HOMEWORK 6: HILBERT-NULLSTELLENSATZ
DUE: APRIL 8TH

1) Let K be an algebraically closed field and let X_1, X_2 be 2 proper varieties in K^n. Let $U_i = X_i^c = K^n \setminus X_i$ be its open complement. Show that $U_1 \cap U_2 \neq \emptyset$. This shows that K^n is irreducible and also demonstrates (and is stronger) that the Zariski Topology is not *Hausdorff*.

Solution: Note that the desired result is equivalent to saying that $X_1 \cup X_2 \neq K^n$. But if $X_1 = V(J_1)$ and $X_2 = V(J_2)$, then

$$K^n = X_1 \cup X_2 = V(J_1 \cap J_2)$$

The only way this is possible is if $J_1 \cap J_2 = 0$, as varieties are in 1-1 correspondence with radical ideals, and 0 is itself radical since $K[x_1, \ldots, x_n]$ is reduced (even a domain). But if $J_1 \cap J_2 = 0$, then either $J_1 = 0$ or $J_2 = 0$ (otherwise we could multiply two non-zero elements). This contradicts properness of X_1 and X_2.

2) Given a field extension $K \subseteq L$, and a variety $X = V(J) \in K^n$, call $(a_1, \ldots, a_n) \in L^n$ an L-valued point of X if $f(a_1, \ldots, a_n) = 0$ for every $f \in J$. Make and prove an analogue of Hilbert-Nullstellensatz using all L-valued points where L/K is algebraic.

Solution: Consider the algebraic closure of K. In the ring $\bar{K}[x_1, \ldots, x_n]$, we have the traditional form of Hilbert-Nullstellensatz from problem 1. Thus if J is a proper ideal of $K[x_1, \ldots, x_n]$, then we have that $J \cdot \bar{K}[x_1, \ldots, x_n]$ is a proper ideal of $\bar{K}[x_1, \ldots, x_n]$. Therefore

$$\emptyset \neq V(J \cdot \bar{K}[x_1, \ldots, x_n]) \subseteq \bar{K^n}$$

Let $(a_1, \ldots, a_n) \in V(J \cdot \bar{K}[x_1, \ldots, x_n])$. Then the field $L = K[a_1, \ldots, a_n]$ is a finite extension field of K, and $(a_1, \ldots, a_n)$ is an L-valued point of J.

Additionally,

$$I(V(J \cdot \bar{K}[x_1, \ldots, x_n])) = \sqrt{J \cdot \bar{K}[x_1, \ldots, x_n]} = \bigcap_{\mathfrak{m} \supseteq J} \mathfrak{m}$$

So as a result, we have

$$I(V(J)) = I(V(J \cdot \bar{K}[x_1, \ldots, x_n])) \cap K[x_1, \ldots, x_n] = \bigcap_{\mathfrak{m} \supseteq J} \mathfrak{m}$$

The last intersection is the intersection over all L-valued points of J! So the statement should be as follows:

Theorem 0.1. *If $J \subsetneq K[x_1, \ldots, x_n]$ is an ideal, then J has an L-valued point for some finite extension L. Additionally,*

$$I(V(J)) = \left(\bigcap_{\substack{(a_1,\ldots,a_n) \\ L\text{-valued for } J}} \langle x_1 - a_1, \ldots, x_n - a_n \rangle \right) \cap K[x_1, \ldots, x_n]$$

3) Suppose $R \subseteq S$ is an integral extension of Noetherian rings. Given $\mathfrak{p} \in \operatorname{Spec}(R)$, show that there are only finitely many prime ideals $\mathfrak{q} \in \operatorname{Spec}(S)$ lying over $\mathfrak{p}$.

Solution: Consider the extension of $\mathfrak{p}$ to S, namely $\mathfrak{p} \cdot S$. Therefore, we can consider the integral extension $R/\mathfrak{p} \subseteq S/\mathfrak{p}S$. Additionally, we can localize at the set $W = R \setminus \mathfrak{p}$. Doing so produces an integral extension

$$\operatorname{Frac}(R/\mathfrak{p}) \subseteq W^{-1}S/\mathfrak{p}S$$

$W^{-1}S/\mathfrak{p}S$ is a localization of a quotient of Noetherian rings, and therefore is Noetherian. Thus it contains only finitely many minimal primes $\mathfrak{q}_1, \cdots, \mathfrak{q}_n$. These minimal primes are actually also maximal, because if we mod out by them, we have an integral extension where $\operatorname{Frac}(R/\mathfrak{p})$ is a field. Finally, examining what $\operatorname{Spec}(W^{-1}S/\mathfrak{p}S)$ is, we see it is the primes of S containing $\mathfrak{p}S$ which do not intersect $R \setminus \mathfrak{p}$. Therefore, each $\mathfrak{q}_i$ are exactly the primes with the property that $\mathfrak{q}_i \cap R = \mathfrak{p}$, as desired.

4) We know that $\operatorname{Spec}(W^{-1}R)$ can be view as a subset of $\operatorname{Spec}(R)$. Show that $\operatorname{Spec}(R_f)$ is exactly the complement of $V(f)$ in $\operatorname{Spec}(R)$.

Solution: $\operatorname{Spec}(R_f)$ is the set of primes $\mathfrak{p} \in \operatorname{Spec}(R)$ which don't intersect $W = \{1, f, f^2, \ldots\}$. This is equivalent to saying $f \notin \mathfrak{p}$, since $1 \notin \mathfrak{p}$ and $f^n \in \mathfrak{p}$ implies $f \in \mathfrak{p}$. But $V(f)$ is exactly the set of primes which contain f. This concludes the proof.

5) Consider the ring

$$R = K[x_{1,1}, x_{2,1}, x_{2,2}, x_{3,1}, \ldots] = K[x_{i,j}]_{i \geq j}$$

This is a non-Noetherian ring, since it has very natural ascending chains of ideals that never stabilize. Consider now the multiplicative set W which is defined as the complement of

$$W^c = \langle x_{1,1} \rangle \cap \langle x_{2,1}, x_{2,2} \rangle \cap \langle x_{3,1}, x_{3,2}, x_{3,3} \rangle \cap \cdots$$

Show that $W^{-1}R$ is a Noetherian ring. What does $\operatorname{Spec}(W^{-1}R)$ look like? [1]

Solution: By our perspective on what Spec of a localization looks like, we know that primes are in bijection with $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} \cap W = \emptyset$. This is to say that $\mathfrak{p}$ lies in W^c. The maximal ideals with this property are

[1] This yields an example of a Noetherian topological space of 'infinite dimension'.

$\langle x_{1,1}\rangle, \langle x_{2,1}, x_{2,2}\rangle, \langle x_{3,1}, x_{3,2}, x_{3,3}\rangle, \cdots$. So every prime must be contained in one of these ideals. Therefore, we can identify

$$\operatorname{Spec}(W^{-1}R) = \coprod_{i=1}^{\infty} \operatorname{Spec}(K[x_{i,1}, \ldots, x_{i,i}]) = \coprod_{i=1}^{\infty} \mathbb{A}_K^i$$

Finally, we need to conclude $W^{-1}R$ is Noetherian. It suffices to check that every prime is finitely generated. But this follows from the above analysis; if $\mathfrak{p} \in \mathbb{A}_K^i \cong \operatorname{Spec}(K[x_1, \ldots, x_i])$, then it is necessarily finitely generated by Hilbert's Basis Theorem.

6) If R is a Noetherian ring, we know it has finitely many minimal primes. Can you describe how to find them geometrically?

What about algebraically? Show that minimal primes of R contain only zero divisors.

Solution: We can reduce to the reduced case (☺) by considering the ring homomorphism $R \to R/nil(R) =: R_{red}$, and noting that this is a bijection on Spec.

There are many ways to argue from here, as this question is mostly philosophical. One way to find such a minimal prime is by considering $\operatorname{Spec}(R)$. Since R is Noetherian, $\operatorname{Spec}(R)$ is a Noetherian topological space. Thus there exists only finitely many irreducible components $X_1, \ldots, X_n$. If we consider $X_i = V(\mathfrak{p}_i) = \operatorname{Spec}(R/\mathfrak{p}_i)$ (since X_i is irreducible, its ideal is prime!), then the $\mathfrak{p}_i$ are the minimal primes of R.

On the algebraic front, first one should note that every minimal prime ideal contains only zero divisors. Indeed, if $x \in \mathfrak{q} \subseteq R$, then localizing R at $\mathfrak{q}$ would yield a local Noetherian ring with a unique prime ideal $\mathfrak{q} = nil(R_{\mathfrak{q}})$. Thus $x^n = 0$ for some minimal n. In the localization, this implies there exists $b \in R \setminus \mathfrak{q}$ such that $x^n b = 0$. But this means $x \cdot x^{n-1} b = 0$, implying x must be a zero divisor (by induction), since $b \neq 0$.

As a result, to find minimal primes not containing x, we could consider $\operatorname{Ann}(x)$. Modding out by this makes x a NZD in $R/\operatorname{Ann}(x)$. As a result, a minimal prime not containing x is simply any of the minimal primes of $R/\operatorname{Ann}(x)$.