HOMEWORK 7: MANIFOLDS & TYCHONOFF
DUE: NOVEMBER 2

1) Recall that for $Y \subseteq Z$, a continuous map $r : Z \to Y$ is called a **retraction** if $r(y) = y$ for all $y \in Y$. Show that if Z is Hausdorff, then Y is a closed subset of Z. Also, show that there exists no retraction $r : \mathbb{R}^2 \to \{x, y\}$, but there is a retraction $R : \mathbb{R}^2 \setminus \{0\} \to S^1$.

Solution: Suppose $z \in Y^c$. Since Z is Hausdorff, we can consider U, V open disjoint sets such that U contains z and V contains $r(z)$. But note that $V \cap Y$ is an open neighborhood of $r(z) \in Y$. I claim that

$$z \in U \cap r^{-1}(V \cap Y)$$

is the desired open neighborhood. Indeed, if $x \in r^{-1}(V \cap Y)$, then either $x \in V \cap Y$ or $x \in Y^c$. On the otherhand, $x \in U \subseteq V^c$, so $x \in Y^c$.

For the second part, note that if $r : \mathbb{R}^2 \to \{x, y\}$ existed, then $\mathbb{R}^2 = r^{-1}(x) \cup r^{-1}(y)$ would be open disjoint subsets. Since $\mathbb{R}^2$ is connected, no such thing can exist.

On the other hand, we have a natural radial projection map

$$R((x, y)) = \left(\frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \right)$$

This is a continuous map as a result of the theory of metric spaces (and trig!). For a fixed (x, y) and $\sqrt{x^2 + y^2} > \epsilon > 0$, $\delta = \sin(\epsilon) \cdot \epsilon$ suffices (a picture will help clear this up).

2) Suppose $X_1 \subseteq X_2 \subseteq \ldots$ are a sequence of subspaces such that $X_i \subseteq X_{i+1}$ is a closed subset. We can put a topology on $X = \bigcup_{i=1}^{\infty} X_i$, called the **direct limit topology**, by requiring that $U \subseteq X$ is open if and only if $U \cap X_i \subseteq X_i$ is open. Show this is a topology upon which $X_i \subseteq X$ is a subspace. Additionally, show that $f : X \to Y$ is continuous if $f|_{X_i}$ is continuous for all i.

Solution: The direct limit topology τ is a topology, since

1) $X, \emptyset$ have the property that $X \cap X_i = X_i$ and $\emptyset \cap X_i = \emptyset$, which are open since X_i is a topological space.
2) If $U_\alpha \in \tau$, then $\bigcup_\alpha U_\alpha \in \tau$ since

$$\left(\bigcup_\alpha U_\alpha \right) \cap X_i = \bigcup_\alpha (U_\alpha \cap X_i)$$

3) Similarly, if $U_1, \ldots, U_n \in \tau$, then

$$(U_1 \cap \cdots \cap U_n) \cap X_i = (U_1 \cap X_i) \cap \ldots \cap (U_n \cap X_i)$$

The statement about subspaces is automatic by the definition. For continuity, let $U \subseteq Y$ be an open set, and consider $f^{-1}(U) \subseteq X$. This set is open, since

$$f^{-1}(U) \cap X_i = f|_{X_i}^{-1}(U)$$

Note that I haven't yet used the fact that the subspaces are closed.

3) With the set up of the previous problem, show that X_i is normal for all i, then so is X with the direct limit topology. (**Hint**: Use Tietze iteratively.)

Solution: Suppose A, B are two disjoint closed subsets of X. Since A, B are closed in X with the direct limit topology, $A_i = A \cap X_i$ and $B_i = B \cap X_i$ are closed subsets of X_i.

We will use induction: Since X_1 is normal, we can use Tietze to extend the function $f_1' : A_1 \cup B_1 \to [0, 1]$ with $f_1'(A_1) = 0$ and $f_1'(B_1) = 1$ to a function f_1 on all of X_1.

For the inductive step, suppose we have produced a function $f_n : X_n \to [0, 1]$ which is 0 on A_n and 1 on B_n, and agrees with f_i on X_i for all $i \leq n$. It goes to produce $f_{n+1} : X_{n+1} \to [0, 1]$ with these properties. We note that

$$f_{n+1}' : X_n \cup A_{n+1} \cup B_{n+1} \to [0, 1] : x \mapsto \begin{cases} f_n(x) & x \in X_n \\ 0 & x \in A_{n+1} \\ 1 & x \in B_{n+1} \end{cases}$$

This function is continuous since it is well defined on the overlaps of the intermediate domains. Therefore, using Tietze, we can produce $f_{n+1} : X_{n+1} \to [0, 1]$ with the desired properties.

Define $f : X \to [0, 1]$ by $f(x) = f_n(x)$ if $x \in X_n$. This makes sense since every $x \in X$ is in some X_n. Futhermore, once it is one, $f_m(x) = f_n(x)$ for all $m \geq n$. So it is well defined. It is continuous by the previous problem, and is 0 on A and 1 on B by design.

Therefore, $U = f^{-1}([0, \frac{1}{2}))$ and $V = f^{-1}((\frac{1}{2}, 1])$ are the desired disjoint neighborhoods of A and B respectively.

4) Show that every manifold is T3, and therefore is also metrizable. Is the Hausdorff condition used?

Solution: If $x \in X$, where X is an m-manifold, and U is a neighborhood of x, then there exists a chart containing x:

$$\psi : U' \to U'' \subseteq \mathbb{R}^n$$

By intersecting U with U', we may assume that $U \subseteq U'$. As a result, we can consider $\psi(x) \in \psi(U) \subseteq \mathbb{R}^n$. Since ψ is a homeomorphism, $\psi(U)$ is an open neighborhood of $\psi(x)$. Therefore, since $\mathbb{R}^n$ is normal, there exists $V = B(x, r) \subseteq U$ such that

$$\psi(x) \subseteq V \subseteq \bar{V} \subseteq \psi(U) \subseteq U''$$

Note that $\bar{V}$ is therefore compact. Now it suffices to invert the procedure, and note that in $U \subseteq X$

$$x \in \psi^{-1}(V) \subseteq \overline{\psi^{-1}(V)} \cap U = \psi^{-1}(\bar{V}) \subseteq U$$

To see that $\psi^{-1}(V)$ has a closure in X (as opposed to U in the previous line) staying within U, we use the Hausdorff condition: Since $\psi^{-1}(\bar{V})$ is a compact in a Hausdorff space, it is itself closed:

$$\overline{\psi^{-1}(V)} = \psi^{-1}(\bar{V})$$

This completes the proof.

5) Show that if X is a connected, second-countable, locally Euclidean Hausdorff space, then the dimension m in the definition of locally Euclidean is the same on any chart.[1] You may assume the following beautiful result from algebraic topology[2]: If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open subsets, and $U \cong V$, then $n = m$.

Solution: Suppose the charts of X are $\psi_i : U_i \to \mathbb{R}^{m_i}$, and a result, $X = \bigcup_{i\in\mathbb{N}} U_i$. As a lemma, suppose $x \in U_i \cap U_j \neq \emptyset$. Then we may consider $U_{i,j} = U_i \cap U_j$ and the following:

$$\mathbb{R}^{m_i} \supseteq \psi_i(U_{i,j}) \cong U_{i,j} \cong \psi_j(U_{i,j}) \subseteq \mathbb{R}^{m_j}$$

Therefore, by invariance of dimension, we have that $m_i = m_j$.

Now, we want to bring in the notion of connectedness to complete the proof. Consider

$$X = U_1 \cup \bigcup_{i\neq 1} U_i$$

Since X is connected, these two sets intersect. Say $\emptyset \neq U_1 \cap U_{i_1}$ where $i_1 \in \mathbb{N}$ is chosen minimally. Therefore, we can move U_{i_1} to the other side:

$$X = (U_1 \cup U_{i_1}) \cup \bigcup_{i\neq 1, m_1} U_i$$

and note that $m_1 = m_{i_1}$. Iterate this process countably many times, and let $S \subseteq \mathbb{N}$ be the collection of indices on the left side. Then

$$X = \left(\bigcup_{i\in S} U_i\right) \cup \left(\bigcup_{i\notin S} U_i\right)$$

are 2 open sets covering X, and since X is connected, this implies that there exists

$$x \in \left(\bigcup_{i\in S} U_i\right) \cap \left(\bigcup_{i\notin S} U_i\right)$$

However, this would contradict the minimality of the chosen index: If $x \in U_i \cap U_j$ for $i \in S$ and $j \notin S$. Then U_j should have been moved at worst $|j - i|$-steps after U_i. A contradiction.

6) Show that if X is a locally Euclidean Hausdorff space, then if X is compact, each connected component of X is an m-manifold for some $m \in \mathbb{N}$.

Solution: By the previous problem, we know that m in the definition of locally Euclidean is constant on a connected set. Therefore, it suffices to check that X is second-countable. Given X is compact, the collection of charts may be assumed to

[1]This demonstrates that calling X an m-manifold is well-defined.

[2]Usually referred to as **Invariance of Dimension**.

be finite (since it represents an open cover); say $\psi_i : U_i \to U_i' \subseteq \mathbb{R}^m$ for $i = 1, \ldots, n$. Moreover, since $\mathbb{R}^m$ is a second countable space, so are each of the U_i. Let their countable basis be $U_{i,1}, U_{i,2}, \ldots$.

As a result, a basis for the topology is given by

$$\mathcal{B} = \{\psi_i^{-1}(U_{i,j}) \mid i = 1, \ldots, n,\ j \in \mathbb{N}\}$$

This is countable since it is a finite union of countable sets. It is a basis, since every $x \in U_i$ for some i, and therefore, $\psi(x) \in U_{i,j}$ since it is a basis. Therefore, $x = \psi^{-1}(\psi(x)) \in \psi^{-1}(U_{i,j})$. Similar results prove implication 2 upon restricting to the chart that $x \subseteq U$ is in.

Alternate Solution (by student(s)): You can check that the embedding theorem for compact manifolds doesn't require the second countable condition; finite partitions of unity exist simply for compact Hausdorff spaces. Therefore, $X \hookrightarrow \mathbb{R}^N$ for some $N > 0$, which is second-countable, and a subspace of a second-countable space is second-countable.

7) Let $\mathcal{M}$ be a maximal collection of subsets of X with the finite intersection property, as in the proof of Tychonoff's Theorem.
   - Show $x \in \bar{A}$ for each $A \in \mathcal{M}$ if and only if every neighborhood U of x is in $\mathcal{M}$. Which direction(s) requires maximality?
   - Show that if $A \subseteq B$ for some $A \in \mathcal{M}$, then $B \in \mathcal{M}$.
   - Show that if X is T1, then $\bigcap_{A \in \mathcal{M}} \bar{A}$ is either empty or a single point.

   Solution:
   - ($\Rightarrow$): If $x \in \bar{A}$ for every $A \in \mathcal{M}$, then $A \cap U \neq \emptyset$ (otherwise, $x \in \bar{A}^c$). Therefore, Lemma 22.4 part 2 implies the result, which requires $\mathcal{M}$ to be maximal.
     ($\Leftarrow$): If $x \notin \bar{A}$ for some $A \in \mathcal{M}$, then $\exists U \subseteq \bar{A}^c$ a neighborhood of x. Therefore, $U \cap \bar{A} = \emptyset$, so $U \notin \mathcal{M}$. This direction certainly doesn't use maximality.
   - Since $A \cap C \neq \emptyset$ for any $C \in \mathcal{M}$, we note that
     $$\emptyset \neq A \cap C \subseteq B \cap C.$$
     So again, Lemma 22.4 proves the result.
   - Suppose $x, y \in \bigcap_{A \in \mathcal{M}} \bar{A}$, and $x \neq y$. Note that the set $\{x\}$ is closed, since X is T1. Therefore, we can add it to the set $\mathcal{M}$. Let $\mathcal{M}'$ be the resulting set. Then clearly
     $$\bigcap_{A \in \mathcal{M}'} \bar{A} = \{x\} \cap \bigcap_{A \in \mathcal{M}} \bar{A} = \{x\} \cap \{x, y\} = \{x\}$$
     This contradicts the maximality of $\mathcal{M}$.