CLASS 1, SEPTEMBER 9TH: COMPLEX NUMBERS

We will begin our course by describing complex numbers and various representations of them based on our existent knowledge of the reals.

Definition 1.1. A **complex number** is a symbol of the form $a+ib$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$ is the imaginary unit. The collection of all complex numbers is denoted $\mathbb{C}$.

For referencing a complex number, $z = a + ib$, we write $Re(z) = a$ and $Im(z) = b$, the **real** and **imaginary** parts of z.

We also have two natural operations on $\mathbb{C}$:

$$(a+ib) + (c+id) := (a+c) + i(b+d)$$

$$(a+ib) \cdot (c+id) := (ac-bd) + i(ad+bc)$$

This yields a few clear identification of $\mathbb{C}$ with other familiar objects:

1) As real vector spaces, $\mathbb{C}$ is exactly $\mathbb{R}^2$.
2) $\mathbb{C}$ is a commutative ring; it is an Abelian group under addition, closed under multiplication (also commutative), and a quick check shows that multiplication distributes over addition.
3) As a more complicated (though often fruitful) representation, we can view complex numbers $a+ib$ as matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

This is compatible with matrix multiplication!

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}$$

We won't focus on this representation so much, but in complex analysis it is often beneficial to see how it fits into the broad framework of mathematics.

Focusing on the identification of $\mathbb{C}$ with $\mathbb{R}^2$, there is another useful method for viewing points: by their distance from the origin r and the angle they form with the x-axis θ. This is often referred to as **polar coordinates**, and written $z = re^{i\theta}$.

Note: θ is only determined up to 2π, so it is often convenient to restrict θ to $(-\pi, \pi]$ and choose only non-negative values for r.

As in Calculus, there is a natural way to transfer between the 2 coordinate systems: $z = a + ib = re^{i\theta}$, then

$$a = r\cos(\theta) \qquad\qquad b = r\sin(\theta) \qquad\qquad r = \sqrt{a^2+b^2}$$

Considerations for θ, as in trigonometry, are a little more bizarre. If for example $a > 0$, then it is concise:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right).$$

Otherwise, some care should be applied (c.f. homework).

Definition 1.2. We call θ the **argument** of z and write $\theta = Arg(z)$. r is called the **absolute value** of z, written $r = |z|$.

The question is then why would we care about polar coordinates? Well, for one, they make multiplication quite a bit easier!

$$z \cdot w = re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)}.$$

This will be checked as one of your homework assignments. Note in particular that multiplication behaves exactly as you might expect based on your knowledge of the exponential function. This is no coincidence.

The absolute value is exactly the Euclidean norm, the standard measure of distance, on $\mathbb{R}^2$. A simple way to determine it is through the use of the complex conjugate.

Definition 1.3. If $z = a + ib$ is a complex number, then the **complex conjugate** of z, denoted $\bar{z}$, is $\bar{z} = a - ib$.

The complex conjugate can be viewed as the reflection of z across the real axis, making it so that if $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$. An immediate conclusion regarding it is

$$|z|^2 = r^2 = z \cdot \bar{z}.$$

Additionally, the following equalities are easily verified:

$$Re(z) = \frac{z + \bar{z}}{2} \qquad\qquad Im(z) = \frac{z - \bar{z}}{2}$$

The complex conjugate also immediately realizes the following:

Theorem 1.4. *Given, $z \neq 0$ a complex number, there exists $z^{-1} \in \mathbb{C}$ such that $z \cdot z^{-1} = 1$. This is to say that $\mathbb{C}$ is a field.*

Proof. Let $z^{-1} = \frac{1}{|z|^2}\bar{z}$. □

Example 1.5. Considering the complex number $z = 1 + 2i$ (it is a strange but convenient notation to list real numbers in front of i, and variables after i), it is clear that $Re(z) = 1$ and $Im(z) = 2$. Using our rules from above, we can further compute $|z| = \sqrt{5}$ and $Arg(z) = \tan^{-1}(2) \approx 1.10714872$ radians, or roughly 63.4 degrees. This should be confirmed by our intuition. So we could write

$$1 + 2i \approx \sqrt{5}e^{1.10714872i}$$

Finally, since $\mathbb{C}$ with the absolute value is nothing but $\mathbb{R}^2$ with the Euclidean norm, we have:

Theorem 1.6. *$\mathbb{C}$ is a complete metric space.*

To be a **complete metric space** means that every **Cauchy sequence** will **converge**. These words should be familiar to you from real analysis. Next time we will talk about sets in $\mathbb{C}$ and begin talking about complex functions.