CLASS 22, NOVEMBER 4TH: FOURIER SERIES

Today we will finish our discussion of the logarithm and move onto a study of Fourier Series (preceding the chapter on the Fourier transform).

Recall that last time we ended with the following unproven theorem:

Theorem. *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists another holomorphic function $g(z)$ such that*

$$f(z) = e^{g(z)}$$

Proof. Fix z_0 in Ω, and define

$$g(z) = \int_\gamma \frac{f'(w)}{f(w)} dw + c_0$$

where γ is a path connecting z_0 to z, and $c_0 \in \mathbb{C}$ is such that $e^{c_0} = f(z_0)$. Immediately it should be stated that this is well defined due to simple connectedness. As expected, g is a primative for $\frac{f'(z)}{f(z)}$. Moreover,

$$\frac{\partial}{\partial z}\left[f(z)e^{-g(z)}\right] = f'(z)e^{-g(z)} - f(z)e^{-g(z)}g'(z) = 0$$

Thus the function itself is constant. Checking the equation at z_0 shows the desired result. $\square$

This gives an interesting presentation of f for any f satisfying the assumptions above. We now switch gears to Fourier Series.

Let f be holomorphic on $B(z_0, R)$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be its power series expansion.

Theorem 22.1. *The coefficients of f are defined by*

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta} d\theta$$

for all $n \geq 0$ and any $0 < r < R$. Additionally,

$$\frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta} d\theta = 0$$

for any $n < 0$.

Proof. We already have that $a_n = \frac{f^{(n)}(z_0)}{n!}$ for $n \geq 0$. Applying Cauchy's integral theorem now yields

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{1}{2\pi i} \int_C \frac{f(z_0 + re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}} re^{i\theta} d\theta = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta})e^{-in\theta} d\theta$$

Now, it goes to show the statement for $n < 0$. But this is an integral of a holomorphic function! So it is 0 by Goursat. $\square$

The last part of Theorem 22.1 gives the idea that this calculation could also work for meromorphic functions with mild modification. In particular, if g has a pole of order m at z_0, then

$$g(z) = \frac{b_{-m}}{(z-z_0)^m} + \ldots + \frac{b_{-1}}{(z-z_0)} + h(z)$$

where $h(z)$ is a holomorphic function. Therefore, we can consider $f(z) = (z-z_0)^m g(z)$, which is a nice holomorphic function. Writing $f(z) = \sum_{n=0} a_n(z-z_0)^n$ and applying Theorem 22.1, we would produce

$$\begin{aligned} b_{n-m} = a_n &= \frac{1}{2\pi r^n}\int_0^{2\pi} f(z_0+re^{i\theta})e^{-in\theta}d\theta \\ &= \frac{1}{2\pi r^n}\int_0^{2\pi} g(z_0+re^{i\theta})r^m e^{im\theta}e^{-in\theta}d\theta \\ &= \frac{1}{2\pi r^{n-m}}\int_0^{2\pi} g(z_0+re^{i\theta})e^{-i(n-m)\theta}d\theta \end{aligned}$$

for $n \geq 0$. Substituting $n-m$ with n yields that

$$b_n = \frac{1}{2\pi r^n}\int_0^{2\pi} g(z_0+re^{i\theta})e^{-in\theta}d\theta$$

for all $n \geq -m$, which naturally generalizes the previous result.

2 other neat corollaries of the result which might go under the radar are the following:

Theorem 22.2 (Mean Value Property)**.** *If f is holomorphic on $B(z_0,R)$, then*

$$f(z_0) = \frac{1}{2\pi}\int_C f(z)dz = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})d\theta$$

If we consider this statement for its real and imaginary parts, we conclude the following:

Corollary 22.3. *If $f = u+iv$ is holomorphic on $B(z_0,R)$, then*

$$u(z_0) = \frac{1}{2\pi}\int_0^{2\pi} u(z_0+re^{i\theta})d\theta$$

This property holds for any harmonic function u.[1] This can be deduced from the fact that every harmonic function is the real part of some holomorphic function. This is exercise 12 in chapter 2, and goes as follows: consider $2\dfrac{\partial u(w)}{\partial z}$. Then consider $f(z) = \int_\gamma 2\dfrac{\partial u(w)}{\partial w}dw$, where γ is a path connecting 0 to z. Then $f'(z) = 2\dfrac{\partial u(w)}{\partial w}$. This is always true for a holomorphic function by our analysis of the CR equations.

[1]Recall from an ancient homework that u is harmonic if and only if $\Delta u = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.