CLASS 14, OCTOBER 9: THE RESIDUE THEOREM

Last time we studied how zeroes become poles when you invert the function, and how the order of a pole is an important quantity telling you how you can multiply away the pole. Today we will use this machinery to study integrals of such functions.

Lemma 14.1. *Suppose f is a function holomorphic on $\bar{B}(w,r)$ except at some point $z_0 \in B(w,r)$ where it has a pole or order m. Then*

$$\int_C f(z)dz = 2\pi i \cdot res_{z_0} f$$

This result is only a slight deviation from what we've already proved.

Proof. Consider the keyhole contour connecting C to a circle C_ϵ of radius ϵ about z_0. By our assumptions, f is holomorphic here so the integral along the keyhole is 0. Sending the width of the corridor to 0 produces

$$\int_C f(z)dz = \int_{C_\epsilon} f(z)dz$$

where C_ϵ is oriented clockwise. Expressing f here, we see

$$\int_{C_\epsilon} f(z)dz = \sum_{n=2}^{m} \int_{C_\epsilon} \frac{a_n}{(z-z_0)^n}dz + \int_{C_\epsilon} \frac{\mathrm{res}_{z_0} f}{z-z_0}dz + \int_{C_\epsilon} g(z)dz$$

where g is holomorphic. Now using Cauchy's integral theorem, we have

$$\int_{C_\epsilon} \frac{a_n}{(z-z_0)^n}dz = \left(\frac{\partial}{\partial z}\right)^{n-1} (a_n)_{z_0} = 0$$

$$\int_{C_\epsilon} f(z)dz = \int_{C_\epsilon} \frac{\mathrm{res}_{z_0} f}{z-z_0}dz = 2\pi i \cdot \mathrm{res}_{z_0} f$$

□

The same result, as usual, holds for any toy contour. This is because CIT does as well. We can use this to produce a more general result using this observation:

Theorem 14.2 (The Residue Formula)**.** *Suppose f is a function holomorphic on Ω except at some points $z_1, \ldots, z_n \in B(w,r)$ where it has a poles. If C is a toy contour enclosing $z_1, \ldots, z_n$, then*

$$\int_C f(z)dz = 2\pi i \cdot \sum_{i=1}^{n} res_{z_i} f$$

Proof. We can proceed by induction. The base case is the toy contour version of Lemma 14.1. By a finiteness argument, we can produce a keyhole contour around z_n with the corridor paths and circle avoiding $z_1, \ldots, z_{n-1}$. Our inductive hypothesis ensures that

$$\int_K f(z)dz = 2\pi i \cdot \sum_{i=1}^{n-1} \mathrm{res}_{z_i} f$$

Examining the left hand side, we see

$$\int_K f(z)dz = \int_C f(z)dz - \int_{C_\epsilon} f(z)dz$$

So again applying Lemma 14.1 yields the desired result:

$$\int_C f(z)dz = \int_{C_\epsilon} f(z)dz + 2\pi i \cdot \sum_{i=1}^{n-1} \text{res}_{z_i} f = 2\pi i \cdot \sum_{i=1}^{n} \text{res}_{z_i} f.$$

□

So far in evaluating several of the real integrals we needed to invoke Cauchy's theorem and ensure holomorphicity. This improves this technique dramatically by allowing finitely many poles!

Example 14.3. Consider the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

This can be evaluated by methods of trigonometric integrals. But we can simplify this quite a bit! Consider the upper half semi-circle C. The integral along the bottom portion is exactly what we want. Now noticing

$$\int_C \frac{dz}{z^2+1} = \int_C \frac{dz}{(z+i)(z-i)}$$

So this function has a pole at i and $-i$. However, our semi-circle doesn't enclose $-i$, so we have

$$\int_C \frac{dz}{z^2+1} = 2\pi i \text{res}_i f = 2\pi i \frac{1}{i+i} = \pi$$

Thus it only goes to show that the upper portion C_R doesn't contribute anything. But this is easy!

$$|\int_{C_R} \frac{dz}{z^2+1}| \leq \pi R \cdot \sup |f(z)| = \pi R \frac{1}{R^2-1} \to 0$$

It should be noted that the choice is yours to integrate over the upper or lower half semicircles in the previous case. Note $\text{res}_{-i} f = \frac{1}{z-i}|_{z=-i} = -\frac{1}{2i}$. But this is accounted for because we have gone around clockwise instead of counterclockwise. The rest of the story goes through as expected.

This naturally generalizes to any rational function $\frac{p(z)}{q(z)}$ without poles on the real line. Assuming $\deg(p(z)) + 1 < \deg(q(z)$, then we know the integral converges (this is actually an if and only if statement. We would simply choose either the upper or lower semicircles (note that if p and q have real coefficients, then both will have the same number of enclosed poles), and apply the residue theorem. The same result will show C_R contributes nothing to the integral, so we would derive

$$\int_{-\infty}^{\infty} \frac{p(z)}{q(z)} dz = \sum_{\substack{z_i \\ q(z_i)=0}} \text{res}_{z_i} \left(\frac{p(z)}{q(z)} \right)$$

This is amazing! You never need to evaluate integrals of rational functions with simple poles again. Instead in suffices to simply plug in values.