HOMEWORK 5: COUNTABLILITY AND SEPARATIONS
DUE: OCTOBER 19

1) Show that if (X, τ) is second countable, then any basis $\mathcal{B}$ contains a subset $\mathcal{B}'$ which is countable and still a basis for τ.

Solution: Given X is second countable, there exists a countable basis $\mathcal{B}''$. Now it goes to compare the bases. Given $U_i \in \mathcal{B}''$, we know it is open in $\mathcal{B}$, since they generate the same topology. Therefore,$U_i = \bigcup_\alpha V_\alpha$ for some $V_\alpha \in \mathcal{B}$. This implies in particular that $V_\alpha \subseteq U_i$. Moreover, $V_\alpha = \bigcup_{j \in I \subseteq \mathbb{N}} U_j$, so $U_j \subseteq V_\alpha$. Call a V_α obtained in this way $V_{i,j} = V_\alpha$, so that

$$U_j \subseteq V_{i,j} \subseteq U_i$$

Note that this can't be done for all i, j, since j implicitly depends on i. However, I claim

$$\mathcal{B}'' = \{V_{i,j} \mid U_j \subseteq V_{i,j} \subseteq U_i\}$$

is the desired basis. It is of course countable, since it is surjected onto by $\mathbb{Z}^2$. Furthermore, $\mathcal{B}''$ generates the same topology as $\mathcal{B}$, since there is always a $V_{i,j} \subseteq U_i$ for any choice of i, so $\tau_{\mathcal{B}''} \supseteq \tau_{\mathcal{B}}$. Futhermore, $\mathcal{B}'' \subseteq \mathcal{B}'$, so $\tau_{\mathcal{B}''} \subseteq \tau_{\mathcal{B}'} = \tau_{\mathcal{B}}$.

Finally, note that $\mathcal{B}'$ is still a basis. Given $x \in X$, there exists some $x \in U_i$ since $\mathcal{B}''$ is a basis. But then $U_i = \bigcup V_\alpha = \bigcup U_j$ as before, so $x \in U_j$ for some j represented in the union. Therefore, $x \in U_j \subseteq V_{i,j}$. Similarly, if $x \in V_{i,j} \cap V_{i',j'}$, then we note that $x \in U_{j_0} \cap U_{j_0'}$ for some j_0, j_0' obtained from the cover. As a result, there is $x \in U_k \subseteq U_{j_0} \cap U_{j_0'}$. Finally, similar to the first part, there exists $V_{k,l}$ such that

$$x \in V_{k,l} \subseteq U_k \subseteq U_{j_0} \cap U_{j_0'} \subseteq V_{i,j} \cap V_{i',j'}$$

as desired.

2) Let $f : X \to Y$ be a continous open map. Show that if X is first (or second)-countable, then so is $f(X)$.[1]

Solution: Suppose that X is first-countable. Let $y = f(x)$, and let $U_1, \ldots$ be a countable neighborhood base at x. I claim that $f(U_i)$ is the desired neighborhood base of y.

If V is an open neighborhood of y, then $f^{-1}(V)$ is an open neighborhood of x. Therefore, it contains some U_i such that $x \in U_i \subseteq f^{-1}(V)$. As a result,

$$f(x) = y \in f(U_i) \subseteq f(f^{-1}(V)) \subseteq V.$$

3) Let $Y \subset \mathbb{R}^{\mathbb{N}}$ with the box topology be the set of sequences $(x_1, x_2, \ldots)$ such that $x_n = 0$ for $n \geq N$ for some N, and $x_i \in \mathbb{Q}$. Show that Y has closed points. Find which separation axioms T(1-4) Y possesses.

[1]Note this is not a statement about Y.

Solution: First note that Y has closed points since $\mathbb{R}^{\mathbb{N}}$ does. Moreover, by virtue of the fact that $\mathbb{R}^{\mathbb{N}}$ with the box topology is Hausdorff/T2 (finer than the product topology, apply Theorem 16.5 and the first sentence of exercise 6). Therefore it is also T1.

Now, I claim Y is regular/T3. We can do this by checking that $\mathbb{R}^{\mathbb{N}}$ itself is T3 and apply Theorem 16.5. Let $x \in \mathbb{R}^{\mathbb{N}}$ and U be a neighborhood of x. We may assume

$$U = \prod_i (-\epsilon_i, \epsilon_i)$$

WLOG (it certainly contains such basis element, so make U smaller if necessary). Now, note that if we take

$$V = \prod_i (-\frac{\epsilon_i}{2}, \frac{\epsilon_i}{2})$$

Then since $\prod_i [-\frac{\epsilon_i}{2}, \frac{\epsilon_i}{2}]$ is a closed set containing V, contained within U, we see $\bar{V} \subseteq U$.

The T4 portion is a (hard) extra credit problem.

4) Given a metric space (X, d) and a closed subset $Z \subseteq X$, show that the function

$$f(x) = d(x, Z) = \inf\{d(x, z) \mid z \in Z\}$$

is a continuous function $f : X \to \mathbb{R}$. Furthermore, show that $f(x) = 0$ if and only if $x \in A$.

Solution: Note that the function is well defined since we are taking the infimum of a collection of real numbers bounded below by 0.

Fix $\epsilon > 0$. It goes to find a neighborhood $U = B(x, \delta)$ of x for which every $y \in B(x, \delta)$ has the property that $d(y, Z) \in (d(x, Z) - \epsilon, d(x, Z) + \epsilon)$. I claim that $\delta = \epsilon$ suffices:

$$d(y, Z) \leq d(y, z) \leq d(x, z) + d(x, y)$$
$$d(x, Z) \leq d(x, z) \leq d(y, z) + d(x, y)$$

Taking the infimum over all such $z \in Z$ preserves the inequality, as the set of lower bounds is closed. As a result,

$$|d(x, Z) - d(y, Z)| \leq d(x, y) < \epsilon$$

as desired. Finally, since Z is closed, its complement is open. So for a fixed $x \in Z^c$ there exists $\epsilon > 0$ such that $B(x, \epsilon) \subseteq Z^c$. However, this is the set of all points of distance less than ϵ from x, and it doesn't intersect Z. Therefore $d(x, Z) \geq \epsilon > 0$. The reverse direction is trivial.

5) Use the previous problem to show that every metric space is T4/normal.

Solution: Take Z, Z' two closed subsets of X. Note that $D(x) = d(x, Z) - d(x, Z')$ is a continuous function in x by the previous problem. Therefore

$$U = \{x \in X \mid d(x, Z) < d(x, Z')\} = D^{-1}((-\infty, 0))$$
$$U' = \{x \in X \mid d(x, Z) > d(x, Z')\} = D^{-1}((0, \infty))$$

are open disjoint sets, with $Z \subseteq U$ and $Z' \subseteq U'$.

6) Given $\tau \subseteq \tau'$, it is easy enough to check that if X_τ is Hausdorff then so is also $X_{\tau'}$. Is the same true for T3 and T4? Justify your answer.

Solution: I claim that regular and normal don't necessarily behave this way. The easiest way to see this is as follows: The indiscrete topology is always T3 and T4. The reason being there do not even exist non-empty closed subsets of X that are not X itself. So this is vacuously true. Now it only goes to say that there exist non-T3 and T4 subsets. ☺

There also exist more natural examples. Since $\mathbb{R}^n$ with the Euclidean topology is an example of a space which is a metric space, thus T4 and T3, we only need to find finer topologies which violate T3 and T4. Examples of this include the slit disc and tangent disc topologies from example 16.3. They also generalize naturally to higher dimensions.

7) If Y is Hausdorff and $f, g : X \to Y$ are continuous maps, show that $Z = \{x \in X \mid f(x) = g(x)\}$ is a closed set.[2]

Solution: As usual, I will show instead that Z^c is open. Suppose x is such that $f(x) \neq g(x)$. Then there exists U, V disjoint open sets such that $f(x) \in U$ and $g(x) \in V$. Continuity of f and g imply that there exist neighborhoods of x, say U' and V', such that $f(U') \subseteq U$ and $f(V') \subseteq V$. This implies that that open neighborhood $N_x = U' \cap V'$ has the property that $f(y) \in U$ and $g(y) \in V$ for all $y \in N_x$, therefore they are never equal on N_x, and thus $N \subseteq Z^c$.

8) Let $p : X \to Y$ be a continuous, closed, surjective map. Show that if X is normal, then so is Y.

Solution: Let Z be a closed subset of Y and U be an open neighborhood of Z. It goes to find $V \subseteq U$ open, containing Z, such that $\bar{V} \subseteq U$ (again by Theorem 16.4). Consider $Z' = f^{-1}(Z)$ and $U' = f^{-1}(U)$. These also satisfy the preceding properties; Z' is closed, U' is an open neighborhood of Z' (by continuity of f). Moreover, surjectivity of f implies $f(Z') = Z$ and $f(U') = U$.

Since X is normal, we know there exists $V' \subseteq U'$ an open neighborhood of Z' such that $\bar{V}' \subseteq U$. Let $V = f(\bar{V}')$ be a closed subset of U. It also contains $Z = f(f^{-1}(Z)) = f(Z')$. It only goes to show that V also contains an open neighborhood containing Z.

Recall the hint from last homework. If $p^{-1}(z) \subseteq V'$, then there exists a neighborhood U_z of z such that $p^{-1}(U_z) \subseteq V$. Therefore,

$$Z \subseteq \bigcup_{z \in Z} U_z \subseteq \overline{\bigcup_{z \in Z} U_z} \subseteq V \subseteq U$$

So $\bigcup_{z \in Z} U_z$ is the desired neighborhood.

[2]Note we cannot subtract in a generic topological space, though this generalizes such an idea.