CLASS 27, NOVEMBER 14: PARACOMPACTNESS

Today we will explore an extraordinarily useful generalization of compactness, known as paracompactness. This notion is ubiquitous throughout topology and differential geometry, with many theorems about manifolds (in particular) relying on a partition of unity subordinate to a locally finite cover.

Definition 27.1. A space is **paracompact** if every open cover $X = \bigcup_{\alpha\in\Lambda} X_\alpha$ can be refined to a locally finite open cover $= \bigcup_{\alpha\in\Lambda'} X'_\alpha$.

Example 27.2. $\mathbb{R}^n$ is paracompact. Indeed, we can cover $\mathbb{R}^n$ by $B(x, \sqrt{n})$, where $x \in \frac{1}{n}\mathbb{Z}^n$. This is a cover of $\mathbb{R}^n$ by spaces whose closures are compact. Enumerate the balls $B_1, B_2, \ldots$. For a given cover $\mathbb{R}^n = \bigcup U_\alpha$ Choose finitely many U_α covering $\overline{B_1}$ (by compactness). Call the collection $\mathfrak{B}_1$. Now inductively, choose finitely many U_α covering the compact space $\overline{B_n}$ and create

$$U'_\alpha = U_\alpha \setminus (\overline{B_1} \cup \cdots \cup \overline{B_{n-1}})$$

Call the set of these $\mathfrak{B}_n$. Now I claim that $\mathfrak{B} = \bigcup_{i\in\mathbb{N}} \mathfrak{B}_i$ is the desired cover. Note it is a cover, since

$$X = \bigcup_n B_n = \bigcup_n (B_1 \cup \cdots \cup B_n) = \bigcup_n \bigcup_{m\leq n} \bigcup_{X_\alpha\in\mathfrak{B}_m} X_\alpha$$

and it is locally finite, since only sets within $\mathfrak{B}_n$ can only intersect sets $X_\alpha \in \mathfrak{B}_m$ for $m \leq n$.

Now we can generalize our previous realization about compact Hausdorff spaces.

Theorem 27.3. *Every paracompact Hausdorff space X is T4.*

Proof. First, I prove X is T3. Let $x \in X$ and $Z \subseteq X$ a closed set with $x \notin Z$. For each $z \in Z$, we can find an open set U_z not containing z whose closure doesn't contain x. We can cover X by $\{U_z\}$ and $V = Z^c$. Applying paracompactness, we note that there exists a locally finite refinement covering X. Call it $\mathfrak{B}$ and let $\mathfrak{B}' \subseteq \mathfrak{B}$ consist of elements of $\mathfrak{B}$ intersecting Z.

By our assumptions, we know that $A \in \mathfrak{B}'$ implies $x \notin \bar{A}$. Consider

$$V = \bigcup_{A\in\mathfrak{B}'} A$$

Because $\mathfrak{B}'$ is locally finite, we have $\bar{V} = \bigcup_{A\in\mathfrak{B}'} \bar{A}$. But this implies V and $\bar{V}^c$ is the desired separation. Normality follows by the same argument. □

Additionally, paracompact spaces behave well with respect to closed subspaces.

Theorem 27.4. *Every closed subspace $Z \subseteq X$ of a paracompact space is itself paracompact.*

Proof. Similar to the proof for compactness, we can take for any cover $\{U_\alpha\}$ of Z a corresponding collection of open sets in X, say $\{X'_\alpha\}$, and add to it Z^c. This is a covering so a locally finite refinement exists. Intersect each element of the refinement with Z to verify Z is paracompact. □

Unlike the case of compactness however, paracompactness doesn't usually imply closed.

Example 27.5. Note $(0,1)$ is paracompact despite being open in $\mathbb{R}$.

On the other hand, we can consider the compact space $\bar{S_\omega} = \{1, 2, \ldots, \omega\}$. This is a compact set, since an open set containing ω necessarily contains a basis element $(n, \omega]$. The remaining space is finite, thus compact. Therefore, $\bar{S_\omega} \times \bar{S_\omega}$ is also compact. However, $S_\omega \times \bar{S_\omega}$ is a Hausdorff space which is non-T4. Therefore, it couldn't be paracompact by Theorem 27.3.

Next, I prove a useful characterization of open covers of a T3 space X.

Lemma 27.6. *Let X be T3. Then TFAE for every open covering $X = \bigcup_\alpha U_\alpha$:*

1) There exists a countably locally finite open covering refinement.
2) There exists a covering refinement which is locally finite.
3) There exists a closed covering refinement which is locally finite.
4) There exists an open covering refinement which is locally finite.

Proof. 1) $\Rightarrow$ 2): Let $\mathfrak{B} = \bigcup_{n\in\mathbb{N}} \mathfrak{B}_n$ be an open countably locally finite refinement of some cover $\mathcal{A}$. For a given $i \in \mathbb{N}$, let $V_i = \bigcup_{U\in\mathfrak{B}_i} U$. Then we can consider for a given $U \in \mathfrak{B}_n$, let

$$U' = U \setminus (V_1 \cup \ldots \cup V_{n-1})$$

This is an intersection of an open set and a closed set, so it likely doesn't have either property. If we let $\mathfrak{B}'_n$ be the refinement of U' obtained in this way, then we can obtain $\mathfrak{B}' = \bigcup_n \mathfrak{B}'_n$ a refinement of $\mathfrak{B}$.

I claim that this is the desired refinement. Note that this collection still covers the space, since we only removed points which were in earlier covers. It goes to show that it is locally finite. For a given $x \in X$, we know that there exists an n minimal such that $x \in U \in \mathfrak{B}'_n$. For each $m \leq n$, choose a neighborhood V_m of x intersecting only finitely many elements of $\mathfrak{B}'_m$ by local finiteness. I claim the desired neighborhood is then

$$V = V_1 \cap \ldots \cap V_n \cap U$$

V intersects only finitely many elements of each $\mathfrak{B}'_m$ with $m < n$. Furthermore, since $V \subseteq U$, it doesn't intersect ANY element of $\mathfrak{B}'_m$ for $m > n$. Thus it is locally finite.

2) $\Rightarrow$ 3): Now it goes to show we can upgrade a locally refinement to a closed one. Let $\mathfrak{B}$ be the collection of all open sets $U \subseteq X$ whose closure lies in some U_α. Since X is T3, the neighborhood criteria concludes that $\mathfrak{B}$ is still a cover. One can refine $\mathfrak{B}$ by 2) to a locally finite cover. Now let $\mathfrak{B}' = \{\bar{Z} \mid Z \in \mathfrak{B}\}$. It is locally finite by Lemma 25.3, part 2.

3) $\Rightarrow$ 4): Let $\mathfrak{B}$ be a refinement of U_α which is locally finite. For each $x \in X$, choose U_x an open neighborhood intersecting only finitely many elements of $\mathfrak{B}$. U_x forms an open cover of X. Let $\mathfrak{B}'$ be a closed refinement of $\{U_x\}$ which is locally finite. Therefore, each element of $\mathfrak{B}'$ intersects only finitely many elements of $\mathfrak{B}$.

For $B \in \mathfrak{B}$, let $\mathfrak{B}'_B = \{C \in \mathfrak{B}' \mid C \subseteq X \setminus B\}$. We can then produce

$$C_B = X \setminus \bigcup_{C\in\mathfrak{B}'_B} C$$

Again by Lemma 25.3, part 3), we can conclude that this set is open and contains B. Finally, we can choose an element $U_{\alpha(B)}$ which contains B and let $\mathfrak{B}'' = \{C_B \cap U_{\alpha(B)}\}$. This is an open refinement of $\mathcal{A}$ covering X.

It goes to show $\mathfrak{B}''$ is locally finite. Given $x \in X$, choose V_x intersecting finitely many $C_1, \ldots, C_n \in \mathfrak{B}'$. Note $V_x \subseteq C_1 \cup \ldots \cup C_n$. Given $C \in \mathfrak{B}'$, if $C \cap C_B \cap U_{\alpha(B)} \neq \emptyset$, then $C \cap C_B \neq \emptyset$, so in particular $C \not\subseteq B^c$ or $C \cap B \neq \emptyset$. However, this is true for only finitely many B, so is the case for $\mathfrak{B}''$!

4) $\Rightarrow$ 1): Trivial. Finite implies countable. $\square$