CLASS 9, SEPTEMBER 26: CONNECTEDNESS

So far we have managed to show that any topological space can be decomposed into a union of disjoint *connected components*. We also showed that $\mathbb{Q} \subseteq \mathbb{R}$ with the subspace topology is an example of a space where this decomposition is silly:

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$

However, there are some things that can be said about the decomposition:

Proposition 9.1. *Every connected component U_x of a space X is closed. If X has a finite decomposition into connected components, then each of them is open.*

To prove this, we need a lemma:

Lemma 9.2. *If A is a connected subset of X and $A \subseteq B \subseteq \bar{A}$, then B is connected. In particular $\bar{A}$ is connected.*

Proof. Suppose B is separated by C, D. Then by Proposition 8.4 from the last class, we know that $A \subseteq C$ or $A \subseteq D$. WLOG, assume $A \subseteq C$. Therefore, $\bar{A} \subseteq \bar{C}$. Since D is open, $\bar{C} \subseteq D^c$, so we see $\bar{C} \cap D = \emptyset$. Therefore, $B \subseteq C$ and $D = \emptyset$, a contradiction. □

Proof. of Proposition 9.1: Lemma 9.2 implies that $\overline{U_x}$ is a connected set containing x. But U_x is defined to be the largest open set containing x. Therefore, $U_x = \overline{U_x}$ implying that U_x is closed. Therefore, if we are able to write

$$X = U_1 \cup U_2 \cup \ldots \cup U_n$$

with each U_i a connected component (and thus closed), we have

$$U_i = (U_1 \cup \ldots U_{i-1} \cup U_{i+1} \cup \ldots \cup U_n)^c$$

Since a finite union of closed sets is closed, we see U_i is open. □

Before moving to a few examples, I include some useful properties of connectedness relative to continuity and products.

Proposition 9.3. *Let $f : X \to Y$ be a continuous map of topological spaces and $Z \subseteq X$ be a connected subset. Then $f(Z)$ is connected.*

Proof. Let $f(Z) = U \cup V$ be a separation. Then

$$f^{-1}(Z) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

is a separation of the preimage, and since $Z \subseteq f^{-1}(f(Z))$, we can conclude

$$Z = (f^{-1}(U) \cap Z) \cup (f^{-1}(V) \cap Z)$$

but Z is connected, so WLOG we can assume $f^{-1}(V) \cap Z = \emptyset$. But this implies $f(Z) \cap V = \emptyset$, contradicting the separation. □

Note that the same cannot be said about the pre-image.

Example 9.4. Consider $f : \mathbb{R} \coprod \mathbb{R} \to \mathbb{R}$ given by sending each copy of $\mathbb{R}$ in the domain to $\mathbb{R}$ via the identity. Than the preimage of any non-empty connected subset is *not* connected.

For a less obvious example, consider the connected subset $\Gamma \subseteq \mathbb{R}^2$ which is the graph of $y = x^2$. Γ is a connected subset, since it is the image of $\mathbb{R}$ under the continuous map $x \mapsto (x, x^2)$. Now, if we consider the projection map $\pi : \Gamma \to \mathbb{R} : (x, y) \mapsto y$. This is continuous since it is a projection map. Taking the connected subset $(0, \infty) \subseteq \mathbb{R}$, it's preimage is given by

$$\{(\sqrt{y}, y) \mid y > 0\} \cup \{(\sqrt{y}, y) \mid y < 0\}$$

which forms a separation of Γ, thus disconnected.

Proposition 9.5. *If $X_1, X_2, \ldots, X_n$ are connected spaces, then so is $X_1 \times \cdots \times X_n$.*

Proof. I proceed by induction. For the base case, $n = 1$ is trivial so we consider the product of 2 connected spaces $X \times Y$. Consider for a specific choice of $x \in X$ and $y \in Y$, the T-shaped set

$$T_{x,y} = \{x\} \times Y \cup X \times \{y\}$$

Since Y (resp. X) is connected and homeomorphic to $\{x\} \times Y$ (resp $X \times \{y\}$), it is also connected. This is because it is the image of the map you showed is continuous in Homework 2! Finally, since $(x, y) \in \{x\} \times Y \cap X \times \{y\}$, we see $T_{x,y}$ is connected by Proposition 8.5. Now, if we let $x \in X$ vary, we get

$$X \times Y = \bigcup_{x \in X} T_{x,y}$$

Note that for fixed $y \in Y$ every $T_{x,y}$ contains (x, y) for any $x \in X$. Therefore, again by Proposition 8.5, we can conclude their union is connected.

For induction, we view $X_1 \times \cdots \times X_n$ as $(X_1 \times \ldots \times X_{n-1}) \times X_n$, and apply the logic of the base case. $\square$

The natural question arises; can we extend this to arbitrary products? The answer is as usual is yes, but only in the product topology. The reason is that we can write every *2* open separating sets U_1, U_2 as a union of

$$U_i = U_i^{\alpha_1} \times \cdots \times U_i^{\alpha_n} \times \prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_i^{\alpha}$$

Then project onto $X_1 \times \cdots \times X_n$ without loss of information. In the box topology, the same cannot be said.

Example 9.6. Consider $\mathbb{R}^{\mathbb{N}}$ with the box topology. Then we can let U be the set of bounded sequences and V be the set of unbounded sequences. Note that a sequence is bounded if there exists N such that $|a_i| < N$ for all $i \in \mathbb{N}$. Therefore, U and V are necessarily disjoint. Furthermore, if $a = (a_1, a_2, \ldots) \in U$, then so is

$$(a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \ldots \in U$$

These sequences can be bounded by $N + 1$ if a is bounded by N. The same applies to V with the same choice of open neighborhood. This shows $\mathbb{R}^{\mathbb{N}} = U \cup V$ is a separation, and thus $\mathbb{R}^{\mathbb{N}}$ is disconnected.

This is part of a general phenomenon; namely if $\tau \supset \tau'$ is a finer topology, than τ-connectedness implies τ'-connectedness, but not vice-versa.