CLASS 5, WEDNESDAY FEBRUARY 14TH: LOCALIZATION

Today, to simplify matters, we will focus exclusively on commutative rings R.

When studying the properties of the ring, sometimes having possibly uncountably many maximal ideals can be a burden. Therefore, we often can use a process called **localization** of a ring to make the ring have only a single maximal ideal. Such a ring is called **local**.

The basic idea is as follows;

1) We can make it so that any element is a unit by adjoining an inverse of it to the ring. For example, with $\mathbb{Z}$, we can make 2 into a unit by adjoining $\frac{1}{2}$: $\mathbb{Z}_2 = \mathbb{Z}[\frac{1}{2}]$.
2) The effect of this is the following: $\langle 2 \rangle$ was a prime (maximal) ideal of $\mathbb{Z}$. However, in $\mathbb{Z}_2$ we have made it so that $\mathbb{Z}$ retains all of its prime ideals except $\langle 2 \rangle$.
3) We can "continue" to adjoin inverses to remove other prime ideals.

But how can this be generalized?

Definition 0.1. A **multiplicatively closed set** $W \subseteq R$ is a subset of R such that it is closed under multiplication. We assume $1 \in W$ and $0 \notin W$ for simplicity, though the theory can be developed more broadly.

If W is a multiplicatively closed set, then we define the **localization of** R **at** W, denoted $W^{-1}R$ to be the following ring: As a set,

$$W^{-1}R = \{(w,r) \ : \ w \in W, \ r \in R\}/\sim$$

where $\sim$ is the equivalence relation defined by $(w,r) \sim (w',r')$ if there exists $s \in W$ such that

$$s(wr' - w'r) = 0$$

The multiplication operation is $(w,r) \cdot (w',r') = (ww', rr')$. For addition, we declare

$$(w,r) + (w',r') = (ww', rw' + r'w)$$

Finally, we get a ring homomorphism $R \to W^{-1}R$ given by $r \mapsto (1,r)$. This is usually called the **localization map**.

It is worthwhile to check that this is a ring. Note that even though the operations in for a localized ring are complicated, they are inspired by something quite simple:

Example 0.2 (The Good)**.** Suppose that R is an integral domain, and W is a multiplicatively closed set. We will switch between the following two notations freely:

$$(w,r) = \frac{r}{w}$$

Then, as one may expect,

$$(w,r) \sim (w',r') \Leftrightarrow wr' = w'r \Leftrightarrow \frac{r'}{w'} = \frac{r}{w}$$

$$(w,r) \cdot (w',r') = \frac{r}{w} \cdot \frac{r'}{w'} = \frac{rr'}{ww'} = (ww', rr')$$

$$(w,r) + (w',r') = \frac{r}{w} + \frac{r'}{w'} = \frac{rw' + r'w}{ww'} = (ww', rw' + r'w)$$

So the motivation for localization is very simply **fractions**. However, fractions make far less sense when you are outside of an integral domain. In particular, we know division by 0 is problematic, but what about division by a zero divisor?

Example 0.3 (The Bad)**.** Suppose $z \in W$ is a zero divisor for R. Note that $(1, 0) \sim (w, 0)$ for any $w \in W$. Therefore, whenever $r \cdot z = 0$, we have that $(1, r) \sim 0$, since $z(r - 0) = 0$. Therefore, any r multiplying with z to 0 **becomes** 0 in $W^{-1}R$.

Example 0.4 (The Ugly)**.** Consider the ring $R = k[x, y, z]/\langle xy, xz\rangle$. If we localize at the multiplicative set $W = \{1, x, x^2, \ldots\}$, we see that $y = z = 0$ in the new ring. So $W^{-1}R = k[x, x^{-1}]$.

Lemma 0.5 (The Beautiful)**.** *If R is a commutative ring, and $\mathfrak{p}$ is a prime ideal, then $R \setminus \mathfrak{p}$ is a multiplicatively closed set.*

Proof. See homework. □

As a result, we can make the following definition.

Definition 0.6. For a ring R and prime ideal $\mathfrak{p}$, we define the localization of R at $\mathfrak{p}$ to be the ring

$$R_{\mathfrak{p}} := (R \setminus \mathfrak{p})^{-1}R$$

We think of this ring as describing the geometry of the ring R near the prime $\mathfrak{p}$. This will be made rigorous later on. Here is the reason localization is so powerful:

Theorem 0.7. *The collection of prime ideals of $W^{-1}R$ is exactly the collection of prime ideals of R not intersecting W:*

$$\{\mathfrak{p} \subset R \text{ a prime ideal, } W \cap \mathfrak{p} = \emptyset\} \leftrightarrow \{\mathfrak{p} \in W^{-1}R \text{ a prime ideal}\}$$

Corollary 0.8. *The prime ideals of $R_{\mathfrak{p}}$ are in natural bijection with the primes of R contained in $\mathfrak{p}$. In particular, the unique maximal ideal of $R_{\mathfrak{p}}$ is $\mathfrak{p} \cdot R_{\mathfrak{p}}$.*

Proof. Prime ideals cannot contain units. Therefore, if we consider a prime ideal $\mathfrak{q}$ of $W^{-1}R$, we know that $\varphi(w) = (1, w) \notin \mathfrak{q}$, where $\varphi : R \to R_{\mathfrak{p}}$ is the localization map. Therefore, $\varphi^{-1}(\mathfrak{q})$ is a prime ideal of R by the result of the homework.

Moreover, if $\mathfrak{q}$ is a prime of R, then I claim $\mathfrak{q} \cdot R_{\mathfrak{p}}$ is a prime ideal of $R_{\mathfrak{p}}$. Indeed, if $(w, r) \cdot (w', r') \in \mathfrak{q} \cdot R_{\mathfrak{p}}$, then $r \cdot r' \in \mathfrak{q}$ by clearing denominators. Finally, we see r or r' must have been in $\mathfrak{q}$ to begin with, and therefore either (w, r) or (w', r') was in $\mathfrak{q} \cdot R_{\mathfrak{p}}$. This completes the proof. □

Example 0.9. Let's examine what the prime ideals of $W^{-1}\mathbb{Z}$ are where $W = \{1, \underline{2}, \underline{3}, 4, \underline{5}, 6, 8, 9, 10, \ldots\}$. By the Theorem, we have that they are in bijection with the primes primes of $\mathbb{Z}$ not intersecting W. So those primes are exactly primes not divisible 2, 3, or 5. So they are $0, \langle 7\rangle, \langle 11\rangle, \langle 13\rangle, \ldots$.

Next time we will do some homework presentations and talk about the biggest possible localization: the ring of fractions.