CLASS 26, NOVEMBER 13TH: JENSEN'S FORMULA

We will now transfer our studies to entire functions. We have proved Liouville's theorem, which says that interesting entire functions are unbounded. We also know that polynomials have finitely many zeroes, equal to their degree. What about other entire functions? How are they distributed?

We will build up the following comparison formally: the faster a function grows as $|z| \to \infty$, the more zeroes it will have.

Let $\mathbb{D}_R = B(0, R)$, and C_R be D_R's boundary circle. Assume f is not the zero function.

Theorem 26.1 (Jensen's Formula)**.** *Let Ω contain $\bar{\mathbb{D}_R}$. Assume f is holomorphic in Ω, $f(0) \neq 0$, and f doesn't vanish on C_R. If $z_1, \ldots, z_N$ are the zeroes of f inside $\mathbb{D}_R$, then*

$$\log|f(0)| = \sum_{k=1}^{N} \log\left(\frac{|z_k|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\theta})|d\theta$$

It should be noted that this formula is a little unintuitive. Here is an interpretation of each part. The sum is representative of a logarithmic measure of the number of zeroes in the disc. The integral is measure the average logarithmic magnitude of the function on the circle. The LHS is the strangest of all, but you should think of this as playing a role as it does in the MVT.

Jensen's Formula will be very important in our total study of analytic functions.

Proof. **Step 1)** First, note that if f_1 and f_2 satisfy the hypothesis and conclusion of the theorem, so does $f_1 \cdot f_2$. This is simply using the property of log for real valued functions.

Step 2) Consider $g(z) = \frac{f(z)}{(z-z_1)\cdots(z-z_n)}$. This function is bounded near each z_j, and thus only has removable discontinuities at z_j. So

$$f(z) = g(z)(z - z_1)\cdots(z - z_n)$$

Now, g is nowhere vanishing on $\mathbb{D}_R$. Thus by 1), it suffices to check the result for functions like g and $z - w$.

Step 3) Consider g as in 2). We must establish the identity

$$\log|g(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|g(Re^{i\theta})|d\theta$$

In a slightly larger disc, we can write $g = e^{h(z)}$ for some holomorphic function h. Now, note that

$$|g(z)| = |e^{h(z)}| = |e^{u+iv}| = e^u$$

Thus $\log|g(z)| = u(z)$. The above formula is simply the mean value property for u.

Step 4) It goes to check the fomula for $z - w$. We need to show

$$\log|w| = \log\left(\frac{|w|}{R}\right) + \frac{1}{2\pi}\int_0^{2\pi} \log|Re^{i\theta} - w|d\theta$$

Since $\log(\frac{w}{R}) = \log|w| - \log(R)$ and $\log|Re^{i\theta} - w| = \log(R) + \log|e^{i\theta} - \frac{w}{R}|$, it is enough to show that

$$\int_0^{2\pi} \log|e^{i\theta} - a| d\theta = 0$$

if $|a| < 1$. This is equivalently

$$\int_0^{2\pi} \log|1 - ae^{i\theta}| d\theta = 0$$

We can use $F(z) = 1 - az$, which vanishes nowhere in the disc. Therefore, we can write $F(z) = e^{G(z)}$. Again, $|F| = e^{Re(G)}$, so $\log|F| = Re(G)$. Since $F(0) = 1$, $\log|F(0)| = 0$. Now, again the mean value theorem applied to $\log|F(z)|$ proves the desired result. □

Let $\mathfrak{n}_f(r)$ denote the number of zeroes, with multiplicity, inside $\mathbb{D}_r$. Following through with Theorem 26.1, I claim that the following result holds given the assumptions on f above hold:

Corollary 26.2. *If f is as in Theorem 26.1, then*

$$\int_0^R \frac{\mathfrak{n}_f(r)}{r} dr = \frac{1}{2\pi} \int_0^{2\pi} \log|f(Re^{i\theta})| d\theta - \log|f(0)|$$

This formula is better; the left shows the growth of the number of zeroes, and the right how fast it is growing. It is immediately realized from Theorem 26.1 and the following Lemma:

Lemma 26.3. *If $z_1, \ldots, z_N$ are the zeroes of f inside $\mathbb{D}_R$, then*

$$\int_0^R \frac{\mathfrak{n}_f(r)}{r} dr = \sum_{k=1}^N \log\left|\frac{R}{z_k}\right|$$

Proof. Notice that

$$\sum_{k=1}^N \log\left|\frac{R}{z_k}\right| = \sum_{k=1}^N \int_{|z_k|}^R \frac{dr}{r}$$

If we define η_k to the the indicator function on $r - |z_k|$, then then $\sum_{k=1}^N \eta_k(r) = \mathfrak{n}_f(r)$. The lemma is now realized by

$$\sum_{k=1}^N \int_{|z_k|}^R \frac{dr}{r} = \sum_{k=1}^N \int_0^R \eta_k(r) \frac{dr}{r} = \int_0^R \sum_{k=1}^N \eta_k(r) \frac{dr}{r} = \int_0^R \frac{\mathfrak{n}_f(r)}{r} dr$$

□

Corollary 26.4. *If f is entire with $f(0) = 1$, and $M(r) = \sup\{|f(re^{i\theta})|\}$, then $\mathfrak{n}_f(r)\log(2) \leq \log(M(2r))$.*

Proof.

$$\mathfrak{n}_f(r)\log(2) \leq \int_r^{2r} \frac{\mathfrak{n}_f(s)}{s} ds \leq \int_0^{2r} \frac{\mathfrak{n}_f(s)}{s} ds = \frac{1}{2\pi} \int_0^{2\pi} \log|f(2re^{i\theta})| d\theta \leq \log(M(2r))$$

where $z_1, \ldots, z_m \in B(0, r)$ are the zeroes. □