CLASS 5, SEPTEMBER 18: POWER SERIES

Last time, we left off with the following result:

Theorem. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then f is holomorphic in its disc of convergence. Furthermore,*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

has the same radius of convergence R.

Proof. First note that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. As a result, if we can prove the other assertions, Hadamard's formula will ensure that they share the same radius of convergence.

Let $|z_0| < r < R$ for some fixed r. Let $S_N(z)$ denote the N^{th} partial sum of the series $\sum_{n=0}^{N} a_n z^n$, and $E_n(z)$ denote their difference (i.e. the error in this estimation). Let $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$. If h is chosen such that $|z_0 + h| < r$, then

$$\frac{f(z_0+h) - f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right) + (S_N'(z_0) - g(z_0)) + \left(\frac{E_N(z_0+h) - E_N(z_0)}{h}\right)$$

Now, using the equality $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})$, letting $a = z_0 + h$ and $b = z_0$ yields

$$\left|\frac{E_N(z_0+h) - E_N(z_0)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}$$

the last equality is due to our choices of $|z_0| < r$ and $|z_0 + h| < r$. On the right we have the end of a convergent series by the first part. Thus it can be made arbitrarily small: Given $\epsilon > 0$, we can choose N_1 such that $\frac{E_N(z_0+h) - E_N(z_0)}{h} < \epsilon$ for all $N > N_1$.

Additionally, since $\lim_N S_N'(z_0) = g(z_0)$, we can choose N_2 such that $|S_N'(z_0) - g(z_0)| < \epsilon$ for $N > N_2$.

If we choose N larger than both of them, and choose $|h| < \delta$ so that $\left|\frac{S_N(z_0+h) - S_N(z_0)}{h} - S_N'(z_0)\right| < \epsilon$ we get that the left hand side is bounded above by 3ϵ, which can be made arbitrarily small. Therefore $g(z_0) = f'(z_0)$. □

By induction, we achieve the following corollary:

Corollary 5.1. *Power series are infinitely complex differentiable in their disc of convergence, with derivatives obtained via term-wise differentiation.*

One should also note that we have been focusing on power series centered at the origin. In general, a **power series centered at** z_0 has the form

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

It's disc of convergence is centered at z_0. We can even transfer between the two notions with the transform $g(z) = f(z - z_0)$ assuming z_0 is in the disc of convergence of f.

Definition 5.2. A complex function $f : \Omega \to \mathbb{C}$ is **analytic at** $z_0 \in \Omega$ if there exists a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with positive radius of convergence R agreeing with f in $B(z_0, R) \cap \Omega$. f is **analytic** if it is analytic at all its points.

As a result of the theorem above, we note immediately that all analytic functions are holomorphic. Using Cauchy's Theorem from chapter 2, we will prove that the converse holds as well. This demonstrates something much stronger than the result stated in Class 0.

We now switch gears to the question of integration. Unlike with $\mathbb{R}$, where we can usually simply specify the bounds of integration, in $\mathbb{C}$ we need to specify a curve with which to integrate along. Note we are implicitly doing this in Calculus I & II.

Definition 5.3. A **path** is a function $\gamma : [a, b] \to \mathbb{C}$ for $a < b$ real numbers. A path is **smooth** if $\gamma'(t)$ exists and is a non-zero continuous function. At the boundaries, we make special notations

$$\gamma'(a) = \lim_{t \to a^+} \frac{\gamma(t) - \gamma(a)}{t - a} \qquad \gamma'(b) = \lim_{t \to b^-} \frac{\gamma(t) - \gamma(b)}{t - a}$$

We also call a path **piecewise-smooth** if there exist $a = a_0 < a_1 < \ldots < a_n = b$ such that $\gamma|_{[a_i, a_{i+1}]}$ is a smooth curve for each $i = 0, \ldots, n-1$.

Intuitively, we can break up a piecewise smooth curve into several smooth curves. We call $\gamma_1 \simeq \gamma_2$ **equivalent**, where $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ if there exists a continuously differentiable bijection $\sigma : [a, b] \to [c, d]$ with $\sigma'(t) > 0$[1] such that $\gamma_1(t) = \gamma_2(\sigma(t))$.

Definition 5.4. We will call a piecewise smooth path simply a **curve**.

All equivalent paths yield $C \subseteq \mathbb{C}$ given by the image of γ with an orientation. We also have C traversed in the opposite direction, denote $\bar{C}$, which is determined by the path $\bar{\gamma}(t) = (b + a - t)$.

Definition 5.5. A path is **closed** (sometimes called a **loop**) if $\gamma(a) = \gamma(b)$. It is **simple** if γ is an injective map, i.e. it doesn't self intersect except perhaps at its endpoints.

Example 5.6. We can easily produce a curve using a circle:

$$C_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$$

To give this a parameterization, we can use the exponential:

$$\gamma : [0, 2\pi] \to \mathbb{C} : t \mapsto z_0 + re^{it}$$

γ is said to have **positive orientation** (counterclockwise). To produce its negative counterpart (clockwise), we have naturally

$$\bar{\gamma} : [0, 2\pi] \to \mathbb{C} : t \mapsto z_0 + re^{-it}$$

You can also easily produce piecewise smooth paths by considering polygons (i.e. triangles, squares, etc). Next time, we will setup the familiar idea of a path-integral using these notions.

[1] This condition preserves orientation of γ_1, so that you don't traverse it backwards and yield negative signs.