CLASS 3, SEPTEMBER 13: HOLOMORPHIC FUNCTIONS

Today we will focus what it means for a complex function to be differentiable. Though the definition is exactly the one from real analysis, the repercussions are far deeper.

Definition 3.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset of the complex plane, and $f : \Omega \to \mathbb{C}$ a function. If $z \in \Omega$, then f is said to be **holomorphic at** z if

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. The value of the above limit is denoted $f'(z)$, the **derivative of** f **at** z.

f is called **holomorphic** if $f'(z)$ exists at each point $z \in \Omega$.

If Ω is not open, f is **holomorphic on** Ω if there exists an open set $\Gamma \supseteq \Omega$ for which f is defined and holomorphic on.

If $\Omega = \mathbb{C}$, we call f an **entire** function.

A key difference to note here is that $h \in \mathbb{C}$, which differs from the case of real analysis in the sense that the limit is not a 2-sided approach, but instead a circular/∞-sided one.

Just to show you that much of you intuition is correct about differentiation, we have the following examples:

Example 3.2. If $f(z) = a_0 + a_1 z + \ldots + a_n z^n$ is a polynomial, then f is holomorphic on any domain Ω, thus entire. As you may expect,

$$f'(z) = a_1 + \ldots + n a_n z^{n-1}.$$

This follows by the binomial theorem.

Example 3.3. $f(z) = \frac{1}{z^n}$ is holomorphic for any Ω not containing 0 (thus f is not entire). One can compute

$$f'(z) = \lim_{h\to 0} \frac{\frac{1}{(z+h)^n} - \frac{1}{z}}{h} = \lim_{h\to 0} \frac{z^n - (z+h)^n}{hz^n(z+h)^n} = -\frac{1}{z^n} \lim_{h\to 0} \frac{(z+h)^n - z^n}{h(z+h)^n}$$

$$= -\frac{1}{z^n} \lim_{h\to 0} \frac{\binom{n}{1} h z^{n-1} + \binom{n}{2} h^2 z^{n-2} + \ldots + \binom{n}{n} h^n}{h(z+h)^n} = -\frac{1}{z^n} \lim_{h\to 0} \frac{\binom{n}{1} z^{n-1} + \binom{n}{2} h z^{n-2} + \ldots + \binom{n}{n} h^{n-1}}{h(z+h)^n}$$

$$= \frac{-nz^{n-1}}{z^n \cdot z^n} = \frac{-n}{z^{n+1}}$$

Note here that my limits only allow values of h in $B(0, |z|)$, to avoid the possibility of $z+h = 0$. This is fine because the definition of a limit only requires some open neighborhood of the desired point $h = 0$.

Example 3.4. The assignment $f(z) = \bar{z}$ is a perfectly well defined continuous function. It is however nowhere holomorphic. Indeed, noting that conjugation commutes with sums ($\overline{z+h} = \bar{z} + \bar{h}$), we have

$$\frac{f(z+h) - f(z)}{h} = \frac{\bar{h}}{h}$$

This has no limit as $h \to 0$. For example, if we approach 0 along the real axis, we would get 1. On the other hand, approaching 0 along the imaginary axis yields -1. Distinct limits imply that the function cannot be differentiated.

The last example shows that real and complex derivatives are distinct. Indeed, viewed as a function $F : \mathbb{R}^2 \to \mathbb{R}^2$, complex conjugation is differentiable: There is a linear transformation (**the Jacobian**) $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{|F(P+H) - F(P) - J(H)|}{H} \to 0$$

as $H \to 0$. Equivalently, we can write

$$F(P+H) - F(P) = J(H) + |H|\Psi(H)$$

with $\Psi(H) \to 0$ with H (i.e. using little o notation, they are equal $o(|H|)$). Given coordinate-wise, $F(x,y) = (x, -y)$. Thus the corresponding matrix J is

$$J(x,y) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the case of the complex derivative, the result is a complex number and not a matrix. This yields a reduction of 2 degrees of freedom from real differentiation!

This can be realized explicitly. Assume f is holomorphic at $z = x + iy$. Then we know $f'(z)$ exists. Let's let h approach along the real and imaginary axes:

$$f'(z) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} = \frac{\partial f}{\partial x}(z)$$
$$f'(z) = \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{ih} = \frac{1}{i}\frac{\partial f}{\partial y}(z)$$

Therefore, if f is holomorphic, these 2 limits agree:

$$\frac{\partial f}{\partial x}(z) = \frac{1}{i}\frac{\partial f}{\partial y}(z)$$

Viewing f in terms of 2 real valued functions $f = u(x,y) + iv(x,y)$, we produce

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = -i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y)$$

Relating the real and imaginary parts of this equation gives us the **Cauchy-Riemann equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These equations fundamentally link real and complex analysis. We can clarify the situation further by defining two differential operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad\qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

These results can be summed up as follows:

Theorem 3.5. *If f is holomorphic at $z_0 \in \mathbb{C}$, then f satisfies the Cauchy-Riemann equations. Additionally,*

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \qquad\qquad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$$

Finally, f is differentiable in the sense of real analysis, and

$$|f'(z)|^2 = \det(J(x_0, y_0))$$

where J is the Jacobian as above.