Prop: Additionally, if two lifts agree at 1 point, they agree everywhere. (e.g. unique).

Oct 30: Classification of Covering Sp:

Note: Beutiful Prop from last time:

$$p_*: \pi_1(\tilde{X}, \tilde{x}_0) \hookrightarrow \pi(X, x_0)$$

Thus $p_* \pi_1(\tilde{X}, \tilde{x}_0) \subseteq \pi_1(X, x_0)$ is a subgroup.

The question is; Does every subgroup $\subseteq \pi_1(X,x)$ correspond to a covering space?

Let's answer the simpler question can 0 be realized?
We assume spaces are
- locally path connected (so we can divide into path components)
- Semilocally ~~path~~ simply -connected: $\pi_1(U) \xrightarrow{i_* = 0} \pi_1(X)$ for some open neighborhood U of any point $x \in X$.

If $p: \tilde{X} \to X$ is a c.s., then $\forall x \in X$ $\exists U \ni x$ w/ $p^{-1}(U) \cong \coprod_{\alpha} U$. If

$\tilde{X}$: simply connected

$\gamma \in \pi_1(U)$, then $\tilde{\gamma} \in \pi_1(p^{-1}(U))$ is trivial, an $p_* \tilde{\gamma} = \gamma$ is trivial.

(It is a slight weakening of locally simply connected)

Prop: $\exists\ p:\tilde{X}\to X$ w/ $\tilde{X}$ simply connected,
Defn | if X is PC, loc PC, SLSC. $\tilde{X}$ is called the universal cover of X.

Pf: Let $\tilde{X} = \{[\gamma] \mid \gamma$ is a path starting @ $x_0\}$
Note: γ is taken up to homotopy of paths.

- $p:\tilde{X}\to X : [\gamma] \mapsto \gamma(1)$ is well defined, surjective (PC)
- Topology for $\tilde{X}$: Let $\mathcal{U} = \{U \overset{\text{open}}{\subseteq} X \mid i_{U*}\pi_1(U) = 0\}$

If $V \subseteq U$ is open, this also holds $\Rightarrow$ basis for a topology X

If $x \in U$, γ connects x_0 to x,
$U_{[\gamma]} = \{[\gamma\cdot\eta] \mid \eta$ ~~connects~~ in U starting at $\gamma(1)\}$

$U_{[\gamma]} \twoheadrightarrow U$ since U is PC and inj since $i_{U*}\pi_1(U)$ is trivial.

$U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]} : \gamma' = \gamma\cdot\eta \Rightarrow$

$U_{[\gamma']} = \{[\gamma\cdot\eta\cdot\mu]\}$ $\quad U_{[\gamma]} = \{[\gamma\cdot\mu] = [\gamma\cdot\eta\cdot\bar{\eta}\cdot\mu]\}$ $\quad \overset{[\gamma'\cdot\bar{\eta}\mu]}{=}$

$\Rightarrow \tilde{\mathcal{U}} = \{U_{[\gamma]}\}$ is a basis for the topo on $\tilde{X}$. $p|_{U_{[\gamma]}} : U_{[\gamma]} \to U$ is a homeo by construction, and thus p is continuous, w/ $p^{-1}(U) = \coprod_{[\gamma]} U_{[\gamma]} \Rightarrow p$ is a cov sp

disjoint since if $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']} \Rightarrow$ allequal

Lastly, WTS $\tilde{X}$ is simply connected. $[\gamma] \in \tilde{X}$
Let γ_t be $\gamma|_{[0,t]}$. $t \mapsto \gamma_t$ is a path in $\tilde{X}$ lifting γ $[x_0]$ to γ. Thus $\tilde{X}$ is PC. WTS

$p_*\pi_1(\tilde{X},\tilde{x}_0)=0$, since p_* is injective. If γ is in the image, it lifts to a loop $\tilde{\gamma}$ in $\tilde{X}$. but $t \mapsto \gamma_t$ is a homotopy $\Rightarrow$ trivial $\boxtimes$

Why care?: This shows

$$\left\{\begin{matrix}\text{Cov Sp of } X \\ p:(\tilde{X},\tilde{x}_0)\to(X,x_0)\end{matrix}\right\}/\cong \;\longleftrightarrow\; \left\{\text{Subgroups } H \le \pi_1(X,x_0)\right\}$$

$$p \longmapsto p_*\pi_1(\tilde{X},\tilde{x}_0)$$

$$\tilde{X}/\sim_H \longleftarrow H$$

$[\gamma] \sim_H [\gamma']$ if $\gamma(1)=\gamma'(1)$ and $[\gamma\gamma'] \in H$.

$$p \cong p' \iff \begin{tikzcd} \tilde{X} \arrow[rr, shift left, "f"] \arrow[dr, "p"'] & G & \tilde{X}' \arrow[ll, shift left, "g=f^{-1}"] \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$ w/ $f(\tilde{x}_0)=\tilde{x}_0'$ and vice versa

Prop: $p \cong p' \iff p_*\pi_1(\tilde{X},\tilde{x}_0) = p_*\pi_1(\tilde{X}',\tilde{x}_0')$

$$\begin{tikzcd} & X^u \arrow[dl] \arrow[dr] & \\ \tilde{X} \arrow[rr, shift left, "f"] \arrow[dr] & & \tilde{X}' \arrow[ll, shift left, "g"] \arrow[dl] \\ & X & \end{tikzcd}$$

Prop: Let X be PC, LPC, SLSC. Then

$$\left\{ \begin{array}{c} p:(\tilde{X},\tilde{x}_0)\to(X,x_0) \\ \text{preserving b.p.} \end{array} \right\} \Big/ \underset{\text{preserving b.p.}}{\cong} \longleftrightarrow \left\{ H \subseteq \pi_1(X,x_0) \right\}$$

$$\left\{ p:\tilde{X}\to X \right\}/\cong \;\longleftrightarrow\; \left\{ gHg^{-1} \subseteq \pi_1(X,x_0) \right\}$$

Nov 1: Deck Transformations: We Know , which tell us the set corresponding to $\pi_1(X,x_0)$, namely (as sets) $\pi_1(X,x_0) \longleftrightarrow p_u^{-1}(x_0)$

Moreover subgroups $\longleftrightarrow$ Quotient covers of X_u.

Now, we want to study the gp structure.

$$\begin{tikzcd} \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & \tilde{X} \arrow[dl, "p'"] \\ & X & \end{tikzcd}$$

f is called a deck transform. They form a group under Comp: $G(\tilde{X})$

$p:\tilde{X}\to X$ is called normal ~~if $p_*\pi_1(\tilde{X})$ is a normal subgp.~~ if $\forall x\in X$, and $\tilde{x}, \tilde{x}' \in \tilde{X}$ mapping to x, $\exists f:\tilde{X}\to\tilde{X} : \tilde{x}\mapsto\tilde{x}'$

(Note: Sometimes called a regular cover).

Prop: Let $(\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ be path connected cov sp of X LPC, PC, and let $H \leq \pi_1(X, x_0)$ w/ $H = p_* \pi_1(\tilde{X}, \tilde{x}_0)$. Then

a) p is normal $\Longleftrightarrow$ H is a normal subgroup
b) $G(\tilde{X}) \cong N(H)/H$, where $N(H)$ is the normalizer subgroup of H.

__Recall__: $N(H) = \{ g \in G \mid gHg^{-1} \subseteq H \}$

Cor: If p is a normal cover, then $G(\tilde{X}) = \pi_1(X)/p_* \pi_1(\tilde{X})$.

In particular $G(\tilde{X}_u) \cong \pi_1(X, x_0)$

Pf: Change of basepoint: $\pi_1(\tilde{X}, \tilde{x}_0) \xrightarrow{\beta_{\tilde{h}}} \pi_1(\tilde{X}, \tilde{x}_1)$ where $\tilde{h}$ connects $\tilde{x}_0$ to $\tilde{x}_1$ in $\tilde{X}$ is a lift of a loop in X at x_0. Thus $h \in N(H)$ iff

$$\begin{tikzcd} \pi_1(\tilde{X}, \tilde{x}_0) \arrow[rr, "\beta_h"] \arrow[dr, "p_*"'] & & \pi_1(\tilde{X}, \tilde{x}_1) \arrow[dl, "p_*'"] \\ & \pi_1(X, x_0) & \end{tikzcd}$$

$\mathrm{Im}(p_*) = \mathrm{Im}(p_*')$. By the lifting criterion $\Rightarrow \exists\, f_h \in G(\tilde{X})$ taking $\tilde{x}_0$ to $\tilde{x}_1$. $\Rightarrow$ (a) $\boxtimes$

Thus, $\varphi : N(H) \to G(\tilde{X}) : [\gamma] \mapsto \tilde{\gamma}$ above.

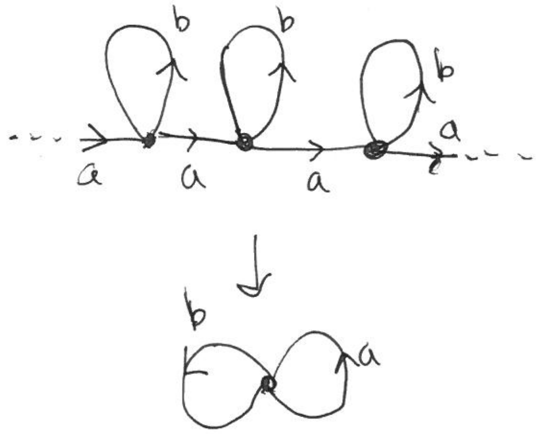
φ is a hom, and $\mathrm{Ker}(\varphi) = \{ \gamma \text{ lifting to loops } \tilde{\gamma} \}$ $= H$. $\boxtimes$

Ex/
$$\begin{tikzcd} S^n \arrow[rr, "x \mapsto -x"] \arrow[dr] & & S^n \arrow[dl] \\ & \mathbb{R}P^n & \end{tikzcd}$$

is a deck Transformation since $\pi_1(S^n) = 0$, this implies $\uparrow n \geq 2$

$$\pi_1(\mathbb{R}P^n) = \mathbb{Z}/2\mathbb{Z}$$

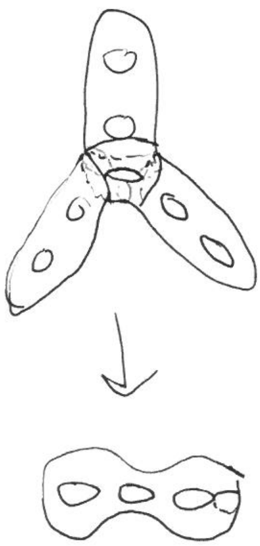
Ex/



A Deck transformation of this space is a translation by a certain $\mathbb{Z}$-number of a translations

This is a normal cover

$$\Rightarrow G(\tilde{X}) = \pi_1(\infty) / p_*\pi_1(\cdots\!\circ\!\circ\!\circ\cdots)$$
$$= \underset{a}{\mathbb{Z}} * \underset{b}{\mathbb{Z}} / \underset{b}{\mathbb{Z}}$$
$$= \mathbb{Z}$$



$m=3$, $n=2$

$$\begin{tikzcd} M_7 \arrow[d] & \text{or} & M_{mn+1} \arrow[d] \\ M_3 & & M_{m+1} \end{tikzcd}$$

We have
$G(\tilde{X}) = \mathbb{Z}/3\mathbb{Z}$
given by rotation