CLASS 28, NOVEMBER 16: PARACOMPACTNESS

Today we will exploit the result about refinements of open coverings from last class to produce some interesting examples of paracompact spaces. In addition, we will prove that paracompact spaces have partitions of unity subordinate to any cover. This builds the machinery to prove the Smirnov-Metrization Theorem next time, which also classifies metrizability of spaces.

Proposition 28.1. *Every metric space is paracompact.*

Proof. By Lemma 25.6, we know there exists a countably locally finite open refinement of any given open cover. By Lemma 27.6, we can conclude (since every metric space is T3) that this implies there exists a locally finite open refinement. Therefore it is paracompact. □

Munkres also includes a result about T3 Lindelof/σ-compact spaces being paracompact, which also utilizes Lemma 27.6 in its proof.

Example 28.2. One can check that $\mathbb{R}$ with the lower limit topology (generated by $[a, b)$) is paracompact (it is T3 and σ-compact). However, $\mathbb{R}\times\mathbb{R}$ with the product lower-limit topology is not, as it is non-T4. Therefore products of paracompact spaces aren't necessarily paracompact.

On the other hand, since $\mathbb{R}^{\mathbb{N}}$ with the product (or uniform) topology is metrizable, we know that it is necessarily paracompact. The statement for the box topology is open.

Next I prove the existence of partitions of unity for paracompact Hausdorff spaces. This signifies the importance of paracompactness as a condition.

Lemma 28.3. *Given X a paracompact Hausdorff space,, and U_α an open covering X. Then there exists a locally finite open collection V_α (with the same indexing set) covering X such that $\forall\alpha$, $\bar{V_\alpha} \subseteq U_\alpha$.*

The condition in the conclusion is sometimes called a **precise refinement**.

Proof. Let $\mathfrak{B}$ be the collection of ALL open sets V such that $\bar{V} \subseteq U_\alpha$ for some $\alpha \in \Lambda$. Since X is T4+T1, it is T3, so these sets cover X. We can choose a locally finite refinement of $\mathfrak{B}$ by open sets covering X. Call it $\mathfrak{B}'$. Then let $\mathfrak{B}' = \{V_\beta\}_{\beta\in\Lambda'}$. Utilizing the axiom of choice, we can define a choice function $f : \Lambda' \to \Lambda$ such that $\bar{V_\beta} \subseteq U_{f(\beta)}$. Let

$$V_\alpha = \bigcup_{f(\beta)=\alpha} V_\beta$$

Since the collection on the right is locally finite, we can conclude

$$\bar{V_\alpha} = \bigcup_{f(\beta)=\alpha} \bar{V_\beta} \subseteq U_\alpha$$

Finally, it suffice to check that V_α forms a locally finite collection. For a given $x \in X$, choose U intersecting only finitely may $V_{\beta_1}, \ldots, V_{\beta_n}$. Then U intersects at most $V_{f(\beta_1)}, \ldots, V_{f(\beta_n)}$. □

Theorem 28.4. *Let X be a paracompact Hausdorff space and U_α a cover. Then there exists a partition of unity subordinate to U_α.*

This proof has notable similarities to that of the same statement for compact spaces.

Proof. Recall that by the shrinking lemma used to prove partitions of unity for compact manifolds allows us to find

$$W_\alpha \subseteq \bar{W_\alpha} \subseteq V_\alpha \subseteq \bar{V_\alpha} \subseteq U_\alpha$$

with W_α and V_α open locally finite covers of X. Since X is T4, we can choose $\psi_\alpha : X \to [0,1]$ such that $\psi(\bar{W_\alpha}) = 1$ and $\psi(V_\alpha^c) = 0$. Therefore,

$$\mathrm{Supp}(\psi_\alpha) = \overline{\{x \in X \mid \psi_\alpha(x) \neq 0\}} \subseteq \bar{V_\alpha} \subseteq U_\alpha$$

Now we can consider $\Psi : X \to \mathbb{R} : x \mapsto \sum_\alpha \psi_\alpha(x)$. Even though this sum is potentially uncountable, we note that for a given value of x, the collection $\mathrm{Supp}(\psi_\alpha)$ is locally finite. Therefore, for a fixed neighborhood of any point, the sum is merely finite and bounded below by 1. Therefore, it is continuous on these neighborhoods, which cover X, so we note that Ψ is continuous as well. Therefore, a partition of unity is given by

$$\varphi_\alpha(x) = \frac{\psi_\alpha(x)}{\Psi(x)}$$

□

Our characterization of manifolds always implies that it is paracompact. Indeed, they are second countable and T3 (by homework). Therefore, every cover has a countable subcover and therefore is countably locally finite! Therefore, we obtain the following sometimes useful corollary.

Corollary 28.5. *If X is a manifold, then $\exists \iota : X \to \mathbb{R}^\Lambda$ an embedding, where Λ is at most countable. Therefore, every manifold is metrizable and hence normal.*

Proof. The same proof goes through as in the compact case, namely we can find countably many charts (by second-countability) $\psi_i : U_i \to U_i' \subseteq \mathbb{R}^{m_i}$ and a partition of unity $\varphi_i : X \to [0,1]$ subordinate to U_i. Then the desired map is

$$\iota : X \to \mathbb{R}^{\mathbb{N}} : x \mapsto (\psi_1(x)\varphi_1(x), \varphi_1(x), \psi_2(x)\varphi_2(x), \varphi_2(x), \ldots)$$

This is an embedding by the embedding theorem.

The fact that it is metrizable follows from the fact that $\mathbb{R}^{\mathbb{N}}$ is metrizable. □

Example 28.6. A perfectly reasonable (!) manifold arises as follows:

$$X = \mathbb{R} \coprod \mathbb{R}^2 \coprod \mathbb{R}^3 \coprod \cdots$$
$$Y = S^1 \coprod S^2 \coprod S^3 \coprod \cdots$$

Note here $\coprod$ denotes the disjoint union, which is to say the objects have no overlap.

These are certainly Hausdorff, since each connected component is. Furthermore, they are second countable since each component is and countable unions of countable sets are countable. Finally, each component is locally Euclidean, so X and Y are themselves. Therefore, X and Y are manifolds.

Of course, it is impossible to embed such spaces into finite dimensional vector spaces, since this would imply

$$\mathbb{R}^n \hookrightarrow X \hookrightarrow \mathbb{R}^m$$
$$S^n \hookrightarrow Y \hookrightarrow \mathbb{R}^m$$

for $n > m$, and this is impossible by invariance of dimension. But we can still conclude that these spaces are metrizable by embedding them into $\mathbb{R}^{\mathbb{N}}$ and applying the corresponding metric.