# HOMEWORK 6: SINGULARITIES
# DUE: WEDNESDAY, OCTOBER 30TH

(1) Show that if $u \in \mathbb{R} \setminus \mathbb{Z}$, then

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{\sin(\pi u)^2}.$$

This can be done by integrating $f(z) = \frac{\pi \cot(\pi z)}{(u+z)^2}$ on the circle of radius $N + \frac{1}{2}$ with $N \in \mathbb{Z}$, and sending $N \to \infty$. Show why.[1]

**Solution:** Note that

$$f(z) = \frac{\pi \cot(\pi z)}{(u+z)^2} = \frac{\pi \cos(\pi z)}{(u+z)^2 \sin(\pi z)}$$

has a pole of order 1 at $z \in \mathbb{Z}$ and of order 2 at $z = -u$. Therefore, by the residue theorem, we can calculate

$$\begin{aligned}
\int_{C_N} f(z)dz &= 2\pi i \sum_{n=-N}^{N} res_n(f(z)) + 2\pi i \cdot res_{-u}(f(z)) \\
&= 2\pi i \sum_{n=-N}^{N} \lim_{z\to n} \frac{\pi}{\pi(u+z)^2} + 2\pi i \cdot \lim_{z\to -u} \frac{\partial}{\partial z} \pi \cot(\pi z) \\
&= 2\pi i \left( \sum_{n=-N}^{N} \lim_{z\to n} \frac{1}{(n+z)^2} - \pi^2 \lim_{z\to -u} \csc^2(\pi z) \right)
\end{aligned}$$

Sending $N \to \infty$ produces

$$\lim_{N\to\infty} \int_{C_N} f(z)dz = 2\pi i \left( \sum_{n=-\infty}^{\infty} \frac{1}{(u+z)^2} - \pi^2 \lim_{z\to -u} \csc^2(\pi u) \right)$$

So it only suffices to check

$$0 = \lim_{N\to\infty} \int_{C_N} f(z)dz$$

Per usual, note that

$$\cot(\pi z) = -i \frac{e^{i\theta} + e^{-i\theta}}{e^{i\theta} - e^{-i\theta}} = -2i \left( 1 + 2 \frac{e^{-i\theta}}{e^{i\theta} - e^{-i\theta}} \right)$$

In absolute value, I claim that on $C_N$ this function is bounded above by 3 (due especially to the $\frac{1}{2}$). One needs to note that

$$|\cot(z)|^2 = 1 + \frac{\cos(2x)}{\sin^2(x) + \sinh^2(y)}$$

[1]This is a sort of shifted $\zeta$-function at $s = 2$.

If $|x| \geq N\pi + \frac{\pi}{2} - \frac{\pi}{4} = R\pi - \frac{\pi}{4}$, then $\sin^2(x) \geq \frac{1}{2}$. Similarly, if $|x| \leq R\pi - \frac{\pi}{4}$, then $\sinh^2(y) \geq 1$. Thus in total $\cot(z)$ is bounded above by 3.

(2) Suppose $f$ is holomorphic in $B_*(0,1)$ and that

$$|f(z)| \leq A|z|^{-1+\epsilon}$$

for some $\epsilon > 0$ and all $z_0$ near 0. Show that $f$ has a removable singularity at 0.

**Solution:** Since $|f(z)| \leq A|z|^{-1+\epsilon}$, we have that $|zf(z)| \leq A|z|^{\epsilon}$. Thus $\lim_{z\to 0} zf(z) = 0$ has a removable singularity at $z = 0$. This is to say $zf(z)$ is holomorphic at the origin. Therefore, we can consider its power series:

$$zf(z) = a_0 + a_1 z + a_2 z + \ldots$$

But since it takes value 0 at $z = 0$, we have $a_0 = 0$. Thus

$$zf(z) = a_1 z + a_2 z + \ldots = z(a_1 + a_2 z + \ldots)$$

dividing by $z$ implies $f$ is itself holomorphic at the origin, or that the singularity at $z = 0$ is removable.

(3) Show that all entire functions which are also injective ($f(z) = f(w)$ if and only if $z = w$) are linear:

$$f(z) = az + b \qquad\qquad a \neq 0$$

(**hint:** Use Casorati-Weirstrass on $f(\frac{1}{z})$, and apply the open mapping theorem).

**Solution:** Given $f(z)$ is entire, $f(\frac{1}{z})$ is holomorphic everywhere except 0 and injective. Casorti-Weirstrass implies $f(\frac{1}{z})$ necessarily cannot have an essential singularity at 0 since the image is dense for any $\epsilon$-neighborhood (it couldn't possibly be injective if this were the case). If $f(\frac{1}{z})$ had a removable singularity at 0, then it would be bounded in a neighborhood. But then it would be bounded everywhere, and thus constant. Again injectivity is contradicted.

Thus $f(\frac{1}{z})$ has a pole of some order at 0: $f(\frac{1}{z}) = \frac{a_{-m}}{z^m} + \frac{a_{-m+1}}{z^{m-1}} + \ldots + \frac{a_{-1}}{z} + g(z)$, where $g(z)$ is holomorphic. I claim first $g(z)$ is constant. Indeed, if it weren't than it would be entire and thus unbounded. But

$$f(z) = a_{-m}z^m + \ldots + a_{-1}z + g\left(\frac{1}{z}\right)$$

But this would imply $\lim_{z\to 0}(g(\frac{1}{z}))$ is either $\infty$ (by unboundedness) or doesn't exist. Both are impossible. As a result, $f(z)$ is polynomial. The only injective non-constant polynomials are linear.

(4) Suppose $f$ and $g$ are holomorphic on $\bar{B}(0,1)$, and that $f$ has only a simple zero at $z = 0$. Show that

$$f_\epsilon(z) = f(z) + \epsilon g(z)$$

has exactly one zero on $\bar{B}(0,1)$, and if we call it $z_\epsilon$, then $z_\epsilon$ varies continuously in $\epsilon$.

**Solution:** We want to use the Rouche's Theorem. Since $f$ is holomorphic and non-zero on the circle $|z| = 1$, we have that $|f(z)| > \delta$ for all $z$ and some $\delta > 0$.

Additionally, $|g(z)| < M$ for some $M$ for all $z \in \bar{B}(0,1)$. Both of these observations follow from continuity and compactness.

As a result, we can choose $\epsilon \le \delta \cdot \frac{1}{M}$, so that

$$|\epsilon g(z)| < \delta \cdot \frac{1}{M}|g(z)| < \delta < |f(z)|$$

Thus Rouche implies that $f_\epsilon$ and $f$ have the same number of zeroes in $\bar{B}(0,1)$.

It remains to show that $z_\epsilon$, the zero, varies continuously for $\epsilon$ near 0 Let $\epsilon \in [0, \frac{\delta}{2M}]$. I claim it is continuous here.

First, I demonstrate this fact at $\epsilon = 0$. Suppose it doesn't. That is to say there exists $\epsilon > 0$ such that for any $\delta > 0$

$$|z_{\delta'}| \ge \epsilon \qquad\qquad \text{for some } \delta' < \delta$$

Again, we can use compactness. $f$ doesn't have a zero on $\epsilon \le |z| \le 1$, so it is bounded below by some $\epsilon' > 0$. Thus we can choose $\delta < \epsilon' \cdot \frac{1}{M}$ to force $f_{\delta'}(z) > 0$ for $\epsilon \le |z| \le 1$. This contradicts our assumption about $z_\epsilon$.

The same logic applies for other $\epsilon$ in this range: We could consider

$$0 = f_{\epsilon'}(z_{\epsilon'}) = f_\epsilon(z_{\epsilon'}) + (\epsilon' - \epsilon)g(z)$$

This puts us in exactly the same framework as the case of 0, and we can conclude for $0 < \delta < \frac{\epsilon'-\epsilon}{M}$, $z_{\epsilon'}$ must be $(\epsilon -' \epsilon)$-close to $z_\epsilon$.

(5) Let $f$ be non-constant holomorphic in $\Omega \supseteq \bar{B}(0,1)$. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then $\bar{B}(0,1) \subseteq f(\Omega)$.

If instead $|f(z)| \ge 1$ whenever $|z| = 1$ and there is some $z_0 \in \bar{B}(0,1)$ with $|f(z_0)| < 1$, then $\bar{B}(0,1) \subseteq f(\Omega)$.

(**hint:** for the first part, show that it suffices to check that $f(z)$ has a root. Then apply the maximum modulus principle).

**Solution:** It suffices to check that $f(z) = w_0$ has a solution for every $w_0 \in \bar{B}(0,1)$.

First note that we can find $z$ such that $f(z) = 0$. If we couldn't, then one would have $g(z) = \frac{1}{f(z)}$ is a holomorphic function on the disc with $|g(z)| > 1$ inside the disc and $= 1$ on the disc. This is a violation of the maximum modulus principle.

Now we can consider the function

$$h(w) = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z) - w}$$

which counts the number of zeroes of $f(z) - w$ in $C = \{|z| = 1\}$. This is a function continuous in $w$. Thus, since it is integer valued, we have that it is constant on $B(0,1)$. Thus all values are attained.

The same proof holds for the second part, using instead $z_0$ as the constant number of zeroes.