HOMEWORK 4: CUACHY'S INTEGRAL COROLLARIES
DUE: WEDNESDAY, OCTOBER 9TH

1) If $f : \mathbb{R} \times i \cdot (-1, 1) \to \mathbb{C}$ is a holomorphic function on the real strip, with
$$|f(z)| \leq A(1 + |z|)^n$$
for some n fixed and all z, show that for each integer m we have
$$|f^{(m)}(x)| \leq A_m(1 + |x|)^n$$
for some $A_m > 0$ and all $x \in \mathbb{R}$.

Solution: By Cauchy's inequality, we may consider the ball of radius $1 - \epsilon$ centered at $x \in \mathbb{R}$. Doing so produces
$$|f^{(m)}(x)| \leq \frac{m!\|f\|_{C_\epsilon}}{(1-\epsilon)^n}$$
where C_ϵ is the circle of radius $1 - \epsilon$ centered at x. But we also have
$$|f(z)| \leq A(1 + |z|)^n$$
for every $z \in \mathbb{C}$, which of course specializes to $\mathbb{R}$. This shows
$$|f^{(m)}(x)| \leq \frac{m!\|f\|_C}{(1-\epsilon)^n} \leq \frac{m!A(1 + \sup_{z\in C_\epsilon} |z|)^n}{(1-\epsilon)^n} =$$
Noticing $\sup_{z\in C_\epsilon} |z| \leq |x| + 1$ yields
$$|f^{(m)}(x)| \leq \frac{m!A(2 + |x|)^n}{(1-\epsilon)^n} \to m!A(2 + |x|)^n = A \cdot m! \cdot 2^n(1 + \frac{|x|}{2})^n$$
This is bounded above by $A \cdot m! \cdot 2^n(1 + |x|)^n$, thus $A_m = A \cdot m! \cdot 2^n$ will suffice.

2) Weirstrass's theorem asserts every continuous function of $[0, 1]$ can be approximately uniformly by polynomials. Is the same true for continuous complex valued functions on the unit disc?

Solution: The answer is no. If it can be uniformly approximated by polynomials, then it must be analytic on the disc since
$$p_n(z) = a_n z^n + \ldots + a_0 \to \sum_{n=0}^{\infty} a_n z^n = f(z)$$
as $n \to \infty$. So it must be a holomorphic function. So it suffices to take a continuous but not differentiable function. Our favorite example is $f(z) = \bar{z}$. This provides a counter example.

Indeed, if $\bar{z} = \sum_{n=0}^{\infty} a_n z^n$, then
$$Re^{-i\theta} = \sum_{n=0}^{\infty} a_n R^n e^{in\theta}$$

$$R = \sum_{n=0}^{\infty} a_n R^n e^{i(n+1)\theta}$$

The right is a power series, thus holomorphic, with imaginary part constant. Thus it must be constant.

3) The following function are analytic on the unit disc but cannot be extended outside this domain. If $f : B(z_0, r) \to \mathbb{C}$ is holomorphic, and $|z - z_0| = r$, then z is called **regular** if there is a power series centered at z agreeing with the one for f on points of intersection. Thus a function cannot be analytically continued outside the circle if no point is regular along the boundary.

Let $\alpha > 0$. Show that the following have radius of convergence 1:

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \qquad\qquad g(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n}$$

Additionally, show the second extends continuously to the boundary circle $|z| = 1$, and that neither can be analytically extended beyond the disc.

Solution: First, lets consider the radius of convergence:

$$\frac{1}{R_f} = \limsup |a_n|^{\frac{1}{n}} = \lim |1|^{\frac{1}{2^n}} = 1$$

$$\frac{1}{R_g} = \limsup |a_n|^{\frac{1}{n}} = \lim |2^{-n\alpha}|^{\frac{1}{2^n}} = \lim 2^{-\frac{n\alpha}{2^n}} = 2^{-\lim \frac{n\alpha}{2^n}} = 2^0 = 1$$

For the second statement, if $|z| = 1$, then $\sum_{n=0}^{\infty} |2^{-n\alpha} z^n| = \sum_n (2^\alpha)^{-n} = \frac{1}{1-2^\alpha}$.

Finally, suppose there is a regular point on the boundary of the unit circle for $f(z)$ and $g(z)$. Suppose z_0 is one such. Notice that the points $e^{\frac{2\pi i}{2^n}}$ fill up the circle for $n \gg 0$, so any radius of convergence with contain some such point say w_0 for some fixed n. Then notice that

$$f(w_0) = \sum_{m=0}^{\infty} w_0^{2^n} = \sum_{m=0}^{n-1} w_0^{2^m} + \sum_{m=n}^{\infty} w_0^{2^m} = C + \sum_{m=n}^{\infty} 1$$

which clearly diverges! Similarly, choosing $|w_0| = r = 1+\epsilon > 1$ in the neighborhood produces

$$g(w_0) = \sum_{m=0}^{\infty} 2^{-m\alpha} w_0^{2^m} = C + \sum_{m=n}^{\infty} 2^{-m\alpha} r^{2^m} = C + \sum_{m=n}^{\infty} \left(\frac{r^{\frac{2^m}{m}}}{2^{-\alpha}} \right)^m$$

The last sum diverges for any $r > 1$ since $2^{-\alpha}$ is just a constant.

4) Suppose $f : \mathbb{C} \to \mathbb{C}$ is an analytic function:

$$f(z) = \sum_{n=0} a_n (z - z_0)^n$$

Show that if one coefficient $a_N = 0$, then f is a polynomial function.

Solution: If $a_N = 0$ for some N, then this is saying that for every $w \in \mathbb{C}$, there exists m such that $f^{(m)}(w) = 0$. This follows by converting f to a power series in $(z-w)$ using the binomial theorem.

Let $A_m = \{w \in \mathbb{C} \mid f^{(m)}(w) = 0\}$. Then f is polynomial iff A_m is not countable for some m. The if part of this result is to say that an uncountable set must have a limit point. Then by Lemma 11.1, we have $f^{(m)}(z) = 0$. So f must be polynomial.

Finally, since every $w \in \mathbb{C}$ fits into one A_m, there must be one that is uncountable.

5) Suppose f is holomorphic in Ω an open set except at $z_0 \in \Omega$ where $|z_0| = 1$. If $\bar{B}(0,1) \subseteq \Omega$ and

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is a power series for f in $B(0,1)$, show $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$.

Solution: Suppose that z_0 is a simple pole. This yields that

$$(z-z_0)f(z) = g(z) = \sum_{n=0}^{\infty} b_n z^n$$

is holomorphic in Ω and in particular converges around $z = z_0$. This implies in particular that $b_n \to 0$.

Now we compare coefficients:

$$(z-z_0)f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)z^n = \sum_{n=1}^{\infty} a_{n-1}z^n - \sum_{n=0}^{\infty} a_n z_0 z^n = \sum_{n=0}^{\infty}(a_{n-1} - a_n z_0)z^n$$

Thus $b_n = a_{n-1} - a_n z_0 \to 0$. This is only possible if $\frac{a_{n-1}}{a_n} \to z_0$.

6) If $f : \bar{B}(0,1) \to \mathbb{C}$ is non-vanishing and continuous, holomorphic on $B(0,1)$, then show that if $|f(z)| = 1$ for all $|z| = 1$, then f is constant. (**hint:** Show that f can be extended to all of $\mathbb{C}$ by $1/\overline{f(\frac{1}{\bar{z}})}$ as in the Schwarz reflection principle.)

Solution: Following the hint, define $g(z)$ piecewise by f on $\bar{B}(0,1)$ and $1/\overline{f(\frac{1}{\bar{z}})}$ for $|z| \geq 1$. Note that this is well defined when $|z| = 1$:

$$\frac{1}{\bar{z}} = \frac{1}{e^{-i\theta}} = e^{i\theta} = z$$

Applying the same logic to f we get that they are equal on the boundary

Now, it goes to show that g is holomorphic. It suffices to check that $\frac{1}{\overline{f(\frac{1}{\bar{z}})}}$ satisfies the polar CR equations and has continuously differentiable partials. Notice that we can write $\frac{1}{\overline{f(\frac{1}{\bar{z}})}}$ as $b(o(f(b(o(z)))))$, where $o(z) = \frac{1}{z}$ and $b(z) = \bar{z}$. Then we know $b_x(z) = 1$ and $b_y(z) = -1$. As a result, after applying the chain rule CR for f, the 2 negative signs cancel. This shows g is holomorphic.

Now the piecewise function is again holomorphic on the boundary by Morera's Theorem, since it is bounded on the interior of any triangle by compactness and continuity. So having a bounded + entire function yields constant by Louiville's Theorem.