CLASS 12, OCTOBER 4: FURTHER APPLICATIONS

Last time we saw a few very important corollaries of Cuachy's integral theorem, such as the idea of analytic continuation and Morera's theorem. Today we will study sequences of Holomorphic functions and the Schwarz reflection principle.

The first thing I want to tackle is when the limit of holomorphic functions is holomorphic. It turns out it is sufficient to assume uniform convergence on compact subsets:

Theorem 12.1. *If $f_n : \Omega \to \mathbb{C}$ is a sequence of holomorphic functions such that $f_n \to f$ uniformly for every compact subset $K \subseteq \Omega$, then f is holomorphic on Ω.*

Note that finite collections of points are always compact. So this is a much stronger type of convergence than pointwise convergence.

Proof. Let $T \subseteq B(z_0, r) \subseteq \Omega$ be a triangle inside an open disc with z_0 in its interior. Since f_n is holomorphic, we have

$$\int_T f_n(z)dz = 0$$

by Goursat's theorem. But by uniform convergence, we can choose $N \gg 0$ such that $|f(z) - f_n(z)| < \frac{\epsilon}{l(T)}$ for any $\epsilon > 0$, where $l(T)$ is the length of the triangle. This shows

$$\int_T f_n(z)dz \to \int_T f(z)dz = 0$$

Now as a result of the fact that a uniform limit of continuous functions is continuous, Morera's theorem implies f is holomorphic in $B(z_0, r)$. But these cover Ω, so the same is true for Ω. $\square$

There are many examples of real valued functions where this property is false. For example,

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \sin(2^n x)$$

is uniformly approximatable by partial sums, but isn't differentiable at 0.

In the complex setting, we can do one better:

Theorem 12.2. *With the assumptions of Theorem 12.1, $f_n'(z)$ converges uniformly to $f'(z)$ on every compact subset $K \subseteq \Omega$.*

Proof. Let $\Omega_\delta = \{z \in \Omega \;:\; \bar{B}(z, \delta) \subseteq \Omega\}$. I claim that $f_n'(z) \to f(z)$ uniformly for each $\delta > 0$, which would prove the result. We claim for any holomorphic function F on Ω,

$$\sup_{z \in \Omega_\delta} |F'(z)| \leq \frac{1}{\delta} \sup_{w \in \Omega} |F(w)|$$

To see this, consider CIT:

$$F'(z) = \frac{1}{2\pi i} \int_C \frac{F(w)}{(w-z)^2} dw$$

where C is the circle of radius δ about z. Now our standard approximation yields

$$|F'(z)| \leq \frac{1}{2\pi} \int_C \frac{|F(w)|}{|w-z|^2} |dw| = \frac{1}{2\pi\delta^2} \int_C |F(w)||dw| \leq \frac{2\pi\delta}{2\pi\delta^2} \sup_{w \in C} |F(w)|$$

This demonstrates the claim. Now if we consider the known holomorphic function $F = f - f_n$, we have bounded $|f' - f_n'|$ by a constant times something which approaches 0 as $n \gg 0$. Therefore they must agree. $\square$

Corollary 12.3. *If $f_n : \Omega \to \mathbb{C}$ is a sequence of holomorphic functions such that $f_n \to f$ uniformly for every compact subset $K \subseteq \Omega$, then $f_n^{(m)} \to f^{(m)}$ uniformly on every compact set.*

Proof. By induction. $\square$

Now we turn to Schwarz reflection principle. This gives us an idea for extending a holomorphic function to a larger domain. This is easier in real analysis, since differentiability is such a weaker condition.

Let Ω be an open set which is symmetric over the real line: $z \in \Omega \iff \bar{z} \in \Omega$. Call Ω^+ and Ω^- the open subsets with positive and negative imaginary parts respectively. Additionally, let $I = \Omega \cap \mathbb{R}$.

Theorem 12.4 (Symmetry Principle)**.** *If f^+ and f^- are holomorphic functions on Ω^+ and Ω^- respectively that can be continuously extended to I and agree with one another, then f defined piecewise by these functions is holomorphic.*

Proof. Let $B(iy, r) \subseteq \Omega$ be a disc centered along I, and $T \subseteq B(iy, r)$ be a triangle with a vertex or side on I. In the case of a side, we can consider an $\epsilon > 0$ shift of the triangle upward or downward to place it entirely in Ω^+ or Ω^-. Doing so ensures

$$\int_{T_\epsilon} f(z)dz = 0$$

but since f is continuous in Ω, this converges to $\int_T f(z)dz$. In the case of a vertex, we can subdivide the triangle to reduce to the previous case. The same holds for the general case! Thus again by Morera's theorem we can conclude f is holomorphic. $\square$

We can rephrase Theorem 12.4 slightly to yield a similar result with only the task of defining a function in the lower half-plane.

Theorem 12.5 (Schwarz Reflection Principle)**.** *Suppose f is holomorphic on Ω^+ and can be extended continuously to a real valued function on I. Then f can be extended to a holomorphic function on the whole region Ω.*

Proof. It goes to define the function holomorphically on Ω^- and use Theorem 12.4. A good choice is $f^-(z) = \overline{f(\bar{z})}$. This makes it immediate that f^- agrees with f on I. Now it only goes to show that it is holomorphic.

Given $z_0 \in \Omega^-$, we have $\bar{z}_0 \in \Omega^+$. Therefore, there is a power series expansion for f near $\bar{z}_0$:

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n(\bar{z} - \bar{z}_0)^n$$

Applying the complex conjugate, we get that

$$f^-(z) = \sum_{n=0}^{\infty} \bar{a_n}(z - z_0)^n$$

This power series has the same radius of convergence as the original, so f^- is holomorphic in this radius. This holds for every point, so we are done. $\square$

It is fun to think about the importance of the conditions in Theorem 12.4 & Theorem 12.5 .