CLASS 7, FEBRUARY 20TH: MODULES

As with many fields of the mathematics, many times the objects of interest are really the structures you can put on top of another more common object. This immediately makes modules an intellectually profitable realm of study.

Definition 7.1. A **module** over a commutative ring R is an abelian group $(M, +)$ with multiplication $\cdot : R \times M \to M$, such that for all $r, s \in R$ and $m, m' \in M$:

1) R**-Distributive:** $(r + s) \cdot m = rm + sm$.
2) M**-Distributive:** $r \cdot (m + m') = rm + rm'$.
3) **Associative:** $(rs)m = r(sm)$.
4) **Unital:** $1 \cdot m = m$.

M is often referred to as an R**-module**.

The next few examples show the prevalence of R-modules:

Example 7.2 (Vector Spaces). If V is a vector space over a field K, then V is also a module over K. So you can view $\mathbb{R}^n$ as a $\mathbb{R}$-module. In fact, every K-module is a vector space!

More generally, every vector space is a free K-module:

Definition 7.3. A module M of R is called **free** if

$$M = R^{\oplus\Lambda} = R^{\Lambda} = \{(r_\lambda)_{\lambda\in\Lambda} \mid r_\lambda \in R,\ r_\lambda = 0 \text{ for all but finitely many } \lambda\}$$

Modules also unify the notions of this class so far!

Example 7.4 (Ideals). If I is an ideal of R, then I is naturally an R-module. In fact, it inherits all of the above properties from R! In particular, R is an R-module. We can say I is a **submodule** of R if we want to keep track of where it lives.

Example 7.5 (Ring Homomorphisms). Let $\varphi : R \to S$ be a ring homomorphism. Then S can be viewed as an R-module via the following action:

$$r \cdot s = \varphi(r)s$$

where the second multiplication is simply multiplication in S. One checks respectively:

1) $(r + r') \cdot s = \varphi(r + r')s = (\varphi(r) + \varphi(r'))s = \varphi(r)s + \varphi(r')s = r \cdot s + r' \cdot s$.
2) $r \cdot (s + s') = \varphi(r)(s + s') = \varphi(r)s + \varphi(r)s' = rs + rs'$
3) It is associative since S-multiplication is.
4) $\varphi(1_R) = 1_S$.

More generally, this shows that every S-module can naturally be made into an R-module.

An additional example is in fact a subset of a major theorem.

Example 7.6 (Abelian groups). There is a natural bijection between the set of Abelian groups and the set of $\mathbb{Z}$-modules. Given an Abelian group G, we have a $\mathbb{Z}$-action given by

$n \cdot g = ng \in G$, given by applying the G group operation n times to g: $ng = g + g + \ldots + g$. In addition, the conditions of being a module at all are to be an abelian group under $+$.

Because for any unital ring R we have a natural map $\mathbb{Z} \to R : 1 \mapsto 1_R$, every R-module is a $\mathbb{Z}$-module by Example 7.5.

Next up, we study maps of modules, and what the compatible structure should be.

Definition 7.7. Let M and N be R-modules. Then an **R-module homomorphism** from M to N is a map $\varphi : M \to N$ such that

- $\varphi(m + m') = \varphi(m) + \varphi(m')$
- $\varphi(rm) = r\varphi(m)$

In addition, we define the following quantities to a module homomorphism:

- $\ker(\varphi)$ to be the set of $m \in M$ such that $\varphi(m) = 0$.
- $\mathrm{im}(\varphi)$ is the set of $n \in N$ such that there exists $m \in M$ with $\varphi(m) = n$.

In the case that $\ker(\varphi) = 0$ and $\mathrm{im}(\varphi) = N$, we call φ an **isomorphism**. Finally, we call the group of module homomorphisms $\mathrm{Hom}_R(M, N)$.

We can immediately say even more:

Proposition 7.8. *The set $\mathrm{Hom}_R(M, N)$ has the structure of an R-module.*

Proof. We give it the structure of an R-module as follows: We define

$$\varphi + \psi : M \to N : m \mapsto \varphi(m) + \psi(m)$$

$$r\varphi : M \to N : m \mapsto r\varphi(m)$$

One quickly verifies the axioms of a module based on that of M and N. □

Example 7.9. There is a natural isomorphism of R-modules between $\mathrm{Hom}_R(R, M)$ and M, given by $\varphi \mapsto \varphi(1)$ and $m \mapsto (\varphi : R \to M : 1 \mapsto m)$.

Next up, I summarize a few results which are very similar to the case of rings.

Definition 7.10. A subset $N \subseteq M$ is called a **submodule** of M if N is a module in it's own right. That is, $rn_1 + n_2 \in N$ if $n_1, n_2 \in N$ and $r \in R$.

We can then consider M/N to be the set of cosets of N inside M (as abelian groups). This is a R-module in it's own right.

Proposition 7.11. *If $N, N' \subseteq M$ are submodules, then $N + N'$ and $N \cap N'$ are also submodules.*

Proof. Homework 3, #6. □

Finally, this allows us to write down the module isomorphism theorems. The proofs of each are almost identical to the case of rings/groups.

Theorem 7.12. *1) If $\varphi : M \to N$, then $M/\ker(\varphi) \cong Im(\varphi)$.*

2) If $N, N' \subseteq M$ are submodules, then

$$(N + N')/N' \cong N/N \cap N'$$

3) If $N \subseteq N' \subseteq M$ are a chain of submodules, then

$$(M/N)/(N'/N) \cong M/N'$$

4) If $N \subseteq M$, then there is a natural bijection

$$\{\text{submodules of } M \text{ containing } N\} \leftrightarrow \{\text{submodules of } M/N\}$$