CLASS 19, OCTOBER 24: THE URYSOHN METRIZATION THEOREM

Our objective today is to demonstrate that a large class of topological spaces are in fact metric spaces. As a result, I will convince you that the beautiful Urysohn Lemma was actually intended to be a lemma by its creator, to prove a corresponding result about *metrizability*. We will use this notion of metrizability so I define it here. But the core idea is the following:

$$(X, d) \overset{\text{Metric Topo}}{\Longrightarrow} (X, \tau) \overset{X \text{ Metrizable}}{\Longrightarrow} (X, d_\tau)$$

Definition 19.1. A topological space (X, τ) is called **metrizable** if there exists d a metric on X such that τ is the metric-topology associated to (X, d).

As a further application of Urysohn's Lemma, we have the following theorem:

Theorem 19.2 (Urysohn Metrization Theorem)**.** *Every second-countable T3 + T1 topological space (X, τ) is metrizable.*

Note that by Theorem 17.1, we already know that X is normal. However, not all normal spaces are metrizable. So this can be viewed as a strengthening of Theorem 17.1 as well as a partial converse to the result of Homework 5, Exercise 5: Metric spaces are normal. Before this, I demonstrate a claim from Class 14:

Lemma 19.3. *The space $\mathbb{R}^{\mathbb{N}}$ with the product topology is metrizable.*

Proof. I claim that if $x = (x_1, x_2, \ldots), y = (y_1, y_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$, and we let $d_i = \min\{|x_i - y_i|, 1\}$, then

$$d(x, y) = \sup_{i \in \mathbb{N}} \left\{ \frac{d_i(x, y)}{n} \right\}$$

is the required metric. Note that

$$B(x, \epsilon) = \prod_{i=1}^{n} (x_i - i \cdot \epsilon, x_i + i \cdot \epsilon) \times \prod_{i > n} \mathbb{R}$$

where n is chosen minimal such that $\frac{1}{n} < \epsilon$. Therefore, the metric topology of d is contained within the product topology. On the other hand, given a basis element of the product topology

$$U = \prod_{i=1}^{n} (a_i, b_i) \times \prod_{i > n} \mathbb{R}$$

and $x \in U$, we get that $B(x, r) \subseteq U$, where

$$r = \min \left\{ \frac{x_i - a_i}{i}, \frac{b_i - x_i}{i} \mid i = 1, 2, \ldots, n \right\}$$

$\square$

Now, to prove the metrization theorem, we aim to prove that a second countable metric space may be embedded[1] into $\mathbb{R}^{\mathbb{N}}$ with the product topology. This, combined with

[1]A topological embedding is short for a map $f : X \to Y$ which is a homeomorphism of X to $f(X)$.

Lemma 19.3 yields the desired result, since a subspace of a metric space is a metric space. First we need another lemma:

Lemma 19.4. *If (X, τ) is a second-countable T3 space, then there exists a countable collection $f_n : X \to [0,1]$ such that for any $x \in X$ and neighborhood U of x, there exists n such that $f(x) \neq 0$ and $f(U^c) = 0$.*

Proof. Let $U_1, U_2, \ldots$ be a countable basis for τ. For any pair of indices n, m such that $\overline{U_n} \subseteq U_m$, apply Urysohn's Lemma to produce a function $f_{n,m} : X \to [0,1]$ such that $f_{n,m}(\overline{U_n}) = 1$ and $f_{n,m}(U_m^c) = 0$. This is a countable collection of functions since it is surjected onto by $\mathbb{Z}^2$.

Now, given U a neighborhood of x, there exists $U_m \subseteq U$ containing x by definition of a basis. Now applying Theorem 16.4, we know there is some open subset $V \subseteq U_m$ containing x whose closure lies within U_m. But since it is open, it contains some U_n containing x!. The function $f_{n,m}$ then satisfies $f_{n,m}(x) = 1$ since $x \in U_n$, and $f(U^c) = 0$ since $U^c \subseteq U_m^c$. ☐

We can now prove the main result.

Proof. (of Theorem 19.2): Let $f_1, f_2, \ldots$ be the collection of functions guaranteed by Lemma 19.4. Take $F : X \to \mathbb{R}^{\mathbb{N}} : x \mapsto (f_1(x), f_2(x), \ldots)$.

First, I claim F is an embedding. Note that F is continuous:

$$F^{-1}\left(U_1 \times \ldots \times U_n \times \prod_{i \geq n+1} \mathbb{R}\right) = f_1^{-1}(U_1) \cap \ldots \cap f_n^{-1}(U_n)$$

Now, it goes to show that F is injective. Suppose $x \neq y$. Then $\exists U$ an open neighborhood of x not containing y (e.g. $\{y\}^c$ using T1). Therefore, there exists U_n for which $x \in U_n$ and $y \notin U_n$, implying $f_n(x) > 0 = f_n(y)$. Therefore, $F(x) \neq F(y)$.

Now, it goes to prove that X is homeomorphic to its image. We know this is equivalent to checking F is an open map. Let $U \subseteq X$ be an open set, and let $V = F(U)$. Letting $v \in V$, it suffices to construct an open neighborhood V' of v contained within V. Let $x \in X$ be such that $F(x) = v$. Since U is an open neighborhood of x, there exists f_n such that $f_n(x) > 0$, and $f_n(U^c) = 0$. That is to say $f_n^{-1}((0, \infty)) \subseteq U$.

Therefore, we note that $\pi_n^{-1}(0, \infty)$ is an open of $\mathbb{R}^{\mathbb{N}}$, and therefore $V' = \pi_n^{-1}(0, \infty) \cap F(X)$ is open in the subspace topology of $F(X)$. Notice that $v \in V'$:

$$\pi_n(v) = \pi_n(F(x)) = f_n(x)$$

Lastly, we need to show $V' \subseteq V$. For $v' \in V'$, notice that $v' = F(x')$ for some x', and $\pi_n(z) \in (0, \infty)$. But $\pi_n(v') = f_n(x')$, and therefore $x' \in U$ since $f_n(U^c) = 0$. Therefore, $v' = F(x) \in F(U)$. This completes the proof. ☐

It should be noted that in Munkres, they prove the same fact with the uniform topology. The last order of business is to note that we proved something slightly stronger:

Corollary 19.5 (Embedding Theorem)**.** *If X is T1, then if $f_\alpha : X \to \mathbb{R}$ for $\alpha \in \Lambda$ are continuous functions such that every point x and neighborhood U has some f_α with $f_\alpha(x) > 0$ and $f_\alpha(U^c) = 0$, then $F : X \to \mathbb{R}^\Lambda : x \mapsto (f_\alpha(x))_{\alpha \in \Lambda}$ is an embedding. We can also restrict the domain to $[0,1]^\Lambda$ if each f_α is bounded in $[0,1]$.*