CLASS 31, MAY 3RD: EQUIVALENT CONDITIONS FOR DVRS

Today we will approach DVRs from a different angle (or two). This gives some equivalent characterizations and produces an easy way to see when a ring is a DVR. I will first note a specific case of the much broader Krull Intersection Theorem.

Lemma 31.1. *If R is a Noetherian integral domain, and $0 \neq t \in R$ is a non-unit, then $\cap_{n=1}^{\infty} \langle t^n \rangle = 0$.*

Proof. Note that

$$\langle t \rangle \supsetneq \langle t^2 \rangle \supsetneq \cdots \supsetneq \langle t^n \rangle \supsetneq \langle t^{n+1} \rangle \supsetneq \cdots$$

Each inclusion is strict: If $t^{n+1}g = t^n$. This implies that $t^n(tg-1) = 0$. Therefore, $tg-1 = 0$, since $t \neq 0$ in a domain. But t is not a unit! So no such g exists.

Suppose $x \in \langle t \rangle$. Then $x = tx_1$. Continue in this process with x_1 to produce

$$\langle x \rangle \subseteq \langle x_1 \rangle \subseteq \langle x_2 \rangle \subseteq \ldots \subseteq \langle x_n \rangle$$

This process must stop by the Noetherian property. But this means $x_n \notin \langle t \rangle$, which is to say $x \notin \langle t^{n+1} \rangle$. $\square$

Theorem 31.2. *Let R be a local domain with $\mathfrak{m} = \langle t \rangle$, $t \neq 0$. Assume $\bigcap_{i=1}^{\infty} \langle t^n \rangle = 0$. Then*

(a) *Every $0 \neq x \in R$ has the form $x = ut^n$ for u a unit and some n.*
(b) *Define $v(x) = n$, with x as in (a). For $\frac{a}{b} \in \mathrm{Frac}(R)$, define $v(\frac{a}{b}) = v(a) - v(b)$. Then v is a discrete valuation on $\mathrm{Frac}(R)$.*
(c) *Every non-zero ideal $I = \langle t^n \rangle$ for some n.*

Proof. (a) Since $\mathfrak{m} = \langle t \rangle$ is maximal, everything indivisble by t is a unit. Thus x is either a unit ($x = u$) or $x = tx_1$. Continuing inductively with x_1 to produce $x = t^n x_n$ whenever possible. Since $\bigcap_{i=1}^{\infty} \langle t^n \rangle = 0$, we have that at some point x_n is not divisible by t, and is thus a unit. Therefore, $x = u \cdot t^n$, with $u = x_n$.

(b) v is well defined by the previous step. $v(xx') = v(ut^n u' t^{n'}) = v(uu' t^{n+n'}) = n + n'$. Similarly, WLOG assuming $n \geq n'$

$$v(x + x') = v(ut^n + u't^{n'}) = v(t^{n'}(ut^{n-n'} + u'))) = n' + v(ut^{n-n'} + u') \geq n' = \min\{n, n'\}$$

(c) Choose $n = \min\{n' \mid x = ut^{n'} \in I\}$. Then clearly $I = \langle x^n \rangle$.

$\square$

This shows that ALL Noetherian local integral domains with principal maximal ideal are DVRs. In fact, even non-Noetherian rings with the above description satisfy many of the desirable properties of a DVR.

Now I move to the so-called *Main Theorem on DVRs*. It says that DVRs are exactly Noetherian normal domains with exactly 2 pointed spectra.

Theorem 31.3. *The following sets are in natural bijection:*

$$\{R \textit{ a DVR}\} \longleftrightarrow \{R \textit{ a Noetherian normal ring with } \mathrm{Spec}(R) = \{0, \mathfrak{m}\}\}$$

This can be said concisely by DVRs are precisely 1-dimension local normal domains. We will use this slight abstraction of what a DVR to great effect later on, when we consider localizations of normal domains. This will reduce many local questions in Commutative Algebra, Number Theory, and Algebraic geometry to rather simple questions regarding DVRs.

Proof. ($\Longrightarrow$): Since every ideal of a DVR R has the form $I = \langle t^n \rangle$, it is clearly finitely generated. Thus R is Noetherian. It is a domain, since it is a subring of a field (domain). Lastly, $\mathfrak{m} = \langle t \rangle$ is the only non-zero prime ideal, thus it is maximal.

To check that R is normal, we use the fact that R is a PID, thus a UFD. All UFD are themselves normal (c.f. Exam 1).

($\Longleftarrow$): Choose $x \in \mathfrak{m} \setminus \mathfrak{m}^2$. Note that this is possible because of Nakayma's Lemma! I claim that $\mathfrak{m} = \langle x \rangle$. Consider the R-module $M = \mathfrak{m}/\langle x \rangle$. If $M \neq 0$, then M has an associated prime $\mathfrak{p} = \mathrm{Ann}(y)$! Note that since $\langle x \rangle \subseteq \mathfrak{p}$, we have that $\mathfrak{p} = \mathfrak{m}$. Therefore, there exists $y \in \mathfrak{m} \setminus \langle x \rangle$ such that $y \cdot \mathfrak{m} \subseteq \langle x \rangle$.

We now move to $K = \mathrm{Frac}(R)$. Then $\frac{y}{x} \in K$, but $\frac{y}{x} \notin R$ since $y \notin \langle x \rangle$. However, $\frac{y}{x}\mathfrak{m} \subseteq R$, and is an ideal of R.

Case 1: $\frac{y}{x}\mathfrak{m} = R$. This implies the existence of $y' \in \mathfrak{m}$ such that $\frac{yy'}{x} = 1$, or equivalently $x = yy'$. But $y, y' \in \mathfrak{m}$ imply that $yy' = x \in \mathfrak{m}^2$, contradicting our choice of x.

Case 2: $\frac{y}{x}\mathfrak{m} \subseteq \mathfrak{m}$. In this case, I claim that $\frac{y}{x}$ is integral over R. Indeed, we can view $\varphi : \mathfrak{m} \to \mathfrak{m} : m \mapsto \frac{y}{x}m$ as an endomorphism of finitely generated (Noetherian!) R-modules. Therefore, the determinant trick implies that φ satisfies a monic polynomial relation $\varphi^n + r_1\varphi^{n-1} + \ldots + r_n = 0$ with $r_i \in R$. Applying this relation to $r \neq 0$ in $\mathfrak{m}$ yields the desired result since R is a domain.

Finally, since we assumed R was normal, we have that $\frac{y}{x} \in R$, which implies $y \in \langle x \rangle$. This is a contradiction to our choice of y.

Therefore $\mathfrak{m} = \langle x \rangle$, which implies by Lemma 31.1 and Theorem 31.2 that R is a DVR. $\square$

Example 31.4. We have shown that $K[x,y]/\langle y^2 - x^3 \rangle$ and $K[x,y]/\langle y^2 - x^3 - x^2 \rangle$ are non-normal, even upon localization at $\langle x, y \rangle$. In each case it can be shown that $\frac{y}{x}$ is integral over these rings. Therefore, they are non-normal and thus immediately not DVRs.

Example 31.5. The ring $R = K[x,y]/\langle y^2 - x^3 - x \rangle$ is normal (for a field K not of characteristic 2). As a result, we have that for any (a, b) satisfying $b^2 - a^3 - a$ in K, the ring $R_{\langle x-a, y-b \rangle}$ is a DVR. Spec(R) is an example of an elliptic curve (missing one 'projective' point at infinity).

Next time, we will discuss what to do if our valuation is not discrete, i.e. doesn't take values in $\mathbb{Z}$. This will lead us to a similar notion to discrete valuation rings, but will allow us to consider rings such as germs of functions on an algebraic variety:

$$K[x,y]_{\mathfrak{m}}/\sqrt{J}$$