CLASS 16, OCTOBER 17: SEPARATION AXIOMS

Today, I will expand the idea of a space being Hausdorff to more generic settings. These come in a variety of T-conditions.

Definition 16.1. The following are called the separation axioms:

0) A space X is called **T0** if for every 2 distinct points $x, y \in X$, there exists U an open neighborhood of 1 of the points but not the other.
1) A space X is called **T1** if for every 2 distinct points $x, y \in X$, there exists U an open set such that $x \in U$ and $y \notin U$.
2) A space X is called **T2** or **Hausdorff** if for every 2 distinct points $x, y \in X$, there exists U, V disjoint open sets such that $x \in U$ and $y \in V$.
3) A space X is called **T3** or **Regular** if for every points $x \in X$ and closed set $Z \subseteq X$, there exists U, V disjoint open sets such that $x \in U$ and $Z \subseteq V$.
4) A space X is called **T4** or **Normal** if for every 2 closed sets $Z, Z' \subseteq X$, there exists U, V disjoint open sets such that $Z \subseteq U$ and $Z' \subseteq V$.

It should be clear that we call these separation axioms because we are finding separations of 2 types of sets in X. There are even more of these axioms, $T(\frac{N}{2})$ for $N = 0, \ldots, 12$.[1]

Proposition 16.2. *If X is a topological space with one point sets closed (equivalently T1), then if X is $T(a)$ for some $a > 0$, then X is also $T(b)$ for any $b < a$:*[2]

$$T4 \Rightarrow T3 \Rightarrow T2 \Rightarrow T1 \Rightarrow T0$$

As a result, it is sometimes assumed that every T2-4 space is T1. Here are some examples showing that the above concepts are distinct. Note that as a result of the homework, all metric spaces are T4 (and T0, therefore all of the separation axioms hold). Therefore, coming up with examples will mostly be outside the realm of metric spaces.

Example 16.3.

- The 2 point space $X = \{a, b\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, b\}\}$ is T0 but not T1 (because b can't be in an open set not containing a). This is also an example of a T4 space (almost vacuously) which is not T3!
- The finite complement topology on an infinite set X is always T1, but never T2.
- The **slit disc topology** on $\mathbb{R}^2$ is an example of a T2 but not T3 space. This is defined as follows:
  - If (x, y) is such that $y \neq 0$, then let (x, y) have a neighborhood base given by $B((x, y), r)$.
  - Let $(x, 0)$ have a neighborhood base given by $\{(x, 0)\} \cup B((x, 0), r) \setminus (\mathbb{R} \times \{0\})$ for any $x', y' \in \mathbb{R}$.

  This is a basis for a topology. It is T2 since any 2 points can be separated by discs. However, L is a closed subset of $\mathbb{R}^2$ with this topology. Also, as a subspace it has the discrete topology. Therefore $Z = L \setminus \{(0, 0)\}$ is a closed subset as well. Moreover, $P = (0, 0)$ is closed (in fact all 1 point sets are closed). But there exist no sepation of Z and P! (Pictures!)
- Another example from the book is $\mathbb{R}$ with the topology (a, b) and $(a, b) \setminus \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ from the first homework (maybe with slight modification). This is T2 but not T3.

[1] You can check these out at `https://en.wikipedia.org/wiki/Separation_axiom#Main_definitions`.

[2] It should also be noted that the words regular and normal are some of the most abused/overused terms in mathematics. For example, in the world of commutative algebra, regular is a far more stringent condition than normal. Here, that realization is flipped!

- The **tangent disc topology** of $\mathbb{R}^2$ is a topology for which
  - If (x, y) is such that $y \neq 0$, then let (x, y) have a neighborhood base given by $B((x, y), r)$.
  - Let $(x, 0)$ have a neighborhood base given by $\{(x, 0)\} \cup B((x, r), r) \cup B((x, -r), r)$ for any $x', y' \in \mathbb{R}$.

  This space can be shown (with some work!) to be T3 but not T4. Also, all points are closed.

Now we can break into some properties of T3 and T4 spaces. Here are some equivalent formulations:

Theorem 16.4. *Let X be a topological space.*

1) X is T3 if and only if every neighborhood U of a point $x \in X$ has a smaller neighborhood V such that $\bar{V} \subseteq U$.

2) X is T4 if and only if every neighborhood U of a closed set $Z \subseteq X$ has a smaller neighborhood V such that $\bar{V} \subseteq U$.

Proof. 1) $(\Rightarrow)$: Suppose X is T3. Given U, we note that we may assume by shrinking U that it is open. Therefore, U^c is closed. Therefore, there exists V containing x and V' containing U^c open disjoint sets. But this implies

$$\bar{V} \subseteq V'^c \subseteq (U^c)^c = U$$

$(\Leftarrow)$: Suppose $x \in X$ and A is a closed subset of X. Taking $U = A^c$ produces an open neighborhood of x. Therefore, applying the property, we see that V is a neighborhood of x disjoint from the open neighborhood $\bar{V}^c$ of A.

2) The argument when X is T4 uses identical methods.

□

Finally, a quick statement about subspaces and products of T2 and T3 spaces.

Proposition 16.5. *If X is a T2 (resp. T3) space, then so is any $Y \subseteq X$ with the subspace topology. If X_α are T2 (resp. T3) for all α, then so is $\prod_\alpha X_\alpha$ with product topology.*[3]

Proof. I will only prove the statements for T3. The T2 statements are similar but strictly easier. Suppose X is T3. Let $y \in Y$ and $Z \subseteq Y$ be a closed subset. This implies that there exists Z' a closed subset of X such that $Z = Z' \cap Y$. Since X is T3, we have U, V open subsets of X such that U contains y and V contains Z. Intersecting these open sets with Y yields a separation of y, Z in Y.

Now, let X_α be T3, and suppose $x = (x_\alpha) \in X = \prod_\alpha X_\alpha$ and $Z \subseteq X$ is a closed subset. Choose an open neighborhood $U = \prod_\alpha U_\alpha$ of x disjoint from Z. Note this is possible since Z^c is open. Now, by Theorem 16.4, we see that there exists $V_\alpha \in U_\alpha$ an open neighborhood of x_α with $\bar{V}_\alpha \subseteq U_\alpha$. If $U_\alpha = X_\alpha$, we may assume $V_\alpha = X_\alpha$ to stay within the product topology. Therefore, $V = \prod_\alpha V_\alpha$ is an open neighborhood of x disjoint from $\bar{V}^c \supseteq Z$. □

Example 16.6. Recall the topology τ on $\mathbb{R}$ generated by $[a, b)$. You have shown this is strictly finer than the Euclidean topology. Note that this space is T4 and points are closed. This follows from Theorem 16.4. However, I claim the product of 2-copies is non-T4: $X = \mathbb{R} \times \mathbb{R}$. Consider $\Gamma \subseteq X$ given as the graph of $f(x) = -x$. This is closed in the Euclidean topology, thus also closed in τ. Note Γ with the subspace topology is discrete; $(x, y) \in [x, x + \epsilon) \times [y, y + \epsilon)$. Therefore every subset of Γ is closed.

Let Z be the set of points on Γ with rational coordinates, and $Z' = \Gamma \setminus Z$ be the irrational coordinates. Then there exist no open subsets in X separating these two sets.

[3]These statements are false for normal spaces. The points play an important role in the proof.