# WORKSHEET: WEIRSTRASS INFINITE PRODUCTS

Today we will return to the question posed in Class 27; given a non-accumulating sequence $a_n$, can we find an entire function vanishing precisely at these points? We pointed out that the naive guess is

$$f(z) = (z - a_1) \cdot (z - a_2) \cdots$$

but we would need to deal with convergence issues. This issue was tackled by Weirstrass. And now we will prove it in steps.

**Theorem 0.1** (Weirstrass Infinite Products)**.** *Suppose $a_n$ is a sequence with $|a_n| \to \infty$. There exists $f$ entire such that $f(a_n) = 0$ and $f(z) \neq 0$ for $z \neq a_n$. Any other function with these properties has the form $f(z)e^{g(z)}$ for some entire function $g$.*

Note that since $a_n$ and $a_m$ can agree for various $n, m$, we can achieve zeroes of any order as well!

(1) Start by proving the second statement: Suppose $f_1$ and $f_2$ are functions satisfying Theorem 0.1. What can you say about $\frac{f_1}{f_2}$?

(2) Consider the functions $E_k$, called **the canonical factors**, defined by

$$E_k(z) := (1-z)e^{z+\frac{z^2}{2}+\cdots+\frac{z^k}{k}}$$

for each $k \geq 0$, with $E_0(z) = 1 - z$. Prove the following lemma:

**Lemma 0.2.** *If $|z| \leq \frac{1}{2}$, then $|1 - E_k(z)| \leq c|z|^{k+1}$ for some constant $c > 0$.*

As a hint, write $(1-z) = e^{\log(1-z)}$ and consider the resulting power series in the exponential.

(3) Reduce to the case where none of the $a_n$ are zero. Now we will show that

$$f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right)$$

is the desired function. For a specific $R > 0$, let $z \in B(0, R)$ consider separately the case of $|a_n| \leq 2R$ and $|a_n| > 2R$.

For the first case, note that there is a partial product for $f$ which vanishes at each of the points.

For the second case, find an appropriate bound for each $z \in B(0, R)$ for the function $1 - E_n\left(\frac{z}{a_n}\right)$.

Finally, note that

$$f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{z}{a_n}\right) = \prod_{n=1}^{\infty} 1 - \left(1 - E_n\left(\frac{z}{a_n}\right)\right)$$

converges for each $z \in B(0, R)$. Deduce the result.

Note the following corollary, which is valuable for the generation of universes ;-)

**Corollary 0.3.** *If $a_n \to \infty$ and $b_n \to \infty$, then there exists $f$ a meromorphic function with zeroes at $a_n$ and poles at $b_n$ (precisely).*

Combining several of the results and notions of the past 3 classes, Hadamard (our friend with the formula) proved the following improvement of Weirstrass's theorem:

**Theorem 0.4** (Hadamard)**.** *Suppose $f$ is entire and has growth order $\rho_0$. Set $k = \lfloor \rho_0 \rfloor$. If $0 \neq a_1, a_2, \ldots$ are the zeroes of $f$, then*

$$f(z) = e^{P(z)} z^m \prod_{n=1}^{\infty} E_k\left(\frac{z}{a_n}\right)$$

*where $P$ is a polynomial of degree $\leq k$, and some $m$.*

Hadamard proved this by showing that the degree of the canonical factors can be taken to be constant. The proof is illustrated in chapter 5, section 5 of the book (pgs 147-153) if you are interested. But since only 5 classes remain, we will move instead to conformal mappings.

Notes will be posted on the details of this argument. If you would like to turn this in (before the end of class), I will weight the worksheet as half of your second worst homework to date.