HOMEWORK 3: MODULES, TENSORS, AND LOCALIZATION
DUE: FRIDAY, MARCH 16TH

1) Show that $\mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Solution: The first isomorphism is true generally when tensoring by R over R: $\mathbb{Q} \to \mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} : a \mapsto a \otimes 1$ and $a \otimes b = ab \otimes 1 \mapsto ab$.

Therefore, it is sufficient to show $\mathbb{Q} \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$. Consider again the map $a \mapsto a \otimes 1$. This is an injective homomorphism, since the tensor product plays well with sums. It only goes to show that it is surjective, which in turn we only need to show $\frac{a}{b} \otimes \frac{c}{d}$ is in the image. Note that

$$\frac{a}{b} \otimes \frac{c}{d} = d \cdot \frac{a}{bd} \otimes \frac{c}{d} = \frac{a}{bd} \otimes c = \frac{ac}{bd} \otimes 1$$

Therefore, the map is surjective. This completes the proof.

2) If $\mathfrak{q}$ is a minimal prime ideal of a reduced ($\sqrt{0} = 0$) Noetherian ring R show that $R_{\mathfrak{q}}$ is a field. (**Hint:** A minimal prime ideal is a prime ideal not contained in any other prime ideal. In a Noetherian ring, a minimal prime is composed entirely of zero divisors. You may assume this. What does the localization set to 0?)

Solution: First note that a ring S is a field if and only if it is a commutative domain with a unique prime ideal. Commutative comes for free, and by the 4th isomorphism theorem

$$\{\text{prime ideals of } R_{\mathfrak{q}}\} \leftrightarrow \{\text{prime ideals of } R \text{ contained in } \mathfrak{q}\}$$

Since $\mathfrak{q}$ is assumed to be minimal, the only prime ideal containing $\mathfrak{q}$ is $\mathfrak{q}$ itself. Therefore, $\mathfrak{q}R_{\mathfrak{q}}$ is the unique prime ideal of $R_{\mathfrak{q}}$. So it only goes to show it is a domain. As demonstrated in class, when localizing at a multiplicative set containing 0-divisors a, if $a \cdot b = 0$, then $b = 0$. This is true of every element of $\mathfrak{q}$. Therefore, $\mathfrak{q}R_{\mathfrak{q}} = 0$, and thus $R_{\mathfrak{q}}$ is a domain. This completes the proof.

3) Show that if I, J are ideals of R, then $R/I \otimes_R R/J \cong R/(I + J)$.

Solution: By the Proposition proved in class,

$$R/I \otimes_R R/J \cong (R/I)/(J + I/I)$$

By the 2nd isomorphism theorem, this is isomorphic to $R/I + J$.

Alternatively, we can construct the map explicitly:

$$R \to R/I \otimes_R R/J : r \mapsto r \otimes 1 = 1 \otimes r$$

This is naturally a homomorphism of R-modules. In addition, it is surjective, since $rr' \mapsto rr' \otimes 1 = r \otimes r'$ for any $r, r' \in R$. Therefore, it suffices to compute the kernel by the 1st module isomorphism theorem. If $r = i + j \in I + J$, we see that

$$r \mapsto r \otimes 1 = i \otimes 1 + j \otimes 1 = i \otimes 1 + 1 \otimes j = 0 + 0 = 0$$

Similarly, if $r \notin I+J$, one can see that $r \mapsto r \otimes 1 \neq 0$, since the product of quotient maps $R/I \times R/J \to R/(I+J)$ is non-zero on $(r+I, r+J)$.

4) Show that the following conditions are equivalent for a short exact sequence

$$0 \to M' \xrightarrow{\varphi} M \xrightarrow{\psi} M'' \to 0$$

i. $M \cong M' \oplus M''$.
ii. There is a homomorphism $\varphi' : M \to M'$ such that $\varphi' \circ \varphi = Id_{M'}$.
iii. There is a homomorphism $\psi' : M'' \to M$ such that $\psi \circ \psi' = Id_{M''}$.

Solution: If i. is true, we can easily deduce ii. or iii. by the projection or inclusion operators.

Suppose ii. Then I claim $M \to M' \oplus M'' : m \mapsto (\varphi'(m), \psi(m))$ is the desired map. First, note that it is injective since if $m \mapsto (0,0)$, then $\psi(m) = 0$ implies that there exists $m' \in M'$ such that $\varphi(m') = m$. But $m' = \varphi'(\varphi(m')) = \varphi'(m) = 0$. Therefore, m was 0 to begin with. On the other-hand, the map is surjective: Consider $(m', m'') \in M' \oplus M''$. There exists $m \in M$ mapping to m'' under ψ. Consider the element $m + \varphi(m' - \varphi'(m)) \in M$. Under this map, this goes to

$$\begin{aligned}
&(\varphi'(m + \varphi(m' - \varphi'(m))), \psi(m + \varphi(m' - \varphi'(m)))) \\
&= (\varphi'(m) + \varphi'(\varphi(m' + \varphi'(m)))), \psi(m) + \psi(\varphi(m' - \varphi'(m)))) \\
&= (\varphi'(m) + m' - \varphi'(m), \psi(m) + \psi(\varphi(m' - \varphi'(m)))) \\
&= (m', m'')
\end{aligned}$$

Now suppose iii. We can construct a map

$$M' \oplus M'' \to M : (m', m'') \mapsto \varphi(m') + \psi'(m'')$$

This map is surjective: if $m \in M$, then

$$m - \psi'(\psi(m)) \mapsto 0$$

So $\exists m' \in M'$ with $\varphi(m') = m - \psi'(\psi(m))$, or rephrased, $(m', \psi(m)) \mapsto m$. Additionally, suppose that $(m', m'') \mapsto 0$. Then

$$0 = \psi(\varphi(m') + \psi'(m'')) = (\psi \circ \varphi)(m') + \psi \circ \psi'(m'') = m''$$

Additionally, φ is injective, so $m' \mapsto 0$ implies $m' = 0$.

5) Show that if P and P' are projective modules, so is $P \otimes_R P'$.

Solution: By Hom-$\otimes$ adjointness,

$$\mathrm{Hom}_R(P \otimes P', -) = \mathrm{Hom}_R(P, \mathrm{Hom}_R(P', -))$$

Therefore, as functors, the left is a composition of 2 exact functors $\mathrm{Hom}_R(P', -)$ and $\mathrm{Hom}_R(P, -)$.

6) Prove the following lemma from class 12:

Lemma 0.1. *For modules M_λ, N, $\lambda \in \Lambda$, we have isomorphisms*

$$\mathrm{Hom}_R(\bigoplus_{\lambda \in \Lambda} M_\lambda, N) \cong \prod_\lambda \mathrm{Hom}_R(M_\lambda, N)$$

$$\mathrm{Hom}_R(N, \bigoplus_{\lambda \in \Lambda} M_\lambda) \cong \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_R(N, M_\lambda)$$

Solution: One can construct the maps explicitly. Let $q_\lambda : \bigoplus_{\lambda \in \Lambda} M_\lambda \to M_\lambda$ and $i_\lambda : M_\lambda \to \bigoplus_{\lambda \in \Lambda} M_\lambda$ be the quotient and inclusion maps respectively. Then for

$$\mathrm{Hom}_R(\bigoplus_{\lambda \in \Lambda} M_\lambda, N) \cong \prod_\lambda \mathrm{Hom}_R(M_\lambda, N)$$

we have the explicit isomorphism $\psi \mapsto \prod_\lambda \psi \circ i_\lambda$. The inverse of this map is

$$\prod_\lambda \psi_\lambda \mapsto (\psi : m_{\lambda_1} + \ldots + m_{\lambda_n} \mapsto \psi_{\lambda_1}(m_{\lambda_1}) + \ldots + \psi_{\lambda_n}(m_{\lambda_n}))$$

On the other hand, for the isomorphism

$$\mathrm{Hom}_R(N, \bigoplus_{\lambda \in \Lambda} M_\lambda) \cong \bigoplus_{\lambda \in \Lambda} \mathrm{Hom}_R(N, M_\lambda)$$

We similarly take $\psi \mapsto \oplus_\lambda q_\lambda \circ \psi$ with inverse

$$\oplus \psi_\lambda \mapsto \sum_\lambda i_\lambda \circ \psi$$

Note that one is a direct product because it can send each entry to something non-zero (because it takes only finitely many non-zero entries at once). In the second case, it cannot map to a non-zero entry for infinitely many λ, so it is a direct sum.

7) Prove the following universal property of the tensor product:

Theorem 0.2. *The pair $(M \otimes_R N, \otimes : M \times N \to M \otimes_R N)$ satisfies the following: Given a bilinear map $\varphi : M \times N \to P$, then $\exists! \varphi' : M \otimes_R N \to P$ a homomorphism factoring φ. That is to say $\varphi = \varphi' \circ q$. If T is any other module with this factorization property, then T is uniquely isomorphic to $M \otimes_R N$.*

Solution: The desired map is

$$\otimes : M \times N \to M \otimes_R N : (m, n) \mapsto m \otimes n$$

For a given bilinear map $\varphi : M \times N \to P$, we take

$$\varphi' : M \otimes_R N \to P : m \otimes n \to \varphi(m, n)$$

Since φ was bilinear, we have that

$$\varphi(rm, n) = \varphi(m, rn)$$

$$\varphi(m + m', n) = \varphi(m, n) + \varphi(m', n)$$

$$\varphi(m, n + n') = \varphi(m, n) + \varphi(m, n')$$

These exactly give that φ' is a homomorphism of R-modules.

Now suppose that T is another module satisfying this property. Note that $\otimes$ is itself a bilinear map. Therefore, we get a unique map $T \to M \otimes_R N$, and vice-versa. These are inverse to one another and unique by assumption.

8) Show that for a given ring R, every R-module is projective if and only if every R-module is injective.

Solution: $\Rightarrow$: Let M be a module and $I \subseteq R$ an inclusion of ideals. Suppose $I \to M$ is a map. Since R/I is projective, we see that $R \cong I \oplus R/I$. Therefore, we can create the desired map $R \to M$ by

$$R \cong I \oplus R/I \to I \to M$$

where the last map is the original map.

$\Leftarrow$: Let M be a module and $\varphi : N \to Q$ be a surjection. Suppose $M \to Q$ is a map. Since $\ker(\phi)$ is injective, we see that $N \cong \ker(\phi) \oplus Q$. Therefore, we can create the desired map $M \to N$ by

$$M \to Q \hookrightarrow \ker(\phi) \oplus Q \cong N$$

where the first map is the original map. This completes the proof (Note that in this case every sequence splits! They are called **semi-simple Artinian rings**).