HOMEWORK 10: CONFORMAL MAPPINGS
DUE: FRIDAY DECEMBER 6TH

(1) Prove that the following product converges and the result is $\frac{\sin(z)}{z}$:

$$\cos\left(\frac{z}{2}\right)\cos\left(\frac{z}{4}\right)\cdots = \prod_{n=1}^{\infty}\cos\left(\frac{z}{2^n}\right)$$

As a hint, recall the double angle identity for sin.

(2) If $|z| < 1$, show that

$$(1+z)(1+z^2)(1+z^4)\cdots = \prod_{n=1}^{\infty}(1+z^{2^n}) = \frac{1}{1-z}$$

(3) Assuming the result of Hadamard, stated as Theorem 29.4 in the notes, show Picard's Little Theorem:

Theorem 0.1. *If f is an entire function of finite order that omits 2 values, then f is constant.*

Picard's 'big theorem' is the one about essential singularities having infinite sheeted coverings nearby missing perhaps 1 point.

(4) Show that if $f : U \to V$ is a conformal map, then if U is connected or simply connected, then V is also. Therefore these properties are preserved by conformal equivalence.

(5) Is there a holomorphic surjection from the disc onto $\mathbb{C}$?

(6) Suppose F is holomorphic at 0, and $F(0) = F'(0) = 0$, but $F''(0) \neq 0$. Show that there exist two curves $\gamma_1, \gamma_2 : [-1, 1] \to \mathbb{C}$ with $\gamma_i(0) = 0$ and such that $F \circ \gamma_1$ is real valued with a minimum at 0 and $F \circ \gamma_2$ is real valued with a maximum at 0. (**hint:** Write $F(z) = (g(z))^2$ for some g, and consider g and its inverse)[1]

(7) If $F : \mathbb{H} \to \mathbb{C}$ is holomorphic satisfying

$$|F(z)| \leq 1 \qquad\qquad F(i) = 0$$

Prove that

$$|F(z)| \leq \left|\frac{z-i}{z+i}\right| \qquad\qquad \forall z \in \mathbb{H}$$

[1]This is an analog of a saddle point in calculus and real analysis.