HOMEWORK 9: ADVANCED HOMOLOGY
DUE: FRIDAY, DECEMBER 8

1) Compute the homology groups $H_i(S^2, A)$ and $H_i(S^1 \times S^1, A)$ when A is a finite collection of points.

2) Compute $H_1(\mathbb{R}, \mathbb{Q})$. Note that it is NOT a good pair.

3) Show that $H_1(X, A) \not\cong \tilde{H}_1(X/A)$, where $X = I = [0, 1]$ and $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$. Note that X/A is naturally homeomorphic to the shrinking wedge of circles.

4) Prove the following theorem about the suspension of X using the pair (CX, X), where CX is the cone of X:
$$H_i(X) = H_{i+1}(SX) \ \ \forall i \geq 1$$

5) Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ have isomorphic homology groups. However, show that the universal covers of each space do not have isomorphic homology groups.[1] You can use the fact that the universal cover of $S^1 \vee S^1 \vee S^2$ is homotopically equivalent to $\vee_{\mathbb{N}} S^2$.

6) Given a map $f : S^{2n} \to S^{2n}$, show that there is some point $x \in S^{2n}$ with $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{RP}^{2n} \to \mathbb{RP}^{2n}$ has a fixed point. You may use the following fact freely: If $\mathbb{RP}^i \to \mathbb{RP}^i$ is a map without fixed points, then $\bar{f}_* \pi_1(\mathbb{RP}^i) = 0$. This then becomes a question of whether you even **lift**.

   On the other hand, consider a surjective linear map $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ with no eigenvectors (no real eigenvalues).[2] You may assume it's existence. Show that this induces a map without fixed points on $\mathbb{RP}^{2n-1} \to \mathbb{RP}^{2n-1}$.

[1] **Note:** By the midterm, this is an example of non-homotopic spaces having identical homology groups.

[2] Such a thing could be described by a rotation by a fixed angle θ_i in $\mathbb{R}^2_i$ applied to $\mathbb{R}^{2n} = \mathbb{R}^2_1 \times \ldots \times \mathbb{R}^2_n$.

Some extra problems for practice (not graded)

7) A complex polynomial $f(z) \in \mathbb{C}[z]$ can be viewed as a map $\mathbb{C} \to \mathbb{C}$ given by evaluation. A map such as this can always be extended to a continuous map from $\mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$, which are homeomorphic (in fact quite more) to the 'Riemann sphere' S^2. Show that the degree of f as a polynomial is the same as the degree of the map $\hat{f} : S^2 \to S^2$. **Note:** It may be beneficial to look up local degree in Hatcher, pg 136.

8) Compute the Homology groups of the space X obtained as the quotient of S^2 with points on the equator identified via $x \sim -x$. That is
$$X = S^2/(x, y, 0) \sim (-x, -y, 0)$$
Do the same for S^3 with equatorial points of S^2 identified in this way. You may use the fact that $\tilde{H}_i(\mathbb{RP}^3)$ is $\mathbb{Z}$ if $i = 3$, $\mathbb{Z}/2\mathbb{Z}$ when $i = 1$, and is 0 otherwise.

9) Use Meyer-Vietoris to show that if (X, x) and (Y, y) are good pairs, then $\tilde{H}_n(X \vee Y)$ is isomorphic to $\tilde{H}_n(X) \oplus \tilde{H}_n(Y)$. This gives a easier proof of an earlier result proved by excision.

10) Show $\tilde{H}_n(SX) \cong \tilde{H}_{n-1}(X)$ for all $n \geq 0$ via Meyer-Vietoris.

11) Suppose X is a space covered by open sets $A_1, \ldots, A_n$ with $\tilde{H}_l(A_{i_1} \cap \ldots \cap A_{i_j}) = 0$ for each $l \geq 0$. Show that $\tilde{H}_l(X) = 0$ for $l \leq n - 1$. Furthermore, show that this bound is strict.