HOMEWORK 4: CAUCHY'S INTEGRAL COROLLARIES
DUE: WEDNESDAY, OCTOBER 9TH

1) If $f : \mathbb{R} \times i \cdot (-1, 1) \to \mathbb{C}$ is a holomorphic function on the real strip, with
$$|f(z)| \leq A(1 + |z|)^n$$
for some n fixed and all z, show that for each integer m we have
$$|f^{(m)}(x)| \leq A_m(1 + |x|)^n$$
for some $A_m > 0$ and all $x \in \mathbb{R}$.

2) Weirstrass's theorem asserts every continuous function on $[0, 1]$ can be approximated uniformly by polynomials. Is the same true for continuous complex valued functions on the unit disc $\bar{B}(0, 1)$?

3) The following function are analytic on the unit disc but cannot be extended outside this domain. If $f : B(z_0, r) \to \mathbb{C}$ is holomorphic, and $|z - z_0| = r$, then z is called **regular** if there is a power series centered at z agreeing with the one for f on points of intersection. Thus a function cannot be analytically continued outside the circle if no point is regular along the boundary.

   Let $\alpha > 0$. Show that the following have radius of convergence 1:
$$f(z) = \sum_{n=0}^{\infty} z^{2^n} \qquad\qquad g(z) = \sum_{n=0}^{\infty} 2^{-n\alpha} z^{2^n}$$
Additionally, show the second extends continuously to the boundary circle $|z| = 1$, and that neither can be analytically extended beyond the disc.

4) Suppose $f : \mathbb{C} \to \mathbb{C}$ is an analytic function ($R = \infty$) and for each expansion
$$f(z) = \sum_{n=0} a_n (z - z_0)^n$$
that one coefficient $a_N = 0$. Show that f is a polynomial function.

   (**hint:** Consider the sets $A_m = \{z \in \mathbb{C} \mid f^{(m)}(z) = 0\}$. Show f is polynomial if and only if some A_m is uncountable.).

5) Suppose f is holomorphic in Ω an open set except at a pole $z_0 \in \Omega$ where $|z_0| = 1$. If $\bar{B}(0, 1) \setminus \{z_0\} \subseteq \Omega$ and
$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$
is a power series for f in $B(0, 1)$, show $\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = z_0$.

6) If $f : \bar{B}(0, 1) \to \mathbb{C}$ is non-vanishing and continuous, holomorphic on $B(0, 1)$, then show that if $|f(z)| = 1$ for all $|z| = 1$, then f is constant. (**hint:** Show that f can be extended to all of $\mathbb{C}$ by $1/\overline{f(\frac{1}{\bar{z}})}$ as in the Schwarz reflection principle.)