CLASS 13, OCTOBER 7: ZEROES AND POLES

Now that we have many of the underpinnings of what a holomorphic function is, we will ask how far we can deviate without losing everything. A deviation from holomorphic readily arrives at the singularity.

Definition 13.1. If f is a complex valued function defined in a neighborhood of a point z_0, but not defined at z_0, then z_0 is called an **isolated singularity for** f.

A canonical example of this is $f(z) = \frac{1}{z}$. It is necessarily singular at $z = 0$ because it can't be continuously extended to $z = 0$. An even 'worse' singularity would occur if you study the function $g(z) = e^{\frac{1}{z}}$. $f(z)$ at the very least can be made holomorphic by multiplying by z, where as $\lim_{z\to 0} |z^n e^{\frac{1}{z}}| = \infty$.

Definition 13.2. If $z_0 \in \mathbb{C}$ is such that $f(z_0) = 0$, we call z_0 a **zero** of f. The **order** of a zero of f is a positive integer m, if it exists, such that

$$f(z) = (z - z_0)^m \cdot g(z)$$

in a neighborhood of z_0, and $g(z_0) \neq 0$. We label it $\mathrm{ord}_{z_0}(f) = m$.

If f is holomorphic, then our analytic continuation result shows the zeroes of f are isolated (they can have no accumulation point). There is something to show here (namely that the order makes sense!):

Proposition 13.3. *If f is a holomorphic function with an isolated zero at z_0, then the order exists uniquely.*

Proof. We know that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ in some disc $B(z_0, r)$ around z_0. Since f is not identically 0 near z_0, by the analytic continuation results we have that $\exists m$ minimal such that $a_m \neq 0$:

$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \ldots = (z - z_0)^m(a_m + (z - z_0)h(z))$$

As a result, $g(z) = a_m + (z - z_0)h(z)$ demonstrates the result. Now, assume that there is another integer n such that $f(z) = (z - z_0)^m l(z)$ where $l(z_0) \neq 0$ in a neighborhood of z_0. We may assume by intersecting that they are equal and defined on a given set. Then we can divide by $(z - z_0)^m$:

$$(z - z_0)^{-m} f(z) = g(z) = (z - z_0)^{n-m} l(z)$$

If $n - m > 0$, then $z \to z_0$ would yield that $g(z_0) = 0$, a contradiction. Similarly, if $n - m < 0$, then $l(z_0) = 0$, another contradiction. So they must agree. $\square$

The key point here is that we can now define the idea of a pole. If $f(z)$ is holomorphic in a neighborhood U of z_0 with an isolated zero (in U) at z_0, then we can consider $\frac{1}{f(z)}$ in $U \setminus \{z_0\}$.

Definition 13.4. If $\mathrm{ord}_{z_0}(f) = m$, then we say $\frac{1}{f}$ has a **pole of order** m at z_0. If $m = 1$, then it is sometimes referred to as a **simple pole**. We write $\mathrm{ord}_{z_0}(\frac{1}{f}) = -m$.

Example 13.5. If we let C be the counterclockwise oriented unit circle, then we know

$$\int_C \frac{dz}{z^m} = \begin{cases} 2\pi i & m = 1 \\ 0 & \text{otherwise} \end{cases}$$

This already shows the importance of the simple pole.

A different way to classify this, identical to Proposition 13.3 is the following result:

Proposition 13.6. *If f has a pole at $z_0 \in \Omega$ of order m, then in a neighborhood of z_0, there is a holomorphic function h such that*

$$f(z) = (z - z_0)^{-m} h(z)$$

and such that $h(z_0) \neq 0$.

I leave it to you to prove this result. The following corollary arises immediately:

Corollary 13.7. *If f is a function with a pole of order m at z_0, then*

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \ldots + \frac{a_{-1}}{z - z_0} + g(z)$$

where $g(z)$ is a holomorphic function at z_0.

Proof. Consider the function $h(z)$ in Proposition 13.6. Since h is holomorphic, non-vanishing, we have

$$h(z) = b_0 + b_1(z - z_0) + \ldots + b_{m-1}(z - z_0)^{m-1} + b_m(z - z_0)^m + \ldots$$
$$(z - z_0)^{-m} h(z) = (z - z_0)^{-m} b_0 + b_1(z - z_0)^{-m+1} + \ldots + b_{m-1}(z - z_0)^{-1} + b_m + \ldots$$

As a result, if we let $a_{-n} = b_{m-n}$ for $n = 1, \ldots, m$ and $g(z) = b_m + b_{m+1}(z - z_0) + \ldots$ then we acquire the desired result. ☐

Definition 13.8. We call $\frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \ldots + \frac{a_{-1}}{z-z_0}$ the **principal part of** f at z_0. a_1 is called the **residue** and we write $a_1 = res_{z_0} f$.

As a result of the example above and Corollary 13.7, given any function $f(z)$ with a pole of order m at z_0, in a neighborhood of z_0 where z_0 is the only pole, we have

$$\int_C f(z)dz = \int_C \left(\frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \ldots + \frac{a_{-1}}{z - z_0} + g(z) \right) dz = 2\pi i a_{-1}$$

where C is a circle centered at z_0 in the neighborhood. So the residue is quite an important concept for our purposes. It can be computed quite easily for **simple poles**:

$$\text{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z)$$

If f has a pole of order m, then we can use the following more general result:

Theorem 13.9. *If f has a pole of order m at z_0, then*

$$res_{z_0} f = \lim_{z \to z_0} \frac{1}{(m-1)!} \left(\frac{\partial}{\partial z} \right)^{m-1} ((z - z_0)^m f(z))$$

Proof. Notice that due to Corollary 13.7,

$$(z - z_0)^m f(z) = a_{-m} + a_{-m+1}(z - z_0) + \ldots + a_{-1}(z - z_0)^{m-1} + (z - z_0)^m g(z)$$

A simple computation now proves the result. ☐