HOMEWORK 2: PRODUCTS AND CONTINUITY
DUE: FRIDAY, SEPTEMBER 21

1) Show that if $A \subseteq X$ and $B \subseteq Y$ are topological subspaces, then $A \times B$ with the product topology is equivalent to $A \times B \subseteq X \times Y$ with the subspace topology. That is to say, a product of subspaces is the subspace of the product.

   Solution: We can go about this by comparing the bases of each topologies. First, for the subspace topology $A \times B \subseteq X \times Y$, we note that $X \times Y$ has a basis given by $U \times V$, where U is open in X and V is open in Y. Thus the basis for $A \times B$ is given by $U \times V \cap A \times B$. But this is simply $(U \cap A) \times (V \cap B)$. And this is *exactly* the basis for the product of subspace topologies.

2) Let a_i be a sequence of positive real numbers, and b_i be a sequence of real numbers. Show that the map

$$f : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}} : (x_1, x_2, \ldots) \mapsto (a_1 x_1 + b_1, a_2 x_2 + b_2, \ldots)$$

   is a homeomorphism in the product topology. Is the same true in the box topology?

   Solution: First, it is necessarily bijective since it has an inverse

$$(y_1, y_2, \ldots) \mapsto (\frac{y_1 - b_1}{a_1}, \frac{y_2 - b_2}{a_2}, \ldots)$$

   This is even of the same form as the original map, so it only suffices to check continuity of the original map to prove it is a homeomorphism.

   For the product topology, an open set of $\mathbb{R}^{\mathbb{N}}$ is of the form (since in the product topology there are only finitely many non-$\mathbb{R}$ opens in the product)

$$U = U_1 \times U_2 \times \ldots \times U_n \times \mathbb{R} \times \mathbb{R} \times \ldots$$

   where $U_i \subseteq \mathbb{R}$ is open. It suffices to check continuity on a basis, since preimage and union commute. Therefore we may assume

$$U = (c_1, d_1) \times (c_2, d_2) \times \ldots \times (c_n, d_n) \times \mathbb{R} \times \mathbb{R} \times \ldots$$

   Since a_i are all positive, we have an increasing linear function. Therefore, the preimage is given by

$$f^{-1}(U) = (\frac{c_1 - b_1}{a_1}, \frac{d_1 - b_1}{a_1}) \times \ldots \times (\frac{c_n - b_n}{a_n}, \frac{d_n - b_n}{a_n}) \times \mathbb{R} \times \ldots$$

   The result is also true if both copies of $\mathbb{R}^{\mathbb{N}}$ are endowed with the box topology by an identical argument.

3) Consider $X = \mathbb{R}^{\mathbb{N}}$, the space of sequences of real numbers with the product topology. Inside it is a subset Y given by all sequences that are eventually zero. What is the closure of Y within X?

   Solution: In class we spoke of the closure as

$$\bar{Y} = ((Y^c)^\circ)^c$$

So it suffices to compute the interior of the complement. Suppose that $a = (a_1, a_2, \ldots) \in (Y^c)^\circ$. By definition, that implies that there exists an open subset U containing a such that $U \subseteq Y^c$. Therefore, given the base of the topology of $\mathbb{R}^{\mathbb{N}}$, we know

$$a \in (b_1, c_1) \times (b_2, c_2) \times \ldots \times (b_n, c_n) \times \mathbb{R} \times \mathbb{R} \times \ldots \subseteq U$$

However, for any n, we can find a sequence $(a_1, a_2, a_3, \ldots, a_n, a_{n+1}, 0, 0, \ldots) \in Y$ and thus also in U. But $U \subseteq Y^c$, which contradicts our assumption that $a \in (Y^c)^\circ$, and proves $(Y^c)^\circ = \emptyset$. Therefore $\bar{Y} = X$.[1]

4) Show that if X, Y are two topological spaces, then for a fixed $y_0 \in Y$, the map

$$i : X \to X \times Y : x \mapsto (x, y_0)$$

is continuous.[2]

Solution: Again, the basis for the topology of the product is given by $U \times V$, U open in X and V open in Y. Therefore, if $y_0 \in V$ we have

$$i^{-1}(U \times V) = U$$

which is open in X, and if $y_0 \notin V$, the preimage is empty. Taking the union of these subsets proves the claim.

5) Suppose that X_α with $\alpha \in \Lambda$ is a collection of subsets of X for which $X = \bigcup_\alpha X_\alpha$. Let $f : X \to Y$ be a map such that $f|_{A_\alpha} : A_\alpha \to Y$ is continuous for each α.
   - Suppose that A_α are closed and Λ is a finite set. Show f is continuous.
   - Show the same is not true if we relax the finiteness of Λ.
   - We call the A_α a **locally finite** collection if for any $x \in X$, there is a neighborhood U_x of x such that are only finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n$ for which $U \cap A_{\alpha_i} \neq \emptyset$. Show that if A_α are closed and locally finite, then f is continuous.

Solution:

1) It suffices to show that the preimage of a closed set is closed. Let $Z \subseteq Y$ be a closed set, and $\Lambda = \{1, 2, \ldots, n\}$ (by renaming some elements). Since $f|_{X_i}$ is a continuous function, we have that

$$f|_{X_i}^{-1}(Z) = f^{-1}(Z) \cap X_i$$

is a closed subset of X_i. But in the subspace topology, this just means that there is a closed subset A_i of X such that

$$f|_{X_i}^{-1}(Z) = A_i \cap X_i$$

We can take the union of this to produce the whole picture for X:

$$f^{-1}(Z) = f^{-1}(Z) \cap X = f^{-1}(Z) \cap \bigcup_{i=1}^{n} X_i = \bigcup_{i=1}^{n} \left(f^{-1}(Z) \cap X_i\right) = \bigcup_{i=1}^{n} A_i \cap X_i$$

which is closed. This completes the proof.

[1] This formalizes that topologically infinite sequences can be approached by finite sequences.
[2] An injective continuous map is called an embedding.

2) Take $Id : \mathbb{Q}_{fc} \to \mathbb{Q}_d$, where the fc and d indicate respectively the finite complement and discrete topologies. This map is definitely not continuous since the discrete topology has $\{0\}$ as an open subset, but finite complement does not. However, $\mathbb{Q}$ is countable, so we can take each X_i to be a 1-point set. This is closed, since it's complement has finite complement! And of course, any constant map is continuous, so $Id|_{X_i}$ is continuous.
3) We may assume these neighborhoods of x are open, since they must contain an open set. Let U_x be the neighborhood of x, and let $\alpha_1, \ldots, \alpha_n$ be the indices which intersect U_x. By part 1), we have $f|_{X_1 \cup \ldots \cup X_n}$ is continuous (finite unions of closed subsets are closed). But $U_x \subseteq X_1 \cup \ldots \cup X_n$, so $f|_{U_x}$ is also continuous. Finally, if $V \subseteq Y$ is an open set, then $f|_{U_x}^{-1}(V) = f^{-1}(V) \cap U_x$ is open in X. Therefore $f^{-1}(V)$ is open in X:

$$f^{-1}(V) = \bigcup_{x \in X} f|_{U_x}^{-1}(V)$$

6) Find a function $f : \mathbb{R} \to \mathbb{R}$ (Euclidean topologies) continuous at only a single point.

Solution: The standard example is $f : \mathbb{R} \to \mathbb{R}$ defined piecewise as

$$f(x) = \begin{cases} -x & x \in \mathbb{Q} \\ x & x \notin \mathbb{Q} \end{cases}$$

We see that $f^{-1}(B(0,r)) = B(0,r)$ for any $r > 0$. However, for any $x \neq 0$ we get

$$f^{-1}(B(x, |\frac{x}{2}|)) = \left(\mathbb{Q} \cap B(-x, |\frac{x}{2}|)\right) \cup \left((\mathbb{R} \setminus \mathbb{Q}) \cap B(-x, |\frac{x}{2}|)\right)$$

which is not open.