarrow{\Gamma} I^2 \xrightarrow{\Gamma\times\Gamma} I^4 \xrightarrow{\Gamma\times\Gamma\times\Gamma} \cdots \xrightarrow{\Gamma^{n-1}} I^{2^m}$$

Since this is a composition of surjective continuous maps, it is itself continuous and surjective. Finally, we can project onto the desired number of copies of I to produce our map. $\square$

We can also upgrade this argument to $\mathbb{R}$.

Corollary 31.3. *There exists a surjective continuous map $\Gamma'' : \mathbb{R} \to \mathbb{R}^n$ for any $n \in \mathbb{N}$.*

Proof. Tile $\mathbb{R}^n$ with I^n, and enumerate them by $2\mathbb{Z}$. We can apply Γ' by the previous corollary modified by $[0,1] \cong [2n, 2n+1]$. Call their aggregate Γ''_0:

$$\Gamma''_0 : \bigcup_{n\in\mathbb{Z}} [2n, 2n+1] \to \mathbb{R}^n$$

This map is already surjective and continuous. For $[2n-1, 2n]$, use the straight-line path from $\Gamma''_0(2n-1)$ to $\Gamma''_0(2n)$. The combined map is continuous by the (locally finite) pasting lemma, and is of course surjective since it is on a subset of the domain. $\square$

Just as a final note, the function Γ is self-intersecting. You can already see this at the level of $P(P(\gamma))$ in the above illustration. This tells us that at minimum $P^3(\gamma)$ has 4 intersection points, and $P^{n+2}(\gamma)$ has at least (in fact many more!) 4^n intersection points.

Of course, if you believe invariance of dimension, no bijection could possibly exist. Indeed, it would be a continuous bijection from a compact space to a Hausdorff space, therefore a homeomorphism.