Theorem 0.1. *If $f \sim g : X \to Y$, then $f_* = g_* : H_n(X) \to H_n(Y)$.*

Proof. Where we were: We constructed the map

$$P : C_n(X) \to C_{n+1}(Y) : \sigma \mapsto \sum_{i=0}^{n} (-1)^i (F \circ (\sigma \times Id))|_{\Delta_i^{n+1}}$$

and I was in the process of showing that

$$* \qquad P \circ \partial_n^X + \partial_{n+1}^Y \circ P = g_\# - f_\#$$

as functions $C_n(X) \to C_n(Y)$. Here is a diagram:

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1}(X) \arrow[r, "\partial_{n+1}^X"] \arrow[d, "f_\#, g_\#"] \arrow[dl, "P"'] & C_n(X) \arrow[r, "\partial_n^X"] \arrow[d, "f_\#, g_\#"] \arrow[dl, "P"'] & C_{n-1}(X) \arrow[r, "\partial_{n-1}^X"] \arrow[d, "f_\#, g_\#"] \arrow[dl, "P"'] & \dots \arrow[dl, "P"'] \\
\dots \arrow[r] & C_{n+1}(Y) \arrow[r, "\partial_{n+1}^Y"] & C_n(Y) \arrow[r, "\partial_n^Y"] & C_{n-1}(Y) \arrow[r, "\partial_{n-1}^Y"] & \dots
\end{tikzcd}$$

Now, I will show this explicitly:

$$P \circ \partial_n^X(\sigma) = P\left(\sum_{j=0}^{n} (-1)^j \sigma_{[v_0, \dots, \hat{v}_j, v_n]}\right) = \sum_{i<j} (-1)^i (-1)^j F \circ (\sigma \times Id)_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]}$$
$$+ \sum_{i>j} (-1)^{i-1} (-1)^j F \circ (\sigma \times Id)_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]}$$

Similarly,

$$\partial_{n+1}^Y \circ P(\sigma) = \sum_{j \geq i} (-1)^i (-1)^{j+1} F \circ (\sigma \times Id)_{[v_0, \dots, v_i, w_i, \dots, \hat{w}_j, \dots, w_n]}$$
$$+ \sum_{i \geq j} (-1)^i (-1)^j F \circ (\sigma \times Id)_{[v_0, \dots, \hat{v}_j, \dots, v_i, w_i, \dots, w_n]}$$

When adding them together, you are left only with the terms $i = j$ from the second sum. Then most of those terms cancel as well, except for 2 terms; $i = j = 0$ and $i = j = n$:

$$F \circ (\sigma \times Id)_{[\hat{v}_0, w_0, \dots, w_n]} - F \circ (\sigma \times Id)_{[v_0 \dots, v_n, \hat{w}_n]}$$

This is exactly $g_\#(\sigma) - f_\#(\sigma)$.

Now, as claimed in class, it goes to show that for $\sigma \in Z_n(X)$, $P \circ \partial_n^X(\sigma) + \partial_{n+1}^Y \circ P(\sigma)$ is a boundary in $C_n(Y)$, which will complete the proof that $f_*\sigma = g_*\sigma$.

Since σ is a cycle, $\partial_X(\sigma) = 0$, so $P(\partial_X(\sigma)) = 0$. In addition, $\partial_{n+1}^Y(P(\sigma))$ is clearly a boundary. $\square$

Algebra: As noted at the end of class, P is called a homotopy operator (for reasons of algebraic topology). If you have one of these on ANY chain complexes or groups, satisfying $*$ above, induces an isomorphism on homology groups.