CLASS 13, MARCH 6TH: HILBERT BASIS THEOREM

Finally, we come to the standard tool to detect whether a ring is a Noetherian ring; The Hilbert Basis Theorem. It allows us to ensure that many of our common rings of interest are in fact Noetherian. Therefore, we needn't worry about infinitely generated ideals in these cases!

Theorem 13.1 (Hilbert Basis Theorem)**.** *If R is a Noetherian ring, then $R[x]$ is also Noetherian.*

Example 13.2. By induction, we can conclude that $K[x_1, \dots, x_n]$, $\mathbb{Z}[x_1, \dots, x_n]$, and any quotient or localization of such a ring is a Noetherian ring, where K is a field. This is most of the rings we have encountered which weren't specifically labeled as non-Noetherian!

It should be noted that the proof indicated here is a much cleaner version of the proof that Hilbert originally demonstrated:

Proof. Suppose $I \subseteq R[x]$ is an ideal. It suffices to check that I is finitely generated. We can define an auxiliary ideal of R as follows:

$$J_n = \{r \in R \mid f(x) = rx^n + a_{n-1}x^{n-1} + \ldots + a_0 \in I\}$$

That is to say J is the set of leading coefficients of elements of I. This is an ideal of R. In addition, this gives a chain of ideals in R:

$$I \cap R = J_0 \subseteq J_1 \subseteq J_2 \subseteq \ldots$$

This is due to the fact that we can multiply $f(x)$ as above by x^{m-n} to get the desired leading coefficient in J_m.

Now, we can use the fact that R is assume to be Noetherian to conclude that $J_N = J_{N+1} = \cdots$ for some $N \in \mathbb{N}$. Additionally, each J_i is finitely generated, say by $r_1^{(i)}, \dots, r_{m_i}^{(i)}$. Let $f_{j,i}$ for $i = 1, \dots, n$ and $j = 1, \dots, m_i$ be an element with leading coefficient $r_j^{(i)}$.

Finally, I claim that the collection $f_{j,i}$ generate I. Assume $f \in I$ has degree M.

Assume $M \geq N$. Then f has leading coefficient in $J_M = J_N$, so we can choose $f_{j,N}$ and $r_j \in R$ such that

$$f - \sum_{j=1}^{m_N} r_j x^{M-N} f_{j,N}$$

has degree smaller than M. But we can do this for ANY $M \geq N$. So we may assume that $M < N$ by continued reduction.

If $M < N$, then we know our leading coefficient is in J_M. But as a result, we can do the same trick, selecting $r_j \in R$ such that

$$f - \sum_{j=1}^{m_M} r_j f_{j,M}$$

has degree smaller than M. Once we reduce to the case of a 0 degree polynomial, we can conclude that the element is in $I \cap R = J_0$, and conclude that it is a finite sum of our generators $f_{j,0}$.

This procedure implies that any f is expressible as a finite sum of the elements $f_{j,i}$ showing $I = \langle f_{j,i} \rangle$. $\square$

Corollary 13.3. *Any finitely generated R-algebra A, with R Noetherian, can be expressed as*

$$A \cong R[x_1, \dots, x_n]/I$$

Additionally, the corresponding relations are finitely generated! I.e. the free resolution of A as an $R[x_1, \dots, x_n]$-module is given as

$$\dots \to R[x_1, \dots, x_n]^{\oplus m_1} \to R[x_1, \dots, x_n] \to A \to 0$$

Proof. Note that when we say generated as an R-algebra, we mean we allow multiplication of elements to generate new elements like x_1^2 or $x_1^{m_1} \cdots x_n^{m_n}$.

The only thing to say here is that since $R[x_1, \dots, x_n]$ is a Noetherian ring by Theorem 13.1, I is a finitely generated ideal. So the kernel of the generation map needs only finitely many generators.[1] $\square$

One can also perform a similar style of proof for power series rings. To finish off our study of the Noetherian property, we prove a result of I.S. Cohen:

Theorem 13.4. *If R is a ring in which every prime ideal is finitely generated, then R is Noetherian.*

This is pretty neat, since it seems like a vast weakening of the condition, but is enough to conclude the desired result.

Proof. Suppose R is not Noetherian. Consider the set

$$\mathcal{S} = \{I \subseteq R \mid I \text{ is not finitely generated}\}$$

Since R is not Noetherian, this set is non-empty. Suppose that

$$I_1 \subseteq I_2 \subseteq \cdots$$

is an ascending chain of ideals in $\mathcal{S}$. Then we already know that $I = \bigcup_{i \geq 1} I_i$ is an ideal, and furthermore it is not finitely generated. If it were, by $r_1, \dots, r_n$, then we could find i_j such that $r_j \in I_{i_j}$, and conclude that $I_{\max\{i_1, \dots, i_n\}}$ is a finitely generated ideal equal to I.

Therefore, $I \in \mathcal{S}$ is an upper bound, and therefore $\mathcal{S}$ contains a maximal element $\mathfrak{p}$ by Zorn's Lemma. I claim $\mathfrak{p}$ is a prime ideal.

Suppose $a \cdot b \in \mathfrak{p}$ but $a, b \notin I$. Then it must be true that $\langle a \rangle + \mathfrak{p}$ is a finitely generated ideal, say by a and $f_1, \dots, f_n \in \mathfrak{p}$. Moreover, if we consider

$$\mathfrak{p} : a = \{r \in R \mid ra \in \mathfrak{p}\}$$

is an ideal which contains b and $\mathfrak{p}$, so must also be finitely generated, say by $g_1, \dots, g_m$. I claim that this implies

$$\mathfrak{p} = \langle f_1, \dots, f_n, ag_1, \dots, ag_n \rangle$$

which would contradict $\mathfrak{p} \in \mathcal{S}$. Indeed, all of these elements are chosen inside $\mathfrak{p}$. Moreover, if $f \in \mathfrak{p} \subseteq \mathfrak{p} + \langle a \rangle$, then we know

$$f = r_1 f_1 + \ldots + r_n f_n + r_0 a$$

But $f_1, \dots, f_n \in \mathfrak{p}$, so $r_0 a \in \mathfrak{p}$, implying $r_0 \in \mathfrak{p} : a = \langle g_1, \dots, g_n \rangle$, which proves the claim. $\square$

[1] One should note that this continues to higher parts of the resolution.