# CLASS 22, APRIL 12TH: THE ZARISKI TOPOLOGY

Today we will take a dive into topology to develop the fundamentals of a geometric space. This will allow us to properly frame our notion of a variety in the broader geometric landscape. Recall that we left off with

**Proposition 1.** *$X$ is irreducible if and only if $I(X)$ is prime:*

$$\textit{Spec}(K[x_1,\dots,x_n]) = \{J \subseteq K[x_1,\dots,x_n] \text{ prime}\} \longleftrightarrow \{V = V(J) \subseteq K^n \text{ irreducible}\}$$

*Proof.* Suppose $I(X)$ is not prime. Then there exist $f, g \notin I(X)$ such that $fg \in I(X)$. Consider the ideals $I(X) + \langle f\rangle$ and $I(X) + \langle g\rangle$. Then we get that

$$X = V(I(X)) \supseteq V(I(X) + \langle f\rangle) \cup V(I(X) + \langle g\rangle)$$

On the other hand, if $\mathfrak{m}$ is a maximal (or even prime) ideal containing $I(X)$, then it necessarily contains either $f$ or $g$ by primality. As a result, we get the reverse inclusion and realize $X$ as a union of 2 proper subvarieties. Note they are proper since $f, g \notin I(X)$.

If $X = X_1 \cup X_2$ is reducible. Let $I = I(X)$ and $I_i = I(X_i)$. Then the Lemma 22.1 (following this proof) allows us to conclude the result. Indeed,

$$(I(X) + \langle f\rangle) \cdot (I(X) + \langle g\rangle) = I(X)^2 + fI(X) + gI(X) + \langle fg\rangle \subseteq I(X)$$

or rephrased

$$V(I(X) + \langle g\rangle) \cup V(I(X) + \langle g\rangle) = V((I(X) + \langle f\rangle) \cdot (I(X) + \langle g\rangle)) \supseteq V(I(X)) = X$$

□

**Lemma 22.1.** *If $I$ and $J$ are ideals of $K[x_1,\dots,x_n]$, then*

$$V(I \cdot J) = V(I \cap J) = V(I) \cup V(J)$$

$$V(I + J) = V(I) \cap V(J)$$

*Proof.* For the first part, '$\supseteq$' follows by the inclusion reversing property of $V$. On the other hand, if a maximal ideal $\mathfrak{m}$ containing $I \cdot J$ does not contain $I$, then there exists $f \in I \setminus \mathfrak{m}$. For every $g \in J$, this implies $f \cdot g \in \mathfrak{m}$, and therefore $g \in \mathfrak{m}$ by primality. This is to say $\mathfrak{m} \supseteq J$, thus yielding the '$\subseteq$' containment.

For the second part, I will actually prove that the result holds for general sums (not only of 2 ideals). Recall that $\sum_\alpha I_\alpha$ is the smallest ideal containing each $I_\alpha$. Therefore, by the inclusion reversing property of $V$ we see that $V(\sum_\alpha I_\alpha) \subseteq V(I_\alpha)$ for all $\alpha$. This gives the '$\subseteq$' direction.

Now suppose $\mathfrak{m} \in \bigcap_\alpha V(I_\alpha)$. This states that $\mathfrak{m}$ is an ideal containing all of the $I_\alpha$. Thus by our description above, we get that $\mathfrak{m} \in V(\sum_\alpha I_\alpha)$, as desired. □

**Corollary 22.2.** *Every variety is of the form*

$$V(I) = V(\langle f_1,\dots,f_m\rangle) = V(\langle f_1\rangle) \cap \ldots \cap V(\langle f_m\rangle)$$

Sometimes $V(\langle f_i\rangle)$ is abbreviated to $V(f_i)$. It is commonly called a **hypersurface**.

**Example 22.3.** Let's check out what our tools imply about $V(I) \subseteq K^2$ for $I = \langle x(y-1), x^2 - 5y^2\rangle$. Let's assume $K$ is algebraically closed (or at least contains $\sqrt{5}$). We see that

$$V(I) = V(x(y-1)) \cap V(x^2-5y^2) = (V(x) \cup V(y-1)) \cap \Big(V(x-\sqrt{5}y) \cup V(x+\sqrt{5}y)\Big)$$

As a result, we see that $V(I) = \{(0,0), (-\sqrt{5}, 1), (\sqrt{5}, 1)\}$ is composed of 3 points.

Lemma 22.1 gives us a natural relationship with topological spaces from geometry:

**Definition 22.4.** A **topological space** is a set $X$ together with a collection $\tau \subseteq \mathcal{P}(X)$ (the power set of $X$, i.e. the set of all subsets of $X$) satisfying the following criteria:

(a) $X, \emptyset \in \tau$
(b) If $X_1, \ldots, X_n \in \tau$, then $X_1 \cup \ldots \cup X_n \in \tau$.
(c) If $\{X_\alpha \mid \alpha \in \Lambda\}$ is any collection of elements of $\tau$ ($\Lambda$ is any set, be it uncountable or not), then so is their intersection

$$\bigcap_\alpha X_\alpha \in \tau$$

An element of $\tau$ is called a **closed subset** of $X$.

The conditions can be labeled as such: (a) is a minimum size criteria for a topology, making it so that there are at least 2 closed sets. (b) states finite unions of closed sets are closed, and (c) states any intersection of closed sets remains closed. This should pair well with your intuition of closed sets from real analysis.

There are also corresponding ways to define a topology based on the **open sets** of $X$, which are precisely the complements of closed sets.

**Corollary 22.5.** *The set of varieties*

$$\tau = \{V(J) \mid J \subseteq K[x_1, \ldots, x_n] \text{ an ideal}\}$$

*induces a topology on* $K^n$.

*Proof.* Properties (b) & (c) are guaranteed by Lemma 22.1. The only thing to note is that $X = V(0)$ and $\emptyset = V(K[x_1, \ldots, x_n])$. □

**Definition 22.6.** The topology described in Corollary 22.5 is the **Zariski Topology**.

One thing to note is that this gives a VERY different interpretation of closed sets from that of Real Analysis. In particular, an 'open ball' of some radius is very much not open in the Zariski Topology. The following example allows us to see that open sets in the Zariski topology are open in the Euclidean Topology:

**Example 22.7.** If $(a_1, \ldots, a_n) \notin V(I)$, then this is simply saying that if $I = \langle f_1, \ldots, f_m\rangle$, then there is some $f_i$ such that $f_i(a_1, \ldots, a_n) \neq 0$. But since polynomials are continuous in the Euclidean topology, we know that there is a ball $B$ of some radius about $(a_1, \ldots, a_n)$ such that $f_i(b_1, \ldots, b_n) \neq 0$ for every $(b_1, \ldots, b_n) \in B$. As a result, we can conclude that $V(I)$ is closed in the Euclidean topology.