HOMEWORK 5: COHEN-MACAULAY AND CHARACTERISTIC $p > 0$
DUE: MONDAY, APRIL 23

1) Show that the ring $R = K[x, y, u, v]/\langle x, y\rangle \cap \langle u, v\rangle$ from class 18 is non-CM. In particular, show that $\dim(R) = 2$ (what are its prime ideals?) and that modding out by any NZD leaves a ring with only units and zero-divisors.

   Solution: We note that the prime ideals of R are exactly the prime ideals of $K[x, y, u, v]$ containing $\langle x, y\rangle \cap \langle u, v\rangle$. Therefore, we may conclude that the prime ideals are those either containing $\langle x, y\rangle$ or $\langle u, v\rangle$. Thus, a longest chain of primes is
   $$\langle x, y\rangle \subseteq \langle x, y, u\rangle \subseteq \langle x, y, u, v\rangle$$
   Therefore, $\dim(R) = 2$. It goes to prove the claim about depth. In the local case, everything outside of the maximal ideal $\langle x, y, u, v\rangle$ is inverted. Consider the NZD $x - u$. Modding out by $x - u$ sets $x = u$. Then
   $$R/\langle x - u\rangle = K[x, y, v]/\langle x, y\rangle \cap \langle x, v\rangle = K[y, v]/\langle yv\rangle$$
   (all localized). Therefore, there are no non-unit NZD remaining. So depth = 1.

2) Show that $\mathrm{Tor}_i^R(M \oplus M', N) \cong \mathrm{Tor}_i^R(M, N) \oplus \mathrm{Tor}_i^R(M', N)$.

   Solution: If we direct sum a free resolution of M with M', we get one for $M \oplus M'$. Tensoring by N commutes with the direct sum. Thus the two agree.

3) Compute $\mathrm{Tor}_i^R(M, M)$ where $R = \mathbb{Z}/6\mathbb{Z}$ and $M = \mathbb{Z}/3\mathbb{Z}$.

   Solution: Consider the SES
   $$0 \to 3\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/3\mathbb{Z} \to 0$$
   Tensoring up by M, we get
   $$0 = \mathrm{Tor}_1(R, M) \to \mathrm{Tor}_1(M, M) \to 3\mathbb{Z}/6\mathbb{Z} \otimes_R M \to \mathbb{Z}/6\mathbb{Z} \otimes_R M \to \mathbb{Z}/3\mathbb{Z} \otimes_R M \to 0$$
   The middle term is $R \otimes_R M \cong M$, which holds in general. Because $R/I \otimes_R R/J = R/(I + J)$ by homework 3, problem 3, we have that the last term is again M. Finally, $3\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/2\mathbb{Z}$ as $\mathbb{Z}/6\mathbb{Z}$-modules, so this is 0 by the previous argument. So we conclude the exact sequence for $i \geq 1$ we have $\mathrm{Tor}_i^R(M, M) = 0$ (by induction) and $\mathrm{Tor}_0^R(M, M) = \mathbb{Z}/3\mathbb{Z}$.

4) Prove the following Proposition from class:

   Proposition 0.1. *A ring R is equal characteristic 0 if and only if $\mathbb{Q} \subseteq R$. A ring is characteristic $p > 0$, where p is a prime number, if and only if $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \subseteq R$. All other rings are mixed characteristic, which holds if and only if $\mathrm{char}(R) \neq \mathrm{char}(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$.*

   Solution: If a ring is equal characteristic, then it contains a field. The smallest field containing $\mathbb{Z}$ is $\mathbb{Q}$, therefore the first statement holds naturally.

For the second, we know $\mathbb{Z}/p\mathbb{Z} \to R : 1 \mapsto 1$ by definition. This is $\mathbb{F}_p$! Therefore we are done.

For the third, if R is a ring not containing a field, then $\mathbb{Z} \to R$. For every p prime, we may consider its image in R. If $p \mapsto 0$, then $R \supseteq \mathbb{F}_p$ and we would contradict our assumption. On the other hand, if every p is a unit in R, then $R \supseteq \mathbb{Q}$ necessarily. Therefore, $\exists p$ prime such that $p \in R$ is not a unit nor 0. Therefore, we can take $\mathfrak{m}$ containing $p > 0$ maximal. Therefore, $\mathbb{Z}/p\mathbb{Z} \subseteq R/\mathfrak{m}$ has characteristic $p > 0$. However, R is observed to be characteristic 0. This completes the proof.

5) Show that if $R \subseteq S$, M is an S-module, and $Hom_R(S, R) \cong S$ as S-modules, then the map given by composition is surjective:

$$\mathrm{Hom}_S(M, S) \times \mathrm{Hom}_R(S, R) \to \mathrm{Hom}_R(M, R)$$

Solution: Given $\varphi \in \mathrm{Hom}_R(M, R)$, it goes to find $\varphi' \in \mathrm{Hom}_S(M, S)$ and $\psi \in \mathrm{Hom}_R(S, R)$ with $\psi \circ \varphi' = \varphi$.

By Hom-Tensor Adjointness, we have that

$$Hom_R(M, R) \cong \mathrm{Hom}_R(M \otimes_S S, R) \cong \mathrm{Hom}_S(M, \mathrm{Hom}_R(S, R)) \cong \mathrm{Hom}_S(M, S)$$

Where the maps are given by

$$\xi \mapsto \psi(m \otimes s) = \xi(sm) \mapsto \xi(m)(s) = \psi(m \otimes s) \mapsto \varphi(m) = A(\xi(m))$$

where A is the isomorphism stated in the question. Therefore,

$$(\xi, A^{-1}(1)) \mapsto \varphi$$

Thus the stated map is a surjection.

6) Prove (or recall) the following proposition from class:

Proposition 0.2. *If R is a ring of characteristic $p > 0$, then $F : R \to R$ is injective if and only if R is reduced (e.g. the nilradical $\mathcal{N} = 0$).*

Solution: Suppose that F is injective. Then $0 \neq r \mapsto r^p \neq 0$. If $r \neq 0$ but $r^n = 0$ for some $n > 1$ minimal, we know that $(r^{n-1})^p = r^{p(n-1)} = r^n r^{(p-1)(n-1)} = 0$. This contradicts the assumption.

On the other hand, if $r^p = 0$ for some $r \neq 0$, $r \in \mathcal{N}$. This completes the proof.

7) Show that localization commutes with F_*. That is to say $F_* W^{-1} R \cong W^{-1} F_* R$.

Solution: I claim that the desired isomorphism is

$$F_* W^{-1} R \to W^{-1} F_* R : F_*(w, r) = F_*(w^p, w^{p-1} r) \mapsto (w, F_* w^{p-1} r)$$

For the inverse inclusion, we send $(w, F_* r)$ to $F_*(w^p, r)$. These are mutually inverse functions, so this proves the desired statement.

8) Find an example of a field which is not F-finite.

Solution: $\mathbb{F}_p(x_1, x_2, \ldots)$ is such a field. Note that this follows from the previous statement. If $R = \mathbb{F}_p[x_1, \ldots]$ then this has a free basis $\{x_{i_1}^{a_1} \cdots x_{i_n}^{a_n} \mid i_j, n \in \mathbb{N},\ a_i = 0, \ldots, p-1\}$. Therefore, $F_* R \cong R^{\mathbb{Z}}$. Therefore, localizing at $W = R \setminus \{0\}$, we get the field above.

9) Find an example of an F-finite field which isn't perfect.

   Solution: Similar to the last problem, $F_*\mathbb{F}_p(x)$ is p-dimensional vector space over $\mathbb{F}_p$.

10) Let $R = \mathbb{F}_q[x_1, \ldots, x_n]$ for some $q = p^e$. Show that F_*R is a free R-module and calculate its rank.

    Solution: The basis consists of

$$\{F_*x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid 0 \leq \alpha_i < p\}$$

This is because if $f = \sum k_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, then

$$\begin{aligned} F_*f &= \sum_{alpha} F_*k_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \\ &= \sum_{0 \leq \alpha_i < p} k_\alpha^{\frac{1}{p}} x_1^{\lfloor \frac{\alpha_1}{p} \rfloor} \cdots x_n^{\lfloor \frac{\alpha_1}{p} \rfloor} F_*x_1^{p\{\frac{\alpha_1}{p}\}} \cdots x_n^{p\{\frac{\alpha_1}{p}\}} \end{aligned}$$

Where the braces indicate the fractional part.