CLASS 12, OCTOBER 3: COMPACTNESS OF PRODUCTS

I now state a corollary of the results from last class.

Corollary 12.1. *If $f : X \to Y$ is a continuous bijective map with X compact and Y Hausdorff, then f is a homeomorphism.*

Proof. It suffices to check that f is a closed map. This is the content of homework #3. $\square$

Next we check that finite products behave well with respect to compactness:

Theorem 12.2. *If $X_1, \ldots, X_n$ are compact spaces, then so is $X_1 \times \ldots \times X_n$.*

Proof. By induction it suffices to check that the result is true for a product of 2 spaces, say X and Y. Furthermore, given a cover of $X \times Y$, say $\bigcup_\alpha U_\alpha$, we know that each U_α has the structure of a collection of products:

$$U_\alpha = \bigcup_\beta U^X_{\alpha,\beta} \times U^Y_{\alpha,\beta}$$

So if we can prove it for covers of the form

$$X \times Y = \bigcup_\alpha U_\alpha \times V_\alpha$$

we are done. So consider such a cover and fix $x_0 \in X$. Let α^{x_0} run through a subset of indices α for which $(x_0, y) \in U_{\alpha^{x_0}} \times V_{\alpha^{x_0}}$. Then the fiber of x_0 has the property that

$$\{x_0\} \times Y \subseteq \bigcup_{\alpha^{x_0}} U_{\alpha^{x_0}} \times V_{\alpha^{x_0}}$$

Projecting onto Y, we see that their images $V_{\alpha^{x_0}}$ cover Y. Therefore, we can find finitely many to cover Y by compactness: $Y = V_{\alpha_1^{x_0}} \cup \ldots \cup V_{\alpha_{n_{x_0}}^{x_0}}$. Let $U_{x_0} = U_{\alpha_1^{x_0}} \cap \ldots \cap U_{\alpha_{n_{x_0}}^{x_0}}$, which is open by finiteness. Then we have the chain of inclusions

$$\{x_0\} \times Y \subseteq U_{x_0} \times \left(V_{\alpha_1^{x_0}} \cup \ldots \cup V_{\alpha_{n_{x_0}}^{x_0}}\right) \subseteq \left(U_{\alpha_1^{x_0}} \times V_{\alpha_1^{x_0}}\right) \cup \ldots \cup \left(U_{\alpha_{n_{x_0}}^{x_0}} \times V_{\alpha_{n_{x_0}}^{x_0}}\right)$$

Now we can use our standard trick combined with compactness:

$$X = \bigcup_{x \in X} U_x = U_{x_1} \cup \ldots \cup U_{x_n}$$

But this gives us our refinement:

$$\begin{aligned} X \times Y =& (U_{x_1} \cup \ldots \cup U_{x_n}) \times Y = & \bigcup_{i=1}^n (U_{x_i} \times Y) \\ =& \bigcup_{i=1}^n \left(U_{x_i} \times \bigcup_{j=1}^{n_{x_i}} V_{\alpha_j^{x_0}} \right) = & \bigcup_{i=1}^n \bigcup_{j=1}^{n_{x_i}} \left(U_{x_i} \times V_{\alpha_j^{x_0}} \right) \end{aligned}$$

This shows compactness, proving the claim. $\square$

This shows for example that $\mathbb{T}^n$ is a compact space, since $S^1 \subseteq \mathbb{R}$ is closed. Many more examples, such as n-cubes are also compact as a result: $B = [a_1, b_1] \times \ldots \times [a_n, b_n]$. Of course, this doesn't extend to infinite products with the box topology:

Example 12.3. Consider $Y = [0,1]^{\mathbb{N}} \subseteq \mathbb{R}^{\mathbb{N}} = X$, where we give X the box topology and Y the subspace topology. Consider the covering of $\mathbb{R}$ given by $U = [0, \frac{2}{3})$ and $V = (\frac{1}{3}, 1]$, which are both open in the subspace $[0,1]$. We can consider the countable cover given by infinite products of either U or V. There is no finite refinement.

Later on we will prove that the same holds for arbitrary products of topological spaces with the product topology. This is the famous Tychonoff Theorem. To conclude, I would like to present the corresponding statements for closed sets.

Definition 12.4. A collection $\mathcal{C}$ of subsets of X is said to have the **finite intersection property** if any finite subset Λ of $\mathcal{C}$, the intersection of elements of Λ is non-empty.

This may seem strange, but it gives a closed classification of compactness:

Theorem 12.5. *Let X be a topological space. Then X is compact if and only if any subset $\mathcal{C} \subset \mathcal{P}(X)$ containing only closed subsets and having the finite intersection property also has the intersection property:*

$$\bigcap_{C \in \mathcal{C}} C \neq \emptyset$$

Proof. $(\Leftarrow)$: Suppose X is compact. Then given such a $\mathcal{C}$, assume $\mathcal{C}$ doesn't have the intersection property:

$$\bigcap_{C \in \mathcal{C}} C = \emptyset$$

Then consider $\mathcal{C}^c = \{U^c \mid U \in \mathcal{C}\}$. This is a set of open subsets, and indeed $\mathcal{C}^c$ forms an open cover of X:

$$X = \emptyset^c = \left(\bigcap_{C \in \mathcal{C}} C\right)^c = \bigcup_{C \in \mathcal{C}} C^c = \bigcup_{C \in \mathcal{C}^c} C$$

But this implies a finite subcover exists by compactness:

$$X = C_1 \cup \ldots \cup C_n$$

But this implies $\emptyset = C_1^c \cup \ldots \cup C_n^c$, contradicting the finite intersection property of $\mathcal{C}$.

$(\Rightarrow)$: Suppose X is not compact. Taking a cover with no finite subcover, and $\mathcal{C}$ the collection of complements, produces as example of a set $\mathcal{C}$ with the finite intersersection property but not the intersection property. $\square$

This shows a nice thing about nested closed subsets in a compact space X:

Corollary 12.6. *If $C_1 \supseteq C_2 \supseteq \ldots$ is a nested collection of non-empty closed subsets of a compact topological space, then*

$$\emptyset \neq \bigcap_{i=1}^{\infty} C_i$$

This isn't true for non-compact spaces. Indeed, consider the sequence $C_n = [n, \infty)$.