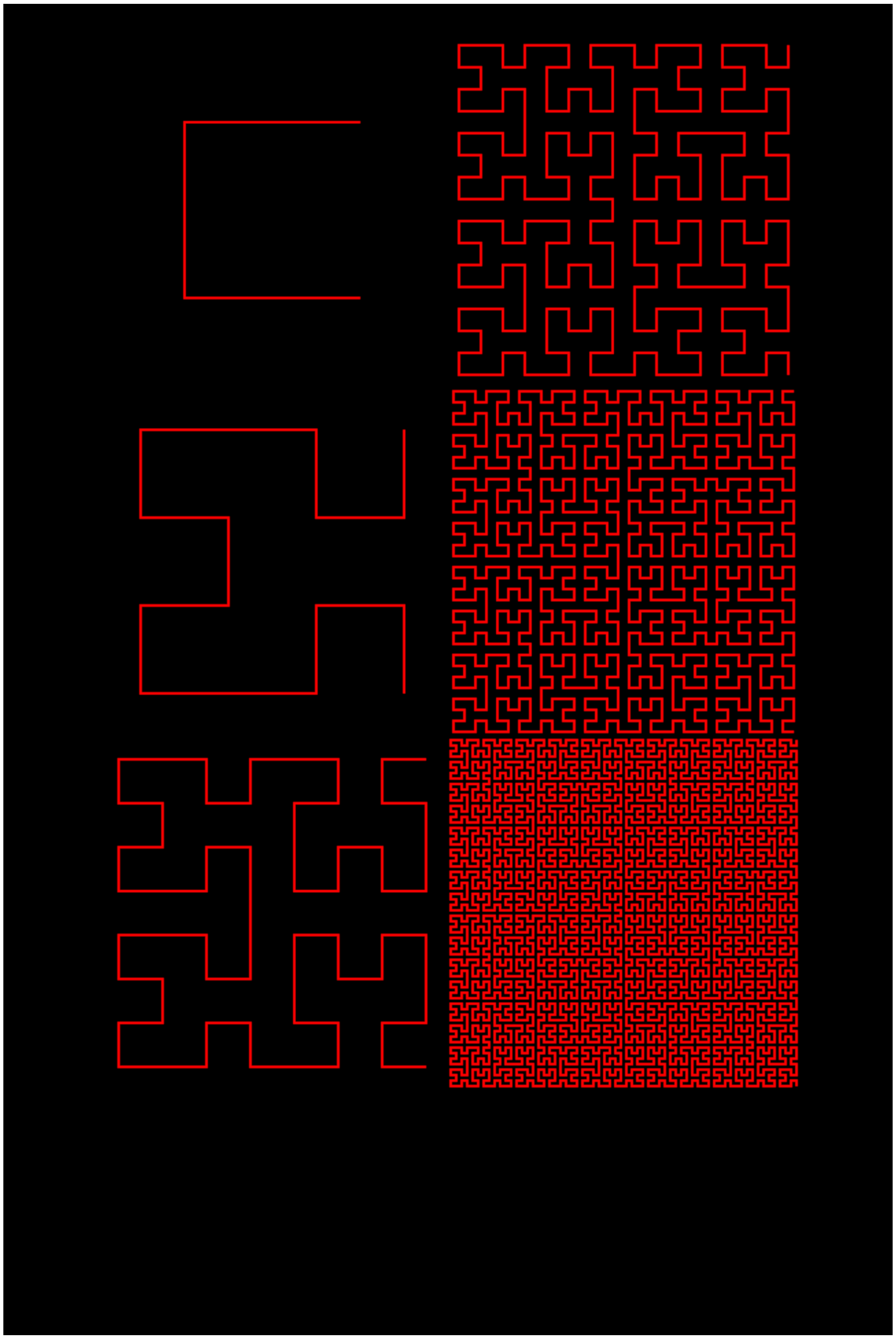
COURSE NOTES FOR MATH 374: GENERAL TOPOLOGY
WILLIAMS COLLEGE

ANDREW BYDLON



CLASS 1, SEPTEMBER 7: COMPARISON WITH METRIC SPACES

From real analysis, one of the most prominent objects is that of a metric space. This vastly generalizes many of the spaces you have seen in Calculus and even elementary geometry, and gives a way to measure 'how far apart' 2 points are in your space. Just to recall, here is a definition:

Definition 1.1. Let S be a set. A function $d : S \times S \to \mathbb{R}_{\geq 0}$ is called a **metric** if it satisfies the following conditions:

1) **Separation Axiom:** $d(x, y) = 0$ if and only if $x = y$
2) **Symmetry:** $d(x, y) = d(y, x)$
3) **Triangle Inequality:** $d(x, y) + d(y, z) \leq d(x, z)$

A pair (S, d) as above is called a **metric space**.

Example 1.2.
- $\mathbb{R}$ equipped with the metric $d(x, y) = |y - x|$ is a metric space.
- Let V be a finite dimensional vector space. Then $d_1(u, v) = |v_1 - u_1| + \ldots |v_n - u_n|$, $d_2(u, v) = \sqrt{(v_1 - u_1)^2 + \ldots + (v_n - u_n)^2)}$, and $d_\infty(u, v) = \max\{|v_i - u_i|\}$ all produce (equivalent!) metric spaces. These metrics are called the Manhattan, the Euclidean, and the Chebyshev metrics respectively.
- On a sphere S^2 (or S^n for any $n \geq 0$) is a metric space. This can be seen since it sits within $\mathbb{R}^3$ (or $\mathbb{R}^{n+1}$) which are metric spaces with a (or many) choices of d.
- Vertices on connected graphs have a metric, defined by how many edges one needs to travel to get between two vertices.

So many of the object you hold close have a notion of distance. This brings about the notion of an open or closed set in a natural way:

Definition 1.3. If (S, d) is a metric space, a subset $U \subseteq S$ is called **open** if for every point $x \in U$, there exists $\epsilon > 0$ (depending on x) such that $B(x, \epsilon) \subset U$, where

$$B(x, \epsilon) := \{y \in S \mid d(x, y) < \epsilon\}$$

This is commonly called an ϵ-ball around x or ϵ-neighborhood of x.

A subset $Z \subseteq S$ is called **closed** if its complement $Z^c = S \setminus Z$ is open.

Thus objects such as *open* intervals $(a, b) \subseteq \mathbb{R}$ are also open in a metric sense. Phrased differently, a set is called open if it is a union of ϵ-neighborhoods:

$$U = \bigcup_{x \in U} B(x, \epsilon_x)$$

Here are some nice properties of open sets (which you can transfer to corresponding statements for closed sets):

Proposition 1.4. *Let (S, d) be a metric space.*

1) S and $\emptyset$ are open sets.
2) If $U_\alpha \subseteq S$ are any collection of open sets indexed by $\alpha \in \Lambda$, then so is

$$U = \bigcup_{\alpha \in \Lambda} U_\alpha$$

3) If $U_1, \ldots, U_n$ are open sets, then so is $V = U_1 \cap \ldots \cap U_n$

Proof. 1) Obvious. Take any ϵ your heart desires.

2) If $x \in U$, then $x \in U_\alpha$ for some $\alpha \in \Lambda$, and therefore, $B(x, \epsilon_x) \subseteq U_\alpha \subseteq U$.

3) If $x \in V$, then $x \in U_i$ for each $i = 1, \ldots, n$. Therefore, there is $\epsilon_i > 0$ such that $B(x, \epsilon_i) \subseteq U_i$. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\} > 0$. Then $B(x, \epsilon) \subseteq B(x, \epsilon_i) \subseteq U_i$, and thus $B(x, \epsilon) \subseteq V$.

□

Note that the collection of open sets is *heavily* dependent on the choice of metric:

Example 1.5. Let (V, d_2) be a finite dimensional vector space with the Euclidean metric as above. We can also define $d_0 : V \times V \to \mathbb{R}$ by

$$d_0(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

With some thought, you can check that this is a metric on V (or any set) and every subset is open. Of course this is not the case for d_2; c.f. $[a, b]$. d_0 is called the **discrete** metric.

This brings about the following question: How can we tell when the collections of open sets induced by two metrics are the same?

Definition 1.6. Two metrics $d_1, d_2 : V \times V \to \mathbb{R}_{\geq 0}$ are said to be **strongly equivalent** if there exists $a, b > 0$ such that for every pair of points $x, y \in V$, we have

$$a \cdot d_1(x, y) \leq d_2(x, y) \leq b \cdot d_1(x, y)$$

That is to say the ratio $\frac{d_2}{d_1}$ is uniformly bounded above and below by positive numbers. The following proposition realizes the importance of this definition.

Proposition 1.7. *If d_1 and d_2 are strongly equivalent, then they share the exact same collection of open sets.*

Proof. Suppose U is d_1-open. Then if $B_1(x, \epsilon) \subseteq U$ (shorthand for ball in the d_1-metric), then $B_2(x, b \cdot \epsilon) \subseteq U$. This follows, as by definition $d_2(x, y) \leq b \cdot d_1(x, y) < b \cdot \epsilon$. Similarly, if V is d_2-open, and $B_2(x, \epsilon) \subseteq V$, then $B_1(x, \frac{\epsilon}{a}) \subseteq V$. Thus being open in one metric is *equivalent* to being open in the other. □

There is also a notion of (non-strongly) equivalent metrics, which give not only a sufficient, but also a necessary condition! It simply takes away the uniformity of a and b in the above definition. In particular, it says that for a fixed x and $r > 0$, we can find r', r'' such that

$$B_1(x, r') \leq B_2(x, r) \leq B_1(x, r'')$$

Now, the collection of open sets determine most of the important data about metric spaces, e.g. continuity of functions, differentiability of functions, completions, etc. Topology peals away the rigidity of a metric and dealing all of the numerics, and instead focuses simply on the collection of open sets. This yields a broadened and rich field of study which we will embark on next class!

CLASS 2, SEPTEMBER 10: TOPOLOGICAL SPACES

We now have the required properties to define a topological space, following the principals from last class and in particular Proposition 1.4:

Definition 2.1. A **Topological Space** is a pair (X, τ), where X is a set and τ is a collection of subsets (concisely, $\tau \subseteq \mathcal{P}(X)$) with the following properties:

1) $\emptyset, X \in \tau$.
2) If $U_\alpha \in \tau$, with $\alpha \in \Lambda$ any indexing set, then
$$U = \bigcup_{\alpha \in \Lambda} U_\alpha \in \tau.$$
3) If $U_1, \ldots, U_n \in \tau$, then $V = U_1 \cap \ldots \cap U_n \in \tau$.

Oftentimes people refer to X as a topological space, suppressing τ. The elements of τ are called **open sets**. Their complements are called **closed sets**. τ itself is called a **topology**.

Therefore, a topological space is simply a set with a selection of open sets, where we want the open sets to satisfy axioms like the ones we know best. This gives us a canonical example of topological spaces: metric spaces! The induced topology is often referred to as the **metric** topology. In fact, many common objects in mathematics can be realized topologically.

Example 2.2. Let X be any set.

a) If $\tau = \{X, \emptyset\}$, then τ satisfies the axioms of a topology and therefore gives X a topological structure. This is called the **indiscrete** or **trivial topology**.
b) If $\tau = \mathcal{P}(X)$, then τ is also a topology, called the **discrete topology**.
c) If $\tau = \{Y \subseteq X \mid Y^c \text{ is finite}\}$, then τ is a topology. It is (cleverly) named the **finite complement topology**.
d) If $\tau = \{Y \subseteq X \mid Y^c \text{ is at most countable}\}$, then τ is a topology.

Example 2.3 (2 points)**.** If $X = \{a, b\}$, there there are 4 possible topologies; the discrete topology, the indiscrete topology, $\tau = \{\emptyset, \{a\}, X\}$, and $\tau = \{\emptyset, \{b\}, X\}$. Up to renaming a and b, there are thus 3 topologies.

The 3 point case, for which there are 29 total topologies or 9 up to reordering of points, is illustrated on page 76 on Munkres (or on wikipedia, c.f. finite topological spaces).

Topologies are often compared to one another, and we have words to make there discussions more seamless:

Definition 2.4. Let X be a set, and let τ and σ be 2 topologies on X. Suppose $\tau \supseteq \sigma$ (so that every set open in the σ-topology is open in the τ-topology as well). Then we say τ is **finer** than σ, or σ is **coarser** than τ. In either of these case, we say the topologies are **comparable**. Otherwise, we call them **incomparable**.

Therefore, the indiscrete topology is coarser than every other possible topology. Similarly, the discrete topology is the finest topology.

We can also produce a notion of neighborhoods to add to our favorable comparison with metric spaces:

Definition 2.5. If x is a point in X, and V is any subset containing x, then V is called a **neighborhood of** x if there exists an open subset U such that $x \in U \subseteq V$.

If $S \subset X$ is any subset, then we say V is a **neighborhood of** S if V is a neighborhood of any point of S.

Similar to the case of metric spaces, neighborhoods can be refined:

Proposition 2.6. *If X is a topological space, $x \in X$, and V, V' are 2 neighborhoods of x, then so is $V \cap V'$.*

The proof is left to the reader. Next, we discuss some of the operations that can be performed on sets in a topological space.

Definition 2.7. Let $S \subseteq X$ be a subset of a topological space. We define the **interior** of S, denoted S°, to be the largest open set contained within S. Similarly, we define the **closure** of S, denoted $\bar{S}$, to be the smallest closed set containing S.

Proposition 2.8. *S° and $\bar{S}$ are well defined objects.*

Proof. Note that there is always an open set contained within S, namely the empty set. Therefore, we can simply define

$$S^\circ = \bigcup_{\substack{U \subseteq S \\ U \text{ open}}} U$$

Because this is a union of open sets, it is itself open by the second axiom of a topological space. Of course, any open subset contained within S is inside this union, so this is necessarily the largest such set.

For $\bar{S}$, we can use this operation along with the complement to produce the result:

$$\bar{S} = ((S^c)^\circ)^c$$

That is to say $\bar{S}$ is the complement of the interior of the complement of S. Note that this is closed since the interior is open by definition and we take its complement. It goes to show this is the smallest closed set containing S. Suppose there is a closed Z such that $S \subseteq Z \subseteq \bar{S}$. Taking complements, we retrieve

$$S^c \supseteq Z^c \supseteq \bar{S}^c = (S^c)^\circ$$

Since $(S^c)^\circ$ is the largest open set inside S^c, we see that $Z^c = (S^c)^\circ$ and thus $Z = \bar{S}$. $\square$

Proposition 2.9. *If A, B are 2 subsets of X, then $(A \cap B)^\circ = A^\circ \cap B^\circ$. However, generally $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.*

Proof. By the third axiom, $A^\circ \cap B^\circ$ is an open set, contained within $A \cap B$. Therefore, $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$. Additionally, $(A \cap B)^\circ \subseteq A, B$. Therefore, since it is open, we have $(A \cap B)^\circ \subseteq A^\circ, B^\circ$. Therefore, $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$ and thus equality is shown.

For the second claim, consider $A = [0, 1]$ and $B = [1, 2]$ inside $\mathbb{R}$ with the (standard) metric topology. It is easy to check $A^\circ = (0, 1)$ and $B^\circ = (1, 2)$, but $(A \cup B)^\circ = (0, 2)$. $\square$

CLASS 3, SEPTEMBER 12: BASES AND SUBSPACES

As with many things in mathematics, it's often nice to think of an object based on some significantly smaller defining objects. For example, in the study of finite dimensional linear algebra, we refine an uncountable collection of vectors to only finitely many basis vectors without reducing information. A similar notion exists for topology; instead of listing every open set in a topology, why not only list some building blocks?

Definition 3.1. Let X be a set. A **basis for a topology on X** is a subset $\mathcal{B} \subseteq \mathcal{P}(X)$ such that

1) For each $x \in X$, there exists $U \in \mathcal{B}$ such that $x \in U$.
2) If $x \in B_1, B_2$ for $B_1, B_2 \in \mathcal{B}$, then there exists $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$.

Notice that we can also generate (or span) a topology given a basis:

Proposition 3.2. *Given any collection $\mathcal{B} \subseteq \mathcal{P}(X)$ satisfying properties 1) and 2), there exists a unique smallest topology τ containing $\mathcal{B}$. In this case,* $\mathcal{B}$ **generates** τ.

Proof. Let

$$\tau = \{\bigcup_{\alpha \in \Lambda} U_\alpha \mid U_\alpha \in \mathcal{B}\} \cup \{\emptyset\}.$$

Of course, this contains $\mathcal{B}$ and is contained within any topology containing $\mathcal{B}$ by axiom 2) of a topology. Additionally, it contains X by property 1) above.

Finally, if $V_1, \ldots, V_n \in \tau$, then for each $x \in V_1 \cap \ldots, \cap V_n$, we can (inductively using property 2)) find a set $V_x \in \mathcal{B}$ such that $V_x \subseteq V_1 \cap \ldots \cap V_n$. This results in

$$V = V_1 \cap \ldots \cap V_n = \bigcup_{x \in V} V_x$$

and thus $V \in \tau$. $\square$

Example 3.3. Consider $\mathbb{R}^n$ with the metric topology τ. Letting $\mathcal{B}$ be any of the following yields a basis for τ:

- τ itself.
- $\{B_x(d) \mid x \in \mathbb{R}^n,\ d > 0\}$.
- $\{(a_1, b_1) \times \ldots \times (a_n, b_n) \mid a_i, b_i \in \mathbb{R}, a_i < b_i\}$.
- $\{B_x(d) \mid x \in \mathbb{Q}^n,\ d \in \mathbb{Q}_+\}$. That is to say we consider open balls of rational radius centered at points with rational coordinates.
- $\{B_x(d) \mid x \in \mathbb{Q}^n,\ d \in \mathbb{Q} \text{ with } 0 < d < 1\}$

The point here is that unlike in the case of (finite-dimensional) vector spaces, bases can have completely different sizes! Note that the first 3 bases above are uncountable, where as the last 2 are countable (since they are in particular finite products of copies of $\mathbb{Q}$).

Next, we check that a basis does determine enough information to make statements about the topology itself.

Lemma 3.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for τ and τ' respectively. Then TFAE:*

- *τ' is finer than τ.*
- *For each $x \in X$ and $B \in \mathcal{B}$ containing x, there exists $B' \in \mathcal{B}'$ containing x such that $B' \subseteq B$.*

Proof. ($\Rightarrow$) : Suppose τ' is finer than τ. Then B is an open set in the τ' topology. Therefore, by definition of generation, there exists $B'_\alpha \in \mathcal{B}'$ such that $B = \cup_{\alpha\in\Lambda} B'_\alpha$. Since $x \in B$, there exists some $\alpha_0 \in \Lambda$ for which $x \in B'_{\alpha_0}$. Letting $B' = B'_{\alpha_0}$ completes the proof.

($\Rightarrow$) : Suppose U is open in the τ-topology. Then $U = \cup_{\alpha\in\Lambda} B_\alpha$ for some $B_\alpha \in \mathcal{B}$. Now, the second property implies that for each $x \in B_\alpha$, there exists $B'_{\alpha,x} \in \mathcal{B}'$ such that $x \in B'_{\alpha,x} \subseteq B_\alpha$. This then implies that

$$U = \bigcup_{\alpha\in\Lambda} \bigcup_{x \in B_\alpha} B_{x,\alpha}$$

and that U is open in the $\mathcal{B}'$-topology. This completes the proof. $\square$

It is left as a homework exercise to use this to show that some of the topologies listed above are equivalent.

Next up, we will start a trend of producing new topological spaces from old. The first method of doing so is by taking the subspace topology:

Definition 3.5. Let (X, τ) be a topological space, and $Y \subseteq X$ be any subset. Define

$$\tau_Y = \tau|_Y = \{V \subseteq Y \mid V = Y \cap U \text{ for some } U \in \tau\}$$

This is called the **subspace topology** on Y.

It is a quick check (left for the reader) that this is in fact a topology on Y. Many of the common objects, such as a space of interest sitting inside of $\mathbb{R}^n$, can thus naturally receive a topology. These spaces include spheres, tori, Mobius Strips, Klein Bottles, space filling curves, etc.

Example 3.6. Let $\mathbb{R}$ be given with the Euclidean topology.

- The integers $\mathbb{Z} \subseteq \mathbb{R}$ with the subspace topology yields the discrete topology.
- Giving $[0, 1]$ the subspace topology of $\mathbb{R}$ produces open sets of the form (a, b), $[0, a)$, $(b, 1]$, and $[0, 1]$ (as well as unions of such things) where $0 < a < b < 1$. Note in particular these sets are not all open in $\mathbb{R}$ itself!
- If Y is an open (respectively closed) subset of X, and $U \subset Y$ is open (resp. closed) in the subspace topology, then $U \subseteq X$ is also open (resp. closed). Compare this with the previous bullet.

Relating back to bases, we have the following result:

Lemma 3.7. *If $\mathcal{B}$ is a basis for a topology on X, and we let*

$$\mathcal{B}_Y = \{B \cap Y \mid B \in \mathcal{B}\}$$

Then $\mathcal{B}_Y$ forms a basis for the subspace topology on Y.

This proof is again left for the curious reader, though both axioms follow very naturally. Next time we will talk about the product topology.

CLASS 4, SEPTEMBER 14: THE PRODUCT TOPOLOGY

When one has 2 different topological spaces X and Y, it may be typical (and productive) to ask if there is a natural topological space encapsulating the information of both simultaneously. This is where the product topology enters.

Definition 4.1. If (X, τ) and (Y, σ) are 2 topological spaces, we define the **product topology** on the cartesian product

$$X \times Y = \{(x, y) \mid x \in X,\ y \in Y\}$$

to be defined by the basis $\mathcal{B}$ consisting of $U \times V$, where $U \in \tau$ and $V \in \sigma$. For simplicity, we denote this by $\tau \times \sigma$.

This should really be a definition-proposition because it isn't immediately clear that this is a basis. Axiom 1) of a basis is immediate, since $(x, y) \in X \times Y$ (which is in $\mathcal{B}$. Furthermore, if $x \in U_1 \times V_1$ and $(u, v) \in U_2 \times V_2$ are 2 elements of $\mathcal{B}$, we know that $(u, v) \in (U_1 \cap U_2) \times (V_1 \cap V_2)$ is in $\mathcal{B}$ and contained within the intersection. This verifies axiom 2).

Example 4.2. We can give $\mathbb{R}^n$ the product topology of n-copies of $\mathbb{R}$ inductively. In homework 1 question 3, you have demonstrated that in fact this topology is equivalent to the standard metric topology.

Note. *An important consideration, and common mistake, is that not all elements of the product topology are 'squares'; that is of the form $U \times V$. In particular, if we consider $(1, 2) \times (1, 2) \cup (3, 4) \times (3, 4)$ in $\mathbb{R}^2$, we get a set not expressible in this form.*

Though this is a fairly common misconception, the idea does work on bases:

Proposition 4.3. *If $\mathcal{B}_\tau$ is a basis for τ on X, and $\mathcal{B}_\sigma$ is a basis for σ on Y, then*

$$\mathcal{B} = \mathcal{B}_\tau \times \mathcal{B}_\sigma = \{U \times V \mid U \in \mathcal{B}_\tau,\ V \in \mathcal{B}_\sigma\}$$

is a basis for the product topology $\tau \times \sigma$.

So in particular we can make the basis much smaller than initially demonstrated.

Proof. By definition of the product topology, we can express any open set $A \in \tau \times \sigma$ by

$$A = \bigcup_\alpha U_\alpha \times V_\alpha$$

where each of U_α and V_α are open in τ and σ respectively. By definition of a bases, each U_α and V_α can be expressed as

$$U_\alpha = \bigcup_\beta U_{\alpha,\beta}$$

$$V_\alpha = \bigcup_\gamma V_{\alpha,\gamma}$$

where $U_{\alpha,\beta} \in \mathcal{B}_\tau$ and $V_{\alpha,\gamma} \in \mathcal{B}_\sigma$. Noting that

$$U_\alpha \times V_\alpha = \bigcup_{\beta,\gamma} U_{\alpha,\beta} \times V_{\alpha,\gamma}$$

we can conclude that

$$A = \bigcup_{\alpha,\beta,\gamma} U_{\alpha,\beta} \times V_{\alpha,\gamma}.$$

$\square$

Next, just for context, I introduce the projection maps:

Definition 4.4. Given $X \times Y$ with the product topology, we define $\pi_X : X \times Y \to X$ where $\pi_X(x, y) = x$. A similar definition is given to $\pi_Y : X \times Y \to Y$. These are called the **projection maps**.

These maps allow us to later formalize the statement in the introduction; namely that $X \times Y$ encodes all of the topological data of X and Y simultaneously. We will return to this once we develop the notion of a continuous map.

Finally, our definition allows us to construct a topology on a finite Cartesian product by induction. However, how to handle the infinite case? This arises typically when studying function spaces, such as $C(\mathbb{R})$ (continuous functions $\mathbb{R} \to \mathbb{R}$) or $C^\infty(\mathbb{R})$ (infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$).

Definition 4.5. Given (X_α, τ_α) a collection of topological spaces, the **product topology** on $\prod_\alpha X_\alpha$ is defined by a basis

$$\mathcal{B} = \{\prod_\alpha U_\alpha \mid \emptyset \neq U_\alpha \in \tau_\alpha, \text{all but finitely many } U_\alpha = X_\alpha\}$$

The clear ambiguous feature is in the finiteness requirement, which lives implicitly in finite products. If we drop this requirement, we are left with the (much finer) **box topology**. Though this may make more intuitive sense, there is good reason to define the product topology in this way.

Example 4.6. Consider $X = \mathbb{R}^\mathbb{N}$, which is a countable product of copies of $\mathbb{R}$ (one for each natural number). If we assign the box topology to X, an example of an open set is

$$U = \prod_{n=1}^\infty (-\frac{1}{n}, \frac{1}{n}) = (-1, 1) \times \left(-\frac{1}{2}, \frac{1}{2}\right) \times \left(-\frac{1}{3}, \frac{1}{3}\right) \times \dots$$

Even if we pick the most simple function, $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = x$, then took the product of these functions,

$$F : \mathbb{R} \to \mathbb{R}^\mathbb{N} : x \mapsto (x, x, x, x, \dots)$$

the only element which can map into U is 0. Since $\{0\}$ is a closed (non-open) set, F doesn't behave well with respect to the topologies of the respective spaces.

However, if we instead consider the product topology, we couldn't have this shrinking interval problem, so the elements which can map to a given open set will be the intersection of *finitely many* open intervals, thus open by axiom 3) of a topological space!

CLASS 5, SEPTEMBER 17: CONTINUITY I

In almost all regions of math, objects become far more interesting once you give them an idealized version of a function. In the theory of metric spaces, this comes in the form of an ϵ-δ notion of a continuous function. Here we generalize this to arbitrary topological spaces and simplify matters simultaneously.

Definition 5.1. If (X, d) and (X', d') are metric spaces, then a function $f : X \to X'$ is said to be **continuous at a point** $x \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$, then we have that $d'(f(x), f(y)) < \epsilon$. f is said to be **continuous** if it is continuous at all of its points.

This is a cumbersome thing to check in practice, usually resulting in the production of a function δ depending on x and ϵ. The corresponding notion for topological spaces is as follows:

Definition 5.2. Let (X, τ) and (X', τ') be topological spaces. Then a function $f : X \to X'$ is said to be continuous if for every open set $U \subseteq X'$, the preimage is also open:

$$f^{-1}(U) = \{x \in X \mid f(x) \in U\}$$

Example 5.3.
1) If X is any set with the discrete topology, Y any topological space, then *any* function $f : X \to Y$ is continuous!
2) If Y is any set with the indiscrete topology, X any topological space, then *any* function $f : X \to Y$ is continuous!
3) If τ, τ' are topologies on X, then the identity function $Id_X : (X, \tau) \to (X, \tau')$ is continuous if and only if τ' is finer than τ.
4) Give $\mathbb{R}$ the Euclidean/metric topology. Considering the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$, we expect this function to *not* be continuous. This is in fact the case, because $U = (-1, 1)$ is open in $\mathbb{R}$ with
$$f^{-1}(U) = (-\infty, -1) \cup \{0\} \cup (1, \infty)$$
which is *not* open.

Of course, we wouldn't want continuity to mean completely different things in different overlapping contexts. A proposition corrects these doubts.

Proposition 5.4. *Let $f : (X, d) \to (X', d')$ be a map between metric spaces. Then f is continuous in the sense of metrics if and only if f is a continuous map of topological spaces with the metric topologies.*

Proof. ($\Rightarrow$): Suppose that U is an open set of X', and let $x' \in U$ be such that $f(x) = x'$ for some $x \in X$. For any fixed $\epsilon > 0$, $\exists \delta > 0$ such that $d(x, y) < \delta$ implies $d(x', f(y)) < \epsilon$. Since we know $B(x, r)$ form a basis for the metric topology, there exists $\epsilon_{x'}$ such that $B(x', \epsilon_{x'}) \subseteq U$. But this implies $B(x, \delta_{x'}) \subseteq f^{-1}(U)$. Therefore

$$f^{-1}(U) = \bigcup_{x' \in X} \bigcup_{f(x) = x'} B(x, \delta_{x'})$$

is an open set.

($\Leftarrow$): We know that $f^{-1}(B(x',\epsilon))$ is open. Therefore, if $f(x) = x'$, there exists some $\delta > 0$ such that $B(x,\delta) \subseteq f^{-1}(B(x',\epsilon))$. This again implements the fact that the metric topology has a basis of open balls. This δ then satisfies the conditions for continuity at x for the given (arbitrary) ϵ. □

The notion of a continuous function also gives a nice formulation for the product topology:

Proposition 5.5. *The product topology on $\prod_\alpha X_\alpha$ is the smallest/coarsest topology such that each of the projection maps $\pi_\alpha : \prod_\alpha X_\alpha \to X_\alpha$ are continuous.*

Proof. Given $U_{\alpha'} \subset X_{\alpha'}$, we need $\pi_{\alpha'}^{-1}(U_{\alpha'})$ to be open. But this is exactly $U_{\alpha'}$ product with the remaining X_α:

$$\pi_{\alpha'}^{-1}(U_{\alpha'}) = U_{\alpha'} \times \prod_{\alpha \neq \alpha'} X_\alpha$$

To ensure we have a topology, we need to ensure finite intersections of such sets are included:

$$\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) = U_{\alpha_1} \times U_{\alpha_2} \times \ldots \times U_{\alpha_n} \times \prod_{\alpha \neq \alpha'} X_\alpha$$

But this is exactly the basis described for the product topology! □

To finish up, I want to include two other notions which are distinct from being continuous that illustrate a common confusion:

Definition 5.6. A map $f : X \to Y$ of topological spaces is said to be **open** (respectively **closed**) if for every open (resp. closed) subset $U \subseteq X$, the set $f(U)$ is open (resp. closed).

Notice that in general, we can only say

$$U \subseteq f^{-1}(f(U))$$
$$f(f^{-1}(V)) \subseteq V$$

for $U \subseteq X$ and $V \subseteq Y$. Here are some examples showing some examples which satisfy one property but not the other.

Example 5.7.
- If $f : X \to Y$ is a constant map, meaning $f(x) = y$ for all $x \in X$ and some $y \in Y$, then f is certainly continuous; the preimage of an open set is either X or $\emptyset$ depending on whether or not it contains y. However, it is generally only an open/closed mapping if $\{x\}$ is itself open or closed (or if X has the indiscrete topology).
- The mapping f from Example 5.3 4) is non-continuous, but it is closed! This is because if $a < 0 < b$, we have
$$f([a,b]) = (-\infty, \frac{1}{a}] \cup \{0\} \cup [\frac{1}{b}, \infty) = \left((\frac{1}{a}, 0) \cup (0, \frac{1}{b}) \right)^c$$
- The projection map $\pi : \mathbb{R}^2 \to \mathbb{R} : (x,y) \mapsto y$ is an example of an open map which is not closed. Take for example the closed set which is the graph of $y = \tan^{-1}(x)$. The image of this set is $(-\frac{\pi}{2}, \frac{\pi}{2})$, which is open! Showing it's open is just the realization that $\pi(B((x,y),r) = B(y,r) = (y-r, y+r)$.

CLASS 6, SEPTEMBER 19: CONTINUITY II

Now that we have defined continuous functions, some properties and related notions can be demonstrated.

Theorem 6.1. *Let X and Y be topological spaces and $f : X \to Y$ be a function. Then the following are equivalent (forevermore, TFAE):*

1) *f is continuous*
2) *If $Z \subseteq Y$ is a closed set, $f^{-1}(Z)$ is closed.*
3) *For any subset $A \subseteq X$, $f(\overline{A}) \subseteq \overline{f(A)}$.*
4) *For each $x \in X$ and neighborhood $f(x) \subseteq V \subseteq Y$, we can find a neighborhood $U \subseteq X$ with $x \in U$ and $f(U) \subseteq V$.*

Condition 4) is markedly similar to the case of metric spaces.

Proof. 1) $\Rightarrow$ 2): Note that $(f^{-1}(Z^c))^c = f^{-1}(Z)$. This follows since every point of X must map to either Z or Z^c. Therefore, since Z^c is open, continuity of f implies $f^{-1}(Z^c)$ is open, and thus $f^{-1}(Z)$ is closed.

2) $\Rightarrow$ 3): By 2), we realize that $f^{-1}(\overline{f(A)})$ is a closed set. Moreover, as previously mentioned, for *any* set

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\overline{f(A)}).$$

Taking closures, we see

$$\overline{A} \subseteq \overline{f^{-1}(\overline{f(A)})} = f^{-1}(\overline{f(A)}).$$

or

$$f(\bar{A}) \subseteq f(f^{-1}(\overline{f(A)})) \subseteq \overline{f(A)}.$$

3) $\Rightarrow$ 4): Given V is a neighborhood of $f(x)$, we know that it contains an open set V' containing $f(x)$. Therefore, $f(x) \notin \overline{V^c}$. Applying 3) to $f^{-1}(V^c)$, we see that

$$f(\overline{f^{-1}(V^c)}) \subseteq \overline{f(f^{-1}(V^c))} \subseteq \overline{V^c} \subseteq V'^c$$

Finally, since $\overline{f^{-1}(V^c)}$ contains all points (and potentially more) mapping to V^c, letting $U = (\overline{f^{-1}(V^c)})^c$ produces the desired neighborhood.

4) $\Rightarrow$ 1): Take $V \subseteq Y$ to be open. If $x \in X$ is such that $f(x) \in V$, we can find a neighborhood $U_x \subseteq X$ containing x such that $f(U_x) \subseteq V$. Since it is a neighborhood, its interior also contains x, allowing us to write

$$f^{-1}(V) = \bigcup_{x \in f^{-1}(V)} U_x^\circ$$

a union of open sets, which is thus open.

$\square$

Recall that a function $f : X \to Y$ is said to be **bijective** if for every $y \in Y$, there exists (surjective) a unique (injective) point $x \in X$ such that $f(x) = y$. We give a special name to spaces to a collection of such special continuous maps which plays a similar role as isomorphisms in group theory (or algebra generally).

Definition 6.2. A continuous bijective map $f : X \to Y$ is said to be a **homeomorphism** if f^{-1} is also continuous.

Using some of the properties of functions demonstrated so far, we can go further with more assumptions:

Proposition 6.3.
- *If $f : X \to Y$ is a surjective map, then $f(f^{-1}(S)) = S$ for any subset $S \subseteq Y$.*
- *If $f : X \to Y$ is an injective map, then $f^{-1}(f(S)) = S$ for any subset $S \subseteq X$.*
- *A bijective continuous map $f : X \to Y$ is a homeomorphism if and only if f is open if and only if f is closed.*

Proof.
- We always have $f(f^{-1}(S)) \subseteq S$. If $s \in S$, then there exists $x \in X$ such that $f(x) = s$. Of course, since $x \in f^{-1}(s) \subseteq f^{-1}(S)$, we see $s \in f(f^{-1}(S))$.
- We always have $f^{-1}(f(S)) \supseteq S$. If $s \in f^{-1}(f(S))$, then $f(s) \in f(S)$. But there is only one $x \in X$ mapping to any give $y \in Y$, which implies $s \in S$.
- This is a combination of the two previous statements, which together imply $(f^{-1})^{-1}(U) = f(U)$. If U was open (resp closed) then so is one of the above equal sets by one of the given assumptions on f^{-1}.

Example 6.4. Consider the mapping $\tan : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ with the metric topology. This is a homeomorphism because it is continuous and $\tan^{-1}$ is its continuous inverse. Equivalently, $\tan((a,b)) = (\tan(a), \tan(b))$, which is open.

This yields a nice more general result: A continuous function $f : \mathbb{R} \to \mathbb{R}$ which is strictly increasing or decreasing is a homeomorphism onto its image.

□

Example 6.5. Examples of continuous bijective functions which are not homeomorphisms are easy to construct. Take $Id : (X, \tau) \to (X, \tau')$ where τ is strictly finer than τ'. Then the map is continuous but not open.

Next is a list of easy to show properties of continuous functions:

Proposition 6.6.
1) *If $f : X \to Y$ and $g : Y \to Z$ are two continuous functions, then so is $g \circ f$.*
2) *If $f : X \to Y$ is continuous and $A \subseteq X$ has the subspace topology, so is $f|_A : A \to Y$.*
3) *If $X = \bigcup_\alpha U_\alpha$ where U_α are open sets, then $f : X \to Y$ is continuous if and only if $f|_{U_\alpha} : U_\alpha \to Y$ is for all α.*
4) **Pasting Lemma:** *If $X = A \cup B$ for A and B open (or closed), and $f : A \to Y$ and $g : B \to Y$ are two continuous functions which agree on $A \cap B$, then so is the piecewise function*

$$h = \begin{cases} f(x) & x \in A \\ g(x) & x \in B \end{cases}$$

CLASS 7, SEPTEMBER 21: THE QUOTIENT TOPOLOGY

Next we study a topology that can be induced by any surjective map $f : X \to Y$ from a topological space X to a set Y. This adds a lot of depth to our knowledge of existent topological spaces and compares to the idea of a quotient group in abstract algebra.

Definition 7.1. Let X be a topological space and Y be a set. We define the **quotient topology** induced by a surjective map $p : X \to Y$ by the following property:

$$U \subseteq Y \text{ is open} \Leftrightarrow p^{-1}(U) \subseteq X \text{ is open}$$

In such a case, p is called a **quotient map**.

Note that this is stronger than continuity, which only implies the $\Rightarrow$ implication.

Proposition 7.2. *The quotient topology is a topology.*

Proof. Let $\tau = \{U \subseteq Y \mid p^{-1}(U) \subseteq X \text{is open}\}$.

1) $p^{-1}(Y) = X$ and $p^{-1}(\emptyset) = \emptyset$ are open subsets of X, so $Y, \emptyset \in \tau$.
2) If $U_\alpha \in \tau$, then
$$p^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha p^{-1}(U_\alpha).$$
Thus $\bigcup_\alpha U_\alpha \in \tau$.
3) Similarly,
$$p^{-1}(U_1 \cap U_2 \cap \ldots \cap U_n) = p^{-1}(U_1) \cap p^{-1}(U_2) \cap \ldots \cap p^{-1}(U_n)$$

Finally, since p is surjective, this satisfies the condition in the definition. That is to say, if $p^{-1}(U)$ is open, we know $U = p(p^{-1}(U))$, and thus $p^{-1}(U)$ being open implies U is. $\square$

Example 7.3. Consider $\mathbb{R}^2$ with the Euclidean topology. We can construct a map $f : \mathbb{R}^2 \to [0, 1)^2$

$$f(x, y) = (x \pmod 1, y \pmod 1)$$

Equivalently, take the decimal portion of a positive real number (or its complement if negative). This places a topology on the set $[0, 1)^2$ via the quotient topology. This is the Torus, the boundary of a doughnut, with its standard topology.

Example 7.4. Consider the subspace $X = [0, 1] \cup [2, 3] \subseteq \mathbb{R}$ and $Y = [0, 2] \subseteq \mathbb{R}$. Then we can construct the map $p : X \to Y$ with

$$p(x) = \begin{cases} x & 0 \leq x \leq 1 \\ x - 1 & 2 \leq x \leq 3 \end{cases}$$

This map does satisfy the axioms of a quotient map; it is surjective and open intervals pull back to open sets. However, this map is *not* open. In particular, $p([0, 1]) = [0, 1]$, which is not open even though $[0, 1] \subseteq X$ is.

Another particularly common example of a quotient mapping is found by 'collapsing' a subspace to a point.

Definition 7.5. Let $A \subseteq X$. Let Y be the set $Y = (X \setminus A) \cup \alpha$, where α is a distinguished element. We can define a map $p : X \to Y$ by the relation

$$p(x) = \begin{cases} x & x \notin A \\ \alpha & x \in A \end{cases}$$

Then Y with the quotient topology is called the quotient of X by A.

Example 7.6. Take $\mathbb{D}^2 = \bar{B}(0,1)$ the unit ball in $\mathbb{R}^2$ (or for the brave, $\mathbb{R}^n$). Take A to be the boundary subspace, defined by the points exactly of distance 1 from the origin. A can be naturally viewed as S^1 (resp. S^{n-1}). Then quotienting X by A produces a S^2 (resp. S^n) with its usual topology.

Theorem 7.7. *Let $p : X \to Y$ be a quotient map, and let $f : X \to Z$ be any map with the property that for a fixed $y \in Y$, and all $x, x' \in p^{-1}(y)$, $f(x) = f(x')$. Then there exists a map $g : Y \to Z$ with $g \circ p = f$. Furthermore, g is a continuous (resp. quotient) map if and only if f is continuous (resp. a quotient).*

Proof. If $x \in p^{-1}(y)$, we define $g(y) := f(x)$. This is well defined by the constancy condition of the theorem, and thus produces a map of sets. with $g \circ p = f$.

Now, to show the statement about continuity, suppose $U \subseteq Z$ is an open subset. Then by definition of p being a quotient map, we see

$$p^{-1}(g^{-1}(U)) = f^{-1}(U) \subseteq X \text{ is open} \Leftrightarrow g^{-1}(U) \subseteq Y \text{ is open}$$

Similarly, given p is surjective, it follows that g is surjective if and only if f is. Finally, again using the fact that p is a quotient map, we see that $U \subseteq Z$ is open would be equivalent to the following equivalent conditions:

$$f^{-1}(U) \subseteq X \text{ is open} \Leftrightarrow g^{-1}(U) \subseteq Y \text{ is open}$$

□

This allows us to realize that a quotient space is simply many quotients by individual subspaces.

Corollary 7.8. *Let $f : X \to Z$ be a continuous surjective map. Let $X^* = \{f^{-1}(z) \mid z \in Z\}$ be the set of fibers of the map f. Note $p : X \to X^* : x \mapsto f^{-1}(f(x))$ is a surjective map. Give X^* the quotient topology by p.*

Given this setup, the map $g : X^* \to Z$ from Theorem 7.7 is a continuous bijective map which is a homeomorphism if and only if f is a quotient map.

Proof. Indeed, if g is a homemorphism, then it is in particular a quotient map, thus f is as well by Theorem 7.7. On the flip side, if f is a quotient map, so is g. Therefore, g is a bijective map that is open: $g(U)$ is open since $f^{-1}(g(U)) = p^{-1}(U)$ is open. Thus g is a homeomorphism. □

CLASS 8, SEPTEMBER 24: CONNECTEDNESS

We now enter the realm where we have all of the basic players in the field; topological spaces and continuous maps. Now the questions start to arise; what kind of conditions can we put on such objects to make them 'nice'? Do these theorems encapsulate many of the main properties from real analysis or calculus in a more formal way?

The first of these definitions is that of connectedness, which generalizes the notion of an interval in $\mathbb{R}$.

Definition 8.1. A topological space X is called **connected** if for any open subsets U, V covering X ($X = U \cup V$) we have that $U \cap V \neq \emptyset$. Otherwise, the set is called **disconnected**, and in this case the sets U and V are called a **separation** of X.

This can be rephrased in terms of open sets; X is connected if and only if the only subsets of X which are closed and open (clopen) are X and $\emptyset$. This can be seen by taking U open and $V^c = U$ in the definition of connected.

Example 8.2. 1) Any set with the indiscrete topology is a connected space.

2) Any set with more than 1 point and the discrete topology is disconnected.
3) The following gives a common example of connected subsets of $\mathbb{R}$: intervals!

   Proposition 8.3. $(0, 1)$ *is a connected subset of* $\mathbb{R}$ *with the Euclidean topology.*

   Proof. Suppose not. Then there exists U, V open non-empty, not intersecting, and covering $(0, 1)$. By our basis for the topology of $\mathbb{R}$, and combining overlapping intervals, we know
$$U = \bigcup_{\alpha} (a_\alpha, b_\alpha)$$
   We can then form a new covering given a specific choice of α, say α_0:
$$U' = (a_{\alpha_0}, b_{\alpha_0})$$
$$V' = V \cup \left(\bigcup_{\alpha \neq \alpha_0} (a_\alpha, b_\alpha) \right)$$
   Both of these sets are open since they are unions of basis elements. However, the assumptions for a separation yield that V' has one of the following forms:
$$V' = (0, a_{\alpha_0}] \cup [b_{\alpha_0}, 1) \textbf{ or } (0, a_{\alpha_0}] \textbf{ or } [b_{\alpha_0}, 1)$$
   In any of these cases, V' is not open since either $B(\alpha_0, \epsilon) \not\subseteq V$ or $B(\beta_0, \epsilon) \not\subseteq V$. ∎

4) The same argument can be applied to the intervals (a, b), $(a, b]$, $[a, b)$ and $[a, b]$.
5) $\mathbb{Q} \subset \mathbb{R}$ is a *totally* disconnected space, meaning it's only connected components are single points! Suppose $a \neq b \in U \subset \mathbb{Q}$ a connected subset. There exists $c \in \mathbb{R}$ with $a < c < b$ and c irrational. Therefore
$$(a, b) \cap \mathbb{Q} = ((a, c) \cap \mathbb{Q}) \cup ((c, b) \cap \mathbb{Q})$$

Intersecting these sets with U produces a separation of U, contradicting our assumption that 2 distinct points can be a in a connected subset of $\mathbb{Q}$.

6) Let X be the union of the x-axis and the graph of $y = \frac{1}{x}$ for $x > 0$ in $\mathbb{R}^2$. Then this space is disconnected. Indeed, each of the subsets can be enclosed in open disjoint sets in $\mathbb{R}$, and therefore under the subspace topology they remain open individually:

$$U = \{(x, y) \mid y > \frac{1}{2x},\ x > 0\}$$

$$V = \{(x, y) \mid y < \frac{1}{2x}\} \cup (-\infty, 1) \times (-\frac{1}{2}, \frac{1}{2})$$

Now we can produce some properties under which connectedness is preserved.

Proposition 8.4. *If X is separated by two open subsets U, V, and $Y \subseteq X$ is connected, then $Y \subseteq U$ or $Y \subseteq V$.*

Proof. Suppose $Y \not\subseteq V$. Then since Y is connected and

$$Y = (Y \cap U) \cup (Y \cap V)$$

is a union of 2 open subsets, we find that $Y \cap V = \emptyset$, or $Y \subseteq U$. $\square$

Proposition 8.5. *If $x \in \bigcap_\alpha U_\alpha$ where each U_α is connected, then $\bigcup_\alpha U_\alpha$ is also connected.*

Proof. Suppose $\bigcup_\alpha U_\alpha$ is separated by V, V'. Then by Proposition 8.4, each U_α is contained in either V or V'. If $U_\alpha \subseteq V$ and $U_{\alpha'} \subseteq V'$, then $x \in V \cap V'$, contradicting the fact that they form a separation. So all U_α live in either V or V', implying the other is empty. $\square$

Continuing with these ideas, we can represent a space by its so called *connected components.*

Definition 8.6. For a given $x \in X$, there exists a largest connected subset U_x (not necessarily open) such that U_x contains x and U_x. U_x is called the **connected component of** x.

We can of course cheat using Proposition 8.5 to show it exists:

$$U_x = \bigcup_{\substack{x \in U \\ U \text{ connected}}} U$$

Theorem 8.7. *A space X can be decomposed into its connected components in a disjoint way*

$$X = \coprod_\alpha U_\alpha$$

where each U_α is connected and disjoint from any $U_{\alpha'}$ for $\alpha \neq \alpha'$.

Proof. We can create an equivalence relation on X, which says $x \sim y$ if and only if there exists a connected subset $Y \subseteq X$ such that $x, y \in Y$. Call such an equivalence class $[x]$, and the set of all such equivalence classes $X/\sim$. Then x and y share a connected component: $U_x = U_y$ with the terminology of Definition 8.6. We can then form

$$X = \bigcup_{[x] \in X/\sim} U_x$$

This covers X since every $x \in X$ is in U_x. Furthermore, $U_x \cap U_y = \emptyset$ for each $[x] \neq [y]$ by Proposition 8.5; if they shared a point their union would be a larger connected set containing x, contradicting the definition of U_x. $\square$

CLASS 9, SEPTEMBER 26: CONNECTEDNESS

So far we have managed to show that any topological space can be decomposed into a union of disjoint *connected components*. We also showed that $\mathbb{Q} \subseteq \mathbb{R}$ with the subspace topology is an example of a space where this decomposition is silly:

$$\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$$

However, there are some things that can be said about the decomposition:

Proposition 9.1. *Every connected component U_x of a space X is closed. If X has a finite decomposition into connected components, then each of them is open.*

To prove this, we need a lemma:

Lemma 9.2. *If A is a connected subset of X and $A \subseteq B \subseteq \bar{A}$, then B is connected. In particular $\bar{A}$ is connected.*

Proof. Suppose B is separated by C, D. Then by Proposition 8.4 from the last class, we know that $A \subseteq C$ or $A \subseteq D$. WLOG, assume $A \subseteq C$. Therefore, $\bar{A} \subseteq \bar{C}$. Since D is open, $\bar{C} \subseteq D^c$, so we see $\bar{C} \cap D = \emptyset$. Therefore, $B \subseteq C$ and $D = \emptyset$, a contradiction. $\square$

Proof. of Proposition 9.1: Lemma 9.2 implies that $\overline{U_x}$ is a connected set containing x. But U_x is defined to be the largest open set containing x. Therefore, $U_x = \overline{U_x}$ implying that U_x is closed. Therefore, if we are able to write

$$X = U_1 \cup U_2 \cup \ldots \cup U_n$$

with each U_i a connected component (and thus closed), we have

$$U_i = (U_1 \cup \ldots U_{i-1} \cup U_{i+1} \cup \ldots \cup U_n)^c$$

Since a finite union of closed sets is closed, we see U_i is open. $\square$

Before moving to a few examples, I include some useful properties of connectedness relative to continuity and products.

Proposition 9.3. *Let $f : X \to Y$ be a continuous map of topological spaces and $Z \subseteq X$ be a connected subset. Then $f(Z)$ is connected.*

Proof. Let $f(Z) = U \cup V$ be a separation. Then

$$f^{-1}(Z) = f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$$

is a separation of the preimage, and since $Z \subseteq f^{-1}(f(Z))$, we can conclude

$$Z = (f^{-1}(U) \cap Z) \cup (f^{-1}(V) \cap Z)$$

but Z is connected, so WLOG we can assume $f^{-1}(V) \cap Z = \emptyset$. But this implies $f(Z) \cap V = \emptyset$, contradicting the separation. $\square$

Note that the same cannot be said about the pre-image.

Example 9.4. Consider $f : \mathbb{R} \coprod \mathbb{R} \to \mathbb{R}$ given by sending each copy of $\mathbb{R}$ in the domain to $\mathbb{R}$ via the identity. Than the preimage of any non-empty connected subset is *not* connected.

For a less obvious example, consider the connected subset $\Gamma \subseteq \mathbb{R}^2$ which is the graph of $y = x^2$. Γ is a connected subset, since it is the image of $\mathbb{R}$ under the continuous map $x \mapsto (x, x^2)$. Now, if we consider the projection map $\pi : \Gamma \to \mathbb{R} : (x, y) \mapsto y$. This is continuous since it is a projection map. Taking the connected subset $(0, \infty) \subseteq \mathbb{R}$, it's preimage is given by

$$\{(\sqrt{y}, y) \mid y > 0\} \cup \{(\sqrt{y}, y) \mid y < 0\}$$

which forms a separation of Γ, thus disconnected.

Proposition 9.5. *If $X_1, X_2, \ldots, X_n$ are connected spaces, then so is $X_1 \times \cdots \times X_n$.*

Proof. I proceed by induction. For the base case, $n = 1$ is trivial so we consider the product of 2 connected spaces $X \times Y$. Consider for a specific choice of $x \in X$ and $y \in Y$, the T-shaped set

$$T_{x,y} = \{x\} \times Y \cup X \times \{y\}$$

Since Y (resp. X) is connected and homeomorphic to $\{x\} \times Y$ (resp $X \times \{y\}$), it is also connected. This is because it is the image of the map you showed is continuous in Homework 2! Finally, since $(x, y) \in \{x\} \times Y \cap X \times \{y\}$, we see $T_{x,y}$ is connected by Proposition 8.5. Now, if we let $x \in X$ vary, we get

$$X \times Y = \bigcup_{x \in X} T_{x,y}$$

Note that for fixed $y \in Y$ every $T_{x,y}$ contains (x, y) for any $x \in X$. Therefore, again by Proposition 8.5, we can conclude their union is connected.

For induction, we view $X_1 \times \cdots \times X_n$ as $(X_1 \times \ldots \times X_{n-1}) \times X_n$, and apply the logic of the base case. $\square$

The natural question arises; can we extend this to arbitrary products? The answer is as usual is yes, but only in the product topology. The reason is that we can write every *2* open separating sets U_1, U_2 as a union of

$$U_i = U_i^{\alpha_1} \times \cdots \times U_i^{\alpha_n} \times \prod_{\alpha \neq \alpha_1, \ldots, \alpha_n} X_i^{\alpha}$$

Then project onto $X_1 \times \cdots \times X_n$ without loss of information. In the box topology, the same cannot be said.

Example 9.6. Consider $\mathbb{R}^{\mathbb{N}}$ with the box topology. Then we can let U be the set of bounded sequences and V be the set of unbounded sequences. Note that a sequence is bounded if there exists N such that $|a_i| < N$ for all $i \in \mathbb{N}$. Therefore, U and V are necessarily disjoint. Furthermore, if $a = (a_1, a_2, \ldots) \in U$, then so is

$$(a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 1) \times \ldots \in U$$

These sequences can be bounded by $N + 1$ if a is bounded by N. The same applies to V with the same choice of open neighborhood. This shows $\mathbb{R}^{\mathbb{N}} = U \cup V$ is a separation, and thus $\mathbb{R}^{\mathbb{N}}$ is disconnected.

This is part of a general phenomenon; namely if $\tau \supset \tau'$ is a finer topology, than τ-connectedness implies τ'-connectedness, but not vice-versa.

CLASS 10, SEPTEMBER 28: A GENERALIZED MVT

Today I will introduce a new topology that can be put on any totally ordered set. This, together with connectedness can drastically improve the applicability of the Intermediate Value Theorem from calculus to more geometric contexts.

Definition 10.1. A **totally ordered set** is a pair $(X, \leq)$ where X is a set $\leq$ is a binary operation on X such that

1) $a \leq b$ and $b \leq a$ implies $a = b$.
2) $a \leq b$ and $b \leq c$ implies $a \leq c$.
3) $a \leq b$ or $b \leq a$.

Let X be a totally ordered set. Then the order topology τ is the topology generated by the basis

$$\{(a, b), (-\infty, b), (a, \infty)\}$$

where $(a, b) = \{c \mid a < c < b\}$, and the ∞ omit one of the inequalities.

Canonical examples of this are already known to us:

Example 10.2.

1) $\mathbb{R}$ with the Euclidean topology is exactly $\mathbb{R}$ with the order topology given by inequalities of real numbers.
2) $\mathbb{R}^2$ can be endowed with the **dictionary topology**; $(x_1, x_2) \leq (y_1, y_2)$ if either $x_1 < y_1$ or $x_1 = y_1$ and $x_2 < y_2$.
3) By our usual method of induction, the previous example can be bootstrapped to $\mathbb{R}^n$. This is fine than (thanks Ben) the Euclidean topology for $n \geq 2$.
4) Any set of cardinal or ordinal numbers has a natural ordering by size.

Now, we can recall a classical result from Calculus and see what we can tweak;

Theorem 10.3 (The Intermediate Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < f(b)$ (or flipped). Then for any $c' \in R$ such that $f(a) < c' < f(b)$, there exists $c \in (a, b)$ such that $f(c) = c'$.*

Theorem 10.4 (The *Improved* Intermediate Value Theorem)**.** *Let $f : X \to Y$ be a continuous map between X a connected topological space and Y an ordered set with the order topology. Assume $a, b \in X$ are such that $f(a) < r < f(b)$. Then there exists $x \in X$ such that $f(x) = r$.*

Notice that here X has just 1 property of $[a, b]$, and Y 1 property of $\mathbb{R}$.

Proof. We have two open sets in the order topology of interest: $U_< = (-\infty, r)$ and $U_> = (r, \infty)$. These are disjoint open subsets of Y, and thus their preimages are as well. But X is connected, so $X \neq f^{-1}(U_<) \cup f^{-1}(U_<)$. This implies there exists $x \in X \setminus (f^{-1}(U_<) \cup f^{-1}(U_>))$, which of course imples $f(x) = r$. $\square$

Example 10.5. The n-sphere S^n is a connected space, and we can endow $\mathbb{R}$ with the order/Euclidean topology. For any given continuous map $t : S^n \to \mathbb{R}$ a corresponding map $T : S^n \to \mathbb{R}$ such that $T(\mathbf{x}) = t(\mathbf{x}) - t(-\mathbf{x})$. This map is either 0 constantly or has some value $(y_1, y_2) > (0, 0)$. Moreover, this function is odd: $T(\mathbf{x}) = -T(-\mathbf{x})$. The Improve

Intermediate Value Theorem thus implies (in either case) that there exists $\mathbf{x} \in S^n$ such that $-y < T(\mathbf{x}) < y$. This demonstrates for example the following fact:

There exists antipodal points on Earth that share exactly the same temperature.[1]

To finish up, I want to give a slightly stronger notion of connected and compare it with the original notion.

Definition 10.6. A space X is called **path connected** if for any 2 points $x, y \in X$, there exists a continuous function (path) $\gamma : [0, 1] \to X$ such that $\gamma(0) = x$ and $\gamma(1) = y$.

Rephrased, a space is path connected if all of its points can be connected by paths.

Proposition 10.7. *A path connected space X is also connected.*

Proof. Suppose that X is separated by U and V. Let $x \in U$ and $y \in V$. Consider a path γ connecting these points. We can consider the open disjoint sets $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$. Their union is $[0, 1]$, which is connected. Therefore, we may assume $\gamma^{-1}(V) = \emptyset$. But $1 \in \gamma^{-1}(V)$, a contradiction.

Alternatively: The image of γ is connected. Since $Im(\gamma) = \gamma([0, 1])$ is a connected subset of $X = U \cup V$, it must belong to one or the other. $\square$

However, a connected space need not be path connected.

Example 10.8. Consider the following set with the subspace topology:

$$X = \{(x, \sin(\frac{1}{x})) \mid 0 < x \leq 1\} \cup \{0\} \times (-1, 1) \subseteq \mathbb{R}^2$$

I claim this set is connected. Indeed, if U, V separate X, then the 2 components (which are images of intervals, thus connected) must belong to one or the other. Call the graph Γ and the interval I. WLOG, suppose $\Gamma \subseteq U$ and $I \subseteq V$. But V is open in the subspace topology, therefore for $x \in V$, $B(x, r) \subseteq V$ for some $r > 0$. However, this would imply that

$$\emptyset \neq \Gamma \cap V \subseteq U \cap V = \emptyset$$

an immediate contradiction. So X is connected.

To show it is not path connected, suppose we have a path from $(0, 0$ to $(1, \sin(1))$. By connectedness and closedness, there exists some largest $b < 1$ for which $\gamma([0, b]) \subseteq I$. By continuity, we have that $\gamma(b) = \lim_{x \to b} \gamma(x)$. However, there exist infinitely many x near b for which $\sin(x) = 1$ and $\sin(x) = -1$, so the limit can't exist. Therefore, X is not path connected.

[1]With some stronger algebraic topological methods, we can show any map $S^n \to \mathbb{R}^n$ also has this property. Thus in particular, we have antipodal points of earth with the same temperature and pressure!

CLASS 11, OCTOBER 1: COMPACTNESS

In metric spaces, a notion of being a small space makes sense. For general topological spaces not so much. However, we have a basis for the topology of X, dividing not only X but every open subset of X into simple components. As we've seen on the homework, when there are finitely many such components, the local to global behavior is often well understood. Today, we formalize this with the notion of compactness.

Definition 11.1. A topological space X is said to be **covered** by a collection of open sets $U_\alpha \in \tau$ if

$$X = \bigcup_\alpha X_\alpha$$

X is called **compact** if any open cover can be made finite; there exists $\alpha_1, \dots, \alpha_n$ such that

$$X = X_{\alpha_1} \cup \ldots \cup X_{\alpha_n}$$

Example 11.2.
1) $\mathbb{R}$ is not compact. Take $B(x,1)$ for every $x \in \mathbb{R}$. If finitely many cover it, their length (measure) is bounded above by $2n$ where n is as in the definition of compactness. This is a contradiction, because the length of $\mathbb{R}$ is ∞.
2) The same argument extends to $\mathbb{R}^m$.
3) Any finite space is compact, As is any space with the finite complement topology.
4) The subspace $X = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$ is compact! Indeed, if U_α cover X, then there exists U_{α_0} containing 0. But then $B(0,\epsilon) \subseteq U_{\alpha_0}$ for some ϵ. So U contains $\frac{1}{n}$ for any $n > \frac{1}{\epsilon}$. This leaves only finitely many points outside of U_{α_0}, which at worst can be covered individually by the open set containing them.

   A good exercise is to prove that removing $\{0\}$ from X makes it non-compact.
5) The set $[a,b) \subseteq \mathbb{R}$ (or any interval not of the form $[a,b]$) is not compact. It can be covered, for example, by $[a, b - \frac{1}{n})$. However, every finite refinement will miss some points close to b.

A nice thing to note is that if $\tau \subseteq \tau'$, and X is τ'-compact, then it is also τ-compact. Furthermore, we can create many nice examples from *closed* subspaces:

Theorem 11.3. *If $Y \subseteq X$ is a closed subset, and X is compact, then Y is compact.*

Proof. Suppose U_α covers Y. Then there exists V_α open in X such that $V_\alpha \cap Y = U_\alpha$ (by definition of the subspace topology). Then we can consider the open covering of X given by

$$X = (\bigcup_\alpha V_\alpha) \cup (X \setminus Y)$$

Because X is compact, finitely many will do (for which we adjoin $(X \setminus Y)$ to avoid cases):

$$Y \subseteq X = V_{\alpha_1} \cup \ldots \cup V_{\alpha_n} \cup (X \setminus Y)$$

But this implies $Y \subseteq V_{\alpha_1} \cup \ldots \cup V_{\alpha_n}$, or equivalently

$$Y = U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}$$

□

Corollary 11.4 (Of the proof)**.** *A subspace $Y \subseteq X$ is compact if and only if any open covering of Y in X can be refined to a finite collection.*

Next, I would like to provide a partial converse to Theorem 11.3. Note that the converse is not true in general;

Example 11.5. A finite proper subset of X with the trivial topology is compact but not closed.

Therefore, we need an extra condition that we will study in some detail later; Hausdorff.

Definition 11.6. A topological space X is **Hausdorff** if for any 2 points $x, y \in X$, there exists open disjoint set U, V containing x, y respectively.

Examples, similar to that of homework 1, are metric spaces: Let $U = B(x, \frac{d(x,y)}{2})$ and $V = B(y, \frac{d(x,y)}{2})$. This allows us to find the desired converse statement.

Theorem 11.7. *If X is Hausdorff, and $Y \subseteq X$ is compact, then Y is closed.*

Proof. We will prove Y^c is open. Assume it is non-empty (or we are done). Choose $x \in Y^c$ and $y \in Y$. Since X is Hausdorff, there exist U, V as in the definition. Fix $x \in X$ and label these opens U_y and V_y. This produces a covering of Y; $Y = \bigcup_{y \in Y} V_y$. But by compactness we see that $Y = V_{y_1} \cup \ldots \cup V_{y_n}$. But this set is thus disjoint from the open neighborhood of x:

$$U_x = U_{y_1} \cap \ldots \cap U_{y_n}$$

Therefore, $Y^c = \bigcup_{x \in Y^c} U_x$ is an open set! □

Finally, similar to the notion of connected, we see that images of compact sets remain compact:

Proposition 11.8. *If $f : X \to Y$ is a continuous map, and $A \subseteq X$ is a compact set, so is $f(A)$.*

Proof. Let $f(A) = \bigcup_\alpha U_\alpha$ be an open cover. Then

$$A \subseteq f^{-1}(f(A)) \subseteq f^{-1}(\bigcup_\alpha U_\alpha) = \bigcup_\alpha f^{-1}(U_\alpha)$$

and thus a finite collection of these will do:

$$A \subseteq f^{-1}(U_{\alpha_1}) \cup \ldots \cup f^{-1}(U_{\alpha_n})$$

This of course implies

$$f(A) \subseteq f\left(f^{-1}(U_{\alpha_1}) \cup \ldots \cup f^{-1}(U_{\alpha_n})\right) = f\left(f^{-1}(U_{\alpha_1})\right) \cup \ldots \cup f\left(f^{-1}(U_{\alpha_n})\right) \subseteq U_{\alpha_1} \cup \ldots \cup U_{\alpha_n}.$$

□

The same again does not hold for preimages: consider any compact set inside $\mathbb{R}$ and consider the projection map $\pi : \mathbb{R}^2 \to \mathbb{R}$. Its preimage will not be compact.

CLASS 12, OCTOBER 3: COMPACTNESS OF PRODUCTS

I now state a corollary of the results from last class.

Corollary 12.1. *If $f : X \to Y$ is a continuous bijective map with X compact and Y Hausdorff, then f is a homeomorphism.*

Proof. It suffices to check that f is a closed map. This is the content of homework #3. □

Next we check that finite products behave well with respect to compactness:

Theorem 12.2. *If $X_1, \ldots, X_n$ are compact spaces, then so is $X_1 \times \ldots \times X_n$.*

Proof. By induction it suffices to check that the result is true for a product of 2 spaces, say X and Y. Furthermore, given a cover of $X \times Y$, say $\bigcup_\alpha U_\alpha$, we know that each U_α has the structure of a collection of products:

$$U_\alpha = \bigcup_\beta U^X_{\alpha,\beta} \times U^Y_{\alpha,\beta}$$

So if we can prove it for covers of the form

$$X \times Y = \bigcup_\alpha U_\alpha \times V_\alpha$$

we are done. So consider such a cover and fix $x_0 \in X$. Let α^{x_0} run through a subset of indices α for which $(x_0, y) \in U_{\alpha^{x_0}} \times V_{\alpha^{x_0}}$. Then the fiber of x_0 has the property that

$$\{x_0\} \times Y \subseteq \bigcup_{\alpha^{x_0}} U_{\alpha^{x_0}} \times V_{\alpha^{x_0}}$$

Projecting onto Y, we see that their images $V_{\alpha^{x_0}}$ cover Y. Therefore, we can find finitely many to cover Y by compactness: $Y = V_{\alpha_1^{x_0}} \cup \ldots \cup V_{\alpha_{n_{x_0}}^{x_0}}$. Let $U_{x_0} = U_{\alpha_1^{x_0}} \cap \ldots \cap U_{\alpha_{n_{x_0}}^{x_0}}$, which is open by finiteness. Then we have the chain of inclusions

$$\{x_0\} \times Y \subseteq U_{x_0} \times \left(V_{\alpha_1^{x_0}} \cup \ldots \cup V_{\alpha_{n_{x_0}}^{x_0}}\right) \subseteq \left(U_{\alpha_1^{x_0}} \times V_{\alpha_1^{x_0}}\right) \cup \ldots \cup \left(U_{\alpha_{n_{x_0}}^{x_0}} \times V_{\alpha_{n_{x_0}}^{x_0}}\right)$$

Now we can use our standard trick combined with compactness:

$$X = \bigcup_{x \in X} U_x = U_{x_1} \cup \ldots \cup U_{x_n}$$

But this gives us our refinement:

$$\begin{aligned} X \times Y =& (U_{x_1} \cup \ldots \cup U_{x_n}) \times Y = & \bigcup_{i=1}^n (U_{x_i} \times Y) \\ =& \bigcup_{i=1}^n \left(U_{x_i} \times \bigcup_{j=1}^{n_{x_i}} V_{\alpha_j^{x_0}}\right) = & \bigcup_{i=1}^n \bigcup_{j=1}^{n_{x_i}} \left(U_{x_i} \times V_{\alpha_j^{x_0}}\right) \end{aligned}$$

This shows compactness, proving the claim. □

This shows for example that $\mathbb{T}^n$ is a compact space, since $S^1 \subseteq \mathbb{R}$ is closed. Many more examples, such as n-cubes are also compact as a result: $B = [a_1, b_1] \times \ldots \times [a_n, b_n]$. Of course, this doesn't extend to infinite products with the box topology:

Example 12.3. Consider $Y = [0,1]^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}} = X$, where we give X the box topology and Y the subspace topology. Consider the covering of $\mathbb{R}$ given by $U = [0, \frac{2}{3})$ and $V = (\frac{1}{3}, 1]$, which are both open in the subspace $[0,1]$. We can consider the countable cover given by infinite products of either U or V. There is no finite refinement.

Later on we will prove that the same holds for arbitrary products of topological spaces with the product topology. This is the famous Tychonoff Theorem. To conclude, I would like to present the corresponding statements for closed sets.

Definition 12.4. A collection $\mathcal{C}$ of subsets of X is said to have the **finite intersection property** if any finite subset Λ of $\mathcal{C}$, the intersection of elements of Λ is non-empty.

This may seem strange, but it gives a closed classification of compactness:

Theorem 12.5. *Let X be a topological space. Then X is compact if and only if any subset $\mathcal{C} \subset \mathcal{P}(X)$ containing only closed subsets and having the finite intersection property also has the intersection property:*

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset$$

Proof. $(\Leftarrow)$: Suppose X is compact. Then given such a $\mathcal{C}$, assume $\mathcal{C}$ doesn't have the intersection property:

$$\bigcap_{C \in \mathcal{C}} C = \emptyset$$

Then consider $\mathcal{C}^c = \{U^c \mid U \in \mathcal{C}\}$. This is a set of open subsets, and indeed $\mathcal{C}^c$ forms an open cover of X:

$$X = \emptyset^c = \left(\bigcap_{C \in \mathcal{C}} C \right)^c = \bigcup_{C \in \mathcal{C}} C^c = \bigcup_{C \in \mathcal{C}^c} C$$

But this implies a finite subcover exists by compactness:

$$X = C_1 \cup \ldots \cup C_n$$

But this implies $\emptyset = C_1^c \cup \ldots \cup C_n^c$, contradicting the finite intersection property of $\mathcal{C}$.

$(\Rightarrow)$: Suppose X is not compact. Taking a cover with no finite subcover, and $\mathcal{C}$ the collection of complements, produces as example of a set $\mathcal{C}$ with the finite intersersection property but not the intersection property. $\square$

This shows a nice thing about nested closed subsets in a compact space X:

Corollary 12.6. *If $C_1 \supseteq C_2 \supseteq \ldots$ is a nested collection of non-empty closed subsets of a compact topological space, then*

$$\emptyset \neq \bigcap_{i=1}^{\infty} C_i$$

This isn't true for non-compact spaces. Indeed, consider the sequence $C_n = [n, \infty)$.

CLASS 13, OCTOBER 5: LOCAL PROPERTIES

Today we will double our list of nice properties for a topological space by adding one definition; **local**. All of our properties so far have been about X globally as a topological space. Even very nice topological spaces don't satisfy these, such as $\mathbb{R}$ not being compact. But small (closed) neighborhoods do satisfy this property. This is comparable to studying rings instead of algebraic varieties in algebra.

Definition 13.1. Let $\mathcal{P}$ be a property of topological spaces. Then we say X is **locally-$\mathcal{P}$** if for any $x \in X$ and neighborhood Z of x, there exists a neighborhood $Z' \subseteq Z$ of x such that Z' is $\mathcal{P}$.

Examples of $\mathcal{P}$ are connectedness, path connectedness, and compactness. We will study these properties here.

Example 13.2. I recommend as an exercise to draw the resulting spaces.

1) An example of a connected space which is not locally connected is the topologist's sin-curve:
$$X = (\{0\} \times [-1, 1]) \bigcup \left\{(x, \sin(\frac{1}{x})) \mid 0 < x < 1\right\} \subseteq \mathbb{R}^2$$

2) To produce an example of a locally (path) connected but not (path) connected space, take any space with finitely many connected components, each of which is locally (path) connected. An example of this is $X = \mathbb{R} \coprod \mathbb{R}$, the disjoint union of two copies of $\mathbb{R}$. Any neighborhood of x (in say the first copy of $\mathbb{R}$) necessarily contains $(x - \epsilon, x + \epsilon)$, which is connected since it is homeomorphic to $(0, 1)$.

3) To produce an example of a path connected but not locally path connected space, we us the 'topologist's comb':
$$X = ([0, 1] \times \{0\}) \bigcup \{(x, y) \mid x \in \mathbb{Q} \cap (0, 1),\ y \leq x\} \subseteq \mathbb{R}^2$$
This space is path connected, since we can always travel down the comb to $[0, 1] \times \{0\}$, over to the correct value of x, and then up to a given point. On the otherhand, note that $X \cap B((1, 0), \frac{1}{2})$ is not even connected!

I now state an equivalent definition of locally connected.

Proposition 13.3. *X is locally connected if and only if $\forall U \subseteq X$ open, each connected component of U is also open.*

Proof. $(\Rightarrow)$: If X is locally connected, let $V \subseteq U$ be a connected component of U. Then for any $x \in V$ there exists a connected neighborhood C_x of x within V. But neighborhoods contain open subsets V_x containing x, so we see
$$V = \bigcup_{x \in V} U_x$$

$(\Leftarrow)$: If V is a neighborhood of x, we can consider V_x the connected component of x in V. This is a connected open neighborhood by assumption. $\square$

Now I will switch gears and study the notion of **local compactness**.[1]

Example 13.4. 1) $\mathbb{R}^n$ is a locally compact space which is not itself compact. Indeed, every neighborhood contains $B(x,r)$ for some $r > 0$, and thus $\bar{B}(x, \frac{r}{2})$.

2) On the otherhand, $\mathbb{R}^{\mathbb{N}}$ with the product (or box) topology is not locally compact. Indeed, there contain no compact neighborhoods containing ANY of the basis elements $U_1 \times \ldots \times U_n \times \mathbb{R}^{\{n+1,n+2,\ldots\}}$.

Proposition 13.5. *If X is compact and Hausdorff, then X is locally compact.*

Proof. Suppose U is a (open WLOG) neighborhood of some point $x \in X$. For every point $y \in U^c$, let U_y and V_y be open sets containing x and y respectively that are disjoint. Since $U^c \subseteq X$ is a closed subset it is compact, so finitely many will do:

$$U^c \subseteq V_{y_1} \cup \ldots \cup V_{y_n} = V$$

Its complement V^c is a closed subset containing the open set $U_{y_1} \cap \ldots \cap U_{y_n}$, thus a compact neighborhood of x. ☐

Corollary 13.6. *If X is a Hausdorff locally compact space, and $U \subseteq X$ is an open or closed subset of X, then U is locally compact.*

Proof. If U is a closed subset, then U is itself compact and Hausdorff. Therefore, Proposition 13.5 implies U is also locally compact.

On the otherhand, if U is open and V is a neighborhood in U of x, then V is also a neighborhood of $x \in X$. By Proposition 13.5, there is a compact set contained within V in X. This remains compact in the subspace topology of U since U is open, and thus open covers in U are also open in X. ☐

An immediate corollary of Corollary 13.6 is the following:

Corollary 13.7. *Let X be a Hausdorff space that is not itself compact. Then X is locally compact if and only if X is homeomorphic to an open subset of a compact Hausdorff space.*

Finally, I introduce the 1-point compactification of a locally Hausdorff space X. Let $Y = X \cup \{\infty\}$, where ∞ is just a name for a new distinguished point. It goes to define a topology. A subset $U \subseteq Y$ is open if either

- $\infty \notin U$ (or equivalently $U \subseteq X$) and U is open in the topology of X.
- $\infty \in U$ and $U^c \subseteq X$ is a compact subset.

Note that this is in fact a topology. X has the second property and $\emptyset$ has the first. The other 2 facts follow from the fact that arbitrary intersections of closed subsets are closed and finite unions of compact sets are compact. Y is called the **one-point compactification** of X.

1) If $X = \mathbb{R}$, then $Y = \mathbb{R} \cup \{\infty\} \cong S^1$.
2) If $X = \mathbb{C}$, then Y is the Riemann Sphere.
3) If $X = \mathbb{R}^n$, then $Y \cong S^n$.

[1]The book specifies local compactness as every point has a compact neighborhood. This is a less stringent condition in general than what I have defined, but equivalent when X is Hausdorff. This notion makes Proposition 13.5 obvious and not require the Hausdorff condition, but is more esoteric generally speaking.

CLASS 14, OCTOBER 12: COUNTABILITY AND CONVERGENCE

In the coming days, we will introduce various stronger and weaker versions of the Hausdorff condition already stated. Before studying these notions, we need to add the notions of countability and convergent sequences to our list of definitions.

Definition 14.1. Let X be a topological space. A sequence of points $\mathbf{x} = (x_1, x_2, \ldots) \in X^{\mathbb{N}}$ is said to **converge** to a point $x \in X$ if for every (open) neighborhood U of x intersects the sequence for all but finitely many terms. Alternatively, there exists $N = N(U)$ such that $x_n \in U$ for $n \geq N$. We write $x_n \to x$ in this case.

This should of course be familiar from the world of metric spaces. It allows us to define notions of limits. However, as we know, the world of topology adds a lot of depth to our study.

Example 14.2. Let X be any infinite set with the finite complement topology. Let $(x_1, x_2, x_3, \ldots)$ be a sequence of distinct points of X. Then $x_n \to x$ for *any* point $x \in X$. Indeed, any open neighborhood is given by $U = X \setminus \{y_1, y_2, \ldots, y_n\}$, and we have assumed that $x_i \neq x_j$ for every j. So eventually $x_n \neq y_i$!

On the other hand, metric spaces are Hausdorff (you essentially checked this on hwk 1). This is enough to conclude that a sequence as above cannot simultaneously converge to multiple points.

Proposition 14.3. *If X is a Hausdorff space, and $(x_1, x_2, \ldots)$ converges to a point x, then x is unique.*

Proof. Suppose the sequence in question also converges to $y \neq x$. By the Hausdorff condition, there exists open disjoint sets (neighborhoods) of x and y, say U and V respectively, such that $U \cap V = \emptyset$. But then $x_n \in U$ for $n \geq N$ and $x_m \in V$ for $m \geq M$. But this implies in particular that

$$x_{\max\{M,N\}} \in U \cap V = \emptyset$$

which is a contradiction. $\square$

So the importance of avoiding pathologies with the Hausdorff condition is apparent. We can also reverse this by putting a local size cap on our topology. Here we define the notion of first-countable.

Definition 14.4. We say X has a **countable basis** at a point $x \in X$ if there exist a countable collection of neighborhoods of x, say $X_1, X_2, \ldots$, such that any neighborhood U of x contains one of these sets: $U \supseteq X_i$. If X has a countable basis at every point, X is said to be **first-countable**.

Example 14.5. Every metric space satisfies this property. Indeed, the basis for the metric topology is given by

$$\mathcal{B} = \{B(x, d) \mid x \in X,\ d > 0\}$$

However, for a fixed x, we can restrict to only positive rational d. This gives a countable basis at x.

Theorem 14.6. *Let X be a topological space.*

1) *Let $A \subseteq X$. If there exists $(a_1, a_2, \ldots)$ such that $a_n \to x$, then $x \in \bar{A}$.* [1]
2) *If $f : X \to Y$ is a continuous function, then $x_n \to x$ implies $f(x_n) \to f(x)$.*

In either case, if X is first-countable, the converse of each statement holds.

Proof.
1) Let x and a_n be as in the theorem. Since $a_n \to x$, we note that every neighborhood U of x intersects A. If $x \notin \bar{A}$, then $x \in \bar{A}^c = (A^c)^\circ$, and thus there is some neighborhood of x in $\bar{A}^c \subseteq A^c$. This contradicts the first statement.
2) Let V be an open neighborhood of $f(x)$. Then $f^{-1}(V)$ is an open neighborhood of x. Therefore, there exists N such that $x_n \in f^{-1}(V)$ for $n \geq N$. But this implies
$$f(x_n) \in f(f^{-1}(V)) \subseteq V$$

Now lets assume $X_1, X_2, \ldots$ is a countable base for X at x.

1) If $x \in \bar{A}$, choose a sequence $(a_1, a_2, \ldots)$ where $a_n \in X_1 \cap \ldots \cap X_n$. Note this is still an open neighborhood of x. Now, $a_i \in X_n$ for all $i \geq n$. Moreover, if U is any neighborhood of x, $U \supseteq X_n$ for some n. Therefore, it contains all but finitely many points of the sequence.
2) Note that to show the continuity of f, we need that for any set $A \subseteq X$, $f(\bar{A}) \subseteq \overline{f(A)} \subseteq Y$. Given $x \in \bar{A}$, the previous statement implies that there exists $(x_1, x_2, \ldots) \in A^{\mathbb{N}}$ converging to x. But this implies $f(x_n) \to f(x)$ by our assumption. By the first part, this implies $f(x) \in \overline{f(A)}$, as desired.

□

Now I provide a partial converse to Proposition 14.3.

Theorem 14.7. *If X is a first-countable topological space and $x_n \to x$ implies x is unique, then X is Hausdorff.*

Proof. Let $x \neq y$ be 2 distinct points in X. Suppose there do not exist open disjoint neighborhoods of x and y, or equivalently, every 2 neighborhoods of x and y intersect. Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be bases at x and y. Then we note that $X(n) = X_1 \cap \ldots X_n$ is an open neighborhood of x and $Y(n) = Y_1 \cap \ldots Y_n$ is a neighborhood of y. Therefore, they must intersect. Choose
$$a_i \in X(i) \cap Y(i)$$
Then any neighborhood U and V contain $X(n)$ and $Y(n)$ for some $n > 0$. Therefore,
$$U \cap V \supseteq X(n) \cap Y(n) \supseteq \{a_n, a_{n+1}, \ldots\}$$
But this implies the sequence $a_n \to x, y$, contradicting our assumption and proving the claim. □

[1]This characterizes closed sets!

CLASS 15, OCTOBER 15: COUNTABILITY AXIOMS

Recall from last time the notion of first countable, which played an essential role in relating Hausdorffness to convergence of sequences. Today we will discuss first and second countability in greater detail.

Definition 15.1. A topological space X is said to be **second-countable** if there exists a countable basis for the topology.

Therefore, we note immediately that second-countable implies first-countable, as we can take the subset of the basis illustrated in the definition of open sets containing a given point $x \in X$. The following example illustrates that they are not equivalent in general.

Example 15.2.
- $X = \mathbb{R}^n$ with the Euclidean topology is a second countable space. As previously discussed (in ancient times, Class 3), we can take

$$\mathcal{B} = \{B(\mathbf{x}, d) \mid \mathbf{x} = (x_1, \ldots, x_n),\ x_i \in \mathbb{Q},\ d \in \mathbb{Q}_+\}$$

  This is countable, since we have a finite $n+1$-fold product of countable sets, thus countable.
- Let $X = \mathbb{R}^{\mathbb{N}}$ with the uniform topology. Last time, we demonstrated that every metric space is first-countable by taking rational-radii. I claim X is not second-countable. Using the lemma that follows, Lemma 15.3, we note that $A = \mathbb{Z}^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}}$ has the discrete topology, taking $U_a = B(a, \frac{1}{2})$ for example. On the other hand, note that we have a surjection

$$A \to \mathbb{R} : (a_1, a_2, \ldots) \mapsto \ldots \bar{a}_5\bar{a}_3\bar{a}_1.\bar{a}_2\bar{a}_4 \ldots$$

  where we are taking a decimal expansion of a number, and letting $\bar{a}_i$ is the singles digit (remainder $\pmod{10}$) of a_i. Since $\mathbb{R}$ is uncountable, so is A. Therefore, we note X is not second-countable by the contrapositive of Lemma 15.3.

Lemma 15.3. *If X is a second-countable space, then if $A \subseteq X$ has the discrete topology (as a subspace), then A is countable.*

Proof. Since A has the discrete topology, for any point $a \in A$, there exists U_a an open basis element such that $U_a \cap A = \{a\}$. However, this induces an injection

$$\iota : A \to \mathcal{B} : a \mapsto U_a.$$

Therefore, since $\mathcal{B}$ is countable, so is A. □

We now note that our standard operations on spaces maintain the countability axioms:

Theorem 15.4.
1) *If X is first (or second) countable, then so is $Y \subseteq X$ with the subspace topology.*
2) *If $X_1, \ldots, X_n$ is first (or second) countable, then so is $X_1 \times \ldots \times X_n$.*

Just in case you haven't seen the typical logical tricks, note that a countable union of countable sets is countable. Also, a finite product of countable sets is countable.

Proof. 1) The subspace topology has a basis given (by Lemma 3.7) as

$$\mathcal{B}_Y = \{U \cap Y \mid U \in \mathcal{B}\}$$

So if X is first countable, then the countable neighborhood base of $y \in Y \subseteq X$ will do. Similarly, if $\mathcal{B}$ is countable, then $\mathcal{B} \to \mathcal{B}_Y : U \mapsto U \cap Y$ is a surjection.

2) The product topology has a basis given (by Proposition 4.3) as
$$\mathcal{B} = \{U_1 \times \ldots \times U_n \mid U_i \in \mathcal{B}_i\}$$
If X_i is first countable for all i, then the product of the countable neighborhood bases will do. Similarly, if $\mathcal{B}$ is countable, then
$$\mathcal{B}_1 \times \ldots \times \mathcal{B}_n \to \mathcal{B} : (U_1, \ldots, U_n) \mapsto U_1 \times \ldots \times U_n$$
is a surjection.

□

Finally, we proceed to some statements on density of subsets. This plays an essential role in many of our countability axioms in practice; namely that $\mathbb{Q} \subseteq \mathbb{R}$ is dense.

Definition 15.5. $A \subseteq X$ is said to be **dense** if $\bar{A} = X$.

Theorem 15.6. *Suppose X is second-countable.*

1) Every open cover of $Y \subseteq X$ has a countable refinement.
2) There exists a countable dense subset of X.

In this theorem, we are saying that every subset of a second countable space is something weaker than compact, but still has controllable covers. The second statement generalizes the idea $\mathbb{Q} \subseteq \mathbb{R}$ to any second countable space!

Proof. Let $X_1, X_2, \ldots$ be a countable base for X.

1) It suffices to prove the statement for X by Theorem 15.4, part 1. Let $X = \bigcup_\alpha U_\alpha$. Choose $U_{\alpha_i} \supset X_i$ (which exists by definition of basis). Then
$$X = \bigcup_\alpha U_\alpha \supseteq \bigcup_{i=1}^\infty U_{\alpha_i} \supseteq \bigcup_{i=1}^\infty X_i = X$$
where the last equality again follows by definition of a basis noting X is open. Therefore, everything above is equal. Thus $X = \bigcup_{i=1}^\infty U_{\alpha_i}$ is a countable refinement.

2) Choose $x_i \in X_i$ arbitrarily. I claim the set $D = \{x_1, x_2, \ldots\}$ is the desired dense subset. Indeed, if $x \in \bar{D}^c$, then there exists some X_i such that $x \in X_i \in \bar{D}^c \subseteq D^c$. But $x_i \in D \cap X_i$. This is a contradiction.

□

This leads us to a pretty startling realization:

Example 15.7. I claim $\mathbb{R}^\mathbb{N}$ is second-countable with the product topology.[1] Recall that the product topology is given by
$$\mathcal{B} = \{\left(\prod_{i=1}^n U_i\right) \times \mathbb{R}^{\mathbb{N}\setminus\{1,\ldots,n\}} \mid U_i \subseteq \mathbb{R} \text{ is open.}, n > 0\}$$
If we fix a specific n, the elements of the form $(\prod_{i=1}^n U_i) \times \mathbb{R}^{\mathbb{N}\setminus\{1,\ldots,n\}}$ are in bijection with the countable basis of the topology $\mathbb{R}^n$. Call this set $\mathcal{B}_n$. Then
$$\mathcal{B} = \bigcup_{n>0} \mathcal{B}_n$$
Since a countable union of countable sets is countable, $\mathcal{B}$ is countable!

Therefore, there exists a countable dense subset of $\mathbb{R}^\mathbb{N}$ with the Euclidean topology.

[1]This follows directly from the following more general statement: A countable product of second-countable spaces is second countable. The proof is identical.

CLASS 16, OCTOBER 17: SEPARATION AXIOMS

Today, I will expand the idea of a space being Hausdorff to more generic settings. These come in a variety of T-conditions.

Definition 16.1. The following are called the separation axioms:

0) A space X is called **T0** if for every 2 distinct points $x, y \in X$, there exists U an open neighborhood of 1 of the points but not the other.
1) A space X is called **T1** if for every 2 distinct points $x, y \in X$, there exists U an open set such that $x \in U$ and $y \notin U$.
2) A space X is called **T2** or **Hausdorff** if for every 2 distinct points $x, y \in X$, there exists U, V disjoint open sets such that $x \in U$ and $y \in V$.
3) A space X is called **T3** or **Regular** if for every points $x \in X$ and closed set $Z \subseteq X$, there exists U, V disjoint open sets such that $x \in U$ and $Z \subseteq V$.
4) A space X is called **T4** or **Normal** if for every 2 closed sets $Z, Z' \subseteq X$, there exists U, V disjoint open sets such that $Z \subseteq U$ and $Z' \subseteq V$.

It should be clear that we call these separation axioms because we are finding separations of 2 types of sets in X. There are even more of these axioms, $T(\frac{N}{2})$ for $N = 0, \ldots, 12$.[1]

Proposition 16.2. *If X is a topological space with one point sets closed (equivalently T1), then if X is $T(a)$ for some $a > 0$, then X is also $T(b)$ for any $b < a$:*[2]

$$T4 \Rightarrow T3 \Rightarrow T2 \Rightarrow T1 \Rightarrow T0$$

As a result, it is sometimes assumed that every T2-4 space is T1. Here are some examples showing that the above concepts are distinct. Note that as a result of the homework, all metric spaces are T4 (and T0, therefore all of the separation axioms hold). Therefore, coming up with examples will mostly be outside the realm of metric spaces.

Example 16.3.
- The 2 point space $X = \{a, b\}$ with topology $\tau = \{\emptyset, \{a\}, \{a, b\}\}$ is T0 but not T1 (because b can't be in an open set not containing a). This is also an example of a T4 space (almost vacuously) which is not T3!
- The finite complement topology on an infinite set X is always T1, but never T2.
- The **slit disc topology** on $\mathbb{R}^2$ is an example of a T2 but not T3 space. This is defined as follows:
  - If (x, y) is such that $y \neq 0$, then let (x, y) have a neighborhood base given by $B((x, y), r)$.
  - Let $(x, 0)$ have a neighborhood base given by $\{(x, 0)\} \cup B((x, 0), r) \setminus (\mathbb{R} \times \{0\})$ for any $x', y' \in \mathbb{R}$.

  This is a basis for a topology. It is T2 since any 2 points can be separated by discs. However, L is a closed subset of $\mathbb{R}^2$ with this topology. Also, as a subspace it has the discrete topology. Therefore $Z = L \setminus \{(0, 0)\}$ is a closed subset as well. Moreover, $P = (0, 0)$ is closed (in fact all 1 point sets are closed). But there exist no sepation of Z and P! (Pictures!)
- Another example from the book is $\mathbb{R}$ with the topology (a, b) and $(a, b) \setminus \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ from the first homework (maybe with slight modification). This is T2 but not T3.

[1] You can check these out at `https://en.wikipedia.org/wiki/Separation_axiom#Main_definitions`.

[2] It should also be noted that the words regular and normal are some of the most abused/overused terms in mathematics. For example, in the world of commutative algebra, regular is a far more stringent condition than normal. Here, that realization is flipped!

- The **tangent disc topology** of $\mathbb{R}^2$ is a topology for which
  - If (x, y) is such that $y \neq 0$, then let (x, y) have a neighborhood base given by $B((x, y), r)$.
  - Let $(x, 0)$ have a neighborhood base given by $\{(x, 0)\} \cup B((x, r), r) \cup B((x, -r), r)$ for any $x', y' \in \mathbb{R}$.

  This space can be shown (with some work!) to be T3 but not T4. Also, all points are closed.

Now we can break into some properties of T3 and T4 spaces. Here are some equivalent formulations:

Theorem 16.4. *Let X be a topological space.*

1) *X is T3 if and only if every neighborhood U of a point $x \in X$ has a smaller neighborhood V such that $\bar{V} \subseteq U$.*
2) *X is T4 if and only if every neighborhood U of a closed set $Z \subseteq X$ has a smaller neighborhood V such that $\bar{V} \subseteq U$.*

Proof. 1) ($\Rightarrow$): Suppose X is T3. Given U, we note that we may assume by shrinking U that it is open. Therefore, U^c is closed. Therefore, there exists V containing x and V' containing U^c open disjoint sets. But this implies

$$\bar{V} \subseteq V'^c \subseteq (U^c)^c = U$$

($\Leftarrow$): Suppose $x \in X$ and A is a closed subset of X. Taking $U = A^c$ produces an open neighborhood of x. Therefore, applying the property, we see that V is a neighborhood of x disjoint from the open neighborhood $\bar{V}^c$ of A.

2) The argument when X is T4 uses identical methods.

$\square$

Finally, a quick statement about subspaces and products of T2 and T3 spaces.

Proposition 16.5. *If X is a T2 (resp. T3) space, then so is any $Y \subseteq X$ with the subspace topology. If X_α are T2 (resp. T3) for all α, then so is $\prod_\alpha X_\alpha$ with product topology.*[3]

Proof. I will only prove the statements for T3. The T2 statements are similar but strictly easier. Suppose X is T3. Let $y \in Y$ and $Z \subseteq Y$ be a closed subset. This implies that there exists Z' a closed subset of X such that $Z = Z' \cap Y$. Since X is T3, we have U, V open subsets of X such that U contains y and V contains Z. Intersecting these open sets with Y yields a separation of y, Z in Y.

Now, let X_α be T3, and suppose $x = (x_\alpha) \in X = \prod_\alpha X_\alpha$ and $Z \subseteq X$ is a closed subset. Choose an open neighborhood $U = \prod_\alpha U_\alpha$ of x disjoint from Z. Note this is possible since Z^c is open. Now, by Theorem 16.4, we see that there exists $V_\alpha \in U_\alpha$ an open neighborhood of x_α with $\bar{V}_\alpha \subseteq U_\alpha$. If $U_\alpha = X_\alpha$, we may assume $V_\alpha = X_\alpha$ to stay within the product topology. Therefore, $V = \prod_\alpha V_\alpha$ is an open neighborhood of x disjoint from $\bar{V}^c \supseteq Z$. $\square$

Example 16.6. Recall the topology τ on $\mathbb{R}$ generated by $[a, b)$. You have shown this is strictly finer than the Euclidean topology. Note that this space is T4 and points are closed. This follows from Theorem 16.4. However, I claim the product of 2-copies is non-T4: $X = \mathbb{R} \times \mathbb{R}$. Consider $\Gamma \subseteq X$ given as the graph of $f(x) = -x$. This is closed in the Euclidean topology, thus also closed in τ. Note Γ with the subspace topology is discrete; $(x, y) \in [x, x + \epsilon) \times [y, y + \epsilon)$. Therefore every subset of Γ is closed.

Let Z be the set of points on Γ with rational coordinates, and $Z' = \Gamma \setminus Z$ be the irrational coordinates. Then there exist no open subsets in X separating these two sets.

[3]These statements are false for normal spaces. The points play an important role in the proof.

CLASS 17, OCTOBER 19: T4/NORMAL SPACES

Today, we will study the notion of normalcy. On Homework 5, we have shown that every metric space is normal. On the other hand, given a specific space (such as problem 3 of the same homework), it is often difficult 'by hand' to show that a space is normal. Here we give two additional criteria to ensure normality.

Theorem 17.1. *A T3/regular second-countable space X is normal.*

Since regularity can be checked using open neighborhoods of points, whereas normality requires open neighborhoods of closed sets, this is a vast simplification. Also, combined with Theorem 16.5, this implies subspaces and finite products of regular and second countable spaces are themselves normal!

Proof. Suppose $\mathcal{B} = \{U_1, U_2, \dots\}$ is a countable basis for X, and A and B are closed disjoint subsets. For any fixed point $a \in A$, there exists U_a and V_a disjoint open sets such that $a \in U_a$ and $B \subseteq V$. By the neighborhood criteria, we can choose $U_a' \subseteq U_a$ a neighborhood of a such that $\bar{U_a'} \subseteq U_a$. Furthermore, since $\mathcal{B}$ is a basis, we note that there is some $U_{i(a)} \subseteq U_a'$ lying in $\mathcal{B}$. This allows us to choose a countable cover of A in $\mathcal{B}$ such that the closure of each set is disjoint from B.

Symmetrically, choose $V_{j(a)}$ covering B whose closures are disjoint from A.

$$A \subseteq \bigcup_{i(a)} U_{i(a)} = U \qquad\qquad B \subseteq \bigcup_{j(b)} V_{j(b)} = V$$

These are countable covers of their respective sets that need not be disjoint. Enumerate the $i(a)$ with $i_1, i_2, \dots$ and the $j(b)$ by $j_1, j_2, \dots$. By subtracting closed sets (e.g. intersecting with their open complements), we can form the desired open sets:

$$U_{i_k}' = U_{i_k} \setminus \bigcup_{l=1}^{i_k} \bar{V}_{j_l} \qquad\qquad V_{j_k}' = V_{j_k} \setminus \bigcup_{l=1}^{j_k} \bar{U}_{i_l}$$

Let U' and V' be the unions of the U_{i_k}' and V_{j_k}' respectively. Note these new sets still cover their respective spaces, since $a \notin \bar{V}_{i_k}$ and $b \notin \bar{U}_{i_k}$ for any $k \in \mathbb{N}$, $a \in A$, and $b \in B$. Furthermore, they are disjoint. If $x \in U' \cap V'$, then $x \in U_{i_k}' \cap V_{j_{k'}}'$. But one of these sets was removed from the other! This completes the proof. $\square$

Next up, we can also upgrade T2/Hausdorff to normal if we assume the space is compact.

Theorem 17.2. *Every compact Hausdorff space X is normal.*

Proof. You've actually proved a more general version of this on the midterm. Indeed, if A, B are closed subsets of a compact space, they are themselves compact. Therefore, by problem 9 on the midterm, you can separate A, B by open sets. $\square$

Finally, I add one statement about normality of the order topology:

Theorem 17.3. *If X is totally ordered set, then X with the order topology is normal*[1].

[1] In fact, it is T5.

This can be viewed as a generalization of the fact that $\mathbb{R}$ is normal.

Proof. Let A and B be closed subsets of X. We may assume WLOG that no element of A or B is an endpoint of X, i.e. A and B don't contain a largest or smallest element of X (If it does, add ∞ and $-\infty$ to X to enlarge the set). For $a \in A$, choose (invoking the axiom of choice) p_a, q_a satisfying the following conditions:

1) $p_a < a < q_a$.
2) $(p_a, q_a) \cap B \neq \emptyset$.
3) $(a, q_a) = \emptyset$ **or** $q_a \in A$ **or** ($q_a \notin B$ and $(a, q_a) \cap A = \emptyset$).
4) $(p_a, a) = \emptyset$ **or** $p_a \in A$ **or** ($p_a \notin B$ and $(p_a, a) \cap A = \emptyset$).

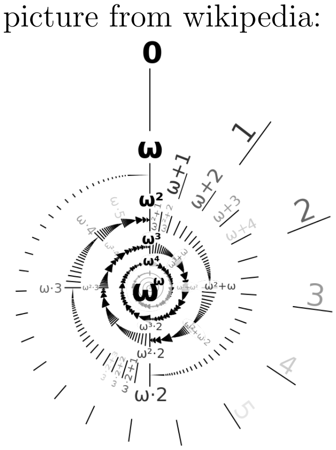
It goes to verify such points exist. 1) is satisfied by our assumption of non-max/minimality of A. 2) is by virtue of the fact that B^c is open and $a \in B^c$. For 3, we proceed as follows. Let $q > a$ satisfy the 2 previous properties. If $(a, q) = \emptyset$, let $q_a = q$. If $(a, q) \cap A \neq \emptyset$, choose $q_a \in (a, q) \cap A$. Lastly, if $(a, q) \neq \emptyset$ but is disjoint from A, choose $q_a \in (a, q)$. A similar argument shows p_a exists.

Now, we may consider $U = \bigcup_{a \in A}(p_a, q_a)$. This open set necessarily contains A. We can furthermore construct an open set $V = \bigcup_{b \in B}(p_b, q_b)$ containing B. Consider the intersection:

$$\begin{aligned}
U \cap V &= \left(\bigcup_{a \in A}(p_a, q_a)\right) \bigcap \left(\bigcup_{b \in B}(p_b, q_b)\right) \\
&= \bigcup_{a \in A} \bigcup_{b \in B} (p_a, q_a) \cap (p_b, q_b) \\
&= \bigcup_{a \in A} \bigcup_{b \in B} ((p_a, a) \cup \{a\} \cup (a, q_a)) \cap ((p_b, b) \cup \{b\} \cup (b, q_b)) \\
&= \bigcup_{a \in A} \bigcup_{b \in B} ((p_a, a) \cap (p_b, b)) \cup ((p_a, a) \cap (b, q_b)) \cup ((a, q_a) \cap (p_b, b)) \cup ((a, q_a) \cap (b, p_b))
\end{aligned}$$

Conditions 3/4 imply that each pairwise intersection must be empty. In particular, the last of the **or** conditions is the only one that isn't completely obvious. ☐

Example 17.4. The space of *ordinal numbers* is naturally ordered by size. We have notions of $0, 1, 2, 3, \ldots$, but then we reach countable infinity: $\omega, \omega + 1, \omega + 2, \ldots$. Next we reach $2\omega, 2\omega + 1, \ldots, n\omega$ for all integers n, ω^2 as their limit, etc. This is excellently illustrated by a picture from wikipedia:



The set of such things, even up to ω^2, is in bijection with $\mathbb{R}$, thus the whole space is horribly uncountable. However, there is some interesting topology/geometry here. Endowing it with the order topology, we get a set which is normal while not being second countable or compact (even if we restrict to $[0, \omega]$).

CLASS 18, OCTOBER 22: THE URYSOHN LEMMA

Today, we will begin a study of some very important theorems for normal topological spaces. The first of which is called Urysohn's Lemma. The proof is substantial and brings in an idea that we haven't seen before. Today I will state the theorem, provide one corollary/application, and finally prove the theorem.

Theorem 18.1 (Urysohn Lemma). *Let X be a normal subspace, and A and B be two disjoint closed subspaces. Then there exists a continuous function $f : X \to [a, b] \subseteq \mathbb{R}$ such that $f(a') = a$ and $f(b') = b$ for all $a' \in A$ and $b' \in B$.*

Therefore, we can separate any two closed sets, not only with open sets, but with a function! This seems more general ($A \subseteq f^{-1}((a, \frac{a+b}{2})) = U$ and $B \subseteq f^{-1}((\frac{a+b}{2}, b)) = U$), but Urysohn shows that they are equivalent notions. The same is not true for T_3, which brings about the notion of $T_{3\frac{1}{2}}$:

Definition 18.2. A space X is $T_{3\frac{1}{2}}$/**completely regular** if for every point $x \in X$ and closed subset A, there exists $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$.

So again, if X has closed points, we see

$$\text{T4} \Rightarrow T_{3\frac{1}{2}} \Rightarrow \text{T3}$$

Here is a beautiful application of this result, allowing us to produce something called a *partition of unity* on a compact Hausdorrf space (or Manifold).

Corollary 18.3. *If X is a compact Hausdorff space, and $U_1, \ldots, U_n$ is an open cover, then there exists functions $f_i : X \to [0, 1]$ such that $f = f_1 + \ldots + f_n$ has the property that $f(x) = 1$ for all $x \in X$, and $f_i^{-1}((0, 1]) \subseteq U_i$.*[1]

Note that any open cover can be refined as such, so this implies something for general covers as well.

Proof. By virtue of Theorem 17.2 from last class, we see that X is a normal space. I claim we can choose V_i with $\bar{V_i} \subseteq U_i$ still covering X. Indeed, consider $A = (U_2 \cup \ldots \cup U_n)^c \subseteq U_1$. Now, as a result of our equivalent definition of T4 (Theorem 16.4), we note that there exists $V_1 \subseteq U_1$ with

$$A \subseteq V_1 \subseteq \bar{V_1} \subseteq U_1$$

So $V_1, U_2, \ldots, U_n$ still covers X. Repeat this process finitely many times.

Given this tiny lemma, we now have $V_i \subseteq \bar{V_i} \subseteq U_i$, with the V_i covering X. Repeat this procedure with the V_i to produce $W_i \subseteq \bar{W_i} \subseteq V_i$ covering X. Now we can apply Urysohn's Lemma to the pair of closed sets $(\bar{W_i}, V_i^c)$: $\exists g_i : X \to [0, 1]$ with $g_i(W_i) = 1$ and $g_i(V_i^c) = 0$. Note that since the W_i cover X, the function $g_1 + g_2 + \ldots + g_n$ is never 0 (in fact always ≥ 1). Therefore, the desired function is

$$f_i(x) = \frac{g_i(x)}{g_1(x) + \ldots + g_n(x)}$$

[1]We say f_i is **supported** in U_i. *Supp*$(f) = \overline{\{x \in X \mid f(x) \neq 0\}} \subseteq U_i$.

Note that $f_i(x) = 0$ for $x \in U_i^c \subseteq \bar{V}_i^c$. □

Proof. (of Theorem 18.1) It suffices to prove the statement when $a = 0$ and $b = 1$, since $[0, 1] \cong [a, b]$ by a linear homeomorphism. We proceed with this proof in several steps:

Step 1) We will construct $U_r \subseteq X$ an open subset for each $r \in \mathbb{Q} \cap [0, 1]$ such that for $p < q$, we have

$$p < q \implies \bar{U}_p \subseteq U_q$$

This goes as follows: Choose $U_1 = X \setminus B$. Now choose U_0 satisfying this property and containing A by Theorem 16.4. Now, enumerate $\mathbb{Q} \cap (0, 1)$, say by $p_1, p_2, \ldots$. We construct U_{p_i} inductively by taking $p_j < p_i < p_k$, where $j, k < i$, and p_j, p_k are the largest and smallest numbers with this property so far constructed[2]. Then let U_{p_i} again be the open set as in Theorem 16.4:

$$Z = \overline{U_{p_j}} \subseteq U_{p_i} \subseteq \overline{U_{p_i}} \subseteq U_{p_k}$$

This produces the desired sets U_r.

Step 2) Next, we construct the desired function. Let $f : X \to [0, 1]$ be the function

$$f(x) = \begin{cases} 1 & x \in B \\ \inf\{r \in \mathbb{Q} \cap [0, 1] \mid x \in U_r\} & x \notin B \end{cases}$$

Note first that this is a well defined function since if $x \notin B$, $x \in U_1$. Therefore, the infimum set is non-empty and bounded below by 0. Furthermore, it is easy to check that $f(b) = 1$ and $f(a) = 0$ for $b \in B$ and $a \in A$.

Step 3) Finally, it goes to check the continuity of f. This can be verified by showing that for any neighborhood $y \in N_y \subseteq [0, 1] \cap \mathbb{Q}$, and for any $x \in X$ with $f(x) = y$, there exists a neighborhood $x \in N_x \subseteq X$ with $f(N_x) \subseteq N_y$. This follows by noting the following two facts:

- If $x \in U_r$, then $f(x) \leq r$.
- If $x \notin \bar{U}_r$, then $f(x) \geq r$.

The first claim is obvious, since the set $S_x = \{r \in \mathbb{Q} \cap [0, 1] \mid x \in U_r\}$ has the property that $r \in S_x$. The infimum is the largest lower bound, so we have $f(x) \leq r$.

For the second claim, we note that if $x \notin \bar{U}_r$, then

$$S_x = \{r \in \mathbb{Q} \cap [0, 1] \mid x \in U_r\} = \{r \in \mathbb{Q} \cap (r, 1] \mid x \in U_r\}$$

and thus r is a (potentially not largest) lower bound for $f(x)$.

- Given $y = 1$, and a neighborhood $N_y = (a, 1]$, we can take $N_x = X \setminus \overline{U_{\frac{a+1}{2}}}$. The notes above imply $f(N_x) \subseteq [\frac{a+1}{2}, 1] \subseteq N_y$.
- Given $y = 0$, and a neighborhood $N_y = [0, b)$, we can take $N_x = U_{\frac{b}{2}}$. The notes above imply $f(N_x) \subseteq [0, \frac{b}{2}] \subseteq N_y$.
- Finally, if $0 < y < 1$ and $N_y = (a, b)$, set $N_x = U_{\frac{b+y}{2}} \setminus \overline{U_{\frac{a+y}{2}}}$, and note

$$f(N_x) \subseteq [\frac{a+y}{2}, \frac{b+y}{2}] \subseteq N_y$$

This completes the proof. □

[2] $S = \{0, 1, p_1, p_2, \ldots, p_{i-1}\}$, then $p_j = \max(S \cap [0, p_i))$ and $p_k = \min(S \cap (p_i, 1])$.

CLASS 19, OCTOBER 24: THE URYSOHN METRIZATION THEOREM

Our objective today is to demonstrate that a large class of topological spaces are in fact metric spaces. As a result, I will convince you that the beautiful Urysohn Lemma was actually intended to be a lemma by its creator, to prove a corresponding result about *metrizability*. We will use this notion of metrizability so I define it here. But the core idea is the following:

$$(X, d) \overset{\text{Metric Topo}}{\Longrightarrow} (X, \tau) \overset{X \text{ Metrizable}}{\Longrightarrow} (X, d_\tau)$$

Definition 19.1. A topological space (X, τ) is called **metrizable** if there exists d a metric on X such that τ is the metric-topology associated to (X, d).

As a further application of Urysohn's Lemma, we have the following theorem:

Theorem 19.2 (Urysohn Metrization Theorem)**.** *Every second-countable T3 + T1 topological space (X, τ) is metrizable.*

Note that by Theorem 17.1, we already know that X is normal. However, not all normal spaces are metrizable. So this can be viewed as a strengthening of Theorem 17.1 as well as a partial converse to the result of Homework 5, Exercise 5: Metric spaces are normal. Before this, I demonstrate a claim from Class 14:

Lemma 19.3. *The space $\mathbb{R}^{\mathbb{N}}$ with the product topology is metrizable.*

Proof. I claim that if $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$, and we let $d_i = \min\{|x_i - y_i|, 1\}$, then

$$d(x, y) = \sup_{i \in \mathbb{N}} \left\{ \frac{d_i(x, y)}{n} \right\}$$

is the required metric. Note that

$$B(x, \epsilon) = \prod_{i=1}^{n} (x_i - i \cdot \epsilon, x_i + i \cdot \epsilon) \times \prod_{i > n} \mathbb{R}$$

where n is chosen minimal such that $\frac{1}{n} < \epsilon$. Therefore, the metric topology of d is contained within the product topology. On the other hand, given a basis element of the product topology

$$U = \prod_{i=1}^{n} (a_i, b_i) \times \prod_{i > n} \mathbb{R}$$

and $x \in U$, we get that $B(x, r) \subseteq U$, where

$$r = \min \left\{ \frac{x_i - a_i}{i}, \frac{b_i - x_i}{i} \mid i = 1, 2, \ldots, n \right\}$$

$\square$

Now, to prove the metrization theorem, we aim to prove that a second countable metric space may be embedded[1] into $\mathbb{R}^{\mathbb{N}}$ with the product topology. This, combined with

[1] A topological embedding is short for a map $f : X \to Y$ which is a homeomorphism of X to $f(X)$.

Lemma 19.3 yields the desired result, since a subspace of a metric space is a metric space. First we need another lemma:

Lemma 19.4. *If (X, τ) is a second-countable T3 space, then there exists a countable collection $f_n : X \to [0,1]$ such that for any $x \in X$ and neighborhood U of x, there exists n such that $f(x) \neq 0$ and $f(U^c) = 0$.*

Proof. Let $U_1, U_2, \ldots$ be a countable basis for τ. For any pair of indices n, m such that $\overline{U_n} \subseteq U_m$, apply Urysohn's Lemma to produce a function $f_{n,m} : X \to [0,1]$ such that $f_{n,m}(\overline{U_n}) = 1$ and $f_{n,m}(U_m^c) = 0$. This is a countable collection of functions since it is surjected onto by $\mathbb{Z}^2$.

Now, given U a neighborhood of x, there exists $U_m \subseteq U$ containing x by definition of a basis. Now applying Theorem 16.4, we know there is some open subset $V \subseteq U_m$ containing x whose closure lies within U_m. But since it is open, it contains some U_n containing x!. The function $f_{n,m}$ then satisfies $f_{n,m}(x) = 1$ since $x \in U_n$, and $f(U^c) = 0$ since $U^c \subseteq U_m^c$. □

We can now prove the main result.

Proof. (of Theorem 19.2): Let $f_1, f_2, \ldots$ be the collection of functions guaranteed by Lemma 19.4. Take $F : X \to \mathbb{R}^{\mathbb{N}} : x \mapsto (f_1(x), f_2(x), \ldots)$.

First, I claim F is an embedding. Note that F is continuous:

$$F^{-1}\left(U_1 \times \ldots \times U_n \times \prod_{i \geq n+1} \mathbb{R}\right) = f_1^{-1}(U_1) \cap \ldots \cap f_n^{-1}(U_n)$$

Now, it goes to show that F is injective. Suppose $x \neq y$. Then $\exists U$ an open neighborhood of x not containing y (e.g. $\{y\}^c$ using T1). Therefore, there exists U_n for which $x \in U_n$ and $y \notin U_n$, implying $f_n(x) > 0 = f_n(y)$. Therefore, $F(x) \neq F(y)$.

Now, it goes to prove that X is homeomorphic to its image. We know this is equivalent to checking F is an open map. Let $U \subseteq X$ be an open set, and let $V = F(U)$. Letting $v \in V$, it suffices to construct an open neighborhood V' of v contained within V. Let $x \in X$ be such that $F(x) = v$. Since U is an open neighborhood of x, there exists f_n such that $f_n(x) > 0$, and $f_n(U^c) = 0$. That is to say $f_n^{-1}((0,\infty)) \subseteq U$.

Therefore, we note that $\pi_n^{-1}(0,\infty)$ is an open of $\mathbb{R}^{\mathbb{N}}$, and therefore $V' = \pi_n^{-1}(0,\infty) \cap F(X)$ is open in the subspace topology of $F(X)$. Notice that $v \in V'$:

$$\pi_n(v) = \pi_n(F(x)) = f_n(x)$$

Lastly, we need to show $V' \subseteq V$. For $v' \in V'$, notice that $v' = F(x')$ for some x', and $\pi_n(z) \in (0,\infty)$. But $\pi_n(v') = f_n(x')$, and therefore $x' \in U$ since $f_n(U^c) = 0$. Therefore, $v' = F(x) \in F(U)$. This completes the proof. □

It should be noted that in Munkres, they prove the same fact with the uniform topology. The last order of business is to note that we proved something slightly stronger:

Corollary 19.5 (Embedding Theorem)**.** *If X is T1, then if $f_\alpha : X \to \mathbb{R}$ for $\alpha \in \Lambda$ are continuous functions such that every point x and neighborhood U has some f_α with $f_\alpha(x) > 0$ and $f_\alpha(U^c) = 0$, then $F : X \to \mathbb{R}^\Lambda : x \mapsto (f_\alpha(x))_{\alpha \in \Lambda}$ is an embedding. We can also restrict the domain to $[0,1]^\Lambda$ if each f_α is bounded in $[0,1]$.*

CLASS 20, OCTOBER 26: THE TIETZE EXTENSION THEOREM

Finally, we come to the Tietze Extension Theorem. This provides a tool which is applicable to things such as metric spaces, but also far more broadly. The formal statement is as follows:

Theorem 20.1 (Tietze Extension Theorem)**.** *Let A be a closed subspace of a normal topological space X. Then if $f : A \to \mathbb{R}$ is a continuous map, then there exists a continuous map $f' : X \to \mathbb{R}$ such that $f(a) = f'(a)$ for all $a \in A$. Moreover, if f is bounded in $[a, b]$, we may assume f' is as well.*

Note that this produces Urysohn as a corollary. Take the subspace $A \cup B$ and the continuous function f to be constantly 0 on A and 1 on B (it is continuous since A, B are open and closed in the subspace topology by virtue of their separation). Teitze allows us to extend f to a continuous function on all of X! Since we use Urysohn's Lemma in the proof of Tietze, we actually show that they are equivalent theorems.

We will use one tool from real analysis here, since our image is a metric space.

Lemma 20.2. *If $f_n : X \to \mathbb{R}$ are a sequence of continuous functions converging uniformly to $f : X \to \mathbb{R}$ ($\forall \epsilon > 0$, $\exists N > 0$ such that $|f(x) - f_n(x)| < \epsilon \ \forall n \geq N$). Then f is a continuous function.*

Proof. This follows by the triangle inequality. For a given $\epsilon > 0$, consider

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f_n(x)|$$

Therefore, we can intersect the 3 open neighborhoods of x resulting from $\frac{\epsilon}{3}$ in the definition of continuity of f_n, and uniform convergence of $f_n \to f$. $\square$

Proof. (of Theorem 20.1): I will begin by proving the bounded version: $f : A \to [-m, m]$ where $m = \sup\{|f(a)| \mid a \in A\}$. Consider the subsets $Z_0^- = f^{-1}\big(\big(-\infty, -\frac{m}{3}\big]\big)$ and $Z_0^+ = f^{-1}\big(\big(\frac{m}{3}, \infty\big)\big)$. They are closed and disjoint subsets of X (since A is closed). Therefore, we can apply Urysohn's Lemma to find a continuous function $g_0 : X \to [-\frac{m}{3}, \frac{m}{3}]$ which is constantly $-\frac{m}{3}$ on Z_0^- and $\frac{m}{3}$ on Z_0^+.

Now we may consider the function $f_1 = f - g_0 : A \to \mathbb{R}$ which is still bounded, since it is a difference of bounded functions. Note that since $|g_0(a)| \leq \frac{m}{3}$, we have $|f_1(a)| \leq \frac{2m}{3}$ by the triangle inequality. Let $Z_1^- = f_1^{-1}\big(\big(-\infty, -\frac{2m}{9}\big)\big)$ and $Z_1^+ = f^{-1}\big(\big(\frac{2m}{9}, \infty\big)\big)$. Repeating the procedure, we retrieve a function $g_1 : X \to [-\frac{2m}{9}, \frac{2m}{9}]$ constantly $-\frac{2m}{9}$ on Z_1^- and $\frac{2m}{9}$ on Z_1^+. Therefore, we obtain $|(f - g_0 - g_1)(a)| < \frac{4m}{9}$ for all $a \in A$.

Continuing in this manner, we can produce g_n with

$$|(f - g_0 - \ldots - g_n)(a)| < \frac{2^{n+1}m}{3^{n+1}}$$

Therefore, the functions $f_n = g_0 + g_1 + \ldots + g_n$ converges uniformly to f on A. Letting $f' = \lim_{n\to\infty} f_n$ produces a function on X agreeing with f on A.

Now suppose that $f : A \to \mathbb{R}$ isn't necessarily bounded. We note that $\mathbb{R} \cong (-\frac{\pi}{2}, \frac{\pi}{2})$ by virtue of the continuous function $h = \tan^{-1}$. Therefore, note that

$$h \circ f : A \to (-\frac{\pi}{2}, \frac{\pi}{2})$$

So we can apply the previous result to obtain an extended function $f' : X \to (-\frac{\pi}{2}, \frac{\pi}{2})$. Finally, note that $h^{-1} \circ f' : X \to \mathbb{R}$ has the property that $h^{-1} \circ f'(x) = f(x)$ for all x. Indeed,

$$f' = h \circ f \quad \Rightarrow h^{-1} \circ f' = h^{-1} \circ h \circ f = f$$

This completes the proof. $\square$

As an immediate consequence, we get the following result from analysis;

Corollary 20.3. *A metric space (X, d) is compact if and only if every continuous function $f : X \to \mathbb{R}$ is bounded.*

Proof. ($\Rightarrow$): If X is compact, we get that $f^{-1}((n, n+1))$ forms a cover as n varies. Therefore, a finite subcover will do, and the maximum $|n| + 1$ from this sub will do.

($\Leftarrow$): Suppose every continuous function is bounded. Note that in the world of metric spaces, X is compact if and only if it is sequentially compact. Assume (aiming for a contradiction) $(x_1, x_2,)$ is a sequence in X with no convergent subsequence. Then the set $A = \{x_1, x_2, \ldots\}$ is closed (since it contains all of its limit points). Therefore, we can define the function $f : A \to \mathbb{R}$ by $x_n \mapsto n$. This is continuous since A has the discrete topology as a subspace. Therefore, we can extend f to an unbounded function $f : X \to \mathbb{R}$ by Tietze. This contradicts are assumption. $\square$

It should be noted that the proof does not extend to arbitrary normal spaces, as we only have compactness implies sequential compactness. An example to show that the reverse is not true is the normal subspace of ordinals $< \Omega := \omega^\omega$, denoted in the book by S_Ω.

We can also extend Tietze's Extension Theorem to more general ranges. Here is a formal statement:[1]

Theorem 20.4 (Tietze Extension V2)**.** *Suppose $A \subseteq X$ is a closed subspace of a normal space X and Z is a retraction of the space $\mathbb{R}^\Lambda$ for some indexing set Λ. Then if $f : A \to Z$ is a continuous map, there exists $f' : X \to Z$ extending f.*

Proof. Since Z is a retraction of $\mathbb{R}^\Lambda$, there exists a continuous function $r : \mathbb{R}^\Lambda \to Z$ which is the identity when restricted to $Z \subseteq \mathbb{R}^\Lambda$. Therefore, given f as in the theorem, it suffices to construct $f' : X \to \mathbb{R}^\Lambda$ agreeing with f on A and apply r: letting $f'(a) = z$, note that $r \circ f : X \to Z$ has the property that

$$r(f(a)) = r(f'(a)) = r(z) = z$$

Therefore, it suffices to check the statement when $Z = \mathbb{R}^\Lambda$. But $f : A \to \mathbb{R}^\Lambda$ is continuous if and only if each of its coordinate functions $f_\alpha = \pi_\alpha \circ f : A \to \mathbb{R}$ are continuous. Therefore, we can apply the original Tietze Extension Theorem to produce $f'_\alpha : X \to \mathbb{R}$ agreeing with f_α. Then let $f' = (f_\alpha) : X \to \mathbb{R}^\Lambda$ be the desired extension. $\square$

[1]In Munkres, this is phrased as the **Universal Extension Property**, introduced in $\oint$35, problem 5.

CLASS 21, OCTOBER 29: EMBEDDINGS OF MANIFOLDS

Today, we move on to one of the central topics of general and differential topology; Manifolds. As we have seen so far in this course, the world of topological spaces at large can be quite daunting; incredibly wild and non-intuitive things can happen. So we have specialized to things like normal spaces and showed that they exhibit many of the desirable properties of metric spaces. Here is a similar style of specialization that produces more well behaved topological spaces:

Definition 21.1. A topological space is called an m-**manifold** if X is a second-countable Hausdorff space which is **locally Euclidean**. That is to say there exists a neighborhood U of any point $x \in X$ such that $U \cong U' \subseteq \mathbb{R}^m$, where U' is an open subset of $\mathbb{R}^n$.[1]

Often times, 1-manifolds are called **curves**, 2-manifolds are called **surfaces**, and m-manifolds for $m \geq 3$ are shortened to m-**folds**. In addition, the maps representing the homeomorphisms $\varphi_i : U \to U' \subseteq \mathbb{R}^m$ are referred to as **charts**, and the collection $\{\varphi_i\}$ are called an **atlas**.

Example 21.2. It may seem bizarre that we have neighborhoods of any point homeomorphic to a Hausdorff space, but require that X be Hausdorff. This example shows the importance of the Hausdorff condition.

Let X be the quotient of $\mathbb{R} \coprod \mathbb{R}$ by the relation that if $x \neq 0$, then $x_1 \sim x_2$, where the subscripts are denoting which copy of $\mathbb{R}$ the point is being viewed in. The space X is referred to as the real line with the origin doubled, since $0_1 \not\sim 0_2$.

Now, note that X is not Hausdorff: there do not exist open disjoint sets U, V separating 0_1 from 0_2. Indeed, they look like open neighborhoods of 0 in $\mathbb{R}$ containing either 0_1 or 0_2, and therefore for some $\epsilon > 0$,

$$(-\epsilon, 0) \cup (0, \epsilon) \subseteq U \cap V.$$

On the other hand, this space *is* locally Euclidean. Indeed, $X \setminus \{0_1\} \cong \mathbb{R} \cong X \setminus \{0_2\}$.

Recall that in Corollary 18.3 we proved that every compact Hausdorff space in fact has a finite partition of unity. I define this notion here just to reiterate.

Definition 21.3. Given a locally finite open cover $X = \bigcup_\alpha U_\alpha$ (c.f. Homework 1), a collection of functions $f_\alpha : X \to [0, 1]$ is said to be a **partition of unity subordinate to** $\{U_\alpha\}$ if $\mathrm{Supp}(f_\alpha) \subseteq U_\alpha$, and $\sum_\alpha f_\alpha(x) = 1$ for every $x \in X$.[2]

This result allows us to show a baby version of the famous Whitney Embedding Theorem.

Theorem 21.4. *Let X be a compact m-manifold. Then there exists an $n \gg 0$ such that*

$$\iota : X \hookrightarrow \mathbb{R}^n$$

where ι is an embedding, e.g. an injective map which is a homeomorphism onto its image.

[1] Sometimes these objects are referred to as **topological manifolds** to avoid confusion with their differential version; **smooth manifolds**.

[2] Note the locally finite condition makes this sum a finite sum! So we needn't worry about convergence.

So ANY compact manifold is a subspace of some Euclidean $\mathbb{R}^n$!

Proof. Let $\varphi_i : U_i \to \mathbb{R}^m$ be charts for X, where $i = 1, 2, \ldots, n$ since X is compact. Since X is compact and Hausdorff, it is normal by Theorem 17.2. Applying Corollary 18.3, we know that a partition of unity f_i subordinate to U_i exists. Let

$$A_i = Supp(f_i) = \overline{\{x \in X \mid \varphi_i(x) \neq 0\}}.$$

Then for $i = 1, \ldots, n$, define a new function

$$h_i(x) = \begin{cases} f_i(x) \cdot \varphi_i(x) & x \in U_i \\ 0 & x \in A_i^c \end{cases}$$

Note this function is well defined, because if $x \in U_i \setminus A_i$, the $f_i(x) = 0$. So the 2 internal functions agree on the overlaps of their domains. Additionally, h_i is continuous, because it is continuous when restricted to the open sets U_i and A_i^c, and therefore the preimage of an open is a union of 2 open sets.

Now we may consider the desired function:

$$\iota : X \to (\mathbb{R})^n \times (\mathbb{R}^m)^n \cong \mathbb{R}^{n(m+1)} : x \mapsto (f_1(x), \ldots, f_n(x), h_1(x), \ldots, h_n(x))$$

ι is a product of continuous functions, therefore continuous. Next, I claim ι is injective. Suppose $\iota(x) = \iota(y)$. Then $h_i(x) = h_i(y)$ and $f_i(x) = f_i(y)$ for all $i = 1, 2, \ldots, n$. Since

$$1 = (\sum_{i=1}^{n} f_i)(x) = (\sum_{i=1}^{n} f_i)(y)$$

we know there is some i s.t. $f_i(x) = f_i(y) > 0$. But this implies

$$f_i(x)\varphi_i(x) = f_i(y)\varphi_i(y)$$
$$\varphi_i(x) = \varphi(y)$$

But φ_i are charts, therefore injective. This implies $x = y$.

Finally, it goes to show X is homeomorphic to its image. This follows from Corollary 12.1, restated here for convenience: If $f : X \to Y$ is a continuous bijective map with X compact and Y Hausdorff, then f is a homeomorphism. The result then follows by virtue of the fact that $\iota(X) \subseteq \mathbb{R}^{n(m+1)}$, and subspaces of T2 spaces are T2. $\square$

The following example demonstrates the inefficiencies of Theorem 21.4.

Example 21.5. S^n is a manifold. Indeed, we can identify $\{N\}^c$ and $\{S\}^c$ with $\mathbb{R}^n$, when N and S are the north and south pole respectively. This identification can be made by a process of stereographic projection. Theorem 21.4 tells us we can embed $S^n \hookrightarrow \mathbb{R}^{2n+2}$. However, we know we can embed S^n in $\mathbb{R}^{n+1}$.

Similarly, we can give the n-dimensional torus $\mathbb{T}^n = S^1 \times \ldots \times S^1$ the structure of a manifold with $2n$-many charts. Theorem 21.4 allows us to embed $\mathbb{T}^n \hookrightarrow \mathbb{R}^{n^2+n}$, whereas in reality $\mathbb{R}^{n+1}$ suffices.

However, we are still embedding a manifold in a finite dimensional vector space, which is a huge advantage.

Just to complement our theorem, the Whitney's Embedding Theorem tells us that we can embed any smooth n-dimensional manifold in $\mathbb{R}^{2n-1}$. This requires a lot of machinery (its own class worth).

CLASS 22, OCTOBER 31: THE TYCHONOFF THEOREM

Now we have the desired tools to return to the question of arbitrary products of compact sets. The statement of the Tychonoff Theorem is as follows:

Theorem 22.1 (Tychonoff Theorem)**.** *If X_α is a collection of compact topological spaces, then so is $X = \prod_\alpha X_\alpha$ endowed with the product topology.*

As we've already seen, the box topology has far too many open sets to be viable for such a theorem even in the countable product case. This should further illustrate the desirability of the product topology. Recall the following equivalent definition of compactness for closed sets (Theorem 12.5 from our notes).

Theorem. *X is compact if and only if every collection $\mathcal{C}$ of closed sets such that $\emptyset \neq C_1 \cap C_2 \cap \cdots \cap C_n$ for any choice of $C_i \in \mathcal{C}$,[1] then*

$$\emptyset \neq \bigcap_{C \in \mathcal{C}} C.$$

Just to reiterate, this is proved by taking the complements the sets in an open cover, and vice-versa. It is in fact the contrapositive statement. To prove Theorem 22.1, we will show instead that X has the finite intersection property. To do this, we require a few lemmas. The first is actually a very common tool which is equivalent to the axiom of choice (which is always assumed in topology).

Lemma 22.2 (Zorn's Lemma)**.** *If $(\mathcal{S}, <)$ is a non-empty partially ordered set in which every chain of elements of $\mathcal{S}$ has an upper bound $B \in \mathcal{S}$:*

$$B_1 < B_2 < B_3 < \ldots < B$$

Then $\mathcal{S}$ contains a maximal element $M \in S$, e.g. $\nexists M' \in \mathcal{S}$ such that $M < M'$.

The proof is a deep dive into the language of set theory. A nice accessible proof of this fact is given here: `https://arxiv.org/pdf/1207.6698.pdf`. We will instead choose to apply it in our specific scenario:

Lemma 22.3. *Let X be any set, and let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a collection of subsets with the finite intersection property. Then there exists a maximal collection $\mathcal{M} \subseteq \mathcal{P}(X)$ containing $\mathcal{A}$ with the property that $\mathcal{M}$ has the finite intersection property.*

Proof.

$$\mathcal{S} := \{\mathcal{B} \subseteq \mathcal{P}(X) \mid \mathcal{A} \subseteq \mathcal{B}, \text{ and } \mathcal{B} \text{ has the finite intersection property}\}.$$

Make it into a poset by inclusion. It suffices to check the condition of Zorn's Lemma for $\mathcal{S}$. Suppose $\mathcal{B}_i \in \mathcal{S}$ is such that

$$\mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \ldots$$

It suffices to check that $\mathcal{B} = \bigcup_{i=1}^\infty \mathcal{B}_i$. I claim $\mathcal{B} \in \mathcal{S}$. Suppose $C_1, \ldots, C_n \in \mathcal{B}$. Since $\mathcal{B}$ is a union of sets, we have that $C_i \in \mathcal{B}_{j_i}$ for some $j_i \in \mathbb{N}$ depending on i. Therefore, letting

[1] This is referred to as the finite intersection property.

$N = \max\{j_1, \ldots, j_n\}$, we have that $C_1, \ldots, C_n \in \mathcal{B}_N$. But $\mathcal{B}_N$ has the finite intersection property, so

$$C_1 \cap \ldots \cap C_n \neq \emptyset.$$

Zorn's lemma immediately implies the existence of the desired $\mathcal{M}$. □

Next up, we prove some nice properties of the resulting set $\mathcal{M}$.

Lemma 22.4. *Let $\mathcal{M}$ be the set obtained in Lemma 22.3. Then*

- *If $C_1, \ldots, C_n \in \mathcal{M}$, then $C_1 \cap \cdots \cap C_n \in \mathcal{M}$*
- *If $A \subseteq X$ has the property that $C \cap A \neq \emptyset$ for all $C \in \mathcal{M}$, then $A \in \mathcal{M}$.*

Proof.

- Suppose the statement is false, and consider $\mathcal{M}' = \mathcal{M} \cup \{C_1 \cap \ldots \cap C_n\}$. I claim $\mathcal{M}'$ has the finite intersection property. Indeed, if $C_i' \in \mathcal{M}$, then

$$(C_1 \cap \ldots \cap C_n) \cap C_1' \cap \ldots \cap C_m' = C_1 \cap \cdots \cap C_n \cap C_1' \cap \cdots \cap C_m' \neq \emptyset$$

  where the last equality follows by virtue of the fact that we intersected finitely many sets in $\mathcal{M}$. But $\mathcal{M} \subsetneq \mathcal{M}'$, which contradicts the maximality of $\mathcal{M}$.
- Similarly, suppose the statement is false, and consider $\mathcal{M}' = \mathcal{M} \cup \{A\}$. Note again that $\mathcal{M}'$ has the finite intersection property:

$$A \cap C_1 \cap \cdots C_n = A \cap (C_1 \cap \cdots C_n) \neq \emptyset$$

  where the last equality follows by virtue of the fact that $C_1 \cap \cdots \cap C_n \in \mathcal{M}$ by the first statement. Again, this contradicts the maximality of $\mathcal{M}$.

□

Proof. (of Theorem 22.1): Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a collection of subsets with the finite intersection property. It suffices to check that

$$\bigcap_{A \in \mathcal{A}} \bar{A} \neq \emptyset$$

as applying Theorem shows X is compact. Choose $\mathcal{M}$ according to Lemma 22.3. Note that by adding more sets, we have made the intersection only smaller. Therefore, it suffices to check

$$\bigcap_{A \in \mathcal{M}} \bar{A} \neq \emptyset.$$

For a given α, consider the projection map $\pi_\alpha : X \to X_\alpha$. Now, consider the collection $\mathcal{M}_\alpha = \{\pi_\alpha(A) \mid A \in \mathcal{M}\}$. By the compactness of X_α, we note that $\mathcal{M}_\alpha$ having the finite intersection property implies $\exists\ x_\alpha \in \bigcap_{A_\alpha \in \mathcal{M}_\alpha} \bar{A}_\alpha$. I claim $\mathbf{x} = (x_\alpha) \in \bar{A}$ for every $A \in \mathcal{M}$.

First I claim that if $x_\alpha \in U_\alpha \subseteq X_\alpha$ is an open set, then $\pi^{-1}(U_\alpha) \cap \bar{A} \neq \emptyset$. Since $x \in U_\alpha$ is an open neighborhood, and $x_\alpha \in \pi_\alpha(\bar{A}) \subseteq \overline{\pi_\alpha(A)}$ by assumption[2], $\exists y_\alpha = \pi_\alpha(\mathbf{y}) \in U_\alpha \cap \pi_\alpha(A)$. Therefore, $\mathbf{y} \in \pi_\alpha^{-1}(U_\alpha) \cap A$. This demonstrates the claim.

Finally, by the second part of Lemma 22.4, we see that $\pi_\alpha^{-1}(U_\alpha) \in \mathcal{M}$ for all U_α open neighborhoods of x_α in X_α. Therefore, by part 1 of Lemma 22.4, taking the intersection of finitely many such sets is in $\mathcal{M}$. But these are the basis elements of the product topology. But $\mathcal{M}$ has the finite intersection property, and every basis element containing $\mathbf{x}$ intersects every $A \in \mathcal{M}$. Therefore, $\mathbf{x} \in \bar{A}$ (it is a limit point) for all $A \in \mathcal{M}$, and thus $\mathcal{M}$ has the infinite intersection property. □

[2]They are in fact equal by HWK 4, #4.

CLASS 23, NOVEMBER 2: STONE-ČECH COMPACTIFICATION

Today we will add to our repertoire of methods to compactify a space. Since we didn't cover the 1-point compacfication in class (it is in Class 13 notes), I recall the procedure here.

Definition 23.1. A space Y is called a **compactification** of another space X if $\exists \iota : X \hookrightarrow Y$ an embedding such that Y is compact and Hausdorff and $\bar{X} = Y$. Two such compactications are **equivalent** if there is a homeomorphism $f : Y \to Y'$ such that $f(\iota(x)) = \iota'(x)$ for every $x \in X$:

Example 23.2 (1-point compactification)**.** Assume X is a locally Hausdorff space. Let $Y = X \cup \{\infty\}$, where ∞ is just a name for a new distinguished point. It goes to define a topology. A subset $U \subseteq Y$ is open if either

- $\infty \notin U$ (or equivalently $U \subseteq X$) and U is open in the topology of X.
- $\infty \in U$ and $U^c \subseteq X$ is a compact subset.

Note that this is in fact a topology. Y has the second property and $\emptyset$ has the first. The other 2 facts follow from the fact that arbitrary intersections of closed subsets are closed and finite unions of compact sets are compact. Y is called the **one-point compactification** of X.

1) If $X = \mathbb{R}$, then $Y = \mathbb{R} \cup \{\infty\} \cong S^1$.
2) If $X = \mathbb{C}$, then Y is the Riemann Sphere.
3) If $X = \mathbb{R}^n$, then $Y \cong S^n$.

Lemma 23.3. *Let X be a space, and $f : X \to Z$ be an embedding where Z is a compact Hausdorff space. Then there exists Y a compactification of X such that $\exists \iota : Y \to Z$ an embedding such that $f(x) = \iota(x)$ for each $x \in X$. Y is unique up to equivalence.*

Proof. Let $Y = \overline{f(X)} \subseteq Z$, and ι represent the inclusion as a map. Since Y is a closed subset of a compact Hausdorff space, Y with the subspace topology is compact and Hausdorff. Moreover, ι is still an embedding, since all we did was take a subspace of the range of f. Therefore, Y is a compactification of X.

Suppose Y' is another compactification with the desired properties, and let $\iota' : Y' \to Z$ be its embedding into Z. Note that we have $\iota(x) = f(x) = \iota'(x)$ for all $x \in X$. It suffices to show that Y and Y' are homeomorphic.

Note that since $f(X) \subseteq \iota'(Y')$, and $f(\bar{X}) = Y \subseteq Z$, we must have that $\iota'(Y') \subseteq \iota(Y)$. On the other hand, $\iota(Y')$ is the image of a compact set, and therefore is itself compact in a Hausdorff space. Therefore, it is closed. But $\iota(Y) = f(\bar{X})$. Therefore, $\iota'(Y') = \iota(Y)$.

Finally, since $\iota : Y \to \iota(Y) = \iota'(Y')$ and $\iota' : Y' \to \iota(Y')$ are homeomorphisms, we see that

$$\iota^{-1} \circ \iota' : Y' \to Y$$

is also a homeomorphism, with $\iota^{-1}(\iota'(x)) = \iota^{-1}(x) = x$. ∎

There are many ways non-homeomorphic ways to compactify a space in general:

Example 23.4. Above we noted that the 1-point compactification of $\mathbb{R}$ is homeomorphic to S^1. Another compactification is by adding 2-points; $-\infty, \infty$. This yields the extended real line which is sometimes written $\bar{\mathbb{R}}$. Consider the corresponding embedding: $\iota : \mathbb{R} \to$

$[-\infty,\infty]$. Note $[-\infty,\infty] \cong [0,1]$ by use of a piecewise defined $\tan^{-1}$-function. Of course, $\iota(\bar{\mathbb{R}}) = [-\infty,\infty]$. This should be viewed as the 2-point compactification of a space.

On the other hand, we can embed $\mathbb{R} \cong (0,1)$ into $[0,1]\times[0,1]^2 \subseteq \mathbb{R}^2$ via the topologist's sin curve: $x \mapsto (x, \sin\left(\frac{1}{x}\right))$. The resulting compactification adds an entire line segment (uncountable set) to $\mathbb{R}$!

The 1-point compactification is somehow the smallest possible compactification of a space X; indeed, at least one point must be added to make the space compact when X itself isn't compact. The main theorem for today introduces a new way to compactify a space, which should be thought of as the *largest* compactification.

Theorem 23.5 (Stone-Čech Compactification Theorem)**.** *Let X be a T3.5 space*[1]*. Then there exists Y a compactification of X such that every bounded continuous map $f : X \to \mathbb{R}$ extends uniquely to a continuous map $f' : Y \to \mathbb{R}$.*

Example 23.6. Continuing with the previous example, consider the extended real line $\bar{\mathbb{R}}$. If $f : \mathbb{R} \to \mathbb{R}$ is a continuous bounded function, then we can consider $a = \lim_{x\to-\infty} f(x)$ and $b = \lim_{x\to\infty} f(x)$. f extends to $\bar{\mathbb{R}}$ if and only if both limits exist. Therefore, a function like $f(x) = x\sin(x)$ tells us that $\bar{B}$ is not the compactification Y in Theorem 23.5.

With our topologists sin curve example, the set of functions which can be extended increases. If both limits exist, we define $f(x) = \lim_{y\to\infty} f(y)$ for each y in the newly adjoined line segment. On the other hand, we can extend the function $f(x)\sin(\frac{1}{x})$ (viewed as $(0,1) \to [0,1]$) as well! We would take $f((0,y)) = y$.

Proof. (of Theorem 23.5). Let $C_0(X) = \{f : X \to \mathbb{R} \mid f \text{ continuous, bounded}\}$. Additionally, for each $f \in C_0(X)$, let $I_f = [\inf f, \sup f]$. Then we have a (ultimate) function

$$F : X \to \prod_{f\in C_0(X)} I_f : x \mapsto (f(x))$$

By the Tychonoff Theorem, $\prod_{f\in C_0(X)} I_f$ is a compact space. Moreover, since X is T3.5, we have that functions separates points from closed sets. Therefore, by The Embedding Theorem (Corollary 19.5), we know that F is an embedding.

Therefore, by Lemma 23.3 we have that there exists a subspace $\iota : Y \hookrightarrow \prod_{f\in C_0(X)} I_f$ such that Y is a compactification of X. It suffices to show that $f \in C_0(X)$ extends to Y. Note that $f : X \to I_f \subseteq \mathbb{R}$ is the composition $\pi_f \circ F$. Therefore, I claim that $\pi_f \circ \iota : Y \to I_f$ is the desired extension of f. It suffices to check that it is unique, which follows by the following lemma: □

Lemma 23.7. *If $A \subseteq X$, and $f : A \to Z$ is a continuous map to a Hausdorff space Z, then if f extends to $\bar{A}$, it extends uniquely.*

Proof. This follows from our standard trick; suppose $f', f'' : \bar{A} \to Z$ are two extensions. Then if $f'(x) \neq f''(x)$, we can separate them by open sets U, V respectively. Choose $x \in U', V' \subseteq \bar{A}$ such that $f'(U') \subseteq U$ and $f''(V') \subseteq V$. Then $\exists a \in A$ such that $a \in U$. Therefore, $f'(a) = f''(a)$. But this implies

$$f'(a) \in f(U') \cap f'(V') \subseteq U \cap V = \emptyset$$

a contradiction. □

[1]To maximize generality. You may assume T4+T1 for comforts sake.

CLASS 24, NOVEMBER 5: STONE-ČECH COMPACTIFICATION

Recall the excellent compactification result from last class:

Theorem 24.0 (Stone-Čech Compactification Theorem)**.** *Let X be a T3.5 space*[1]*. Then there exists Y a compactification of X such that every bounded continuous map $f : X \to \mathbb{R}$ extends uniquely to a continuous map $f' : Y \to \mathbb{R}$.*

It's sheer existence is an extraordinarily powerful result, as it's a rare phenomenon to even be able to extend a single function to a larger even densely populated space Y, let alone a compact space. Today, we will peer deeper into this compactification.

Theorem 24.1. *Let X be a T3.5 space, and let Y be the resulting compactification of Theorem 24.0. Given C compact and Hausdorff, and $f : X \to C$ a continuous function, f extends uniquely to a function $f' : Y \to C$.*

This should be thought of as a mild generalization of Theorem 24.0, since the range need not be $[a, b] \subseteq \mathbb{R}$.

Proof. Note that C is itself T4 and T1, therefore T3.5. Therefore we may embed C into $[0, 1]^\Lambda$ for some Λ by The Embedding Theorem. So we may assume $C \subseteq [0, 1]^\Lambda$, and $f : X \to [0, 1]^\Lambda$ with range in C. Then each component function

$$f_\alpha = \pi_\alpha \circ f : X \to [0, 1]$$

is a bounded continuous function. Therefore, f_α extends to $g_\alpha : Y \to [0, 1]$ by Theorem 24.0. Therefore, we can define

$$G : Y \to [0, 1]^\Lambda : y \to (g_\alpha(y))$$

G is automatically continuous, because with the product topology G is continuous iff g_α are for each α. Finally, it goes to show $G(Y) \subseteq C$. This goes as follows, by continuity:

$$G(Y) = G(\bar{X}) \subseteq \overline{G(X)} = \overline{F(X)} \subseteq \bar{C} = C$$

where the last equality follows by compact + Hausdorff implies closed. Uniqueness follows by Lemma 23.7. □

Finally, we show that such an extension Y is unique up to equivalence, justifying it being called *The* Stone-Čech Compactification.

Proposition 24.2. *If X is T3.5, and Y and Y' are two compactifications satisfying Theorem 24.0, then Y is equivalent to Y'.*

Proof. Since Y and Y' are themselves compact, with corresponding embeddings ι, ι' of X, we can apply Theorem 24.1 to produce maps $f : Y \to Y'$ and $g : Y' \to Y$ such that $\iota' = f \circ \iota$ and $\iota = g \circ \iota$.

I claim $g = f^{-1}$. Indeed, note that $g \circ f : Y \to Y$ is an extension of the identity map $Id_X : X \to X \subseteq Y$ to Y. Similarly, the identity $Id_Y : Y \to Y$ is an extension of Id_X. By uniqueness of extensions, $g \circ f = Id_Y$. Similarly, $f \circ g = Id_Y$. Therefore, $f = g^{-1}$, as claimed, and f and g are therefore homeomorphisms which are the identity on X. □

[1] To maximize generality. You may assume T4+T1 for comforts sake.

Definition 24.3. We call the *unique* choice of Y in Theorem 24.0 the **Stone-Čech Compactification** of X. It is often labeled $\beta(X)$, and is established in the following way: $\beta(X)$ is the unique compactification of X such that every continuous function $f : X \to C$ to a compact Hausdorff space C extends to $\beta(C)$.

One can also note that this correspondence between completely regular spaces and their compactifications (compact Hausdorff spaces) is **functorial**: Given $f : X \to Y$ a continuous function between 2 completely regular spaces, we can extend the range to view $f : X \to \beta(Y)$. Now by Theorem 24.0, this implies that there is an extension $f' : \beta(X) \to \beta(Y)$ agreeing with f on X. Call this function $\beta(f) = f'$. As already noted, $\beta(Id_X) = Id_{\beta(X)}$, and it is a quick exercise to check $\beta(f \circ g) = \beta(f) \circ \beta(g)$.

It would be nice to see that some particularly desirable properties pass between X and $\beta(X)$. Here is one example of such a property:

Proposition 24.4. *A completely regular space X is connected if and only if $\beta(X)$ is connected.*

Proof. ($\Rightarrow$): Note that X as a subpace of $\beta(X)$ has the property that $\bar{X} = \beta(X)$. Therefore, we can go through our ancient notes, find Lemma 9.2, and realize this implies $\beta(X)$ is necessarily connected.

($\Leftarrow$): Suppose $X = A \cup B$ is a separation of X. This implies that there exists a continuous function $f : X \to \{0, 1\}$ which is 0 on A and 1 on B. By Theorem 24.1, we know that f extends to a continuous function $f' : \beta(X) \to \{0, 1\}$. This implies $\beta(X)$ is disconnected. □

This statement can yield some very strange results. However, the easier ($\Rightarrow$) direction actually implies every compactification of a connected space X is itself connected. The beauty is that the Stone-Čech compactification actually reverses this. The same is not true for general compactifications:

Example 24.5. If $X = [a, c) \cup (c, b]$, then the one point compactification is exactly $[a, b]$, by adding c into the interval. It should be clear that $\bar{X}$ is connected, but X is not.

Example 24.6. Suppose X is a topological space with the discrete topology. Let's consider some properties of $\beta(X)$. Note that in this case ANY function $f : X \to [0, 1]$ is continuous, so we expect $\beta(X)$ to have cardinality like that of $[0, 1]^{|X|}$! Additionally, the discrete topology is the least compact topology! So we expect $\beta(X)$ to be huge.

- If $A \subseteq X$, then $\bar{A}$ and $\overline{X \setminus A}$ are disjoint. This follows from Proposition 24.4. $X = A \cup X \setminus A$ is a separation of X, so we can find $f : \beta(X) \to \{0, 1\}$ with $f(A) = 0$ and $f(X \setminus A) = 1$. Then $f^{-1}(0)$ is closed and disjoint form $f^{-1}(1)$, and therefore the statement follows.
- If U is open in $\beta(X)$, then $\bar{U}$ is also open. Note that $\overline{X \cap U} = \overline{U}$. This follows since $\bar{X} = \beta X$. In particular, if $x \in \bar{U}$ and V is a neighborhood of x, then $U \cap V \neq \emptyset$. But this implies $V \cap U \cap X$ is non-empty, since $V \cap U$ is open and X is dense. But this implies $x \in U \cap X$, since V is any neighborhood.
- $\beta(X)$ is totally disconnected (but not in general discrete!). If $x, y \in V$, and V is connected, consider U an open set containing x but such that $y \notin \bar{U}$. Then $\bar{U}$ is a clopen subset of Y by the previous part, a contradiction. Of course, any infinite discrete space, such as $\mathbb{Z}$, can not be compact. So $\beta(X)$ is usually not itself discrete. On the other hand, in the finite case $\beta(X) = X$ since X is automatically compact.

CLASS 25, NOVEMBER 9: LOCAL FINITENESS

We have already demonstrated that there exist partitions of unity subordinate to a finite cover $X = U_1 \cup \ldots \cup U_n$. I defined a partition of unity in a broader setting of locally finite covers. For a given cover, we can refine it to a finite cover (in general) only in a compact space. For locally finite, we only need need the space to be **locally compact**. This includes things like $\mathbb{R}^n$, broadening our generality quite a lot. Today we will explore this notion further which will lead to great results.

Definition 25.1. Let X be any space and $\mathcal{A}$ be a collection of subsets of X. $\mathcal{A}$ is said to be **locally finite** if for any given point $x \in X$, there exists a neighborhood U of x such that only finitely many $A_1, \ldots, A_n$ intersect U: $A_i \cap U \neq \emptyset$.

We have seen the power of such a condition in Homework 1. I note here the importance of the neighborhood condition:

Example 25.2. Consider the collection $\mathcal{A} = \{(0, \frac{1}{n})\}$. This is a locally finite collection in $(0, 1)$, as for any α we can choose n large enough such that $\frac{1}{n} \leq \alpha$. On the other hand, if we consider this collection in $\mathbb{R}$, we realize it is no longer locally finite. Indeed, a neighborhood of 0 has the form $(-\epsilon, \epsilon)$. And this intersects every element of $\mathcal{A}$. The same is true if we restrict even more to $\mathcal{A}' = \{(\frac{1}{n+1}, \frac{1}{n})\}$.

Of course, no element of $\mathcal{A}$ contains 0.

Here I list some nice properties of locally finite collections.

Lemma 25.3. *Let $\mathcal{A}$ be a locally finite collection of subsets of X.*

1) *If $\mathcal{A}' \subseteq \mathcal{A}$, then $\mathcal{A}'$ is locally finite.*
2) *If $\mathcal{B} = \{\bar{A} \mid A \in \mathcal{A}\}$, then $\mathcal{B}$ is locally finite.*
3) $\bigcup_{A \in \mathcal{A}} \bar{A} = \overline{\bigcup_{A \in \mathcal{A}} A}$.

This last property is pretty strange generally speaking, and shows how nice the locally finite property is. In general, we only have that $\bigcup_\alpha \bar{U_\alpha} \subseteq \overline{\bigcup_\alpha U_\alpha}$. This follows since $V \subset U$ implies $\bar{V} \subseteq \bar{U}$. In general however the left side isn't even closed!

Proof.
1) Trivial.
2) To see this, note that every open set which intersects $\bar{A}$ also intersects A (contrapositively, $U \subseteq A^c$ implies $U \subseteq (A^c)^\circ$, whose complement is $\bar{A}$). Therefore, we can take the same neighborhood and elements of $\mathcal{A}$ that show it is locally finite.
3) It suffices to check that $\bigcup_{A \in \mathcal{A}} \bar{A}$ is a closed subset. Let $x \in \overline{\bigcup_{A \in \mathcal{A}} A}$. Let $U \subseteq X$ be an open neighborhood of x intersecting only finitely many sets in $\mathcal{A}$, say $A_1, \ldots, A_n$. We claim $x \in \bar{A}_i$ for some i. If not, then $x \in U' = U \setminus (\bar{A}_1 \cup \ldots \cup \bar{A}_n)$ which is an open set. But this implies U' doesn't intersect *any* element of $\mathcal{A}$, and thus is in the complement's interior. This contradicts our choice of x.

□

Definition 25.4. A collection $\mathcal{A}$ is said to be **countably locally finite** if $\mathcal{A}$ can be decomposed into a countable collection $\mathcal{A}_1, \mathcal{A}_2, \ldots$ such that each $\mathcal{A}_i$ is locally finite.

Definition 25.5. Given $\mathcal{A}$ and $\mathcal{B}$ two collections of subsets of X, we say $\mathcal{B}$ **refines** (or is a **refinement** of) $\mathcal{A}$ if every element $B \in \mathcal{B}$ has a $A \in \mathcal{A}$ such that $B \subseteq A$. If $\mathcal{B}$ is composed of open (or closed) sets, it is called an **open** (or **closed**) **refinement**.

To relate the previous notions and a key lemma to a future metrization theorem, we have the following:

Lemma 25.6. *If X is a metric space, and $\mathcal{A}$ is an open cover, then there exists $\mathcal{B}$ an open covering refinement which is countably locally finite.*

Proof. We will using the well ordering theorem, which states that given any set we can well order the elements. This is actually easy to check, since we can inject any *set* into the space of ordinals. Applying the natural ordering there to its image produces such an ordering.

Let $U \in \mathcal{A}$. We can define $U_n \subseteq U$ to be the $\frac{1}{n}$-shrinking of U;

$$U_n = \{x \in X \mid B\left(x, \frac{1}{n}\right) \subseteq U\}$$

Note eventually U_n is non-empty, since U is open. We can define a further refinement of U:

$$U_n' = U_n \setminus \bigcup_{V<U} V$$

Again, one of these must be non-empty by the previous part. This collection of subsets is disjoint, since either $V < U$ or $U < V$. Even stronger, for a fixed n they are separated by distance at least $\frac{1}{n}$. Indeed, if $V < U$, then V_n has the property that $B(x, \frac{1}{n}) \subseteq V \subseteq U_n^c$ for each $x \in V_n$.

The only issue left to resolve is that these sets are not open (they are even closed!). Indeed, we subtracted a bunch of open sets from an arbitrary (closed) set, so they likely are not open. However, we may consider

$$U_n'' = \bigcup_{x \in U_n'} B\left(x, \frac{1}{3n}\right)$$

This is a union of open sets and therefore is itself open. Furthermore, if $x' \in B\left(x, \frac{1}{3n}\right) \subseteq U_n''$ and $y' \in B\left(y, \frac{1}{3n}\right) \subseteq V_n''$, then the triangle inequality implies

$$d(x', y') \geq d(x, y) - d(x', x) - d(y, y') \geq \frac{1}{n} - \frac{2}{3n} = \frac{1}{3n} > 0$$

so $x' \neq y'$ and therefore $U_n'' \cap V_n'' = \emptyset$.

Finally, I claim that if we let $\mathfrak{B}_n = \{U_n'' \mid U \in \mathcal{A}\}$, and let $\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathfrak{B}_n$ we are done. Indeed, $U_n'' \subseteq U$ for every n by design, and

$$\bigcup_{U \in \mathcal{A}} \bigcup_{n \in \mathbb{N}} U_n'' \supseteq \bigcup_{U \in \mathcal{A}} \bigcup_{n \in \mathbb{N}} U_n' = \bigcup_{U \in \mathcal{A}} \bigcup_{n \in \mathbb{N}} \left(U_n \setminus \bigcup_{V<U} V\right) = \bigcup_{U \in \mathcal{A}} \left(U \setminus \bigcup_{V<U} V\right) = \bigcup_{U \in \mathcal{A}} U = X$$

So $\mathcal{B}$ covers X. Finally, each $\mathfrak{B}_n$ is composed of pairwise disjoint sets! So they are automatically locally finite. This completes the proof. $\square$

Corollary 25.7. *Metric spaces have a countably locally finite basis.*

CLASS 26, NOVEMBER 12: NAGATA-SMIRNOFF METRIZATION

Today we will explore the relationship between the T3/regularity of a space and its metrizability. It turns out that they are fairly close if we are willing to assume our new condition of being countably locally finite.

Lemma 26.1. *Let X be a topological space which is T3 and has a countably locally finite basis. Then X is T6.*

Proof. It goes to show X is normal and every closed set $Z \subseteq X$ is a G_δ-set.

Let W be an open set. I claim that W can be written as

$$W = \bigcup_{i=1}^{\infty} U_i = \bigcup_{i=1}^{\infty} \bar{U}_i$$

for some open subsets $U_i \subseteq W$. Divide a basis $\mathcal{B}$ into a countable collection of locally finite subsets $\mathcal{B}_n$. Let $\mathcal{C}_n \subseteq \mathcal{B}_n$ be the subset of elements U which have the property that $\bar{U} \subseteq W$. Define $U_n = \bigcup_{U \in \mathcal{C}_n} U$. By Lemma 25.3 part 3), we have that

$$\bar{U_n} = \overline{\bigcup_{U \in \mathcal{C}_n} U} = \bigcup_{U \in \mathcal{C}_n} \bar{U}$$

So in particular $\bar{U_n} \subseteq W$. On the other hand, $\mathcal{B}$ is a basis of a regular space, so there always exists $x \in U \subseteq \bar{U} \subseteq W$. So equality holds as asserted.

We now show that this implies every closed set $C \subseteq X$ is a G_δ set. Consider $C^c = U$. By the previous part, we know that

$$C = \left(\bigcup_{i=1}^{\infty} \bar{U_n}\right)^c = \bigcap_{i=1}^{\infty} \bar{U_n}^c$$

Finally, it goes to show that X is T4. Let Z, Z' be closed disjoint subsets. By the previous step, we know $Z = \bigcap_{i=1}^{\infty} U_n = \bigcap_{i=1}^{\infty} \bar{U}_n$ and $Z' = \bigcap_{i=1}^{\infty} U_n' = \bigcap_{i=1}^{\infty} \bar{U}_n{}'$ for some open subsets U_n, U_n'. As a result, we can repeat the proof that a second-countable regular space is normal (Theorem 17.1). That is to say, create a new collection $V_n = U_n \setminus (\overline{U_1'} \cup \ldots \cup \overline{U_n'})$ (similarly V_n') which cover Z, Z' and are disjoint. □

Also recall Homework 6, #5.

Lemma 26.2. *In a normal space, a closed subset $A \subseteq X$ is a G_δ if and only if there is a continuous function $f : X \to [0, 1]$ with $f^{-1}(0) = A$.*

These 2 results combine to give our main theorem of today:

Theorem 26.3 (Nagata-Smirnoff Metrization Theorem)**.** *A topological space X is metrizable if and only if it is regular and has a countably locally finite basis.*

Compare this with homework 6, #7. It generalizes it! NST gives an equivalent condition to metrizable spaces in general, whereas the exercise only gives the compact metric spaces. Furthermore, compact + countably locally finite basis implies second countable.

Proof. ($\Rightarrow$): We know that metric spaces are T0-6, so it only goes to show they are countably locally finite. Consider the basis resulting from our analysis in the exercise; namely

$$\mathcal{B}_n = \{B(x, \frac{1}{n}) \mid x \in X\}$$

By Lemma 25.6 from last class, we have that there exists a $\mathcal{B}'_n$ a refinement which is countably locally finite. Let $\mathcal{B} = \bigcup_{i=1}^{n} \mathcal{B}'_n$ be their union. Since each a countable union of countable sets is countable, we note that this set is also countably locally finite. Furthermore, in the exercise you have shown that this is a basis for the topology.

($\Leftarrow$) : Similar to the proof of Urysohn's Metrization Theorem, we show that the space is metrizable by embedding it into a metric space; namely $(\mathbb{R}^J, \rho)$, where ρ is the uniform metric.

Let $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_n$ be the countably locally finite basis, with each $\mathcal{B}_n$ being locally finite. Lemma 26.1 allows us to conclude X is normal. Therefore, we can apply Urysohn to construct for each $B \in \mathcal{B}_n$ a continuous function

$$f_{n,B} : X \to \left[0, \frac{1}{n}\right]$$

with $f_{n,B}^{-1}(0) = B^c$. Note that this collection of functions separates points from closed sets. Indeed, being a basis ensures there is an open set containing every point x whose complement contains a neighborhood of Z.

Consider $J = \{(n, B) \mid B \in \mathcal{B}_n\} \subseteq \mathbb{N} \times \mathcal{B}$. Then we can define a function

$$F : X \to [0, 1]^J : x \mapsto (f_{n,B}(x))$$

Note that by the the embedding theorem, Corollary 19.5, we know that if we give the right side the product topology this is already an embedding! However, the product topology on an uncountable product of copies of $\mathbb{R}$ is not-metrizable. Therefore, we need to show it is still an embedding under the finer uniform topology. Since F is an open mapping to its image in the product topology, it also true in the uniform topology.

Therefore, it only goes to show that F is continuous. Note that on $[0, 1]$, we may ignore the min; $\rho(x, y) = \sup_{j \in J}(|x_j - y_j|)$. Consider for $f(x) = y$ the open set $B(y, \epsilon) \subseteq [0, 1]^J$. It goes to show that there exists some neighborhood V of x in X such that $F(V) \subseteq B(y, \epsilon)$.

Fix n and choose U_n a neighborhood of x with the locally finite condition of $\mathcal{B}_n$. This implies that there are only finitely many functions with non-zero values on U_n among $f_{n,B}$. Additionally, we can choose $V_n \subseteq U_n$ a neighborhood of x such that the finitely many remaining functions vary by $\leq \frac{\epsilon}{2}$.

Choose $N \gg 0$ such that $\frac{1}{N} \leq \frac{\epsilon}{2}$ and choose $W = V_1 \cap V_2 \cap \cdots \cap V_N$. If $w \in W$, and $n < N$, we have

$$|f_{n,B}(x) - f_{n,B}(w)| \leq \frac{\epsilon}{2}$$

But this inquality is also true if $n > N$, since then $f(w) \leq \frac{1}{n} \leq \frac{\epsilon}{2}$. But this implies $\rho(F(x), F(w)) \leq \frac{\epsilon}{2} < \epsilon$, and therefore the supremum is also bounded above by $\frac{\epsilon}{2}$. $\square$

CLASS 27, NOVEMBER 14: PARACOMPACTNESS

Today we will explore an extraordinarily useful generalization of compactness, known as paracompactness. This notion is ubiquitous throughout topology and differential geometry, with many theorems about manifolds (in particular) relying on a partition of unity subordinate to a locally finite cover.

Definition 27.1. A space is **paracompact** if every open cover $X = \bigcup_{\alpha\in\Lambda} X_\alpha$ can be refined to a locally finite open cover $= \bigcup_{\alpha\in\Lambda'} X'_\alpha$.

Example 27.2. $\mathbb{R}^n$ is paracompact. Indeed, we can cover $\mathbb{R}^n$ by $B(x, \sqrt{n})$, where $x \in \frac{1}{n}\mathbb{Z}^n$. This is a cover of $\mathbb{R}^n$ by spaces whose closures are compact. Enumerate the balls $B_1, B_2, \ldots$. For a given cover $\mathbb{R}^n = \bigcup U_\alpha$ Choose finitely many U_α covering $\overline{B_1}$ (by compactness). Call the collection $\mathfrak{B}_1$. Now inductively, choose finitely many U_α covering the compact space $\overline{B_n}$ and create

$$U'_\alpha = U_\alpha \setminus (\overline{B_1} \cup \cdots \cup \overline{B_{n-1}})$$

Call the set of these $\mathfrak{B}_n$. Now I claim that $\mathfrak{B} = \bigcup_{i\in\mathbb{N}} \mathfrak{B}_i$ is the desired cover. Note it is a cover, since

$$X = \bigcup_n B_n = \bigcup_n (B_1 \cup \cdots \cup B_n) = \bigcup_n \bigcup_{m\leq n} \bigcup_{X_\alpha \in \mathfrak{B}_m} X_\alpha$$

and it is locally finite, since only sets within $\mathfrak{B}_n$ can only intersect sets $X_\alpha \in \mathfrak{B}_m$ for $m \leq n$.

Now we can generalize our previous realization about compact Hausdorff spaces.

Theorem 27.3. *Every paracompact Hausdorff space X is T4.*

Proof. First, I prove X is T3. Let $x \in X$ and $Z \subseteq X$ a closed set with $x \notin Z$. For each $z \in Z$, we can find an open set U_z not containing z whose closure doesn't contain x. We can cover X by $\{U_z\}$ and $V = Z^c$. Applying paracompactness, we note that there exists a locally finite refinement covering X. Call it $\mathfrak{B}$ and let $\mathfrak{B}' \subseteq \mathfrak{B}$ consist of elements of $\mathfrak{B}$ intersecting Z.

By our assumptions, we know that $A \in \mathfrak{B}'$ implies $x \notin \bar{A}$. Consider

$$V = \bigcup_{A\in\mathfrak{B}'} A$$

Because $\mathfrak{B}'$ is locally finite, we have $\bar{V} = \bigcup_{A\in\mathfrak{B}'} \bar{A}$. But this implies V and $\bar{V}^c$ is the desired separation. Normality follows by the same argument. ∎

Additionally, paracompact spaces behave well with respect to closed subspaces.

Theorem 27.4. *Every closed subspace $Z \subseteq X$ of a paracompact space is itself paracompact.*

Proof. Similar to the proof for compactness, we can take for any cover $\{U_\alpha\}$ of Z a corresponding collection of open sets in X, say $\{X'_\alpha\}$, and add to it Z^c. This is a covering so a locally finite refinement exists. Intersect each element of the refinement with Z to verify Z is paracompact. ∎

Unlike the case of compactness however, paracompactness doesn't usually imply closed.

Example 27.5. Note $(0,1)$ is paracompact despite being open in $\mathbb{R}$.

On the other hand, we can consider the compact space $\bar{S_\omega} = \{1, 2, \ldots, \omega\}$. This is a compact set, since an open set containing ω necessarily contains a basis element $(n, \omega]$. The remaining space is finite, thus compact. Therefore, $\bar{S_\omega} \times \bar{S_\omega}$ is also compact. However, $S_\omega \times \bar{S_\omega}$ is a Hausdorff space which is non-T4. Therefore, it couldn't be paracompact by Theorem 27.3.

Next, I prove a useful characterization of open covers of a T3 space X.

Lemma 27.6. *Let X be T3. Then TFAE for every open covering $X = \bigcup_\alpha U_\alpha$:*

1) There exists a countably locally finite open covering refinement.
2) There exists a covering refinement which is locally finite.
3) There exists a closed covering refinement which is locally finite.
4) There exists an open covering refinement which is locally finite.

Proof. 1) $\Rightarrow$ 2): Let $\mathfrak{B} = \bigcup_{n \in \mathbb{N}} \mathfrak{B}_n$ be an open countably locally finite refinement of some cover $\mathcal{A}$. For a given $i \in \mathbb{N}$, let $V_i = \bigcup_{U \in \mathfrak{B}_i} U$. Then we can consider for a given $U \in \mathfrak{B}_n$, let

$$U' = U \setminus (V_1 \cup \ldots \cup V_{n-1})$$

This is an intersection of an open set and a closed set, so it likely doesn't have either property. If we let $\mathfrak{B}'_n$ be the refinement of U' obtained in this way, then we can obtain $\mathfrak{B}' = \bigcup_n \mathfrak{B}'_n$ a refinement of $\mathfrak{B}$.

I claim that this is the desired refinement. Note that this collection still covers the space, since we only removed points which were in earlier covers. It goes to show that it is locally finite. For a given $x \in X$, we know that there exists an n minimal such that $x \in U \in \mathfrak{B}'_n$. For each $m \leq n$, choose a neighborhood V_m of x intersecting only finitely many elements of $\mathfrak{B}'_m$ by local finiteness. I claim the desired neighborhood is then

$$V = V_1 \cap \ldots \cap V_n \cap U$$

V intersects only finitely many elements of each $\mathfrak{B}'_m$ with $m < n$. Furthermore, since $V \subseteq U$, it doesn't intersect ANY element of $\mathfrak{B}'_m$ for $m > n$. Thus it is locally finite.

2) $\Rightarrow$ 3): Now it goes to show we can upgrade a locally refinement to a closed one. Let $\mathfrak{B}$ be the collection of all open sets $U \subseteq X$ whose closure lies in some U_α. Since X is T3, the neighborhood criteria concludes that $\mathfrak{B}$ is still a cover. One can refine $\mathfrak{B}$ by 2) to a locally finite cover. Now let $\mathfrak{B}' = \{\bar{Z} \mid Z \in \mathfrak{B}\}$. It is locally finite by Lemma 25.3, part 2.

3) $\Rightarrow$ 4): Let $\mathfrak{B}$ be a refinement of U_α which is locally finite. For each $x \in X$, choose U_x an open neighborhood intersecting only finitely many elements of $\mathfrak{B}$. U_x forms an open cover of X. Let $\mathfrak{B}'$ be a closed refinement of $\{U_x\}$ which is locally finite. Therefore, each element of $\mathfrak{B}'$ intersects only finitely many elements of $\mathfrak{B}$.

For $B \in \mathfrak{B}$, let $\mathfrak{B}'_B = \{C \in \mathfrak{B}' \mid C \subseteq X \setminus B\}$. We can then produce

$$C_B = X \setminus \bigcup_{C \in \mathfrak{B}'_B} C$$

Again by Lemma 25.3, part 3), we can conclude that this set is open and contains B. Finally, we can choose an element $U_{\alpha(B)}$ which contains B and let $\mathfrak{B}'' = \{C_B \cap U_{\alpha(B)}\}$. This is an open refinement of $\mathcal{A}$ covering X.

It goes to show $\mathfrak{B}''$ is locally finite. Given $x \in X$, choose V_x intersecting finitely many $C_1, \ldots, C_n \in \mathfrak{B}'$. Note $V_x \subseteq C_1 \cup \ldots \cup C_n$. Given $C \in \mathfrak{B}'$, if $C \cap C_B \cap U_{\alpha(B)} \neq \emptyset$, then $C \cap C_B \neq \emptyset$, so in particular $C \not\subseteq B^c$ or $C \cap B \neq \emptyset$. However, this is true for only finitely many B, so is the case for $\mathfrak{B}''$!

4) $\Rightarrow$ 1): Trivial. Finite implies countable. $\square$

CLASS 28, NOVEMBER 16: PARACOMPACTNESS

Today we will exploit the result about refinements of open coverings from last class to produce some interesting examples of paracompact spaces. In addition, we will prove that paracompact spaces have partitions of unity subordinate to any cover. This builds the machinery to prove the Smirnov-Metrization Theorem next time, which also classifies metrizability of spaces.

Proposition 28.1. *Every metric space is paracompact.*

Proof. By Lemma 25.6, we know there exists a countably locally finite open refinement of any given open cover. By Lemma 27.6, we can conclude (since every metric space is T3) that this implies there exists a locally finite open refinement. Therefore it is paracompact. □

Munkres also includes a result about T3 Lindelof/σ-compact spaces being paracompact, which also utilizes Lemma 27.6 in its proof.

Example 28.2. One can check that $\mathbb{R}$ with the lower limit topology (generated by $[a, b)$) is paracompact (it is T3 and σ-compact). However, $\mathbb{R} \times \mathbb{R}$ with the product lower-limit topology is not, as it is non-T4. Therefore products of paracompact spaces aren't necessarily paracompact.

On the other hand, since $\mathbb{R}^{\mathbb{N}}$ with the product (or uniform) topology is metrizable, we know that it is necessarily paracompact. The statement for the box topology is open.

Next I prove the existence of partitions of unity for paracompact Hausdorff spaces. This signifies the importance of paracompactness as a condition.

Lemma 28.3. *Given X a paracompact Hausdorff space,, and U_α an open covering X. Then there exists a locally finite open collection V_α (with the same indexing set) covering X such that $\forall \alpha$, $\bar{V_\alpha} \subseteq U_\alpha$.*

The condition in the conclusion is sometimes called a **precise refinement**.

Proof. Let $\mathfrak{B}$ be the collection of ALL open sets V such that $\bar{V} \subseteq U_\alpha$ for some $\alpha \in \Lambda$. Since X is T4+T1, it is T3, so these sets cover X. We can choose a locally finite refinement of $\mathfrak{B}$ by open sets covering X. Call it $\mathfrak{B}'$. Then let $\mathfrak{B}' = \{V_\beta\}_{\beta \in \Lambda'}$. Utilizing the axiom of choice, we can define a choice function $f : \Lambda' \to \Lambda$ such that $\bar{V_\beta} \subseteq U_{f(\beta)}$. Let

$$V_\alpha = \bigcup_{f(\beta)=\alpha} V_\beta$$

Since the collection on the right is locally finite, we can conclude

$$\bar{V_\alpha} = \bigcup_{f(\beta)=\alpha} \bar{V_\beta} \subseteq U_\alpha$$

Finally, it suffice to check that V_α forms a locally finite collection. For a given $x \in X$, choose U intersecting only finitely may $V_{\beta_1}, \ldots, V_{\beta_n}$. Then U intersects at most $V_{f(\beta_1)}, \ldots, V_{f(\beta_n)}$. □

Theorem 28.4. *Let X be a paracompact Hausdorff space and U_α a cover. Then there exists a partition of unity subordinate to U_α.*

This proof has notable similarities to that of the same statement for compact spaces.

Proof. Recall that by the shrinking lemma used to prove partitions of unity for compact manifolds allows us to find

$$W_\alpha \subseteq \bar{W_\alpha} \subseteq V_\alpha \subseteq \bar{V_\alpha} \subseteq U_\alpha$$

with W_α and V_α open locally finite covers of X. Since X is T4, we can choose $\psi_\alpha : X \to [0,1]$ such that $\psi(\bar{W_\alpha}) = 1$ and $\psi(V_\alpha^c) = 0$. Therefore,

$$\text{Supp}(\psi_\alpha) = \overline{\{x \in X \mid \psi_\alpha(x) \neq 0\}} \subseteq \bar{V_\alpha} \subseteq U_\alpha$$

Now we can consider $\Psi : X \to \mathbb{R} : x \mapsto \sum_\alpha \psi_\alpha(x)$. Even though this sum is potentially uncountable, we note that for a given value of x, the collection $\text{Supp}(\psi_\alpha)$ is locally finite. Therefore, for a fixed neighborhood of any point, the sum is merely finite and bounded below by 1. Therefore, it is continuous on these neighborhoods, which cover X, so we note that Ψ is continuous as well. Therefore, a partition of unity is given by

$$\varphi_\alpha(x) = \frac{\psi_\alpha(x)}{\Psi(x)}$$

□

Our characterization of manifolds always implies that it is paracompact. Indeed, they are second countable and T3 (by homework). Therefore, every cover has a countable subcover and therefore is countably locally finite! Therefore, we obtain the following sometimes useful corollary.

Corollary 28.5. *If X is a manifold, then $\exists \iota : X \to \mathbb{R}^\Lambda$ an embedding, where Λ is at most countable. Therefore, every manifold is metrizable and hence normal.*

Proof. The same proof goes through as in the compact case, namely we can find countably many charts (by second-countability) $\psi_i : U_i \to U_i' \subseteq \mathbb{R}^{m_i}$ and a partition of unity $\varphi_i : X \to [0,1]$ subordinate to U_i. Then the desired map is

$$\iota : X \to \mathbb{R}^{\mathbb{N}} : x \mapsto (\psi_1(x)\varphi_1(x), \varphi_1(x), \psi_2(x)\varphi_2(x), \varphi_2(x), \ldots)$$

This is an embedding by the embedding theorem.

The fact that it is metrizable follows from the fact that $\mathbb{R}^{\mathbb{N}}$ is metrizable. □

Example 28.6. A perfectly reasonable (!) manifold arises as follows:

$$X = \mathbb{R} \coprod \mathbb{R}^2 \coprod \mathbb{R}^3 \coprod \cdots$$
$$Y = S^1 \coprod S^2 \coprod S^3 \coprod \cdots$$

Note here $\coprod$ denotes the disjoint union, which is to say the objects have no overlap.

These are certainly Hausdorff, since each connected component is. Furthermore, they are second countable since each component is and countable unions of countable sets are countable. Finally, each component is locally Euclidean, so X and Y are themselves. Therefore, X and Y are manifolds.

Of course, it is impossible to embed such spaces into finite dimensional vector spaces, since this would imply

$$\mathbb{R}^n \hookrightarrow X \hookrightarrow \mathbb{R}^m$$
$$S^n \hookrightarrow Y \hookrightarrow \mathbb{R}^m$$

for $n > m$, and this is impossible by invariance of dimension. But we can still conclude that these spaces are metrizable by embedding them into $\mathbb{R}^{\mathbb{N}}$ and applying the corresponding metric.

CLASS 29, NOVEMBER 19: SMIRNOFF'S METRIZATION THEOREM

The idea of paracompactness has already produced far reaching consequences. Today, we will prove one residual theorem about paracompact spaces and complement Nagata-Smirnoff with an additional set of conditions to ensure a space is metrizable.

Theorem 29.1. *Let X be a paracompact Hausdorff space, and $\mathfrak{B}$ be a collection of subsets of X. For each $B \in \mathfrak{B}$, choose $\epsilon_B > 0$. If $\mathfrak{B}$ is locally finite, then there exists a continuous function $f : X \to (0, \infty)$ such that $f(x) \leq \epsilon_B$ for $x \in B$.*

You can think of this theorem as giving a nice continuous function with boundable values on any locally finite collection.

Proof. As usual, let U_x be an open neighborhood of x intersecting only finitely many elements of $\mathfrak{B}$. For a given U_x, choose $\epsilon_x = \min_{B \cap U_x \neq \emptyset}\{\epsilon_B\}$. This results in a positive number, since it is a minimum of finitely many positive numbers.

Also, choose $\varphi_x : X \to [0, 1]$ a partition of unity subordinate to the cover U_x. Then I claim the desired function is

$$f : X \to (0, \infty) : y \mapsto \sum_x \epsilon_x \cdot \varphi_x(y)$$

As usual, the sum on the right is finite for any choice of y and thus continuous. If $y \in B$

$$f(y) = \sum_x \epsilon_x \cdot \varphi_x(y) \leq \epsilon_B \sum_x \varphi_x(y) = \epsilon_B$$

□

Theorem 29.2 (Smirnoff Metrization Theorem)**.** *A space X is metrizable if and only if it is paracompact, Hausdorff, and locally metrizable.*

Recall the following definition which came up on Exam 2:

Definition 29.3. A space X is called **locally metrizable** if every point $x \in X$ has a neighborhood which is metrizable.

On the exam, it was demonstrated that a compact Hausdorff space which is locally metrizable is metrizable. Theorem 29.2 generalizes this result naturally.

Example 29.4. We have brought up the existence of this 'long line' when speaking of the importance of second-countability in the definition of a manifold. Here we can see it also yields a nice example of a space which is locally metrizable but not metrizable.

Recall S_{ω^ω} or S_Ω is the set of ordinals less than ω_1, the first uncountable ordinal. It consists of

$$S_\Omega = \{1, 2, \ldots, \omega, \omega+1, \ldots, 2\omega, \ldots, n\omega, \ldots, \omega^\omega, \ldots, \omega^{\omega^\omega}, \ldots, \epsilon_0 = \omega^{\omega^{\omega^{\cdot^{\cdot^{\cdot}}}}}, \ldots\}$$

Because it was asked, each of these are countable ordinals by induction: It is clear that $\{1, 2, \ldots\}$ is countable, which is the collection $\leq \omega$. But this shows that the collection less than ω^2 is countable, since

$$[0, \omega^2) = \bigcup_{i=0}^{\infty} [i\omega, (i+1)\omega]$$

and $[i\omega, (i+1)\omega] \leftrightarrow [0, \omega]$. Similarly,

$$[0,\omega^3) = \bigcup_{i=0}^{\infty} [i\omega^2, (i+1)\omega^2] \qquad [0,\omega^\omega) = \bigcup_{i=0}^{\infty} [0,\omega^i]$$

$$[0,\epsilon_0) = \bigcup_{i=0}^{\infty} [0, \omega^{\cdot^{\cdot^{\cdot^{\omega}}}}]$$

and since a countable union of countable sets is countable, we see each is countable inductively.

Let $X = S_\Omega \times [0,1)$, equipped with the dictionary topology. Then X is called the long line, because it is intuitively obtained by adjoining an uncountable collection of intervals together. I claim that X is locally metrizable, and in fact every point (a,x) has a neighborhood homeomorphic to an open subset of $\mathbb{R}$.

For a given ordinal, there exist only countably many occuring before it (by definition of ω_1). As a result, we can see that

$$((1,0),(a,0)) \cong \mathbb{R} \cong (0,1)$$

But as a result, we can adjoint $((a,0),(a,1))$ to obtain a space homeomorphic to $(0,2)$ with the dictionary topology.

Lastly, note that X is non-metrizable. This is because it is not second-countable, which in the case of connected locally Euclidean spaces is equivalent to paracompactness. Therefore, it violates Theorem 29.2.

Now I will prove the metrization theorem. First, recall the Nagata-Smirnoff Metrization Theorem:

Theorem (Nagata-Smirnoff Metrization Theorem)**.** *A topological space X is metrizable if and only if it is regular and has a countably locally finite basis.*

This will be used in the proof.

Proof. (of Theorem 29.2) ($\Rightarrow$): If X is metrizable, then it is paracompact, Hausdorff, and locally metrizable. The first 2 have been proved, and the last statement is immediate from the definition.

($\Leftarrow$): Assume X is paracompact, locally metrizable, and Hausdorff. Since a paracompact Hausdorff space is T4, it is also regular, so by Nagata-Smirnoff we only need to check that there is a countably locally finite basis.

Choose U_x metrizable neighborhoods of each point $x \in X$. U_x cover X, and since X is paracompact we can refine this to a locally finite collection with the same property. Call it V_α, and let d_α be the metric on V_α. Since V_α is an open set, we note that $B_\alpha(x,r) \subseteq V_\alpha$ is also open in X.

Define

$$\mathfrak{B}_n = \{B_\alpha(x, \frac{1}{n}) \mid x \in V_\alpha \text{ and any } \alpha\}$$

and let $\mathfrak{B}'_n$ be a locally finite refinement of this collection which covers X. Of course, this implies that $\mathfrak{B} = \bigcup_{i\in\mathbb{N}} \mathfrak{B}'_i$ is a countably locally finite set.

It suffices to check $\mathfrak{B}$ is a basis for the topology of X. Given $x \in X$ and a neighborhood U which intersects only finitely many V_α, say $V_{\alpha_1}, \ldots, V_{\alpha_n}$ (WLOG, by shrinking). For each of these, $U \cap V_{\alpha_i}$ is open so contains some $B_{\alpha_i}(x,\epsilon_i)$. Therefore, we can choose $0 < \frac{2}{n} < \min\{\epsilon_1,\ldots,\epsilon_n\}$. Since $\mathfrak{B}'_n$ is a cover, there exists y such that $x \in B(y,\frac{1}{n}) \in \mathfrak{B}'_n$. But note that for $z \in B_\alpha(y,\frac{1}{n})$

$$d_\alpha(x,z) \le d_\alpha(y,x) + d_\alpha(y,z) \le \frac{2}{n} < \min\{\epsilon_1,\ldots,\epsilon_n\}$$

So $B_\alpha(y,\frac{1}{n}) \subseteq B_{\alpha_i}(x,\epsilon_i)$. But this implies $\alpha = \alpha_i$, and furthermore, $x \in B_\alpha(y,\frac{1}{n}) \subseteq B_{\alpha_i}(x,\epsilon_i)$, as desired.

Now suppose that $x \in B_\alpha(x,\frac{1}{n}) \cap B_{\alpha'}(x',\frac{1}{n'})$. We can take $n'' = 2\cdot\max\{n,n'\}$ as before. Then since $\mathfrak{B}_{n''}$ is again a covering, there exists $x'' \in X$ such that $x \in B_{\alpha''}(x'',\frac{1}{n''})$. Therefore, $\mathfrak{B}$ is a basis. $\square$

CLASS 30, NOVEMBER 26: COMPLETIONS OF METRIC SPACES

Recall the notion of completeness of a metric space:

Definition 30.1. A metric space (X, d) is said to be **complete** if every Cauchy sequence converges. A sequence (x_n) is said to be **Cauchy** if for every $\epsilon > 0$, $\exists N \gg 0$ such that

$$d(x_N, x_n) < \epsilon$$

for every $n > N$.

Example 30.2. $\mathbb{R}^n$ is a complete metric space with the Euclidean metric d_2 (and thus d_∞ or d_1 since they are all equivalent metrics). Indeed, let (x_n) be a Cauchy sequence. For a given ϵ, choose N_ϵ such that $x_n \in B(x_{N_\epsilon}, \epsilon)$ for all $n > N_\epsilon$. Then note that the collection

$$\{\bar{B}(x_{N_\epsilon}, \epsilon) \mid 0 < \epsilon < 1\}$$

has the finite intersection property, and is contained within a compact set $\bar{B}(x_{N_1}, 1)$. Therefore there is an element

$$x \in \bigcap_{0<\epsilon<1} B(x_{N_\epsilon}, \epsilon)$$

and x is the limit of x_n.

Now we note that there is an extension of Example 30.2 to $\mathbb{R}^{\mathbb{N}}$.

Lemma 30.3. *If X_α are topological spaces, and $X = \prod_\alpha X_\alpha$ with the product topology, then if x_n is a sequence in X, $x_n \to x$ if and only if $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for all α.*

Proof. $(\Rightarrow)$: Since π_α is continuous, this follows from Theorem 14.6.

$(\Leftarrow)$: Let $x \in U = U_{\alpha_1} \times \ldots \times U_{\alpha_m} \times \prod_{\alpha \neq \alpha_i} X_\alpha$ is an open set. Since $\pi_{\alpha_i}(x_n) \to \pi_\alpha(x)$, for some $n > N_i$ we have $\pi_{\alpha_i}(x_n) \in U_i$ for each $n > N_i$. Choosing $N = \max\{N_1, \ldots, N_m\}$, we have this condition uniformly satisfied for $n > N$. As a result, $x_n \in U$ for $n > N$. Thus $x_n \to x$ as asserted. ☐

Corollary 30.4. $\mathbb{R}^{\mathbb{N}}$ *with the product topology is a complete metric space.*

Examples of non-complete metric spaces are as folows:

Example 30.5. $(\mathbb{Q}, d)$ with the Euclidean metric is non-complete. Indeed, we can find a sequence of rational numbers in $\mathbb{R}$ converging to an irrational number. The same sequence will not converge in $\mathbb{Q}$.

More generally, given a complete metric space and a convergent sequence $x_n \to x$ such that $x_n \neq x$, then $X \setminus \{x\}$ is no longer complete. This follows since sequences converge *uniquely* in Hausdorff spaces. $\mathbb{Q}$ is an example of this happening uncountably many times, and $(0, \infty) \subseteq [0, \infty) \subseteq \mathbb{R}$ is another such example.

We can similarly state the same result for the uniform topology (with bigger products):

Theorem 30.6. *If (X, d) is a metric space, we can put the* **uniform metric** *ρ onto $Y = X^\Lambda$:*

$$\rho(x, y) = \sup_{\alpha \in \Lambda} \{\min\{d(\pi_\alpha(x), \pi_\alpha(x)), 1\}\}$$

If (X, d) is complete, so is (Y, ρ).

Proof. Let x_n be a Cauchy sequence in Y. This implies $\forall \epsilon > 0$, there exists $N \gg 0$ such that $\sup_{\alpha\in\Lambda} \{d(\pi_\alpha(x_N), \pi_\alpha(x_n))\} < \frac{\epsilon}{2}$. But this implies that the statement is true for each coordinate, thus we have a Cauchy sequence $\pi_\alpha(x_n)$ in X. Let x_α be its limit. Then it is immediate that $d(\pi_\alpha(x_n), x_\alpha) \leq \frac{\epsilon}{2}$. As a result,

$$\rho(x_n, (x_\alpha)) = \sup\{d(\pi_\alpha(x_n), x_\alpha)\} \leq \frac{\epsilon}{2} < \epsilon$$

$x_n \to (x_\alpha)$. □

Recall that we can view the set of all (not necessarily continuous) functions $f : X \to Y$ as Y^X. Since Theorem 30.6 applies to any generic *set* Λ, it also applies to Y^X. This allows us to prove an interesting theorem about the resulting product.

Theorem 30.7. *If X is a topological space and (Y, d) is a metric space. The subsets $\mathcal{C}, \mathcal{B} \subseteq Y^X$ of continuous and bounded (resp.) functions from X to Y is closed under ρ.*

Corollary 30.8. *If Y is a complete metric space, then so are $\mathcal{C}, \mathcal{B}$.*

Proof. (of Theorem 30.7). The result for continuous functions follows from Lemma 20.2 (whose proof extends naturally to any metric space); A sequence of functions $f_n \in Y^X$ converges to a function f under ρ if and only if it converges *uniformly*. This is just unravelling definitions:

$$\rho(f_n, f) = \sup_{x\in X}\{d(f_n(x), f(x))\}$$

Therefore, it only goes to show that a uniform limit of bounded functions is bounded. But being ϵ away from f_n means $d(f_n(x), f(x)) < \epsilon$ for *every* x! □

As an application, we have the notion of a **completion** of a metric space!

Theorem 30.9. *Given a metric space (X, d), there exists an isometric (distance preserving) embedding ι of X into a complete metric space $\hat{X}$ with $\overline{\iota(X)} = \hat{X}$.*

Proof. We can consider $\mathcal{B} \subseteq \mathbb{R}^X$ to be the set of bounded functions $X \to \mathbb{R}$. Since $\mathbb{R}$ is complete, Corollary 30.8 allows us to conclude that $\mathcal{B}$ is complete with the uniform metric.

Given $a, b \in X$, we can define

$$\phi_b(x) = d(x, a) - d(x, b)$$

By the triangle inequality, ϕ_b is bounded in $[-d(a, b), d(a, b)]$. Therefore, we can define

$$\iota : X \to \mathcal{B} : b \mapsto \phi_b$$

Note that this map is injective, since

$$\phi_b(b') - \phi_{b'}(b') = d(b, b') - d(b', b') = d(b, b') = 0 \Longleftrightarrow b = b'$$

Furthermore, distances are preserved:

$$\rho(\iota(b), \iota(b')) = \rho(\phi_b, \phi_{b'}) = \sup\{|d(x, b) - d(x, b')| \mid x \in X\} \leq d(b, b')$$

On the other hand, lettings $x = b'$ or $x = b$, we see that equality is achieved. Therefore they are equal. The closure statement results by replacing $\mathcal{B}$ with $\overline{\iota(X)}$. □

$\hat{X}$ is called the **completion** of X, and plays a similar role to a compactification.

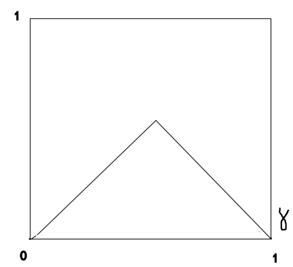
CLASS 31, NOVEMBER 28: SPACE FILLING CURVES

We can employ our results about complete metric spaces to realize some very interesting facts about the so-called 'Peano Curve'.

Theorem 31.1 (Peano)**.** *There exists a surjective continuous map* $\Gamma : [0,1] \to [0,1]^2$.

Note that this doesn't violate invariance of dimension, as the map we will construct is highly non-injective. I will make this precise after producing the construction.

Proof. The proof uses 2 critical notions; a fractal construction (employing the fact that $[0, \frac{1}{2^n}] \cong [0,1]$) and the completeness of continuous functions to $[0,1]$.

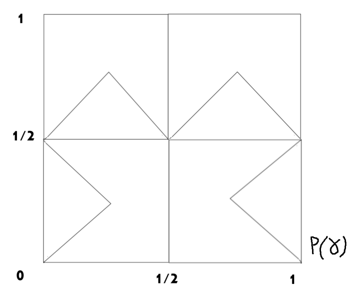


Step 1: Let $I = [0,1]$. Given a path $\gamma : I \to I^2$ with $\gamma(0) = (0,0)$ and $\gamma(1) = (1,0)$, we can construct a new path $P(\gamma)$ with the same properties as follows: Let

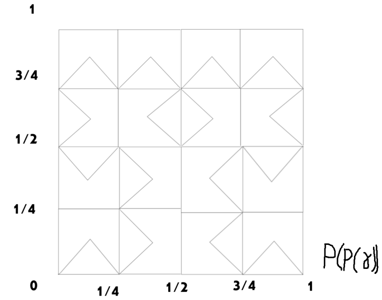
$$T : I^2 \to I^2 : (x,y) \mapsto (y,x)$$

this is a homeomorphism. Now we can write

$$P(\gamma)(t) := \begin{cases} T(\frac{1}{2} \cdot \gamma(4t)) & t \in [0, \frac{1}{4}] \\ (\frac{1}{2}, 0) + \frac{1}{2} \cdot \gamma(4t-1) & t \in [\frac{1}{4}, \frac{1}{2}] \\ (\frac{1}{2}, \frac{1}{2}) + \frac{1}{2} \cdot \gamma(4t-2) & t \in [\frac{1}{2}, \frac{3}{4}] \\ (1, \frac{1}{2}) - T(\frac{1}{2} \cdot \gamma(4t-3)) & t \in [\frac{3}{4}, 1] \end{cases}$$



As illustrated on the right, the effect of this is to iterate the same curve 4 times with different orientations. $\gamma^{(2)}$ is continuous by the pasting lemma: It is defined by continuous functions defined on 4 closed sets which agree on their overlap.



Step 2: We can apply this construction iteratively. Let $f_0 : I \to I^2$ be the function[1] defined by

$$f_0(t) = \begin{cases} (t,t) & t \leq \frac{1}{2} \\ (t, 1-t) & t \geq \frac{1}{2} \end{cases}$$

Now let f_n be obtained by applying either P (if the curve ends in the lower right corner) or $T \circ P \circ T$ (if it ends in the upper left) to each of the piecewise functions making up f_{n-1} inductively.

---

[1]This choice of function is NOT important in the grand scheme of the proof. You may in fact replace it with any choice of γ as above.

Step 3: Consider d_∞ on I^2 defined by $d_\infty((x,y),(x',y')) = \max\{|x-x'|, |y-y'|\}$. Let ρ be the uniform metric on $(I^2)^I$. By Theorem 30.7 this is a complete metric space, and therefore we can conclude that the subset of continuous functions $I \to I^2$ is as well. Therefore, to produce a limit of the f_n, it suffices to demonstrate that they form a Cauchy sequence.

The idea is that for a given $t \in [0,1]$, we can specify which 2^{-n}-square the function will be in and note that this is preserved when n is increased.

Given $\epsilon > 0$, choose N such that $2^{-N} < \epsilon$. By the previous observation, we note that for $n > N$ and a given t, we have that $f_N(t)$ and $f_n(t)$ lie in the same 2^{-N}-square. As a result

$$d_\infty(f_n(t), f_N(t)) \leq 2^{-N}$$

Since ρ is exactly the supremum of these numbers, $\rho(f_n, f_N) \leq 2^{-N} < \epsilon$, as desired. Let Γ be its limit.

Step 4: Surjectivity of Gamma! Since f_n has values in each square of size 2^{-n} which tile $[0,1]$, we note that for any given $x \in I^2$, there exists t such that $d(x, f_n(t)) \leq 2^{-n}$.

Let $\epsilon > 0$. I claim $B(x,\epsilon) \cap \Gamma(I) \neq \emptyset$. Choose $N \gg 0$ such that $\rho(\Gamma, f_N) < \frac{\epsilon}{2}$ and $\frac{1}{2^N} < \frac{\epsilon}{2}$. Fix such a t as in the previous paragraph. This implies

$$d(\Gamma(t), x) \leq d(\Gamma(t), f_N(t)) + d(f_N(t), x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} < \epsilon$$

So the asserted claim is proved.

Since every neighborhood of x intersects $\Gamma(I)$, we know that $x \in \overline{\Gamma(I)}$. But $\Gamma(I)$ is the image of a compact set, therefore itself compact, in a Hausdorff space. Therefore it is closed, proving surjectivity. $\square$

Note that the 2 dimensional cube was not important at all.

Corollary 31.2. *There exists a surjective continuous map $\Gamma' : [0,1] \to [0,1]^n$ for any $n \in \mathbb{N}$.*

Proof. Choose $2^m > n$. Then we can define a surjective function

$$I \xrightarrow{\Gamma} I^2 \xrightarrow{\Gamma\times\Gamma} I^4 \xrightarrow{\Gamma\times\Gamma\times\Gamma} \cdots \xrightarrow{\Gamma^{n-1}} I^{2^m}$$

Since this is a composition of surjective continuous maps, it is itself continuous and surjective. Finally, we can project onto the desired number of copies of I to produce our map. $\square$

We can also upgrade this argument to $\mathbb{R}$.

Corollary 31.3. *There exists a surjective continuous map $\Gamma'' : \mathbb{R} \to \mathbb{R}^n$ for any $n \in \mathbb{N}$.*

Proof. Tile $\mathbb{R}^n$ with I^n, and enumerate them by $2\mathbb{Z}$. We can apply Γ' by the previous corollary modified by $[0,1] \cong [2n, 2n+1]$. Call their aggregate Γ''_0:

$$\Gamma''_0 : \bigcup_{n\in\mathbb{Z}} [2n, 2n+1] \to \mathbb{R}^n$$

This map is already surjective and continuous. For $[2n-1, 2n]$, use the straight-line path from $\Gamma''_0(2n-1)$ to $\Gamma''_0(2n)$. The combined map is continuous by the (locally finite) pasting lemma, and is of course surjective since it is on a subset of the domain. $\square$

Just as a final note, the function Γ is self-intersecting. You can already see this at the level of $P(P(\gamma))$ in the above illustration. This tells us that at minimum $P^3(\gamma)$ has 4 intersection points, and $P^{n+2}(\gamma)$ has at least (in fact many more!) 4^n intersection points.

Of course, if you believe invariance of dimension, no bijection could possibly exist. Indeed, it would be a continuous bijection from a compact space to a Hausdorff space, therefore a homeomorphism.

CLASS 32, NOVEMBER 30: (BABY) ASCOLI'S THEOREM

Ascoli's Theorem produces an excellent peek into functional analysis, where you study the space of operators from a metric space to $\mathbb{R}$ subject to various (continuity, differentiability, integrability) conditions. To state and prove the theorem, we need some prerequisites from real.

Recall the following theorem from real analysis:

Theorem 32.1. *If X is a metric space, then TFAE:*

1) X is compact.
2) X is sequentially compact (e.g. every sequence has a convergent subsequence).
3) X is complete and totally bounded (e.g. $\forall \epsilon > 0$, $\exists x_1, \dots, x_n$ such that $X = \bigcup_{i=1}^n B(x_i, \epsilon)$).

There is a proof in Munkres of 1) $\Leftrightarrow$ 3), but this is a foundational result in Real analysis so I omit the proof here.

Next, I add some definition which are very important in functional analysis. Equicontinuous gives a kind of uniform continuity of a *collection* of functions, and pointwise-bounded means almost exactly what it says.

Definition 32.2. Let Y be a metric space, and X a topological space. Let $C(X, Y)$ be the set of continuous functions $X \to Y$. A subset $D \subseteq C(X, Y)$ is said to be **equicontinuous** at x if for each $\epsilon > 0$, there exists a neighborhood U of x such that $\forall y \in U$,

$$d(f(x), f(y)) < \epsilon$$

D is **equicontinuous** if it is for every point $x \in X$.

D is said to be **pointwise bounded** if for each *point* $x \in X$, the set

$$D_x = \{f(x) \mid f \in D\}$$

is bounded in Y.

Lemma 32.3. *If X is a space and Y is a metric space, then if $\mathcal{Z} \subseteq C(X, Y)$ is totally bounded, then $\mathcal{Z}$ is equicontinuous.*

Proof. Choose $\epsilon > 0$. Given $\mathcal{Z}$ is totally bounded, choose finitely-many $f_1, \dots, f_n$ such that balls of radius $\frac{\epsilon}{3}$ cover the space. This implies that every function $f \in \mathcal{Z}$ has

$$d(f_i(x), f(x)) < \frac{\epsilon}{3}$$

for some i and *all* $x \in X$. Choose U a neighborhood of x such that $d(f_i(y), f_i(x)) < \frac{\epsilon}{3}$ for $y \in U$. Again by the triangle inequality, we see that for $y \in U$, $d(f(y), f(x)) < \epsilon$. □

In the compact case, a partial converse exists.

Lemma 32.4. *If X is compact and Y is a compact metric space, and $\mathcal{Z} \subseteq C(X, Y)$ is equicontinuous, then $\mathcal{Z}$ is totally bounded in the uniform topology.*

Proof. Note that the uniform metric $\rho(f, g) = \sup_x\{d(f(x), g(x))\}$ is well-defined since X is compact and therefore so is it's image. Thus its image is totally bounded.

For a given $\epsilon > 0$, it suffices to cover $\mathcal{Z}$ by finitely many ϵ-balls.

Choose neighborhoods $U_x \subseteq X$ of x such that $d(f(x), f(y)) < \frac{\epsilon}{3}$ for each $y \in U_x$ and $f \in \mathcal{Z}$. Since X is compact, finitely many will do, say $U_{x_1}, \ldots, U_{x_n}$. This is possible by equicontinuity. We can similarly cover Y by finitely many $\frac{\epsilon}{3}$-balls, say $C_1, \ldots, C_m$.

For a given pair $(i,j) \in [n] \times [m]$, choose a function $f_{i,j} : U_{x_i} \to C_j$ with $f_{i,j} \in \mathcal{Z}$ if it exists. This is a finite collection of functions. Then I claim $B(f_{i,j}, \epsilon)$ cover $\mathcal{Z}$, proving total boundedness.

This follows again by the triangle inequality. Suppose $f \in \mathcal{Z}$ and $f(x_i) \in C_j$. Then

$$d(f(x), f_{i,j}(x)) \leq d(f(x), f(x_i)) + d(f(x_i), f_{i,j}(x_i)) + d(f_{i,j}(x_i), f_{i,j}(x)) < \epsilon$$

But $x \in X$ was arbitrary, showing $\rho(f, f_{i,j}) < \epsilon$, as the supremum is a maximum given X is compact and Y is Hausdorff. □

Theorem 32.5 (Ascoli's Theorem). *Let X be a compact topological space. Let $\mathcal{C} = C(X, \mathbb{R}^n)$ be the set of continuous functions from X to $\mathbb{R}^n$, with the uniform topology. Then $\mathcal{Z} \subseteq \mathcal{C}$ has compact closure if and only if $\mathcal{Z}$ is equicontinuous and pointwise bounded.*

Proof. $(\Rightarrow)$: Suppose $\overline{\mathcal{Z}}$ is compact. It suffices to check $\overline{\mathcal{Z}}$ is equicontinuous and pointwise bounded.

By virtue of Lemma 32.3, we can can conclude that since $\overline{\mathcal{Z}}$ is compact, it is totally bounded, and therefore equicontinuous. Totally bounded implies bounded. Indeed, choose

$$R = \max_{i=2,\ldots,n} \{d(x_1, x_i) + \epsilon \mid X = \bigcup_{i=1}^{n} B(x_i, \epsilon)\}$$

then $\overline{\mathcal{Z}} \subseteq \bar{B}(x_1, R)$. But this implies $\overline{\mathcal{Z}}$ is pointwise-bounded since $\rho(f,g) = \sup\{f(x), g(x)\}$.

$(\Leftarrow)$: First, I want to show that if $\mathcal{Z}$ is equicontinuous and pointwise bounded, then so is $\overline{\mathcal{Z}}$. Given $x \in X$, $\epsilon > 0$, there is a neighborhood of x, say U, such that $d(f(x), f(x_0)) < \frac{\epsilon}{3}$ for each $x \in U$ and $f \in \mathcal{Z}$. We can furthermore choose $f \in \mathcal{Z} \cap B(g, \frac{\epsilon}{3})$ for some $g \in \overline{\mathcal{Z}}$. As a result, for $x \in U$,

$$d(g(x_0), g(x)) \leq d(g(x_0), f(x_0)) + d(f(x_0), f(x)) + d(f(x), g(x)) < \epsilon$$

A similar argument shows pointwise boundedness.

Next, I show that if $\overline{\mathcal{Z}}$ is equicontinuous and pointwise bounded, then there exists a compact space $Y \subseteq \mathbb{R}^n$ such that $ev(X \times \overline{\mathcal{Z}}) \subseteq Y$.

For each $x \in X$, choose U_x a neighborhood such that for each $y \in U$, $d(g(x), g(y)) < 1$ for every $g \in \overline{\mathcal{Z}}$. Since X is compact, there exists a finite covering $U_{x_1}, \ldots, U_{x_n}$. Since each $\overline{\mathcal{Z}}_{x_i} = ev(U_{x_i} \times \bar{Z})$ is bounded by pointwise boundedness, so is their union. Thus $\exists R > 0$ s.t.

$$\overline{\mathcal{Z}}_{x_i} \subseteq \bar{B}(0, R)$$

But this implies $\overline{\mathcal{Z}}_x \subseteq B(0, R+1) \subseteq \bar{B}(0, R+1) = Y$.

Finally, I prove the Theorem. It suffices to check that $\overline{\mathcal{Z}}$ is complete & totally bounded in $\mathcal{C}$.

Since $\overline{\mathcal{Z}} \subseteq \mathcal{C}$ is closed, and $\mathcal{C}$ is complete, $\overline{\mathcal{Z}}$ is automatically complete. Additionally, the previous parts show that $\overline{\mathcal{Z}} \subseteq C(X, Y)$ is equicontinuous when Y is compact. Thus Lemma 32.4 shows that it is totally bounded in $\mathcal{C}$. This completes the proof. □

Rephrasing a few things, we note the following:

Corollary 32.6. *If X is compact, and $C(X, \mathbb{R}^n)$ has the uniform topology, then $\mathcal{Z} \subseteq C(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded, and equicontinuous.*

CLASS 33, DECEMBER 3: HOMOTOPY THEORY

One of the central objectives of mathematics is to classify all objects of a given type, be it groups, rings, metric spaces, topological spaces, varieties, categories, etc. This problem is immense, and if solved would essentially complete a branch of mathematics.

In our world of topological spaces, we would like to classify spaces up to homeomorphism. Again, this is highly intractable. Even in specific cases, the problem is one of the most difficult ever solved:

Theorem 33.1 (The Poincaré Conjecture, Perelman's Theorem(?))**.** *If X is a simply-connected, compact, 3-manifold, then $X \cong S^3$.*

This was solved in 2005 and was awarded a Fields Medal as well as a Millennium Prize.

Instead of trying to prove results like this, it is often easier and more reasonable to develop tools to say when two spaces are non-homeomorphic.

An initial step in this direction is to develop a weaker notion of 2 topological spaces being equivalent. This is the idea of homotopic spaces. To do this, we need to develop a few intermediate notions.

Definition 33.2. Let $f, g : X \to Y$ be continuous maps. f and g are said to be **homotopic** if there exists a continuous map

$$F : X \times I \to Y$$

with $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$. This is usually written $f \simeq g$. We say that the maps are **homotopic relative** Z, where $Z \subseteq X$, if additionally $F(z, t) = f(z)$ for all $t \in I$ and $z \in Z$. This is written $f \simeq g$ rel Z.

The idea of such a thing is as follows: we can continuously deform f to g by an interval worth of intermediate maps $f_t(x) = F(x, t)$. The idea of a relative homotopy is to keep a particular region fixed as t varies.

Example 33.3. Let S^1 be parameterized by $t \in [0, 1]$ with $0 = 1$. Then we can consider maps

$$\theta_m : S^1 \to S^1 : s \mapsto ms$$

$$\theta_{m,n} : S^1 \to S^1 : s \mapsto ms + n \sin(2\pi s)$$

This is the map which winds S^1 around m-times. These maps are homotopic relative to $\{0, 1\} \subseteq [0, 1]$. The homotopy can be given explicitly as

$$\Theta_{m,n} : S^1 \times I \to S^1 : (s, t) \mapsto ms + tn \sin(2\pi s)$$

Note that $\Theta_{m,n}(s, 0) = \theta_m(s)$, $\Theta_{m,n}(s, 1) = \theta_{m,n}(s)$, $\Theta_{m,n}(0, t) = 0$, and $\Theta_{m,n}(1, t) = 0$.

After describing homotopic maps, we can describe spaces as being homotopic:

Definition 33.4. If X and Y are topological spaces, then X is said to be **homotopic** to Y if there exists maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq Id_X$ and $f \circ g \simeq Id_Y$.

Note now that it is now automatic that two homeomorphic spaces are homotopic. Indeed, homeomorphisms require that $f \circ g$ and $g \circ f$ are equal to the identity. Therefore, we can take the **constant** homotopy maps $F(x, t) = g(f(x))$ and $G(x, t) = f(g(x))$.

Example 33.5. I claim that $S^1 \simeq \mathbb{R}^2 \backslash \{0\}$. We have the natural inclusion map $f : S^1 \to \mathbb{R}^2 \backslash \{0\}$ and the map $g : \mathbb{R}^2 \setminus \{0\} \to S^1 : x \to \frac{x}{\|x\|}$ was constructed on a homework as a retraction. Note that $g \circ f = Id_{S^1}$. On the other hand, the map $f(g(x)) = \frac{x}{\|x\|}$.

It suffices to construct a homotopy $\frac{x}{\|x\|} \simeq Id_{\mathbb{R}^2 \backslash \{0\}}$. We can construct this using a typical idea of a **linear homotopy**:

$$F : \mathbb{R}^2 \setminus \{0\} \times I \to \mathbb{R}^2 \setminus \{0\} : (x,t) \mapsto t \cdot x + (1-t) \cdot \frac{x}{\|x\|} = \frac{(t\|x\| + (1-t))x}{\|x\|}$$

In general, a linear homotopy between $f : X \to Y$ and $g : X \to Y$ is

$$F : X \times I \to Y : (x,t) \mapsto t \cdot f(x) + (1-t) \cdot g(x)$$

where we are implicitly assuming Y is (a subspace of) a real vector space.

Proposition 33.6. *$\simeq$ and $\simeq$ rel Z are equivalence relations.*

Proof.

- **Reflexive:** The constant homotopy (independent of t) shows $f \simeq f$ and $f \simeq f$ rel Z for any Z.
- **Symmetric:** Suppose $F : X \times I \to Y$ is a homotopy connecting f to g (rel Z). Then we can consider $G(x,t) = F(x, 1-t)$. This shows $g \simeq f$ (rel Z).
- **Transitive:** Suppose $f \simeq g$ (rel Z) by F and $g \simeq h$ (rel Z) by G. We can create a third homotopy

$$H(x,t) = \begin{cases} F(x,2t) & t \leq \frac{1}{2} \\ G(x,2t-1) & t \geq \frac{1}{2} \end{cases}$$

This is continuous by the pasting lemma:

$$F(x,1) = G(x,0) = g(x)$$

□

The transitive portion of this result gives the idea for how the group operation in the fundamental group will be developed.

Definition 33.7. A **path** from x to y in X is a continuous map $\gamma : I \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. If $x = y$, then γ is said to be a **loop based at** x.

Given 2 paths γ_1 from x to y and γ_2 from y to z, we define (perhaps abusively) the **composition** of the paths to be

$$\gamma_1 * \gamma_2 : I \to X : t \mapsto \begin{cases} \gamma_1(2t) & t \leq \frac{1}{2} \\ \gamma_2(2t-1) & t \geq \frac{1}{2} \end{cases}$$

This is a path from x to z.

Note that this is not itself a group operation; there exists no identity, inverses, nor is the operation associative. However, we can correct this by restricting our attention to loops and consider two homotopic loops to be equal. This is the idea of the fundamental group, which we will start next time.

CLASS 34, DECEMBER 5: THE FUNDAMENTAL GROUP

Today we introduce the fundamental group, which provides a tool to easily distinguish some non-homotopic spaces via abstract algebra. It is a surprising achievement of the 20th century that relates distinct regions of math.

Proposition 34.1. *Let $\Omega(X, x)$ be the set of loops based at x. Define*

$$\pi_1(X, x) := (\Omega(X, x)/ \simeq \ \textit{rel}\ \{0, 1\}\ ,\ *)$$

That is to say our elements are equivalence classes of homotopic loops relative to the basepoint x with the operation of composition of paths. Then $\pi_1(X, x)$ is a group.

Proof. ◦ **Identity:** The identity element for $*$ is the constant path $e : I \to X : t \to x$. Indeed,

$$F : I \times I \to X : (s, t) \mapsto \begin{cases} x & s \leq \frac{t}{2} \\ \gamma\left(\frac{s-\frac{t}{2}}{1-\frac{t}{2}}\right) & s \geq \frac{t}{2} \end{cases}$$

This is an explicit homotopy $\gamma \simeq e * \gamma$ (since $\frac{s-\frac{1}{2}}{1-\frac{1}{2}} = 2s - 1$). Additionally, $F(s, 0) = x$ and $F(s, 1) = \gamma(1) = x$ for all s. So in fact $\gamma \simeq e * \gamma$ rel $\{0, 1\}$, as desired.

A similar homotopy show $\gamma \simeq \gamma * e$ rel $\{0, 1\}$

Existence of inverses: I claim that for a given loop γ, the inverse is given by

$$\bar{\gamma}(s) = \gamma(1 - s)$$

(note the bar here is used to avoid confusion with the inverse image under γ). Again, we demonstrate this with an explicit homotopy operator:

$$F : I \times I \to X : (s, t) \mapsto \begin{cases} \gamma(2s) & s \leq \frac{t}{2} \\ \gamma(t) & \frac{t}{2} \leq s \leq 1 - \frac{t}{2} \\ \bar{\gamma}(2s - 1) & s \geq 1 - \frac{t}{2} \end{cases}$$

The idea here is quite simple; run through γ for a shorter and shorter period of time, stop at whatever point you get to, and then go back. Note that clearly $F(s, 0) = e(x) = x$, and additionally that $F(s, 1) = \gamma * \bar{\gamma}(s)$. Finally, note that this is a continuous map by the pasting lemma: $\gamma(2\frac{t}{2}) = \gamma(t) = \bar{\gamma}(2(1 - \frac{t}{2}) - 1) = \bar{\gamma}(1 - t)$.

Associative: One can compute

$$\gamma_1 * (\gamma_2 * \gamma_3) : s \mapsto \begin{cases} \gamma_1(2s) & s \in [0, \frac{1}{2}] \\ \gamma_2(4s - 2) & s \in [\frac{1}{2}, \frac{3}{4}] \\ \gamma_3(4s - 3) & s \in [\frac{3}{4}, 1] \end{cases} \qquad (\gamma_1 * \gamma_2) * \gamma_3 : s \mapsto \begin{cases} \gamma_1(4s) & s \in [0, \frac{1}{4}] \\ \gamma_2(4s - 1) & s \in [\frac{1}{4}, \frac{1}{2}] \\ \gamma_3(2s - 1) & s \in [\frac{1}{2}, 1] \end{cases}$$

The homotopy can be explicitly constructed as follows:

$$\Gamma : (s, t) \mapsto \begin{cases} \gamma_1\left(\frac{4}{1+t}s\right) & s \leq \frac{1}{4} + \frac{t}{4} \\ \gamma_2(4s - 1 - t) & \frac{1}{4} + \frac{t}{4} \leq s \leq \frac{1}{2} + \frac{t}{4} \\ \gamma_3\left(\frac{4}{2-t}s - \frac{t+2}{2-t}\right) & \frac{1}{2} + \frac{t}{4} \leq s \end{cases}$$

I leave it to you to check this is the desired homotopy. □

Therefore, $\pi_1(X,x)$ is a group! This is a very exciting result. Now we can proceed to discover some neat features of it. Recall the following definition:

Definition 34.2. A space X is called **path connected** if for every 2 points x, y, there exists a path γ such that $\gamma(0) = x$ and $\gamma(1) = y$. In such a case we write $\pi_1(X)$ instead of $\pi_1(X,x)$.

The reason we can do this is the following proposition:

Proposition 34.3. *If X is path connected, then for any two points x, y,*

$$\pi_1(X,x) \cong \pi_1(X,y)$$

Proof. Let γ be as in the definition of path connected. Then we can construct an explicit group isomorphism:

$$\Gamma : \pi_1(X,x) \to \pi_1(X,y) : \sigma \mapsto \bar{\gamma} * \sigma * \gamma$$

Note that this is a loop based at y, so it as at least a function. It is furthermore a group homomorphism:

$$\Gamma(\sigma * \sigma') = \bar{\gamma} * \sigma * \sigma' * \gamma \simeq \bar{\gamma} * \sigma * \gamma * \bar{\gamma} * \sigma' * \gamma = \Gamma(\sigma) * \Gamma(\sigma')$$

Of course, the inverse of this is given by a map of the same type:

$$\Gamma^{-1} : \pi_1(X,y) \to \pi_1(X,x) : \sigma \mapsto \gamma * \sigma * \bar{\gamma}$$

□

Now onto some examples (with nice definitions):

Example 34.4.
- $\pi_1(\mathbb{R}^n) = 0$. This is because $\mathbb{R}^n$ is a **contractible space**: $\mathbb{R}^n \simeq pt$. Indeed, we can use the simple homotopy $F(x,t) = (1-t) \cdot x$ to show this. Now, given a loop γ based at 0 (WLOG by Proposition 34.3), then we can consider

  $$G(s,t) = F(\gamma(s), t)$$

  This shows that $e \simeq \gamma$ rel $\{0,1\}$, and thus $\pi_1(\mathbb{R}^n) = 0$. A path-connected space with trivial fundamental group is called **simply-connected** (c.f. the Poincaré conjecture). Note that since $\mathbb{R}^n \ncong \mathbb{R}^m$ for $n \neq m$, we have that the fundamental group doesn't distinguish non-homeomorphic spaces.
- Additionally, not all spaces with trivial fundamental group are contractible. This is demonstrated by S^n for $n \geq 2$. All of these spaces bound an n-dimensional space, so are non-contractible.

  If $\gamma : I \to S^n$ is a path, then either γ is surjective (space filling) or it isn't. If it isn't surjective, then $\gamma : I \to S^n \setminus \{pt\} \cong \mathbb{R}^n$, and since $\mathbb{R}^n$ is contractible, so is the curve. If γ is surjective, we can take a small open neighborhood of a non-basepoint, and modify γ homotopically by taking the curve through the disc instead along the boundary, making it non-surjective. This shows $pi_1(S^n) = 0$.
- $\pi_1(S^1) = \mathbb{Z}$. This takes some work to show, but intuitively is quite easy to visualize. We can count the number of times a curve loops around the circle counterclockwise (the winding number of γ). This is a homotopy invariant (cf Example 33.3), and therefore establishes a well defined map $\pi_1(S^1) \to \mathbb{Z}$, treating clockwise rotations as negative. The difficult part is showing this is injective.

CLASS 35, DECEMBER 7: COVERING SPACES

One final example of a fundamental group should be noted:

Example 35.1. Consider the space $X = S^1 \vee S^1$, obtained by identifying 1 point on 2 distinct circles (looks like an 8). This space has a non-abelian fundamental group. Indeed, the group can be described explicity as $\mathbb{Z} * \mathbb{Z}$, the free product on 2 copies of $\mathbb{Z}$.

Without going into too much detail, the non-abelianess of this group can be realized as follows: let γ_1 be once around the left circle, and let γ_2 be once around the right circle. Then

$$\gamma_1 * \gamma_2 \not\simeq \gamma_2 * \gamma_1 \text{ rel } \{0, 1\}$$

Proposition 35.2 (Functoriality of π_1)**.** *Given a continuous map $f : X \to Y$ of topological spaces, we can induce a group homomorphism*

$$\pi_1(f) = f_* : \pi_1(X, x) \to \pi_1(Y, f(x)) : \gamma \to f \circ \gamma$$

Moreover, if $g : Y \to Z$ is another, $(g \circ f)_* = g_* \circ f_* : \pi_1(X, x) \to \pi_1(Z, g(f(z)))$.

Proof. It only goes to show that $f_*(\gamma_1 * \gamma_2) \simeq f_*(\gamma_1) * f_*(\gamma_2)$. They are in fact equal!

$$f_*(\gamma_1 * \gamma_2) \simeq f_*(\gamma_1) * f_*(\gamma_2) = \begin{cases} f(\gamma_1(2t)) & t \leq \frac{1}{2} \\ f(\gamma_2(2t-1)) & t \geq \frac{1}{2} \end{cases}$$

The second statement follows immediately from the definition of f_*. □

Now we come to a geometric analogy with Galois Theory:

Definition 35.3. A covering space of X is a surjective map $p : X' \to X$ such that for each $x \in X$, there exists U a neighborhood of x such that

$$p^{-1}(U) \cong \coprod_{\alpha} U$$

that is to say that the preimage of U is many copies of U in X'.

Some examples of this phenomenon we have already seen are as follows:

Example 35.4. The map $\theta_n : S^1 \to S^1 : t \mapsto nt$ is a covering space. Indeed, for a given x, we can choose a sufficiently small $\epsilon < \frac{1}{2n}$ and then for $t \in S^1$, consider $U = (t - \epsilon, t + \epsilon)$. In this case,

$$\theta_n^{-1}(U) = \bigcup_{i=0}^{n-1} \left(\frac{t + i - \epsilon}{n}, \frac{t + i + \epsilon}{n} \right) \cong \coprod_{i=0}^{n-1} U$$

such a thing is called an n-sheeted covering space. Note that this induces the (injective!) map

$$(\theta_n)_* : \pi_1(S^1) \to \pi_1(S^1) : m \mapsto n \cdot m$$

Another example of a covering space of the circle is given by $p : \mathbb{R} \to S^1 : t \to \lfloor t \rfloor$. Of course, given $\epsilon < \frac{1}{2}$, we see that

$$p^{-1}((t - \epsilon, t + \epsilon)) = \bigcup_{i \in \mathbb{Z}} (t + i - \epsilon, t + i + \epsilon) \cong \coprod_{\mathbb{Z}} (t - \epsilon, t + \epsilon)$$

Lemma 35.5. *If X satisfies some mild conditions*[1]*, then there exists a simply connected covering space $\tilde{X} \to X$.*

$\tilde{X}$ is called the **Universal Cover** of X. $\mathbb{R}$ is the universal cover of S^1. It is constructed by taking the space of all equivalence classes of paths in X which start at a selected basepoint x_0. The difficultly in the proof of this lemma is showing that it is simply connected. This gets us to the classification of covering spaces:

Theorem 35.6 (Classification of Covering Spaces)**.** *If X satisfies the same mild conditions, then there is a bijection between basepoint preserving path connected covering spaces and*

$$\{p : X' \to X \mid p \text{ is a covering space, } p(x_0') = x_0\}/\cong \longleftrightarrow \{H \subseteq \pi_1(X, x_0) \mid H \text{ a subgroup}\}$$
$$p \mapsto p_*\pi_1(X', x_0')$$

If we forget about the choice of basepoint, then we get

$$\{p : X' \to X \mid p \text{ is a covering space}\}/\cong \longleftrightarrow \{H \subseteq \pi_1(X, x_0) \mid H \text{ a subgroup}\}/\sim$$

where $\sim$ denotes conjugacy equivalence.

Finally, if $H' \subseteq H \subseteq \pi_1(X)$, and $X_{H'}$ and X_H are their associated path connected covering spaces, then

$$\exists p'' : X_{H'} \to X_H$$

a covering space which factors the covering space $p' : X_{H'} \to X$ and $p : X_H \to X$; $p \circ p'' = p'$.

There is also a notion of a *normal* cover, which corresponds exactly to the notion of a normal subgroup, and a notion of **deck transformations**, which plays a nearly identical role to the automorphisms fixing the base field in Galois Theory:

Theorem 35.7. *If L/K is a Galois extension of fields, then*

$$\{K' \mid K \subseteq K' \subseteq L\}/ \longleftrightarrow \{H \subseteq Gal(L/K) \mid H \text{ a subgroup}\}$$

[1] X is path connected, locally path connected, and semilocally simply-connected.