FINAL EXAM QUESTIONS

1) **Fundamental Groups**
   a) **Definition:** Path Connected
   b) **Question:** Why are path components a well defined notion?
   c) **Definition:** Homotopy of paths
   d) **Definition:** Fundamental Group
   e) **Question:** What is the change of base point morphism and why is it an isomorphism of fundamental groups?
   f) **Question:** Why does a homotopy equivalence induce an isomorphism on the fundamental group?
   g) **Theorem:** State Van-Kampen's Theorem.
   h) **Question:** How does this apply to $\pi_1(X \vee Y)$?
   i) **Computation:** Compute $\pi_1(M_g)$ using $\pi_1(\mathbb{T}^2)$.
   j) **Definition:** What is a retraction.
   k) **Question:** What is the push forward by a retraction?
   l) **Theorem:** If X is a CW complex, how is it's fundamental group computed?
   m) **Computation:** Suppose I take a cylinder and attach one circle to the other via an n-sheeted cover: $\gamma : S^1 \to S^1 : \theta \mapsto n\theta$. What is the fundamental group of the resulting space?
   n) **Question:** What is $\pi_1(X \times Y)$? (Does the same hold for homology?)

2) **Covering Spaces**
   a) **Theorem:** What is the homotopy lifting property?
   b) **Theorem:** What is the lifting criterion?
   c) **Definition:** What is a normal covering space?
   d) **Question:** How would you construct such a non-normal covering space?
   e) **Definition:** What is the universal cover?
   f) **Question:** How can you obtain other covers from the universal cover?
   g) **Computation:** Attach a Klein Bottle to the torus by it's boundary circle. What is $\pi_1(X)$. How would its universal cover look.
   h) **Question:** What can you say about p_* of any covering space?
   i) **Definition:** What is a deck transformation?
   j) **Theorem:** What correspondences are there with:
      - The number of sheets of a covering space?
      - The number of PC covering spaces (of a PC, LPC, SLSC space X)?
      - The group of deck transformations of a covering space?
   k) **Computation:** What are the (path connected) covering spaces of S^1? Of the Torus?
   l) **Computation:** What are the 2-sheeted and 3-sheeted PC covering spaces of $S^1 \vee S^1$?

3) Homology

a) **Definition:** What is a chain? Cycle? Boundary?
b) **Definition:** Boundary map?
c) **Definition:** Simplicial or Singular Homology?
d) **Computaion:** Compute $H_1(K)$ and/or $H_1(\mathbb{RP}^2)$.
e) **Question:** What is $H_0(X)$ for any space X?
f) **Definition:** How is $f_* : H_i(X) \to H_i(Y)$ defined?
g) **Computation:** Why is f_* well defined?
h) **Computation:** What is r_* where r is a retraction. Why?
i) **Definition:** What is a good pair?
j) **Question:** What are the homologies for a good pair?
k) **Definition:** How is $H_n(X, A)$ defined?
l) **Theorem:** What is the LES of a pair (X, A)?
m) **Computation:** Compute the homology of the space where you identify the north and south pole of S^2
n) **Computation:** Compute the homology groups of $S^1 \times (S^1 \vee S^1)$.
o) **Computation:** If $f : X \to Y$ is a homotopy equivalence of pairs (X, A) and (Y, B), then why is $H_n(X, A) \cong H_n(Y, B)$? (Hint: 5-lemma)
p) **Computation:** Let X be a Δ-complex. Given $X^n/X^{n-1} \simeq \vee_{\alpha_n} S^n$, show the following (using induction):
   - If $X = X^n$, $H_i(X) = 0$ for $i > n$.
   - If $X = X^n$, $H_n(X) = \mathbb{Z}^{\alpha_n}$.
   - If X has k-many n-simplices, $H_n(X)$ has at most k-generators.

q) **Theorem:** Excision (either version).
r) **Computation:** Compute $H_i(\mathbb{T}^2, \mathbb{T}^2 \setminus x)$.
s) **Computation:** Why is it true that $\tilde{H}_n(\vee_\alpha X_\alpha) \cong \oplus_\alpha \tilde{H}_n(X_\alpha)$ (if (X_α, x_α) is a good pair).
t) **Theorem:** State Meyer-Vietoris.
u) **Computation:** Homologies of M_2 (M_g?).
v) **Definition:** What is the Euler Characteristic? Of M_g?
w) **Definition:** What is the degree of a map between spheres?
x) **Computation:** Given $H_n(\mathbb{RP}^n)$ is $\mathbb{Z}$ when n is odd and 0 when n is even, show that if $f : S^n \to S^n$ is an even map, it's degree is necessarily even. If n is even, the map is necessarily 0.
y) **Question:** Can any Abelian group be made a (any) homology group of a space X?