CLASS 8, SEPTEMBER 24: CONNECTEDNESS

We now enter the realm where we have all of the basic players in the field; topological spaces and continuous maps. Now the questions start to arise; what kind of conditions can we put on such objects to make them 'nice'? Do these theorems encapsulate many of the main properties from real analysis or calculus in a more formal way?

The first of these definitions is that of connectedness, which generalizes the notion of an interval in $\mathbb{R}$.

Definition 8.1. A topological space X is called **connected** if for any open subsets U, V covering X ($X = U \cup V$) we have that $U \cap V \neq \emptyset$. Otherwise, the set is called **disconnected**, and in this case the sets U and V are called a **separation** of X.

This can be rephrased in terms of open sets; X is connected if and only if the only subsets of X which are closed and open (clopen) are X and $\emptyset$. This can be seen by taking U open and $V^c = U$ in the definition of connected.

Example 8.2. 1) Any set with the indiscrete topology is a connected space.

2) Any set with more than 1 point and the discrete topology is disconnected.

3) The following gives a common example of connected subsets of $\mathbb{R}$: intervals!

Proposition 8.3. $(0,1)$ *is a connected subset of* $\mathbb{R}$ *with the Euclidean topology.*

Proof. Suppose not. Then there exists U, V open non-empty, not intersecting, and covering $(0,1)$. By our basis for the topology of $\mathbb{R}$, and combining overlapping intervals, we know

$$U = \bigcup_{\alpha} (a_\alpha, b_\alpha)$$

We can then form a new covering given a specific choice of α, say α_0:

$$U' = (a_{\alpha_0}, b_{\alpha_0})$$

$$V' = V \cup \left(\bigcup_{\alpha \neq \alpha_0} (a_\alpha, b_\alpha) \right)$$

Both of these sets are open since they are unions of basis elements. However, the assumptions for a separation yield that V' has one of the following forms:

$$V' = (0, a_{\alpha_0}] \cup [b_{\alpha_0}, 1) \textbf{ or } (0, a_{\alpha_0}] \textbf{ or } [b_{\alpha_0}, 1)$$

In any of these cases, V' is not open since either $B(\alpha_0, \epsilon) \not\subseteq V$ or $B(\beta_0, \epsilon) \not\subseteq V$.

$\square$

4) The same argument can be applied to the intervals (a,b), $(a,b]$, $[a,b)$ and $[a,b]$.

5) $\mathbb{Q} \subset \mathbb{R}$ is a *totally* disconnected space, meaning it's only connected components are single points! Suppose $a \neq b \in U \subset \mathbb{Q}$ a connected subset. There exists $c \in \mathbb{R}$ with $a < c < b$ and c irrational. Therefore

$$(a,b) \cap \mathbb{Q} = ((a,c) \cap \mathbb{Q}) \cup ((c,b) \cap \mathbb{Q})$$

Intersecting these sets with U produces a separation of U, contradicting our assumption that 2 distinct points can be a in a connected subset of $\mathbb{Q}$.

6) Let X be the union of the x-axis and the graph of $y = \frac{1}{x}$ for $x > 0$ in $\mathbb{R}^2$. Then this space is disconnected. Indeed, each of the subsets can be enclosed in open disjoint sets in $\mathbb{R}$, and therefore under the subspace topology they remain open individually:

$$U = \{(x,y) \mid y > \frac{1}{2x},\ x > 0\}$$

$$V = \{(x,y) \mid y < \frac{1}{2x}\} \cup (-\infty, 1) \times (-\frac{1}{2}, \frac{1}{2})$$

Now we can produce some properties under which connectedness is preserved.

Proposition 8.4. *If X is separated by two open subsets U, V, and $Y \subseteq X$ is connected, then $Y \subseteq U$ or $Y \subseteq V$.*

Proof. Suppose $Y \not\subseteq V$. Then since Y is connected and

$$Y = (Y \cap U) \cup (Y \cap V)$$

is a union of 2 open subsets, we find that $Y \cap V = \emptyset$, or $Y \subseteq U$. $\square$

Proposition 8.5. *If $x \in \bigcap_\alpha U_\alpha$ where each U_α is connected, then $\bigcup_\alpha U_\alpha$ is also connected.*

Proof. Suppose $\bigcup_\alpha U_\alpha$ is separated by V, V'. Then by Proposition 8.4, each U_α is contained in either V or V'. If $U_\alpha \subseteq V$ and $U_{\alpha'} \subseteq V'$, then $x \in V \cap V'$, contradicting the fact that they form a separation. So all U_α live in either V or V', implying the other is empty. $\square$

Continuing with these ideas, we can represent a space by its so called *connected components*.

Definition 8.6. For a given $x \in X$, there exists a largest connected subset U_x (not necessarily open) such that U_x contains x and U_x. U_x is called the **connected component of** x.

We can of course cheat using Proposition 8.5 to show it exists:

$$U_x = \bigcup_{\substack{x \in U \\ U \text{ connected}}} U$$

Theorem 8.7. *A space X can be decomposed into its connected components in a disjoint way*

$$X = \coprod_\alpha U_\alpha$$

where each U_α is connected and disjoint from any $U_{\alpha'}$ for $\alpha \neq \alpha'$.

Proof. We can create an equivalence relation on X, which says $x \sim y$ if and only if there exists a connected subset $Y \subseteq X$ such that $x, y \in Y$. Call such an equivalence class $[x]$, and the set of all such equivalence classes $X/\sim$. Then x and y share a connected component: $U_x = U_y$ with the terminology of Definition 8.6. We can then form

$$X = \bigcup_{[x] \in X/\sim} U_x$$

This covers X since every $x \in X$ is in U_x. Furthermore, $U_x \cap U_y = \emptyset$ for each $[x] \neq [y]$ by Proposition 8.5; if they shared a point their union would be a larger connected set containing x, contradicting the definition of U_x. $\square$