CLASS 21, OCTOBER 29: EMBEDDINGS OF MANIFOLDS

Today, we move on to one of the central topics of general and differential topology; Manifolds. As we have seen so far in this course, the world of topological spaces at large can be quite daunting; incredibly wild and non-intuitive things can happen. So we have specialized to things like normal spaces and showed that they exhibit many of the desirable properties of metric spaces. Here is a similar style of specialization that produces more well behaved topological spaces:

Definition 21.1. A topological space is called an m**-manifold** if X is a second-countable Hausdorff space which is **locally Euclidean**. That is to say there exists a neighborhood U of any point $x \in X$ such that $U \cong U' \subseteq \mathbb{R}^m$, where U' is an open subset of $\mathbb{R}^n$.[1]

Often times, 1-manifolds are called **curves**, 2-manifolds are called **surfaces**, and m-manifolds for $m \geq 3$ are shortened to m**-folds**. In addition, the maps representing the homeomorphisms $\varphi_i : U \to U' \subseteq \mathbb{R}^m$ are referred to as **charts**, and the collection $\{\varphi_i\}$ are called an **atlas**.

Example 21.2. It may seem bizarre that we have neighborhoods of any point homeomorphic to a Hausdorff space, but require that X be Hausdorff. This example shows the importance of the Hausdorff condition.

Let X be the quotient of $\mathbb{R} \coprod \mathbb{R}$ by the relation that if $x \neq 0$, then $x_1 \sim x_2$, where the subscripts are denoting which copy of $\mathbb{R}$ the point is being viewed in. The space X is referred to as the real line with the origin doubled, since $0_1 \not\sim 0_2$.

Now, note that X is not Hausdorff: there do not exist open disjoint sets U, V separating 0_1 from 0_2. Indeed, they look like open neighborhoods of 0 in $\mathbb{R}$ containing either 0_1 or 0_2, and therefore for some $\epsilon > 0$,

$$(-\epsilon, 0) \cup (0, \epsilon) \subseteq U \cap V.$$

On the other hand, this space *is* locally Euclidean. Indeed, $X \setminus \{0_1\} \cong \mathbb{R} \cong X \setminus \{0_2\}$.

Recall that in Corollary 18.3 we proved that every compact Hausdorff space in fact has a finite partition of unity. I define this notion here just to reiterate.

Definition 21.3. Given a locally finite open cover $X = \bigcup_\alpha U_\alpha$ (c.f. Homework 1), a collection of functions $f_\alpha : X \to [0, 1]$ is said to be a **partition of unity subordinate to** $\{U_\alpha\}$ if $\mathrm{Supp}(f_\alpha) \subseteq U_\alpha$, and $\sum_\alpha f_\alpha(x) = 1$ for every $x \in X$.[2]

This result allows us to show a baby version of the famous Whitney Embedding Theorem.

Theorem 21.4. *Let X be a compact m-manifold. Then there exists an $n \gg 0$ such that*

$$\iota : X \hookrightarrow \mathbb{R}^n$$

where ι is an embedding, e.g. an injective map which is a homeomorphism onto its image.

[1]Sometimes these objects are referred to as **topological manifolds** to avoid confusion with their differential version; **smooth manifolds**.

[2]Note the locally finite condition makes this sum a finite sum! So we needn't worry about convergence.

So ANY compact manifold is a subspace of some Euclidean $\mathbb{R}^n$!

Proof. Let $\varphi_i : U_i \to \mathbb{R}^m$ be charts for X, where $i = 1, 2, \ldots, n$ since X is compact. Since X is compact and Hausdorff, it is normal by Theorem 17.2. Applying Corollary 18.3, we know that a partition of unity f_i subordinate to U_i exists. Let

$$A_i = Supp(f_i) = \overline{\{x \in X \mid \varphi_i(x) \neq 0\}}.$$

Then for $i = 1, \ldots, n$, define a new function

$$h_i(x) = \begin{cases} f_i(x) \cdot \varphi_i(x) & x \in U_i \\ 0 & x \in A_i^c \end{cases}$$

Note this function is well defined, because if $x \in U_i \setminus A_i$, the $f_i(x) = 0$. So the 2 internal functions agree on the overlaps of their domains. Additionally, h_i is continuous, because it is continuous when restricted to the open sets U_i and A_i^c, and therefore the preimage of an open is a union of 2 open sets.

Now we may consider the desired function:

$$\iota : X \to (\mathbb{R})^n \times (\mathbb{R}^m)^n \cong \mathbb{R}^{n(m+1)} : x \mapsto (f_1(x), \ldots, f_n(x), h_1(x), \ldots, h_n(x))$$

ι is a product of continuous functions, therefore continuous. Next, I claim ι is injective. Suppose $\iota(x) = \iota(y)$. Then $h_i(x) = h_i(y)$ and $f_i(x) = f_i(y)$ for all $i = 1, 2, \ldots, n$. Since

$$1 = (\sum_{i=1}^{n} f_i)(x) = (\sum_{i=1}^{n} f_i)(y)$$

we know there is some i s.t. $f_i(x) = f_i(y) > 0$. But this implies

$$f_i(x)\varphi_i(x) = f_i(y)\varphi_i(y)$$
$$\varphi_i(x) = \varphi(y)$$

But φ_i are charts, therefore injective. This implies $x = y$.

Finally, it goes to show X is homeomorphic to its image. This follows from Corollary 12.1, restated here for convenience: If $f : X \to Y$ is a continuous bijective map with X compact and Y Hausdorff, then f is a homeomorphism. The result then follows by virtue of the fact that $\iota(X) \subseteq \mathbb{R}^{n(m+1)}$, and subspaces of T2 spaces are T2. $\square$

The following example demonstrates the inefficiencies of Theorem 21.4.

Example 21.5. S^n is a manifold. Indeed, we can identify $\{N\}^c$ and $\{S\}^c$ with $\mathbb{R}^n$, when N and S are the north and south pole respectively. This identification can be made by a process of stereographic projection. Theorem 21.4 tells us we can embed $S^n \hookrightarrow \mathbb{R}^{2n+2}$. However, we know we can embed S^n in $\mathbb{R}^{n+1}$.

Similarly, we can give the n-dimensional torus $\mathbb{T}^n = S^1 \times \ldots \times S^1$ the structure of a manifold with $2n$-many charts. Theorem 21.4 allows us to embed $\mathbb{T}^n \hookrightarrow \mathbb{R}^{n^2+n}$, whereas in reality $\mathbb{R}^{n+1}$ suffices.

However, we are still embedding a manifold in a finite dimensional vector space, which is a huge advantage.

Just to complement our theorem, the Whitney's Embedding Theorem tells us that we can embed any smooth n-dimensional manifold in $\mathbb{R}^{2n-1}$. This requires a lot of machinery (its own class worth).