CLASS 30, MAY 1ST: DISCRETE VALUATIONS

We will now move into the final chapter of Reid, which talks about valuation rings and normal domains. These are important classes of rings in commutative algebra, algebraic geometry, and number theory.

We have seen many examples of discrete valuation rings previously. I now recall some of them to motivate the definition that will follow.

Example 30.1 (Localizations of PIDs)**.** The rings $\mathbb{Z}_{\langle p \rangle}$ and $K[x]_{\langle x \rangle}$ are both local domains. A further thing to note is the following: the maximal ideal is principal, generated by $t = p$ in the first case and $t = x$ in the latter. This gives us a very interesting interpretation for what elements of these rings look like; $f = u \cdot t^n$ for some $n \geq 0$ and u a unit.

Example 30.2 (Power Series in 1 Variable)**.** The same story applies to $K[\![x]\!]$, or the subring of convergent power series. We showed in Homework 3, #4, that every element of a power series ring in any number of variables is a unit if and only if it has non-zero constant coefficient. Here this means that every element can be expressed as $f = u \cdot x^n$ for some n and u a unit.

The enjoyment of rings with such properties brings us to the following definition;

Definition 30.3. Let K be a field. A **discrete valuation** is a surjective map $v : K^\times \to \mathbb{Z}$ such that the following properties hold $\forall x, y \in K$:

(a) $v(x \cdot y) = v(x) + v(y)$.
(b) $v(x \pm y) \geq \min\{v(x), v(y)\}$.

As a convention, we let $v(0) = \infty$ to extend the valuation to all of K, and to further ensure that the above properties are satisfied.

Example 30.4. Consider the field $L = K((x))$ of formal Laurent series with coefficients in K. It's elements look like

$$f = \sum_{i=m}^{\infty} a_i x^i$$

where $m \in \mathbb{Z}$. L is a field. This allows us to consider the valuation

$$v : L \to \mathbb{Z} : f \mapsto \inf\{m \mid a_m \neq 0\}$$

This is a valuation; the smallest non-zero coefficient of a product $f \cdot g$ is $a_m b_n x^{m+n}$, where $f = \sum_{i=m}^{\infty} a_i x^i$ and $g = \sum_{i=n}^{\infty} b_i x^i$. For addition, the smallest coefficient of $f + g$ is either a_m, b_n, or $a_m + b_m$ if $m = n$ and $a_m \neq b_m$ (otherwise it is larger degree than m).

Definition 30.5. The **valuation ring** associated to a valuation $v : K^\times \to \mathbb{Z}$ is

$$R = R_v = \{a \in K \mid v(a) \geq 0\}$$

Proposition 30.6. *R_v is a local ring with maximal ideal*

$$\mathfrak{m}_v = \{a \in K \mid v(a) > 0\}$$

Proof. First note that R is a ring. $0 \in R$ by our convention, and $1 \in R$ since

$$v(1) = v(1 \cdot 1) = v(1) + v(1) \qquad \Longrightarrow \qquad v(1) = 0$$

It is also closed under addition and multiplication by the axioms for a valuation, and distribute since it is a subset of a field.

Finally, it goes to show $\mathfrak{m}_v$ is the unique maximal ideal. Every unit in R_v is a unit in K (duh!). But we have that

$$v(x) + v(x)^{-1} = v(x \cdot x^{-1}) = v(1) = 0$$

which is to say that $v(x^{-1}) = -v(x)$. So $x, x^{-1} \in R_v$ if and only if $v(x) = 0$. This shows everything outside of $\mathfrak{m}_v$ in R_v is a unit. $\square$

Example 30.7. Continuing with Example 30.4, we see that $R_v = K[\![x]\!]$ and $\mathfrak{m}_v = \langle x \rangle$. This is the unique maximal ideal of R_v by the previous discussion.

Example 30.8. We can consider

$$v : \mathbb{Q}^\times \to \mathbb{Z} : \frac{a}{b} = p^m \frac{a'}{b'} \mapsto m \qquad \text{where } p \nmid a', b'$$

This is also a valuation, typically called the p-adic valuation on $\mathbb{Q}$. One can verify that $R_v = \mathbb{Z}_{\langle p \rangle}$ and $\mathfrak{m}_v = \langle p \rangle$. One can also extend v to $\mathbb{Q}_p$, the p-adic rationals, to yield R_v the p-adic integers and $\mathfrak{m}_v = \langle p \rangle$.

Proposition 30.9. *For a discrete valuation $v : K^\times \to \mathbb{Z}$, $\mathfrak{m}_v = \langle t \rangle$ is a principal ideal. In fact every non-zero ideal is of the form $I = \langle t^n \rangle$ for some $n \in \mathbb{N}$. In particular, DVRs are Noetherian.*

Proof. Pick any element $t \in R_v$ such that $v(t) = 1$. If $s \in \mathfrak{m}_v$ is another, then in $K^\times$ we have that there exists u such that $t = us$. But then

$$1 = v(t) = v(us) = v(u) + v(s) \leq v(u) + 1$$

which is to say that $v(u) \geq 0$, i.e. $u \in R_v$. Therefore $s \in \langle t \rangle$.

The statement for I is identical, taking t^n to be your element with valuation n the smallest among elements of I. $\square$

This is actually very strong. It says that the proper ideals of R_v are in bijection with positive integers $n \in \mathbb{N}$.

Definition 30.10. t as in Proposition 30.9 is called a (sometimes **uniformizing**)**parameter** for v.

As we can see, there are some natural choices of a uniformizing parameter in the above examples, such as p and x. However, such a choice is clearly non-unique, as we can multiply it by any unit that exists within our ring (not the field).

Next time, we will study some equivalent formulations of being a DVR. For the interested student, I encourage you to check out the opening of the wiki page:

`https://en.wikipedia.org/wiki/Discrete_valuation_ring`