CLASS 17, OCTOBER 23: ESSENTIAL SINGULARITIES

We now have produced a way to subclassify all types of singularities. They must fall into one of the following buckets:

1) **Removable singularities**: Fixable without modification
2) **Poles**: Fixable by multiplication by $(z - z_0)^m$
3) **Essential Singularities**: Not fixable $(z - z_0)^m$.

These words are quite loose, but Corollary 16.4 from last class really firms up this understanding. As a result, one may ask what can we say about essential singularities. One interesting observation is the following:

Theorem 17.1 (Casorati-Weierstass)**.** *If f is holomorphic near z_0 and has an essential singularity at z_0, then $f(B_*(z_0, r)) \subseteq \mathbb{C}$ is dense, where the $*$ indicates without its center.*

$B_*(z_0, r)$ is often called the **punctured disc**. This shows just how wild these essential singularities are: any neighborhood however small with fill up the entire complex plane up to closure!

Proof. Suppose the assertion is false. This is equivalent to saying there exists w and δ such that

$$B(w, \delta) \subseteq \mathbb{C} \setminus f(B(z_0, r))$$

This allows us to consider a new function on $B_*(z_0, r)$:

$$g(z) = \frac{1}{f(z) - w}$$

Note that $g(z)$ is bounded above by $\frac{1}{\delta}$ and is furthermore holomorphic on its domain. As a result of Riemann's theorem (Theorem 16.2, we get that $g(z)$ has a removable singularity at z_0. If $g(z_0) \neq 0$, then $f(z) - w$ is holomorphic at z_0. This is impossible since f has an essential singularity there and w is just a constant. If $g(z_0) = 0$, then $\lim_{z \to z_0}(|f(z) - w|) = \infty$, which implies it is a pole by Corollary 16.4. We have reach a contradiction. $\square$

It should be noted that Picard proved in fact that f takes on each complex value infinitely often with the exception of a single point. So at the very least, the image of the punctured disc misses a single point!

Example 17.2. Examining again our essential singularity $e^{\frac{1}{z}}$, we claim it hits every point except 0. Suppose $w = re^{i\theta} \in \mathbb{C}$ with $r > 0$. Then we note

$$re^{i\theta} = e^{\frac{1}{z}} = e^{\frac{1}{R}e^{-i\phi}} = e^{\frac{1}{R}\cos(\phi)}e^{-i\frac{1}{R}\sin(\phi)}$$

This gives us a set of 2 real valued equations:

$$r' = \ln(r) = \frac{1}{R}\cos(\phi)$$

$$\theta = \frac{1}{R}\sin(\phi) \pmod{2\pi}$$

For any fixed choice of $m \in \mathbb{N}$, and r', we can find a unique $R > 0$ and $\phi \in (-\pi, \pi]$ such that the equations above hold with $\theta + 2\pi m = \frac{1}{R}\sin(\phi)$ (think of them as points on a circle centered at the origin). Therefore there are infinitely many elements in the preimage of any non-zero complex number, as Picard expects.

We can now turn to the function which I would deem best without being holomorphic.

Definition 17.3. $f : \Omega \to \mathbb{C}$ is called **meromorphic** if there exist at most countably many points $z_1, z_2, \dots$ without a limit point such that f is holomorphic for $z \neq z_i$, and f has a pole at z_i.

There is also a natural way to view the idea of being meromorphic on the **extended complex plane**. This is similar to the case where we adjoin ∞ to $\mathbb{R}$ and make it into a circle.

Definition 17.4. We define the extended complex plane, or the **Riemann sphere** to be $\mathbb{C} \cup \{\infty\}$. It is denoted $\mathbb{C}_\infty$.

It 'looks like' a sphere since we can do the stereographic projection to the complex plane for all values $\neq \infty$, which allows us to identify $\mathbb{C} \subseteq \mathbb{C}_\infty$ in a nice geometric way.

If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic for all large values of z, then we can study $F(z) = f(\frac{1}{z})$. This function is now holomorphic in a neighborhood of 0 with an isolated singularity at 0. Therefore we can say that f has a pole, or essential singularity, or a removable singularity (thus is holomorphic) at ∞ if F has those properties at 0.

The following is an excellent classification of the meromorphic functions on $\mathbb{C}_\infty$:

Theorem 17.5. *f is meromorphic on $\mathbb{C}_\infty$ if and only if f is a rational function.*

Proof. It is clear that rational functions are meromorphic (by clearing denominators). So suppose f is meromorphic. Then f must be either holomorphic or have a pole at ∞. In either case, it is holomorphic in a neighborhood of ∞. Therefore, f can only have finitely many poles in the plane since removing a neighborhood of ∞ yields a compact set. Call them $z_1, \dots, z_n$.

For each z_i, we can write

$$f(z) = p_i(z) + f_i(z)$$

where p_i is the principal part of f at z_i and f_i is holomorphic at z_i. Similarly, we can write

$$f\left(\frac{1}{z}\right) = \tilde{p}_\infty(z) + f_\infty(z)$$

Additionally, let $p_\infty(z) = \tilde{p}(\frac{1}{z})$. Combining this information, we assert that

$$H(z) = f(z) - p_\infty(z) - \sum_{i=1}^{n} p_i(z)$$

is a entire and bounded, thus constant by Liouville. Note first that subtracting off the principal part ensures that H has removable singularities at each z_i, so in particular is holomorphic there.

Additionally, subtracting off the principal part in each neighborhood yields that f is bounded in those neighborhoods, since f is continuous on a compact set. Finally, $\mathbb{C}$ without all these neighborhoods is a closed and bounded set, thus compact. As a result, f is everywhere bounded as claimed. $\square$