CLASS 25, MONDAY APRIL 30: FEDDER'S CRITERION

A key point from Friday's lecture is that F-split, though a very nice criterion, is slightly tricky to check by hand. So the question becomes how can we simiplify this procedure? This led Fedder to prove a beautiful theorem.

To state the theorem, we need some machinery. We note that by criteria 4 for being F-split, R is F-split if and only if

$$ev_1 : \mathrm{Hom}_R(F_*R, R) \to R : \psi \mapsto \psi(1)$$

is surjective. This naturally motivates studying $\mathrm{Hom}(F_*R, R)$ as an object.

Theorem 0.1. *Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over an R-finite field K of characteristic $p > 0$. Then there exists $\Phi_S \in \mathrm{Hom}_S(F_*S, S)$ such that Φ is a F_*S-module generator.*

Proof. We can begin by localizing at the origin $\langle x_1, \ldots, x_n\rangle$ without loss of generality. First, note that $\mathrm{Hom}_S(F_*S, S)$ has the natural structure of an S-module and F_*S-module:

$$(s \cdot \Psi)(F_*m) := s\Psi(F_*m) = \Psi(F_*s^p m)$$

$$(F_*s \cdot \Psi)(F_*m) := \Psi(F_*sm)$$

Now, by Kunz Theorem, we know that $F_*S \cong S^l$ for some $l = m \cdot p^n$, where m is the dimension of F_*K over K. We may assume one copy of S is generated by $F_*(x_1^{p-1} \cdots x_n^{p-1})$. I claim that projecting from this copy of S is the desired generator. Call it Φ_S. Indeed, suppose that $\Psi : F_*S \to S$. Ψ is determined uniquely by where it sends the basis:

$$\{F_*k_i x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 1 \leq i \leq m,\ 0 \leq \alpha_j < p\rangle$$

Now, note that

$$F_*x_1^{p-1} \cdots x_n^{p-1} = F_*\left(x_1^{p-1-\alpha_1} \cdots x_n^{p-1-\alpha_n}\right) \cdot F_*\left(k_i x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right)$$

So if $\Psi(F_*\left(k_i x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right)) = s_{\alpha,i}$, we see

$$\Psi(-) = \sum_{i,\alpha} s_{\alpha,i}\Phi_S(F_*\left(k_i x_1^{p-1-\alpha_1} \cdots x_n^{p-1-\alpha_n}\right) \cdot -) = \Phi_S(F_*\left(\sum_{i,\alpha} s_{\alpha,i}^p k_i x_1^{\alpha_1} \cdots x_n^{\alpha_n}\right) \cdot -)$$

This completes the proof. □

Note that this theorem extends naturally to the case of $\mathrm{Hom}_S(F_*^e S, S)$. So there in particular is a homomorphism generating the others as a F_*S-module.

Definition 0.2. Let I, J be ideals of a ring R. Then we define the **colon ideal**

$$I :_R J := \{r \in R \mid r \cdot J \subseteq I\}$$

sometimes the R is omitted. This is also sometimes called the **ideal quotient**.

This allows us to setup Fedder's Criterion:

Theorem 0.3 (Fedder's criterion)**.** *If $R = S/I$, where $S = K[x_1, \ldots, x_n]$ and K is F-finite,*

$$\mathrm{Hom}_R(F_*R, R) \cong \left(F_*(I^{[p]} : I)/F_*(I^{[p]})\right) \cdot \mathrm{Hom}_S(F_*S, S)$$

Proof. I divide this proof into several parts.

(1) There is a map $\Lambda : F_*(I^{[p]} : I) \cdot \mathrm{Hom}_S(F_*S, S) \to \mathrm{Hom}_R(F_*R, R)$.
(2) Λ is a surjective map.
(3) $\ker(\Lambda)$ is exactly $IF_*R = F_*I^{[p^e]}$.

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(1) Suppose that $x \in I^{[p]} : I$. Then for a given map $\varphi : F_*S \to S$, I define

$$\Lambda_x(\varphi)(F_*r) = \overline{\varphi(F_*xr)} \in R = S/I$$

It goes to show that this is a well defined homomorphism. Suppose that $r - r' \in I$. Then the problem is equivalent to showing that $\Lambda_x(\varphi)(F_*r) = \Lambda_x(\varphi)(F_*r')$:

$$\Lambda_x(\varphi)(F_*r) - \Lambda_x(\varphi)(F_*r') = \overline{\varphi(F_*xr)} - \overline{\varphi(F_*xr')} = \overline{\varphi(F_*x(r - r'))}$$

But by definition of the colon ideal, we see $x \cdot (r - r') = a^p \cdot y \in I^{[p]}$. Therefore,

$$\overline{\varphi(F_*x(r - r'))} = \overline{\varphi(aF_*y)} = \overline{a\varphi(F_*y)} = 0$$

as desired.

(2) Next, it goes to show that for any map $\psi \in \mathrm{Hom}_R(F_*R, R)$, we can find $\varphi \in F_*(I^{[p]} : I) \cdot \mathrm{Hom}_S(F_*S, S)$ corresponding to it. Since S is assumed regular, we know that F_*S is a projective (free) S-module. Therefore, given the map $F_*S \xrightarrow{F_*q} F_*R \xrightarrow{\psi} R$, and the surjection $q : S \to R$, we get that there exists a map $\varphi : F_*S \to S$ such that $q \circ \varphi = \psi \circ F_*q$.

It only goes to show that $\varphi \in F_*(I^{[p]} : I) \cdot \mathrm{Hom}_S(F_*S, S)$. Suppose not. Then $\varphi(-) = \Phi(F_*r \cdot -)$ with $r \notin I^{[p]} : I$. Then $ri \notin I^{[p^e]}$ for some $i \in I$, and therefore $\Phi(F_*ri) \notin I$. On the other hand,

$$(\varphi \circ q)(ri) = \psi(F_*q(ri)) = \psi(0) = 0$$

Therefore φ is not well defined, a contraction.

(3) It is clear that $IF_*R \subseteq \ker(\Lambda)$, by R-linearity. On the other hand, if $\varphi \in \ker(\Lambda)$, then $\varphi(-) = \Phi(F_*r \cdot -)$. But then φ applied to the basis is 0 necessarily. This implies precisely that $F_*r \in IF_*R$. This completes the proof.

$\square$

Example 0.4. Last time we showed that $R = K[x_1, \ldots, x_n]/\langle x_1 \cdots x_n\rangle$ is an F-split ring. Here is a quick proof. Applying Fedder's criterion, and the fact that $I^{[p]} : I = \langle x_1^{p-1} \cdots x_n^{p-1}\rangle$, we see that

$$\varphi(-) = \Phi_S(F_*x_1^{p-1} \cdots x_n^{p-1} \cdot -) \in \mathrm{Hom}_R(F_*R, R)$$

But this implies $\varphi(F_*1) = 1$. Therefore, we are done!

As a Corollary of Theorem 0.3, we have the following.

Corollary 0.5. *If R is a local ring, then R is F-split if and only if $I^{[p]} : I \not\subseteq \mathfrak{m}^{[p]}$*

Example 0.6. If $R = K[x, y]/\langle f = x^2 + y^2\rangle$. This is also a non-regular ring. However, $I^{[p]} : I = \langle f^{p-1}\rangle$. Notice

$$f^{p-1} = \sum_{i+j=p-1} c_{ij}x^{2i}y^{2j}$$

If we consider the corollary, we have that this is F-split if and only if $\exists i, j$ satisfying $2i, 2j < p$ and $i + j = p - 1$. If p is odd, then $\frac{p-1}{2}$ is an integer, and i, j can be set to it making R F-split. If $p = 2$, then $f \in \mathfrak{m}^{[2]}$, so R is NOT F-split.