CLASS 0, DATE SEPTEMBER 6: INTRODUCTION

There are many reasons to study complex analysis.

1) **It is beautiful in its own right:** It is a common misconception that complex analysis would be more complicated than real analysis, due to the fact that $\mathbb{C}$ is more complicated and even contains $\mathbb{R}$. However there are many things that make it far simpler to study.

   One result we will prove in the first half of the course is the following:

   Theorem 0.1 (The Fundamental Theorem of Algebra). *If $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is a polynomial with complex coefficients, then $p(x)$ factors into linear terms.*

   Another question we will formalize within the first half of the class is the question of differentiability of complex functions. It turns out that something quite magical happens which greatly differs from the case of real analysis:

   Theorem 0.2 (Informal Version). *If $f : \mathbb{C} \to \mathbb{C}$ is a complex valued function, then the first derivative $f'(z)$ exists if and only if the* **all** *derivatives $f^{(n)}(z)$ exist.*

   It is easy to conceive of real valued functions for which derivatives exist up to an arbitrary order, and then fail to exist for higher orders (via, for example, integrating $|x|$).

2) **Signal Processing:** Historically (since roughly the 17^{th} century), complex analysis was developed to study waves in physics. There is a particularly easy way to study a wave as a complex valued function; using the exponential! It simplifies a great deal of the difficult trigonometric identities and unifies other forces of nature.

   One of the central things we aim to study in this course is the Fourier Transform. Its definition is given as follows: if $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function, its **Fourier Transform** is given by

   $$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi ixy}dx$$

   For now (until next week) you can think of the exponential factor as making f into a wave.

   The more precise advantage of doing this is that it has the ability to transfer complex differential equations to easier to deal with algebraic (polynomial) equations. Moreover, there is an inverse Fourier Transform which can transfer all of the data back, making for a very simple transference of a solution back to the original space.

3) **Number Theory:** It may be surprising (it certainly was to me the first time I saw these types of results!) that complex analysis can relate to something arithmetic. It does so however via some of the deepest mathematical results and conjectures.

One of the most baffling problems in mathematics is measuring the growth rate of the prime numbers. Several estimates are easily generated: Let $\pi(x)$ denote the number of primes less than a given real number x. Then

$$\pi(x) \sim \frac{x}{\ln(x)} \sim \int_2^x \frac{dt}{\ln(t)}$$

This is the renowned **prime number theorem**, one of the most foundation results about the primes. Now the question becomes 'they are asymptotically the same, i.e. the limit of their quotients are 1 as $x \to \infty$, but how far off are we along the way?'

Denote by $Li(x)$ the integral on the far right. It is known that

$$\pi(x) = Li(x) + O\left(x \cdot e^{-a\sqrt{\ln(x)}}\right)$$

for some $a > 0$, where the O notation implies that they differ by at most a constant times the quantity within.

One of the most famous open problems in mathematics is the **Riemann Hypothesis**, which states that the non-trivial complex roots of (an analytic continuation of) $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ all lie on the line $s = \frac{1}{2} + iy$.

If (and only if) RH is true, this would improve our approximation of the above error to

$$\pi(x) = Li(x) + O\left(\sqrt{x}\log(x)\right)$$

The constant is even estimated to be $\frac{1}{8\pi}$!

We will study the ζ function later on and time permitting touch on number theory near the end of the course.

4) **Algebra, Topology, etc:** The list goes on :)