CLASS 12, FRIDAY MARCH 9TH: PROJECTIVE MODULES

Projective modules play an incredibly important role in commutative algebra, similar to that of vector bundles in algebraic or differential geometry. As we will see, they are closely related to free modules in a precise sense.

Definition 0.1. A module P is said to be **projective** if whenever $\varphi : M \to N$ is a surjective homomorphism, and $\psi : P \to N$ is any homomorphism, there exists $\psi' : P \to M$ such that $\psi = \varphi \circ \psi'$.

Before moving to some equivalent formulations of a projective module, I state an important notes about the Hom_R operation (functor).

Theorem 0.2. *Suppose* $0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$ *is a short exact sequence. Then for a fixed module* N*, we get exact sequences*

$$0 \to \mathrm{Hom}_R(N, M') \to \mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, M'')$$

$$0 \to \mathrm{Hom}_R(M'', N) \to \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M', N)$$

Proof. First, note that if $\varphi : A \to B$ is a map of R-modules, then we get the following homomorphisms for free via composition:

$$\varphi_* : Hom(N, A) \to \mathrm{Hom}(N, B) : \psi \mapsto \varphi \circ \psi$$

$$\varphi^* : \mathrm{Hom}(B, N) \to \mathrm{Hom}(A, N) : \psi \mapsto \psi \circ \varphi$$

So in each case, we take these R-module homomorphisms to be our maps in the asserted SESs. It goes to show exactness.

Suppose $\psi \in \mathrm{Hom}_R(N, M')$ maps to to $0 \in \mathrm{Hom}_R(N, M)$. Then $\varphi(\psi(n)) = 0$ for every $n \in N$. But φ was assumed injective, so $\psi(n) = 0$ which is to say $\psi = 0$. Therefore, the first map is injective.

The only things left to show for the first SES is that it is exact at $\mathrm{Hom}_R(N, M)$, or

$$\mathrm{im}(\mathrm{Hom}_R(N, M') \to \mathrm{Hom}_R(N, M)) = \ker\left(\mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(N, M'')\right)$$

I denote these submodules im and ker for sake of brevity. Note im $\subseteq$ ker, since

$$\psi(\varphi(\xi(m)) = (\psi \circ \varphi)(\xi(m)) = 0(\xi(m)) = 0.$$

In addition, suppose $\xi \in \ker$. Then $\psi(\xi(n)) = 0$ for all $n \in N$. For each n, we know that $\xi(n) \in \ker(\psi) = \mathrm{im}(\varphi)$. Therefore, we can see that $\xi(n) = \varphi(m')$ for some $m' \in M'$. Define $\xi' : N \to M' : n \mapsto m'$. This is well defined by the previous argument, and a homomorphism because ξ, φ are. Therefore, $\xi' \mapsto \xi \in \ker$, which completes the proof.

The case of the second SES is left as an exercise. □

What I have just proved is that $\mathrm{Hom}_R(N, -)$ and $\mathrm{Hom}_R(-, N)$ are **left exact**, e.g. they take short exact sequences to left exact sequences. We note that the Hom-map on the right is not surjective in general. Here is an example:

Example 0.3. There is a natural SES of $\mathbb{Z}$-modules

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

If we apply $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, -)$ to this sequence, we get

$$0 \to 0 \to 0 \to \mathbb{Z}/n\mathbb{Z} \to 0$$

Of course, there is no way $\mathbb{Z}/n\mathbb{Z}$ can be surjected on by 0.

However, for projective modules P, this is an exact sequence. This can be seen in the following characterization:

Proposition 0.4. *The following conditions are equivalent:*

1) P is a projective R-module.
2) $\mathrm{Hom}_R(P, -)$ is **exact***: Applying it maintains exactness of a SES.*
3) Every exact sequence

$$0 \to A \to B \to P \to 0$$

is a split exact sequence.
4) P is a direct summand of a free module: $F = R^\Lambda \cong P \oplus M$.

Proof. 1) $\Leftrightarrow$ 2): It goes to show exactness given left exactness, which equates to showing $\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, M'')$ is surjective. P being projective states that every $\xi : P \to M''$ factors as $\psi \circ \xi' : P \to M \to M''$. Therefore, $\xi' \mapsto \xi$ and surjectivity ensues.

Similarly, if $\mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, M'')$ is surjective, we can simply take $\xi' \mapsto \xi$ to show projectivity.

1) $\Rightarrow$ 3): By Homework 3, or Class 11 Definition 0.6, the sequence is split exact if there exists $P \to B$ composing with the given map to the identity on P. Take the identity map on P. By projectivity of P, this map lifts to the desired splitting map $P \to B$.

3) $\Rightarrow$ 4): As we have talked about, any module has a surjection from a free module $F = R^\Lambda$. This fits into a SES

$$0 \to \ker(\psi) \to F \xrightarrow{\psi} P \to 0$$

Being split exact, again by Homework 3, a split exact sequence implies that $F = R^\Lambda \cong P \oplus \ker(\psi)$.

4) $\Rightarrow$ 1): If $R^\Lambda = P \oplus M$, and $M \to M''$ is a surjection, then R^Λ is certainly projective. Therefore, given a map $P \to M''$, we can precompose with $F \to P$ to get a map $F \to M$. This yields

$$P \to R^\Lambda \to M$$

which maps to the original map $P \to M''$. □

Notice the following fact about Hom_R:

Lemma 0.5. *For modules M_λ, N, $\lambda \in \Lambda$, we have isomorphisms*

$$\mathrm{Hom}_R\Big(\bigoplus_{\lambda\in\Lambda} M_\lambda, N\Big) \cong \prod_{\lambda} \mathrm{Hom}_R(M_\lambda, N)$$

$$\mathrm{Hom}_R\Big(N, \bigoplus_{\lambda\in\Lambda} M_\lambda\Big) \cong \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_R(N, M_\lambda)$$

As a corollary, we get the following nice statement:

Corollary 0.6. *If P_λ are projective modules, so is $\oplus P_\lambda$.*