CLASS 33, MAY 8TH: NORMAL IN CODIMENSION 1

Today we will study what normal tells you about the singular locus of a given algebraic variety. This is one of the essential results that make normal rings/varieties a nice class of objects to study. First, one last example of a valuation ring in general.

Example 33.1. If R is a DVR with parameter t, we can construct a new valuation ring by adjoining all the roots of t. Let $A = R[t^{\frac{1}{2}}, t^{\frac{1}{3}}, \ldots]_{\mathfrak{m}}$. It is easy to check that

$$L = \mathrm{Frac}(A) = \mathrm{Frac}(R)(t^{\frac{1}{2}}, t^{\frac{1}{3}}, \ldots)$$

Therefore, we can define a valuation $v : L \to \mathbb{Q} : t^\alpha \mapsto \alpha$, which extends the valuation for R. Then A is exactly the set of elements for which $v(a) \geq 0$.

We can do a similar thing with $\mathbb{R}$ to produce a valuation ring with value group $\mathbb{R}$.

Now we turn to the condition of being normal. Recall the definition:

Definition 33.2. An integral domain R is **normal** if R is integrally closed in $K = \mathrm{Frac}(R)$.

First, we will start with a local characterization:

Lemma 33.3. *If R is an integral domain, then $R_\mathfrak{p} \subseteq K$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Furthermore,*

$$R = \bigcap_{\mathfrak{p} \in \mathrm{Spec}(R)} R_\mathfrak{p} = \bigcap_{\mathfrak{m} \in m\text{-}\mathrm{Spec}(R)} R_\mathfrak{m}$$

Note these intersections are all happening in K by the first part.

Proof. For $x \in K$, define the **ideal of denominators** of x to be

$$D(x) = \{r \in R \mid rx \in R\} = \{0\} \cup \{s \in R \mid x = \frac{r}{s} \text{ for some } r \in R\}$$

This is an ideal. For $x \in K$, we have $x \in R$ iff $D(x) \neq R$. If $D(x)$ is proper, it is contained within some maximal ideal $\mathfrak{m}$, and therefore $x \notin R_\mathfrak{m}$. $x \notin \bigcap_{\mathfrak{m} \in \mathrm{m\text{-}Spec}(R)} R_\mathfrak{m}$ as desired. □

Now we can prove that being normal is a local condition, much like being 0 as a module.

Proposition 33.4. *TFAE:*

(a) *R is normal*
(b) *$R_\mathfrak{p}$ is normal for all $\mathfrak{p} \in \mathrm{Spec}(R)$.*
(c) *$R_\mathfrak{m}$ is normal for all $\mathfrak{m} \in m\text{-}\mathrm{Spec}(R)$.*

Proof. For (a)⇒(c)⇒(b), I prove instead that $W^{-1}R$ is normal if R is normal for any multiplicative set W. If $\alpha \in K$ satisfies some monic polynomial with coefficients in $W^{-1}R$, then

$$\alpha^n + \frac{a_1}{b_1}\alpha^{n-1} + \ldots + \frac{a_n}{b_n} = 0$$

Therefore, we can multiply the whole equation by $(b_1 \cdots b_n)^n$ to produce a non-monic relation in R:

$$(b_1 \cdots b_n\alpha)^n + a_1 b_2 \cdots b_n (b_1 \cdots b_n \alpha)^{n-1} + \ldots + b_{n-1}^{-1}(b_1 \cdots b_{n-1})^n a_n = 0$$

Therefore, since R is normal, we note $b_1 \cdots b_n \alpha \in R$. As a result, we can conclude that $\alpha \in S^{-1}R$, as desired.

Since (b)$\Rightarrow$(c) is a triviality, it suffices to check (c) implies (a). If x is integral over R, then x is clearly integral over $R_{\mathfrak{m}}$ for each $\mathfrak{m}$. As a result, $x \in R_{\mathfrak{m}}$ by normality. But then $x \in \cap_{\mathfrak{m}} R_{\mathfrak{m}} = R$ by Lemma 33.3. $\square$

This brings us to the central theorem about Normal domains:

Theorem 33.5. *Let R be a normal Noetherian domain. If $\mathfrak{p} \neq 0$ is minimal among non-zero primes, then $R_{\mathfrak{p}}$ is a DVR. Furthermore, if $I \neq 0$ is principal, $\mathfrak{p} \in Ass(R/I)$ are among the minimal non-zero prime ideals of R.*

Proof. By Lemma 33.3, $R_{\mathfrak{p}}$ is a normal Noetherian domain. Moreover, $\mathrm{Spec}(R_{\mathfrak{p}})$ contains only $\mathfrak{p}$ and 0 by assumption of minimality of $\mathfrak{p}$. Therefore, by Theorem 31.3, we have $R_{\mathfrak{p}}$ is a DVR.

For the second statement, we first reduce to the local case. Given $\mathfrak{p} \in \mathrm{Ass}(R/I)$, consider the local ring $(R_{\mathfrak{p}}, \mathfrak{p}R_{\mathfrak{p}})$. Let $I' = xR_{\mathfrak{p}}$ be the extension of I. Since $R_{\mathfrak{p}}/I' = (R/I)_{\mathfrak{p}}$, we have by Corollary 29.7 that $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{Ass}(R_{\mathfrak{p}}/I')$. If we can show $\mathfrak{p}R_{\mathfrak{p}}$ is a minimal non-zero prime ideal of $R_{\mathfrak{p}}$, then $\mathfrak{p}$ is minimal non-zero in R by our relationships of ideals through localization.

Let $\mathfrak{m} = \mathfrak{p}R_{\mathfrak{p}}$. Since $\mathfrak{m} \in \mathrm{Ass}(R_{\mathfrak{p}}/I')$, there exists $y \notin I'$ such that $y \cdot \mathfrak{m} \subseteq I'$. As a result, there exists $x \in I'$ such that $\frac{y}{x} \in \mathrm{Frac}(R)$ satisfies $\frac{y}{x} \cdot \mathfrak{m} \subseteq R_{\mathfrak{p}}$. Therefore, we can mirror the proof of Theorem 31.3:

Case 1: $\frac{y}{x}\mathfrak{m} \subseteq \mathfrak{m}$. In this case, $\frac{y}{x}$ is integral over $R_{\mathfrak{p}}$. But $R_{\mathfrak{p}}$ is normal, so $\frac{y}{x} \in R_{\mathfrak{p}}$. But this would contradict that $y \notin I'$.

Case 2: $\frac{y}{x}\mathfrak{m} = R_{\mathfrak{p}}$. This implies the existence of $y' \in \mathfrak{m}$ such that $\frac{yy'}{x} = 1$, or equivalently $x = yy'$. But then we would have for all $z \in \mathfrak{m}$, $z\frac{y}{x} = \frac{z}{y'} \in R_{\mathfrak{p}}$, or $z \in \langle y' \rangle$.

In this case, we know $\mathfrak{m} = \langle y' \rangle$. As a result, $R_{\mathfrak{p}}$ is a DVR and $\mathfrak{m}$ is a minimal non-zero prime ideal. $\square$

Geometrically, what we have just ascertained is that the singular points of a normal variety must exist in codimension ≥ 2. This is a major result, as it gives us the following information:

(a) Normal curves are non-singular.
(b) Normal surfaces have isolated point singularities.
(c) Normal 3-folds have at worst singularities lying on a curve.
(d) Normal n-folds have at worst singularities lying on a $(n-2)$-fold.

Example 33.6. We have already shown that the cusp $R = K[x,y]/\langle y^2 - x^3 \rangle$ is non-normal; $\tilde{R}$ contains $\frac{y}{x}$ as a root of $t^2 - x$.

Additionally, on the Midterm it was demonstrated that $R = K[x,y]/\langle y^2 - x^3 - x \rangle$ is non-normal for the same reason. These 2 facts are special cases of (a) in this classification.

The Whitney umbrella is an example of a surface with a singular curve $C = V(x,y)$ (or worse if $char(K) = 2$); $R = K[x,y,z]/\langle x^2 z - y^2 \rangle$. Thus we immediately know R cannot be normal. This is further verified by noting

$$\left(\frac{y}{x}\right)^2 = \frac{x^2 z}{x^2} = z \qquad \text{or} \qquad \left(\frac{xz}{y}\right)^2 = \frac{y^2 z}{y^2} = z$$

This condition of being regular in codimension 1 is insufficient to verify normalcy. You also need a condition known as S_2, Serre's second condition. This is beyond our scope!