CLASS 14, OCTOBER 12: COUNTABILITY AND CONVERGENCE

In the coming days, we will introduce various stronger and weaker versions of the Hausdorff condition already stated. Before studying these notions, we need to add the notions of countability and convergent sequences to our list of definitions.

Definition 14.1. Let X be a topological space. A sequence of points $\mathbf{x} = (x_1, x_2, \ldots) \in X^{\mathbb{N}}$ is said to **converge** to a point $x \in X$ if for every (open) neighborhood U of x intersects the sequence for all but finitely many terms. Alternatively, there exists $N = N(U)$ such that $x_n \in U$ for $n \geq N$. We write $x_n \to x$ in this case.

This should of course be familiar from the world of metric spaces. It allows us to define notions of limits. However, as we know, the world of topology adds a lot of depth to our study.

Example 14.2. Let X be any infinite set with the finite complement topology. Let $(x_1, x_2, x_3, \ldots)$ be a sequence of distinct points of X. Then $x_n \to x$ for *any* point $x \in X$. Indeed, any open neighborhood is given by $U = X \setminus \{y_1, y_2, \ldots, y_n\}$, and we have assumed that $x_i \neq x_j$ for every j. So eventually $x_n \neq y_i$!

On the other hand, metric spaces are Hausdorff (you essentially checked this on hwk 1). This is enough to conclude that a sequence as above cannot simultaneously converge to multiple points.

Proposition 14.3. *If X is a Hausdorff space, and $(x_1, x_2, \ldots)$ converges to a point x, then x is unique.*

Proof. Suppose the sequence in question also converges to $y \neq x$. By the Hausdorff condition, there exists open disjoint sets (neighborhoods) of x and y, say U and V respectively, such that $U \cap V = \emptyset$. But then $x_n \in U$ for $n \geq N$ and $x_m \in V$ for $m \geq M$. But this implies in particular that

$$x_{\max\{M,N\}} \in U \cap V = \emptyset$$

which is a contradiction. □

So the importance of avoiding pathologies with the Hausdorff condition is apparent. We can also reverse this by putting a local size cap on our topology. Here we define the notion of first-countable.

Definition 14.4. We say X has a **countable basis** at a point $x \in X$ if there exist a countable collection of neighborhoods of x, say $X_1, X_2, \ldots$, such that any neighborhood U of x contains one of these sets: $U \supseteq X_i$. If X has a countable basis at every point, X is said to be **first-countable**.

Example 14.5. Every metric space satisfies this property. Indeed, the basis for the metric topology is given by

$$\mathcal{B} = \{B(x, d) \mid x \in X,\ d > 0\}$$

However, for a fixed x, we can restrict to only positive rational d. This gives a countable basis at x.

Theorem 14.6. *Let X be a topological space.*

1) *Let $A \subseteq X$. If there exists $(a_1, a_2, \ldots)$ such that $a_n \to x$, then $x \in \bar{A}$.* [1]
2) *If $f : X \to Y$ is a continuous function, then $x_n \to x$ implies $f(x_n) \to f(x)$.*

In either case, if X is first-countable, the converse of each statement holds.

Proof. 1) Let x and a_n be as in the theorem. Since $a_n \to x$, we note that every neighborhood U of x intersects A. If $x \notin \bar{A}$, then $x \in \bar{A}^c = (A^c)^\circ$, and thus there is some neighborhood of x in $\bar{A}^c \subseteq A^c$. This contradicts the first statement.

2) Let V be an open neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open neighborhood of x. Therefore, there exists N such that $x_n \in f^{-1}(V)$ for $n \geq N$. But this implies

$$f(x_n) \in f(f^{-1}(V)) \subseteq V$$

Now lets assume $X_1, X_2, \ldots$ is a countable base for X at x.

1) If $x \in \bar{A}$, choose a sequence $(a_1, a_2, \ldots)$ where $a_n \in X_1 \cap \ldots \cap X_n$. Note this is still an open neighborhood of x. Now, $a_i \in X_n$ for all $i \geq n$. Moreover, if U is any neighborhood of x, $U \supseteq X_n$ for some n. Therefore, it contains all but finitely many points of the sequence.
2) Note that to show the continuity of f, we need that for any set $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)} \subseteq Y$. Given $x \in \bar{A}$, the previous statement implies that there exists $(x_1, x_2, \ldots) \in A^{\mathbb{N}}$ converging to x. But this implies $f(x_n) \to f(x)$ by our assumption. By the first part, this implies $f(x) \in \overline{f(A)}$, as desired.

□

Now I provide a partial converse to Proposition 14.3.

Theorem 14.7. *If X is a first-countable topological space and $x_n \to x$ implies x is unique, then X is Hausdorff.*

Proof. Let $x \neq y$ be 2 distinct points in X. Suppose there do not exist open disjoint neighborhoods of x and y, or equivalently, every 2 neighborhoods of x and y intersect. Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be bases at x and y. Then we note that $X(n) = X_1 \cap \ldots X_n$ is an open neighborhood of x and $Y(n) = Y_1 \cap \ldots Y_n$ is a neighborhood of y. Therefore, they must intersect. Choose

$$a_i \in X(i) \cap Y(i)$$

Then any neighborhood U and V contain $X(n)$ and $Y(n)$ for some $n > 0$. Therefore,

$$U \cap V \supseteq X(n) \cap Y(n) \supseteq \{a_n, a_{n+1}, \ldots\}$$

But this implies the sequence $a_n \to x, y$, contradicting our assumption and proving the claim. □

[1] This characterizes closed sets!