CLASS 30, NOVEMBER 22TH: CONFORMAL MAPPINGS

For the remainder of the semester, we will move to more geometrically motivated questions. The primary question in most of these fields is which types of objects are equivalent, where the notion of equivalence varies. For our sake, we are now experts on the notion of holomorphic, so we will use this as our starting place.[1]

Let $U, V \subseteq \mathbb{C}$ be open sets.

Definition 30.1. U and V are **conformal**, or **conformally equivalent**, or **biholomorphic** if there exists a holomorphic bijection $f : U \to V$. f is called a **conformal map**.

Currently, this doesn't seem like an equivalence relation (symmetry?). The following proposition fixes this for us:

Proposition 30.2. *If $f : U \to V$ is a conformal map, then $f'(z) \neq 0$ for any $z \in U$. In particular, $f^{-1} : V \to U$ is also holomorphic/conformal.*

Proof. Suppose that $f'(z_0) = 0$ for some $z_0 \in U$. Then

$$f(z) = f(z_0) + a(z - z_0)^k + G(z)$$

for z sufficiently close to z_0, and G some function vanishing to order $> k$ at z_0. For w small, let $F(z) = a(z - z_0)^k - w$ so that

$$f(z) - f(z_0) - w = F(z) + G(z)$$

Notice that since $|F| > |G|$ for $|z - z_0|$ small enough, we can apply Rouche's Theorem to deduce

$$f(z) - f(z_0) - w = 0$$

has k-solutions. Since $f'(z) \neq 0$ near z_0, these solutions are distinct. But this is only possible if f isn't injective.

Now, if $w = f(z)$ and $w_0 = f(z_0)$, we can compute

$$\frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{1}{\frac{w - w_0}{f^{-1}(w) - f^{-1}(w_0)}} = \frac{1}{\frac{f(z) - f(z_0)}{z - z_0}} \to \frac{1}{f'(z_0)}$$

This is the calculus formula $(f^{-1})'(z_0) = \frac{1}{f'(f^{-1}(z_0))}$. But in particular, f^{-1} is holomorphic since $f'(z_0) \neq 0$ lets that expression make sense. □

It should also be noted that some authors allow for non-injective conformal mappings, only requiring $f'(z) \neq 0$. This would allow for maps that are surjective with many sheets, and behave like covering spaces. That is to say f is locally injective, and thus locally like one of our maps.

Our definition has the benefit of being an equivalence relation. Look at some examples of conformal spaces yields surprising results!

[1] For those of you who have studied differential topology of real manifolds, this is a stronger notion than diffeomorphism. Again, this follows by virtue of the existence of infinitely differentiable non-analytic functions.

Example 30.3. Consider the upper half-plane $\mathbb{H} = \{z = x + iy \mid y > 0\}$ and the disc $\mathbb{D} = B(0,1)$. I claim that these are conformal!

Consider the maps $F(z) = \frac{i-z}{i+z}$ and $G(w) = i\frac{1-w}{1+w}$.

Theorem 30.4. *$F : \mathbb{H} \to \mathbb{D}$ is a conformal map with inverse $G : \mathbb{D} \to \mathbb{H}$.*

Proof. Notice that F is bijective with inverse G:

$$F(G(w)) = \frac{i - i\frac{1-w}{1+w}}{i + i\frac{1-w}{1+w}} = \frac{w+1-1+w}{1+w+1-w} = \frac{2w}{2} = w$$

$$G(F(z)) = i\frac{1 - i\frac{i-z}{i+z}}{1 + \frac{i-z}{i+z}} = i\frac{i+z-i+z}{i+z+i-z} = i\frac{2z}{2i} = z$$

Also note that $Im(F) = \mathbb{D}$: First

$$|i+z| > |i-z|$$

shows that $Im(F) \subseteq \mathbb{D}$. Similarly, $Im(G) \subseteq \mathbb{H}$:

$$G(w) = i\left(1 - \frac{2w}{1+w}\right)$$

and $Re(\frac{2w}{1+w}) < 1$ by the following computation: let $w = x + iy$.

$$Re\left(\frac{2(x+iy)}{1+x+iy}\right) = Re\left(\frac{2(x+iy)((1+x)-iy)}{(1+x)^2+y^2}\right) = \frac{2x(1+x)+2y^2}{(1+x)^2+y^2}$$

$$\frac{2x^2+2y^2+2x}{x^2+y^2+2x+1} < \frac{x^2+y^2+2x+1}{x^2+y^2+2x+1} = 1$$

So it only goes to check that $F'(z) \neq 0$ in the upper half plane:

$$F'(z) = \frac{-(i+z)-(i-z)}{(i+z)^2} = \frac{-2i}{(i+z)^2} \neq 0$$

□

You can think about where points are going under the map F: $F(i) = 0$, as $z \to \infty$ in any direction, we'll get limit -1. As $z \to 0$, the value is 1, and finally for any $\xi \in \mathbb{R}$,

$$\lim_{z\to\xi} F(z) = \frac{1}{\xi^2+1}((\xi^2) - 1 - i2\xi)$$

These values fill in the remainder of the boundary circle.

So you can think of the conformal mapping as placing 0 at 1, and wrapping the real line around the circle.

Next time we will go through other examples of conformal mappings and the Schwarz Lemma.