CLASS 17, APRIL 1ST: GENERAL LOCALIZATION

So far we have only been able to localize integral domains at their prime ideals (or even multiplicative sets). This was ensured by viewing these objects within their fraction field and allowing only denominators in the multiplicative set. This has several advantages, such as the map from a ring to its localization being injective. The general case is not so much more intense to construct when viewed formally, and allows us to also localize modules as well!

Definition 17.1. Let R be a ring and $W \subseteq R$ be a multiplicative set (recall we require $1 \in W$ and $0 \notin W$). Then the **localization of R at W** is the ring

$$W^{-1}R := W \times R/\sim$$

where $(w, r) \sim (w', r')$ if and only if there exists $s \in W$ such that $s(rw' - wr') = 0$ in R. Furthermore, the operations are defined as

$$(w, r) + (w', r') = (ww', rw' + r'w)$$

$$(w, r) \cdot (w', r') = (ww', rr')$$

Lastly, there is also a **localization map**:

$$R \to W^{-1}R : r \mapsto (1, r)$$

Notation is often abused, and we simply write $\frac{r}{s}$ instead of (s, r). It should be noted that often times the elements of the localization do NOT behave like fractions, except perhaps spiritually.

Lemma 17.2. *This new notion of localization is isomorphic to our old notion of localization for integral domains.*

I leave it to you to check this feature of the new method (forever more, simply called localization).

Example 17.3 (Zero Divisors)**.** Suppose $z \in W$ is a zero divisor for R. Note that $(1, 0) \sim (w, 0)$ for any $w \in W$. Therefore, whenever $r \cdot z = 0$, we have that $(1, r) \sim 0$, since $z(r - 0) = 0$. Therefore, any r multiplying with z to 0 **becomes** 0 in $W^{-1}R$. Rephrasing this, if we call $l : R \to W^{-1}R$ the localization map, then

$$\ker(l) = \{r \in R \mid \exists w \in W \text{ such that } rw = 0\}$$

Example 17.4 (Specific Example)**.** Consider the ring $R = K[x, y, z]/\langle xy, xz\rangle$. If we localize at the multiplicative set $W = \{1, x, x^2, \ldots\}$, we see that $(1, y) = (1, z) = 0$ in the localized ring. So $W^{-1}R = K[x, x^{-1}]$.

If we chose instead the multiplicative set $W = \{1, y, z, y^2, yz, z^2, \ldots\}$, then we would get that $x = 0$ in the localization. Therefore

$$W^{-1}R = K[y, y^{-1}, z, z^{-1}]$$

Example 17.5. If we weakened our assumption to allow for multiplicative sets that contain 0, we would simply eliminate every element of the ring: $(0,0) \sim (r,r')$ for any choice of r, r'. As a result, we get

$$W^{-1}R = 0 \iff 0 \in W \iff W \cap nil(R) \neq \emptyset$$

Therefore, since such a case is so boring, I opt to ignore it when declaring a multiplicative set. Some authors don't enforce such a requirement.

We can furthermore upgrade one of our previous homework results without any change to localization:

Proposition 17.6. *If R is a ring and W is a multiplicative set, then*

$$\mathrm{Spec}(W^{-1}R) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \cap W = \emptyset\}$$

I leave it to you to check this result, but note the following example:

Example 17.7. Let's examine what the prime ideals of $W^{-1}\mathbb{Z}$ are where $W = \{1, \underline{2}, \underline{3}, 4, \underline{5}, 6, 8, 9, 10, \ldots\}$. By Proposition 17.6, we have that they are in bijection with the primes primes of $\mathbb{Z}$ not intersecting W. So those prime ideals are exactly those generated by prime numbers not 2, 3, or 5. So they are $0, \langle 7\rangle, \langle 11\rangle, \langle 13\rangle, \ldots$. In fact, it is easy to check that $W^{-1}\mathbb{Z} \cong \mathbb{Z}[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}]$.

Two of the most common examples of multiplicative sets are prime ideals $\mathfrak{p}$ and sets focused around an element $f \in R$: $\{1, f, f^2, \ldots\}$. As a result, we give these two localizations special names: $R_\mathfrak{p}$ and R_f respectively. Geometrically, this should be thought of as focusing locally around the point $\mathfrak{p} \in \mathrm{Spec}(R)$ and focusing away from primes containing f respectively. In fact, we can formally present R_f:

Proposition 17.8. *$R_f \cong R[x]/\langle xf - 1\rangle$.*

Proof. There is a natural surjective map $\psi : R[x] \to R_f$, where $\psi(r) = (1, r)$ and $\psi(x) = (f, 1)$. It only goes to show (by the isomorphism theorems) that $\ker(\psi) = \langle xf - 1\rangle$. Suppose $h(x) \in \ker(\psi)$. That is to say $h(\frac{1}{f}) = 0$ in R_f. Since h has some finite degree, there exists n such that $f^n h(\frac{1}{f}) \in R$. As a result, $f^n h(x) = g(fx) \in R[x]$ satisfies $g(1) = 0$. By the division algorithm, $g = (y-1)g_1(y)$. But as a result,

$$f^n h(x) = g(fx) = (fx - 1)g_1(fx)$$

Finally, noting that $fx - 1$ and f are coprime, we conclude that $h(x) \in \langle xf - 1\rangle$. This is the desired result. $\square$

Finally, I note a universal property of localization.

Theorem 17.9. *If W is a multiplicative set, and $\psi : R \to S$ is a ring homomorphism for which $\psi(w)$ is a unit for every $w \in W$, then ψ factors through the localization map. That is the following diagram commutes: $\psi = \psi' \circ \varphi$*

$$\begin{tikzcd} R \arrow[r, "\psi"] \arrow[dr, "\varphi"'] & S \\ & W^{-1}R \arrow[u, dashed, "\psi'"] \end{tikzcd}$$

Proof. See homework. $\square$