# CLASS 29, NOVEMBER 19: SMIRNOFF'S METRIZATION THEOREM

The idea of paracompactness has already produced far reaching consequences. Today, we will prove one residual theorem about paracompact spaces and complement Nagata-Smirnoff with an additional set of conditions to ensure a space is metrizable.

**Theorem 29.1.** *Let $X$ be a paracompact Hausdorff space, and $\mathfrak{B}$ be a collection of subsets of $X$. For each $B \in \mathfrak{B}$, choose $\epsilon_B > 0$. If $\mathfrak{B}$ is locally finite, then there exists a continuous function $f : X \to (0, \infty)$ such that $f(x) \leq \epsilon_B$ for $x \in B$.*

You can think of this theorem as giving a nice continuous function with boundable values on any locally finite collection.

*Proof.* As usual, let $U_x$ be an open neighborhood of $x$ intersecting only finitely many elements of $\mathfrak{B}$. For a given $U_x$, choose $\epsilon_x = \min_{B\cap U_x \neq \emptyset}\{\epsilon_B\}$. This results in a positive number, since it is a minimum of finitely many positive numbers.

Also, choose $\varphi_x : X \to [0, 1]$ a partition of unity subordinate to the cover $U_x$. Then I claim the desired function is

$$f : X \to (0, \infty) : y \mapsto \sum_x \epsilon_x \cdot \varphi_x(y)$$

As usual, the sum on the right is finite for any choice of $y$ and thus continuous. If $y \in B$

$$f(y) = \sum_x \epsilon_x \cdot \varphi_x(y) \leq \epsilon_B \sum_x \varphi_x(y) = \epsilon_B$$

□

**Theorem 29.2** (Smirnoff Metrization Theorem)**.** *A space $X$ is metrizable if and only if it is paracompact, Hausdorff, and locally metrizable.*

Recall the following definition which came up on Exam 2:

**Definition 29.3.** A space $X$ is called **locally metrizable** if every point $x \in X$ has a neighborhood which is metrizable.

On the exam, it was demonstrated that a compact Hausdorff space which is locally metrizable is metrizable. Theorem 29.2 generalizes this result naturally.

**Example 29.4.** We have brought up the existence of this 'long line' when speaking of the importance of second-countability in the definition of a manifold. Here we can see it also yields a nice example of a space which is locally metrizable but not metrizable.

Recall $S_{\omega^\omega}$ or $S_\Omega$ is the set of ordinals less than $\omega_1$, the first uncountable ordinal. It consists of

$$S_\Omega = \{1, 2, \ldots, \omega, \omega + 1, \ldots, 2\omega, \ldots, n\omega, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots, \epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}, \ldots\}$$

Because it was asked, each of these are countable ordinals by induction: It is clear that $\{1, 2, \ldots\}$ is countable, which is the collection $\leq \omega$. But this shows that the collection less than $\omega^2$ is countable, since

$$[0, \omega^2) = \bigcup_{i=0}^{\infty} [i\omega, (i + 1)\omega]$$

and $[i\omega, (i + 1)\omega] \leftrightarrow [0, \omega]$. Similarly,

$$[0, \omega^3) = \bigcup_{i=0}^{\infty} [i\omega^2, (i+1)\omega^2] \qquad [0, \omega^\omega) = \bigcup_{i=0}^{\infty} [0, \omega^i]$$

$$[0, \epsilon_0) = \bigcup_{i=0}^{\infty} [0, \omega^{\cdot^{\cdot^{\cdot^{\omega}}}}{}^{i}]$$

and since a countable union of countable sets is countable, we see each is countable inductively.

Let $X = S_\Omega \times [0, 1)$, equipped with the dictionary topology. Then $X$ is called the long line, because it is intuitively obtained by adjoining an uncountable collection of intervals together. I claim that $X$ is locally metrizable, and in fact every point $(a, x)$ has a neighborhood homeomorphic to an open subset of $\mathbb{R}$.

For a given ordinal, there exist only countably many occuring before it (by definition of $\omega_1$). As a result, we can see that

$$((1, 0), (a, 0)) \cong \mathbb{R} \cong (0, 1)$$

But as a result, we can adjoint $((a, 0), (a, 1))$ to obtain a space homeomorphic to $(0, 2)$ with the dictionary topology.

Lastly, note that $X$ is non-metrizable. This is because it is not second-countable, which in the case of connected locally Euclidean spaces is equivalent to paracompactness. Therefore, it violates Theorem 29.2.

Now I will prove the metrization theorem. First, recall the Nagata-Smirnoff Metrization Theorem:

**Theorem** (Nagata-Smirnoff Metrization Theorem)**.** *A topological space $X$ is metrizable if and only if it is regular and has a countably locally finite basis.*

This will be used in the proof.

*Proof.* (of Theorem 29.2) ($\Rightarrow$): If $X$ is metrizable, then it is paracompact, Hausdorff, and locally metrizable. The first 2 have been proved, and the last statement is immediate from the definition.

($\Leftarrow$): Assume $X$ is paracompact, locally metrizable, and Hausdorff. Since a paracompact Hausdorff space is T4, it is also regular, so by Nagata-Smirnoff we only need to check that there is a countably locally finite basis.

Choose $U_x$ metrizable neighborhoods of each point $x \in X$. $U_x$ cover $X$, and since $X$ is paracompact we can refine this to a locally finite collection with the same property. Call it $V_\alpha$, and let $d_\alpha$ be the metric on $V_\alpha$. Since $V_\alpha$ is an open set, we note that $B_\alpha(x, r) \subseteq V_\alpha$ is also open in $X$.

Define

$$\mathfrak{B}_n = \{B_\alpha(x, \frac{1}{n}) \mid x \in V_\alpha \text{ and any } \alpha\}$$

and let $\mathfrak{B}'_n$ be a locally finite refinement of this collection which covers $X$. Of course, this implies that $\mathfrak{B} = \bigcup_{i \in \mathbb{N}} \mathfrak{B}'_i$ is a countably locally finite set.

It suffices to check $\mathfrak{B}$ is a basis for the topology of $X$. Given $x \in X$ and a neighborhood $U$ which intersects only finitely many $V_\alpha$, say $V_{\alpha_1}, \ldots, V_{\alpha_n}$ (WLOG, by shrinking). For each of these, $U \cap V_{\alpha_i}$ is open so contains some $B_{\alpha_i}(x, \epsilon_i)$. Therefore, we can choose $0 < \frac{2}{n} < \min\{\epsilon_1, \ldots, \epsilon_n\}$. Since $\mathfrak{B}'_n$ is a cover, there exists $y$ such that $x \in B(y, \frac{1}{n}) \in \mathfrak{B}'_n$. But note that for $z \in B_\alpha(y, \frac{1}{n})$

$$d_\alpha(x, z) \leq d_\alpha(y, x) + d_\alpha(y, z) \leq \frac{2}{n} < \min\{\epsilon_1, \ldots, \epsilon_n\}$$

So $B_\alpha(y, \frac{1}{n}) \subseteq B_{\alpha_i}(x, \epsilon_i)$. But this implies $\alpha = \alpha_i$, and furthermore, $x \in B_\alpha(y, \frac{1}{n}) \subseteq B_{\alpha_i}(x, \epsilon_i)$, as desired.

Now suppose that $x \in B_\alpha(x, \frac{1}{n}) \cap B_{\alpha'}(x', \frac{1}{n'})$. We can take $n'' = 2 \cdot \max\{n, n'\}$ as before. Then since $\mathfrak{B}_{n''}$ is again a covering, there exists $x'' \in X$ such that $x \in B_{\alpha''}(x'', \frac{1}{n''})$. Therefore, $\mathfrak{B}$ is a basis. $\square$