from Top to Gp

Prop:

Oct 16: π_1 is a Functor: If $f: X \to Y$ is continuous, then $\exists \left[f_*: \pi_1(X) \to \pi_1(Y) \right] = \pi_1(f)$ and s.t. $f_* \circ g_* = (f \circ g)_*$.

Pf: Define $\pi_1(f) = f_*$ by

$$\begin{aligned} f_*: \pi_1(X, x) &\to \pi_1(Y, f(x_0)) \\ \gamma &\longmapsto f \circ \gamma : I \to Y \end{aligned}$$

Then $\underbrace{(g \circ f)_* (\gamma)(t)}_{\boxed{\text{Retr, Def Ret.}}} = (g(f(\gamma(t))) = g_*(f_*(\gamma))(t)$

Prop: $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$

Pf:

$$\begin{tikzcd} X \times Y \arrow[r, shift left, "p_Y"] \arrow[d, shift right, "p_X"'] & Y \arrow[l, shift left, "i_Y"] \\ X \arrow[u, shift right, "i_X"'] & \end{tikzcd}$$

$$\begin{aligned} \pi_1(X \times Y) &\to \pi_1(X) \times \pi_1(Y) \\ \gamma &\longmapsto (p_{X*}(\gamma), p_{Y*}(\gamma)) \\ \gamma = (i_{X*}\gamma_1, i_{Y*}\gamma_2) &\longleftarrow (\gamma_1, \gamma_2) \end{aligned}$$

Note: $\gamma \mapsto (i_{X*}\gamma, i_{Y*}\gamma) \mapsto (p_{X*} i_{X*}\gamma, p_{Y*} i_{Y*}\gamma)$

$$= ((p_X i_X)_* \gamma, (p_Y i_Y)_* \gamma)$$

$$= \gamma$$

Sim other inverse!

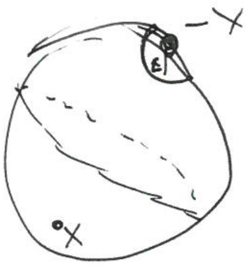
Ex/ $\pi_1(\mathbb{T}^2) = \pi_1(S^1 \times S^1) = \mathbb{Z}^2$

Prop:
$\pi_1(S^n)=0$ for $n \neq 1$

Pf: S^0 is trivial; $S^0 = P_1 \sqcup P_2$. Let γ be a path

If $n>1$, then ~~define~~ consider $B_\varepsilon(-x)\subseteq S^n$

for $0<\varepsilon\ll 1$. I claim we can find



$$\tilde{\gamma}\simeq\gamma \quad \text{s.t.} \quad \tilde{\gamma}: I \to S^n \setminus B_\varepsilon(-x)$$

Note: If we can do this, then $\mathbb{R}^n \cong S^n\setminus B_\varepsilon(x)$ is contractible, so γ is null homotopic.

Let $\mathcal{U}\subseteq I$ be

$$\mathcal{U}=\{x\in I : \gamma(x)\in B_\varepsilon(x)\}$$

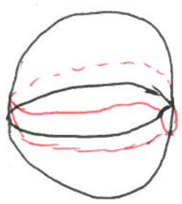
$\mathcal{U}$ is open, so $\mathcal{U}=\bigcup_{i=1}^{n}(a_i,b_i)\subseteq[0,1]$

for $\gamma|_{(a_i,b_i)}$, consider $\tilde{\gamma}_i:(a_i,b_i)$ connecting $\gamma(a_i)$ to $\gamma(b_i)$ on the circle $\partial D^n(-x,\varepsilon)$.

Then $\gamma|_{(a_i,b_i)}\simeq\tilde{\gamma}_i$.

Define $\tilde{\gamma}(t)=\begin{cases}\gamma(t) & t\notin\bigcup(a_i,b_i)\\ \tilde{\gamma}_i(t) & t\in(a_i,b_i)\end{cases}$

Then $\tilde{\gamma}\simeq\gamma$ and we are done.



$\pi_1(X\times Y)$

Cor: $\mathbb{R}^2 \not\cong \mathbb{R}^n$ for $n \neq 2$.

Pf: If $n=1$, $\mathbb{R}^1\setminus pt$ is not connected, whereas $\mathbb{R}^2\setminus pt$ is.

If $n>2$, then consider

$$S^1 \cong \mathbb{R}^2\setminus pt \longrightarrow \mathbb{R}^n\setminus pt \simeq S^{n-1}$$

After

$$\pi_1(\mathbb{R}^2\setminus pt) = \pi_1(S^1) = \mathbb{Z}$$
$$\pi_1(\mathbb{R}^n\setminus pt) = 0$$

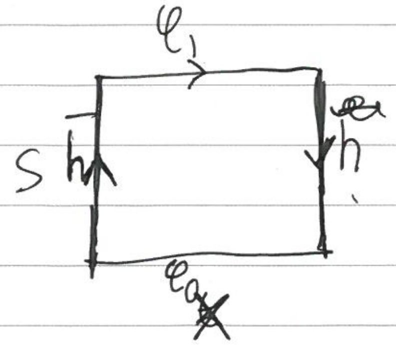
We need the following prop:

Prop: If $\varphi: X \to Y$ is a homotopy eq, then

$$\varphi_*: \pi_1(X, x_0) \longrightarrow \pi_1(Y, \overset{\varphi(x_0)}{\overset{\|}{y_0}})$$ is an isom.

Pf: If φ_t is a homotopy, then

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\varphi_{0*}"] \arrow[dr, "\varphi_{1*}"'] & \pi_1(Y, \varphi_0(x_0)) \arrow[d, "\beta_h"] \\
 & \pi_1(Y, \varphi_1(x_0))
\end{tikzcd}$$



If φ is as above, let ψ be its homotopy inverse. Then

$$\varphi \circ \psi \simeq Id_Y$$
So $(\varphi\circ\psi)_* \gamma \simeq \gamma$
$$\Rightarrow (\varphi\circ\psi)_* = Id_X$$
$$\overset{\|}{\varphi_* \circ \psi_*}$$

Oct 20: If X is a space, can we compute $\pi_1(X)$ by smaller pieces?

Thm: If $A_\alpha \overset{i_\alpha}{\hookrightarrow} X$ are path connected open spaces $A_\alpha \ni x$, then if $A_\alpha \cap A_\beta$ is path connected, then

$$\underset{\alpha}{*}\pi_1(A,x) \xrightarrow{*i_\alpha} \pi_1(X,x) \quad \text{is surjective}$$

If moreover, $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then $\ker(*_\alpha) = \langle i_{\alpha\beta_*}(\gamma)\, i_{\beta\alpha_*}(\gamma)^{-1} \rangle$

$$\Rightarrow \pi_1(X,x) = *\pi_1(A_\alpha,x) / \langle i_{\alpha\beta_*}(\gamma)\, i_{\beta\alpha_*}(\gamma)^{-1} \rangle$$

Particular Case: $X = A \cup B$

$$\begin{tikzcd} & A \arrow[dr] & \\ A\cap B \arrow[ur] \arrow[dr] & & A \cup B \\ & B \arrow[ur] & \end{tikzcd}$$

If $A, B, A\cap B$ are path connected, then

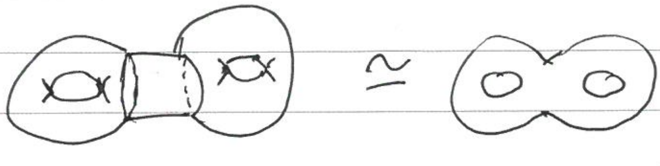
$$\pi_1(X) = \pi_1(A) \underset{\pi_1(A\cap B)}{*} \pi_1(B)$$

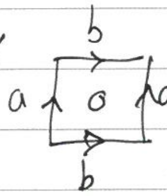
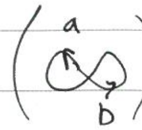
Example: $X \vee Y$ (pc): Take $A = X$, $B = Y$, $A \cap B = pt.$

$$\pi_1(X \vee Y) = \pi_1(X) \underset{\pi_1(pt)}{*} \pi_1(Y) = \pi_1(X) * \pi_1(Y)$$

So $\pi_1(\infty) = \pi_1(S^1 \vee S^1) = \mathbb{Z} * \mathbb{Z}$

Connected Sum of Tori: $\mathbb{T}^2 \# \mathbb{T}^2 =$



$\pi_1(\mathbb{T}^2 \setminus pt) = \pi_1\Big($  $\Big) = \pi_1\Big($  $\Big)$

$= \mathbb{Z} * \mathbb{Z}$

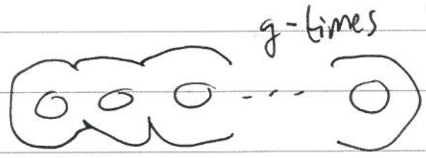
$$\Rightarrow \pi_1(M_2) = (\overset{a}{\mathbb{Z}} * \overset{b}{\mathbb{Z}}) \underset{\mathbb{Z}}{*} (\overset{c}{\mathbb{Z}} * \overset{d}{\mathbb{Z}})$$

$$aba^{-1}b^{-1} \leftarrow\!\shortmid 1 \mapsto c^{-1}d^{-1}cd$$

$$\Rightarrow \pi_1(M_2) = \langle a,b,c,d \rangle / \langle\langle aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle\rangle$$

$$= \mathbb{Z}^{*4} / \langle aba^{-1}b^{-1}cdc^{-1}d^{-1} \rangle$$

This can be generalized: $M_g = \mathbb{T}^2 \# \overset{g}{\cdots} \# \mathbb{T}^2$



$$\pi_1(M_g) = \mathbb{Z}^{*2g} / \langle a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1} \rangle$$

Let's look at the proof: First I will show the surjective part. Given $\gamma \in \pi_1(X,x)$, we have

$\gamma : I \to X = \bigcup A_\alpha$, then we can cover

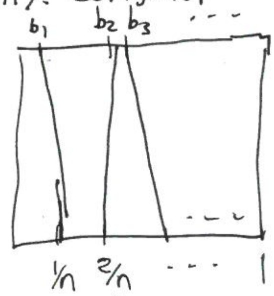
$I = \bigcup_\alpha \gamma^{-1}(A_\alpha)$ take a finite refinement

$I = \bigcup_{i=0}^{n} (a_i, a_{i+1})$ $\quad b_i$ $\quad$ I is connected

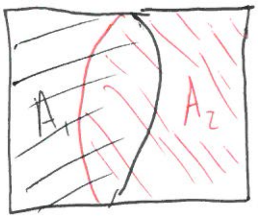
Refine to $I = \bigcup_{i=0}^{\infty} [b_i, b_{i+1}]$ $\quad [b_i, b_{i+1}] \subseteq (a_i, a_{i+1})$

Consider $\gamma_i = \gamma|_{[b_i, b_{i+1}]} : [b_i, b_{i+1}] \to A_{\alpha_i}$.

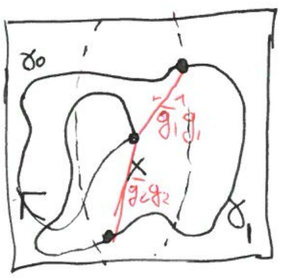
Then, since each $A_{\alpha_i} \cap A_{\alpha_{i+1}}$ is path connected, $\exists$ P_i connecting x to $\gamma(b_{i+1})$. Consider



$$\gamma \simeq \gamma_0 \cdots \gamma_{n-1}$$



$$\simeq (\gamma_0 \cdot \bar{g}_1)(g_1 \gamma_1 g_2)(\bar{g}_2 \gamma_2 g_3) \cdots (g_{n-1} \gamma_n)$$



$$= i_{\alpha_1 *}(\gamma_0 \bar{g}_1) \cdots i_{\alpha_n}(g_{n-1}\gamma_n)$$

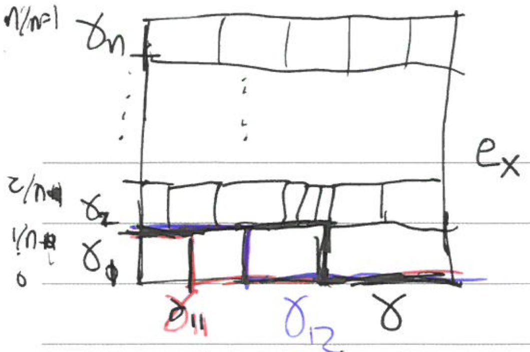
$$\in \mathrm{Im}(*_\alpha i_{\alpha *}) \Rightarrow \text{Surjective.}$$

Now, we assume that every triple intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected. Consider $\mathrm{Ker}(* i_\alpha)$. It clearly contains $\langle i_{\alpha\beta *}(\gamma) i_{\beta\alpha *}(\gamma)^{-1} \rangle$ Since $\gamma \cdot \bar{\gamma} = e$ in X. It goes to show $\mathrm{Ker}(* i_\alpha) \subseteq \langle i_{\alpha\beta *}(\gamma) \cdot i_{\beta\alpha *}(\gamma)^{-1} \rangle$. Let

$$* i_\alpha(\gamma_1 \cdots \gamma_n) = 0$$

w/ $\gamma_i \in A_{\alpha_i}$. That is to say $\gamma_1 \cdots \gamma_n \simeq e_x$.

Let $F : X \times I \to$ be the homotopy.



each $\square$ is s.t. $F(\square) \subseteq A_\alpha = A_{ij}$

We perturb so that each point lies in at most 3 $A_{\alpha_1}, A_{\alpha_2}, A_{\alpha_3}$, Changing only row not top or bottom. ~~we~~ relabel as $R_1, R_2, \cdots, R_{nm}$, n rows, m-columns.

For each corner, if $\overline{\quad F(c_{ij})} = x$, we can replace it with a loop $g_{ij}\,\overline{g_{ji}}$ contained in the 3 (or less) A_{ij}. This replaces A loop on the boundaries into A homotopy inside 1 ~~A_{ij}~~. A_α (or even $A_\alpha \cap A_\beta \cap A_\gamma$)

Doing this one step at a time produces

$$\gamma_1 \simeq \gamma_{11} \text{ in } A_{11},\ \gamma_{11} \simeq \gamma_{12} \text{ in } A_{12},\ \gamma_{12} \simeq \gamma_{13} \cdots$$

$$\Rightarrow N = \mathrm{Ker}\,(*)$$