HOMEWORK 4: HOMOTOPY EXTENSION PROPERTY & FUNDAMENTAL GROUP
DUE: MONDAY, OCTOBER 16

1) Use Corollary .20, stated below for your convenience, to show that if (X, A) has the homotopy extension property, then $X \times I$ **deformation** retracts onto $X \times \{0\} \cup A \times I$.

   Corollary 0.1. *If (X, A) has the homotopy extension property and the inclusion $\iota : A \to X$ is a homotopy equivalence, then A is a deformation retract of X.*

   Deduce that the following theorem (.18) holds for any pair (X, A) with the homotopy extension property:

   Theorem 0.2. *If (X, A) is a CW pair and we have attaching maps $f, g : A \to Y$ that are homotopic, then $X \coprod_f Y \simeq X \coprod_g Y$ rel X.*

2) Show that if (X_1, A) has the HEP, then so does every pair $(X_0 \coprod_f X_1, X_0)$ obtained by attaching X_1 to X_0 via a map $f : A \to X_0$.

3) In this example, we consider a generalization of the free product of groups introduced in class.

   Definition 0.3. Let F, G, H be groups, and let $\phi : F \to G$ and $\psi : F \to H$. We form the amalgamated product of groups $G *_F H$ as

   $$G *_F H = G *_{F,\phi,\psi} H = G * H / \langle \phi(f) * \psi(f)^{-1} \rangle$$

   That is to say we identify the images of the two homomorphisms: $\phi(f) = \psi(f)$.

   Give a presentation for the following 2 amalgamated products:
   ◦ Given $f, g : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = x$ and $g(x) = -x$, $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$.
   ◦ Given $f, g : \mathbb{Z} \to \mathbb{Z}$ with $f(x) = 2x$ and $g(x) = 3x$, $\mathbb{Z} *_{\mathbb{Z}} \mathbb{Z}$.

   Note: This will play an important role later with VanKampen's Theorem.

4) Show the following cancellation property for composition of paths: If $f_0 g_0 \simeq f_1 g_1$, show that

   $$f_0 \simeq f_1 \Leftrightarrow g_0 \simeq g_1$$

5) For a path-connected space X, show that $\pi_1(X)$ is abelian if and only if all basepoint-change homomorphisms (as described in class) depend only on the endpoints.

6) Show that for a path connected space, X being simply connected is equivalent to every map $S^1 \to X$ being homotopic to one another.