CLASS 14, WEDNESDAY MARCH 14TH: FLAT MODULES

The motivation for studying flat modules is to give $M \otimes_R -$ the same treatment as $\mathrm{Hom}_R(M, -)$ and $\mathrm{Hom}_R(-, M)$. Basically, projectives and injectives transform the Hom_R functors from merely left exact to exact functors. A flat module will do the same thing for the tensor product.

Proposition 0.1. *Let N be an R-module. If*

$$0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$$

is a SES of R-modules, then the following sequence is also exact:

$$M' \otimes_R N \xrightarrow{\varphi \otimes Id_N} M \otimes_R N \xrightarrow{\psi \otimes Id_N} M'' \otimes_R N \to 0$$

Proof. First, I will show surjectivity of $M \otimes_R N \xrightarrow{\psi \otimes Id_N} M'' \otimes_R N$. Given $m'' \otimes n$, there is $m \in M$ such that $\psi(m) = m''$. Therefore, surjectivity is achieved:

$$(\psi \otimes Id_N)(m \otimes n) = \psi(m) \otimes n = m'' \otimes n$$

Now, it goes to show the sequence is exact in the middle.

$$(\psi \otimes Id_N) \circ (\varphi \otimes Id_N) = (\psi \circ \varphi) \otimes Id_N = 0 \otimes Id_N$$

So ker $\supseteq$ im. This implies that there is a natural map

$$M \otimes_R N / \operatorname{im}(\varphi \otimes Id_N) \to M'' \otimes_R N$$

and it suffices to prove that this is an isomorphism by construction of an inverse. Given $m'' \otimes n$, we know $\exists m \in M$ such that $\psi(m) = m''$. Thus, we define

$$M'' \times N \to M \otimes_R N / \operatorname{im}(\varphi \otimes Id_N) : (m'', n) \mapsto m \otimes n$$

It goes to show this is well defined. If $m_1, m_2 \mapsto m''$, then $m_1 - m_2 \in \ker(\psi) = \operatorname{im}(\varphi)$. Therefore, $\exists m' \mapsto m_1 - m_2$. Therefore

$$m_1 \otimes n = (m_2 + \varphi(m')) \otimes n = m_2 \otimes n + \varphi(m') \otimes n = m_2 \otimes n$$

This is also seen to be the inverse of the map above, completing the proof. $\square$

As an alternative, one can use the adjointness of Hom and $\otimes$ proven last class. Note that this is not exact on the left:

Example 0.2. Consider the injection $\mathbb{Z} \hookrightarrow \mathbb{Z} : 1 \mapsto n$ of $\mathbb{Z}$-modules. Tensoring by $\mathbb{Z}/n\mathbb{Z}$, we get the map

$$\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z} : 1 \otimes 1 \mapsto n \otimes 1 = 1 \otimes n = 0$$

The zero map between $\mathbb{Z}/n\mathbb{Z}$ and itself clearly isn't injective.

This brings about the definition of a flat R-module:

Definition 0.3. A module F is said to be **flat** if and only if for every injection of R-modules $M \hookrightarrow N$, we get that $M \otimes_R F \hookrightarrow N \otimes_R F$ is injective. Equivalently, $- \otimes_R F$ is an exact functor (taking exact sequences to exact sequences).

Proposition 0.4. *If P is a projective module, then P is also flat.*

Proof. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence. Since P is a projective module, we know that $F = R^n \cong P \oplus P'$. Tensoring by F, we get

$$0 \to (M')^n \to M^n \to (M'')^n \to 0$$

Which is still exact, since the maps are direct sums of exact sequences. On the other hand, this induces

$$0 \to (M' \otimes_R P) \oplus (M' \otimes_R P') \to (M \otimes_R P) \oplus (M \otimes_R P') \to (M'' \otimes_R P) \oplus (M'' \otimes_R P') \to 0$$

Therefore, injectivity holds on the left, and thus $M' \otimes_R P \hookrightarrow M \otimes_R P$ □

Just to keep up the list of definitions we can associate to modules: Free implies Projective, and Projective implies Flat. The following two examples show that the notions are inequivalent.

Example 0.5 (Projective but not free)**.** Let $R = \mathbb{Z}/6\mathbb{Z}$, and $M = \mathbb{Z}/2\mathbb{Z}$ with the quotient module action. Then $R \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ by the Chinese remainder theorem. Therefore,

$$\mathrm{Hom}_R(R, -) \cong \mathrm{Hom}_R(\mathbb{Z}/3\mathbb{Z}, -) \oplus \mathrm{Hom}_R(M, -)$$

Thus we can conclude that M is projective, since the left hand side is clearly exact. However, M cannot be free, since a free R-module has either 6^n elements, or ∞-many elements.

Example 0.6 (Flat but not projective)**.** Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. $\mathbb{Q}$ is flat, since if $\varphi : M \to N$ is injective, and $\frac{1}{d} \otimes m \mapsto \frac{1}{d} \otimes \varphi(m) = 0$, then $c\varphi(m) = 0$. But this implies $cm = 0$, since φ was injective to begin with. Therefore, $\frac{1}{d} \otimes m = \frac{1}{cd} \otimes cm = 0$. However, $\mathbb{Q}$ is not projective: Suppose $F = \mathbb{Z}^\Lambda = \mathbb{Q} \oplus P$. However, $\mathrm{Hom}_\mathbb{Z}(\mathbb{Q}, \mathbb{Z}) = 0$ (since if $\alpha \mapsto n$, $\frac{\alpha}{2n} \mapsto \frac{1}{2} \notin \mathbb{Z}$). Therefore, $\mathbb{Q}$ is not a projective $\mathbb{Z}$-module.

I conclude by showing that every module injects into a injective module:

Lemma 0.7. *Let A be an R-algebra, F a flat A-module, and I an injective R-module. Then $\mathrm{Hom}_R(F, I)$ is also injective as an A-module.*

Proof. Note that $\mathrm{Hom}_A(-, \mathrm{Hom}_R(F, I)) \cong \mathrm{Hom}_R(- \otimes_A F, I)$ is a composition of two exact functors $- \otimes_R F$ and $\mathrm{Hom}_R(-, I)$. Therefore, it is exact itself. □

Theorem 0.8. *Every R-module M injects into an injective module.*

Proof. Any commutative unital ring R can be viewed as a $\mathbb{Z}$-algebra via the canonical map $\mathbb{Z} \to R : 1 \mapsto 1$. Let $I = \mathbb{Q}/\mathbb{Z}$ be the injective $\mathbb{Z}$-module discussed last class, and consider the R-module $\mathrm{Hom}_\mathbb{Z}(M, I)$. We can take a free module $F = R^\Lambda$ surjecting onto it. Applying $\mathrm{Hom}_R(-, \mathbb{Q}/\mathbb{Z})$ to this surjection, we have

$$\mathrm{Hom}_R(\mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = M^{\vee\vee} \hookrightarrow \mathrm{Hom}_R(F, \mathbb{Q}/\mathbb{Z})$$

Lastly, $M \hookrightarrow M^{\vee\vee} : m \mapsto (\psi \mapsto \psi(m))$. This completes the proof. □

Definition 0.9. The smallest such injective module (ordered by inclusions) containing a given module M is called the injective hull of M. It is usually denoted by $E_R(M)$.

There is a very nice summary of many of these results on page 402 of Dummit-Foote.

After spring break, we will talk about Nakayama's Lemma and regular rings, and then move on to positive characteristic commutative algebra.