CLASS 30, NOVEMBER 26: COMPLETIONS OF METRIC SPACES

Recall the notion of completeness of a metric space:

Definition 30.1. A metric space (X, d) is said to be **complete** if every Cauchy sequence converges. A sequence (x_n) is said to be **Cauchy** if for every $\epsilon > 0$, $\exists N \gg 0$ such that

$$d(x_N, x_n) < \epsilon$$

for every $n > N$.

Example 30.2. $\mathbb{R}^n$ is a complete metric space with the Euclidean metric d_2 (and thus d_∞ or d_1 since they are all equivalent metrics). Indeed, let (x_n) be a Cauchy sequence. For a given ϵ, choose N_ϵ such that $x_n \in B(x_{N_\epsilon}, \epsilon)$ for all $n > N_\epsilon$. Then note that the collection

$$\{\bar{B}(x_{N_\epsilon}, \epsilon) \mid 0 < \epsilon < 1\}$$

has the finite intersection property, and is contained within a compact set $\bar{B}(x_{N_1}, 1)$. Therefore there is an element

$$x \in \bigcap_{0<\epsilon<1} B(x_{N_\epsilon}, \epsilon)$$

and x is the limit of x_n.

Now we note that there is an extension of Example 30.2 to $\mathbb{R}^{\mathbb{N}}$.

Lemma 30.3. *If X_α are topological spaces, and $X = \prod_\alpha X_\alpha$ with the product topology, then if x_n is a sequence in X, $x_n \to x$ if and only if $\pi_\alpha(x_n) \to \pi_\alpha(x)$ for all α.*

Proof. $(\Rightarrow)$: Since π_α is continuous, this follows from Theorem 14.6.

$(\Leftarrow)$: Let $x \in U = U_{\alpha_1} \times \ldots \times U_{\alpha_m} \times \prod_{\alpha \neq \alpha_i} X_\alpha$ is an open set. Since $\pi_{\alpha_i}(x_n) \to \pi_\alpha(x)$, for some $n > N_i$ we have $\pi_{\alpha_i}(x_n) \in U_i$ for each $n > N_i$. Choosing $N = \max\{N_1, \ldots, N_m\}$, we have this condition uniformly satisfied for $n > N$. As a result, $x_n \in U$ for $n > N$. Thus $x_n \to x$ as asserted. □

Corollary 30.4. $\mathbb{R}^{\mathbb{N}}$ *with the product topology is a complete metric space.*

Examples of non-complete metric spaces are as folows:

Example 30.5. $(\mathbb{Q}, d)$ with the Euclidean metric is non-complete. Indeed, we can find a sequence of rational numbers in $\mathbb{R}$ converging to an irrational number. The same sequence will not converge in $\mathbb{Q}$.

More generally, given a complete metric space and a convergent sequence $x_n \to x$ such that $x_n \neq x$, then $X \setminus \{x\}$ is no longer complete. This follows since sequences converge *uniquely* in Hausdorff spaces. $\mathbb{Q}$ is an example of this happening uncountably many times, and $(0, \infty) \subseteq [0, \infty) \subseteq \mathbb{R}$ is another such example.

We can similarly state the same result for the uniform topology (with bigger products):

Theorem 30.6. *If (X, d) is a metric space, we can put the* **uniform metric** *ρ onto $Y = X^\Lambda$:*

$$\rho(x, y) = \sup_{\alpha \in \Lambda} \{\min\{d(\pi_\alpha(x), \pi_\alpha(x)), 1\}\}$$

If (X, d) is complete, so is (Y, ρ).

Proof. Let x_n be a Cauchy sequence in Y. This implies $\forall \epsilon > 0$, there exists $N \gg 0$ such that $\sup_{\alpha \in \Lambda} \{d(\pi_\alpha(x_N), \pi_\alpha(x_n))\} < \frac{\epsilon}{2}$. But this implies that the statement is true for each coordinate, thus we have a Cauchy sequence $\pi_\alpha(x_n)$ in X. Let x_α be its limit. Then it is immediate that $d(\pi_\alpha(x_n), x_\alpha) \leq \frac{\epsilon}{2}$. As a result,

$$\rho(x_n, (x_\alpha)) = \sup\{d(\pi_\alpha(x_n), x_\alpha)\} \leq \frac{\epsilon}{2} < \epsilon$$

$x_n \to (x_\alpha)$. □

Recall that we can view the set of all (not necessarily continuous) functions $f : X \to Y$ as Y^X. Since Theorem 30.6 applies to any generic *set* Λ, it also applies to Y^X. This allows us to prove an interesting theorem about the resulting product.

Theorem 30.7. *If X is a topological space and (Y, d) is a metric space. The subsets $\mathcal{C}, \mathcal{B} \subseteq Y^X$ of continuous and bounded (resp.) functions from X to Y is closed under ρ.*

Corollary 30.8. *If Y is a complete metric space, then so are $\mathcal{C}, \mathcal{B}$.*

Proof. (of Theorem 30.7). The result for continuous functions follows from Lemma 20.2 (whose proof extends naturally to any metric space); A sequence of functions $f_n \in Y^X$ converges to a function f under ρ if and only if it converges *uniformly*. This is just unravelling definitions:

$$\rho(f_n, f) = \sup_{x \in X}\{d(f_n(x), f(x))\}$$

Therefore, it only goes to show that a uniform limit of bounded functions is bounded. But being ϵ away from f_n means $d(f_n(x), f(x)) < \epsilon$ for *every* x! □

As an application, we have the notion of a **completion** of a metric space!

Theorem 30.9. *Given a metric space (X, d), there exists an isometric (distance preserving) embedding ι of X into a complete metric space $\hat{X}$ with $\overline{\iota(X)} = \hat{X}$.*

Proof. We can consider $\mathcal{B} \subseteq \mathbb{R}^X$ to be the set of bounded functions $X \to \mathbb{R}$. Since $\mathbb{R}$ is complete, Corollary 30.8 allows us to conclude that $\mathcal{B}$ is complete with the uniform metric.

Given $a, b \in X$, we can define

$$\phi_b(x) = d(x, a) - d(x, b)$$

By the triangle inequality, ϕ_b is bounded in $[-d(a, b), d(a, b)]$. Therefore, we can define

$$\iota : X \to \mathcal{B} : b \mapsto \phi_b$$

Note that this map is injective, since

$$\phi_b(b') - \phi_{b'}(b') = d(b, b') - d(b', b') = d(b, b') = 0 \iff b = b'$$

Furthermore, distances are preserved:

$$\rho(\iota(b), \iota(b')) = \rho(\phi_b, \phi_{b'}) = \sup\{|d(x, b) - d(x, b')| \mid x \in X\} \leq d(b, b')$$

On the other hand, lettings $x = b'$ or $x = b$, we see that equality is achieved. Therefore they are equal. The closure statement results by replacing $\mathcal{B}$ with $\overline{\iota(X)}$. □

$\hat{X}$ is called the **completion** of X, and plays a similar role to a compactification.