HOMEWORK 9: ENTIRE FUNCTIONS
DUE: WEDNESDAY, NOVEMBER 20TH

(1) Find the order of growth of a polynomial $p(z)$, $f(z) = e^{bz^n}$ with $b \neq 0$, and $g(z) = e^{e^z}$.

Solution: The order of growth of a polynomial is 0. Indeed, for any $\rho > 0$, we have that

$$|p(z)| \leq e^{|z|^\rho}$$

for $|z| > R$ sufficiently large. This is simply the fact that $\frac{p(|z|)}{e^{|z|^\rho}} \to 0$ as $z \to \infty$. As a result, we can choose $A = \max_{z \in \bar{B}(0,R)}(p(z))$ and $B = 1$ conclude the desired result:

$$|p(z)| \leq Ae^{B|z|^\rho}$$

Since 0 is the smallest rate of growth allowable, this is the infimal rate of growth.

For f, the answer is obviously n. This follows by

$$\frac{e^{B|z|^n}}{Ae^{B|z|^\rho}} = \frac{1}{A}e^{B(|z|^n - |z|^\rho)} \to \begin{cases} 0 & \rho > n \\ \frac{1}{A} & \rho = n \\ \infty & \rho < n \end{cases}$$

For g, the answer is ∞, or that there is no definable rate of growth. Indeed,

$$\frac{e^{e^z}}{Ae^{B|z|^\rho}} = \frac{1}{A}e^{B(e^z - |z|^\rho)}$$

and $e^z - |z|^\rho \to \infty$ as $z \to \infty$ along the real line.

(2) Show that if τ is fixed with $Im(\tau) > 0$, then the Jacobi function

$$\Theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

is of order 2 in z. (**hint:** Notice that $-n^2 t + 2n|z| \leq -\frac{n^2 t}{2}$ for $t > 0$ and $n \geq 4\frac{|z|}{t}$)

Solution: If $t > 0$ and $n \geq 4\frac{|z|}{t}$, then

$$-\frac{n^2 t}{2} \leq -\frac{\frac{4|z|}{t} n t}{2} = -2n|z|$$

Now consider the sum:

$$\sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n z} = \sum_{n \in \mathbb{Z}} e^{\pi i (n^2 \tau + 2nz)}$$

So in absolute value:

$$|\Theta(z, \tau)| \leq \sum_{n \in \mathbb{Z}} |e^{\pi i (n^2 \tau + 2nz)}| = \sum_{n \in \mathbb{Z}} |e^{\pi i n^2 \tau}| \cdot |e^{\pi i 2nz}| \leq \sum_{n \in \mathbb{Z}} e^{\pi(-n^2 Im(\tau) + 2n|z|)}$$

As a result, for a fixed z and for $n \geq \frac{4|z|}{Im(\tau)}$,

$$e^{\pi(-n^2 Im(\tau) + 2n|z|)} \leq e^{-\frac{\pi}{2} n^2 Im(\tau)}$$

Therefore, if we consider our sum in 2 pieces with $N = \frac{4|z|}{Im(\tau)}$, we have

$$\sum_{|n|<N} e^{\pi i n^2 \tau} e^{2\pi i n z} + \sum_{|n|\geq N} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

The second sum converges by our analysis independent of z. For the first sum, we can approximate

$$\left| \sum_{|n|<N} e^{\pi i n^2 \tau} e^{2\pi i n z} \right| \leq \sum_{|n|<N} |e^{2\pi i n z}| \leq 2N e^{2\pi N |z|} = 2N e^{\frac{8\pi |z|^2}{Im(\tau)}}$$

With A chosen appropriately and $B = \frac{8\pi}{Im(\tau)} + 1$ (+1 to deal with the $|z|$ growth in N), we achieve the desired goal.

The rate of growth is exactly 2 by consideration of $z = -iy$ for $y \in \mathbb{R}$.

(3) For $t > 0$ fixed, consider

$$F(z) = \prod_{n\geq 1} \left(1 - e^{-2\pi n t} e^{2\pi i z}\right)$$

Note that $F(z)$ is entire.

- Show $|F(z)| \leq A e^{a|z|^2}$, hence F is of order 2.
- $F(z) = 0$ exactly when $z = nit + m$, where $n > 1$ and $n, m \in \mathbb{Z}$. Thus if z_n are its zeroes, then

$$\sum_n \frac{1}{|z_n|^2} = \infty \qquad\qquad \sum_n \frac{1}{|z_n|^{2+\epsilon}} < \infty$$

Solution: Note that F is entire since each of the partial products are. Further, the convergence is uniform on compact sets! Indeed, if $|z| < B$, then for $n > \frac{B}{t}$, $|e^{-2\pi n t} e^{2\pi i z}| < 1$ and experiences exponential decay.

- Again, we can choose $N = \frac{2|z|}{t}$ and study

$$\prod_{n\geq N} \left(1 - e^{-2\pi n t} e^{2\pi i z}\right)$$

Just like in the previous exercises, this product converges. As a result, we need only study

$$\prod_{n<N} \left(1 - e^{-2\pi n t} e^{2\pi i z}\right)$$

Examining each term individually yields

$$|1 - e^{-2\pi n t} e^{2\pi i z}| \leq 1 + |e^{-2\pi n t} e^{2\pi i z}| = 1 + e^{-2\pi n t} \cdot e^{2\pi |z|} \leq A e^{B|z|}$$

As a result, again we have

$$\prod_{n<N} \left|1 - e^{-2\pi n t} e^{2\pi i z}\right| \leq A' \left(e^{B|z|}\right)^N = A' e^{2B\frac{|z|}{t}|z|} = A' e^{B|z|^2}$$

◦ The sum reads

$$\sum_n \frac{1}{|z_n|^2} = \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \frac{1}{|m+nit|^2} = \sum_{n=1}^{\infty}\sum_{m\in\mathbb{Z}} \frac{1}{m^2+n^2t^2} = \infty$$

This sum is infinity since the respective integral is:

$$\int_1^\infty \int_{-\infty}^\infty \frac{1}{m^2+n^2t^2} dmdn = \int_1^\infty \frac{1}{nt}\left[\arctan\left(\frac{m}{nt}\right)\right]_{-\infty}^\infty dn = \int_1^\infty \frac{\pi}{nt} dn = \infty$$

Similarly, for any $\epsilon > 0$, the integral above would converge. Thus the rate of growth is exactly 2 by Proposition 28.3.

(4) If $\alpha > 1$, then

$$F_\alpha(z) = \int_{-\infty}^\infty e^{-|t|^\alpha} e^{2\pi izt} dt$$

has order of growth $\frac{\alpha}{\alpha-1}$. (**hint:** Show that $-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leq c|z|^{\frac{\alpha}{\alpha-1}}$ by consideration of $|t|^{\alpha-1} \leq A|z|$ and $|t|^{\alpha-1} \geq A|z|$ for some $A > 0$)

Solution: Following the hint, let $A = \frac{1}{4\pi}$. In the case that $|t|^{\alpha-1} \leq A|z|$, we find that

$$-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leq 2\pi|z||t| \leq 2\pi|z|A^{\frac{1}{\alpha-1}}|z|^{\frac{1}{\alpha-1}} = 2\pi A^{\frac{1}{\alpha-1}}|z|^{\frac{\alpha}{\alpha-1}}$$

In the other case, since $\alpha > 1$, as $t \to \infty$, the limit

$$e^{-|t|^\alpha} e^{2\pi izt} = e^{-|t|^\alpha + 2\pi izt} = e^{|t|(2\pi|z| - |t|^{\alpha-1})} \to 0$$

So again we can conclude that there exists some constant bounding the function in this region, and choose a corresponding c maximizing both of these quantities with respect to z.

As a result, we have

$$|F_\alpha(z)| \leq \int_{-\infty}^\infty |e^{-|t|^\alpha + 2\pi izt}| dt \leq \int_{-\infty}^\infty e^{-|t|^\alpha + 2\pi|z||t|} dt \leq \int_{-A|z|}^{A|z|} e^{c|z|^{\frac{\alpha}{\alpha-1}}} dt + C = 2A|z|e^{c|z|^{\frac{\alpha}{\alpha-1}}} + C$$

This proves the result. The rate of growth is exactly $\frac{\alpha}{\alpha-1}$ since we can consider $z = -iy$ for $y \in \mathbb{R}_{>0}$.

(5) Establish the following identities:

◦ If $\sum |a_n|^2$ converges, and $a_n \neq -1$ for any n, then $\prod(1+a_n)$ converges and is non-zero if and only if $\sum a_n$ converges.

◦ Find an example for which $\sum a_n$ converges, but $\prod(1+a_n)$ diverges.

◦ Find a convergent $\prod(1+a_n)$ where $\sum a_n$ diverges.

Solution:

⋄ Choose N such that $|a_n| \leq \frac{1}{2}$ for $n > N$. Then we may consider

$$\log\left(\prod_{n=N}^{\infty}(1+a_n)\right) = \sum_{n=N}^{\infty} \log(1+a_n)$$

Just like in the second example, if we subtract $\sum_{n=N}^{\infty} a_n$, what we are left with is

$$\sum_{n=N}^{\infty} \log(1+a_n) - a_n \approx -\sum_{n=N}^{\infty} \left(\frac{a_n^2}{2} - \frac{a_n^3}{3} + \cdots \right) \leq \sum_{n=N}^{\infty} a_n^2$$

Our last equality is using the assumption on N, and thus $|a_n^{n+2}| \leq 2^n |a_n|^2$. As a result, our assumption yields that the series on the right converges, so we get that $\sum_{n=N}^{\infty} \log(1+a_n)$ converges if and only if $\sum_{n=N}^{\infty} a_n$ does. Moreover, since limits pass over continuous functions, we have

$$\prod_{n=N}^{\infty} (1+a_n) = \exp(\sum_{n=N}^{\infty} \log(1+a_n)) = \exp(C) \neq 0$$

- ◇ To produce such an example, we must leave the situation of the previous bullet. Doing so can be achieved through the alternating series test: we know $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges. If we consider

$$\prod_{n=1}^{\infty} \left(1 + \frac{(-1)^n}{\sqrt{n}} \right)$$

We note this converges if and only if

$$\sum_{n=2}^{\infty} \log \left(1 + \frac{(-1)^n}{\sqrt{n}} \right)$$

does. We know the power series expansion for this, and since $\sum_{k=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges, we have that

$$\sum_{n=2}^{\infty} \log \left(1 + \frac{(-1)^n}{\sqrt{n}} \right) + \frac{(-1)^n}{\sqrt{n}} \approx \sum_{k=2}^{\infty} \frac{1}{k} + O\left(\frac{1}{k^{\frac{3}{2}}} \right)$$

which naturally diverges.

- ◇ Similarly,

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}} - \frac{1}{\sqrt{2n}+1} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{2n}(\sqrt{2n}+1)} \sim \sum_{n=1}^{\infty} \frac{1}{2n} \to \infty$$

But on the other hand,

$$\left(1 + \frac{1}{\sqrt{2n}} \right) \left(1 - \frac{1}{\sqrt{2n}+1} \right) = 1$$

So the product will converge to 1!