HOMEWORK 2: THE COMPLEX PLANE
DUE: WEDNESDAY, SEPTEMBER 25TH

1) Prove the complex version of the chain rule: if $f : U \to V$ and $g : V \to \mathbb{C}$ are two differentiable functions, and $h = g \circ f$
$$\frac{\partial h}{\partial z} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial z} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial z}$$
$$\frac{\partial h}{\partial \bar{z}} = \frac{\partial g}{\partial z}\frac{\partial f}{\partial \bar{z}} + \frac{\partial g}{\partial \bar{z}}\frac{\partial \bar{f}}{\partial \bar{z}}$$
(**hint:** It is best to consider f, g, h as functions of $z, \bar{z}$ instead of $x + iy$).

2) If $f : \Omega \to \mathbb{C}$ is holomorphic, then assuming any of the following conditions one can conclude f is constant:
   i. $Re(f)$ is constant.
   ii. $Im(f)$ is constant.
   iii. $|f|$ is constant.

3) Verify the Euler relations for $\sin(z)$ and $\cos(z)$:
$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i} \qquad\qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

4) Determine (and prove) the radii of convergence for the following power series:
   i. $\sum_{n=1}^{\infty} (\log(n))^2 z^n$
   ii. $\sum_{n=0}^{\infty} (n!) z^n$
   iii. $\sum_{n=0}^{\infty} \left(\frac{n^2}{4^n+3n}\right) z^n$
   iv. $\sum_{n=0}^{\infty} \left(\frac{(n!)^3}{(3n)!}\right) z^n$

   For iv. it may be helpful to use Sterling's Formula: $n! \sim cn^{n+\frac{1}{2}}e^{-n}$ for some constant $c > 0$

5) Verify that our notion of 2 parameterized curves being equivalent forms an **equivalence relation**. There are 2 statements here: if $\gamma_1 \simeq \gamma_2$, then $\gamma_2 \simeq \gamma_1$. Additionally, show that if $\gamma_2 \simeq \gamma_3$, then $\gamma_1 \simeq \gamma_3$.

6) Suppose $|a| < r < |b|$, and let C be the circle of radius r. Show that
$$\int_C \frac{1}{(z-a)(z-b)} dz = \frac{2\pi i}{a-b}$$

7) Consider the real valued function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x^2}} & x > 0 \end{cases}$$
Show that this function is infinitely differentiable, but the n^{th}-derivative $f^{(n))}(0) = 0$ for every n. Conclude there is no power series for f at $x = 0$.