CLASS 11, MARCH 1ST: NOETHERIAN RINGS

We already discussed how to generate an ideal by select elements. The notation was

$$\langle A\rangle = \langle f_\alpha\rangle$$

We saw the importance of finite generation in the statement of Nakayama's Lemma. And indeed, it (and some of the corollaries on the homework) are not even true if the module isn't finitely generated.

Example 11.1. Consider the ring $R = \mathbb{F}_p[x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \ldots] = K[x]_{perf}$. This ring has a maximal ideal

$$\mathfrak{m} = \langle x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \ldots\rangle$$

So we can produce a local ring $R_\mathfrak{m}$. Note that $\mathfrak{m} = \mathfrak{m}^2$, as $f \in \mathfrak{m}$ has the form

$$f = \sum_{n=1}^{N} c_n x^{\frac{n}{p^e}} = \sum_{n=1}^{N} c_n x^{\frac{np}{p^{e+1}}} = x^{\frac{1}{p^{e+1}}} \sum_{n=1}^{N} c_n x^{\frac{np-1}{p^{e+1}}}$$

and both sides of the product are in $\mathfrak{m} \neq 0$, violating the assertion of Nakayama.

Definition 11.2. A ring R is called **Noetherian** if every ideal is finitely generated.

This is a fantastic property, named after the great mathematician Emmy Noether. We can already see that Example 11.1 is an example of a non-Noetherian ring. Here some equivalent ways to specify it:

Proposition 11.3. *TFAE:*

1) *R is a Noetherian ring.*
2) *Every ascending chain of ideals of R eventually* **stabilizes***: if*

$$I_1 \subseteq I_2 \subseteq \ldots$$

the $\exists n > 0$ such that $I_n = I_{n+1} = I_{n+2} = \ldots$.

3) *Every non-empty collection of Ideals $\{I_\alpha\}_{\alpha\in\Lambda}$ contains a maximal element. That is to say that there exists $\beta \in \Lambda$ such that there are no $\alpha \in \Lambda$ such that $I_\beta \subsetneq I_\alpha$.*

Proof.

- 1) $\Rightarrow$ 2): Suppose $I_1 \subseteq I_2 \subseteq \ldots$ is an ascending chain of ideals. We know by our proof of existence of maximal ideals that $I = \cup_{i=1}^{\infty} I_i$ is an ideal. By 1), $I = \langle f_1, \ldots, f_n\rangle$ is finitely generated. But this implies $f_i \in I_{j_i}$ for some j_i, since I is a union. As a result,

$$I = I_{\max\{j_1,\ldots,j_n\}}$$

i.e. the chain stabilized.

- 2) $\Rightarrow$ 3): 2) states that every ascending chain of ideals has an upper bound; namely where it stabilizes. As a result, Zorn's lemma implies 3) is true.
- 3) $\Rightarrow$ 1): Let I be an ideal of R. Consider the collection

$$\mathcal{S} = \{J \subseteq I \mid J \text{ an ideal, } J \text{ finitely generated}\}.$$

This set is of course non-empty, since it contains the ideal generated by any single element of I. By 3), we see that $\mathcal{S}$ has a maximal element, say $\mathfrak{m}$ (not a maximal ideal). Suppose $f \in I \setminus \mathfrak{m}$. Then $I + \langle f\rangle \in \mathcal{S}$ is a finitely generated ideal since we only added 1 generator to a finite set. This contradicts maximality of $\mathfrak{m}$, and therefore no such f can exist, i.e. $I = \mathfrak{m}$ is finitely generated.

$\square$

Definition 11.4. Property 2) in Proposition 11.3 is called the **ascending chain property**, or sometimes the **A.C.C**.

The opposite property, called the **descending chain property**, or the **D.C.C.**, states that every descending chain of ideals eventually stabilizes:

$$I_1 \supseteq I_2 \supseteq \ldots \supseteq I_n = I_{n+1} = \ldots$$

A ring with this property is called **Artinian**, for Emil Artin.

We will focus on Noetherian rings, as Artinian is a very restrictive condition that can even be shown to imply Noetherian!

Here is a nice property which allows us to generate many Noetherian rings from known ones.

Proposition 11.5. *If R is a Noetherian ring, and $\varphi : R \to S$ is a surjective map, then S is a Noetherian ring. If R is also a domain, and $\mathfrak{p} \in \operatorname{Spec}(R)$, then $R_{\mathfrak{p}}$ is Noetherian.*

Proof. Let $I_1 \subseteq I_2 \subseteq \ldots$ be an ascending chain of ideals of S. Then

$$\varphi(I_1)^{-1} \subseteq \varphi^{-1}(I_2) \subseteq \ldots$$

is an ascending chain of ideals in R. Therefore it stabilizes. But the correspondence of *ideals* of S to that of R is injective, so the same stabilization occurs for the original chain.

The same proof goes through for localizations as well! $\square$

Example 11.6.

1) Every P.I.D., e.g. $K[x]$ or $\mathbb{Z}$, is Noetherian. This follows by the original definition of Noetherian.
2) $K[x_1, x_2, x_3, \ldots]$ is non-Noetherian, since it has the ascending chain
$$\langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \langle x_1, x_2, x_3 \rangle \subsetneq \ldots$$
3) The ring $R = K[x, xy, xy^2, xy^3, \ldots]$ of Homework 4, #1 is a non-Noetherian ring.
4) Consider the ring $C(\mathbb{R})$ of continuous functions from $\mathbb{R}$ to itself. This is certainly non-Noetherian. Even if we localize, i.e. consider functions which are equivalent near 0, we get a quotient of this ring by a relation $f \sim g$ if and only if $f = g$ in an open neighborhood of $0 \in \mathbb{R}$. In fact it is a local ring with maximal ideal those functions for which $f(0) = 0$.

   I claim it is still non-Noetherian. Indeed, suppose $f_1, \ldots, f_n$ generate the maximal ideal. Note that $g = \sum_{i=1}^{n} a_i f_i$ has the property that $g(x) < C \cdot \max\{|f_i(x)|\}$ as $x \to 0$. There are functions vanishing more slowly. I.e. functions like $G(x) = \sqrt{\max\{|f_i(x)|, |x|\}}$. This would imply
$$\frac{G(x)}{\max\{|f_i(x)|, |x|\}} \to \infty$$

Next week we will cover the Hilbert Basis Theorem, which will give a great deal more examples of Noetherian rings.