CLASS 8, MONDAY FEBRUARY 26TH: THE STRUCTURE OF A MODULE: GENERATION AND FREE MODULES

Last time we talked about the sum of 2 submodules $N + N' \subseteq M$. This can be used to establish the idea of generation for modules in an identical way to that of ideals.

Definition 0.1.
- If $N_\lambda \subseteq M$ is a submodule for each λ in an indexing set Λ, then
$$\sum_{\lambda \in \Lambda} N_\lambda = \{n_{\lambda_1} + \ldots + n_{\lambda_m} \mid n_{\lambda_i} \in N_{\lambda_i}\}$$
That is to say the **sum of modules** consists of a finite sum of elements from each. If Λ is a finite indexing set, it is often written as $N_1 + \ldots + N_m$
- If $n \in N$, we let $\langle n \rangle_N$ be the smallest submodule of N containing n. It consists precisely of elements $r \cdot n$ for $r \in R$. We can further write
$$\langle S \rangle = \sum_{s \in S} \langle s \rangle_N$$
for any subset $S \subseteq N$.
- We say a module is **generated** by a subset $S \subseteq M$ if $M = \langle S \rangle$.
- We say M is **finitely generated** if S can be assumed to be a finite set.
- We say M is **cyclic** if S can be assumed to be 1 element.
- If M is finitely generated, we call S a **minimal generating set** if there exists no generating set of smaller cardinality.

Finitely generated modules over Noetherian rings are one of the most well studied objects in commutative algebra.

Example 0.2 (Non-finitely generated modules)**.** Consider $\mathbb{Q}$ is a $\mathbb{Z}$-module. It is fairly easy to see that this is a non-finitely generated $\mathbb{Z}$-module. In particular, if $\mathbb{Q} = \langle \frac{a_1}{b_1}, \ldots, \frac{a_n}{b_n} \rangle$, we can choose a rational number smaller than $|\frac{1}{b_1 \cdots b_n}|$. This number cannot be represented as a sum with integer coefficients.

Additionally, $K[x]$ as a K-module is an infinite dimensional vector space (with basis $1, x, x^2, \ldots$). Therefore it cannot be finitely generated, or it would be a finite dimensional vector space.

Example 0.3 (Many cyclic modules)**.** R viewed as an R-module is a cyclic module, generated by 1. The same holds for R/I for an ideal I, so these are all examples of cyclic modules.

Example 0.4 (Finitely generated modules)**.** Let $R = S = K[x]$. Consider the map $R \to S : x \mapsto x^n$ with K fixed. Then S is a non-cyclic but finitely generated R-module. It has a (minimal) generating set given by $\langle 1, x, \ldots, x^{n-1} \rangle$.

Next up, we can consider the operation of $\oplus$, called the **direct sum**.

Definition 0.5. For 2 modules M, N, we define
$$M \oplus N = \{(m, n) \mid m \in M, n \in N\}$$

where addition and multiplication are defined by $r(m,n) = (rm, rn)$ and $(m,n)+(m',n') = (m+m', n+n')$. We can perform this operation inductively to produce a finite direct sum of modules $M_1 \oplus M_2 \oplus \ldots \oplus M_n$.

There is also a notion of an infinite direct sum, where we consider infinite tuples of elements of each module, but require that all but finitely many of them are 0:

$$\oplus_{\lambda\in\Lambda} M_\lambda = \{(m_\lambda)_{\lambda\in\Lambda} \mid m_\lambda = 0 \text{ for almost all } \lambda \in \Lambda\}$$

This differs from the notion of the **Direct Product**, for which no such restriction is put on almost all m_λ. They are however identical in the case of a finite indexing set.

Definition 0.6. A module F is said to be **free** if $F \cong R^{\oplus\Lambda}$ for some indexing set Λ. If Λ is a finite set, we define the **rank** of F is $\text{rank}(F) = |\Lambda|$.

The rank of a free module is the same as the rank/dimension of a vector space.

One can view minimal generation in terms of free modules. Say M is a module generated minimally by the set $S = \{m_1, \ldots, m_n\}$. We can then consider the homomorphism

$$g : F = R^n \to M : (r_1, \ldots, r_n) \mapsto r_1 m_1 + \ldots + r_n m_n$$

This map is surjective by definition of generation! The kernel of this map can be thought of as an **obstruction** to being free. That is to say $\ker(g) = 0$ if and only if M is free, and larger kernels can be thought of as 'less free' modules.

Aside (Homology)**.** *This produces the idea of the* **Homology** *of a module. Because we can surject onto any module M by a free module F_0, we can form a* **free resolution** *of M by surjecting onto the kernel of the map by a free module F_1, and continue in this fashion:*

$$\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$

The propogation of kernels allows one to measure the complexity of the module. We may return to Homological Algebra later on.

On the opposite end of the spectrum, we have a notion of torsion modules:

Definition 0.7. A module M is said to be **torsion** if for each $m \in M$ there exists a non-zero divisor $r \in R$ (depending on m) such that $r \cdot m = 0$.

Example 0.8 ($\mathbb{Z}$)**.** Any finite $\mathbb{Z}$-module M is a finite Abelian group (as discussed in Class 6). Therefore, if $|M| = n$, we know that $n \cdot M = 0$. Therefore, M is a torsion module!

There also exist infinite torsion groups. Let p_i be the i^{th} prime number. Then

$$M = \oplus_{i=1}^{\infty} \mathbb{Z}/p_i\mathbb{Z}$$

is an infinitely generated (thus infinite) torsion module.

This is part of a much larger theorem, that I will state without proof:

Theorem 0.9 (Finitely Generated Modules over a PID)**.** *Let R be a principal ideal domain (every ideal is principal). Then if M is a finitely generated module,*

$$M \cong F \oplus T$$

where F is a free module and T is a torsion module.

This is not the case if R is not a PID ($R = K[x,y]$, $M = \langle x^2 + y^3, x^4 - y^2\rangle$) or if M is infinitely generated ($R = \mathbb{Z}$, $M = \mathbb{Q}$).