CLASS 11, OCTOBER 2: ANALYTIC CONTINUATION

Cauchy's theory is yielding the following general principle: Holomorphicity is an extremely strong condition. We have seen it yield strong bounds on functions, the analytic property, and only constant bounded functions! Today, we will study the idea of extending a function outside of its domain.

We intend to show that such an extension must be unique. To do this, we need a lemma about accumulation of zeroes of holomorphic functions.

Lemma 11.1. *Suppose f is holomorphic in Ω a connected set. If $z_n \in \Omega$ is a sequence of distinct points which converges to some $z_\infty \in \Omega$, and $f(z_n) = 0$ for all $n \in \mathbb{N}$, then f is the zero function on Ω.*

Proof. We begin by showing f is 0 in a neighborhood of z_∞. We can choose a disc for which f has a power series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_\infty)^n$$

Choosing m minimal such that $a_m \neq 0$, we can write

$$f(z) = a_m (z - z_\infty)^m (1 + (z - z_0) g(z))$$

where $(z - z_\infty) g(z) \to 0$ as $z \to z_\infty$. Choosing z_k sufficiently close to z_∞, we can ensure $|(z - z_0) g(z)| < \frac{1}{2}$. But as a result

$$0 = |f(z_k)| = |a_m (z_k - z_\infty)^m (1 + (z - z_0) g(z))| \geq \frac{1}{2} |a_m| |z_k - z_\infty|^m > 0$$

This is a contradiction to our assumptions. So f is 0 in a neighborhood of z_∞.

Now we finish by use of connectedness. First, let U be the interior of $f^{-1}(0)$. We just showed U is non-empty. Furthermore, since f is continuous, if $z_n \to z$ in U, $f(z) = 0$. Thus U is closed. But to be connected implies that the only open and closed sets are empty or Ω. Thus $U = \Omega$. $\square$

We can use this to prove the desired statement about function's extensions by considering their difference:

Corollary 11.2. *If f and g are holomorphic in a connected region Ω, and $f(z) = g(z)$ in an open subset of Ω, then $f = g$ throughout Ω.*

Here is a way to reinterpret this statement. If $\Omega \subseteq \Omega'$ are two open sets, and $f : \Omega \to \mathbb{C}$ is a holomorphic function, there exists **at most** one extension of f to Ω'. Here an extension is a holomorphic function $g : \Omega' \to \mathbb{C}$ such that $f(z) = g(z)$ for all $z \in \Omega$. Of course, an extension needn't exist. If they do, we call g an **analytic continuation** of f to Ω'.

Example 11.3. The function $f(z) = \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus 0$. However, there cannot exist even a continuous function on $\mathbb{C}$ agreeing with f.

In the case of real valued functions, there can exist as many distinct extensions as one may with.

Example 11.4. Going back to our function $f(x) = e^{-\frac{1}{x^2}}$ on $(0, \infty)$, the homework exercise shows that $f(x)$ can be extended to $\mathbb{R}$ by 0. But you could extend it also by $e^{-\frac{c}{x^2}}$ for any $c > 0$, or more generally any C^∞ function with all derivatives vanishing at the origin.

Example 11.5. We know that the power series $\sum_{n=0}^{\infty} z^n$ has a radius of convergence of $R = 1$. So our earlier result of Theorem 4.5 tells us explicitly that the series diverges for $|z| > 1$. On the other hand, in its radius of convergence this function agress with $\frac{1}{1-z}$. This function can be defined anywhere where $z \neq 1$, and is therefore the analytic continuation of the power series to $\mathbb{C} \setminus \{1\}$.

Example 11.6. The zeta function as discussed on day 1 is $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. This function makes sense whenever $Re(s) > 1$. However, in this domain one can show that

$$\zeta(s) = \frac{\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx}{\Gamma(s)} = \frac{\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx}{\int_0^\infty x^{s-1} e^{-x} dx}$$

We will examine this equality later, but suffice it to say that the RHS makes sense whenever $s \neq 1$. Therefore, we can extend $\sum_{n=1}^{\infty} \frac{1}{n^s}$ to $\mathbb{C} \setminus \{1\}$.

It should be noted that $\Gamma(s)$ is itself a very important function. First of all, if s is an integer, then one can immediately conclude by integration by parts that

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx = \left[-x^{n-1} e^{-x}\right]_{x=0}^{\infty} - \int_0^\infty (s-1) x^{s-2} e^{-x} dx$$

By induction, with base case $\int_0^\infty e^{-x} dx = 1$, shows that $\Gamma(s) = (s-1)!$, the factorial of $s - 1$. Thus as a very cool analogy, $\Gamma(n + 1)$ is continuation of the factorial from the non-negative integers to $\mathbb{C} \setminus (\mathbb{Z}_{n<0})$! The removal of the negative integers is due to the fact that the function isn't defined at thus values (there are **simple poles** there).

To finish up, I want to discuss an important converse to Cauchy's theorem.

Theorem 11.7 (Morera's Theorem)**.** *If f is continuous in an open disc $B(z, r)$ and such that for any triangle T, we have*

$$\int_T f(z) dz = 0$$

then f is holomorphic in $B(z, r)$.

It is actually quite a simple proof:

Proof. By the proof of Cauchy's integral theorem, f has a primative in $B(z, r)$, namely $F(z) = \int_{\gamma_z} f(z) dz$ where γ_z is a path from a chosen point to z. Since $F'(z) = f(z)$, and F is holomorphic, F has infinitely many derivatives. But this implies $F''(z) = f'(z)$, i.e. f is holomorphic. $\square$

Example 11.8. Returning to our previous example, Morera's Theorem shows that $\zeta(s)$ is a holomorphic function. Indeed, given a triangle T inside of the region of complex numbers with real part bigger than 1, then one can show

$$\int_T \zeta(s) = \int_T \sum_{n=1}^{\infty} \frac{1}{n^s} ds = \sum_{n=1}^{\infty} \int_T \frac{1}{n^s} ds = \sum_{n=1}^{\infty} 0 = 0$$

Here, interchanging the $\sum$ and $\int_T$ is a delicate process. One can cite Fubini or Tonelli's theorem, but at the very least one should note the importance of the uniform absolute convergence of the series on the triangle T.

Regardless, this shows that $\zeta(s)$ is a holomorphic function. The same sort of trick can be applied to the gamma function $\Gamma(s)$.