CLASS 10, FEBRUARY 27TH: EXACTNESS AND SPLITTINGS

To finish up with an introduction to modules, we turn to the idea of an exact sequence. This unifies several important notions into one compact clause, including injectivity, surjectivity, and an isomorphism theorem.

Definition 10.1. Let $\varphi : M \to N$ be a module homomorphism.

$$\ker(\varphi) = \{m \in M \mid \varphi(m) = 0\}$$

$$\operatorname{im}(\varphi) = \{n \in N \mid \exists m \in M \text{ such that } \varphi(m) = n\}$$

$\ker(\varphi) \subseteq M$ and $\operatorname{im}(\varphi) \subseteq N$ are submodules, so we can also quotient:

$$\operatorname{coim}(\varphi) = M/\ker(\varphi)$$

$$\operatorname{coker}(\varphi) = N/\operatorname{im}(\varphi)$$

Now for the definition of exactness:

Definition 10.2. If $\varphi : M' \to M$ and $\psi : M \to M''$ are 2 homomorphisms, we say that the **sequence**

$$M' \xrightarrow{\varphi} M \xrightarrow{\psi} M''$$

is **exact** if $\ker(\psi) = \operatorname{im}(\varphi) \subseteq M$. We can do this at infinitum:

$$\ldots \xrightarrow{\varphi_{-2}} M_{-2} \xrightarrow{\varphi_{-1}} M_{-1} \xrightarrow{\varphi_0} M_0 \xrightarrow{\varphi_1} M_1 \xrightarrow{\varphi_2} M_2 \xrightarrow{\varphi_3} \ldots$$

is an **exact sequence** if $\ker(\varphi_i) = \operatorname{im}(\varphi_{i-1})$ for every $i \in \mathbb{Z}$.

This notion gives a proper generalization of several notions we have already spoken about:

Proposition 10.3 (Exactness vs other properties of maps)**.**

1) *A sequence $0 \to M \xrightarrow{\varphi} N$ is exact if and only if φ is injective.*
2) *A sequence $M \xrightarrow{\varphi} N \to 0$ is exact if and only if φ is surjective.*
3) *A sequence $0 \to M \xrightarrow{\varphi} N \to 0$ is exact if and only if φ is an isomorphism.*
4) *A sequence $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ is exact if and only if φ is injective, ψ is surjective, and $M' = \ker(\psi)$ (or equivalently $M'' = \operatorname{coker}(\varphi) = M/M'$). This is special enough to give it's own name, a* **short exact sequence***. We also call M an* **extension** *of M'' by M'.*

Proof.

1) $0 \to M \xrightarrow{\varphi} N$ is exact if and only if $\ker(\varphi) = \operatorname{im}(0 \to M) = 0$ if and only if φ is injective.
2) $M \xrightarrow{\varphi} N \to 0$ is exact if and only if $N = \ker(N \to 0) = \operatorname{im}(\varphi)$ if and only if φ is surjective.
3) This follows directly from the previous 2 parts.
4) The only new piece of information here is that $M'' = M/M'$. Since $M \xrightarrow{\psi} M''$ is a surjective map, we know that

$$M'' \cong M/\ker(\psi) \cong M/\operatorname{im}(\varphi) \cong M/M'.$$

$\square$

Example 10.4.
- Given ANY R-modules M, N, we can form the exact sequence
$$0 \to M \to M \oplus N \to N \to 0$$
where we send m to $(m, 0)$ and (m, n) to n.
- The following is an exact sequence of $\mathbb{Z}$-modules:
$$0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$
- As $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ modules, we can form the SES (by the 2nd isomorphism theorem)
$$0 \to \mathbb{Z}/m\mathbb{Z} \xrightarrow{\psi} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\varphi} \mathbb{Z}/(n/m)\mathbb{Z} \to 0$$
where $m|n$, $\psi(1) = \frac{n}{m}$, and $\varphi(1) = \bar{1}$.
- More generally, given any ideal $I \subseteq R$, we can form the SES
$$0 \to I \to R \to R/I \to 0$$
- By the 1st isomorphism theorem, given any R-module homomorphism $M \xrightarrow{\psi} N$, we have a SES
$$0 \to \ker(\psi) \to M \to \mathrm{im}(\psi) \to 0$$

Finally, we give the definition of a split exact sequence:

Definition 10.5. A SES $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ is said to be **split exact** if one of the following equivalent conditions is met:
1) $M \cong M' \oplus M''$.
2) There is a homomorphism $\varphi' : M \to M'$ such that $\varphi' \circ \varphi = Id_{M'}$.
3) There is a homomorphism $\psi' : M'' \to M$ such that $\psi \circ \psi' = Id_{M''}$.

Proof. 1) $\Rightarrow$ 2) or 3): Given $M \cong M' \oplus M''$, we get a natural projection and inclusion map
$$\varphi' : M \to M' : m = (m', m'') \mapsto m'$$
$$\psi' : M'' \to M : m'' \mapsto (0, m'')$$
These are clearly the desired maps.

2) $\Rightarrow$ 1): We can construct the map $\Phi : M \to M' \oplus M''$ explicitly:
$$\Phi : M \to M' \oplus M'' : m \to (\varphi'(m), \psi(m))$$
It suffices to check that this is injective and surjective.

Injectivity: Suppose $\Phi(m) = 0$. Then $\psi(m) = 0$ and $\varphi'(m) = 0$. But exactness implies $m \in \ker(\psi) = \mathrm{im}(\varphi)$. As a result, $m = \varphi(m')$ for some $m' \in M'$. As a result, we conclude
$$0 = \varphi'(m) = \varphi'(\varphi(m')) = m'$$
So $m' = 0$, and therefore $m = \varphi(0) = 0$.

Surjectivity: Let $m' \in M'$ and $m'' \in M''$. Since ψ is surjective, there exists $m \in M$ such that $\psi(m) = m''$. Furthermore, we can consider $\varphi(m') \in M$. Note that
$$\varphi'(m + \varphi(m')) = \varphi'(m) + m'$$
So we need an adjustment factor: Consider $m_0 = \varphi(m') + m - \varphi(\varphi'(m))$. Then
$$\psi(m_0) = \psi(m) + (\psi \circ \varphi)(m' - \varphi'(m)) = \psi(m) = m''$$
$$\varphi'(m_0) = \varphi'(\varphi(m')) + \varphi'(m) - \varphi'(\varphi(\varphi'(m))) = m' + \varphi'(m) - \varphi'(m) = m'$$
This completes the proof as 3) $\Rightarrow$ 1) is similar. $\square$