CLASS 8, SEPTEMBER 25: CAUCHY'S THEOREM IN A DISC

As a corollary to Goursat's theorem, we can acquire the following result in a disc. Triangles turn out to be quite powerful objects.

Theorem 8.1. *A holomorphic function $f : B(z_0, r) \to \mathbb{C}$ in an open disc has a primative in that disc.*

Proof. Using the change of variables $z \mapsto z - z_0$, we may assume $z_0 = 0$. Let $z \in B(0, r)$. Consider the piecewise smooth path γ_z going from 0 to $Re(z)$, then to z itself. Orient the curve from 0 to z.

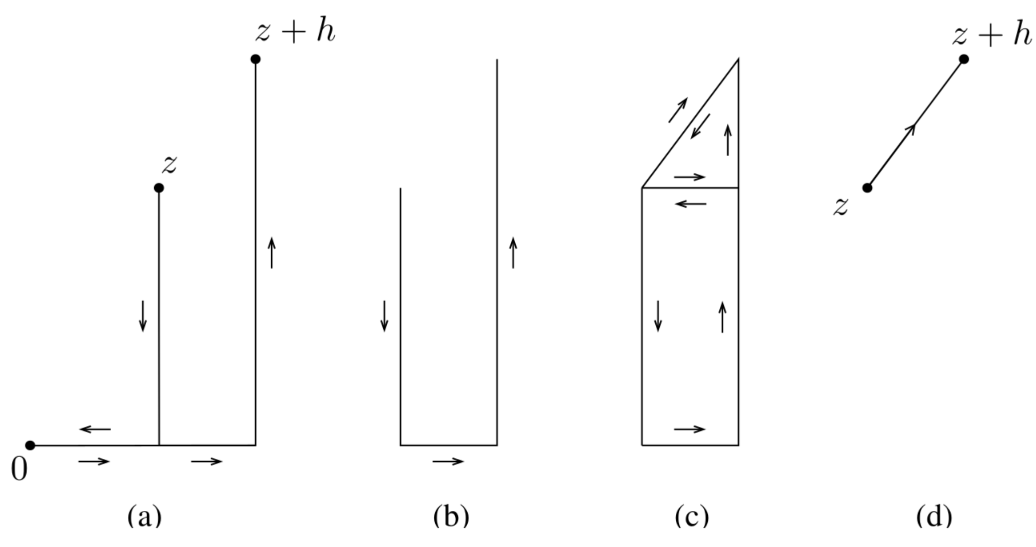
Define

$$F(z) = \int_{\gamma_z} f(w)dw.$$

We assert that $F(z)$ is holomorphic in $B(0, r)$, with $F'(z) = f(z)$. Note that

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)dw - \int_{\gamma_z} f(w)dw$$

It is best to think of this equation geometrically; we can replace the second integral with an integral with the same curve but in the opposite direction. Thus we are able to cancel the trip from 0 to z. This yields (b) in the picture (which is courtesy of pg 38 of Stein/Shakarchi). To obtain (c), we add a curve in both its forwards and backwards orientation, which doesn't change the integral. Finally, notice that all except the curve connecting z to h are parts of a triangular region and a rectangular region. From Goursat we may conclude those integrals are 0, leaving only (d).



All this geometry yields us

$$F(z+h) - F(z) = \int_{\gamma} f(w)dw$$

where γ is simply the straight line connecting z to $z+h$. Since f is continuous, we have that $f(w) - f(z) = \psi(w) \to 0$ as $w \to z$. So as a result,

$$F(z+h) - F(z) = \int_\gamma f(z)dw + \int_\gamma \psi(w)dw = hf(z) + \int_\gamma \psi(w)dw$$

Finally,

$$\left| \int_\gamma \psi(w)dw \right| \leq \sup_{w \in \gamma} |\psi(w)| \cdot |h|.$$

But the supremum goes to 0 as $|h|$ does, so dividing the equality by h and taking the limit yields

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = F'(z) = f(z).$$

□

Theorem 8.2 (Cauchy's Theorem in a Disc)**.** *If f is holomorphic in a disc D, then*

$$\int_\gamma f(z)dz = 0$$

for any loop $\gamma : [a,b] \to D$.

We can also generalize this statement slightly:

Corollary 8.3. *If f is holomorphic in an open set Ω containing a circle C and its interior, then*

$$\int_C f(z)dz = 0$$

Proof. If C is the boundary of $D = \bar{B}(z_0, r)$, there exists $\epsilon > 0$ such that $B(z_0, r+\epsilon) \subseteq \Omega$. This follows by compactness of the disc. As a result, the previous result yields the corollary. □

This corollary actually extends to any loop with a notion of an interior. Fortunately, there is a beautiful result called the "Jordan Curve Theorem" that tells us this is always the case when γ is simple and piecewise smooth[1]. I refer the interested reader to Appendix B of the book. More general versions also exist.

But since we won't have such a result in this class, we will call loops with an obvious interior **toy contours**. These include polygons. One very useful one in complex analysis is the **keyhole contour**. This is the curve that is designed to exclude a certain arc in the complex plane (such as the negative real axis).

The main idea is that when γ is a toy countour and f is holomorphic in an open region containing the interior of γ and γ itself, the

$$\int_\gamma f(z)dz = 0.$$

We will use the keyhole contour and toy contours to great effect later when trying to integrate more interesting functions.

[1]If you're willing to head to the topological world, piecewise smooth is also not needed.

Example 8.4. A difficult to evaluate integral using classical methods is the following:

$$\int_0^\infty \frac{1-\cos(x)}{x^2}dx.$$

Using our new theory, we can show that this is exactly $\frac{\pi}{2}$. To make such an integral appear in the complex world, we consider the function $\frac{1-e^{iz}}{z^2}$. We integrate over a large and small semicircle in the upper halfplane (call their radii R and ϵ respectively), as well as their connecting line segments. Since $f(z)$ is holomorphic everywhere except 0, we have that the total integral of this path is 0. This yields:

$$0 = \int_{-R}^{-\epsilon} \frac{1-e^ix}{x^2}dx + \int_{\epsilon}^{R} \frac{1-e^ix}{x^2}dx - \int_{C_\epsilon} f(z)dz + \int_{C_R} f(z)dz$$

where C_r is the circle of radius r centered at 0 with counterclockwise orientation. Letting $R \to \infty$, we have that $|\frac{1-e^{iz}}{z^2}| \leq \frac{2}{|z|^2}$. As a result, $\int_{C_R} f(z)dz \to 0$ and therefore

$$\int_{|x|\geq\epsilon} \frac{1-e^{ix}}{x^2}dx = \int_{C_\epsilon} f(z)dz$$

We also have a nice power series expansion for e^{iz}:

$$e^{iz} = \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = 1 + iz - \frac{z^2}{2} - \frac{iz^3}{6} + \dots$$

This produces $f(z) = \frac{-iz}{z^2} + E(z)$ where $E(z)$ is bounded as $z \to 0$. Therefore as $\epsilon \to 0$,

$$\int_{C_\epsilon} f(z)dz = \int_{C_\epsilon} \frac{-iz}{z^2} = \int_0^\pi \frac{-i\epsilon e^{i\theta}}{\epsilon^2 e^{2i\theta}} i\epsilon e^{i\theta} d\theta = \int_0^\pi d\theta = \pi$$

Using the fact that we have an even function, we are done!