HOMEWORK 8: FOURIER TRANSFORMS
DUE: WEDNESDAY, NOVEMBER 13TH

(1) We will prove the following: If f is continuous, of moderate descent, and $\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R}$, then $f = 0$.
  - For each t, consider

$$A(z) = \int_{-\infty}^{t} f(x)e^{-2\pi i z(x-t)}dx \qquad\qquad B(z) = -\int_{t}^{\infty} f(x)e^{-2\pi i z(x-t)}dx$$

    Show $A(\xi) = B(\xi)$ for each $\xi \in \mathbb{R}$.
  - Show that F which is equal to A in the upper half plane and B in the lower half plane is entire and bounded. Deduce that $F = 0$.
  - Show that

$$\int_{-\infty}^{t} f(x)dx = 0$$

    for all t, and thus $f = 0$ by continuity.

(2) Show that if $f \in \mathcal{F}_a$, then $f^{(n)} \in \mathcal{F}_b$ for any $0 \leq b < a$.

(3) If $a > 0$ and $\xi \in \mathbb{R}$, show using contour integration that

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{a}{a^2 + x^2}e^{-2\pi i x\xi}dx = e^{-2\pi a|\xi|}$$

Deduce that

$$\int_{-\infty}^{\infty} e^{-2\pi a|\xi|}e^{2\pi i\xi x}d\xi = \frac{1}{\pi}\frac{a}{a^2 + x^2}$$

(4) If P is a polynomial of degree ≥ 2 with simple non-real roots, calculate

$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{P(x)}dx$$

for $\xi \in \mathbb{R}$ in terms of its roots. What if the roots are higher order?
  (**hint:** The cases of positive, negative, and 0 ξ should be treated separately.)

(5) Use the Poisson summation formula to establish the following identities:
  - let $Im(\tau) > 0$. Using $f(z) = (\tau + z)^{-k}$ for $k \geq 2$, show

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau + n)^k} = \frac{(-2\pi i)^k}{(k-1)!}\sum_{m=1}^{\infty} m^{k-1}e^{2\pi i m\tau}$$

  - If $Im(\tau) > 0$, then show

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\tau + n)^2} = \frac{\pi^2}{\sin^2(\pi\tau)}$$

  - Does the previous hold for any non-integer $\tau \in \mathbb{C}$?