CLASS 10, MONDAY MARCH 5TH: TENSOR PRODUCTS II

Recall last time we ended talking about the localization of a module. I will now begin with a few example of such phenomena:

Example 0.1. Consider the $\mathbb{Z}$-module $M = \mathbb{Z}/6\mathbb{Z}$. We can consider the localization at the prime ideal $\langle 2 \rangle$:

$$M_{\langle 2 \rangle} = \mathbb{Z}/6\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}_2$$

Now, 3 is a unit in $\mathbb{Z}$, as $3 \notin \langle 2 \rangle$. Therefore, we conclude

$$2 \otimes 1 = 2 \otimes 3 \cdot \frac{1}{3} = 6 \otimes \frac{1}{3} = 0 \otimes \frac{1}{3} = 0$$

By a similar measure, $n \otimes \frac{2^m}{l} = 0$ for any $n, m > 0$ and $l \notin \langle 2 \rangle$. So we are left exactly with 2 elements (up to the tensor equivalence relations): 0 and $1 \otimes 1 = 3 \otimes 1 = 5 \otimes 1$ (since they all differ by $2 \otimes 1 = 0$). Therefore, we conclude $M_{\langle 2 \rangle} = \mathbb{Z}/2\mathbb{Z}$.

There is an often used and more high level way of seeing facts such as this:

Theorem 0.2 (Universal Property of Localization)**.** *If $\varphi : M \to N$ is a homomorphism of modules, and all $w \in W \subseteq R$ act as invertible elements on N, then φ factors as*

$$M \to W^{-1}M \to N$$

where the first map is the localization map, and $(w, m) \mapsto w^{-1}m$ is the second map.

Proof. See Homework 3. □

As a result, the localization of a module can be thought of as containing all of the information of M when mapping to modules N of the type in the Theorem.

Now I will return to the tensor product. One benefit of the tensor product is that it plays particularly nicely with direct sums of modules. This can be seen as a generalization of Homework 2, number 8.

Theorem 0.3. *Suppose M, N, P are all R-modules. Then there is a natural isomorphism*

$$(M \oplus N) \otimes_R P \cong (M \otimes_R P) \oplus (N \otimes_R P)$$

In addition, $M \otimes_R N \cong N \otimes_R M$ via $m \otimes n \mapsto n \otimes m$.

Proof. Let $\Phi : (M \oplus N) \otimes_R P \to (M \otimes_R P) \oplus (N \otimes_R P)$ be defined by

$$\Phi((m, n) \otimes p) = (m \otimes p, n \otimes p)$$

and extend by linearity. This is well defined since it respects all of the equivalence relations of the tensor product. It is a ring homomorphism for the same reason. It goes to find an inverse mapping:

$$\Psi : (M \otimes_R P) \oplus (N \otimes_R P) \to (M \oplus N) \otimes_R P$$

$$\Psi(m \otimes p, n \otimes p') = (m, 0) \otimes p + (0, n) \otimes p'$$

Extend by linearity and note this is also well defined. Now, we can compute

$$\begin{aligned}\Psi\left(\Phi(\sum_i (m_i, n_i) \otimes p_i\right)) &= \Psi((\sum_i m_i \otimes p, \sum_i n_i \otimes p)) \\ &= \sum_i (m_i, 0) \otimes p_i + (n_i, 0) \otimes p \\ &= \sum_i (m_i, n_i) \otimes p_i\end{aligned}$$

It is left as an exercise to prove $\Phi \circ \Psi = Id$, and that $M \otimes_R N \cong N \otimes_R M$. ☐

An immediate corollary is the exercise:

Corollary 0.4. *If F is a free module, $F \otimes_R M \cong R^n \otimes_R M \cong (R \otimes_R M)^n = M^n$.*

To finish off this lesson, I would like to add a note about tensor products of algebras.

Proposition 0.5. *Let R be a ring, and A, B be R-algebras. Then $A \otimes_R B$ has the structure of an R-algebra.*

Proof. We already know that $A \otimes_R B$ is an R-module. So it only goes to put a multiplicative structure on it, and check that R is in the center.

The desired multiplicative structure is

$$(a \otimes b) \cdot (a' \otimes b') := (aa') \otimes (bb')$$

and extending by linearity (thus it is naturally distributive). It also respects the equivalence relation of the tensor:

$$(ra \otimes b) \cdot (a' \otimes b') = raa' \otimes bb' = aa' \otimes rbb' = (a \otimes rb) \cdot (a' \otimes b')$$

Finally, since R is in the center of A and B (because they are themselves algebras), we see that

$$r \cdot (a \otimes b) = ra \otimes b = ar \otimes b = a \otimes rb = a \otimes br = (a \otimes b) \cdot r$$

Therefore R is in the center of $A \otimes B$. ☐

Example 0.6. Given two finitely generated R-algebras

$$A = R[x_1, \dots, x_n]/I = R[x_1, \dots, x_n]/\langle f_1, \dots, f_k\rangle$$
$$B = R[y_1, \dots, y_m]/J = R[y_1, \dots, y_m]/\langle g_1, \dots, g_l\rangle$$

The tensor product is also an R-algebra given by

$$A \otimes_R B = R[x_1, \dots, x_n, y_1, \dots, y_m]/\langle f_1, \dots, f_k, g_1, \dots, g_l\rangle$$

This can be seen by the isomorphism

$$A \otimes_R B \to R[x_1, \dots, x_n, y_1, \dots, y_m]/\langle f_1, \dots, f_k, g_1, \dots, g_l\rangle : f \otimes g \mapsto f \cdot g$$

extended by linearity. Indeed, we can define an inverse by

$$\sum_{\alpha,\beta} r_{\alpha,\beta} \cdot x^\alpha y^\beta \mapsto \sum_{\alpha,\beta} r_{\alpha,\beta}(x^\alpha \otimes y^\beta)$$

Here $r_{\alpha,\beta} \in R$ and α, β are multi-indexes; $\alpha \in \mathbb{N}^n$ and $\beta \in \mathbb{N}^m$.