CLASS 16, OCTOBER 18: SINGULARITIES

So far we have only talked about poles in the class of isolated singularities. Today we will study the remaining classes.

Definition 16.1. Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function, with z_0 internal to Ω. A singularity $z_0 \in \Omega$ of f is called **removable** if there exists w such that defining $f(z_0) = w$ makes $f : \Omega \to \mathbb{C}$ a holomorphic function.

Thus a singularity is removable if it can be removed from the list of singularities. The following theorem makes this more rigorous:

Theorem 16.2 (Riemann's theorem on removable singularities)**.** *Suppose f is holomorphic on Ω except possibly at a point $z_0 \in \Omega$. If f is bounded near z_0, then z_0 is a removable singularity.*

Proof. We can focus on $\bar{B}(z_0, r) \subseteq \Omega$. Let C be the boundary circle oriented counterclockwise. We want to show that Cauchy's Integral theorem holds:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw$$

for $z \neq z_0$ internal to $\bar{B}(z_0, r)$. We will show that the RHS is a holomorphic function on all of $\bar{B}(z_0, r)$, and it agrees with $f(z)$ whenever $z \neq z_0$. As a result, analytic continuation will yield that the RHS is the desired extension of f to z_0. To show holomorphicity, we use the following lemma:

Lemma 16.3. *Let $F : \Omega \times [0, 1]$ be a function where Ω is open. If*

1) $F(z, t)$ is holomorphic in z for each fixed t.
2) F is continuous.

Then the function $f(z) = \int_0^1 F(z, t)dt$ is holomorphic.

The idea of this lemma is to allow a function to be deformed with respect to a parameter.

Proof. For $n \geq 1$, we can consider the Riemann sum

$$f_n(z) = \frac{1}{n} \sum_{m=1}^{n} F(z, \frac{m}{n})$$

Then $f_n(z)$ is holomorphic by assumption 1). Now we want to show that for any given disc $\bar{B}(z_0, r) \subseteq \Omega$, the sequence f_n converges uniformly to f. Since F is continuous on the compact set $\bar{B}(z_0, r) \times [0, 1]$, we have that it is uniformly continuous: $\forall \epsilon > 0$, there exists $\delta > 0$ such that

$$|s - t| \leq \delta \qquad \Longrightarrow \qquad \sup_{z \in \bar{B}(z_0, r)} |F(z, s) - F(z, t)| < \epsilon$$

Now if $\frac{1}{n} < \delta$, i.e. $n \gg 0$, then

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \sum_{m=1}^{n} \int_{\frac{(m-1)}{n}}^{\frac{m}{n}} F(z, \frac{k}{n}) - F(z,s) ds \right| \\ &\leq \sum_{m=1}^{n} \int_{\frac{(m-1)}{n}}^{\frac{m}{n}} \left| F(z, \frac{k}{n}) - F(z,s) \right| ds \\ &= \sum_{m=1}^{n} \frac{\epsilon}{n} = \epsilon \end{aligned}$$

This shows uniformity of convergence. Finally, since f_n are themselves holomorphic, we have that f is as well by Theorem 12.1 in the notes. □

Returning to the proof of the original result, Lemma 16.3 yields the fact that $\frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw$ is holomorphic everywhere on $B(z_0, r)$.

It now suffices to check that the equality holds. To do this, we will use the 'double keyhole' contour which avoids both z_0 and our point of interest z. Since we are holomorphic on the interior, we yield that

$$\int_C \frac{f(w)}{w-z} dz - \int_{C_z} \frac{f(w)}{w-z} dz - \int_{C_{z_0}} \frac{f(w)}{w-z} dz = 0$$

where C_w is a circle of small radius $\epsilon > 0$ about w oriented clockwise. The residue theorem yields

$$\int_{C_z} \frac{f(w)}{w-z} dz = 2\pi i f(z)$$

Additionally, using the fact that $f(z)$ is bounded near z_0, as in the homework exercise, we may conclude $\int_{C_{z_0}} \frac{f(w)}{w-z} dz = 0$. This shows the desired result. □

A very nice corollary of Theorem 16.2 is the following perhaps expected result is something that you may have initially suspected.

Corollary 16.4. *If f has an isolated singularity at z_0, then z_0 is a pole if and only if $\lim_{z\to z_0} |f(z)| = \infty$.*

Proof. ($\Rightarrow$): If z_0 is a pole of order m, then $\frac{1}{f}$ has a zero of order m at z_0. Thus $|f(z)| \to \infty$ as $z \to z_0$.

($\Leftarrow$): If $\lim_{z\to z_0} |f(z)| = \infty$, then $\frac{1}{f}$ is bounded near z_0 (in fact close to 0). Therefore, $\frac{1}{f}$ has a removable singularity at z_0 necessarily with limit 0. Therefore, writing

$$\frac{1}{f} = a_m(z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \ldots = a_m(z-z_0)(1 + (z-z_0)g(z))$$

we have that f has a pole of order m at z_0. □

Example 16.5. We said $e^{\frac{1}{z}}$ does not have a pole at 0. Corollary 16.4 now can ensure this: Approach 0 from the direction $z = iy$ as $y \to 0$:

$$e^{\frac{1}{iy}} = e^{-i\frac{1}{y}}.$$

This is bounded in absolute value by 1. Thus it cannot have limit ∞ (in fact, it doesn't exist).