HOMEWORK 1: RING THEORY
DUE: MONDAY, FEBRUARY 19TH

1) Determine which of the following are ring homomorphisms from $M_2(\mathbb{Z})$ to $\mathbb{Z}$:

i.
$$\phi : M_2(\mathbb{Z}) \to \mathbb{Z} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a$$

ii.
$$\Phi : M_2(\mathbb{Z}) \to \mathbb{Z} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto a + d$$

iii.
$$\det : M_2(\mathbb{Z}) \to \mathbb{Z} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto ad - bc$$

Solution: Throughout I will use the face that

$$MN = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

i. This is not a ring homomorphism. It does satisfy the additive rules, but not the multiplication rules:

$$\phi(M)\phi(N) = ae \neq ae + bg = \phi(MN)$$

ii. This is not a ring homomorphism, since again it satisfies the additive rules but not the multiplication rules:

$$\phi(M)\phi(N) = (a + d)(e + h) = ae + ah + de + dh \neq ae + bg + cf + dh = \phi(MN)$$

iii. This is not a ring homomorphism, since again it fails to satisfy additive rules (but does satisfy multiplication):

$$\phi(M) + \phi(N) = ad - bc + eh - fg \neq (a + e)(d + h) - (b + f)(c + g) = \phi(M + N)$$

2) Prove Proposition 0.2 from Notes 2:

Proposition 1. *The following conditions are equivalent:*

- *R is a Noetherian ring, e.g. every ideal is finitely generated.*
- *Every ascending chain of ideals eventually* **stabilizes***: if*

$$I_1 \subseteq I_2 \subseteq \dots$$

the $\exists n > 0$ such that $I_n = I_{n+1} = I_{n+2} = \dots$.

- *Every collection of Ideals $\{I_\alpha\}_{\alpha \in \Lambda}$ contains a maximal element. That is to say that there exists $\beta \in \Lambda$ such that there are no $\alpha \in \Lambda$ such that $I_\beta \subsetneq I_\alpha$.*

Solution: 1 → 2: Suppose that $I_1 \subseteq I_2 \subseteq \dots$ is an ascending chain of ideals. Then consider $I = \cup_{i \geq 1} I_i$. This is an ideal, and because R is Noetherian, I is finitely generated. Therefore $I = \langle f_1, \dots, f_n \rangle$. But I is the union of the other ideals, so there is an i_j such that I_{i_j} contains f_j. But then if $i = \max_j(i_j)$, $I_i = I$.

2 → 3: This is directly from Zorn's lemma. Condition 2 implies that every chain is bounded by an element in the chain.

3 → 1: Suppose I. Take $\{I_\alpha\}_{\alpha \in \Lambda}$ to be all the finitely generated ideals contained in I. Then there exists a maximal element of $\{I_\alpha\}_{\alpha \in \Lambda}$, call it J. I claim $J = I$. Indeed, if $I \neq J$, then take $x \in I \setminus J$. Then $J + \langle x \rangle$ is a finitely generated ideal contained in I, contradicting the maximality of J.

3) Show that if $\varphi : R \to S$ is a ring homomorphism, and $\mathfrak{p} \subseteq S$ is a prime ideal, then $\varphi^{-1}(\mathfrak{p})$ is also a prime ideal. As a result, show that if φ is a surjective map, preimages gives an injective map from prime ideals of S to prime ideals of R.

Solution: Let $a, b \notin \varphi^{-1}(\mathfrak{p})$ such that $ab \in \varphi^{-1}(\mathfrak{p})$. Then note that

$$\varphi(ab) = \varphi(a)\varphi(b) \in \varphi(\varphi^{-1}(\mathfrak{p})) \subseteq \mathfrak{p}$$

Therefore, either $\varphi(a)$ or $\varphi(b)$ are in $\mathfrak{p}$. But this implies either a or b are in $\varphi^{-1}(\mathfrak{p})$. This contradicts the hypothesis.

Is the same true in the opposite direction? That is to say, if $\mathfrak{p} \subseteq R$ is a prime ideal, then is $\varphi(\mathfrak{p}) \subseteq S$ a prime ideal? Or perhaps $\langle \varphi(\mathfrak{p}) \rangle \subseteq S$?

Solution: No, it is not true. Consider the ring homomorphism $K[x] \to K[x] : x \mapsto x^2$. Then $\langle x \rangle$ is a prime ideal, but $\langle x^2 \rangle$ is not.

4) Let R be commutative. The **radical** of an ideal I, denoted by $\sqrt{I}$, is the set of elements $r \in R$ such that $r^n \in I$ for some $n \gg 0$. Show that this is an ideal. For every prime ideal $\mathfrak{p}$, show that $\mathfrak{p} = \sqrt{\mathfrak{p}}$.

Solution: If $a, b \in \sqrt{I}$, then $a^n, b^n \in I$ for some $n \gg 0$. Therefore,

$$(a+b)^{2n} = a^{2n} + c_1 a^{2n-1} b + \dots c_n a^n b^n + \dots + c_1 a b^{2n-1} + b^{2n}$$

where $c_i \in \mathbb{Z}$. Each of these elements in the sum are in I, so the whole is. Therefore, $a + b \in \sqrt{I}$.

Moreover, if $r \in R$, $a \in \sqrt{I}$, then $a^n \in I$, and $(ra)^n = r^n a^n \in I$. Therefore, $ra \in \sqrt{I}$.

If $\mathfrak{p}$ is prime, and $a \in \sqrt{\mathfrak{p}}$, then $a^n \in \mathfrak{p}$. But by primality, this implies a or a^{n-1} is in $\mathfrak{p}$. Continue until $n = 1$. Thus $a \in \mathfrak{p}$ to begin with!

In addition, show that the nil-radical (radical of the zero ideal) $\mathcal{N} = \sqrt{0}$ is contained in every other prime ideal.[1]

[1] Later on we will show that $\mathcal{N}$ is precisely the intersection of all prime ideals.

Solution: If $r^n = 0$, we know that $0 \in \mathfrak{p}$, so $r \in \sqrt{\mathfrak{p}} = \mathfrak{p}$.

5) Let R be a commutative ring. An ideal I is called **primary** if whenever $a \cdot b \in I$, we have that $a \in I$ or $b^n \in I$. This is a slight generalization of being prime.
   i. What are the primary ideals of $\mathbb{Z}$?
   ii. I is primary if and only if every element of R/I is either a non-zero-divior or in $\mathcal{N} \subseteq R/I$.
   iii. If $\mathfrak{q}$ is a primary ideal, then $\sqrt{\mathfrak{q}}$ is a prime ideal.

   Solution:
   i. I claim that the primary ideals of $\mathbb{Z}$ are 0 and $\langle p^n \rangle$, where p is a prime number. 0 is definitely primary. If $ab = p^n \cdot c$, then either $p^n | a$ or $p | b$. Therefore, either $a \in \langle p^n \rangle$ or $b^n \in \langle p^n \rangle$, as desired. Finally, if n is composite, $n = pq$ for 2 numbers p, q. Therefore, $pq \in \langle n \rangle$, but $p, q \notin \langle n \rangle$.
   ii. Suppose that I is a primary ideal. Suppose $\bar{r} \in R/I$ is a zero divisor. Then there exists $\bar{q} \in R/I$ such that $\bar{q}\bar{r} = 0$. Consider r, q in R who $\pmod I$ agree with $\bar{r}, \bar{q}$. Then $qr \in I$. Therefore, we can conclude either $q \in I$ or $r^n \in I$. But this implies either $\bar{q} = 0$ or $\bar{r}^n = 0$, implying the result.
   If I is not primary, then $\exists r, q^n \notin I$ with $rq \in I$. Therefore, $\bar{q} \in R/I$ is a zero divisor that is not in $\mathcal{N}$.
   iii. Suppose $ab \in \sqrt{\mathfrak{q}}$. Then $a^n b^n \in \mathfrak{q}$ for some $n \gg 0$. This implies either $a^n \in \mathfrak{q}$ or $b^{nm} \in \mathfrak{q}$ for some $m > 0$ by the definition of primary. This implies directly that $a \in \mathfrak{q}$ or $b \in \mathfrak{q}$. Thus $\mathfrak{q}$ is prime.

6) We will now show that not all powers of prime ideals are primary ideals. Consider the ring $R = K[x, y, z]/\langle xy - z^2 \rangle$. Then show that the ideal $\mathfrak{p} = \langle x, z \rangle$ is prime. However, $\mathfrak{p}^2 = \langle x^2, xz, z^2 = xy \rangle$. Show that $xy \in \mathfrak{p}^2$ implies that $\mathfrak{p}^2$ can NOT be primary.

   Solution: $R/\mathfrak{p} \cong K[y]$, which is an integral domain. Therefore $\mathfrak{p}$ is prime. If we then consider $\mathfrak{p}^2$, we note that $xy \in \mathfrak{p}^2$. However, $x \notin \mathfrak{p}$, and $y^n \notin \mathfrak{p}$. Therefore, $\mathfrak{p}$ is not primary.