CLASS 32, MAY 6TH: VALUATION RINGS

Today we drop the assumption that our value group is discrete, i.e. $\mathbb{Z}$. This allows us to upgrade our list of rings that have several *valuable* properties.

Definition 32.1. Let R be an integral domain with fraction field $L = \mathrm{Frac}(R)$. Then R is said to be a **valuation ring** if for every $0 \neq x \in L$, we have either $x \in R$ or $x^{-1}R$.

This may seem out of the blue compared with our old definition. However, we will give a more favorable definition later on.

Example 32.2. Lets return again to our perfect ring

$$R = \mathbb{F}_p[x]_{perf} = \mathbb{F}_p[x^{\frac{1}{p^\infty}}] = \mathbb{F}_p[x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \ldots]$$

This is an integral domain with fraction field

$$K = \mathbb{F}_p(x^{\frac{1}{p^\infty}}) = \mathbb{F}_p(x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \ldots)$$

R is not a valuation ring, as neither $\frac{x-1}{x-2}$ nor $\frac{x-2}{x-1}$ are in R. We can localize R at $\mathfrak{m} = \langle x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \ldots \rangle$ to fix this problem. Now, given a non-zero fraction $\frac{f}{g}$, we can extract all copies of x:

$$\frac{f}{g} = x^i \frac{f'}{g'}$$

where $f', g' \notin \mathfrak{m}$. Noting that $\frac{f}{g} \in R$ if and only if $i \geq 0$, we immediately yield that either $\frac{f}{g} \in R$ or $\frac{g}{f} \in R$. It should be noted however that i is taking values in $\mathbb{Z}[\frac{1}{p}]$, and in particular is NOT a DVR.

To get to a nice equivalent formulation, I define a total ordering:

Definition 32.3. A set Λ together with a transitive binary operation $<$ with the property that exactly one of $x < y$, $x = y$, or $x > y$ is true is called a **totally ordered set**.

If G is an Abelian group, we call it an **ordered group** if it is totally ordered compatibly with addition: $a \geq b$ and $a' > b'$, then $a + a' > b + b'$.

Given R a domain, we can form $G = K^\times / R^\times$. Note that this is well defined since $R^\times$ is (an automatically normal) subgroup of $K^\times$. It is typical to write G additively, even though the operation is multiplication: $[a \cdot b] = [a] + [b]$.

We can endow G with the **partial order** defined by $[\alpha] > 0$ if and only if $\alpha \in R$. This yields $[\alpha] \geq [\beta]$ if and only if $\frac{\alpha}{\beta} \in R$.

This method brings us to the following equivalent characterization of valuation rings:

Proposition 32.4. *R is a valuation ring if and only if $>$ is a total order on G. In this case, the quotient map $v : K^\times \to G = K^\times / R^\times$ satisfies the following properties:*

(a) $v(xy) = v(x) + v(y)$
(b) $v(x \pm y) \geq \min\{v(x), v(y)\}$

Moreover, if G is any ordered group, and $v : K^\times \to G$ is a surjective map satisfying properties (a) & (b), then the subset

$$R = \{\alpha \in K \mid v(\alpha) \geq 0\} \cup \{0\}$$

is a valuation ring, and $G = K^\times / R^\times$.

Definition 32.5. In the setup of Proposition 32.4, v is called a **valuation** and G is called the **value group** of v.

Example 32.6. In our previous example, Example 32.2, one can verify that

$$v : K^\times \to K^\times / R_{\mathfrak{m}}^\times \cong \mathbb{Z}[\frac{1}{p}] : \frac{f}{g} = x^i \frac{f'}{g'} \mapsto i$$

is a valuation. The stated isomorphism is essentially the given one, since $\frac{f'}{g'} \mapsto 0$ and is exactly representative of the kernel.

Proof. R is a valuation ring if and only if $0 \neq \frac{a}{b} \in K$ implies $\frac{a}{b} \in R$ but not $\frac{b}{a}$, or $\frac{b}{a} \in R$ but not $\frac{a}{b}$, or $\frac{a}{b} \in R^\times$. These are exactly the conditions $v(a) > v(b)$, $v(b) > v(a)$, or $v(b) = v(a)$.

Given R a valuation ring, property (a) is direct from the fact that v is a homomorphism of groups, with multiplication in $K^\times$ and 'addition' in G.

(b) follows, since if $v(x) \geq v(y)$, then $v(xy^{-1}) \geq 0$, implying $xy^{-1} \in R$. Thus

$$v(y^{-1}(x+y)) = v(xy^{-1} + 1) \geq 0$$

or equivalently $v(x+y) \geq v(y)$.

The final sentence (Moreover, ...) is left for the eager reader. □

A cool aftereffect of Proposition 32.4 is the following: If you can come up with a surjective group homomorphism $v : K^\times \to G$, where G is an ordered group, then you have produced a valuation ring.

Definition 32.7. Consider the group $G = \mathbb{Z}^2$ with the lexicographical/dictionary order; $(a,b) > (a',b')$ if and only if $a > a'$ or $a = a'$ and $b > b'$. This yields a natural map

$$v : K(x,y)^\times \to \mathbb{Z}^2 : \frac{f}{g} = x^i y^j \frac{f'}{g'} \mapsto (i,j)$$

where we use the standard that $f', g' \notin \langle x, y\rangle$ to make the map well-defined. Doing this, we can verify that the associated valuation ring is

$$R = K[x,y]_{\langle x,y\rangle} \left[\frac{x}{y}, \frac{x}{y^2}, \frac{x}{y^3}, \ldots\right]$$

and in particular contains $\frac{x^i}{y^j}$ with $i > 0$ and any $j \in \mathbb{Z}$.

This assists with the next realization:

Theorem 32.8. *If R is a valuation ring, then R is Noetherian if and only if R is a DVR.*

Proof. We know DVRs are Noetherian, so it only suffices to check $\Rightarrow$. Assume R is Noetherian. Then all ideals I are finitely generated. If $I = \langle x_1, \ldots, x_n\rangle$, then I claim I is in fact principal! Let x_1 WLOG be the generator with smallest valuation. Then we have $\frac{x_i}{x_1} \in R$ for all i, since

$$v(\frac{x_i}{x_1}) = v(x_i) - v(x_1) \geq 0$$

As a result, $x_1 \frac{x_i}{x_1} = x_i \in \langle x_1\rangle$, meaning every other generator is redundant. □