CLASS 32, DECEMBER 2ND: AUTOMORPHISMS OF $\mathbb{D}$ AND $\mathbb{H}$

Recall that we left off talking about conformal mappings; holomorphic and invertible maps from one open set to another. We then conquered the Schwarz lemma, which gave us a fair amount of information about mappings $\mathbb{D} \to \mathbb{D}$. One such holomorphic map, proven last time to be its own inverse, was

$$\psi_\alpha : \mathbb{D} \to \mathbb{D} : z \mapsto \frac{\alpha - z}{1 - \bar{\alpha} z}$$

The so called **Blaschke maps**. This produces the following classification theorem.

Theorem 32.1. *If $f : \mathbb{D} \to \mathbb{D}$ is an automorphism, then there exists $\theta \in \mathbb{R}$ and $\alpha \in \mathbb{D}$ such that*

$$f(z) = e^{i\theta}\psi_\alpha(z)$$

That is to say it is a Blaschke followed by a rotation.

Proof. Let $\alpha \in \mathbb{D}$ be such that $f(\alpha) = 0$. Now consider $g(z) = f(\psi_\alpha(z))$. $g(0) = 0$, and by the Scwarz lemma, we have $|g(z)| \leq |z|$. Moreover, the same is true for the inverse: $|g^{-1}(w)| \leq |w|$. Applying this to the point $w = g(z)$ shows that $|z| \leq |g(z)|$, so we have that

$$|g(z)| = |z| \qquad\qquad \forall z \in \mathbb{D}$$

So by the second bullet of the Schwarz Lemma, we also have that $g(z)$ is a rotation: $g(z) = e^{i\theta}$ for some θ. But this shows that

$$g = f \circ \psi_\alpha \qquad \implies \qquad g \circ \psi_\alpha = f \circ \psi_\alpha^2 = f$$

So $f(z) = e^{i\theta}\psi_\alpha(z)$. □

Corollary 32.2. *The only automorphisms of the disc fixing the origin are rotations.*

It is also geometrically relevant that automorphisms of the disc are transitive; we can always find one mapping $\alpha \in \mathbb{D}$ to $\beta \in \mathbb{D}$. A simple such example is $\psi = \psi_\beta \circ \psi_\alpha$:

$$\psi_\beta(\psi_\alpha(\alpha)) = \psi_\beta(0) = \beta$$

Additionally, we can view these maps as a subset of the group of 2×2 matrices of non-zero determinant. Note that ψ_α is a special case of a fractional linear transformation: $\psi(z) = \frac{az+b}{cz+d}$. We can associate to this the matrix

$$\psi \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The coolest thing about this is that composition is then transferred to a product of matrices: if $\varphi(z) = \frac{ez+f}{gz+h}$, then

$$\psi \circ \varphi(z) = \frac{a(ez+f) + b(gz+h)}{c(ez+f) + d(gz+h)} = \frac{(ae+bg)z + (af+bh)}{(ce+dg)z + (bf+dh)} \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

Notice that the determinant of such a matrix is 0 if and only if the rows are linearly dependent. This is to say that

$$\frac{az+b}{cz+d} = \frac{az+b}{c'(az+b)} = \frac{1}{c'}$$

which is certainly not a bijective map, since it is constant! If we further note that matrices which are constant multiples of each other yield equivalent maps, we get a bijection between these fractional linear transformations and $\mathbb{P}SL_2(\mathbb{C})$. Now, our class of maps ψ_α are the subset

$$\psi_\alpha \rightsquigarrow \begin{pmatrix} -1 & \alpha \\ -\bar{\alpha} & 1 \end{pmatrix}$$

The collection of such matrices is called $SU(1,1)$, the special unitary group. These are precisely matrices Λ which preserve the hermitian inner product on $\mathbb{C}^2$:

$$\langle (z_1, z_2), (w_1, w_2) \rangle = z_1\bar{w}_1 - z_2\bar{w}_2 = \langle \Lambda(z_1, z_2), \Lambda(w_1, w_2) \rangle$$

We now move to the similar question for $\mathbb{H}$, namely what are the automorphisms of $\mathbb{H}$? Fortunately, since we know that $\mathbb{D}$ and $\mathbb{H}$ are conformal, we can use our previous work! Recall that

$$F : \mathbb{H} \to \mathbb{D} : z \mapsto \frac{i - z}{i + z} \qquad\qquad G : \mathbb{D} \to \mathbb{H} : w \mapsto i\frac{1 - w}{1 + w}$$

are conformal inverses to one another (Theorem 30.4). But this allows us to convert an automorphism of $\mathbb{H}$ to one of $\mathbb{D}$!

$$\begin{tikzcd}
\mathbb{D} \arrow[r, dotted, "e^{i\theta}\cdot\psi_\alpha"] \arrow[d, "G"] & \mathbb{D} \arrow[r, dotted] \arrow[d, shift left, "G"] & \mathbb{D} \\
\mathbb{H} \arrow[r, "\xi"] & \mathbb{H} \arrow[r, "\xi'"] \arrow[u, shift left, "F"] & \mathbb{H} \arrow[u, "F"']
\end{tikzcd}$$

Given an automorphism ξ of $\mathbb{H}$, we can create one of $\mathbb{D}$ by $F \circ \xi \circ G$. But we know what such mappings are; they have the form $e^{i\theta} \cdot \psi_\alpha$. Thus we can recover ξ by the formula

$$\xi = G \circ (e^{i\theta} \cdot \psi_\alpha) \circ F$$

Moreover, this *conjugation* operation Γ behaves well for compositions:

$$\Gamma(\xi' \circ \xi) = G \circ (\xi' \circ \xi) \circ F = (G \circ \xi' \circ F) \circ (G \circ \xi \circ F) = \Gamma(\xi') \circ \Gamma(\xi)$$

In total, this allows us to conclude the following result:

Theorem 32.3. *Every automorphism of $\mathbb{H}$ takes the form*

$$\psi : \mathbb{H} \to \mathbb{H} : z \mapsto \frac{az + b}{cz + d} \rightsquigarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = M$$

where $M \in SL_2(\mathbb{R})$, the matrices of determinant 1 with real coefficients.

Proof. Suppose $\psi \in \mathrm{Aut}(\mathbb{H})$, and that $\psi(\beta) = i$. Choose $N \in SL_2(\mathbb{R})$ such that $N(i) = \beta$. Then we have that

$$F \circ (\psi \circ N) \circ G : \mathbb{D} \to \mathbb{D} : 0 \mapsto 0$$

As a result, we know this map is a rotation by some angle, say -2θ. Expressed as a real matrix, this is

$$F \circ (\psi \circ N) \circ G = F \circ \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \circ G$$

but by inverting F and G, we get that

$$(\psi \circ N) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \qquad \implies \qquad \psi = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \cdot N^{-1} \in SL_2(\mathbb{R})$$

$\square$