CLASS 33, DECEMBER 3: HOMOTOPY THEORY

One of the central objectives of mathematics is to classify all objects of a given type, be it groups, rings, metric spaces, topological spaces, varieties, categories, etc. This problem is immense, and if solved would essentially complete a branch of mathematics.

In our world of topological spaces, we would like to classify spaces up to homeomorphism. Again, this is highly intractable. Even in specific cases, the problem is one of the most difficult ever solved:

Theorem 33.1 (The Poincaré Conjecture, Perelman's Theorem(?))**.** *If X is a simply-connected, compact, 3-manifold, then $X \cong S^3$.*

This was solved in 2005 and was awarded a Fields Medal as well as a Millennium Prize.

Instead of trying to prove results like this, it is often easier and more reasonable to develop tools to say when two spaces are non-homeomorphic.

An initial step in this direction is to develop a weaker notion of 2 topological spaces being equivalent. This is the idea of homotopic spaces. To do this, we need to develop a few intermediate notions.

Definition 33.2. Let $f, g : X \to Y$ be continuous maps. f and g are said to be **homotopic** if there exists a continuous map

$$F : X \times I \to Y$$

with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. This is usually written $f \simeq g$. We say that the maps are **homotopic relative** Z, where $Z \subseteq X$, if additionally $F(z, t) = f(z)$ for all $t \in I$ and $z \in Z$. This is written $f \simeq g$ rel Z.

The idea of such a thing is as follows: we can continuously deform f to g by an interval worth of intermediate maps $f_t(x) = F(x, t)$. The idea of a relative homotopy is to keep a particular region fixed as t varies.

Example 33.3. Let S^1 be parameterized by $t \in [0, 1]$ with $0 = 1$. Then we can consider maps

$$\theta_m : S^1 \to S^1 : s \mapsto ms$$

$$\theta_{m,n} : S^1 \to S^1 : s \mapsto ms + n \sin(2\pi s)$$

This is the map which winds S^1 around m-times. These maps are homotopic relative to $\{0, 1\} \subseteq [0, 1]$. The homotopy can be given explicitly as

$$\Theta_{m,n} : S^1 \times I \to S^1 : (s, t) \mapsto ms + tn \sin(2\pi s)$$

Note that $\Theta_{m,n}(s, 0) = \theta_m(s)$, $\Theta_{m,n}(s, 1) = \theta_{m,n}(s)$, $\Theta_{m,n}(0, t) = 0$, and $\Theta_{m,n}(1, t) = 0$.

After describing homotopic maps, we can describe spaces as being homotopic:

Definition 33.4. If X and Y are topological spaces, then X is said to be **homotopic** to Y if there exists maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq Id_X$ and $f \circ g \simeq Id_Y$.

Note now that it is now automatic that two homeomorphic spaces are homotopic. Indeed, homeomorphisms require that $f \circ g$ and $g \circ f$ are equal to the identity. Therefore, we can take the **constant** homotopy maps $F(x, t) = g(f(x))$ and $G(x, t) = f(g(x))$.

Example 33.5. I claim that $S^1 \simeq \mathbb{R}^2\backslash\{0\}$. We have the natural inclusion map $f : S^1 \to \mathbb{R}^2\backslash\{0\}$ and the map $g : \mathbb{R}^2 \setminus \{0\} \to S^1 : x \to \frac{x}{\|x\|}$ was constructed on a homework as a retraction. Note that $g \circ f = Id_{S^1}$. On the other hand, the map $f(g(x)) = \frac{x}{\|x\|}$.

It suffices to construct a homotopy $\frac{x}{\|x\|} \simeq Id_{\mathbb{R}^2\backslash\{0\}}$. We can construct this using a typical idea of a **linear homotopy**:

$$F : \mathbb{R}^2 \setminus \{0\} \times I \to \mathbb{R}^2 \setminus \{0\} : (x,t) \mapsto t \cdot x + (1-t) \cdot \frac{x}{\|x\|} = \frac{(t\|x\| + (1-t))x}{\|x\|}$$

In general, a linear homotopy between $f : X \to Y$ and $g : X \to Y$ is

$$F : X \times I \to Y : (x,t) \mapsto t \cdot f(x) + (1-t) \cdot g(x)$$

where we are implicitly assuming Y is (a subspace of) a real vector space.

Proposition 33.6. *$\simeq$ and $\simeq$ rel Z are equivalence relations.*

Proof.
- **Reflexive:** The constant homotopy (independent of t) shows $f \simeq f$ and $f \simeq f$ rel Z for any Z.
- **Symmetric:** Suppose $F : X \times I \to Y$ is a homotopy connecting f to g (rel Z). Then we can consider $G(x,t) = F(x, 1-t)$. This shows $g \simeq f$ (rel Z).
- **Transitive:** Suppose $f \simeq g$ (rel Z) by F and $g \simeq h$ (rel Z) by G. We can create a third homotopy

$$H(x,t) = \begin{cases} F(x,2t) & t \leq \frac{1}{2} \\ G(x,2t-1) & t \geq \frac{1}{2} \end{cases}$$

This is continuous by the pasting lemma:

$$F(x,1) = G(x,0) = g(x)$$

□

The transitive portion of this result gives the idea for how the group operation in the fundamental group will be developed.

Definition 33.7. A **path** from x to y in X is a continuous map $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. If $x = y$, then γ is said to be a **loop based at** x.

Given 2 paths γ_1 from x to y and γ_2 from y to z, we define (perhaps abusively) the **composition** of the paths to be

$$\gamma_1 * \gamma_2 : I \to X : t \mapsto \begin{cases} \gamma_1(2t) & t \leq \frac{1}{2} \\ \gamma_2(2t-1) & t \geq \frac{1}{2} \end{cases}$$

This is a path from x to z.

Note that this is not itself a group operation; there exists no identity, inverses, nor is the operation associative. However, we can correct this by restricting our attention to loops and consider two homotopic loops to be equal. This is the idea of the fundamental group, which we will start next time.