CLASS 12, MARCH 4TH: NOETHERIAN MODULES

Today, we will talk about a natural generalization of the Noetherian property to modules. This allows us a greater amount of flexibility as well as allows us to get a handle of finite generation for modules.

Definition 12.1. M an R-module is said to be **Noetherian** if for every ascending chain of submodules

$$M_0 \subseteq M_1 \subseteq \cdots \subseteq M$$

eventually stabilizes. That is to say M has the A.C.C. for submodules.

Clearly a ring R is Noetherian if and only if it is a Noetherian R-module. An identical proof to the case of rings yields the following result:

Corollary 12.2. *TFAE:*

1) *M is Noetherian.*
2) *Every submodule of M is finitely generated.*
3) *Every collection of submodules of M has a maximal element.*

Next up, we can get a neat result for modules connected by a short exact sequence.

Proposition 12.3. *If the following sequence is a S.E.S., then M is Noetherian if and only if M' and M'' are Noetherian:*

$$0 \to M' \xrightarrow{i} M \xrightarrow{j} M'' \to 0$$

Proof. $\Rightarrow$: Ascending chains of submodules in M' and M'' correspond directly to ascending chains in M, and therefore stabilize.

$\Leftarrow$: Let $M_0 \subseteq M_1 \subseteq \ldots \subseteq M$ be an ascending chain of submodules of M. Since $M' \subseteq M$, we can consider

$$M_0 \cap M' \subseteq M_1 \cap M' \subseteq \ldots \subseteq M'$$

an ascending chain of submodules of M'. Since M' is assumed Noetherian, this chain stabilizes, say at $M_n \cap M'$. Similarly, since the image of a module is a module, we can also consider

$$j(M_0) \subseteq j(M_1) \subseteq \ldots \subseteq M''$$

which stabilizes, say at $j(M_m)$.

Let $l = \max\{m, n\}$. I claim the original chain stabilizes at M_l. Indeed, suppose $l' > l$ is such that there exists $m \in M_{l'} \setminus M_l$. Since $j(m) \in j(M_l)$, there exists $n \in M_l$ such that $j(n) = j(m)$. Therefore, $n - m \in \ker(j)$, i.e. $i(m') = n - m$. But since $M_l \cap M' = M_{l'} \cap M'$, we see that $m' \in M_l \cap M'$. But since $n \in M_l \cap M'$, this implies $m \in M_l$, contradicting our choice of m. $\square$

This produces many of the desirable properties of Noetherian Modules:

Corollary 12.4.

1) *If M_i are Noetherian modules, then so is $\oplus_{i=1}^{n} M_i$.*
2) *If R is a Noetherian ring, then M is a Noetherian R-module if and only if it is finitely generated. As a result, if $N \subseteq M$, then N is also Noetherian/finitely generated.*
3) *If R is a Noetherian ring, and $\varphi : R \to S$ is a ring homomorphism such that S is a finitely generated R-module, then S is a Noetherian ring.*

Proof. 1) This follows by induction on n. The case of $n = 1$ is trivial. We also have a natural exact sequence

$$0 \to M_n \to \oplus_{i=1}^{n} M_i \to \oplus_{i=1}^{n-1} M_i \to 0$$

Our inductive hypothesis yields the outer modules to be Noetherian, so the inner is as well by Proposition 12.3.

2) If M is Noetherian, then every submodule is n-generated, including M. On the other hand, if M is finitely generated, we can produce an exact sequence

$$0 \to \ker(\varphi) \longrightarrow R^{\oplus n} \xrightarrow{\varphi} M \to 0$$

By the previous part, R is Noetherian so $R^{\oplus n}$ is Noetherian, so Proposition 12.3 implies M is Noetherian. The second part follows from Corollary 12.2.

3) Note that if $I \subseteq S$ is an ideal, it is also an R-module. This follows since if $\alpha \in I$ and $r \in R$, then $r \cdot \alpha = \varphi(r)\alpha$, and $\varphi(r)$ is simply an element of S. Therefore I is a finitely generated R-module (as every R-submodule of S is necessarily finitely generated), and at worst the same generating set works as an S-module/ideal.

□

So one should think of Noetherian modules as a strengthening of the notion of finitely generated. Of course, the homework due today contains an example of a non-Noetherian ring, namely $R = K[x, xy, xy^2, \ldots]$ which has the property that R is finitely generated, and yet $\mathfrak{m} = \langle x, xy, xy^2, \ldots \rangle$ is not finitely generated. So Noetherian modules are more restrictive than finitely generated modules. Here is another such example:

Example 12.5. Consider our ring $R = \mathbb{F}_p[x]_{perf} = \mathbb{F}_p[x, x^{\frac{1}{p}}, x^{\frac{1}{p^2}}, \ldots]$. One funny thing about this ring is that the natural inclusion map $\mathbb{F}_p[x] \to \mathbb{F}_p[x]_{perf}$ induces a bijection at the level of Spec. On the other hand, every non-zero prime ideal of R_{perf} is not finitely generated. This is because prime ideals are radical, and finitely many $\frac{1}{p^e}$ can also be made smaller.

One final consequence to mention is the following about Homs.

Theorem 12.6. *If R is Noetherian and M and N are finitely generated modules, then $\mathrm{Hom}_R(M, N)$ is also finitely generated.*

Proof. Given $R^n \to M \to 0$ the generating map, we see that any map $M \to N$ yields a map $R^n \to N$ by composition. As a result, $\mathrm{Hom}_R(M, N) \subseteq \mathrm{Hom}_R(R^n, N)$. The latter module can be identified by where it sends each of its coordinates (i.e. $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots)$, $\ldots$). This realization allows us to conclude that

$$\mathrm{Hom}_R(M, N) \subseteq \mathrm{Hom}_R(R^n, N) \cong \mathrm{Hom}_R(R, N)^n \cong N^n$$

Since N is finitely generated, so is N^n. This implies N^n is Noetherian, so $\mathrm{Hom}_R(M, N)$ is finitely generated. □