CLASS 5, FEBRUARY 13TH: RADICALS AND ZERO DIVISORS

Next up we will study the radical of a given ideal, and see how it relates to zero divisors generally speaking. We will also study a particular case of the radical, the nilradical, and see how it relates to prime ideals.

Definition 5.1. Given $I \subsetneq R$ an ideal, the **radical** of I is

$$\sqrt{I} := \{f \in R \mid f^n \in I \text{ for some } n \in \mathbb{N}\}$$

Note the following easy consequences:

- If $\mathfrak{p}$ is prime, then $\sqrt{p} = \mathfrak{p}$.
- If $I \subseteq J$, then $\sqrt{I} \subseteq \sqrt{J}$

There is also a special version of the radical, which gets its own name:

Definition 5.2. The **nilradical** of R is the radical of 0:

$$nil(R) = \sqrt{0} = \{f \in R \mid f^n = 0 \text{ for some } n \in \mathbb{N}\}$$

As its name incurs, it is precisely the set of nilpotent elements of R. It turn out that we only need to study the nilradical of rings to acquire information about the radical of more arbitrary ideals.

Proposition 5.3. *Given the quotient map $\varphi : R \to R/I$, we can compute the radical of I as*

$$\sqrt{I} = \varphi^{-1}(nil(R/I)) = nil(R/I) + I$$

Proof. On the right-hand side, we have elements $f + I$ such that $(f+I)^n = f^n + I = 0 + I$. This is exactly saying the $f^n \in I$. The result is immediately clear. □

As a result, we are able to more easily focus on the nilradical and derive results about the radical under this relationship. The first interesting result concerns how prime ideals relate to the nilradical:

Theorem 5.4. *The nilradical is the intersection of prime ideals:*

$$nil(R) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$$

Proof. The result follows by application of Theorem 4.7 from last class.

For the easy direction, note that if $f^n = 0$, then since $0 \in \mathfrak{p}$ for any *ideal* $\mathfrak{p}$, if $\mathfrak{p}$ is prime we see that either $f \in \mathfrak{p}$ or $f^{n-1} \in p$. Induction allows us to conclude that $f \in \mathfrak{p}$ in either case. Therefore, $f \in nil(R)$ implies $f \in \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$.

Now suppose f is not nilpotent. It suffices to check that there exists $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $f \notin \mathfrak{p}$ as this will imply $f \notin \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$. Note that being non-nilpotent implies that

$$0 \notin S = \{1, f, f^2, f^3, \ldots\}$$

and therefore S satisfies the properties of a multiplicative set. Since 0 is an ideal in any ring, and $S \cap 0 = \emptyset$, there exists a prime ideal $\mathfrak{p}$ of R disjoint from S. Thus $f \notin \mathfrak{p}$ as asserted. □

We can 'upgrade' this to a statement about radicals if we examine more carefully the statement of Homework 1 #2. We know that the map $\mathrm{Spec}(R/I) \hookrightarrow \mathrm{Spec}(R)$ is injective. It's image is exactly

$$\varphi^{\#}(\mathrm{Spec}(R/I)) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid I \subseteq \mathfrak{p}\}$$

This can be realized directly by considering what the preimage of an ideal is.

Corollary 5.5. *The radical of an ideal is the intersection of the prime ideal containment:*

$$\sqrt{I} = \bigcap_{I \subseteq \mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$$

Proof. Following the realization above, we see that by Theorem 5.4

$$\sqrt{I} = \varphi^{-1}(nil(R/I)) = \varphi^{-1}\left(\bigcap_{\mathfrak{p} \in \mathrm{Spec}(R/I)} \mathfrak{p}\right) = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R/I)} \varphi^{-1}(\mathfrak{p}) = \bigcap_{I \subseteq \mathfrak{p} \in \mathrm{Spec}(R)} \mathfrak{p}$$

□

Lastly, I would like to talk about other types of zero divisors than nilpotents.

Example 5.6. The ring $R = K[x,y]/\langle xy \rangle$ is a ring with no nilpotents (called a **reduced ring**). However, we can clearly multiply x and y, both non-zero in the ring, and end up with zero!

By the realization above, $\mathrm{Spec}(R) = \{\mathfrak{p} \in \mathrm{Spec}(K[x,y]) \mid \langle xy \rangle \subseteq \mathfrak{p}\}$. By definition of primality, this implies either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$ (or both). These can be described respectively as $\mathrm{Spec}(K[x,y]/\langle x \rangle) = \mathrm{Spec}(K[y])$ and $\mathrm{Spec}(K[x,y]/\langle y \rangle) = \mathrm{Spec}(K[x])$. By our computation of $\mathrm{Spec}(K[x,y])$ from Class 3, this implies $\mathrm{Spec}(R)$ can be decomposed as follows:

$$\mathrm{Spec}(R) = \mathrm{Spec}(K[y]) \cup \mathrm{Spec}(K[x]) = \mathbb{A}^1_K \cup \mathbb{A}^1_K$$

Their intersection is prime ideals containing both x and y, i.e. $\langle x, y \rangle$!

This example is a specific case of the following Proposition:

Proposition 5.7. *If R is a ring containing zero divisors, then either $nil(R) \neq 0$ or R has more than one minimal prime.*

Finally, a quick word about **idempotent** elements. Recall these are elements $e \in R$ such that $e^2 = e$. The canonical example is a projection operator in linear algebra. The neat thing about these elements is as follows:

Proposition 5.8. *R has an idempotent element $e \neq 0, 1$ if and only if R is a direct sum/cartesian product of 2 rings R_1, R_2.*

Proof. I claim $R \cong eR \oplus (1-e)R$. This is seen by taking homomorphisms

$$\varphi : R \to eR \oplus (1-e)R : r \mapsto (e \cdot r, (1-e)r)$$
$$\psi : eR \oplus (1-e)R \to R : (r, s) \mapsto r + s$$

The only thing to note here is that

$$\varphi(r \cdot s) = (ers, (1-e)rs) = (e^2 rs, (1 - 2e + e^2)rs) = (er \cdot es, (1-e)r(1-e)s)$$
$$= (er, (1-e)r) \cdot (es, (1-e)s) = \varphi(r) \cdot \varphi(s)$$

□