CLASS 20, APRIL 8TH: Spec & ALGEBRAIC VARIETIES

We will now transition to a bit of geometric reasoning. Recall that we left off with the following result before break:

Corollary 1 (Weak Nullstellensatz). *If K/k is a field extension, and K is a finitely generated k-algebra, then K/k is algebraic/integral, and thus is a finite field extension.*

We can view this as a statement about polynomial rings as follows: if $\mathfrak{m} \subsetneq K[x_1, \dots, x_n]$ is a maximal ideal, then $L = K[x_1, \dots, x_n]/\mathfrak{m}$ is a field extension of K. By the Weak Nullstellensatz, we can conclude that L is in fact a finite field extension. This gives us the following beautiful corollary (which simultaneously handles all of the cases we painstakingly dealt with previously).

Theorem 20.1. *If K is an algebraically closed field, then every maximal ideal of $R = K[x_1, \dots, x_n]$ has the form*

$$\mathfrak{m} = \langle x_1 - \alpha_1, \dots, x_n - \alpha_n \rangle$$

where $\alpha_i \in K$. Thus there is a natural bijection of m-Spec(R) with K^n.

Proof. First note that all of the ideals of that form are clearly maximal. Their quotient is K.

Notice that the generators of $L = K[x_1, \dots, x_n]/\mathfrak{m}$ as a K-algebra are the residue classes $\bar{x_1}, \dots, \bar{x_n}$. By the analysis above, we can conclude that L/K is a finite/algebraic extension. But we assume K is algebraically closed! I.e. there exist no non-trivial algebraic extensions of K. That is to say $L = K$. As a result, we note that $\bar{x_i} \in K$. I.e. $\bar{x_i} - \alpha_i = 0$ for some $\alpha_i \in K$ $\square$

A nice corollary of this fact coming from one of the exam questions is as follows:

Corollary 20.2. *Given a polynomial $K[x_1, \dots, x_n]$, we can view*

$$K[x_1, \dots, x_n] \subseteq \bar{K}[x_1, \dots, x_n]$$

This is an integral extension, so every maximal ideal has the form

$$\mathfrak{m} = \langle x_1 - \alpha_1, \dots, x_n - \alpha_n \rangle \cap K[x_1, \dots, x_n]$$

where $\alpha_i \in \bar{K}$.

This brings about the following nice geometric realization of ideals.

Definition 20.3. A K**-variety** is a set $V \subseteq K^n$ such that

$$V = V(I) = \{(a_1, \dots, a_n) \in K^n \mid f(a_1, \dots, a_n) = 0 \ \ \forall f \in I\}$$

where I is an ideal of $K[x_1, \dots, x_n]$.

Example 20.4. Consider the ideal $J = \langle x^2 + y^2 + z^2 - 1 \rangle \subseteq \mathbb{R}[x, y, z]$. The resulting variety $V(J)$ is the sphere S^2. If we considered instead $J = \langle x^2 + y^2 - z^2 \rangle \subseteq \mathbb{R}[x, y, z]$, then $V(J')$ is the cone!

Since $K[x_1, \ldots, x_n]$ is a Noetherian ring, we get that I is a finitely generate ideal:

$$I = \langle f_1, \ldots, f_m \rangle$$

Therefore, $V(I)$ is the set of points for which $f_1(a) = \ldots = f_m(a) = 0$.

Proposition 20.5. *If K is algebraically closed, and $A = K[x_1, \ldots, x_n]/I$ is a finitely generated K-algebra. Then every maximal ideal has the form $\mathfrak{m} = \langle x_1 - \alpha_1, \ldots, x_n - \alpha_n \rangle$, where $(\alpha_1, \ldots, \alpha_n) \in V(I)$. Thus there is a natural bijection between $V(I)$ and m-*Spec(A).

Proof. This is a culmination of several results from the homeworks:

- The preimage of a prime ideal is a prime.
- The preimage of a maximal ideal under a surjection is maximal.
- $\text{Spec}(A) = \{\mathfrak{p} \in \text{Spec}(K[x_1, \ldots, x_n] \mid I \subseteq \mathfrak{p}\}$

$\square$

The final piece of data to speak about today is the ideal/variety correspondence. This gives a map which provides something like an inverse for the map V described above. We will discuss how close it is to an inverse next time.

Definition 20.6. Given *any* subset $X \subseteq K^n$, we can define

$$I(X) = \{f \in K[x_1, \ldots, x_n] \mid f(x) = 0 \ \ \forall x \in X\}$$

This is an ideal of $K[x_1, \ldots, x_n]$ (it is an easy check).

Proposition 20.7. *Both V and I are inclusion reversing maps: If $J' \subseteq J$, then $V(J') \supseteq V(J)$ and if $Y \subseteq X$, then $I(Y) \supseteq I(X)$.*

$$\begin{tikzcd} \{X \subseteq K^n\} \arrow[r, bend right, "I"'] & \{I \subseteq K[x_1, \ldots, x_n] \text{ an ideal.}\} \arrow[l, bend right, "V"'] \end{tikzcd}$$

Proof. For the first statement, if every polynomial $f \in J$ vanishes at some point x, then so does every $f \in J'$! Thus $V(J') \supseteq V(J)$. Similarly, for the second statement, the polynomials which vanish for all $x \in X$ necessarily vanish for all $y \in Y \subseteq X$. $\square$

As an immediate corollary of this fact, we have the following:

Corollary 20.8. *$X \subseteq V(I(X))$ with equality if and only if X is a variety, i.e. $X = V(I)$. Similarly, $J \subseteq I(V(J))$ for any ideal J.*

Proof. $X \subseteq V(I(X))$ is demonstrating by the following; $I(X)$ is the set of all polynomials which vanish on all of X. These functions may vanish elsewhere, but certainly vanish on X! The equality statement follows by definition: $X = V(J)$ is precisely the set of points for which every $f \in J$ vanishes on. Rephrased:

$$V(J) = V(I(V(J)))$$

The other statement also follows via similar analysis; $I(V(J))$ is the set of functions which vanish at all points for which every $f \in J$ vanishes. More functions may exist! $\square$