HOMEWORK 5: COUNTABILITY AND SEPARATIONS
DUE: OCTOBER 19

1) Show that if (X, τ) is second countable, then any basis $\mathcal{B}$ contains a subset $\mathcal{B}'$ which is countable and still a basis for τ.

2) Let $f : X \to Y$ be a continuous open map. Show that if X is first (or second)-countable, then so is $f(X)$.[1]

3) Let $Y \subset \mathbb{R}^{\mathbb{N}}$ with the box topology be the set of sequences $(x_1, x_2, \ldots)$ such that $x_n = 0$ for $n \geq N$ for some N, and $x_i \in \mathbb{Q}$. Show that Y has closed points. Find which separation axioms T(1-3) Y possesses. Is it T4 (for extra credit)?

4) Given a metric space (X, d) and a closed subset $Z \subseteq X$, show that the function
$$f(x) = d(x, Z) = \inf\{d(x, z) \mid z \in Z\}$$
is a continuous function $f : X \to \mathbb{R}$. Furthermore, show that $f(x) = 0$ if and only if $x \in Z$.

5) Use the previous problem to show that every metric space is T4/normal.

6) Given $\tau \subseteq \tau'$, it is easy enough to check that if X_τ is Hausdorff then so is also $X_{\tau'}$. Is the same true for T3 and T4? Justify your answer.

7) If Y is Hausdorff and $f, g : X \to Y$ are continuous maps, show that $Z = \{x \in X \mid f(x) = g(x)\}$ is a closed set.[2]

8) Let $p : X \to Y$ be a continuous, closed, surjective map. Show that if X is normal, then so is Y.

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[1] Note this is not a statement about Y.

[2] Note we cannot subtract in a generic topological space, though this generalizes such an idea.