HOMEWORK 5: FUNDAMENTAL GROUPS, VAN KAMPEN
DUE: MONDAY, OCTOBER 23

1) Given that $\pi_1(X \times Y, (x, y)) = \pi_1(X, x) \times \pi_1(Y, y)$, it follows that loops in $X \times y$ and $x \times Y$ commute in $X \times Y$. Construct an explicit homotopy between $\gamma_X \gamma_Y$ and $\gamma_Y \gamma_X$.

2) Let $\iota : A \subseteq X$ be a path connected subspace of X. Show that ι_* is surjective if and only if every loop in X with basepoint in A is homotopic to a loop in A.

3) Show that there are no retractions $r : X \to A$ in the following settings:
   i. $X = \mathbb{R}^3$ and A any subspace homeomorphic to S^1.
   ii. $X = S^1 \times \mathbb{D}^2$ to $A = S^1 \times S^1$.
   iii. $X = \mathbb{D}^2 \vee \mathbb{D}^2$ to its boundary $A = S^1 \vee S^1$.
   iv. X the Mobius band to A it's boundary circle.

4) Construct infinitely many inequivalent (e.g. non-homotopic) retractions from $S^1 \vee S^1$ to S^1.

5) Suppose that f_t is a homotopy between $f_0 = f_1 = Id_X$. Show that for every $x \in X$, the path $f_t(x) \in \pi_1(X, x)$ is in the center of the group: $\gamma f_t = f_t \gamma$ for every $\gamma \in \pi_1(X, x)$. (**hint:** Apply Lemma 1.19) [1]

6) Construct the space X as follows: take two copies of $\mathbb{T}^2 = S^1 \times S^1$ and union them along the circle $S^1 \times \{0\}$. Use Van Kampen's Theorem to compute $\pi_1(X, x)$.

[1] That is to say, a loop is in the center of $\pi_1(X)$ if it extends to a loop of maps $X \to X$