GENERAL TOPOLOGY: FINAL EXAM

1) [**15 pts**] Prove **the pasting lemma**: Show that if $f : X \to Y$ is a function which is defined piecewise

- By finitely many f_i defined on closed sets $C_1, \dots, C_n$ covering X.
- By any number of f_α defined on open sets U_α covering X.

Such that the piecewise functions agree where their domains overlap (e.g. $f_i(x) = f_j(x)$ for $x \in C_i \cap C_j$, similarly for f_α), then f is a continuous function.

Solution: Suppose Z is a closed subset of Y. It suffices to check that $f^{-1}(Z)$ is closed in X. Note that

$$f^{-1}(Z) = \bigcup_{i=1}^{n} f_i^{-1}(Z)$$

And since C_i is closed, $f_i^{-1}(Z)$ is closed in X. Therefore f is continuous.

Now suppose V is an open subset of Y. By the same reasoning,

$$f^{-1}(V) = \bigcup_{\alpha} f_\alpha^{-1}(V)$$

is an open set. So f is again continuous.

2) **[15 pts]** A **generalized metric space** is a set X with a function $d : X \times X \to [0, \infty]$ satisfying all of the usual metric conditions. Show that the condition $x \sim y$ if $d(x, y) < \infty$ is an equivalence relation[1].

Each equivalence class is itself a metric space, therefore X is a topological space. If there exists $x, y \in X$ with $d(x, y) = \infty$, show X is disconnected.

Solution:

- ◦ Reflexive: $x \sim x$ since $d(x, x) = 0 < \infty$.
- ◦ Symmetric: $x \sim y$ implies $d(y, x) = d(x, y) < \infty$ implies $y \sim x$.
- ◦ Transitive: $x \sim y$ and $y \sim z$ implies

$$d(x, z) \leq d(x, y) + d(y, z) < \infty + \infty = \infty$$

so $x \sim z$.

Denote $[x] = \{y \mid x \sim y\}$. Note that $[x]$ is an open set since

$$[x] = \bigcup_{i=1}^{\infty} B(x, n)$$

Therefore, if $x \not\sim y$, then

$$X = [x] \cup [x]^c = [x] \cup \left(\bigcup_{y \not\sim x} [y] \right)$$

is a union of two open disjoint sets. Therefore X is disconnected.

[1]Each equivalence class is called a **galaxy**, thinking perhaps of X as the universe!

3) **[10 pts]** Show that if X is a second-countable space, then any collection $\mathcal{U} = \{U_\alpha\}$ of open, disjoint sets is countable: $|\mathcal{U}| = |\mathbb{N}|$.

Solution: We can choose for each α an $i(\alpha) \in \mathbb{N}$ such that $B_{i(\alpha)} \subseteq U_\alpha$. Call $c : \mathcal{U} \to \mathcal{B}$ the function which chooses such a basis element. Note that this is an injective function, since $\emptyset = U_\alpha \cap U'_\alpha \supseteq B_{i(\alpha)} \cap B_{i(\alpha')}$. Therefore $\mathcal{U}$ injects into $\mathbb{Z}$, and thus is countable.

4) **[15 pts]** Show that every locally compact Hausdorff space is T3/regular. Conclude that a locally compact second-countable Hausdorff space is metrizable.

Solution: Suppose $x \in X$ and U is an open neighborhood of x. We can choose K a compact neighborhood of x such that

$$x \in K^\circ \subseteq K \subseteq U$$

But then K with the subspace topology is compact Hausdorff, and thus T1+T4, so it is T3.

As a result, Urysohn's Metrization Theorem tells us that a second-countable T3+T1 space is metrizable.

5) [**10 pts**] Prove the following generalization of Urysohn's Lemma: If $A_1, A_2, \ldots, A_n \subseteq X$ are pairwise-disjoint closed subsets of X a normal space. Then there exists a continuous function $f : X \to [1, n]$ such that $f(A_i) = i$. (**hint:** You should not prove this in the same way Urysohn's Lemma was proved, for times' sake).

Solution: Choose a function $f' : A_1 \cup \ldots \cup A_n \to [1, n]$ such that $f(a) = i$ if $a \in A_i$. This is a continuous function, since for any closed set $C \subseteq [1, n]$, we have

$$f^{-1}(C) = \bigcup_{i \in C} A_i$$

Therefore, by Tietze's Extension Theorem, we have the existence of the desired function $f : X \to [1, n]$.

6) [**15 pts**] Recall the Stone-Cech compactification $\beta(X)$ of a T3.5 space X is the unique compactification of X such that if $f : X \to Y$ is a continuous function to a compact Hausdorff space, then f extends uniquely to $\tilde{f} : \beta(X) \to Y$.

Show that if Y is another compactification of X, then there exists a continuous, closed, surjective map $\beta(X) \to Y$ preserving X.

Solution: By our assumptions, we know that $\iota : X \hookrightarrow Y$ yields a continuous map $\tilde{\iota} : \beta(X) \to Y$ preserving X.

Note that since $\tilde{\iota}$ goes from a compact space to a Hausdorff space, it is a closed map. Finally, $\tilde{\iota}$ is surjective since

$$Y = \overline{\iota(X)} \subseteq \overline{\tilde{\iota}(\beta(X))} = \tilde{\iota}(\beta(X)).$$

7) [**15 pts**] Show that any X with the discrete topology is paracompact. Additionally, show that if $f : X \to Y$ is a continuous map, and X is paracompact, $f(X)$ need not be paracompact in Y.

Solution: Note that if X is discrete, then $\{x\}$ for each $x \in X$ is a locally-1 (thus locally finite) open refinement of any open cover. So X is paracompact.

As a result, given X a non-paracompact space (e.g. Hausdorff but not T4 space) has the property that

$$Id_X : (X, discrete) \to X$$

is a continuous map, but $Id_X(X) = X$ is not paracompact.

8) [**10 pts**] Show that every totally ordered set X endowed with the order topology is metrizable if and only if X has a countably locally finite basis.

Solution: Note that an ordered space is always T4. Additionally, it is T1 since

$$\{x\} = ((-\infty, x) \cup (x, \infty))^c$$

Therefore X is T3.

Moreover, by Nigata-Smirnoff, we can conclude that X is metrizable if and only if it has a countably locally finite basis.

9) [**20 pts**] Prove **Banach's Fixed Point Theorem**: If $X \neq \emptyset$ is a complete metric space, and $f : X \to X$ is a contraction, then there exists a *unique* $x \in X$ such that $f(x) = x$. A contraction is a continuous map satisfying:

$$d(f(x), f(y)) \leq \alpha \cdot d(x, y)$$

for some fixed $0 < \alpha < 1$.

- First show that there can't be 2 distinct fixed points.

  Solution: Suppose x, y are fixed points. Then

  $$d(f(x), f(y)) = d(x, y) \leq \alpha \cdot d(x, y)$$

  implying $d(x, y) \leq 0$, or $x = y$.

- Let $x_n = f(x_{n-1})$ inductively. Show that

  $$d(x_{n+1}, x_n) \leq \alpha^n d(x_1, x_0)$$

  Solution: The base case is our assumption. So assume we have proved the claim for $n - 1$. Then

  $$d(x_{n+1}, x_n) \leq \alpha \cdot d(x_n, x_{n-1} \leq \alpha\alpha^{n-1} d(x_1, x_0) = \alpha^n d(x_1, x_0)$$

  As asserted.

- Use the triangle inequality and the previous step to show that x_n is Cauchy.

  Solution: Note that for $m > n$,

  $$\begin{aligned} d(x_m, x_n) &\leq d(x_m, x_{m-1}) + \ldots + \ldots + d(x_{n+1}, x_n) \\ &\leq \alpha^{m-1} d(x_1, x_0) + \ldots + \alpha^n d(x_1, x_0) \\ &\leq \alpha^n (\alpha^{m-n-1} + \ldots + \alpha + 1) d(x_1, x_0) \\ &\leq \alpha^n \frac{1}{1 - \alpha} d(x_1, x_0) \end{aligned}$$

  Since $\frac{1}{1-\alpha} d(x_1, x_0)$ is constant, we can make $n \gg 0$ satisfy $\alpha^n \leq \frac{\epsilon}{\frac{1}{1-\alpha} d(x_1, x_0)}$.

- Use the fact that f is a continuous function to show $f(x) = x$.

  Solution:

  $$f(x) = f(\lim_n x_n) = \lim_n f(x_n) = \lim_n x_{n+1} = x$$

10) **[15 pts]** Show that if X is a compact space, then $\mathcal{Z} \subseteq C(X, \mathbb{R}^n)$ is compact if and only if $\mathcal{Z}$ is closed, bounded, and equicontinuous.

Solution: Ascoli's Theorem tells us that $\bar{\mathcal{Z}} \subseteq C(X, \mathbb{R}^n)$ is compact if and only if $\mathcal{Z}$ is equicontinuous and pointwise-bounded.

Suppose $\mathcal{Z}$ is compact. Then $\mathcal{Z}$ is automatically closed, since $(\mathbb{R}^n)^X$ is a Hausdorff space. Additionally, $\mathcal{Z}$ is equicontinuous by Ascoli. Finally, it is totally bounded and therefore bounded:

$$\mathcal{Z} \subseteq \cup_{i=1}^n B(x_1, r) \Longrightarrow \mathcal{Z} \subseteq B(x_1, d(x_1, x_2) + \ldots + d(x_1, x_n) + r)$$

Now suppose $\mathcal{Z}$ is closed, bounded, and equicontinuous. It suffices to check $\mathcal{Z}$ is pointwise bounded. But being bounded in $\mathcal{Z}$ implies there is an $R > 0$ such that

$$d(f(x), g(x)) < R$$

for every x and every $g \in \mathcal{Z}$. As a result, $g(x) \in B(f(x), R) \subseteq Y$, implying it is pointwise bounded.

11) [**10 pts**] Let $A \subseteq X$ be a subspace, with both X and A path connected. If $\iota : A \hookrightarrow X$ is the inclusion map, show that $i_* : \pi_1(A) \to \pi_1(X)$ is surjective if and only if every loop in X with $\gamma(0) \in A$ is homotopic to a loop in A.

Solution: This is almost definitional. Since X and A are path connected, we can choose our basepoint in A. Call it a. Then TFAE

- i_* is surjective.
- For every $\gamma \in \pi_1(X, a)$, there exists $\gamma' \in \pi_1(A, a)$ such that $\iota_*\gamma' = \gamma$.
- $\iota \circ \gamma' \simeq \gamma$ rel $0, 1$.

This completes the proof.