HOMEWORK 9: SUPPORT & ASSOCIATED PRIMES
DUE: WEDNESDAY, MAY 1ST

1) (From the discussion of 7.2) Given M a finitely generated R-module, construct $\mathcal{M} = \coprod_{\mathfrak{p} \in \mathrm{Spec}(R)} M_{\mathfrak{p}}$. This can be thought of as a copy of M lying over each $\mathfrak{p} \in \mathrm{Spec}(R)$. $\mathrm{Supp}(M)$ now marks the closed subset, by Proposition 25.2 (d), of points that matter in this construction: $\mathcal{M} = \coprod_{\mathfrak{p} \in \mathrm{Supp}(M)} M_{\mathfrak{p}} \cup \coprod_{\mathfrak{p} \notin \mathrm{Supp}(M)} 0$.

   Given $m \in M$, we get a map $f_m : \mathrm{Spec}(R) \to \mathcal{M} : \mathfrak{p} \mapsto \frac{m}{1} \in M_{\mathfrak{p}}$. Show that this identifies M as a collection of sections of $\pi : \mathcal{M} \to \mathrm{Spec}(R) : m \in M_{\mathfrak{p}} \mapsto \mathfrak{p}$, i.e. $\pi \circ f_m = Id_{\mathrm{Spec}(R)}$.

   Additionally, show $W^{-1}M$ represents partially defined sections, for those $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathfrak{p} \cap W = \emptyset$.

2) Verify the claim of Example 25.3; If we consider $M = \oplus_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$, we can conclude that $\mathrm{Supp}(M) \neq V(\mathrm{Ann}(M))$. Show all of your assertions.

3) Consider $M = \mathbb{Z} \oplus \mathbb{Z}/\langle 2 \rangle$ as a $\mathbb{Z}$-module. Find the associated primes of M. Find 2 modules M_1, M_2, both isomorphic to $\mathbb{Z}$, such that $M_1 + M_2 = M$. What does this tell you about $\mathrm{Ass}(M)$ vs. $\mathrm{Ass}(M_1) \cup \mathrm{Ass}(M_2)$?

4) Consider the ring $R = K[x, y, z]/\langle xz - y^2 \rangle$ and the prime ideal $\mathfrak{p} = \langle x, y \rangle$. Let $M = R/\mathfrak{p}^2$. Compute $\mathrm{Ass}(M)$, and find all $m \in M$ for which $\mathrm{Ann}(m) = \mathfrak{p}$ for each $\mathfrak{p} \in \mathrm{Ass}(M)$. Finally, find an ascending chain $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$ such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \mathrm{Ass}(M)$.

5) If $N, N' \subseteq M$, show that
$$\mathrm{Ass}(M/N \cap N') \subseteq \mathrm{Ass}(M/N) \cup \mathrm{Ass}(M/N')$$

6) If $\varphi : R \to S$ is a ring homomorphism, and $\mathfrak{q}$ is $\mathfrak{p}$-primary in S, is it true that $\varphi^{-1}(\mathfrak{q})$ is $\varphi^{-1}(\mathfrak{p})$-primary? In the reverse direction? I.e. is $\varphi(\mathfrak{q}) \cdot S$ primary?