HOMEWORK 3 SUPPLEMENTS TO QUESTIONS 2 AND 8

I wanted to adjoin the results from Friday's presentations as well as one extra result everyone used in problem 8.

Lemma 0.1. *Given a ring R, the nilradical is expressed as*

$$\mathcal{N} = \bigcap_{\mathfrak{p} \text{ prime}} \mathfrak{p} = \bigcap_{\mathfrak{q} \text{ minimal prime}} \mathfrak{q}$$

Proof. The second equality is obvious. We have already shown that $\mathcal{N} \subseteq \mathfrak{p}$ for any prime $\mathfrak{p}$, by virtue of the fact that $a^n = 0 \in \mathfrak{p}$ implies a or $a^{n-1} \in \mathfrak{p}$. Thus induction shows $a \in \mathfrak{p}$.

For the reverse inclusion, suppose $a \notin \mathbb{N}$. Then we have the natural multiplicative set $W = \{1, a, a^2, \ldots, a^n, \ldots\}$. Therefore, if we consider $W^{-1}R$, this ring is non-zero: $1 = (1, 1) \neq 0$. Now, we know that every ring has a prime ideal (e.g. maximal ideals), so $W^{-1}R$ has one. Since a is a unit, a is not in any prime as it generates R. Therefore, taking its corresponding prime ideal $\mathfrak{p}$ in R, we know $a \notin \mathfrak{p}$. $\square$

Lemma 0.2. *If $\mathfrak{q}$ is a minimal prime of a ring R, then $\mathfrak{q}$ is composed entirely of zero-divisors.*

Proof. Let $a \in \mathfrak{q}$. Then $\mathfrak{q} R_{\mathfrak{q}}$ is the unique prime ideal of $R_{\mathfrak{q}}$ by minimality. But $a \in \mathfrak{q} R_{\mathfrak{q}}$, so $a^n = 0 \in R_{\mathfrak{q}}$ with n minimal. By definition of the localization, this implies the existence of $b \in R \setminus \mathfrak{q}$ such that $a^n b = 0$ in R. Therefore, $a \cdot a^{n-1}b = 0$, but neither are 0 by minimality. This implies a is a zero-divisor. $\square$

Lemma 0.3.

$$\{\textit{Zero Divisors of } R\} = \bigcup_{\mathfrak{q} \text{ minimal prime}} \mathfrak{q}$$

Proof. By the previous lemma, if $a, b \notin \mathcal{N}$ is a zero-divisor with $ab = 0$, we know that there exists $\mathfrak{q}$ a minimal prime not containing b. But then $ab \in \mathfrak{q}$, and therefore $a \in \mathfrak{q}$. But a was arbitrary, so this completes the proof. $\square$

All of this information gives some great detail to the structure of minimal primes of a ring R. In particular, all zero divisors are apart of *some* minimal prime $\mathfrak{q}$, but only non-reduced elements are apart of *every* minimal prime. Coupled with the following fact, we get a good handle on minimal primes in a Noetherian ring:

Theorem 0.4. *If R is Noetherian, there exist only finitely many minimal primes $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$.*

The proof of this statement usually requires the theory of **associated primes**.

The following fact is the dual of an easier statement about projective modules:

Proposition 0.5. *I is an injective module if and only if for every inclusion $\iota : I \subseteq M$ of R-modules, $M = I \oplus M/I$. Equivalently, the following is split exact:*

$$0 \to I \to M \to M/I \to 0$$

The corresponding statement proved in class for projectives is

Proposition 0.6. *P is a projective module if and only if for every surjection $\psi : M \to P$ of R-modules, $M = P \oplus \ker(\psi)$. Equivalently, the following is split exact:*

$$0 \to \ker(\psi) \to M \to P \to 0$$

The proof goes as follows:

Proof. It is clear that I injective implies that such a sequence splits by in particular taking the identity map on I which lifts to the desired splitting map $M \to I$.

Now suppose that every injection $\psi : I \to M$ splits. Suppose that $\iota : J \subseteq N$ is an R-submodule and $J \to I$ is a homomorphism. We can consider the **pushout** module

$$I \oplus_J M := I \oplus M / \{(\psi(j), -\iota(j)) \mid j \in J\}$$

The good thing about this module is that it fits naturally in a commutative diagram

$$\begin{array}{ccc} J & \xhookrightarrow{\iota} & M \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi'} \\ I & \xhookrightarrow{\iota'} & I \oplus_J M \end{array}$$

Where ι' and ψ' are defined by their respective inclusions into $I \oplus M$, followed by the quotient. Note that ι' is an inclusion, because if $\iota'(i) = \overline{(i,0)} = 0$, then $(i,0) = (\psi(j), -\iota(j))$. But ι is injective, so $j = 0$, implying $i = \psi(0) = 0$.

Since this is an inclusion, our assumption guarantees that it splits! Therefore, we get $s : I \oplus_J M \to I$ such that $s \circ \iota' = Id_I$. Defining $\psi'' : M \to I$ by $\psi'' = s \circ \psi'$, I claim we get the desired map. It suffices to show that $\psi = \psi'' \circ \iota$:

$$\psi(j) = s(\iota'(\psi(j))) = s(\psi'(\iota(j))) = \psi''(\iota(j))$$

where the middle equality is given by commutativity of the diagram. This shows that I is injective and completes the proof. $\square$