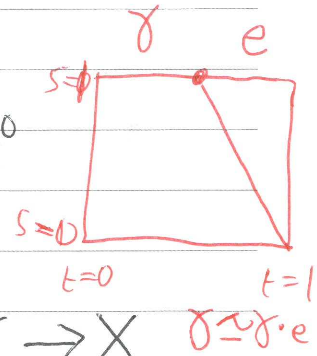
Prop: $\pi_1(X,x_0)$ is a group under composition.

Pf: • Identity: $p = e_{x_0} = t \mapsto x_0$



• Composition of paths is a path:

$p_1 \cdot p_2$ is a continuous map $I \to X$

• Inverses exist: Let p be a path.

Oct 11) Let (suggestively)

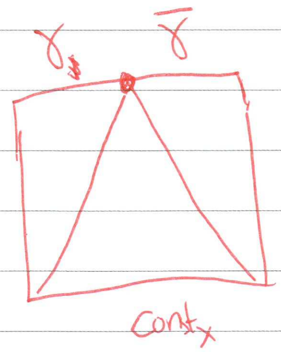
$$p^{-1} = \overline{p} : [0,1] \to X : t \mapsto p(1-t)$$

Note $\overline{p} \in \pi_1(X,x_0)$, as $t \mapsto 1-t$ is continuous and $\overline{p}(0) = \overline{p}(1) = x_0$.

Consider $$\overline{p}\cdot p(t) = \begin{cases} \overline{p}(2t) & t \le \frac{1}{2} \\ p(2t-1) & t \ge \frac{1}{2} \end{cases}$$

Consider

$$\cancel{P : I\times I \to X : \quad P(t,s) = \begin{cases} \overline{p}(2t) & t \le \frac{1}{2}(1-s) \\ p(1-s) & \frac{1}{2}(1-s) \le t \le 1-\frac{1}{2}(1-s) \end{cases}}$$



$$P : I \times I \to X$$

$$P(t,s) = \begin{cases} \overline{p}(2t) & t \le \frac{1}{2}(1-s) \\ \overline{p}(1-s) = p(s) & \frac{1}{2}(1-s) \le t \le 1 - \frac{1}{2}(1-s) \\ p(2t-1) & t \ge 1 - \frac{1}{2}(1-s) \end{cases}$$

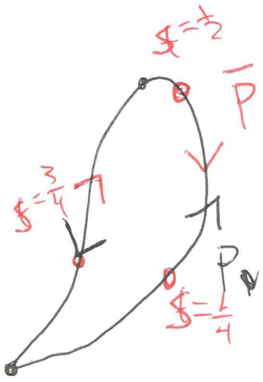
Continuous?

$$\mathcal{P}(\tfrac{1}{2}(1-s), s) = \bar{p}(1-s) = p(s) \quad \checkmark$$

$$\mathcal{P}(1-\tfrac{1}{2}(1-s), s) = \mathcal{P}(\tfrac{1}{2}+\tfrac{s}{2}, s)$$

$$= p(2(\tfrac{1}{2}+\tfrac{s}{2})-1)$$

$$= p(s) \quad \checkmark$$



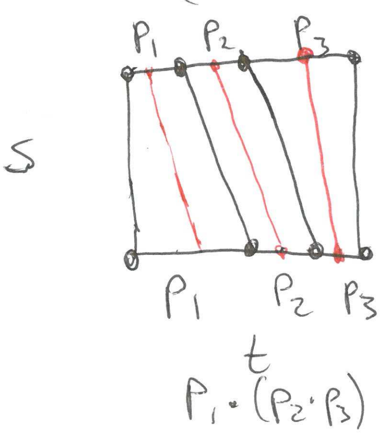
Note: $\mathcal{P}(t,1) = \bar{p}\cdot p(t) \quad \checkmark$

$\mathcal{P}(t,0) = e_{x_0} \quad \checkmark$

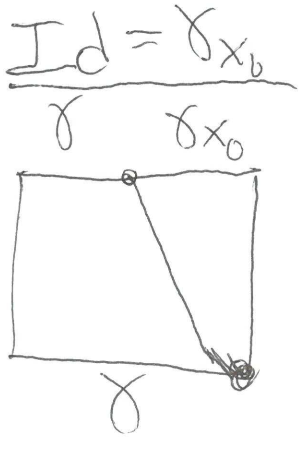
Associative: $(p_1 \cdot p_2)\cdot p_3 = \begin{cases} p_1(4t) & t \le \frac{1}{4} \\ p_2(4t-1) & \frac{1}{4} \le t \le \frac{1}{2} \\ p_3(2t-1) & \frac{1}{2} \le t \end{cases}$

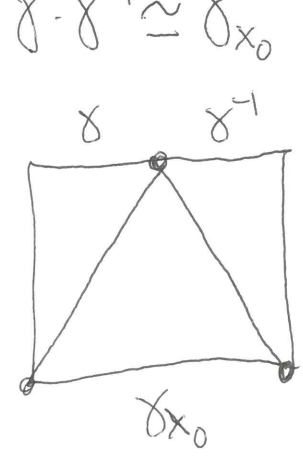
$p_1 \cdot (p_2 \cdot p_3) = \begin{cases} p_1(2t) & t \le \frac{1}{2} \\ p_2(4t-2) & \frac{1}{2} \le t \le \frac{3}{4} \\ p_3(4t-3) & t \ge \frac{3}{4} \end{cases}$

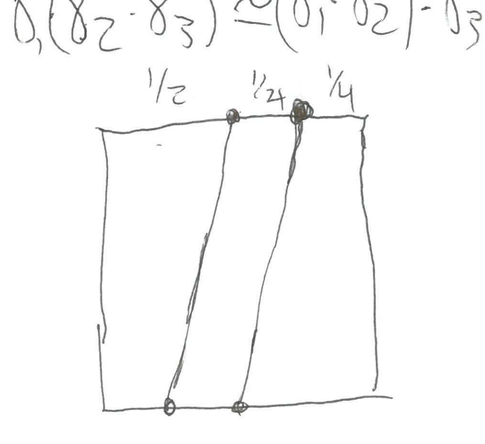
10



$\pi_1(X, x_0)$ is a group $\boxtimes$







Prop: If q is a ~~loop~~ path from $x_0 \to x_1$,
Then

$$\pi_1(X, x_0) \xrightarrow{\phi} \pi_1(X, x_1)$$
$$p \longmapsto \bar{q}\cdot p \cdot q$$

is an isom with inverse $p' \mapsto q \cdot p' \cdot \bar{q}$.

Cor: If X is path connected, $\pi_1(X, x_0)$ is
10 independent of x_0. Thus $\pi_1(X)$ suffices.

Defn: X is called simply connected if

$$\pi_1(X, x_0) = 0 \quad \text{and} \quad X \text{ is PC.}$$

Equivalently, X is simply connected if every $x_0, x_1 \in X$ have a unique ~~homotopy~~ path connecting them up to homotopy.

Contractible

Functoriality: If $\varphi: X \to Y$ is a continuous map, then φ induces a group hom

$$\varphi_* : \pi_1(X, x_0) \to \pi_1(Y, \varphi(x_0))$$
$$\gamma \longmapsto \varphi \circ \gamma$$

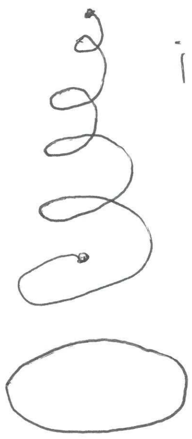
10

Proof: 1) $\varphi_* e_X = e_Y$ is clear

2) φ_* is well defined up to homotopy:
$p_0 \simeq_p p_1 \Rightarrow \pi_* p_0 \simeq_{\pi \circ p} \pi_* p_1$

3) $\pi_*(p_1 \cdot p_2) = \pi_*(p_1) \cdot \pi_*(p_2)$ is by defn of π_*

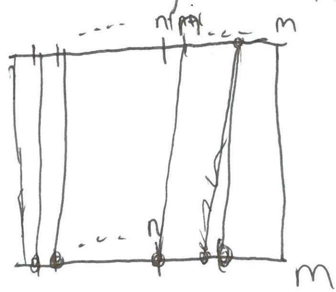
Thm 1.7: The map $\Phi: \mathbb{Z} \to \pi_1(S^1) = \pi_1(\mathbb{R}/\mathbb{Z})$
$n \mapsto \gamma_n : x \mapsto nx$



is an isomorphism.

pf: ~~We can~~ First note Φ is a homomorphism:

$$\Phi(n+m) = \gamma_{n+m} \simeq \gamma_n \cdot \gamma_m$$



5

$$t \in [0,1] \longmapsto \begin{cases} \frac{t(n+m)}{2n} & t \leq \frac{n}{n+m} \\ \frac{1}{2} + \frac{(t - n/(n+m))(n+m)}{2m} & t \geq \frac{n}{n+m} \end{cases}$$

We use two facts: $p: \mathbb{R} \to S^1$

1) For each $\gamma: I \to S^1$, and $\tilde{x}_0 \in p^{-1}(x_0)$ $\exists! \tilde{\gamma}: I \to \mathbb{R}$ a path starting at $\tilde{x}_0$

2) For each $f_t: I \to S^1$ of paths at x_0, and each $\tilde{x}_0 \in p^{-1}(x)$, $\exists! \tilde{f}_t: I \to \mathbb{R}$ starting at $\tilde{x}_0$.

These need to be proved (and generalized) but lets prove the Thm using them first.

Φ is surjective: Let γ be a ~~path~~ loop @ 0 in S^1. Thus γ extends to a path in $\mathbb{R}$, $\tilde{\gamma}$. Consider $\tilde{\gamma}(1) \in \mathbb{Z}$

$\Phi(\tilde{\gamma}(1)) = \gamma$.

Φ is injective: Suppose $\Phi(m) = \Phi(n) = \gamma_m = \gamma_n$

So $\exists F : I \times I \to S^1$ s.t. $F(t,0) = \gamma_n$, $F(t,1) = \gamma_m$

This lifts by (2) to $\mathbb{R}$: $\tilde{F} : I \times I \to \mathbb{R}$.

Thus $F(1,t) = m = n$ for all t (homotopy of paths).

$\boxtimes$

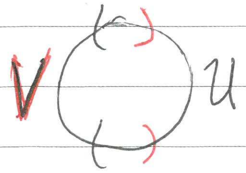
We can prove 1 & 2 together by showing

* If $F : Y \times I \to S^1$ and $\tilde{F}_0 : Y \times \{0\} \to \mathbb{R}$
Then we can find $\tilde{F} : Y \times I \to S^1$ w/

$$\begin{tikzcd} Y \times \{0\} \arrow[r] \arrow[d] & \mathbb{R} \arrow[d, "q"] \\ Y \times I \arrow[r, "F"] \arrow[ur, dashed, "\tilde{F}"] & S^1 \end{tikzcd} \quad \circlearrowleft$$

Ex $Y = pt \Rightarrow (1)$
$Y = I \Rightarrow (2)$

We will do this by writing $S^1 = U \cup V$



w/ $q^{-1}(U) = \coprod_{n \in \mathbb{Z}} U$, $q^{-1}(V) = \coprod q^{-1}(U)$

~~We can construct a lift~~

~~A better cover is $\forall x \in S^1$, take $(x-\varepsilon, x+\varepsilon)$ for $\frac{1}{2} > \varepsilon > 0$. It still has this~~

property.

Let $F(y,0) \in \mathcal{U}_{\alpha_\phi}$, and choose $N_{(y,0)} \overset{\text{open}}{\supseteq} (y,0)$ s.t. $F(N_{(y,0)}) \subseteq \mathcal{U}_{\alpha_\phi}$ (choose connected component of preimage for example). Then for $(z,t) \in N_{(y,0)}$

$$\tilde{F}(z,t) = \tilde{F}_0(z,t)$$

By virtue of the fact that $N_{(y,0)}$ is open $N_{(y,0)} \supseteq V \times [0,t_1)$. Let's now work w/ (y,t_1). same thing $F(y,t_1) \subseteq \mathcal{U}_{\alpha_1}$ and $\exists N_{(y,t_1)}$ s.t. $F(N_{(y,t_1)}) \subseteq \mathcal{U}_{\alpha_2}$....

$\pi^{-1}(\mathcal{U}_{\alpha_2}) \cap \tilde{F}(z,t) \neq \phi$, so define it...

So a partition $t_0 = 0 < \cdots < t_m = 1$ can be refined to one as above. The resulting map is the desired lift. $\boxtimes$

Cor: Every non-constant poly in $\mathbb{C}$ has a root in $\mathbb{C}$.

Cor: Every map $D^2 \to D^2$ has a fixed point $h(x) = x$

Cor: For every continuous map $S^2 \xrightarrow{f} \mathbb{R}^2$, $\exists x$ w/ $f(x) = f(-x)$

Cor: If $S^2 = A_1 \cup A_2 \cup A_3$, closed $\exists i = 1,2,3$ w/ $x, -x \in A_i$

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Pf: [Fundamental Theorem of Algebra] Assume $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_0$ has no roots. Then $\forall r \in \mathbb{R}$

$$\gamma_r(s) = \frac{p(re^{2\pi i s})/p(r)}{\| p(re^{2\pi i s})/p(r) \|}$$

is a loop in $S^1 \subseteq \mathbb{C} \setminus \{0\}$. Varying r is a homotopy of loops. $\gamma_0(s) = \text{cnst}_1$, so

$$\gamma_r = 0 \in \pi_1(S^1) \quad \forall r \in \mathbb{R}$$

Fix $r \gg |a_1| + \dots + |a_n|, 1$ so $|z^n| = r^n > (|a_1| + \dots + |a_n|)$ $|z^{n-1}| \geq |a_{n-1}z^{n-1} + \dots + a_0|$. Thus

$$p_t(z) = z^n + t(a_{n-1}z^{n-1} + \dots + a_0)$$ has no roots on $|z| = r$ for $0 \leq t \leq 1$. This shows

$$\gamma_r \simeq [n] \Rightarrow n = 0. \quad \boxtimes$$

Pf: [Temperature & Pressure] Suppose not. Define $S^2 \xrightarrow{g} S^1$ by $g(x) = f(x) - f(-x) / |f(x) - f(-x)|$

Define $\gamma_s : [0,1] \to S^1 \subseteq S^2$ the map $\gamma_s(t) = s \cdot t \in S^1$. Since

$$g(x) = -g(-x), \text{ if } h = g \circ \gamma_s$$

$h(s + \frac{1}{2}) = -h(s)$. The proof of $\pi_1(S^1) = \mathbb{Z}$ shows

h lifts to $\tilde{h} : I \to \mathbb{R} \Rightarrow \tilde{h}(s) + \frac{q}{2} = \tilde{h}(s + \frac{1}{2})$ for some $q \in 2\mathbb{Z} + 1$.

q depends continuously on s. $\Rightarrow$ constant.

$\tilde{h}(1)=\tilde{h}(\tfrac{1}{2})+q/2=\tilde{h}(0)+q \rightsquigarrow h=[q]$

q odd $\Rightarrow [h]\neq 0$. But

$h=g\circ\eta$ and η is ~~not~~ null-homotopic

$\Rightarrow g\circ\eta$ is null-homotic in S^1 ↯

Pf: [3 Closed sets problem]

Let $d_i(x)=\inf_{y\in A_i}|x-y_i|$. $\exists x\in S^2$ s.t.

$S^2\to\mathbb{R}^2: x\mapsto(d_1(x),d_2(x))$ has $x,-x$ with the same value. If either entry is zero, ~~Case 1: $x\mapsto(0,0)$~~ we are done. otherwise It is in A_3.