HOMEWORK 3: CAUCHY'S THEORY
DUE: WEDNESDAY, OCTOBER 2ND

1) Show that

$$\int_0^\infty \sin(x^2)dx = \int_0^\infty \cos(x^2)dx = \frac{\sqrt{2\pi}}{4}$$

(**hint:** The pie shaped wedge from $\theta = 0$ to $\theta = \frac{\pi}{4}$ may be a useful path to consider. You may assume $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$.)

Solution: Following the hint, Cauchy's Integral Theorem implies

$$0 = \int_0^R e^{-x^2}dx + \int_{\gamma_R} e^{-z^2}dz - \int_0^R e^{-(\frac{t+it}{\sqrt{2}})^2}\frac{1+i}{\sqrt{2}}dt$$

The first term we know to be $\frac{\sqrt{\pi}}{2}$ by symmetry. The second can be shown to be 0 as $R \to \infty$:

$$\left|\int_{\gamma_R} e^{-z^2}dz\right| = \left|\int_0^{\frac{\pi}{4}} e^{-R^2e^{2i\theta}} iRe^{i\theta}d\theta\right| \le \int_0^{\frac{\pi}{4}} Re^{-R^2\sin(2\theta)}d\theta \le \int_0^{\frac{\pi}{4}} Re^{-R^2\frac{4\theta}{\pi}}d\theta$$

As R goes to ∞, this integral is $O(\frac{1}{R})$ and thus tends to 0 as desired. Thus the final integral, which is precisely

$$\int_0^R e^{-it^2}\frac{1+i}{\sqrt{2}}dt = \frac{1+i}{\sqrt{2}}\int_0^R cos(t^2) - i\sin(t^2)dt$$

Multiplying the original equality by $\frac{1-i}{\sqrt{2}}$ yields the result.

2) Evaluate the integral $\int_0^\infty \frac{\sin(x)}{x}$. It may be useful to show it is equal to $\frac{1}{2i}\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x}dx$ and use the upper semi-circle with 0 removed.

Solution: First, it goes to establish that the two integrals are equal:

$$\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x}dx = \int_{-\infty}^{\infty}\frac{\cos(x)+i\sin(x)-1}{x}dx$$

since $\frac{\cos(x)-1}{x}$ is an odd function, we get that its integral is 0. Thus,

$$\int_{-\infty}^{\infty}\frac{e^{ix}-1}{x}dx = \int_{-\infty}^{\infty}\frac{i\sin(x)}{x}dx = 2i\int_0^\infty\frac{\sin(x)}{x}dx$$

as desired. Now, we can consider the function $f(z) = \frac{e^{iz}-1}{z}$ and integrate over the upper semicircle of radius R with inner circle of radius ϵ missed. It is again holomorphic here, so we have that the integrals sum to zero. Rewritten;

$$i\int_{\epsilon<|x|<R}\frac{\sin(x)}{x}dx = \int_{C_\epsilon} f(z)dz - \int_{C_R} f(z)dz$$

where C_r is the upper semicircle oriented clockwise of radius r. It goes to estimate the left hand side. Noting that $e^{-iz} = 1 + iz - \frac{z^2}{2} - \ldots$, we have that

$$\int_{C_\epsilon} f(z)dz = \int_{C_\epsilon} \frac{iz - \frac{z^2}{2} - \ldots}{z} dz \to 0$$

As the whole quantity is bounded, and the length of C_ϵ approaches 0. So it only goes to evaluate

$$\int_{C_R} f(z)dz = \int_0^\pi \frac{e^{iRe^{i\theta}} - 1}{Re^{i\theta}} iRe^{i\theta} d\theta = i\int_0^\pi \left(e^{iRe^{i\theta}} - 1\right) d\theta$$

I claim this equals $-\pi$, or equivalently

$$\int_0^\pi e^{iRe^{i\theta}} d\theta \to 0$$

as $R \to \infty$. This follows naturally from the fact that

$$\left|\int_0^\pi e^{iRe^{i\theta}} d\theta\right| \approx \left|\int_\epsilon^{\pi-\epsilon} e^{iRe^{i\theta}} d\theta\right| \leq \pi \cdot \sup |e^{-R\cdot Im(e^{i\theta})}| = \pi e^{-R\sin(\epsilon)} \to 0.$$

3) If $f(z)$ is continuously complex differentiable in Ω, and $T \subseteq \Omega$ is a triangle, then use Green's Theorem to show that

$$\int_T f(z)dz = 0$$

This proves Goursat's Theorem with stronger assumptions.

Solution: We can rewrite this integral as

$$\int_T f(z)dz = \int_T (u + iv)d(x + iy) = \int_T (udx - vdy) + i(udy + vdx)$$

Now, Green's Theorem implies

$$\int_T f(z)dz = \int_{\mathfrak{T}} (\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})dxdy + i(\frac{\partial v}{\partial y} - \frac{\partial u}{\partial x})dxdy$$

Now the CR equations yield that this is an integral of the 0 function.

4) Let f be a function which is complex differentiable in Ω except possibly at one point w. Let T be a triangle with w in its interior. Show that if f is bounded in a neighborhood of w, then we get the same conclusion from Goursat:

$$\int_T f(z)dz = 0$$

Solution: This follows our usual method just in general. Consider the contour which goes around the original triangle T with a corridor towards the point, then going around a small triangle around the point w. This function is holomorphic on this domain, so Cauchy shows the original integral is equal to the one counterclockwise around the smaller triangle:

$$|\int_T f(z)dz| = |\int_{T_\epsilon} f(z)dz| \leq \sup|f(z)| \cdot 3\epsilon \to 0$$

as $\epsilon \to 0$. This show the desired result.

5) Following the ideas of example 9.1 from our notes, show that for $\xi \in \mathbb{R}$, we have

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{2\pi i x\xi} dx$$

Solution: Consider again the rectangle with endpoints $R, -R, R - i\xi, -R - i\xi$. Orient it clockwise for a change of pace. Integrating over the function $e^{-\pi z^2}$, we again may conclude

$$\int_{-R}^{R} e^{-\pi x^2} dx = 1$$

$$\int_0^{-\xi} e^{-\pi(-R+ix)^2} \cdot idx = \int_{-\xi}^{0} e^{-\pi(-R+ix)^2} \cdot idx = 0$$

So it remains to calculate the integral of the bottom segment. We can reorient to have

$$1 = \int_{-R}^{R} e^{-\pi(x-i\xi)^2} dx = \int_{-R}^{R} e^{-\pi(x^2-2ix\xi-\xi^2)} dx = \int_{-R}^{R} e^{\pi\xi^2} e^{-\pi x^2} e^{2\pi i x\xi} dx$$

Dividing this equation by $e^{\pi\xi^2}$ yields the desired result.

6) If $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function, show that $d = \sup_{z,w\in\mathbb{D}} |f(z) - f(w)|$ satisfies

$$2|f'(0)| \leq d.$$

Moreover, equality holds if and only if f is linear.

Solution: Consider the function

$$g(z) = f(z) - f(-z)$$

This is again a holomorphic function, so we can apply Cauchy's inequality to g:

$$2|f'(0)| = |f'(0) + f'(0)| = |g'(0)| \leq \sup_{z\in C} |g(z)| \leq d$$

It is clear that equality is achieved if $f(z) = az + b$ is linear:

$$2|a| = 2|f'(0)| = \sup|f(z) - f(w)| = |a| \sup|z - w| = 2|a|$$

Now suppose equality is achieved. By Cauchy's integral theorem, we have

$$2f'(0) = \frac{1}{2\pi i} \int_C \frac{f(z) - f(-z)}{z^2} dz$$

Thus

$$d = \frac{1}{2\pi} \left| \int_C \frac{f(z) - f(-z)}{z^2} dz \right| \leq \frac{1}{2\pi} \int_C \left| \frac{f(z) - f(-z)}{z^2} \right| dz \leq \sup|f(z) - f(-z)| \leq d$$

which is to say that $|f(z) - f(-z)| = d$ is constant. This shows it is linear.