# HOMEWORK 8: FOURIER TRANSFORMS
# DUE: WEDNESDAY, NOVEMBER 13TH

(1) We will prove the following: If $f$ is continuous, of moderate descent, and $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$, then $f = 0$.
- For each $t$, consider

$$A(z) = \int_{-\infty}^{t} f(x)e^{-2\pi iz(x-t)}dx \qquad\qquad B(z) = -\int_{t}^{\infty} f(x)e^{-2\pi iz(x-t)}dx$$

  Show $A(\xi) = B(\xi)$ for each $\xi \in \mathbb{R}$.
- Show that $F$ which is equal to $A$ in the upper half plane and $B$ in the lower half plane is entire and bounded. Deduce that $F = 0$.
- Show that

$$\int_{-\infty}^{t} f(x)dx = 0$$

  for all $t$, and thus $f = 0$ by continuity.

**Solution:** The first bullet is obvious, as $B$ is simply the the negatively oriented version of $A$. As a result, applying the symmetry principle to $F$, we get that $F$ is entire. Furthermore, since $f$ is of moderate descent, we have

$$|F(z)| \leq \int_{-\infty}^{t} |f(x)|dx \leq \int_{-\infty}^{t} \frac{A}{1+x^2}dx \leq A\pi$$

As a result $F$ is also bounded, and thus Liouville implies that $F$ is constant. Finally, notice

$$|F(iR)| = |\int_{-\infty}^{t} f(x)e^{2\pi R(x-t)}dx| \leq \int_{-\infty}^{t} Ae^{2\pi R(x-t)}dx = \frac{A}{2\pi R} \to 0$$

as $R \to \infty$. As a result $F = 0$.

For the last bullet, notice that this is $F(0)$. But since $f$ is continuous, if $f$ were ever non-zero, say at $t$, then it would be non-zero in an $\epsilon > 0$ neighborhood. As a result,

$$0 = \int_{-\infty}^{t} f(x)dx = \int_{-\infty}^{t-\epsilon} f(x)dx + \int_{-\infty}^{t-\epsilon} f(x)dx = 0 + \epsilon \cdot m$$

where $m$ bounds $f$ away from 0. This is a contradiction.

(2) Show that if $f \in \mathfrak{F}_a$, then $f^{(n)} \in \mathfrak{F}_b$ for any $0 \leq b < a$.

**Solution:** Note that

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_{C_r} \frac{f(w)}{(w-z)^{n+1}}dw$$

So we can bound $f^{(n)}(z)$ by CIT:

$$|f^{(n)}(x+iy)| \leq \frac{n!\|f\|_C}{r^n}$$

where $r < b - a$. Now notice that

$$\|f\|_C = \sup_{a+ib \in C} |f(z)| \leq \sup_{a+ib \in C} \left| \frac{A}{1+x^2} \right|$$

Now, it's just a matter of geometry. The second quanity is bounded above by $A$ near $x = 0$ and homogeneously so away from 0. So the constant

$$A' = \frac{n!}{(b-a)^n} 2A$$

will do.

(3) If $a > 0$ and $\xi \in \mathbb{R}$, show using contour integration that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x \xi} dx = e^{-2\pi a |\xi|}$$

Deduce that

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|} e^{2\pi i \xi x} d\xi = \frac{1}{\pi} \frac{a}{a^2+x^2}$$

**Solution:** For the first integral, if $\xi \geq 0$, we can study the lower semi-circle. This will as usual demonstrate that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{a}{a^2+x^2} e^{-2\pi i x \xi} dx = -2i \cdot \text{res}_{-ia}(f)$$

(negative because of orientation) and the residue is given by

$$\text{res}_{-ia}(f) = \frac{a}{-ia - ia} e^{-2\pi i i a \xi} = \frac{-1}{2i} e^{-2\pi i i a \xi} = -\frac{1}{2i} e^{2\pi a \xi}$$

Similarly, if $\xi < 0$, we study the upper semicircle and derive the same result studying the residue at $z = ia$.

Finally, the deduction is done through the process of Fourier inversion.

(4) If $P$ is a polynomial of degree $\geq 2$ with simple non-real roots, calculate

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{P(x)} dx$$

for $\xi \in \mathbb{R}$ in terms of its roots. What if the roots are higher order?

(**hint:** The cases of positive, negative, and 0 $\xi$ should be treated separately.)

**Solution:** Let $P$ have leading coefficient 1. If $\xi \geq 0$, we can consider the lower semicircle and the roots $z = a_i$ with negative imaginary part (This ensures that the circular integral goes to 0 since the imaginary part can be assumed negative and $|e^{-2\pi i x \xi}| = |e^{2\pi i \xi(-Im(x))}| < 1$. In this case,

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x \xi}}{P(x)} dx = 2\pi i \sum_i res_{a_i}(f(z))$$

This can naturally be computed as

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{P(x)} dx = 2\pi i \sum_{j \ | \ Im(a_j)<0} e^{-2\pi i a_j \xi} \prod_{k\neq j} \frac{1}{a_j - a_k}$$

A similar computation shows that if $\xi < 0$, we have

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{P(x)} dx = -2\pi i \sum_{j \ | \ Im(a_j)>0} e^{-2\pi i a_j \xi} \prod_{k\neq j} \frac{1}{a_j - a_k}$$

(5) Use the Poisson summation formula to establish the following identities:

- Let $Im(\tau) > 0$. Using $f(z) = (\tau + z)^{-k}$ for $k \geq 2$, show

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^k} = \frac{(-2\pi i)^k}{(k-1)!} \sum_{m=1}^{\infty} m^{k-1} e^{2\pi i m\tau}$$

- If $Im(\tau) > 0$, then show

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}$$

- Does the previous hold for any non-integer $\tau \in \mathbb{C}$?

**Solution:** For the first bullet, it goes to compute $\hat{f}(n)$ for each $n$. If $n \leq 0$,

$$\hat{f}(n) = \int_{-\infty}^{\infty} \frac{e^{-2\pi i n x}}{(\tau + x)^k} dx$$

which we can calculate using the path integral which is a upper semicircle. The overall integral is 0 by Cauchy-Goursat, and the lower half integral is bounded by

$$|\hat{f}(n)| \leq \int_{C_R} \frac{1}{|\tau + z|^k} dz = O\left(\frac{1}{R^{k-1}}\right) \to 0$$

For $n > 0$, we consider the lower semi-circle and conclude

$$\hat{f}(n) = -2\pi i \cdot \text{res}_{-\tau}\left(\frac{e^{-2\pi i n x}}{(\tau + x)^k}\right) = -\frac{2\pi i}{(k-1)!} \frac{\partial^{k-1} e^{-2\pi i n z}}{\partial z^{k-1}} \to \frac{(-2\pi i)^k n^{k-1}}{(k-1)!} e^{2\pi i n\tau}$$

Note the first negative sign is due to the clockwise orientation of the curve. Moving onto the second bullet point, if $k = 2$, we get

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau+n)^2} = -4\pi^2 \sum_{m=1}^{\infty} m e^{2\pi i m\tau}$$

Since the $\pi^2$ is accounted for, it only goes to show

$$\frac{1}{\sin^2(\pi\tau)} = -4 \sum_{m=1}^{\infty} m e^{2\pi i m\tau}$$

If we write $x = e^{2\pi i\tau}$, then we can utilize the formula

$$\sum_{m=1}^{\infty} m q^m = \frac{q}{(1-q)^2}$$

This shows that

$$-4\sum_{m=1}^{\infty} me^{2\pi i m\tau} = -4\frac{e^{2\pi i\tau}}{(1-e^{2\pi i\tau})^2} = \frac{-4}{(e^{-\pi i\tau}-e^{\pi i\tau})^2} = \frac{1}{(i\sin(\pi\tau))^2}$$

Finally, the same formula is not true for $Im(\tau) < 0$. This is because the taylor series used on the previous line does not converge if the absolute value of $e^{2\pi i\tau}$ is greater than or equal to 1.