CLASS 5, SEPTEMBER 17: CONTINUITY I

In almost all regions of math, objects become far more interesting once you give them an idealized version of a function. In the theory of metric spaces, this comes in the form of an ϵ-δ notion of a continuous function. Here we generalize this to arbitrary topological spaces and simplify matters simultaneously.

Definition 5.1. If (X, d) and (X', d') are metric spaces, then a function $f : X \to X'$ is said to be **continuous at a point** $x \in X$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$, then we have that $d'(f(x), f(y)) < \epsilon$. f is said to be **continuous** if it is continuous at all of its points.

This is a cumbersome thing to check in practice, usually resulting in the production of a function δ depending on x and ϵ. The corresponding notion for topological spaces is as follows:

Definition 5.2. Let (X, τ) and (X', τ') be topological spaces. Then a function $f : X \to X'$ is said to be continuous if for every open set $U \subseteq X'$, the preimage is also open:

$$f^{-1}(U) = \{x \in X \mid f(x) \in U\}$$

Example 5.3.
1) If X is any set with the discrete topology, Y any topological space, then *any* function $f : X \to Y$ is continuous!
2) If Y is any set with the indiscrete topology, X any topological space, then *any* function $f : X \to Y$ is continuous!
3) If τ, τ' are topologies on X, then the identity function $Id_X : (X, \tau) \to (X, \tau')$ is continuous if and only if τ' is finer than τ.
4) Give $\mathbb{R}$ the Euclidean/metric topology. Considering the function $f : \mathbb{R} \to \mathbb{R} : x \mapsto \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$, we expect this function to *not* be continuous. This is in fact the case, because $U = (-1, 1)$ is open in $\mathbb{R}$ with
$$f^{-1}(U) = (-\infty, -1) \cup \{0\} \cup (1, \infty)$$
which is *not* open.

Of course, we wouldn't want continuity to mean completely different things in different overlapping contexts. A proposition corrects these doubts.

Proposition 5.4. *Let $f : (X, d) \to (X', d')$ be a map between metric spaces. Then f is continuous in the sense of metrics if and only if f is a continuous map of topological spaces with the metric topologies.*

Proof. ($\Rightarrow$): Suppose that U is an open set of X', and let $x' \in U$ be such that $f(x) = x'$ for some $x \in X$. For any fixed $\epsilon > 0$, $\exists \delta > 0$ such that $d(x, y) < \delta$ implies $d(x', f(y)) < \epsilon$. Since we know $B(x, r)$ form a basis for the metric topology, there exists $\epsilon_{x'}$ such that $B(x', \epsilon_{x'}) \subseteq U$. But this implies $B(x, \delta_{x'}) \subseteq f^{-1}(U)$. Therefore

$$f^{-1}(U) = \bigcup_{x' \in X} \bigcup_{f(x) = x'} B(x, \delta_{x'})$$

is an open set.

($\Leftarrow$): We know that $f^{-1}(B(x',\epsilon))$ is open. Therefore, if $f(x)=x'$, there exists some $\delta>0$ such that $B(x,\delta)\subseteq f^{-1}(B(x',\epsilon))$. This again implements the fact that the metric topology has a basis of open balls. This δ then satisfies the conditions for continuity at x for the given (arbitrary) ϵ. $\square$

The notion of a continuous function also gives a nice formulation for the product topology:

Proposition 5.5. *The product topology on $\prod_\alpha X_\alpha$ is the smallest/coarsest topology such that each of the projection maps $\pi_\alpha : \prod_\alpha X_\alpha \to X_\alpha$ are continuous.*

Proof. Given $U_{\alpha'}\subset X_{\alpha'}$, we need $\pi_{\alpha'}^{-1}(U_{\alpha'})$ to be open. But this is exactly $U_{\alpha'}$ product with the remaining X_α:

$$\pi_{\alpha'}^{-1}(U_{\alpha'}) = U_{\alpha'} \times \prod_{\alpha\neq\alpha'} X_\alpha$$

To ensure we have a topology, we need to ensure finite intersections of such sets are included:

$$\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) = U_{\alpha_1}\times U_{\alpha_2}\times\ldots\times U_{\alpha_n}\times \prod_{\alpha\neq\alpha'} X_\alpha$$

But this is exactly the basis described for the product topology! $\square$

To finish up, I want to include two other notions which are distinct from being continuous that illustrate a common confusion:

Definition 5.6. A map $f:X\to Y$ of topological spaces is said to be **open** (respectively **closed**) if for every open (resp. closed) subset $U\subseteq X$, the set $f(U)$ is open (resp. closed).

Notice that in general, we can only say

$$U\subseteq f^{-1}(f(U))$$
$$f(f^{-1}(V))\subseteq V$$

for $U\subseteq X$ and $V\subseteq Y$. Here are some examples showing some examples which satisfy one property but not the other.

Example 5.7.
- If $f:X\to Y$ is a constant map, meaning $f(x)=y$ for all $x\in X$ and some $y\in Y$, then f is certainly continuous; the preimage of an open set is either X or $\emptyset$ depending on whether or not it contains y. However, it is generally only an open/closed mapping if $\{x\}$ is itself open or closed (or if X has the indiscrete topology).
- The mapping f from Example 5.3 4) is non-continuous, but it is closed! This is because if $a<0<b$, we have

$$f([a,b]) = (-\infty,\frac{1}{a}]\cup\{0\}\cup[\frac{1}{b},\infty) = \left((\frac{1}{a},0)\cup(0,\frac{1}{b})\right)^c$$

- The projection map $\pi:\mathbb{R}^2\to\mathbb{R}:(x,y)\mapsto y$ is an example of an open map which is not closed. Take for example the closed set which is the graph of $y=\tan^{-1}(x)$. The image of this set is $(-\frac{\pi}{2},\frac{\pi}{2})$, which is open! Showing it's open is just the realization that $\pi(B((x,y),r)=B(y,r)=(y-r,y+r)$.