HOMEWORK 6: PERFECT RINGS AND SPLITTINGS
DUE: FRIDAY MAY 4

1) We have already see that the map $R \to F_* R$ induces a bijection on prime ideals. Can you say the same for $R \to R^\infty$?

2) Show that a ring R is F-split (respectively F-regular) if and only if $R_\mathfrak{m}$ is F-split (resp. F-regular) for every maximal ideal.

3) Is $R = K[x, y, z]/\langle x^3 + y^3 + z^3 \rangle$ an F-split ring? Be careful about the characteristic $p > 0$ chosen.

4) Is the Cohen-Macaulay non-regular ring $R = K[x^2, x^3]$ F-split?

5) Show that $R = K[x, y, z]/\langle x^4 + y^4 + z^4 \rangle$ is never F-split.

6) In this problem, we will show that $R = S/I$ in Fedder's Criterion can NOT be weakened to a more arbitrary quotient. Find an example of $S \supseteq J \supseteq I$ such that[1]
$$\operatorname{Hom}(F_* S/J, S/J) \ncong F_*((J/I)^{[p]} : J/I) \operatorname{Hom}(F_* S/I, S/I)$$

7) Suppose that L/K is a finite extension (meaning L is a finite K-module/vector space) of characteristic $p > 0$ fields and $x \in L \setminus K$ but $x^p \in K$. Show that if $\phi : K^{1/p} \longrightarrow K$ extends to $L^{1/p} \longrightarrow L$, then ϕ is the zero map on K.

8) Show that an F-split ring is weakly normal. That is to say that if $r \in K(R) = \prod_\mathfrak{q} R_\mathfrak{q}$, then if $r^p \in R$, then this implies $r \in R$. You may assume R is a domain if desired, though this is not necessary.

9) Prove Lucas's Theorem:

   Theorem 0.1 (Lucas's Theorem)**.** *$\binom{m}{n}$ is divisible by $p > 0$ if and only if expressing $n = \sum_{i=1}^k n_i p^i$ and $m = \sum_{i=1}^l m_i p^i$, for some i, $n_i > m_i$.*

10) A ring R is called F**-pure** if for every R-module M, the map $M \to M \otimes_R F_* R$ is injective. Show that every F-split ring is necessarily F-pure.[2]

[1] Hint: What happens in the case where R/I is not F-split, but R/J is?

[2] In the case where a ring is F-finite, these conditions are in fact equivalent. This can be seen by Matlis Duality.