

## CLASS 34, DECEMBER 6TH: FINALE! (RMT)

We are only left to prove the Riemann Mapping Theorem:

**Theorem** (Riemann Mapping Theorem). *If  $\Omega \subseteq \mathbb{C}$  is a proper, open, and simply-connected subset, then  $\Omega$  is conformally equivalent to  $\mathbb{D}$ .*

*Proof.* **1)** Choose  $\alpha \notin \Omega$ . Then  $f(z) = \log(z - \alpha)$  is a non-vanishing holomorphic function on  $\Omega$ . As a result,  $e^{f(z)} = z - \alpha$  is injective, which implies  $f$  is as well. For  $w \in \Omega$ ,

$$f(z) \neq f(w) + 2\pi i \forall z \in \Omega$$

Otherwise, exponentiating would for  $z = w$ . Using sequences, we can even find a disc about  $f(w) + 2\pi i$  for which no values of  $f$  reach it. Consider

$$F(z) = \frac{1}{f(z) - (f(w) + 2\pi i)}$$

Since  $f$  is conformal, so is  $F$ .  $F(\Omega)$  is also bounded, thus by translation and dilation, we may assume  $0 \in \Omega \subseteq \mathbb{D}$  up to conformal equivalence.

**2)** Consider the uniformly bounded class

$$\mathcal{F} = \{f : \Omega \rightarrow \mathbb{D} \mid f(0) = 0, f \text{ is holomorphic and injective}\}$$

We want to find  $f \in \mathcal{F}$  maximizing  $f'(0)$ . Note they are bounded by Cauchy's inequality. By Montel's theorem, we can choose a sequence  $f_n$  with  $|f'_n(0)|$  approaching this non-zero supremum. Thus by our lemma from last time, the limit  $f$  is injective. Furthermore, by the MMP  $|f(z)| < 1$ . All in all,  $f \in \mathcal{F}$ .

**3)** We need to show that  $f$  is surjective. Suppose  $\alpha \notin f(\Omega)$ . Then  $\psi_\alpha : \mathbb{D} \rightarrow \mathbb{D}$  swaps  $\alpha$  with the origin. If we consider  $U = \psi_\alpha(f(\Omega))$ , then  $0 \notin U$  is simply connected. Therefore we can define a log and thus a square root:  $g(w) = e^{\frac{1}{2} \log(w)}$ . Then I claim the composition

$$F : \Omega \xrightarrow{f} \mathbb{D} \setminus \{\alpha\} \xrightarrow{\psi_\alpha} \mathbb{D} \setminus \{0\} \xrightarrow{g} \mathbb{D} \setminus \{0\} \xrightarrow{\psi_{g(\alpha)}} \mathbb{D} \setminus \{g(\alpha)\}$$

is in  $\mathcal{F}$ . Indeed, it is holomorphic, injective, and each maps 0 to 0. Additionally, if  $h(w) = w^2$ , then

$$f = \psi_\alpha^{-1} \circ h \circ \psi_{g(\alpha)}^{-1} \circ F = \Phi \circ F$$

$\Phi(0) = 0$  is not injective since  $h$  isn't. As a result, Schwarz Lemma (bullet 3) implies that  $|\Phi'(0)| < 1$ , since it can't possibly be a rotation. Finally, the chain rule yields

$$f'(0) = \Phi'(0) \cdot F'(0)$$

but this implies  $f'(0)$  was not maximal! A contradiction. □