**COMPLEX ANALYSIS: FINAL EXAM**

# Instructions

This exam is to be completed within 24 hours. If you need additional pages, staple them to the exam and point to them within the problem.

To attain full points on a given problem, be sure to write clear concise solutions which properly reference the results used (either by name, e.g. "Goursat's Theorem", or by number within the notes, or by statement).

I recommend that you draft your solutions first (on a private board or scratch paper) before transferring them onto the exam.

Things that are accessible for this exam:

- Notes, be they mine or yours.
- Homework assignments.
- The midterm.
- Stein and Shakarchi.

Things that are NOT accessible:

- Other people.
- The internet.
- Other books, math or otherwise.

By signing your name below you assert that you have taken this exam under the above stated rules. Violations are treated as violations of the honor code!

**NAME:** ______________________________

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## Score by page:

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**TOTAL: ______________/150**

1) **(10 points)** Consider the power series expansion about the origin of $f(z) = \frac{1}{(1-z)^m}$:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Find an explicit formula for $a_n$, and show that

$$a_n \sim \frac{1}{(m-1)!} n^{m-1}$$

as $n \to \infty$. Using Hadamard's formula show the radius of convergence is 1.

**Solution:** Consider $g(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$. Notice that

$$g^{(m-1)}(z) = 1 \cdot 2 \cdots (m-1) \frac{1}{(1-z)^m} = \sum_{n=0}^{\infty} n \cdot (n-1) \cdots (n-m+2) z^{n-m+1}$$

since we can differentiate term by term by Theorem 4.5. As a result,

$$f(z) = \sum_{n=0}^{\infty} \frac{n \cdot (n-1) \cdots (n-m+2)}{1 \cdot 2 \cdots (m-1)} z^{n-m+1}$$

which demonstrates $|a_n| \sim \frac{1}{(m-1)!} n^{m-1}$. As a result, since $n^{\frac{m-1}{n}} = (n^{\frac{1}{n}})^{m-1} \to 1$, we have by Hadamard's Theorem that the radius of convergence is 1.

2) **(10 points)** Show explicitly that if $\gamma : [a,b] \to \mathbb{C}$ is a curve, and

$$\bar{\gamma} : [a,b] \to \mathbb{C} : t \mapsto \gamma(b+a-t)$$

is the curve traversed backwards, then

$$\int_{\gamma} f(z)\ dz = -\int_{\bar{\gamma}} f(z)\ dz$$

**Solution:**

$$\begin{aligned} -\int_{\bar{\gamma}} f(z)\ dz &= -\int_a^b f(\gamma(b+a-t)) \cdot \gamma(b+a-t) \cdot (-1) dt \\ &= -\int_b^a f(\gamma(w)) \cdot \gamma(w) \cdot dw \qquad = \int_a^b f(\gamma(w)) \cdot \gamma(w) \cdot dw \end{aligned}$$

Where we did the substitution $w = b+a-t$ and $dw = -dt$.

3) **(20 points)** Given $a > 0$ and $b \in \mathbb{R}$, evaluate the integral

$$\int_0^\infty e^{-ax}\cos(bx)dx \qquad \int_0^\infty e^{-ax}\sin(bx)dx$$

by integrating $e^{-Az}$, where $A = \sqrt{a^2+b^2}$ over a sector with angle $\omega$ such that $\cos(\omega) = \frac{a}{A}$.

**Solution:** We can assume WLOG that $b > 0$ using symmetries of trig functions. Now, following the hint, Cauchy's Theorem yields

$$\int_\gamma e^{-Az}dz = \int_0^R e^{-Ax}dx + \int_{\gamma_R} e^{-Az}dz - \int_0^R e^{-Axe^{iw}}e^{iw}dx$$

Where $\gamma_R$ is the circular piece. This integral is bounded above by

$$\left|\int_{\gamma_R} e^{-Az}dz\right| = \left|\int_0^\omega e^{-ARe^{i\theta}}Rie^{i\theta}d\theta\right| \le we^{-AR\cos(w)}R \to 0$$

As $R \to \infty$. As a result, we have

$$e^{-iw}\int_0^R e^{-Ax}dx = \int_0^R e^{-Axe^{iw}}dx = \int_0^R e^{-Ax(\cos(w)+i\sin(w))}dx = \int_0^R e^{-ax}e^{-ibx}dx$$

This is nothing but

$$\int_0^\infty e^{-ax}\cos(bx)dx - i\int_0^\infty e^{-ax}\sin(bx)dx$$

As a result, notice that

$$e^{-iw}\int_0^R e^{-Ax}dx = (a-ib)\frac{1}{A^2}$$

Comparing Real and imaginary parts produces the desired result:

$$\int_0^\infty e^{-ax}\cos(bx)dx = \frac{a}{A^2}$$

$$\int_0^\infty e^{-ax}\sin(bx)dx = \frac{b}{A^2}$$

4) **(15 points)** Suppose $\Omega$ is an open bounded region and $L$ is a line in $\mathbb{C}$ that intersects $\Omega$ in an interval. Write $\Omega_1$ and $\Omega_2$ be the sections of $\Omega$ on either side of $L$, so that $\Omega = \Omega_1 \cup (\Omega \cap L) \cup \Omega_2$ are all disjoint. Show that if $\Omega_1$ and $\Omega_2$ are simply connected, then so is $\Omega$. Pictures with color may be helpful but are not complete solutions!

**Solution:** This is best done with pictures. Notice that $\Omega_i' = (L \cap \Omega) \cup \Omega_i$ is simply connected. This can be demonstrated by taking $\gamma : [0,1] \to \Omega_i'$ based at $l \in L$. Then $U = \gamma^{-1}(\Omega_i)$ is an open set, which we can large open set within, to perform a homotopy to a constant path. Similarly $L$ is simply connected, so the same can be achieved there. We can then combine each of them to have small paths which traverse back and forth between $L$ and $\Omega_i$, which are easily contracted.

The final conclusion is similar: If $\gamma : [0,1] \to \Omega$ is a loop based in $\Omega_1$, then we can find initial points of an interval (sometimes a point) for which $\gamma(t) \in L$. We can then track adjacent base points, and adjoin a straight line connecting them to form a loop followed by the same line reversed to make it the same homotopically. By the previous part, each of these can be contracted to their basepoint. As a result, our loop $\gamma$ is homotopic to one in $\Omega_i'$, thus trivial!

The same is not true if we weaken our assumptions. The real line cuts the punctured disc in 2 intervals. Each of the opens are SC, but the total is not.

5) **(10 points)** Show that

$$f(z) = z^{10} + 3z^8 + 3z^6 + 2z^4 + 5z - 2$$

has all of its roots in $B(0,2)$.

**Solution:** Notice that $|z|^{10} = 2^{10} = 1024$, and

$$|3z^8 + 3z^6 + 2z^4 + 5z - 2| \leq 3|z|^8 + 3|z|^6 + 2|z|^4 + 5|z| + 2 = 1004$$

So taking $F(z) = z^10$ and $G(z)$ the remaining part yields via Rouche's Theorem that $f(z)$ and $F(z)$ have the same number of zeroes in $B(0,2)$. But by the FTOA, we have $f(z)$ has exactly 10 zeroes!

6) **(30 points)** Here we will formalize how Fourier Transforms are useful for solving differential equations of the form

$$(*) \qquad a_n \frac{\partial^n}{\partial t^n} u(t) + a_{n-1} \frac{\partial^{n-1}}{\partial t^{n-1}} u(t) + \ldots + a_0 u(t) = f(t)$$

for $u$ given $f$ an analytic function.

i. Use induction and integration by parts to deduce that if $f \in \mathcal{F}_a$, then if $g = f^{(n)}$ is the $n^{th}$ derivative of $f$, then

$$\hat{g}(\xi) = (2\pi i \xi)^n \hat{f}(\xi)$$

**Solution:** The base case of $n = 0$ is trivial. So suppose it has been proved for derivatives up to $n - 1$. Then

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} f^{(n)}(x) e^{2\pi i x \xi} dx = \left[ f^{(n-1)}(x) e^{2\pi i \xi x} \right]_{x=-\infty}^{\infty} + \int_{-\infty}^{\infty} f^{(n-1)}(x) 2\pi i \xi e^{2\pi i \xi x} = (2\pi i \xi)^n \hat{f}(\xi)$$

The first equality is by definition, the second by integration by parts, and the third is the induction hypothesis coupled with the fact that $f \in \mathbb{F}_a$ implies that $f$ has moderate descent, which makes the first part $= 0$.

ii. As a result, describe how to find a solution to the differential equation (*) above.

**Solution:** Applying the Fourier transfrom to the diff eq above yields

$$a_n (2\pi i \xi)^n \hat{u}(\xi) + a_{n-1} (2\pi i \xi)^{n-1} \hat{u}(\xi) + \ldots + a_0 \hat{u}(\xi) = \hat{f}(\xi)$$

Which then produces

$$\hat{u}(\xi) = \frac{\hat{f}(\xi)}{a_n (2\pi i \xi)^n + a_{n-1} (2\pi i \xi)^{n-1} + \ldots + a_0}$$

If there are no poles on the real line, then performing Fourier inversion yields

$$u(x) = \int_{\mathbb{R}} \frac{\hat{f}(\xi)}{a_n (2\pi i \xi)^n + a_{n-1} (2\pi i \xi)^{n-1} + \ldots + a_0} e^{2\pi i x \xi} d\xi$$

Otherwise, we would need to go around them with small $\epsilon$-semicircles, and use a line at some positive $b < a$ missing the poles. This can still be done but requires the more difficult techniques of Paley-Weiner.

7) **(10 points)** Suppose $f$ is entire and non-vanishing, and that no derivatives of $f$ vanish. Assuming Hadamard's theorem, prove that if $f$ is of finite order, then $f(z) = e^{az+b}$ for some $a, b \in \mathbb{C}$. As a result, deduce that $f$ has order 1.

**Solution:** Hadamard's theorem immediately implies that $f(z) = e^{P(z)}$, where $P(z)$ is a polynomial. Now notice that

$$f'(z) = P'(z) \neq 0$$

implies that $P'(z)$, again a polynomial, is constant by the FTOA. As a result, we have $\deg(P(z)) = 1$ or 0. This is exactly the first result.

Finally, since $e^{az+b} = e^b e^{az}$, we conclude that its order of growth is 1 (or 0 if it is constant).

8) **(15 points)** Using Picard's big theorem, the stronger version of Riemann's Theorem on essential singularities, show that $f(z) = e^z - z$ has infinitely many zeroes in $\mathbb{C}$.

**Solution:** Notice that $f(z)$ has an essential singularity at $\infty$, since $e^{\frac{1}{z}}$ does. By Picard's Theorem, we have that in a neighborhood of $\infty$, $f(z)$ attains every value except perhaps 1 infinitely many times. So it only remains to show that $f(z)$ does attain 0 at least once. Notice that $f(z + 2\pi i) = f(z) - 2\pi i$. As a result, $f$ cannot miss both 0 and $2\pi i$, so in fact hits both infinitely often.

9) **(15 points)** Show that $f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right)$ is a conformal map from the upper half disc $\mathbb{D} \cap \mathbb{H}$ to the upper half plane $\mathbb{H}$.

**Solution:** $f$ is clearly a holomorphic map, since $0 \notin \mathbb{D} \cap \mathbb{H}$. Moreover,

$$Im(f(z)) = Im\left(-\frac{1}{2}(re^{i\theta} + \frac{1}{r}e^{-i\theta})\right)$$

Since $r < 1$, we have $-Im(\frac{1}{r}e^{-i\theta}) > Im(re^{i\theta}) > 0$, and as a result $Im(f(z)) > 0$. Thus the image of $f$ is in $\mathbb{H}$.

Consider $f(z) = w$. This is satisfied if and only if $z^2 + 2wz + 1 = 0$. Thus to solve this for $z$, achieving an inverse, we can use the quadratic formula: $z = -w \pm \sqrt{w^2 - 1}$, where $\surd$ is interpreted through the principal branch. This is only in $\mathbb{D} \cap \mathbb{H}$ if we consider $-w + \sqrt{w^2 - 1}$, so we have defined an inverse.

10) **(15 points)** A point $z$ is a fixed point for $f : \mathbb{D} \to \mathbb{D}$ if $f(z) = z$. Show that if $f$ is holomorphic with 2 fixed points, then $f$ is the identity map. Does every holomorphic map $f : \mathbb{D} \to \mathbb{D}$ have a fixed point? (**hint:** consider $\mathbb{H}$)

**Solution:** If $f$ has two fixed points, then one must be non-zero. By the Schwarz Lemma 2), $f$ is necessarily a rotation. But non-trivial rotations have only the origin as a fixed point. Thus $f$ is the identity map.

The answer to the second question is no. We have a conformal map $F : \mathbb{D} \to \mathbb{H}$. Notice that $f(z) = z$ if and only if $w = F(z)$ is a fixed point of the conjugated map $F \circ f \circ F^{-1}$. Thus we could consider the maps $z \mapsto 2z$ or $z \mapsto z + 1$ on $\mathbb{H}$ which clearly has no fixed points. As a result, the map $F^{-1} \circ 2z \circ F$ also has no fixed points (and is even bijective!).