GENERAL TOPOLOGY: MIDTERM 1 SOLUTIONS

1) [**7 pts**] Define a topological space (X, τ).

X is a set, and $\tau \subseteq \mathcal{P}(X)$ has the properties
1) $X, \emptyset \in \tau$.
2) If $X_\alpha \in \tau$, so is $\bigcup_\alpha X_\alpha$.
3) If $X_1, \dots, X_n \in \tau$, so is $X_1 \cap \dots \cap X_n$.

2) [**8 pts**] Given X_α topological spaces, define the product topology on $\prod_\alpha X_\alpha$.

The topology is the one generated by the basis

$$\mathcal{B} = \{\prod_\alpha U_\alpha \mid U_\alpha \in \tau_\alpha,\ U_\alpha = X_\alpha \text{for all but finite } \alpha\}$$

3) [**5 pts**] Define a continuous function $f : X \to Y$.

If $U \subseteq Y$ is open, then $f^{-1}(U) = \{x \mid f(x) \in U\} \subseteq X$ is open.

4) [**10 pts**] Given X any topological space, and Y a totally ordered set endowed with the order topology. Show that if $f, g : X \to Y$ are two continuous functions, that $h(x) = \min\{f(x), g(x)\}$ and $H(x) = \max\{f(x), g(x)\}$ are continuous functions as well. (It suffices to check the preimage of (a, b) is open. Pick either h or H to demonstrate the claim, not both.)

Consider

$$h^{-1}((a,b)) = h^{-1}((-\infty, b) \cap (a, \infty)) = h^{-1}((-\infty, b)) \cap h^{-1}((a, \infty))$$

Now, note that $h(x) \in (-\infty, b)$ iff either $f(x)$ is there or $g(x)$ is. Similarly, $h(x) \in (a, \infty)$ iff $f(x)$ and $g(x)$ lie there. Thus

$$h^{-1}((a,b)) = \big(f^{-1}((-\infty, b)) \cup g^{-1}((-\infty, b))\big) \cap \big(f^{-1}((a,\infty)) \cap g^{-1}((a,\infty))\big)$$

which is an open set.

5) [**15 pts**] Is every open, surjective, continuous map $f : X \to Y$ a quotient map? What if we replace *open* with closed? Justify your answer.

The answer in both cases is yes. Note we need to show

$$U \subseteq Y \text{ is open } \Leftrightarrow f^{-1}(U) \subseteq X \text{ is open.}$$

The $\Rightarrow$ implication follows from continuity. For the $\Leftarrow$, we note that $f(f^{-1}(U)) = U$ for any set $U \subseteq Y$. So in the case of an open map, $f^{-1}(U)$ open implies U is. In the case of closed maps, we note that the preimage behaves well with respect to preimages: if $f^{-1}(U)$ is open, note that U^c is closed:

$$f(f^{-1}(U)^c) = f(f^{-1}(U^c)) = U^c$$

6) [**5 pts**] Define the notion of connectedness.

If $X = U \cup V$ for open sets $U, V \subseteq X$, then $U \cap V \neq \emptyset$.

7) [**10 pts**] Prove that if $Z \subseteq X$ is a connected set, so is $f(Z)$ for $f : X \to Y$ a continuous map.

Suppose $f(Z) \subseteq U \cup V$ is a separation in X. Then note

$$Z \subseteq f^{-1}(f(Z)) \subseteq f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

the righthand side is a union of open disjoint sets, implying that $Z \subseteq f^{-1}(U)$ WLOG. But this implies:

$$f(Z) \subseteq f(f^{-1}(U)) \subseteq U$$

contradicting the separation.

8) **[10 pts]** Show that $[a, b)$ and (c, d) are not homeomorphic. (**hint:** Connectedness!)

Suppose $f : [a, b) \to (c, d)$ is a homeomorphism. Then note that $f((a, b)) = (c, f(a)) \cup (f(a), d)$, since f is in particular bijective. However, the image of a connected set is connected by the previous problem.

9) **[15 pts]** Let A, B be two disjoint compact subspaces of X a Hausdorff topological space. Show that there exists U, V open disjoint subspaces of X with $A \subseteq U$ and $B \subseteq V$.

Let $a \in A$ and $b \in B$. Then since X is Hausdorff, there exist $U_{a,b}$ and $V_{a,b}$ separating a and b. Letting a vary, we see $A \subseteq \bigcup_{a \in A} U_{a,b}$. But A is compact, so there exists a finite refinement:

$$A \subseteq U_{a_1,b} \cup \ldots \cup U_{a_n,b} = U_b$$

This yields a neighborhood of $b \in B$ disjoint from U_b:

$$V_b = V_{a_1,b} \cap \ldots \cap V_{a_n,b}$$

Now similarly, we note that if $b \in B$ varies, we cover B:

$$B \subseteq \bigcup_{b \in B} V_b = V_{b_1} \cup \ldots \cup V_{b_m} = V$$

again using compactness. But this set is disjoint from

$$U = U_{b_1} \cap \ldots \cap U_{b_m}$$

Therefore, since each U_{b_i} contains A, we get the desired open separating sets.

10) [**20 pts**] Let $\sim$ be the equivalence relation on $\mathbb{R}^n$, where we say $x \sim y$ if either $\|x\| = \|y\|$ or if $\|x\| = \frac{1}{\|y\|}$. Give $X = \mathbb{R}^n/\sim$ the quotient topology by the surjection $p : \mathbb{R}^n \to X$. Identify whether or not X is connected, Hausdorff, and compact. Justify you answers. (**hint:** You may assume $n = 1$, as all of the spaces obtained this way as n varies are homeomorphic (by use of spherical coordinates))

The hint is realized by consider $\mathbb{R}^n$ in spherical coordinates: $(x_1, \ldots, x_n) = (r, \theta_1, \ldots, \theta_{n-1})$, and then projecting to $\mathbb{R}$ via r. Then we can apply the main theorem of the quotient topology by noting that all fibers of this map (namely spheres) map to the same point. Of course, this may all be assumed!

- **Connectedness:** Note that X is the image of $\mathbb{R}^n$ under p, and therefore is itself connected.
- **Compactness:** Note that X is the image of $\bar{B}(0,1)$ under p, since every equivalence class has some choice of x with $\|x\| \leq 1$. This shows X is compact.
- **Hausdorffness:** Let $x, x' \in X$ be 2 distinct points such that $x = p(a)$ and $y = p(b)$ for $0 \leq a < b \leq 1$ (by the previous step, any equivalence class has this property). It goes to find open sets separating them. Consider $r = \frac{b+a}{2}$. Then we may consider
$$U = (-\infty, \frac{1}{r}) \cup (-r, r) \cup (\frac{1}{r}, \infty)$$
$$V = (U^c)^\circ = (-\frac{1}{r}, -r) \cup (r, \frac{1}{r})$$
These sets are the preimages of their open images, therefore are open since p is a quotient map. Moreover, their images are disjoint, since no point of U is identified with one of V.