HOMEWORK 5: THE RESIDUE THEOREM
DUE: FRIDAY, OCTOBER 18TH

1) Using Euler's formula, show that the complex zeroes of $\sin(\pi z)$ are simple and exactly at the integers. What is their residue if you consider $\frac{1}{\sin(\pi z)}$?

   Solution: Consider $2i\sin(\pi z) = e^{i\pi z} - e^{-i\pi z} = 0$. Breaking this into real and imaginary parts yields

$$0 = e^{i\pi z} - e^{-i\pi z} = e^{i\pi(x+iy)} - e^{-i\pi(x+iy)} = e^{-\pi y}e^{i\pi x} - e^{\pi y}e^{i\pi x}$$

   Comparing absolute values and arguments of the two functions yields

$$e^{y} = e^{-y}$$

$$2\pi x \equiv 0 \,(\mathrm{mod}\, 2)\pi$$

   Thus $y = 0$ and $x \in \mathbb{Z}$. To calculate the residues of $\frac{1}{\sin(\pi z)}$, it suffices to consider

$$\mathrm{res}_m\left(\frac{1}{sin(\pi z)}\right) = \lim_{z\to m}(z-m)\cdot\frac{1}{sin(\pi z) - \sin(\pi)} = \frac{1}{\pi\cos(\pi m)} = (-1)^m\frac{1}{\pi}$$

2) Evaluate the integral

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^4}.$$

   Solution: The roots of $x^4 = -1$ are $e^{\frac{\pi i}{4}}, e^{\frac{3\pi i}{4}}, e^{\frac{5\pi i}{4}}, e^{\frac{7\pi i}{4}}$ by homework 1. Only 2 of which are in the upper half plane. Therefore it goes to compute their residues:

$$\mathrm{res}_{e^{\frac{\pi i}{4}}}\left(\frac{1}{z^4+1}\right) = \frac{1}{e^{\frac{\pi i}{4}} - e^{\frac{3\pi i}{4}}}\frac{1}{e^{\frac{\pi i}{4}} - e^{\frac{5\pi i}{4}}}\frac{1}{e^{\frac{\pi i}{4}} - e^{\frac{7\pi i}{4}}} = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}(1+i)}\frac{1}{\sqrt{2}i} = \frac{1}{2\sqrt{2}(1+i)i}$$

$$\mathrm{res}_{e^{\frac{3\pi i}{4}}}\left(\frac{1}{z^4+1}\right) = \frac{1}{e^{\frac{3\pi i}{4}} - e^{\frac{\pi i}{4}}}\frac{1}{e^{\frac{3\pi i}{4}} - e^{\frac{5\pi i}{4}}}\frac{1}{e^{\frac{3\pi i}{4}} - e^{\frac{7\pi i}{4}}} = \frac{1}{-\sqrt{2}}\frac{1}{\sqrt{2}i}\frac{1}{-\sqrt{2}(1-i)} = \frac{1}{2\sqrt{2}(1-i)i}$$

   Now using conjugates, we get

$$\int_\gamma f(z)dz = 2\pi i\left(\frac{1}{2\sqrt{2}(1+i)i} + \frac{1}{2\sqrt{2}(1-i)i}\right) = 2\pi i\left(\frac{1-i}{4\sqrt{2}i} + \frac{1+i}{4\sqrt{2}i}\right) = 2\pi\frac{2}{4\sqrt{2}} = \frac{\pi}{\sqrt{2}}$$

   Finally, noting that as $R\to\infty$, $\left|\int_{\gamma_R}\frac{1}{1+z^4}dz\right| \le \pi R\frac{1}{R^4-1} \to 0$ yields that the desired integral is $\frac{\pi}{\sqrt{2}}$.

3) Show that

$$\int_{-\infty}^{\infty}\frac{\cos(x)dx}{x^2+a^2} = \pi\frac{e^{-a}}{a}$$

   for any $a > 0$.

Solution: Here the poles occur at $x = \pm ia$. Therefore, if we do the upper semicircle again, we have

$$\int_\gamma \frac{e^{iz}}{z^2+a^2}dz = 2\pi i \cdot \text{res}_{ia}\left(\frac{e^{iz}}{z^2+a^2}\right) = 2\pi i \frac{e^{i^2 a}}{2ia} = \pi\frac{e^{-a}}{a}$$

So per usual it suffices to show that $\int_{\gamma_R} f(z)dz = 0$. Since $im(z) \geq 0$ on this curve, $|e^{iz}| \leq 1$. Thus

$$\left|\int_{\gamma_R} f(z)dz\right| \leq \int_{\gamma_R}\left|\frac{1}{z^2+a^2}\right||dz|0 \leq \pi R\frac{1}{R^2-a^2} \to 0.$$

4) Show that

$$\int_{-\infty}^{\infty}\frac{dx}{(1+x^2)^{n+1}} = \pi\frac{1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$$

Solution: Again, the poles of $\left(\frac{1}{1+z^2}\right)^{n+1}$ occur at $\pm i$. If we do the upper semicircle, the same reasoning as the previous example shows

$$\int_{-\infty}^{\infty}\left(\frac{1}{1+x^2}\right)^{n+1}dx = 2\pi i\cdot\text{res}_i\left(\frac{1}{(1+x^2)^{n+1}}\right)$$

It is clear that $(x-i)^{n+1}\cdot\frac{1}{(1+x^2)^{n+1}} = \frac{1}{(x+i)^{n+1}} \to \frac{1}{(2i)^{n+1}} \neq 0$ So i is a pole of order $n+1$. Therefore, we can apply Theorem 13.9 from the notes:

$$\begin{aligned}
\text{res}_i\left(\frac{1}{(1+x^2)^{n+1}}\right) &= \lim_{z\to i}\frac{1}{n!}\frac{\partial}{\partial z}^n\left((z-i)^{n+1}\frac{1}{(z^2+1)^{n+1}}\right)\\
&= \lim_{z\to i}\frac{1}{n!}\frac{\partial}{\partial z}^n\left(\frac{1}{(z+i)^{n+1}}\right)\\
&= \lim_{z\to i}\frac{1}{n!}\left((-n-1)\cdots(-2n)(z+i)^{-2n-1}\right)\\
&= \frac{1}{n!}\left((-n-1)\cdots(-2n)(2i)^{-2n-1}\right)\\
&= \frac{(-1)^n(2n)!}{((n!)^2\cdot 2^{2n+1}(-1)^n\cdot i}
\end{aligned}$$

Plugging this into our fomula, we see

$$\int_{-\infty}^{\infty}\left(\frac{1}{1+x^2}\right)^{n+1}dx = 2\pi i\frac{(2n)!}{((n!)^2\cdot 2^{2n+1}\cdot i} = \pi\frac{(2n)!}{((n!)^2\cdot 2^{2n}}$$

Comparing terms yields the desired result, noting that

$$\frac{(2n)!}{((n!)^2\cdot 2^{2n}} = \frac{(2n)(2n-1)(2n-2)\cdots 1}{(2n)(2n)(2n-2)(2n-2)\cdots 2\cdot 2} = \frac{(2n-1)(2n-3)\cdots 1}{2n\cdot(2n-2)\cdots 2}$$

5) Show that

$$\int_0^{2\pi}\frac{d\theta}{a+b\cos(\theta)} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

for $a > |b| > 0$ real numbers. (**hint:** Use the Euler Relations on cos and then express this as an integral on the unit circle.)

Solution: Note that

$$\begin{aligned}\int_0^{2\pi} \frac{dx}{a+b\cos(x)} = \int_0^{2\pi} \frac{dx}{a+b\frac{e^{ix}+e^{-ix}}{2}} &= 2\int_0^{2\pi} \frac{e^{ix}dx}{2ae^{ix}+be^{2ix}+b} \\ = \frac{2}{i}\int_C \frac{dz}{bz^2+2az+b} &= \frac{2}{ib}\int_C \frac{dz}{z^2+2\frac{a}{b}z+1} = \frac{2}{ib}\int_C \frac{dz}{(z-z_+)(z-z_-)}\end{aligned}$$

where C is the unit circle, and $z_\pm = -\frac{a}{b} \pm \sqrt{\frac{a^2}{b^2}-1}$. As a result, if $b>0$ we have can quickly derive that z_- is outside the unit circle, so its residue doesn't contribute. Thus we derive

$$\int_0^{2\pi} \frac{dx}{a+b\cos(x)} = 2\pi i \cdot \operatorname{res}_{z_+}(f(z)) = 2\pi i \frac{2}{ib}\frac{1}{z+-z-} = 2\pi i \frac{2}{ib}\frac{1}{2\sqrt{\frac{a^2}{b^2}-1}} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

The case of $b<0$ is nearly identical.