Theorem 0.1. $\pi_1(S^1) = \mathbb{Z}$

Corollary 0.2 (The Fundamental Theorem of Algebra)**.** *Let* $p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_0$ *be a complex polynomial. If* $n > 0$*, then* p *has a complex root. Thus, by the division algorithm, for some* $a_i \in \mathbb{C}$*,*

$$p(z) = \prod (z - a_i)^{n_i}$$

Proof. Suppose $p(z) \neq 0$. Then we can define

$$g(z) = \frac{p(z)}{\|p(z)\|}$$

This is a function of $g : \mathbb{C} \to S^1$. We can also view it as a family of loops based at $x_0 = \frac{p(0)}{\|p(0)\|}$:

$$g(z) = g(re^{i\theta}) = \gamma_r(\theta)$$

This is a loop in S^1 for each r, that varies continuously with r. Therefore,

$$\gamma_r \simeq \gamma_{r'}$$

for any two $r, r' \geq 0$. Note γ_0 is a constant loop, so $\gamma_0 = 0 \in \pi_1(S^1)$. Furthermore, for $r \gg 0$, namely, $r > \max\{1, \|a_{n-1}\| + \ldots + \|a_0\|\}$,

$$\|z\|^n > \|a_{n-1}\| \cdot \|z^{n-1}\| + \ldots + \|a_0\| \geq \|a_{n-1}z^{n-1} + \ldots + a_0\|.$$

Therefore, we can form the homotopy $P(z, t) = z^n + t(a_{n-1}z^{n-1} + \ldots + a_0) + x_0$. This demonstrates $\gamma_r(\theta) \simeq [r^n e^{i\theta n}] = n \in \pi_1(S^1)$. Therefore, $n = 0$, and p was constant to begin with. $\square$

Next class, whenever that may be, we will show the next two excellent corollaries of the Theorem listed above:

Corollary 0.3. *If* $f : S^2 \to \mathbb{R}^2$ *is a continuous function, then there exist antipodal points* $x, -x$ *such that* $f(x) = f(-x)$*.*

Corollary 0.4. *If* $A, B, C \subseteq S^2$ *are 3 closed sets covering* S^2*, then at least one of them contains a pair of antipodal points.*