To the best of my knowledge, these are the tools we have developed so far for computing $\pi_1(X, x)$. You can use any of them without proof:

1° If X is a contractible space, then $\pi_1(X) = 0$.

2° If X is path connected, then for any $x, y \in X$, we have that $\pi_1(X, x) \cong \pi_1(Y, y)$ and the isomorphism is given by 'change of base': $\beta_h(\gamma) = \bar{h}\gamma h$ for a path h connecting x to y. Thus we often write $\pi_1(X)$.

3° (Not mentioned in class, but in office hours): If X is not path connected, then X can be broken up into path-components $X = \coprod X_\alpha$, with each X_α path connected. Therefore, if $x \in X_\alpha$, then

$$\pi_1(X, x) \cong \pi_1(X_\alpha, x) = \pi_1(X_\alpha)$$

4° $\pi_1(S^1) \cong \mathbb{Z}$, and additionally, if $n = 0, 2, 3, 4, \ldots$, then $\pi_1(S^n) = 0$.

5° If $\phi : X \to Y$ is a homotopy equivalence, then $\pi_1(X, x) \cong \pi_1(Y, f(x))$ by ϕ_*. This passes through the lemma that if $\phi_t : X \to Y$ is a homotopy between ϕ_0 and ϕ_1, then $(\phi_0)_* = \beta_h(\phi_1)*$, with β_h as above.

6° **Van Kampen's Theorem:** If X is a topological space with a covering $X = \cup_\alpha A_\alpha$ such that each A_α is **open** and **path connected** and **contains x**, and such that the **intersection of any 2** of them $A_\alpha \cap A_\beta$ is **path connected**, then we can build up a surjection

$$\iota : *_\alpha \pi_1(A_\alpha, x) \to \pi_1(X, x)$$

as follows: Note that for each A_α, we have $\iota_*^\alpha : \pi_1(A_\alpha) \to \pi_1(X)$. Now, define $\iota = *_\alpha \iota^\alpha$, which is defined on words as

$$\iota(\gamma_\alpha * \gamma_\beta * \ldots * \gamma_\eta) = \iota_*^\alpha(\gamma_\alpha) \cdot \iota_*^\beta(\gamma_\beta) \cdots \iota_*^\eta(\gamma_\eta).$$

The theorem is that this map is a **surjective homomorphism**, and that furthermore if any **3-wise intersection** $A_\alpha \cap A_\beta \cap A_\gamma$ is also **path connected**, then the kernel is generated in the intersections:

$$\langle \gamma_\alpha \gamma_\beta^{-1} \mid \gamma_\alpha(t), \gamma_\beta(t) \in A_\alpha \cap A_\beta, \text{ and } \gamma_\alpha \simeq \gamma_\beta \rangle$$

This allows us to avoid over counting the loops of X.

7° **Simplified Van Kampen:** If you are dealing with finite open covers you can simplify the computation substantially by considering 2 at a time and building the group up inductively.

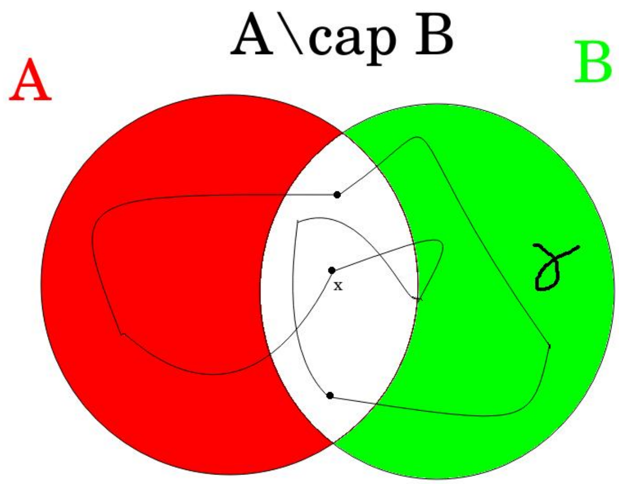
If $A, B \subseteq X$ are **open** and **path connected**, and $A \cap B$ is path connected, then we can compute $\pi_1(X)$ by the following amalgamated free product:

$$\pi_1(X) = \pi_1(A) *_{\pi_1(A \cap B)} \pi_1(B)$$

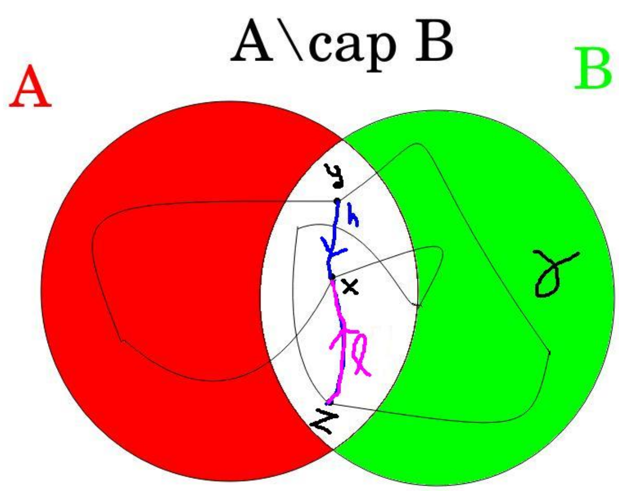
Where the homomorphisms to each space are induced by $i_A : A \cap B \hookrightarrow A$ and $i_B : A \cap B \hookrightarrow B$.

As an excercise, go from full Van Kampen to the simplified version to exercise your definitions. The following picture (produced in something similar to paint) demonstrates how this works:

Given a path γ in $X = A \cup B$



We can decompose it by



Let γ_1 be γ from x to y, γ_2 from y to z and γ_3 from z back to x. Then $\gamma_1 \cdot h \in \pi_1(A)$ and $\bar{h} \cdot \gamma_2 \cdot l, \bar{l} \cdot \gamma_3 \in \pi_1(B)$. Thus γ is in the image of ι by

$$\iota((\gamma_1 \cdot h) * (\bar{h} \cdot \gamma_2 \cdot l) * (\bar{l} \cdot \gamma_3)) = \gamma$$