CLASS 34, MAY 10TH: NORMALS ARE INTERSECTIONS OF DVRS

For the final day, I want to give an example of a non-normal surface with isolated singularities, and then prove all normal Noetherian domains are intersections of DVRs.

Example 34.1. Consider the subring of $K[x, y]$ defined by

$$R = \{f \in K[x, y] \mid f(0, 0) = f(0, 1)\}$$

This can be thought of as taking the affine plane $\mathbb{A}^2_K$ and gluing 2 points together. Note that if we take $\mathrm{Frac}(R)$, it is equal to $\mathrm{Frac}(K[x, y]) = K(x, y)$; note that $x, xy \in R$. As a result, we have $\frac{xy}{x} = y \in \mathrm{Frac}(R)$.

Now, I claim that $y \in \tilde{R}$, meaning $\tilde{R} = K[x, y] \neq R$, implying R is not normal. y satisfies

$$t^2 - t + y(y - 1) = 0$$

Notice that $y(y - 1) \in R$!

R is an example of a non-Normal domain with localizations at minimal non-zero primes DVRs. Therefore it lacks the S_2-condition mentioned last class.

To conclude the course, we prove the following theorem:

Theorem 34.2. *If R is a normal Noetherian integral domain, then*

$$R = \bigcap_{0 \neq \mathfrak{p} \text{ minimal}} R_{\mathfrak{p}}$$

In particular, R is an intersection of DVRs.

Proof. We already know that

$$R = \bigcap_{\mathfrak{p} \in Spec(R)} R_{\mathfrak{p}} \subseteq \bigcap_{0 \neq \mathfrak{p} \text{ minimal}} R_{\mathfrak{p}}$$

So it suffices to check $\supseteq$. Let $x = \frac{r}{s} \in K = \mathrm{Frac}(R)$. Again, let

$$D(x) = \{d \in R \mid dx \in R\}$$

We can note that $D(x) = \{d \in R \mid dr \in \langle s \rangle\} = \langle s \rangle : r = \mathrm{Ann}(\bar{r})$, where we view $\bar{r} \in A/\langle s \rangle$. Suppose $x \notin R$, which is to say $\bar{r} \neq 0$.

I claim $D(x) \subseteq \mathfrak{p} \in \mathrm{Ass}(R/\langle s \rangle)$. $D(x) = \mathrm{Ann}(\bar{r})$, so since R is Noetherian we have $D(x)$ is contained in some maximal element of this form, which is thus associated. By Theorem 33.5, we have any $\mathfrak{p} \in \mathrm{Ass}(R/\langle s \rangle)$ is a minimal non-zero prime. Therefore, $x \notin R_{\mathfrak{p}}$. This completes the proof of $\supseteq$. $\square$