HOMEWORK 1: TOPOLOGICAL SPACES AND CONTINUOUS MAPS
DUE: SEPTEMBER 18, 2017

1) Write down all 9 possible inequivalent Topologies on the set with 3 points: $\{A, B, C\}$. (**hint**: there are 29 in total, avoid rearranging A, B, and C)

2) Let $\mathcal{T}_\alpha$ be a collection of topologies. Is it true that $\bigcap_\alpha \mathcal{T}_\alpha$ is a topology? What about $\bigcup_\alpha \mathcal{T}_\alpha$?

3) A topology $\mathcal{T}$ is called **finer** than a topology $\mathcal{S}$ if $\mathcal{S} \subseteq \mathcal{T}$. That is to say every set which is open in $\mathcal{S}$ is open in $\mathcal{T}$. In this case $\mathcal{S}$ is said to be **coarser** than $\mathcal{T}$. Two topologies are called **comparable** if one is coarser than the other.

   Consider the following topologies on $\mathbb{R}$:
   - $\mathcal{T}_1$ = the standard Euclidean/metric topology.
   - $\mathcal{T}_2$ = the finite complement topology.
   - $\mathcal{T}_3$ = the topology with basis $(a, b]$, where $a, b \in \mathbb{R}$.
   - $\mathcal{T}_4$ = the topology with basis $(-\infty, b)$, where $b \in \mathbb{R}$.
   - $\mathcal{T}_5$ = the topology with basis (a, b) and $(a, b) \setminus K$, where $K = \cup_{n \in \mathbb{Z}} \frac{1}{n}$.

   Compare each of these topologies with one another.

4) Let $X_\alpha \subset X$ be a collection of **closed** subsets that cover X: $X = \cup_\alpha X_\alpha$. Let $f : X \to Y$ be a function (not necessarily continuous) such that $f|_{X_\alpha}$ is continuous.

   - Suppose X_α is a finite collection (e.g. $X_1, \dots, X_n$). Show that f is continuous.
   - Find an example where this is not the case for a countable collection.
   - Suppose X_α is a **locally finite** collection: every $x \in X$ has a neighborhood intersecting at most **finitely many** X_α. In this case, show f is continuous.

5) If $A \subset X$ (with the subspace topology), a retraction is a continuous map $r : X \to A$ with $r|_A = Id_A$. Show that r is a quotient map.

6) Let $\pi : \mathbb{R}^2 \to \mathbb{R}$ be the projection onto the first coordinate. Let

$$A = \{(x, y) \mid x \geq 0 \text{ or } y = 0\}$$

   with the subspace topology. Show that the induce map $\pi|_A$ is a quotient map that is neither closed nor open (a map is closed (resp. open) if the image of a closed (resp. open) set is closed (resp. open)).