# HOMEWORK 10: GRAND FINALE
# DUE: WEDNESDAY MAY 8TH

1) Recall every Noetherian ring $R$ has only finitely many associated primes (Theorem 27.3). Let $W = \{r \in R \mid r \text{ is not a zero-divisor}\}$. Show that $W^{-1}R$ has only finitely many maximal ideals. (**hint:** Use prime avoidance, without proof).

   Work this out in the case $R = K[x, y, z]/\langle xy, y^2, xz\rangle$.

   **Solution:** First note that $\mathrm{Spec}(W^{-1}R)$ can be described as $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $\mathfrak{p} \cap W^{-1}R = \emptyset$. This is to say $\mathfrak{p}$ contains only zero divisors for $R$. We have shown (Corollary 26.5) that the set of zero-divisors for a module, such as $R$, are exactly equal to the union of associated primes:

$$\mathfrak{p} \subseteq \{\text{zero divisors of } R\} = \bigcup_{\mathfrak{q} \in \mathrm{Ass}(R)} \mathfrak{q} = \mathfrak{q}_1 \cup \cdots \cup \mathfrak{q}_n$$

   Now, I claim $\mathfrak{p} \subseteq \mathfrak{q}_i$ for some $i$. This is exactly the statement of prime avoidance. By induction, we can suppose $x_i \in \mathfrak{p} \backslash \mathfrak{q}_1 \cap \ldots \cap \hat{\mathfrak{q}}_i \cap \ldots \cap \mathfrak{q}_n$. Then $x_1 \cdots x_{n-1} + x_n \in \mathfrak{q}_j$ for some $j$. But $x_1 \cdots x_{n-1} \notin \mathfrak{q}_n$ and $x_n \notin \mathfrak{q}_i$ for $i < n$. This contradicts the fact that that $\mathfrak{p} \subseteq \mathfrak{q}_i$.

   Therefore, every prime ideal of $W^{-1}R$ is a subset of an associated prime. So in fact $\mathrm{Spec}(W^{-1}R)$ is finite.

2) Prove the following Lemma using ideas of Lemma 29.4:

   **Lemma 0.1** (Fitting Lemma)**.** *If $\varphi : M \to M$ is a homomorphism, with $M$ a Noetherian Module, show that there exists $n \gg 0$ such that $\mathrm{im}(\varphi^n) \cap \ker(\varphi^n) = 0$.*

   **Solution:** Immediately, we have that

$$\ker(\varphi) \subseteq \ker(\varphi^2) \subseteq \cdots \subseteq \ker(\varphi^N) = \ker(\varphi^{N+1}) = \ldots$$

   since $M$ is a Noetherian module. The inclusions follow since if $\varphi^n(x) = 0$, then clearly

$$\varphi^{n+1} = \varphi(\varphi^n(x)) = \varphi(0) = 0$$

   I claim $\ker(\varphi^N) \cap \mathrm{im}(\varphi^N) = \emptyset$. Suppose not, namely $x$ is in the intersection. Then $\varphi^N(x) = 0$ but $\varphi^N(x') = x$ for some $x' \in M$. However, $\ker(\varphi^N) = \ker(\varphi^{2N})$, so

$$\varphi^{2N}(x') = \varphi^N(x) = 0$$

   So $x' \in \ker(\varphi^N)$, which is to say that $x = 0$.

3) Show that if $R$ is a Noetherian local ring with maximal ideal $\mathfrak{m}$, then $\mathfrak{m}$ is principal if and only if $\mathfrak{m}/\mathfrak{m}^2$ is a 1-dimensional $\mathbb{R}/\mathfrak{m}$-vector space.[1] This allows us to say $R$ is a DVR if and only if $R$ is local Noetherian with $\mathrm{Spec}(R) = \{0, \mathfrak{m}\}$ and $\mathfrak{m}/\mathfrak{m}^2$ a 1-dimensional vector space.

[1] This is what it means to be **regular** for dimension 1 rings.

**Solution:** The $\Rightarrow$ direction is obvious. The $\Leftarrow$ direction is a result of Nakayama's Lemma (v4).

4) If $R = K[x, y]$ with $K$ algebraically closed, let $f$ is an irreducible polynomial of the form
$$f = f' + f''$$
where $f' = ax + by$ and $f'' \in \langle x, y\rangle^2$. Consider $\mathfrak{m} = \langle x, y\rangle \subseteq A = R/\langle f\rangle$. Show that $A_\mathfrak{m}$ is a DVR if and only if $f' \neq 0$.
This shows $f$ is smooth (in the case of $\mathbb{C}$) at $P = (0, 0) \in K^2$ if and only if $A_\mathfrak{m}$ is a DVR.

**Solution:** Suppose first that $f' \neq 0$, which is to say either $a \neq 0$ or $b \neq 0$. We can immediately notice that $R$ is Noetherian implies $A_\mathfrak{m}$ is a Noetherian domain (since $f$ is irreducible). Therefore, by Theorem 31.2 it suffices to check that $\mathfrak{m} = \langle x, y\rangle$ is a principal ideal. By the previous problem, this is equivalent to showing that $\mathfrak{m}/\mathfrak{m}^2$ is a 1-dimensional $A_\mathfrak{m}$ vector space. If $a \neq 0$, I claim $\mathfrak{m}/\mathfrak{m}^2 = \langle y\rangle$. This is because $f = ax + by \equiv 0 \,(\mathrm{mod}\, \mathfrak{m})^2$, so $x = a^{-1}by$. Of course, if $a = 0$, then $\mathfrak{m} = \langle x\rangle$, since $f = by \equiv 0 \,(\mathrm{mod}\, \mathfrak{m})^2$, so only $x$ remains. This completes the proof.
If $f' = 0$, the same argument shows $\mathfrak{m}/\mathfrak{m}^2 = \langle x, y\rangle$ is a 2-dimensional vector space. Therefore $A_\mathfrak{m}$ cannot be a DVR.

5) If $R$ is an intermediate ring $K[x] \subseteq R \subseteq K[\![x]\!]$ which is local, maximal ideal $\langle x\rangle$, show $R$ is a DVR and thus in particular Noetherian.

**Solution:** $R$ is clearly a domain, since it lives in one. $R$ satisfies the properties of Theorem 31.2. Indeed, note that $\bigcap_{n=1}^{\infty}\langle x^n\rangle = 0$ since the same is true in the larger ring $K[\![x]\!]$. We can conclude that $R$ is necessarily a DVR with this valuation. If $\frac{a}{b} \in K$, then $a = ux^n$ and $b = vx^m$ for $u, v$ units (part $a$). Therefore,
$$\frac{a}{b} \in R \iff n \geq m \iff v(\frac{a}{b}) \geq 0$$

6) Given $R$ a DVR with maximal ideal $\mathfrak{m} = \langle t\rangle$, consider the sequence
$$\ldots \overset{\pi_4}{\to} R/\mathfrak{m}^3 \overset{\pi_3}{\to} R/\mathfrak{m}^2 \overset{\pi_2}{\to} R/\mathfrak{m}$$
of ring homomorphisms. Given such an arrangement, we can take the inverse limit:
$$\hat{R} = \varprojlim(R/\mathfrak{m}^n) = \{(a_1, a_2, \ldots) \in \prod_{n=1}^{\infty} R/\mathfrak{m}^n \mid \pi_i(a_i) = a_{i-1}\ \forall i \geq 2\}$$
This is called the **completion** of $R$ with respect to $\mathfrak{m}$. Show $\hat{R}$ is also a DVR with maximal ideal $\mathfrak{m} = \langle\hat{t}\rangle$, where $\hat{t} = (t, t, t, t, \ldots)$.

**Solution:** I claim $\hat{R}$ is local with maximal ideal $\mathfrak{m} = \langle\hat{t}\rangle$. Suppose $a = (a_1, a_2, \ldots) \notin \langle\hat{t}\rangle$. Note that the assumptions on the projection maps imply that $a_{i+1} - a_i \in \mathfrak{m}^i$. But since $R$ is a DVR, we know $\mathfrak{m}^i = \langle t^i\rangle$ and thus we can assume $a_{i+1} - a_i = u_i t^i \,(\mathrm{mod}\, \mathfrak{m}^{i+1})$.

Therefore, the condition $a \notin \langle \hat{t} \rangle$ is equivalent to $a_1 \notin \mathfrak{m}$. In this case, choose $b_0 = a_1^{-1}$, and inductively choose $b_{i+1} - b_i = v_i t^i$ where

$$v_i = -u_0^{-1}(v_{i-1}u_1 + \ldots + v_0 u_i)$$

Then it is a simple computation to check

$$a \cdot b = (a_1 b_1, a_2 b_2, \ldots) = (1, 1, 1, \ldots) = 1 \in \hat{R}$$

To finish, it suffices to check $\cap_{i=1}^{\infty} \langle \hat{t^i} \rangle = 0$. Notice

$$\hat{t^i} = (t^i, t^i, t^i, \ldots) = (0, 0, 0, \ldots, 0, t^i, \ldots)$$

where $t^i$ appears in $\mathfrak{m}^{i+1}$. Therefore, to be in the intersection, elements must be in $\mathfrak{m}^i$ for every $i$. But $R$ is a DVR, thus $\mathfrak{m}^i$ intersect to 0. This shows $\hat{R}$ is a DVR.