CLASS 28, NOVEMBER 18TH: INFINITE PRODUCTS

Today we will try to tackle the question of constructing for a given sequence z_n without limit a corresponding entire function vanishing precisely at this point. To do this, we pass through infinite products.

Consider for a sequence $a_n \in \mathbb{C}$ the product

$$\prod_{n=1}^{\infty}(1+a_n)$$

Definition 28.1. If no $a_n = -1$, say that such a product **converges** if and only if the sequence of partial products converges to a non-zero number.

The following result compares sums with products and produces a very nice technique for determining if the product converges:

Proposition 28.2. *If $\sum_{n=1}^{\infty}|a_n| < \infty$, then $\prod_{n=1}^{\infty}(1+a_n)$ converges. Additionally, if the product is 0, then the factors converge to 0.*

Proof. Assume the sum converges. Then eventually $|a_n| < \frac{1}{2}$. We can assume this for all n if we discard finitely many terms. This allows $1+a_n \in B(1, \frac{1}{2})$ where we can use the logarithm. Thus

$$\prod_{n=1}^{N}(1+a_n) = \prod_{n=1}^{N} e^{\log(1+a_n)} = e^{\sum \log(1+a_n)} = e^{B_N}$$

We can check using the power series that $|\log(1+a_n)| \leq 2|a_n|$. So B_N converges absolutely under our assumptions. Since exp is a continuous function, we have

$$\lim \exp(z_n) = \exp(\lim(z_n))$$

Thus the desired limit is $e^{\lim(B_N)}$. $\square$

Note: There is a nice if and only if version of this result using the sum of squares on the homework.

Proposition 28.2 also generalizes to the case of functions with a bit of care:

Proposition 28.3. *If F_n is a sequence of holomorphic functions on Ω, and there exist constants $c_n > 0$ such that*

$$\sum_n c_n < \infty \qquad\qquad |F_n(z) - 1| \leq c_n \quad \forall z \in \Omega$$

then the following hold:

1. *The product $\prod_n F_n(z)$ converges uniformly in Ω to a holomorphic function F.*
2. *If F_n is non-vanishing on Ω, then*

$$\frac{F'(z)}{F(z)} = \sum_n \frac{F_n'(z)}{F_n(z)}$$

Proof. Note that setting $F_n(z) = 1 + a_n(z)$, we can achieve the first result using Proposition 28.2. Now, we also know that F is holomorphic since we have uniform convergence on the whole set (and thus in particular on every compact subset).

For the second part, let $K \subseteq \Omega$ be compact. Let G_n be the n^{th} partial product. Since by the previous part, $G_N \to F$ as $N \to \infty$ uniformly on K. As a result, by Theorem 12.2 (yes, ancient history) we have that G'_N converges uniformly to F' on K. Since G_N is uniformly bounded below on K, this also implies

$$\frac{G'_N}{G_N} \to \frac{F'}{F}$$

uniformly on K. Thus it converges for any point. As a result,

$$\frac{G'_N}{G_N} = \sum_{n=1}^{N} \frac{F'_n}{F_n} \to \sum_{n=1}^{\infty} \frac{F'_n}{F_n} = \frac{F'}{F}$$

□

Example 28.4. I claim that

$$\frac{\sin(\pi z)}{\pi} = z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

We will derive this fact from the sum expression for $\pi \cot(\pi z) = \pi \frac{\cos(\pi z)}{\sin(\pi z)}$:

$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} \frac{1}{z+n} := \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z+n} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

The first equation holds whenever $z \notin \mathbb{Z}$. Also, it holds if you are willing to examine it center out (as opposed to using the positive and negative terms). It is like a conditionally convergent sequence; rearranging the terms changes the result.

To check this equality, I refer the reader to page 143 to the first half of 144 of Stein/Shakarchi. The proof is very nice, using important properties of each function.

As a corollary of this result, Let $G(z) = \frac{\sin(\pi z)}{\pi}$ and $P(z) = z\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$. $P(z)$ converges by Proposition 28.2. Away from the integers, we have

$$\frac{P'(z)}{P(z)} = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2}$$

by taking log of both sides and the derivative. Since $\frac{G'(z)}{G(z)} = \pi \cot(\pi z)$, the formula above yields

$$\left(\frac{P(z)}{G(z)}\right)' = \frac{P(z)}{G(z)}\left[\frac{P'(z)}{P(z)} - \frac{G'(z)}{G(z)}\right] = 0$$

So P differs from G by a constant. Dividing both by z, we get that

$$\frac{G(z)}{z}, \frac{P(z)}{z} \to 1 \qquad \text{as } z \to 0$$