$\boxed{\text{Nov 27:}}$ Relating Good Pairs to Pairs

If (X,A) is a good pair, then the quotient map $(X, A) \xrightarrow{q} (X/A, A/A)$ induces
$q_*: H_n(X,A) \xrightarrow{\cong} H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$

Pf: Let $V \supseteq A$ be open, def retr to A

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "\cong_1"] \arrow[d, "q_*"] & H_n(X,V) \arrow[d, "q_*"] & H_n(X \setminus A, V \setminus A) \arrow[l, "\sim_1"'] \arrow[d, "q_*"] \\
H_n(X/A, A/A) \arrow[r, "\cong_2"] & H_n(X/A, V/A) & H_n(X/A \setminus A/A, V/A \setminus A/A) \arrow[l, "\sim_2"']
\end{tikzcd}$$

$\cong_1$: Since V def rets to A, we have ~~$Id_X \simeq$ def ret. Therefore~~ $H_n(V,A) = 0$ $\forall n > 0$.
Therefore, via LES for (X,V,A), $\cong$.

$\cong_2$: Same argument

$\sim_1, \sim_2$ are by excision

q_* right: Is an isomorphism, since q is a homeomorphism away from A.

Therefore the other q_* are isos.

Wedges sums: $\vee_\alpha X_\alpha$. Let x_0 be the point in common. We can consider

$$i_\alpha : (X_\alpha, x_0) \hookrightarrow (\bigvee_\alpha X_\alpha, x_0) \quad \forall \alpha$$

Assume good pair

$$i_{\alpha *} : H_n(X_\alpha, x_\alpha) \to H_n(\vee X_\alpha, x_0)$$

$$\begin{array}{cc} \text{\rotatebox{90}{$\cong$}} & \text{\rotatebox{90}{$\cong$}} \\ \tilde{H}_n(X_\alpha) & \tilde{H}_n(X_\alpha) \end{array}$$

Claim: $\bigoplus_\alpha i_{\alpha *}$ is an isomorphism:

$$\tilde{H}_n(\bigvee_\alpha X_\alpha) \cong \bigoplus_\alpha \tilde{H}_n(X_\alpha)$$

Consider $\tilde{H}_n(\underbrace{\amalg X_\alpha}_{= X}, \underbrace{\amalg x_\alpha}_{= A}) = \oplus H_n(X_\alpha)$

$$\text{\rotatebox{90}{$\cong$}}$$

$$\tilde{H}_n(X/A, A/A) = H_n(\bigvee_\alpha X_\alpha, x_0) \cong \tilde{H}_n(\vee X_\alpha)$$

A Slight Generalization of Invariance of dimension:

Thm: If U, V are open sets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and $U \cong V$, then $n = m$.

Pf: $H_n(U, U \setminus \{x\}) \cong H_n(\mathbb{R}^n, \mathbb{R}^n \setminus x)$ by excision. $H_i(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\}) = \mathbb{Z} \iff i = n$. (LES)

Therefore $n=m$. $\boxtimes$

$H_n(X, X \setminus x)$ is called the local homology of a closed point. Not for any open nbhd of x

$$H_n(X, X\setminus x) \cong H_n(\mathcal{U}, \mathcal{U}\setminus x)$$

If $f: X \to Y$, $y=f(x)$, then f is a homeomorphism near x (local homeomorphism) $\Rightarrow$
$$H_n(X, X\setminus x) \cong H_n(Y, Y\setminus y)$$

Nov 29: Equivalence of Singular / Simplicial Homology

If X is a CW complex, lets consider the relation between
$$H_n^{\Delta}(X) \text{ and } H_n(X)$$

Thm: If (X, A) is a Δ-pair, then
$$i: \Delta_n(X,A) \hookrightarrow C_n(X,A)$$
induces $i_*: H_n^{\Delta}(X,A) \to H_n(X,A)$

Pf: I assume X is a finite dimensional space, and for now A is empty.

$$\begin{array}{ccccccccc}
H^{\Delta}_{n+1}(X^K, X^{K-1}) & \to & H^{\Delta}_n(X^{K-1}) & \to & H^{\Delta}_n(X^K) & \to & H^{\Delta}_n(X^K, X^{K-1}) & \to & H^{\Delta}_{n-1}(X \\
* \downarrow & & \downarrow ☆ & & \downarrow & & \downarrow * & & ☆ \downarrow
\end{array}$$

Same thing but singular Homology

* Note $H^{\Delta}_n(X^K, X^{K-1}) \neq 0 \iff n = K$, and is free abelian at K: gen by n-simplices of X, say α.

Similarly. $\Phi : \coprod (\Delta^n, \partial\Delta^n) \longrightarrow (X^n, X^{n-1})$

is s.t. $\coprod_\alpha \Delta^n / \coprod_\alpha \partial\Delta^n \cong X^n / X^{n-1}$

$\Rightarrow H_n(X^K, X^{K-1}) = \mathbb{Z}^{\oplus \alpha}$, so $*$ are isoms for all n.

☆ is an isom: we can proceed by induction: $K=0$ $H^{\Delta}_n(\text{pts}) = H^{\Delta}_n(\text{pts})$. Now the inductive hypothesis implies ☆ is an isomorphism.

We can now apply a wonderful fact from Commutative algebra: Five lemma:

$\exists g'' \mapsto \alpha(g'') = g' \mapsto g \mapsto c(g) = 0$ δ inj

$$\begin{array}{ccccccccc}
G_1 & \xrightarrow{a} & G_2 & \xrightarrow{b} & G_3 & \xrightarrow{c} & G_4 & \xrightarrow{d} & G_5 \\
\alpha\downarrow\cong & & \beta\downarrow\cong & & \gamma\downarrow\cong & & \delta\downarrow\cong & & \varepsilon\downarrow\cong \\
H_1 & \xrightarrow{x} & H_2 & \xrightarrow{y} & H_3 & \xrightarrow{z} & H_4 & \xrightarrow{w} & H_5
\end{array}$$

exact

exact

α surj

$h \mapsto \beta(g') \mapsto 0 \mapsto 0$

(1) (2)

Just for flavor, let's show γ is injective. Show it's surjective as an exercise.

Suppose $g \in G_3$ is s.t. $\gamma(g)=0$. δ is injective

$$\Rightarrow \delta(c(g)) = k'(\gamma(g)) = k'(0) = 0$$

$$\Rightarrow c(g) = 0 \quad g \in \ker(c) = \mathrm{Im}(b)$$

$$\Rightarrow \exists\, g' \in G_2 \text{ w/ } b(g') = g$$

Now $\beta(g')$ is s.t. $y(\beta(g'))=0$.

$\Rightarrow \beta(g') \in \ker(y) = \mathrm{Im}(x)$. So $\exists\, h \in H_1$ s.t. $x(h) = \beta(g')$. But α is surjective, so $\exists\, g'' \in G_1$ s.t. $\alpha(g'') = h$. Now,

$a(g'') = g'$, since β is injective and

$$x(\alpha(g'')) = \beta(a(g'')) \quad \beta(g')$$

$$\| $$

$$x(h)$$

$$\Rightarrow g = b(a(g'')) = 0$$

$\rightsquigarrow H_n^{\Delta}(X) \cong H_n(X)$

~~Some notation: Betti #'s: $\beta_{i,j}$ is then~~

If $A \neq \phi$, 5-lemma w) LES for pairs

Dec 1: Cleaning up and Applications

We have that Δ-Homology and singular agree. Thus if we have a finite Δ-complex X, then

$$H_i(X) \text{ is finitely generated}$$

Nice Algebra result: [Modules over a PID]

Every abelian finitely generated group G is s.t.

$$G \cong \mathbb{Z}^{\oplus n} \oplus T.$$

Where T is a finite abelian group. One can further show (by Chinese Remainder)

$$T \cong \bigoplus_i \mathbb{Z}/p_i^{n_i}$$

Language: $B_i = i^{th}$ betti number is the n above for $H_i(X)$

$\{p_i^{n_i} \mid i\}$ is called the torsion coefficients.

Recall $\tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{otherwise} \end{cases}$

This allows us to study maps $f: S^n \to S^n$ in simpler terms.

$$\deg(f) = n \,\Big|\, [f_*]: \mathbb{Z} \to \mathbb{Z} : 1 \mapsto n$$

1) $\deg(Id) = 1$

2) If f is not surjective, $\deg(f) = 0$:

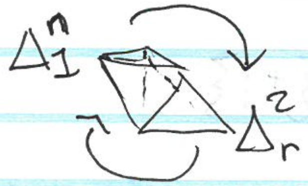
$$f: S^n \xleftrightarrow{f} S^n \setminus x_0 \longrightarrow S^n$$

$$H_n(S^n \setminus x_0) \cong H_n(\mathbb{R}^n) = 0$$

3) $f \simeq g \Rightarrow \deg(f) = \deg(g)$. Hopf showed the converse is also true.

4) $\deg(f \circ g) = \deg(f) \cdot \deg(g)$

5) If f is a reflection, $\deg(f) = -1$.

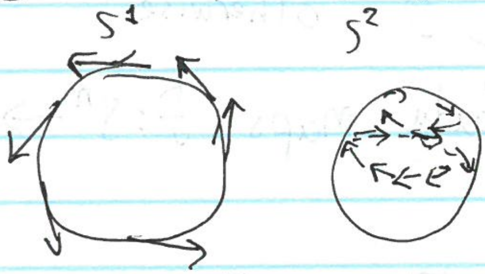


$$H_n(S^n) = \mathbb{Z} = \langle \Delta_1^n - \Delta_2^n \rangle$$

$$\text{ref}: \Delta_1^n \leftrightarrow \Delta_2^n$$

6) $x \mapsto -x$ has degree $(-1)^{n+1}$. It is a composition on $(n+1)$ reflections

Prop: S^n has a continuous tangent, non-zero vector field iff n is odd



Pf: Suppose it does. Let

$$v: S^n \to T(S^n)$$
$$x \mapsto v(x)$$

We can replace it with $v/\|v\|$ so all vectors have norm $= 1$. ~~Therefore we can consider the equatorial circle $S^1 \subseteq S^n$~~

$\cos(t)x + \sin(t)v(x)$ lie on the unit circle in $\langle x, v(x)\rangle$. $\mathrm{Id} \simeq$ Antipodal map

$$\Rightarrow \deg(\mathrm{Id}) = 1 = (-1)^{n+1} \Rightarrow n \text{ is odd}$$

"$\Leftarrow$" $v(x_1, \dots, x_{n+1}) = (x_2, -x_1, \dots, +x_{n+1}, -x_n)$

Thm: $\mathbb{Z}/2\mathbb{Z}$ is the only nontrivial group acting freely (no fixed pts) on S^n if n is even

$$G \to \mathrm{Homeo}(S^n) \xrightarrow{\deg} \{\pm 1\}$$

Action free $\Rightarrow g \neq \mathrm{Id} \rightsquigarrow g \mapsto (-1)^{n+1}$

$$\Rightarrow G \cong \mathbb{Z}/2\mathbb{Z},\ 0$$