CLASS 13, OCTOBER 5: LOCAL PROPERTIES

Today we will double our list of nice properties for a topological space by adding one definition; **local**. All of our properties so far have been about X globally as a topological space. Even very nice topological spaces don't satisfy these, such as $\mathbb{R}$ not being compact. But small (closed) neighborhoods do satisfy this property. This is comparable to studying rings instead of algebraic varieties in algebra.

Definition 13.1. Let $\mathcal{P}$ be a property of topological spaces. Then we say X is **locally-$\mathcal{P}$** if for any $x \in X$ and neighborhood Z of x, there exists a neighborhood $Z' \subseteq Z$ of x such that Z' is $\mathcal{P}$.

Examples of $\mathcal{P}$ are connectedness, path connectedness, and compactness. We will study these properties here.

Example 13.2. I recommend as an exercise to draw the resulting spaces.

1) An example of a connected space which is not locally connected is the topologist's sin-curve:
$$X = (\{0\} \times [-1,1]) \bigcup \left\{ (x, \sin(\frac{1}{x})) \mid 0 < x < 1 \right\} \subseteq \mathbb{R}^2$$
2) To produce an example of a locally (path) connected but not (path) connected space, take any space with finitely many connected components, each of which is locally (path) connected. An example of this is $X = \mathbb{R} \coprod \mathbb{R}$, the disjoint union of two copies of $\mathbb{R}$. Any neighborhood of x (in say the first copy of $\mathbb{R}$) necessarily contains $(x - \epsilon, x + \epsilon)$, which is connected since it is homeomorphic to $(0, 1)$.
3) To produce an example of a path connected but not locally path connected space, we us the 'topologist's comb':
$$X = ([0,1] \times \{0\}) \bigcup \{(x, y) \mid x \in \mathbb{Q} \cap (0,1),\ y \leq x\} \subseteq \mathbb{R}^2$$
This space is path connected, since we can always travel down the comb to $[0, 1] \times \{0\}$, over to the correct value of x, and then up to a given point. On the otherhand, note that $X \cap B((1, 0), \frac{1}{2})$ is not even connected!

I now state an equivalent definition of locally connected.

Proposition 13.3. *X is locally connected if and only if $\forall U \subseteq X$ open, each connected component of U is also open.*

Proof. $(\Rightarrow)$: If X is locally connected, let $V \subseteq U$ be a connected component of U. Then for any $x \in V$ there exists a connected neighborhood C_x of x within V. But neighborhoods contain open subsets V_x containing x, so we see
$$V = \bigcup_{x \in V} U_x$$
$(\Leftarrow)$: If V is a neighborhood of x, we can consider V_x the connected component of x in V. This is a connected open neighborhood by assumption. $\square$

Now I will switch gears and study the notion of **local compactness**.[1]

Example 13.4.
1) $\mathbb{R}^n$ is a locally compact space which is not itself compact. Indeed, every neighborhood contains $B(x,r)$ for some $r > 0$, and thus $\bar{B}(x, \frac{r}{2})$.
2) On the otherhand, $\mathbb{R}^{\mathbb{N}}$ with the product (or box) topology is not locally compact. Indeed, there contain no compact neighborhoods containing ANY of the basis elements $U_1 \times \ldots \times U_n \times \mathbb{R}^{\{n+1,n+2,\ldots\}}$.

Proposition 13.5. *If X is compact and Hausdorff, then X is locally compact.*

Proof. Suppose U is a (open WLOG) neighborhood of some point $x \in X$. For every point $y \in U^c$, let U_y and V_y be open sets containing x and y respectively that are disjoint. Since $U^c \subseteq X$ is a closed subset it is compact, so finitely many will do:

$$U^c \subseteq V_{y_1} \cup \ldots \cup V_{y_n} = V$$

Its complement V^c is a closed subset containing the open set $U_{y_1} \cap \ldots \cap U_{y_n}$, thus a compact neighborhood of x. □

Corollary 13.6. *If X is a Hausdorff locally compact space, and $U \subseteq X$ is an open or closed subset of X, then U is locally compact.*

Proof. If U is a closed subset, then U is itself compact and Hausdorff. Therefore, Proposition 13.5 implies U is also locally compact.

On the otherhand, if U is open and V is a neighborhood in U of x, then V is also a neighborhood of $x \in X$. By Proposition 13.5, there is a compact set contained within V in X. This remains compact in the subspace topology of U since U is open, and thus open covers in U are also open in X. □

An immediate corollary of Corollary 13.6 is the following:

Corollary 13.7. *Let X be a Hausdorff space that is not itself compact. Then X is locally compact if and only if X is homeomorphic to an open subset of a compact Hausdorff space.*

Finally, I introduce the 1-point compactification of a locally Hausdorff space X. Let $Y = X \cup \{\infty\}$, where ∞ is just a name for a new distinguished point. It goes to define a topology. A subset $U \subseteq Y$ is open if either

- ∘ $\infty \notin U$ (or equivalently $U \subseteq X$) and U is open in the topology of X.
- ∘ $\infty \in U$ and $U^c \subseteq X$ is a compact subset.

Note that this is in fact a topology. X has the second property and $\emptyset$ has the first. The other 2 facts follow from the fact that arbitrary intersections of closed subsets are closed and finite unions of compact sets are compact. Y is called the **one-point compactification** of X.

1) If $X = \mathbb{R}$, then $Y = \mathbb{R} \cup \{\infty\} \cong S^1$.
2) If $X = \mathbb{C}$, then Y is the Riemann Sphere.
3) If $X = \mathbb{R}^n$, then $Y \cong S^n$.

[1] The book specifies local compactness as every point has a compact neighborhood. This is a less stringent condition in general than what I have defined, but equivalent when X is Hausdorff. This notion makes Proposition 13.5 obvious and not require the Hausdorff condition, but is more esoteric generally speaking.