HOMEWORK 1: THE COMPLEX PLANE
DUE: WEDNESDAY, SEPTEMBER 18TH

1) Write down a piecewise function to determine the argument of any given complex number $z = a + ib$. Be sure to justify your assertions.

2) Verify the assertion that $re^{i\theta} \cdot se^{i\phi} = rse^{i(\theta+\phi)}$ by using the Cartesian representation of a complex number.

3) Given $w = re^{i\theta}$. $r > 0$, solve the equation $z^n = w$ explicitly. How many solutions are there? To simplify matters, you may give your solutions with $Arg(z) \in [0, 2\pi)$ instead of our usual $(-\pi, \pi)$.

4) Show that it is impossible to define a total ordering $<$ on $\mathbb{C}$ such that
   1) For any $z, w \in \mathbb{C}$, either $z = w$, $z < w$, or $w < z$.
   2) If $a, b, c \in \mathbb{C}$ and $a < b$, then $a + c < b + c$.
   3) If $a, b, c \in \mathbb{C}$ and $0 < a$, then $b < c$ implies $ab < ac$

   (**hint:** What happens when you consider $0 < i$ and $i < 0$?)

5) Show that in polar coordinates, the Cauchy-Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \qquad\qquad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

   Therefore, if we define $\log(z) = \log(r) + i\theta$, where $z = re^{i\theta}$, then log is holomorphic in the region $r > 0$ and $-\pi < \theta < \pi$.

6) Show that the Laplacian

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

   while acting on twice continuously differentiable functions satisfies the following equality:

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar{z}} = 4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}$$

   Why is this assumption necessary? Conclude that if f is holomorphic (with this assumption), then the real and imaginary parts are **harmonic**. That is to say $\Delta f = 0$.

7) Define a function $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = f(x + iy) = \sqrt{|x||y|}$$

   Show that although f satisfies the Cauchy-Riemann equations, f is not holomorphic at 0.