CLASS 24, NOVEMBER 5: STONE-ČECH COMPACTIFICATION

Recall the excellent compactification result from last class:

Theorem 24.0 (Stone-Čech Compactification Theorem)**.** *Let X be a T3.5 space*[1]*. Then there exists Y a compactification of X such that every bounded continuous map $f : X \to \mathbb{R}$ extends uniquely to a continuous map $f' : Y \to \mathbb{R}$.*

It's sheer existence is an extraordinarily powerful result, as it's a rare phenomenon to even be able to extend a single function to a larger even densely populated space Y, let alone a compact space. Today, we will peer deeper into this compactification.

Theorem 24.1. *Let X be a T3.5 space, and let Y be the resulting compactification of Theorem 24.0. Given C compact and Hausdorff, and $f : X \to C$ a continuous function, f extends uniquely to a function $f' : Y \to C$.*

This should be thought of as a mild generalization of Theorem 24.0, since the range need not be $[a, b] \subseteq \mathbb{R}$.

Proof. Note that C is itself T4 and T1, therefore T3.5. Therefore we may embed C into $[0, 1]^\Lambda$ for some Λ by The Embedding Theorem. So we may assume $C \subseteq [0, 1]^\Lambda$, and $f : X \to [0, 1]^\Lambda$ with range in C. Then each component function

$$f_\alpha = \pi_\alpha \circ f : X \to [0, 1]$$

is a bounded continuous function. Therefore, f_α extends to $g_\alpha : Y \to [0, 1]$ by Theorem 24.0. Therefore, we can define

$$G : Y \to [0, 1]^\Lambda : y \to (g_\alpha(y))$$

G is automatically continuous, because with the product topology G is continuous iff g_α are for each α. Finally, it goes to show $G(Y) \subseteq C$. This goes as follows, by continuity:

$$G(Y) = G(\bar{X}) \subseteq \overline{G(X)} = \overline{F(X)} \subseteq \bar{C} = C$$

where the last equality follows by compact + Hausdorff implies closed. Uniqueness follows by Lemma 23.7. □

Finally, we show that such an extension Y is unique up to equivalence, justifying it being called *The* Stone-Čech Compactification.

Proposition 24.2. *If X is T3.5, and Y and Y' are two compactifications satisfying Theorem 24.0, then Y is equivalent to Y'.*

Proof. Since Y and Y' are themselves compact, with corresponding embeddings ι, ι' of X, we can apply Theorem 24.1 to produce maps $f : Y \to Y'$ and $g : Y' \to Y$ such that $\iota' = f \circ \iota$ and $\iota = g \circ \iota$.

I claim $g = f^{-1}$. Indeed, note that $g \circ f : Y \to Y$ is an extension of the identity map $Id_X : X \to X \subseteq Y$ to Y. Similarly, the identity $Id_Y : Y \to Y$ is an extension of Id_X. By uniqueness of extensions, $g \circ f = Id_Y$. Similarly, $f \circ g = Id_Y$. Therefore, $f = g^{-1}$, as claimed, and f and g are therefore homeomorphisms which are the identity on X. □

[1]To maximize generality. You may assume T4+T1 for comforts sake.

Definition 24.3. We call the *unique* choice of Y in Theorem 24.0 the **Stone-Čech Compactification** of X. It is often labeled $\beta(X)$, and is established in the following way: $\beta(X)$ is the unique compactification of X such that every continuous function $f : X \to C$ to a compact Hausdorff space C extends to $\beta(C)$.

One can also note that this correspondence between completely regular spaces and their compactifications (compact Hausdorff spaces) is **functorial**: Given $f : X \to Y$ a continuous function between 2 completely regular spaces, we can extend the range to view $f : X \to \beta(Y)$. Now by Theorem 24.0, this implies that there is an extension $f' : \beta(X) \to \beta(Y)$ agreeing with f on X. Call this function $\beta(f) = f'$. As already noted, $\beta(Id_X) = Id_{\beta(X)}$, and it is a quick exercise to check $\beta(f \circ g) = \beta(f) \circ \beta(g)$.

It would be nice to see that some particularly desirable properties pass between X and $\beta(X)$. Here is one example of such a property:

Proposition 24.4. *A completely regular space X is connected if and only if $\beta(X)$ is connected.*

Proof. ($\Rightarrow$): Note that X as a subpace of $\beta(X)$ has the property that $\bar{X} = \beta(X)$. Therefore, we can go through our ancient notes, find Lemma 9.2, and realize this implies $\beta(X)$ is necessarily connected.

($\Leftarrow$): Suppose $X = A \cup B$ is a separation of X. This implies that there exists a continuous function $f : X \to \{0, 1\}$ which is 0 on A and 1 on B. By Theorem 24.1, we know that f extends to a continuous function $f' : \beta(X) \to \{0, 1\}$. This implies $\beta(X)$ is disconnected. □

This statement can yield some very strange results. However, the easier ($\Rightarrow$) direction actually implies every compactification of a connected space X is itself connected. The beauty is that the Stone-Čech compactification actually reverses this. The same is not true for general compactifications:

Example 24.5. If $X = [a, c) \cup (c, b]$, then the one point compactification is exactly $[a, b]$, by adding c into the interval. It should be clear that $\bar{X}$ is connected, but X is not.

Example 24.6. Suppose X is a topological space with the discrete topology. Let's consider some properties of $\beta(X)$. Note that in this case ANY function $f : X \to [0, 1]$ is continuous, so we expect $\beta(X)$ to have cardinality like that of $[0, 1]^{|X|}$! Additionally, the discrete topology is the least compact topology! So we expect $\beta(X)$ to be huge.

- If $A \subseteq X$, then $\bar{A}$ and $\overline{X \setminus A}$ are disjoint. This follows from Proposition 24.4. $X = A \cup X \setminus A$ is a separation of X, so we can find $f : \beta(X) \to \{0, 1\}$ with $f(A) = 0$ and $f(X \setminus A) = 1$. Then $f^{-1}(0)$ is closed and disjoint form $f^{-1}(1)$, and therefore the statement follows.
- If U is open in $\beta(X)$, then $\bar{U}$ is also open. Note that $\overline{X \cap U} = \overline{U}$. This follows since $\bar{X} = \beta X$. In particular, if $x \in \bar{U}$ and V is a neighborhood of x, then $U \cap V \neq \emptyset$. But this implies $V \cap U \cap X$ is non-empty, since $V \cap U$ is open and X is dense. But this implies $x \in U \cap X$, since V is any neighborhood.
- $\beta(X)$ is totally disconnected (but not in general discrete!). If $x, y \in V$, and V is connected, consider U an open set containing x but such that $y \notin \bar{U}$. Then $\bar{U}$ is a clopen subset of Y by the previous part, a contradiction. Of course, any infinite discrete space, such as $\mathbb{Z}$, can not be compact. So $\beta(X)$ is usually not itself discrete. On the other hand, in the finite case $\beta(X) = X$ since X is automatically compact.