HOMEWORK 9: ENTIRE FUNCTIONS
DUE: WEDNESDAY, NOVEMBER 20TH

(1) Find the order of growth of a polynomial $p(z)$, $f(z) = e^{bz^n}$ with $b \neq 0$, and $g(z) = e^{e^z}$.

(2) Show that if τ is fixed with $Im(\tau) > 0$, then the Jacobi function

$$\Theta(z, \tau) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 \tau} e^{2\pi i n z}$$

is of order 2 in z. (**hint:** Notice that $-n^2 t + 2n|z| \leq -\frac{n^2 t}{2}$ for $t > 0$ and $n \geq 4\frac{|z|}{t}$)

(3) For $t > 0$ fixed, consider

$$F(z) = \prod_{n \geq 1} \left(1 - e^{-2\pi n t} e^{2\pi i z}\right)$$

Note that $F(z)$ is entire.
- Show $|F(z)| \leq Ae^{a|z|^2}$, hence F is of order 2.
- $F(z) = 0$ exactly when $z = nit + m$, where $n > 1$ and $n, m \in \mathbb{Z}$. Thus if z_n are its zeroes, then

$$\sum_n \frac{1}{|z_n|^2} = \infty \qquad\qquad \sum_n \frac{1}{|z_n|^{2+\epsilon}} < \infty$$

(4) If $\alpha > 1$, then

$$F_\alpha(z) = \int_{-\infty}^{\infty} e^{-|t|^\alpha} e^{2\pi i z t} dt$$

has order of growth $\frac{\alpha}{\alpha - 1}$. (**hint:** Show that $-\frac{|t|^\alpha}{2} + 2\pi|z||t| \leq c|z|^{\frac{\alpha}{\alpha-1}}$ by consideration of $|t|^{\alpha-1} \leq A|z|$ and $|t|^{\alpha-1} \geq A|z|$ for some $A > 0$)

(5) Establish the following identities:
- If $\sum |a_n|^2$ converges, and $a_n \neq -1$ for any n, then $\prod(1 + a_n)$ converges and is non-zero if and only if $\sum a_n$ converges.
- Find an example for which $\sum a_n$ converges, but $\prod(1 + a_n)$ diverges.
- Find a convergent $\prod(1 + a_n)$ where $\sum a_n$ diverges.