HOMEWORK 2: PRODUCTS AND CONTINUITY
DUE: FRIDAY, SEPTEMBER 21

1) Show that if $A \subseteq X$ and $B \subseteq Y$ are topological subspaces, then $A \times B$ with the product topology is equivalent to $A \times B \subseteq X \times Y$ with the subspace topology. That is to say, a product of subspaces is the subspace of the product.

2) Let a_i be a sequence of positive real numbers, and b_i be a sequence of real numbers. Show that the map
$$f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}} : (x_1, x_2, \ldots) \mapsto (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots)$$
is a homeomorphism in the product topology. Is the same true in the box topology?

3) Consider $X = \mathbb{R}^{\mathbb{N}}$, the space of sequences of real numbers with the product topology. Inside it is a subset Y given by all sequences that are eventually zero. What is the closure of Y within X?

4) Show that if X, Y are two topological spaces, then for a fixed $y_0 \in Y$, the map
$$i : X \to X \times Y : x \mapsto (x, y_0)$$
is continuous.[1]

5) Suppose that X_α with $\alpha \in \Lambda$ is a collection of subsets of X for which $X = \bigcup_\alpha X_\alpha$. Let $f : X \to Y$ be a map such that $f|_{X_\alpha} : X_\alpha \to Y$ is continuous for each α.
   - Suppose that X_α are closed and Λ is a finite set. Show f is continuous.
   - Show the same is not true if we relax the finiteness of Λ.
   - We call the X_α a **locally finite** collection if for any $x \in X$, there is a neighborhood U_x of x such that are only finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n$ for which $U \cap X_{\alpha_i} \neq \emptyset$. Show that if X_α are a closed and locally finite collection of subsets, then f is continuous.

6) Find a function $f : \mathbb{R} \to \mathbb{R}$ (Euclidean topologies) continuous at only a single point.

[1] An injective continuous map is called an embedding.