CLASS 18, MONDAY APRIL 9TH: REGULAR RINGS AND GLOBAL DIMENSION

We have already seen some nice properties of regular local rings. Today I will extend several notions of the previous lecture to more arbitrary rings. In the process, some hands will be waived for the sake of time. Resources are available for those interested.

Definition 0.1. A Noetherian ring R is said to be **regular** if for every prime ideal $\mathfrak{p} \subseteq R$, $R_\mathfrak{p}$ is a regular local ring (as defined last class).

This is a nice definition, since it makes regularity a local property. However, there is one problem that remains unanswered: Is a regular local ring a regular ring? This is true, yet unclear as of now; how can you ensure every localization at $\mathfrak{p} \subseteq \mathfrak{m}$ is regular?

We will assume throughout all rings are Noetherian. We need a few extra definitions to resolve this question (in the affirmative):

Definition 0.2.

1) A sequence $x_1, x_2, \ldots, x_n$ is called a R-**regular sequence** if each x_i is a NZD for $R/\langle x_1, \ldots, x_{i-1}\rangle$, and $R/\langle x_1, \ldots, x_n\rangle \neq 0$. We can do the analogous thing for a module M.

2) The **depth** of a module M is exactly the length of the longest M-regular sequence.

3) A ring R is called **Cohen-Macaulay** or **CM** if $\text{depth}(R) = \dim(R)$. Note that we always have $\text{depth}(R) \leq \dim(R)$.

4) If R is a ring, and M is an R-module, the **projective dimension** of M is
$$\text{pdim}_R(M) = \inf\{n \mid \exists 0 \to P_n \to \ldots \to P_0 \to M \to 0 \text{ a projective resolution}\}.$$
We can analogously define the **injective dimension**.

5) The **global dimension** of a ring R is
$$\text{gl-dim}(R) = \sup\{\text{pdim}_R(M) \mid M \text{ is an } R\text{-module}\}.$$

Example 0.3.

- Given an RLR $(R, \mathfrak{m})$ with $\mathfrak{m} = \langle x_1, \ldots, x_n\rangle$, then $x_1, \ldots, x_n$ form a regular sequence on R. Thus an RLR is Cohen-Macaulay.
- $R = K[x^2, x^3] \cong K[X, Y]/\langle X^3 - Y^2\rangle$ is Cohen-Macaulay, but not regular. It is not regular because $R_{\langle x^2, x^3\rangle}$ has maximal ideal $\langle x^2, x^3\rangle$, and this can't be generated by 1 element (exercise). But $R \subseteq K[x]$ is an integral extension, so $\dim(R) = 1$.

  On the other hand, $\text{depth}(R) = 1$, since x^2 is a length 1 regular sequence.
- The following ring is not CM:
$$R = K[x, y, u, v]/\langle x, y\rangle \cap \langle u, v\rangle = K[x, y, u, v]/\langle xu, yu, xv, yv\rangle$$
  It is left as a homework exercise to check that $\dim(R) = 2$ and $\text{depth}(R) = 1$.

- If P is projective, $\text{pdim}_R(P) = 0$ since $0 \to P \to P \to 0$ is exact. So it is natural to think lower projective dimension implies more projective.
- Consider the ring $R = K[x, y]/\langle xy \rangle$ and $M = K[x] = R/\langle y \rangle$. We have a projective resolution
$$\ldots \xrightarrow{\cdot x} R \xrightarrow{\cdot y} R \xrightarrow{\cdot x} R \xrightarrow{\cdot y} R \to M \to 0$$
via localization at $\langle x, y \rangle$, we can furthermore see that there is no finite projective resolution of M. Therefore, $\text{pdim}_R(M) = \text{gl-dim}(R) = \infty$.

This last fact is part of an important characterization of RLRs.

Theorem 0.4. *A local ring $(R, \mathfrak{m})$ is regular if and only if R has finite global dimension. In this case,* $\text{gl-dim}(R) = \dim(R)$*.*

Caution: This does NOT hold without the local hypothesis. This is usually shown with the following two lemmas:

Lemma 0.5. *Let $(R, \mathfrak{m})$ be a local ring. Then the following inequality holds:*
$$\text{pdim}_R(R/\mathfrak{m}) \geq \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2) \geq \dim(R)$$

Lemma 0.6. *Let $(R, \mathfrak{m})$ be a local ring. If $R/\mathfrak{m}$ has finite* $\text{pdim} = n$*, then* $\dim(R) \geq n$*. Thus if $(R, \mathfrak{m})$ is regular,* $\dim(R) = \text{pdim}_R(R/\mathfrak{m})$*. Otherwise,* $\text{gl-dim}(R) = \infty$*.*

Finally, coupled with Nakayama's lemma, one can show that $\text{pdim}(M) \leq \text{pdim}(R/\mathfrak{m})$ for any M. These elements completes the proof of Theorem 0.4.

Given these results, one can resolve the question issued above:

Theorem 0.7. *A regular local ring satisfies Definition 0.1, and therefore is itself regular.*

Proof. Let $(R, \mathfrak{m})$ be a local ring, and let $\mathfrak{p}$ be a prime ideal. Let M be an $R_\mathfrak{p}$-module. This gives it the structure of an R-module via the localization map $R \to R_\mathfrak{p}$. Therefore, we can construct a projective resolution of M as an R-module. We know that R has finite projective dimension, so
$$0 \to R^{m_n} \to R^{m_{n-1}} \to \ldots \to R^{m_0} \to M \to 0$$
where $n = \dim(R)$.[1] But we can localize this sequence at $\mathfrak{p}$:
$$0 \to R_\mathfrak{p}^{m_n} \to R_\mathfrak{p}^{m_{n-1}} \to \ldots \to R_\mathfrak{p}^{m_0} \to M_\mathfrak{p} \to 0$$
But from the universal property of localization (Class 10, Theorem 0.2), we have $M_\mathfrak{p}$ is the unique smallest R-module with a homomorphism from M for which elements of $R \setminus \mathfrak{p}$ are inverted. However, note that if $r \in R \setminus \mathfrak{p}$,
$$r \cdot M = (1, r)M = (1, r)(r, 1)M = M$$
So r acts as a unit on M. This shows that $M \cong M_\mathfrak{p}$. Therefore, the above resolution looks like
$$0 \to R_\mathfrak{p}^{m_n} \to R_\mathfrak{p}^{m_{n-1}} \to \ldots \to R_\mathfrak{p}^{m_0} \to M \to 0$$
So M has finite projective dimension, bounded above by $n = \dim(R)$, and M was arbitrary implies $\text{gl-dim}(R_\mathfrak{p}) \leq n$, and thus $R_\mathfrak{p}$ is a RLR as desired. $\square$

We now have most of the desired resources to proceed to some positive characteristic commutative algebra, which we will start Friday.

[1] Every projective is free since R is local!