Exactness is a very strong property for studying subsequent homomorphisms of groups. Here are a few nice properties of Abelian groups that may help you compute things faster.

Theorem 0.1 (Classification of finitely generated modules over a PID). *Let G be a Abelian group (or more generally, a fg module over a principal ideal domain, such as $\mathbb{Z}$). Then*

$$G \cong \mathbb{Z}^n \oplus T$$

where T is a group of finite order. That is we can extract the most copies of $\mathbb{Z}$ from G and what is left is a finite group. We can say a bit more:

$$T \cong \oplus_{i=1}^n \mathbb{Z}/p_i^{n_i}$$

where p_i are prime numbers, and n_i are natural numbers.

The n is called the **free rank** of G. As a result, every Homology group of a finite dimensional Δ-complex has this form. Next up, we can use this to say something nice about a long exact sequence:

Theorem 0.2 (Rank Lemma?). *Suppose that*

$$0 \to G_1 \to G_2 \to \ldots \to G_n \to 0$$

is an exact sequence of abelian groups. Let n_i be the free rank of G_i. Then

$$\sum_{i=1}^n (-1)^i n_i = 0$$

As an immediate corollary, we can say something nice about the free case:

Corollary 0.3. *If*

$$0 \to \mathbb{Z}^{n_1} \to \ldots \to \mathbb{Z}^{n_m} \to 0$$

is an exact sequence, then $0 = n_1 - n_2 + \ldots + (-1)^m n_m$.

Here is a proof of Theorem 0.2.

Proof. Call the maps $f_i : G_i \to G_{i+1}$, and let $G_0 = G_{n+1} = 0$. For each i, we have an exact sequence

$$0 \to \ker(f_i) \to G_i \to \operatorname{im}(f_i) \to 0$$

It is relatively straightforward to see that

$$\operatorname{rk}(G_i) = \operatorname{rk}(\ker(f_i)) + \operatorname{rk}(\operatorname{im}(f_i))$$

By noticing that if $\operatorname{rk}(\ker(f_i)) \leq \operatorname{rk}(G_i)$ since it injects into G_i, and $\operatorname{rk}(\operatorname{im}(f_i)) \leq \operatorname{rk}(G_i)$ since G_i surjects onto the image. On the otherhand, every copy of $\mathbb{Z}$ not in the image of $\ker(f_i)$ is necessarily in $\operatorname{im}(f_i)$, so they are equal.

We see now that the statement is equivalent to showing that

$$\sum_{i=1}^n n_i = \sum_{i=1}^n (-1)^i \left(\operatorname{rk}(\ker(f_i)) + \operatorname{rk}(\operatorname{im}(f_i)\right) = 0$$

Now, by exactness, $\ker(f_{i+1}) = \operatorname{im}(f_i)$ for each $i = 1, \ldots, n$, so adjacent terms cancel:

$$\sum_{i=1}^n (-1)^i \left(\operatorname{rk}(\ker(f_i)) + \operatorname{rk}(\operatorname{im}(f_i)\right) = (-1)^n \operatorname{im}(f_n) - \ker(f_1)$$

Lastly, these objects are both 0 since $f_n : G_n \to 0$ and f_1 is injective. This completes the proof. $\square$

Thus, the following exact sequence is possible, since $3 - 5 + 4 - 2 = 0$

$$0 \to \mathbb{Z}^3 \to \mathbb{Z}^5 \to \mathbb{Z}^4 \to \mathbb{Z}^2 \to 0$$

whereas this sequence is not possible:

$$0 \to \mathbb{Z}^3 \to \mathbb{Z}^5 \to \mathbb{Z}^3 \to 0$$