CLASS 23, APRIL 15TH: NOETHERIAN TOPOLOGIES

Today we will cover decomposition into irreducible subvarieties, an accompanying result to our intersection of hypersurfaces result from last time. To begin, I would like to demonstrate that the notion of Noetherian for rings translates nicely into a statement for topological spaces.

Definition 23.1. A topological space is said to be **Noetherian** if either

- Every descending chain of closed subsets must eventually stabilize:
$$V_1 \supseteq V_2 \supseteq \ldots \supseteq V_N = V_{N+1} = \ldots$$
That is to say that closed subsets have the D.C.C.
- Every non-empty set of closed subsets has a minimal element, one containing no other closed subset properly.

Checking that these 2 properties are equivalent is identical to the methods for Noetherian/Artinian rings previously; use Zorn's Lemma. As a result, we get the following:

Proposition 23.2. *The Zariski topology is Noetherian for K^n.*

Proof. The DCC for varieties translates to the ACC for ideals by the inclusion reversing property of the function (functor) I. Then the Hilbert basis theorem allows us to conclude that $K[x_1, \ldots, x_n]$ is a Noetherian ring, and thus the ascending chain of ideals stabilizes. Applying V again and noting that $V(I(X)) = X$ if X is a variety shows the desired statement. $\square$

Next time we will talk about a slightly different style of decomposition using our new notion of irreducible varieties.

Proposition 23.3. *If $X \subseteq K^n$ is a variety, then X can be decomposed as*
$$X = X_1 \cup \ldots \cup X_n$$
where X_i is an irreducible variety. We can make such a decomposition unique by forcing $X_i \nsubseteq X_j$ for any $i \neq j$.

Proof. Let $\mathcal{S}$ be the set of varieties that have no such decomposition. If $\mathcal{S} = \emptyset$, we are done. By the Noetherian property, if $\mathcal{S} \neq \emptyset$, then $\mathcal{S}$ contains a minimal element X. X clearly can't be irreducible, because then $X = X$ is the correct decomposition. Therefore $X = X_1 \cup X_2$ for 2 smaller varieties X_i. These are not in $\mathcal{S}$ by minimality, so each have a finite decomposition. This contradicts the fact that $X \in \mathcal{S}$.

For the uniqueness statement, suppose $X = X_1 \cup \ldots \cup X_n = X_1' \cup \ldots \cup X_m'$. Since each X_i and X_j' are irreducible, we can conclude $X_i \subseteq X_j' \subseteq X_{i'}$ for some j' depending on i and i' depending on j. This can be seen since
$$X_i \subseteq X = X_1' \cup \ldots \cup X_m' \qquad \Longrightarrow \qquad X_i = (X_1' \cap X_i) \cup \ldots \cup (X_m' \cap X_i)$$
Since the right hand side is a union of closed sets (topology!), this would imply X_i is non-irreducible. But this would imply that $X_i \subseteq X_{i'}$, which would mean $X_i = X_j' = X_{i'}$, as asserted. $\square$

By our correspondence of irreducible varieties with prime ideals, we get the following neat corollary:

Corollary 23.4. *If $J \subseteq K[x_1, \ldots, x_n]$ is a radical ideal, then J is an intersection of* **finitely** *many prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_m$.*

Proof. Applying V yields

$$V(J) = X_1 \cup \cdots \cup X_m = V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_m)$$

Applying I then yields

$$J = I(V(J)) = I(V(\mathfrak{p}_1) \cup \cdots \cup V(\mathfrak{p}_m)) = I(V(\mathfrak{p}_1)) \cap \cdots \cap I(V(\mathfrak{p}_m)) = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_m$$

□

This result is a very important special case of the **associated primary decomposition** which we will discuss in chapter 7 of Reid.

Corollary 23.5. *If J is a radical ideal, and $J = \mathfrak{p}_1 \cap \ldots \cap \mathfrak{p}_m$ is the resulting minimal decomposition from Corollary 23.4, then the set of minimal primes of R/J is exactly $\mathfrak{p}_1 R/J, \ldots, \mathfrak{p}_m R/J$. Therefore the fraction ring of R/J is given as*

$$\mathrm{Frac}(R/J) = \mathrm{Frac}(R/\mathfrak{p}_1) \times \cdots \times \mathrm{Frac}(R/\mathfrak{p}_m)$$

Where each $\mathrm{Frac}(R/\mathfrak{p}_i)$ is simply the fraction field of an integral domain.

By minimal, I mean that there are not redundant choices of prime ideal.

Proof. Since each $\mathfrak{p}_i \supseteq J$, we have $\cup_i V(\mathfrak{p}_i) \subseteq V(J)$. Similarly, if $\mathfrak{p} \supseteq J$ is prime, then $\mathfrak{p} \supseteq \mathfrak{p}_i$ for some i. This follows by our standard trick; By induction, it suffices to prove the case of $m = 2$. Note that

$$\mathfrak{p}_1 \cdot \mathfrak{p}_2 \subseteq \mathfrak{p}_1 \cap \mathfrak{p}_2 \subseteq \mathfrak{p}$$

Then $\mathfrak{p}_1 \subseteq \mathfrak{p}$ or $\mathfrak{p}_2 \subseteq \mathfrak{p}$. Indeed, if $a \in \mathfrak{p}_1 \setminus \mathfrak{p}$, then $ab \in \mathfrak{p}$ for all $b \in \mathfrak{p}_2$ implies $b \in \mathfrak{p}$ by primality. Finally, since $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for all $i \neq j$, this completes the proof.

□

Example 23.6. Consider the ideal $J = \langle x^2yz^n - z^{n+2} \rangle \subseteq K[x, y, z]$ for some $n \geq 1$. It can be checked that $\sqrt{J} = \langle x^2yz - z^3 \rangle$. Its decomposition into primes can be represented by

$$J = \langle x^2y - z^2 \rangle \cap \langle z \rangle$$

The ideal on the left represents the 'Whitney Umbrella'.

Example 23.7. Consider the ideal

$$J = \langle y^2w, x\,y\,w, x^2w, y^2z, x\,y\,z, x^2z \rangle \subseteq K[x, y, z, w]$$

Computing its corresponding decomposition, we can actually note that

$$J = \langle x, y \rangle^2 \cap \langle u, v \rangle \qquad \Longrightarrow \qquad \sqrt{J} = \langle x, y \rangle \cap \langle u, v \rangle$$

Geometrically, this implies that $V(J)$ looks like $V(\langle x, y \rangle) \cup V(\langle z, w \rangle)$. On the right we have the maximal ideals of $K[u, v]$ and $K[x, y]$ respectively, obtained by modding out by the corresponding ideals. Thus it is a union of K planes meeting at a single point $(0, 0, 0, 0) \in K^4$.