HOMEWORK 6: PERFECT RINGS AND SPLITTINGS
DUE: FRIDAY MAY 4

1) We have already see that the map $R \to F_*R$ induces a bijection on prime ideals. Can you say the same for $R \to R^\infty$?

Solution: I claim that it is bijective on spectrum. We will continuously use the fact that $R \subseteq F^e_*R \subseteq R^\infty$. We have an induced map $\varphi : \mathrm{Spec}(R^\infty) \to \mathrm{Spec}(R)$ by taking preimages. It goes to show that this map is bijective.

For surjectivity, let $\mathfrak{p} \subseteq R$ be a prime ideal. Consider

$$\mathfrak{p}^\infty = \left\{x \in R^\infty \mid x^{p^e} \in \mathfrak{p} \text{ for some } e \geq 0\right\}$$

I claim that this is a prime ideal. Let $a, b \in R^\infty$ with $ab \in \mathfrak{p}^\infty$. Then there exists some $e \geq 0$ with $a, b \in F^e_*R$. But this implies $a \cdot b \in F^e_*\mathfrak{p}$, and therefore $a \in F^e_*\mathfrak{p} \subseteq \mathfrak{p}^\infty$ or $b \in F^e_*\mathfrak{p} \subseteq \mathfrak{p}^\infty$. This proves the claim.

Next, I check injectivity. Similarly to the proof of surjectivity, suppose that $\mathfrak{p}^\infty \neq \mathfrak{q}^\infty$, but $\mathfrak{p} = \mathfrak{p}^\infty \cap R = \mathfrak{q}^\infty \cap R = \mathfrak{q}$. Let $f \in \mathfrak{q}^\infty \setminus \mathfrak{p}^\infty$. Then $f \in R^{\frac{1}{p^e}}$ for some $e \geq 0$. Therefore, $f^{p^e} \in \mathfrak{q} \setminus \mathfrak{p}$. This contradicts the assumption.

2) Show that a ring R is F-split (respectively F-regular) if and only if $R_\mathfrak{m}$ is F-split (resp. F-regular) for every maximal ideal.

Solution: ($\Rightarrow$) If R is F-split, we can tensor the splitting with $W^{-1}R$ for any multiplicative set W and get an F-splitting of $W^{-1}R$. The same goes for $R \to F^e_*\langle c\rangle$

($\Leftarrow$) Consider fitting the evaluation map at 1 (or c a NZD) into an exact sequence

$$\mathrm{Hom}_R(F^e_*R, R) \xrightarrow{ev} R \to \mathrm{coker}(ev) \to 0$$

Tensoring this sequence with $R_\mathfrak{m}$ for each $\mathfrak{m}$, we get an exact

$$\mathrm{Hom}_{R_\mathfrak{m}}(F^e_*R_\mathfrak{m}, R_\mathfrak{m}) \xrightarrow{ev} R_\mathfrak{m} \to \mathrm{coker}(ev)_\mathfrak{m} \to 0$$

If we know $R_\mathfrak{m}$ is F-split (resp F-regular) the evaluation map is surjective. So $\mathrm{coker}(ev)_\mathfrak{m} = 0$. But this implies $\mathrm{coker}(ev) = 0$.

3) Is $R = K[x, y, z]/\langle x^3 + y^3 + z^3\rangle$ an F-split ring? Be careful about the characteristic $p > 0$ chosen.

Solution: According to the corollary of Fedder, R is F-split if and only if $f^{p-1} \notin \mathfrak{m}^{[p]}$. So it comes down to considering

$$f^{p-1} = \sum_{i+j+k=p-1} c_{ijk} x^i y^j z^k$$

We note that $c_{ijk} \neq 0$ for any i, j, k, because it is an integer with a rational presentation containing $(p-1)!$ in the numerator (no p factors). The question breaks down into cases.

Case 1; $p = 3$: In this case, $f \in \mathfrak{m}^{[3]}$, so of course f^2 is as well. This implies immediately that R is NOT F-split.

Case 2; $p \equiv 1 \pmod 3$: In this case, $\frac{p-1}{3}$ is an integer. Therefore, we can choose $i = j = k = \frac{p-1}{3}$ to get an $x^{p-1}y^{p-1}z^{p-1}$ term. Therefore R is F-split!

Case 3; $p \equiv 2 \pmod 3$: In this case, $\frac{p-1}{3}$ is NOT an integer. Therefore, either i or j or k is bigger than $\frac{p-1}{3}$. Say i WLOG. This implies

$$3i > 3\frac{p-1}{3} = p - 1$$

But $3i$ is an integer, so $3i \geq p$, implying $x^{3i} \in \mathfrak{m}^{[p]}$. Therefore R is NOT F-split.

4) Is the Cohen-Macaulay non-regular ring $R = K[x^2, x^3]$ F-split?

Solution: I claim it is not for any characteristic. Recall $R \cong K[X,Y]/\langle Y^2 - X^3\rangle$. Therefore, again it suffices to check that whether or not $(Y^2 - X^3)^{p-1} \notin \mathfrak{m}^{[p]}$, or equivalently, if there exists i, j such that $i + j = p - 1$ and $2i, 3j < p$. This implies that $j \leq \frac{p-1}{3}$ and $i \leq \frac{p-1}{2}$. Adding these inequalities together, we see that

$$p - 1 = i + j \leq \frac{5(p-1)}{6} < p - 1$$

Therefore, we see this is impossible.

5) Show that $R = K[x,y,z]/\langle x^4 + y^4 + z^4\rangle$ is never F-split.

Solution: Again, we need to find i, j, k such that $i+j+k = p-1$ and $4i, 4j, 4k < p$. Adding up these equations,

$$p - 1 = i + j + k \leq \frac{3(p-1)}{4} < p - 1$$

6) In this problem, we will show that $R = S/I$ in Fedder's Criterion can NOT be weakened to a more arbitrary quotient. Find an example of $S \supseteq J \supseteq I$ such that

$$\mathrm{Hom}(F_*S/J, S/J) \ncong F_*((J/I)^{[p]} : J/I)\,\mathrm{Hom}(F_*S/I, S/I)$$

Solution: Noting what Fedder's Criterion gives us, we see

$$\mathrm{Hom}(F_*S/J, S/J) = F_*(J^{[p]} : J)\,\mathrm{Hom}_S(F_*S, S)$$

$$\mathrm{Hom}(F_*S/I, S/I) = F_*(I^{[p]} : I)\,\mathrm{Hom}_S(F_*S, S)$$

So comparing the two sides, it suffices to show that

$$F_*(J^{[p]} : J) \neq F_*((J/I)^{[p]} : J/I) \cdot (I^{[p]} : I)$$

Here is the motivation: If we take a prime $\bar{J}$ of R/I non-F-split, and R/J is regular (or more generally F-split), then there is no way $1 \mapsto 1$. As an example, Let $R = K[x,y,z]/\langle x^4 + y^4 + z^4\rangle$ be as in the previous example. Then R is not F-split, and in fact $\mathrm{Hom}_R(F_*R, R) \to \mathfrak{m} \subseteq R$. But then if we take $\mathfrak{m} = J$, we have $R/\mathfrak{m}$ is a field, thus regular, thus F-split. However, if $\psi \in F_*(J^{[p]} : J)\,\mathrm{Hom}(F_*R/I, R/I)$, then $\psi = 0$ on $R/\mathfrak{m}$. Thus this cannot be equal to $\mathrm{Hom}(F_*R/\mathfrak{m}, R/\mathfrak{m})$.

7) Suppose that L/K is a finite extension (meaning L is a finite K-module/vector space) of characteristic $p > 0$ fields and $x \in L \setminus K$ but $x^p \in K$. Show that if $\phi : K^{1/p^e} \longrightarrow K$ extends to $L^{1/p^e} \longrightarrow L$, then ϕ is the zero map on K.

Solution: If $\phi : F_*K \to K$ can be extended to $\phi_L : F_*L \to L$. Note that $x \in F_*K$ by assumption. Therefore, if we take $x \cdot y \in F_*K$, we can consider $x \cdot y \in F_*L$, for which $x \in L$. So

$$\phi(xy) = \phi_L(xy) = x\phi_L(y) \in K$$

But $x \notin K$, so we have that $\phi_L(y) = \phi(y) = 0$.

8) Show that an F-split ring is weakly normal. That is to say that if $r \in K(R) = \prod_{\mathfrak{q}} R_{\mathfrak{q}}$, then if $r^p \in R$, then this implies $r \in R$. You may assume R is a domain if desired, though this is not necessary.

Solution: Suppose R is F-split by $\varphi : F_*R \to R : F_*1 \mapsto 1$. Therefore, tensoring by $K(R)$, we get an F-splitting of $K(R)$:

$$\varphi \otimes 1 : F_*R \otimes K(R) \cong F_*K(R) \to K(R)$$

Note that

$$r \otimes 1 = r^p \otimes (r^{p-1}, 1) = 1 \otimes (r^{p-1}, r^p) = r$$

Under this map,

$$(\varphi \otimes 1)(r \otimes 1) = (\varphi \otimes 1)(1 \otimes r) = r$$

One the other hand, this is the image of $\varphi(r)$ by commutativity. Therefore $r \in R$.

9) Prove Lucas's Theorem:

Theorem 0.1 (Lucas's Theorem)**.** *$\binom{m}{n}$ is divisible by $p > 0$ if and only if expressing $n = \sum_{i=1}^{k} n_i p^i$ and $m = \sum_{i=1}^{l} m_i p^i$, for some i, $n_i > m_i$.*

Solution: Consider in $\mathbb{F}_p$

$$\sum_{n=0}^{m} \binom{m}{n} X^n = (1+X)^m = (1+X)^{\sum_{i=1}^{l} m_i p^i} = \prod_{i=1}^{l} (1+X)^{m_i p^i}$$

$$= \prod_{i=1}^{l} (1+X^{p^i})^{m_i} = \prod_{i=1}^{l} \left(\sum_{n_i=0}^{p-1} \binom{m_i}{n_i} X^{n_i p^i} \right)$$

$$= \sum_{n=0}^{m} \left(\prod_{i=0}^{l} \binom{m_i}{n_i} \right) X^n$$

Comparing coefficients, we conclude that $\binom{m}{n} \neq 0$ if and only if $\binom{m_i}{n_i} \neq 0$ which since $n_i, m_i < p$ is true if and only if $n_i \leq m_i$.

10) A ring R is called F**-pure** if for every R-module M, the map $M \to M \otimes_R F_*R$ is injective. Show that every F-split ring is necessarily F-pure.

Solution: Of course, if R is F-split, then $F_*R \cong R \oplus N$ for some R-module N. Therefore,

$$M \to M \otimes_R F_*R \cong M \otimes (R \oplus N) \cong M \oplus M \otimes N$$

The map is given by the identity on the first term, so it is necessarily injective.