Here is a list of all of the information pertaining to covering spaces, which you can use freely on the midterm.

1º A covering space is a continuous map $p : \tilde{X} \to X$ with the property that for every point $x \in X$, we have an open neighborhood $U$ containing $x$ with $p^{-1}(U) = \coprod_\alpha U$. In this document, I will denote all covering spaces in this way.

2º **Homotopy Lifting Property:** If $f_t : Y \to X$ is a family (homotopy) of maps, and $\tilde{f}_0 : Y \to \tilde{X}$ with $f_0 = p \circ \tilde{f}_0$, then $\exists \tilde{f}_t : Y \to \tilde{X}$ such that $f_t = p \circ \tilde{f}_t$.

Some simple cases are that in which $Y$ is assumed a point, and $f_t(y)$ is thus a path. So paths lift from $X$ to $\tilde{X}$. If $Y = I$, this gives the homotopy of paths lifting property.

3º 2 Corollaries that follow immediately are listed here:
- $p_*$ is an injective map. It's image in $\pi_1(X, x_0)$ are exactly loops at $x_0$ that lift to loops in $\tilde{X}$.
- If $X$ and $\tilde{X}$ are assumed PC, then $n_X = \#$ sheets of p $= [\pi_1(X, x_0) : p_*\pi_1(\tilde{X}, \tilde{x}_0)]$.

4º **The Lifting Criterion:** If $f : Y \to X$ is a map such that $f_*\pi_1(Y, y_0) \subseteq p_*\pi_1(\tilde{X}, \tilde{x}_0)$, then there exists $\tilde{f}$ factoring $f$: $p \circ \tilde{f} = f$.

Moreover, two lifts $\tilde{f}, \hat{f}$ that agree at a single point are equal everywhere.

5º The path connected covering spaces of $X = S^1$ are either $S^1$ with a map $\theta \mapsto \theta \cdot n$ for some integer $n$, or $\mathbb{R}$ the universal cover.

6º For $X$ any PC, LPC, SLSC (think locally simply connected) space, there exists $\tilde{X}_u$ a simply connected space, and a covering space $\tilde{p} : \tilde{X}_u \to X$.

7º This gives us that any subgroup is realized by a path connected covering space. If $H \subset \pi_1(X)$, we can consider $\tilde{X}_H = \tilde{X}_u / \sim_H$ where two path classes (points in $\tilde{X}_u$) are equivalent, $\gamma_0 \sim_H \gamma_1$, if and only if $\gamma_0 \cdot \overline{\gamma_1} \in H$. One can check $p_*\pi_1(\tilde{X}_H) \cong \pi_1(\tilde{X}_H) = H$.

8º Any two path connected covering spaces with the same fundamental group are isomorphic as covering spaces. This means that $\exists q : \tilde{X} \to \tilde{X}'$ a homeomorphism (obtained by the lifting criterion) such that $p = p' \circ q$. Therefore

$$\{\text{PC Covering Spaces } p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)\}/\cong \;\leftrightarrow \{\text{Subgroups of } \pi_1(X, x_0)\}$$

If we ignore basepoints, they differ by a change of basepoint map:

$$\{\text{PC Covering Spaces } p : \tilde{X} \to X\}/\cong \;\leftrightarrow \{\text{Subgroups of } \pi_1(X, x_0)\}/\text{Conjugation}$$

9º A deck transformation is exactly one of the isomorphisms of covering spaces listed above.

10º The group of Deck Transformations (yes, it is a group) is given by

$$G(\tilde{X}) = N(H)/H$$

where $H = \pi_1(\tilde{X})$ as above, and $N(H)$ is the normalizer of $H$ (all elements $g \in \pi_1(X)$ such that $gHg^{-1} \subseteq H$. Therefore, if the covering space is normal,

$$G(\tilde{X}) = \pi_1(X, x_0)/p_*\pi_1(\tilde{X}, \tilde{x}_0)$$

$11^{\text{o}}$ **Andrew's Why Covering spaces Are Great Theorem**

$$\{\text{Sheets of a covering space } p : \tilde{X} \to X\} \leftrightarrow [\pi_1(X, x_0) : p_*\pi_1(\tilde{X}, \tilde{x_0})]$$

$$\{p : (\tilde{X}, \tilde{x_0}) \to (X, x_0)\}/\cong \leftrightarrow \{\text{Subgroups } H \subseteq \pi_1(X, x_0)\}$$

$$\{p : \tilde{X} \to X\}/\cong \leftrightarrow \{\text{Conjugacy classes of Subgroups } gHg^{-1} \subseteq \pi_1(X, x_0)\}$$

$$G(\tilde{X}) = \{\text{Deck Transformations } \phi : \tilde{X} \to \tilde{X}\} = N(\pi_1(\tilde{X}))/\pi_1(\tilde{X})$$

Therefore, in the case that you have a normal cover, and thus a normal subgroup,

$$G(\tilde{X}) = \pi_1(X, x_0)/p_*(\pi_1(\tilde{X}, \tilde{x_0}))$$

So combining several of the above correspondences:

$$\{G(\tilde{X}) \;:\; p : \tilde{X} \to X \text{ a normal cover}\} \leftrightarrow \{\text{Quotients of } \pi_1(X, x_0)\}$$