CLASS 18, APRIL 3RD: GOING UP!

Today we will study how primes behave in integral extensions. We have already seen a case of this subtly introduced before the break:

Proposition 1. *Let $A \subseteq B$ be an integral extension of integral domains. Then*

$$A \text{ is a field} \iff B \text{ is a field}$$

In particular, it is stating that if 0 is the only prime of B, then 0 is also the only prime of A! This goes far deeper.

Theorem 18.1 (Going Up Theorem)**.** *If $A \subseteq B$ is an integral extension of rings, and $\mathfrak{p} \in \operatorname{Spec}(A)$ is a prime ideal of A, then there exists a prime ideal $\mathfrak{q} \in \operatorname{Spec}(B)$ such that $\mathfrak{q} \cap A = \mathfrak{p}$. Furthermore, $\mathfrak{q}$ can be chosen to contain any prime ideal $\mathfrak{q}' \in \operatorname{Spec}(B)$ such that $\mathfrak{q}' \cap A \subseteq \mathfrak{p}$.*

Sometimes the first sentence of this result is known as the **Lying Over Theorem**, and the latter sentence is called **Going Up**. This will be explained in slightly more detail later in the corollaries.

Proof. Given $\mathfrak{q}' \in \operatorname{Spec}(B)$ such that $\mathfrak{q}' \cap A \subseteq \mathfrak{p}$, we can consider instead the integral extension

$$A/\mathfrak{q}' \cap A \subseteq B/\mathfrak{q}'$$

Therefore, without loss of generality we may assume $\mathfrak{q}' = 0$. Relabel A and B as these rings. Let $W = A \setminus \mathfrak{p}$. Then we can consider the localization

$$A_{\mathfrak{p}} = W^{-1}A \subseteq W^{-1}B$$

This allows us to assume A is a local ring with maximal ideal $\mathfrak{p}$. Again replace A and B with $A_{\mathfrak{p}}$ and $W^{-1}B$ respectively.

Given a maximal ideal $\mathfrak{m}$ of B that contains $\mathfrak{p} \cdot B$, it necessarily has the property that $\mathfrak{p} = \mathfrak{m} \cap A$. Therefore, it only suffices to check that $\mathfrak{p} \cdot B \neq B$. If it were equal, then 1 can be written as an B-linear combination of elements of $\mathfrak{p}$:

$$1 = b_1p_1 + \cdots + b_np_n \qquad\qquad b_i \in B,\ p_i \in \mathfrak{p}$$

Let $B' = A[b_1, \ldots, b_n] \subseteq B$. Then $1 \in \mathfrak{p} \cdot B'$ by the previous equality. But B' is a finitely generated A module! So by Nakayama's Lemma, we have that since $B' = \mathfrak{p}B'$, that $B' = 0$. This is impossible since B' contains A which we assumed had a maximal ideal (i.e. is non-zero). □

Corollary 18.2. *If $\iota : A \hookrightarrow B$ is an inclusion which is an integral extension of rings, then the map on* Spec *is a surjection:*

$$\iota^{\#} : \operatorname{Spec}(B) \to \operatorname{Spec}(A) : \mathfrak{q} \mapsto \mathfrak{q} \cap A$$

This is simply put the *lying over* part of Theorem 18.1. We can also do an induction argument to produce a nice statement about ascending chains of ideals:

Corollary 18.3. *Let $A \subseteq B$ be an integral extension of rings. If $\mathfrak{p}_0 \subseteq \mathfrak{p}_1 \subseteq \ldots \subseteq \mathfrak{p}_n$ is an ascending chain of prime ideals in* $\mathrm{Spec}(A)$*, then there exists a corresponding chain of ideals $\mathfrak{q}_0 \subseteq \mathfrak{q}_1 \subseteq \ldots \subseteq \mathfrak{q}_n$ in* $\mathrm{Spec}(B)$ *such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$.*

This result is very important regarding an invariant called dimension of a ring. Reinterpreted, this corollary states that dimension can't drop in an integral extension of rings. There is a more complicated theorem as well called the **Going Down Theorem**. This is currently beyond our scope, but I encourage the aspiring commutative algebraist to at least know the statement.

Example 18.4. Consider the integral extension of rings discussed previously: $\mathbb{Z} \subseteq \mathbb{Z}[\tau]$, where $\tau = \frac{1+\sqrt{5}}{2}$ is the so-called golden ratio[1]. What the going up theorem tells us is that every prime ideal $\mathfrak{p} = \langle p \rangle$ of $\mathbb{Z}$ has a corresponding prime ideal in $\mathbb{Z}[\tau]$ intersecting back to $\mathfrak{p}$.

Note that τ is actually a unit:

$$\tau \cdot (\tau - 1) = (\frac{\sqrt{5}+1}{2}) \cdot (\frac{\sqrt{5}-1}{2}) = 1$$

If we consider the prime ideal $\langle 5 \rangle$ of $\mathbb{Z}$, we notice that its extension to $\mathbb{Z}[\tau] \cong \mathbb{Z}[x]/\langle x^2 - x - 1\rangle$ is NOT prime. This is because $\mathbb{Z}[x]/\langle 5, x^2 - x - 1\rangle$ is not a domain, i.e. $x^2 - x - 1$ factors in $\mathbb{Z}/5\mathbb{Z}[x]$:

$$(x+2)^2 = x^2 + 4x + 4 \equiv x^2 - x - 1 \pmod 5$$

As a result, the prime lying over $\langle 5 \rangle$ in $\mathbb{Z}[\tau]$ is $\langle \tau + 2 \rangle$.

As a final consideration, we can also say a bit more about the primes which lie over a given prime in integral extensions.

Proposition 18.5 (Incomparability)**.** *Suppose $A \subseteq B$ is an integral extension of rings. If $\mathfrak{q}, \mathfrak{q}'$ are 2 prime ideals of B such that $\mathfrak{q} \cap A = \mathfrak{q}' \cap A$, then either $\mathfrak{q} = \mathfrak{q}'$ or $\mathfrak{q} \not\subseteq \mathfrak{q}' \not\subseteq \mathfrak{q}$.*

Proof. Suppose that $\mathfrak{q} \subseteq \mathfrak{q}'$ are prime ideals such that $\mathfrak{p} := \mathfrak{q} \cap A = \mathfrak{q}' \cap A$. We can again consider

$$A/\mathfrak{p} \subseteq B/\mathfrak{q}$$

This is an integral extension of integral domains. Localizing at the multiplicative set $W = R \setminus \mathfrak{p}$, we get an integral extension

$$W^{-1}(A/\mathfrak{p}) = \mathrm{Frac}(A/\mathfrak{p}) \subseteq W^{-1}B/\mathfrak{q}$$

But by Proposition 1, we have that $W^{-1}B/\mathfrak{q}$ is a field. That is to say that $\mathfrak{q}' \cdot W^{-1}B/\mathfrak{q}$ is either 0 or $W^{-1}B/\mathfrak{q}$. In the latter case, we are saying

$$1 = \frac{q'}{a} \qquad\qquad q' \in \mathfrak{q}', a \in W$$

Or equivalently (since $B/\mathfrak{q}$ was a domain), $q' = a$. But $\mathfrak{q}' \notin W$, so this is impossible. $\square$

This result actually shows dimension of rings in an integral extension is *equal*.

[1]Recall that it satisfies the relation $\tau^2 - \tau - 1 = 0$