HOMEWORK 6: NORMALITY AND URYSOHN THEOREMS DUE: OCTOBER 26

1) Show a closed subspace of a normal space is normal.

Solution: Let $Y \subseteq X$ be a closed subset, where X is normal. If $A, B \subseteq Y$ are closed subsets, then A, B are closed in X since Y is closed. Therefore, there exist separating open set U, V in X containing A, B respectively. The desired sets are $U \cap Y$ and $V \cap Y$.

2) Show that if X_α are non-empty topological spaces, and $\prod_\alpha X_\alpha$ is T2 or T3 or T4, then so is each X_α.

Solution: Each of the projection maps are continuous surjective maps. We may consider the inclusion $\iota : X_{\alpha_0} \hookrightarrow \prod_\alpha X_\alpha$ sending $x \in X_{\alpha_0}$ to (x_α), where x_α is fixed for $\alpha \neq \alpha_0$, and $x_{\alpha_0} = x$. This is continuous by a previous homework, and induces a homeomorphism onto its image (in either the product or box topology).

Note that the image under ι of a point is a point (since ι is a function). Furthermore, ι a homeomorphism onto its image, and thus in particular closed. Therefore, $\iota(A) \subseteq \iota(X)$ is a closed set in the subspace topology. This yields a closed subset A' of $\prod_\alpha X_\alpha$ whose intersection with $\iota(X_{\alpha_0})$ is exactly $\iota(A)$). Moreover, we can guarantee that A' and a point (or closed set B') remain disjoint, since $A' = A \times \prod_{\alpha \neq \alpha_0} X_\alpha$ is closed with the given property. In each of the cases above, take U, V the disjoint open subsets of $\prod_\alpha X_\alpha$ separating the given sets of interest for a particular condition T(2-4).

Intersecting U, V with $\iota(X_{\alpha_0})$ produces open subsets of the subspace, which again are disjoint since the original sets were. Therefore, taking their preimages under ι (equivalently, images under $\iota^{-1} : \iota(X_{\alpha_0}) \to X_{\alpha_0}$ produces open disjoint subsets of the required type.

3) Show that the following 2 conditions are equivalent:
   1) Every subspace of X is normal.
   2) For all A, B subsets of X such that $\bar{A} \cap B = A \cap \bar{B} = \emptyset$, there exists U, V open disjoint sets separating A and B; $A \subseteq U$ and $B \subseteq V$.

In such a case, X is said to be **T5**, or **completely normal**.

Solution: 1) $\Rightarrow$ 2): Given every subspace of X is normal, let A, B subsets as in 2). We can consider $A, B \subseteq (\bar{A} \cap \bar{B})^c$. In the subspace topology, these sets have the property that their closures have the form $\bar{A} \setminus \bar{B}$ and $\bar{B} \setminus \bar{A}$ respectively (the overlines are denoting their closure in X). Therefore, these sets are automatically disjoint and closed in the subspace, thus separated by open disjoint sets U, V. But these are open in X as well, since $(\bar{A} \cap \bar{B})^c$ is open. Thus U, V separate A and B in X as well.

2) $\Rightarrow$ 1): Let $Y \subseteq X$ be a subspace. Given A, B closed subsets of Y, note that $\bar{A} \cap \bar{B} \subseteq Y^c$. Therefore, A, B have the property of necessary to apply 2). Letting

U, V be the desired separating sets, we note that $U \cap Y$ and $V \cap Y$ are open in the subspace Y and separate A, B.

4) Show that any connected normal space X containing 2 disjoint non-empty closed sets A,B is uncountable.

Solution: Take A, B closed non-empty disjoint sets. Applying Urysohn, we see that there exists a continuous function $f : X \to [0, 1]$ which is 0 on A and 1 on B. Note that $[0, 1]$ is uncountable. If $a \in [0, 1]$ is such that $f^{-1}(a) =$, then $f^{-1}([0, a))$ and $f^{-1}((a, 0])$ form a separation of X, showing X is not connected. So choosing $x_a \in p^{-1}(a)$ for each $a \in [0, 1]$, we see that

$$[0, 1] \hookrightarrow X : a \mapsto x_a$$

as sets, and therefore X is uncountable.

5) We say $Y \subseteq X$ is a $\mathbf{G}_\delta$ set if Y is an intersection of countably many open sets. Similarly, Y is a $\mathbf{F}_\sigma$ set if it is a countable union of closed sets. Use the techniques of the proof of Urysohn's Lemma to show the following result:

Theorem. *If X is normal, then there exists $f : X \to [0, 1]$ a continuous function such that $f^{-1}(0) = A$ iff A is a closed G_δ set.*[1]

Solution: ($\Rightarrow$): If such an f exists, $A = f^{-1}(0)$ is closed and additionally A is a G_δ:

$$A = \bigcap_{n>0} f^{-1}\left(\left[0, \frac{1}{n}\right)\right)$$

($\Leftarrow$): If A is a G_δ set, let $A = \bigcap_{a\in\mathbb{Q}\cap(0,1)} U_a$. Let $U_1 = X$. With the trick from Urysohn's Lemma, we may assume (by potentially shrinking U_a by intersection and normality) that

$$A \subseteq U_a \subseteq \overline{U_a} \subseteq U_b$$

if $a < b$. To be precise, enumerate $\mathbb{Q} \cap (0, 1)$ with $p_1, p_2, \ldots$. Take V_{p_1} such that

$$A \subseteq V_{p_1} \subseteq \overline{V_{p_1}} \subseteq U_{p_1}$$

Inductively, let V_{p_n} be an open set such that for $p_i < p_n < p_j$ as close as possible,

$$V_{p_i} \subseteq V_{p_n} \subseteq \overline{V_{p_n}} \subseteq V_{p_j} \cap U_{p_n}$$

If no p_i or p_j exist, then let $V_{p_i} = A$ and $V_{p_j} = X$ respectively. Note that since $A \subseteq V_{p_i} \subseteq U_{p_i}$ for each i, we see $A = \cap_i V_{p_i}$. And the V_{p_i} are arranged exactly as in the proof of Urysohn.

Now, we can define $f : X \to [0, 1]$ as in the proof of Urysohn:

$$f(x) = \inf\{a \in \mathbb{Q} \cap [0, 1] \mid x \in V_a\}$$

This again is well defined and continuous by the arguments in Urysohn's lemma. Furthermore, since $A = \bigcap_{a\in\mathbb{Q}\cap(0,1]} V_a$, we have $f(x) = 0$ iff $x \in V_a$ for all a iff $x \in A$. This completes the proof.

[1] Urysohn's Lemma holds exactly when A and B are G_δ-sets.

($\Leftarrow$) (alternative proof by a student): Given X is normal, and $A = \cap_{i\in\mathbb{N}} U_i$ a G_δ-set, Urysohn's Lemma implies that we can find a continuous function $f_i : X \to [0,1]$ which is 0 on A and 1 on U^c. Now, we can form a continuous function

$$f : X \to [0,2] : x \mapsto \sum_{i\in\mathbb{N}} 2^{-i} f_i(x)$$

Note that this function is well defined and continuous, since the sequence of partial sums converges uniformly to f (note $\frac{1}{2^n} < \epsilon$ implies $|f_{n+1}(x) - f(x)| < \epsilon$). Furthermore, if $x \in A$, then $f_i(x) = 0$ for all $i \in \mathbb{N}$, so $f(x) = 0$. On the other hand, if $x \notin A$, then

$$x \in A^c = \left(\bigcap_{i\in\mathbb{N}} U_i\right)^c = \bigcup_{i\in\mathbb{N}} U_i^c$$

so $x \in U_i^c$ for some i, and therefore $f_i(x) = 1$, and therefore $f(x) \geq 2^{-i} > 0$. So $f(x) = 0$ exactly on A.

6) X is **T6** or **perfectly normal** if it is normal and every closed set is a G_δ-set. Show every metric space is T6 and that T6 implies T5.

Solution: Let $A \subseteq X$ be a closed subset of a metric space X. By our previous homework, we know that $f(x) = d(x, A)$ is a continuous function of $x \in X$ with the property that $f^{-1}(0) = A$. Therefore, by the previous problem, A is a G_δ-set.

Let $A, B \subseteq X$ be sets as in definition 2) of problem 3. Assume X is T6. Let f_A and f_B be the functions from the previous problem, for $\bar{A}, \bar{B} \subseteq X$ closed subsets. Since f_A vanishes precisely on $\bar{A}$ and f_B precisely on $\bar{B}$, we see that $f = f_A - f_B : X \to \mathbb{R}$ is a continuous function with the property that $U = f^{-1}((-\infty, 0))$ and $V = f^{-1}((0,\infty))$ are open disjoint sets containing A and B respectively.

7) Show that if X is a compact Hausdorff space, then X is metrizable if and only if X is second-countable.

Solution: ($\Rightarrow$): If X is a compact metric space, then X has a countable basis. Indeed, consider the basis $\mathcal{B}_n = \{B(x, \frac{1}{n}) \mid x \in X\}$. Since this forms an open cover of X finitely many will do, so let $\mathcal{B}'_n$ be the resulting finite collection of opens. Then I claim $\mathcal{B} = \bigcup \mathcal{B}'_n$ is a countable basis for X.

Every $x \in X$ is in some element of each $\mathcal{B}_n$ by definition of a cover. If $x \in B(y, \frac{1}{n}) \cap B(y', \frac{1}{n'})$, then $\exists r > 0$ such that $B(x, r) \subseteq B(y, \frac{1}{n}) \cap B(y', \frac{1}{n'})$ since it is open. On the other hand, if we let $n \gg 0$ such that $\frac{1}{n} \leq \frac{r}{2}$, then $x \in B(y'', \frac{1}{n})$. Furthermore, we see $B(y'', \frac{1}{n}) \subseteq B(x, r)$, since if $z \in B(y'', \frac{1}{n})$, then

$$d(x,z) \leq d(x,y'') + d(y'',z) < \frac{1}{n} + \frac{1}{n} < r$$

It also generates the same topology, but this follows similarly to the previous statement.

($\Leftarrow$): By Theorem 17.2, every compact Hausdorff space is normal. Since X is assumed second-countable, X is metrizable by Urysohn's Metrization Theorem.