CLASS 9, FEBRUARY 25TH: NAKAYAMA'S LEMMA

Last time we proved the determinant trick. This allows us to determine a sufficient condition as to whether a module is annihilated by a given element of R.

Theorem 9.1 (Nakayama's Lemma 1)**.** *If I is an ideal of R and M is a finitely generated module such that $IM = M$, then $\exists r \equiv 1 \pmod{I}$ such that $rM = 0$.*

Proof. Consider the map $\varphi = Id_M$. The assumption of Nakayama's lemma ensures that $Id(M) \subseteq IM$, so we get a polynomial

$$p(1) = 1 + r_1 + \ldots + r_n$$

for which $p(1)m = 0$ for every $m \in M$. But $p(1) \equiv 1 \pmod{I}$, since each $r_i \in I$. This completes the proof. □

As a quick side note, the term **annihilated** above is actually a standard term in commutative algebra.

Definition 9.2. If M is an R-module, then we define the annihilator of M as

$$\operatorname{Ann}_R(M) = \{r \in R \mid r \cdot M = 0\}$$

That is to say $r \in \operatorname{Ann}_R(M)$ iff $r \cdot m = 0$ for every $m \in M$

It should be checked that $\operatorname{Ann}_R(M)$ is a proper ideal of R. In fact, we can naturally give M the structure of an $R/\operatorname{Ann}_R(M)$-module! Thus Nakayama implies the existence of $1 + \alpha \in \operatorname{Ann}_R(M)$ for $\alpha \in I$. Next, I state come other corollaries of Theorem 9.1.

Corollary 9.3 (Nakayama's Lemma 2)**.** *If $(R, \mathfrak{m})$ is a local ring and M is a finitely generated R-module, then $M = \mathfrak{m}M$ implies $M = 0$.*

Proof. We know that $(R, \mathfrak{m})$ is local if and only if everything of the form $1 + m$ is a unit, with $m \in \mathfrak{m}$. As a result, if $M = \mathfrak{m}M$, Theorem 9.1 implies some unit $1 + m$ annihilates M. But this implies

$$M = 1M = (1 + m)^{-1}(1 + m) \cdot M = (1 + m)^{-1} \cdot 0 = 0$$

□

Corollary 9.4 (Nakayama's Lemma 3)**.** *If $(R, \mathfrak{m})$ is a local ring and $N \subseteq M$ are finitely generated R-modules, then $M = \mathfrak{m}M + N$ implies $M = N$.*

The proof of this is left as a Homework 4 #4. As a direct result, we see the following:

Theorem 9.5 (Nakayama's Lemma 4)**.** *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. If $m_1, \ldots, m_n \in M$ are such that $\langle \bar{m}_1, \ldots, \bar{m}_n \rangle = M/\mathfrak{m}M$, then $M = \langle m_1, \ldots, m_n \rangle$.*

Proof. Given the setup, we note that $M = \langle m_1, \ldots, m_n \rangle + \mathfrak{m}M = \mathfrak{m}M + \langle m_1, \ldots, m_n \rangle$. By Corollary 9.4, the result is implied directly: $M = \langle m_1, \ldots, m_n \rangle$. □

One other neat application of Theorem 9.5 is the following, which is known in general due to Vasconcelos.

Proposition 9.6. *If $\varphi : M \to M$ is a surjective R-module homomorphism, then it is also injective.*

This is very similar to the case of finite dimensional vector spaces.

Proof. We can give M the structure of an $R[x]$-module by allowing x to act by φ:

$$(r_n x^n + \ldots + r_1 x + r_0)m := r_n \varphi^n(m) + \ldots + r_1 \varphi(m) + r_0$$

The surjectivity assumption is stating that $I = \langle x \rangle$ has the property that $M = IM$. Theorem 9.1 now implies that $\exists p(x) \in R[x]$ such that $1 - p(x) = x \cdot q(x)$ and $p(x) \cdot m = 0$ for every $m \in M$. Note that $x \cdot q(x)m = m$. Therefore, $x \cdot m = \varphi(m) \neq 0$ for every $m \in M$. This is an equivalent formulation of injectivity! $\square$

This allows us to conclude a wonderful result about finite free R-modules similar to the case of Vector spaces.

Theorem 9.7 (Invariance of Rank)**.** *If R is an integral domain, and $R^n \cong R^m$ as R-modules for $n, m \in \mathbb{N}$, then $n = m$.*

Proof. I start by assuming that R is a local ring with maximal ideal $\mathfrak{m}$. In this case, note that if $R^n \cong R^m$ as R-modules, then we can conclude that $(R/\mathfrak{m})^n \cong (R/\mathfrak{m})^m$ as $R/\mathfrak{m}$-modules. This follows precisely by Theorem 9.5. But this is an isomorphism of finite dimensional vector spaces! As a result, $m = n$.

Now to get to the case of general rings, we use the fact that there exists a maximal ideal for any ring R. So we can localize everything in sight at $\mathfrak{m}$. This reduces us to the previous case and proves the claim. $\square$

Note: Once we develop the notion of localization in full generality (e.g. for non-domains and for arbitrary modules), we can remove the 'domain' condition from the previous result. We can also make similar statements to Nakayama's Lemma for any ring.

Finally, this result allows us to define the rank of a free R-module.

Definition 9.8. The **rank** of a free module $M \cong R^n$ is n.