CLASS 23, NOVEMBER 6TH: FOURIER TRANSFORMS INTRO

Recall previously that we have discussed the idea of the Fourier transform of a function f:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx$$

An important variation on this formula is the **Fourier Inversion Formula**:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-2\pi i x\xi}d\xi$$

Since we are performing an integral over the whole real line, there is a question of convergence. We will work to establish a sufficient condition, called moderate descent. But since we work in the complex world, we will attempt to analytically continue a real valued function with this property to a strip around the real line.

Definition 23.1. A real valued function $f : \mathbb{R} \to \mathbb{R}$ has **moderate descent** if

$$|f(x)|, |\hat{f}(x)| \leq \frac{A}{1+x^2}$$

for some constant A.

This makes it so that the integrals in question in the above formulas are well define, since

$$\left|\hat{f}(\xi)\right| = \left|\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi}dx\right| \leq \int_{-\infty}^{\infty} \left|f(x)e^{-2\pi i x\xi}\right| dx \leq \int_{-\infty}^{\infty} \frac{A}{1+x^2}dx = A\pi$$

Using the fact that the last integral has primative $A\tan^{-1}(x)$. The same goes for the inversion formula.

Now, since we work in the complex plane, we will work to define a class of functions $\mathcal{F}$ for which the same results hold (inversion formula, Poisson summation, etc). We intend to define it analogously to that of moderate descent. But our class will be larger, opening a wider swath of applications.

For each $a > 0$, define $\mathcal{F}_a \subseteq \{f : \mathbb{C} \to \mathbb{C}\}$ satisfying the following properties:

(1) f is holomorphic in the horizontal strip $S_a = \{z = x + iy \mid |y| < a\}$.
(2) There exists $A > 0$ such that for $z = x + iy \in S_a$,

$$f(x+iy) \leq \frac{A}{1+x^2}$$

This is to say $\mathcal{F}_a$ is the set of functions satisfying moderate descent on fixed real lines of S_a uniformly.

Example 23.2. $f(z) = e^{-\pi z^2}$ has $f \in S_a$ for each $a > 0$. This is because

$$|f(z)| = e^{-\pi x^2 + \pi y^2}$$

and since y is bounded, f decays exponentially with f. Note that previously we showed that this function is its own Fourier Transform!

Similarly, $f(z) = \frac{1}{\pi}\frac{c}{c^2+z^2}$ is in $\mathcal{F}_a$ for each $a < c$ (to avoid the poles at $z = \pm ic$). Away from these poles, we get

$$|f(z)| = \left|\frac{1}{\pi}\frac{c}{c^2+z^2}\right| = \frac{c}{\pi}\left|\frac{1}{c^2+z^2}\right|$$

For $x > 2c$, this function experiences the desired rate of decay.

Lastly, we have also shown $\frac{1}{\cosh(\pi z)}$ is its own Fourier transform. It can be shown to be in $\mathcal{F}_a$ for each $a < \frac{1}{2}$.

A nice application of Cauchy's Integral theorem is that if $f \in \mathcal{F}_a$, then so is $f^{(n)}$ for each $n \geq 0$. Indeed, consider

$$\left|f^{(n)}(z)\right| \leq \frac{n!\|f\|_C}{R^n} \leq \frac{n!}{R^n}\sup_{w=x+iy\in C}\left(\frac{A}{1+x^2}\right)$$

$f^{(n)}(z)$ will be bounded near $x = 0$, and for the rest we can choose a constant uniformly.

NOTE: We can allow more functions into $\mathcal{F}_a$ without change if we define moderate decrease by $|f(z)| \leq \frac{C}{1+|x|^{1+\epsilon}}$ for some $\epsilon > 0$.

Definition 23.3. We define $\mathcal{F}$ to be the set of functions f that are in some $\mathcal{F}_a$:

$$\mathcal{F} = \bigcup_{a>0}\mathcal{F}_a$$

Theorem 23.4. *If $f \in \mathcal{F}_a \subseteq \mathcal{F}$, then $|\hat{f}(\xi)| \leq Be^{-2\pi b|\xi|}|$ for all $0 \leq b < a$.*

This result says that if f has moderate decay, then $\hat{f}$ has exponential decay. At the very least, this allows us to say that $\hat{f} \in \mathcal{F}$ given $f \in \mathcal{F}$. But it is a far stronger result than this.

Proof. For $b = 0$, we have that $\hat{f}$ is bounded. This is immediate given its definition. So suppose $0 < b < a$. Begin with the case $\xi > 0$. The idea is to shift the integral down to the line where the imaginary part is $-b$ using contour integration along the rectangle $\mathcal{R} = [-R, R, R - ib, -R - ib]$. Note that the vertical sides of this rectangle have the property that

$$\left|\int_{R-ib}^{R} f(z)e^{-2\pi iz\xi}dz\right| \leq \int_0^b \left|f(R-it)e^{-2\pi i(R-it)}\right| dt \leq \int_0^b \left|\frac{A}{R^2}e^{-2\pi i(R-it)}\right| dt \leq \frac{C}{R^2} \to 0$$

as $R \to \infty$. Therefore, by Cauchy/Goursat, we have that

$$\tilde{f}(\xi) = \int_{-\infty}^{\infty} f(x-ib)e^{-2\pi i(x-ib)\xi}dx$$

and thus

$$|\tilde{f}(\xi)| \leq \left|\int_{-\infty}^{\infty}\frac{A}{1+x^2}e^{-2\pi b\xi}dx\right| \leq A\pi e^{-2\pi b\xi}$$

A very similar argument works for $\xi < 0$, but instead you need to consider the rectangle $\mathcal{R} = [-R, R, R + ib, -R + ib]$. ∎

This at the very least ensures that Fourier Inversion makes sense. Next time we will show that it *is* an inversion formula.