HOMEWORK 7: HOMOTOPY
DUE: WEDNESDAY, NOVEMBER 6TH

(1) Using homotopy, give another proof of the Cauchy integral theorem:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw$$

As a hint, deform the original circle C to a small one around z and note that $\frac{f(w)-f(z)}{w-z}$ is bounded.

(2) Show that there are no holomorphic functions on $B(0, 1)$ which extend continuously to the boundary circle where they equal $f(z) = \frac{1}{z}$.

(3) Show that if f is entire and satisfies

$$\sup_{|z|=R} |f(z)| \leq AR^k + B$$

with $A, B > 0$, then f is a polynomial of degree at most k.

(4) Show if f is holomorphic in the unit disc, is bounded, and converges uniformly to 0 in a sector $\theta_1 < \arg(z) < \theta_2$ as $|z| \to 1$, then $f = 0$.

(5) Let $w_1, \ldots, w_n$ be points on the unit circle. Show that there is a point z on the unit circle with $d = \prod_i |z - w_i| \geq 1$. As a result, conclude that there is a point z_0 for which this product is exactly 1.

(6) Show that every **convex** set is simply-connected. A set is convex if and only if every 2 points in the space have their connecting line in the space.

Additionally, show (more generally) that a **star-shaped** space is simply-connected. This means there is at least 1 point point z_0 such that for any z in the space, the line connecting z_0 to z is in the space.