ALGEBRAIC TOPOLOGY MIDTERM

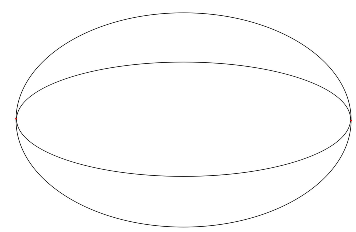
1) [5 pts] Define the fundamental group of a space X at basepoint x_0. Be precise about any equivalence relations involved.

The fundamental group, $\pi_1(X, x_0)$, is defined to be the set of loops γ based at x_0 ($\gamma(0) = \gamma(1) = x_0$) modulo homotopy of loops:

$$\gamma_0 \sim \gamma_1 \text{ rel } 0, 1$$

Expanding out this (which is unnecessary for a complete solution) states $\exists \Gamma : I \times I \to X$ with $\Gamma(0,t) = \Gamma(1,t) = x_0$, and $\gamma_i(t) = \Gamma(t,i)$ for $i = 0, 1$.

2) [10 pts] What is the fundamental group of the space X obtained by taking two circles and identifying 2 distinct points on one circle with 2 distinct points on the other?



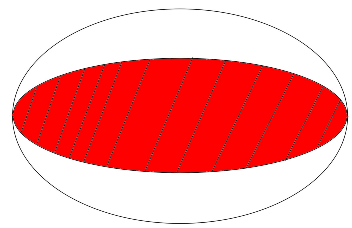
Call the lines between the 2 points a, b, c, d in order. One can contract b to a point homotopically. This makes X homotopic to $S^1 \vee S^1 \vee S^1$, and therefore, $\pi_1(X) \cong \mathbb{Z}^{*3} = F_3 = \langle a, c, d \rangle$.

3) [5 pts] If X is a topological space, and γ is a loop based at x_0 in X, what is the effect of adjoining a 2-cell by $e^2 = \gamma$ to $\pi(X, x_0)$?

By the Theorem from class, the fundamental group of the resulting space Y is

$$\pi_1(Y, x_0) = \pi_1(X, x_0)/\langle\gamma\rangle$$

4) [10 pts] What is the fundamental group of the space obtained from part 2 by filling in one region?



There are 2 approaches to this. One is to use question 3:

$$\pi_1(Y, x_0) = \pi_1(X, x_0)/\langle\gamma\rangle = \langle a, b, c, d|b\rangle/\langle cb^{-1}\rangle = \langle a, d\rangle = \mathbb{Z}^{*2}$$

The other is to say this is homotopic to $S^1 \vee S^1$.

5) [10 pts] State the (simplified) Van Kampen Theorem.

Let $X = A \cup B$ be covered by 2 path connected, open sets A, B with $A \cap B$ path connected. Then for $x_0 \in A \cap B$

$$\pi_1(X, x_0) = \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0)$$

6) [10 pts] Let X be a path connected space. Recall that the suspension of X, called $S(X) = SX$, is defined by

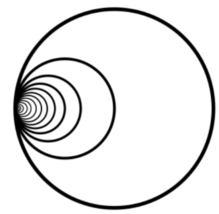
$$S(X) = X \times I / \sim$$

where $(x, 0) \sim (y, 0)$ and $(x, 1) \sim (y, 1)$ for all $x, y \in X$. That is, we pinch the sides of the interval to a point. Find $\pi_1(SX)$.

Let's divide SX into 2 sets A, B: $A = X \times [0, \frac{1}{2} + \epsilon)$ and $B = X \times (\frac{1}{2} - \epsilon, 1]$ for any $0 < \epsilon < \frac{1}{2}$. Now, $A \cap B = X \times (\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon) \simeq X$. So, since X is path connected Van Kampen applies. But each A, B are contractible, by contracting the interval to 0 or 1 respectively. So

$$\pi_1(X, x_0) = \pi_1(A, x_0) *_{\pi_1(A \cap B, x_0)} \pi_1(B, x_0) = 0 * 0 = 0$$

7) [20 pts] **Shrinking Wedge of Circles:** Let X be the subspace of $\mathbb{R}^2$ formed by taking a wedge sum at the origin of all the circles C_n of radius $\frac{1}{n}$ centered at $(0, \frac{1}{n})$. We will show that this easily obtained space X has an uncountable fundamental group:



i. [5 pts] The group $G = \prod_{i=1}^{\infty} \mathbb{Z}$ is the ordered set of infinitely many integers. Show that it is uncountable (by for example, comparing it to $\mathbb{R}$).

Let's show that $\Phi : \mathbb{R} \hookrightarrow G$. Let $a = a_1.a_2a_3a_4 \in \mathbb{R}$ be it's decimal expansion (with a_1 the integer part). Define $\Phi(a) = (a_1, a_2, \ldots)$. This is a well-defined injection. So G is uncountable.

ii. [8 pts] Show that for $\boldsymbol{a} = (a_1, a_2, \ldots) \in G$, there is $\gamma_{\boldsymbol{a}} \in \pi_1(X)$ such that $\gamma_{\boldsymbol{a}}$ loops a_1-times around C_1, then a_2-times around C_2, and so on (say on timescale $[\frac{1}{n+1}, \frac{1}{n}]$). In particular, show continuity.

Define $\gamma_{\boldsymbol{a}}$ as in the statement of the problem. It only goes to show that this operation is continuous as $t = 0$ (otherwise it can be viewed as a standard loop on a circle).
Let $\epsilon > 0$. Then there exists $n \gg 0$ such that $\frac{1}{n} < \frac{\epsilon}{2}$. Then

$$\gamma_a(t) \in \cup_{i=n}^{\infty} C_i \subseteq D_{\frac{1}{n}}(\frac{1}{n})$$

So $|\gamma_a(t)| \leq \frac{2}{n} < \epsilon$.

iii. [7 pts] Show that the $\gamma_{\boldsymbol{a}}$ are non-homotopic, by considering retractions $r_n : X \to C_n$ sending all other circles to the origin.[1]

If $\boldsymbol{a} \neq \boldsymbol{b}$, then $\exists n \in \{1, 2, 3, \ldots\}$ such that $a_n \neq b_n$. Consider $(r_n)_*(\gamma_{\boldsymbol{a}}) = a_n \in \mathbb{Z}$ and $(r_n)_*(\gamma_{\boldsymbol{b}}) = b_n$. These are distinct integers, so they couldn't have been homotopic to begin with (as $(r_n)_*$ is invariant under the class in π_1).

[1]Note this also distinguishes the space X from $\vee_{i=1}^{\infty} S^1$, which has countable fundamental group $\mathbb{Z}^{*\infty}$.

8) [5 pts] Define a covering space.

A covering space is a map $p : \tilde{X} \to X$ such that for every $x \in X$, there exists U open containing x such that

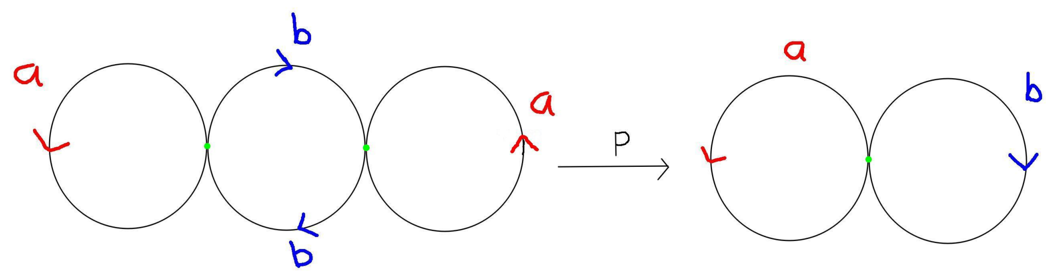
$$p^{-1}(U) = \coprod_{\alpha} U$$

9) [10 pts] Let $X = \cup_\alpha X_\alpha$ be a locally finite open cover of X. If $\tilde{X} = \coprod_\alpha X_\alpha$ is the disjoint union of the open sets in the cover, show that $p : \tilde{X} \to X$ is a covering space.[2]

Let $x \in X$. Then because the cover is locally finite, there exist only finitely many$X_{\alpha_1}, \ldots, X_{\alpha_n}$ containing x. Therefore, $X_{\alpha_1} \cap \ldots \cap X_{\alpha_n} = U_x$ is an open set (because X is topological) and $p^{-1}(U_x) = \coprod_{i=1}^n U_x$. Thus p is a covering space.

[2]Therefore, nice open covers can be viewed as covering spaces.

10) [10 pts] Consider the cover of $X = S^1 \vee S^1$ given by the following picture. Present and describe in words $G(\tilde{X})$, the group of deck transformations of $\tilde{X}$ over X.



p is a 2-sheeted normal (because taking a 180 degree spin of it doesn't change p) covering space. Therefore, $G(\tilde{X}) = \pi_1(X)/p_*(\pi_1(\tilde{X}))$. We note that the images of loops is exactly $\langle a, b^2 \rangle$, and therefore, $G(\tilde{X}) = \mathbb{Z}/2\mathbb{Z}$.

11) [15 pts] Let X and Y be path connected, locally connected spaces, and let $\tilde{X}$ and $\tilde{Y}$ be their respective universal covering spaces (so that $\tilde{X}$ and $\tilde{Y}$ are simply connected). Show that if $X \simeq Y$, then $\tilde{X} \simeq \tilde{Y}$. (**Hint:** lifting properties!)

Given $\tilde{X}, \tilde{Y}$ are simply connected, their fundamental groups are 0. Therefore, by the lifting criterion, there exist maps $\tilde{f}, \tilde{g}$ making the following diagram commute:

$$\begin{tikzcd}
\tilde{X} \arrow[r, shift left, "\tilde{f}"] \arrow[d, "p"'] & \tilde{Y} \arrow[l, shift left, "\tilde{g}"] \arrow[d, "q"] \\
X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"]
\end{tikzcd}$$

This is true because clearly $f_* p_* \pi_1(\tilde{X}) = 0 \subseteq q_* \pi_1(\tilde{(}Y) = 0$ and vice-versa. Here, $g : Y \to X$ is the homotopy inverse to f, so that $f \circ g \simeq Id_Y$.

Now, let's compare $\tilde{g} \circ \tilde{f}$ and $Id_{\tilde{X}}$. If we look at p of these maps, we know that $Id_X \simeq g \circ f$. $Id_{\tilde{X}}$ is a lift of the homotopy at time 0, so the homotopy extends to one above for some lift of Id_X, say $\tilde{Id_X}$. Now, $Id_{\tilde{X}}$ and $\tilde{Id_X}$ differ by a deck transformation. Call it ϕ. Therefore,

$$\tilde{g} \circ \tilde{f} \simeq \tilde{Id_X} = \phi \circ Id_{\tilde{X}}$$

or equivalently,

$$(\phi^{-1} \circ \tilde{g}) \circ \tilde{f} \simeq \phi^{-1} \tilde{Id_X} = Id_{\tilde{X}}$$

Similarly for $\tilde{f} \circ \tilde{g}$, $\exists \psi \in G(\tilde{Y})$ such that

$$\tilde{f} \circ \tilde{g} \simeq \tilde{Id_Y} = \psi \circ Id_{\tilde{Y}} = Id_{\tilde{Y}} \circ \psi$$

$$\tilde{f} \circ \tilde{g} \circ \psi^{-1} \simeq Id_{\tilde{Y}}$$

Applying the listed fact, we are done.