CLASS 17, OCTOBER 19: T4/NORMAL SPACES

Today, we will study the notion of normalcy. On Homework 5, we have shown that every metric space is normal. On the other hand, given a specific space (such as problem 3 of the same homework), it is often difficult 'by hand' to show that a space is normal. Here we give two additional criteria to ensure normality.

Theorem 17.1. *A T3/regular second-countable space X is normal.*

Since regularity can be checked using open neighborhoods of points, whereas normality requires open neighborhoods of closed sets, this is a vast simplification. Also, combined with Theorem 16.5, this implies subspaces and finite products of regular and second countable spaces are themselves normal!

Proof. Suppose $\mathcal{B} = \{U_1, U_2, \ldots\}$ is a countable basis for X, and A and B are closed disjoint subsets. For any fixed point $a \in A$, there exists U_a and V_a disjoint open sets such that $a \in U_a$ and $B \subseteq V$. By the neighborhood criteria, we can choose $U_a' \subseteq U_a$ a neighborhood of a such that $\bar{U_a'} \subseteq U_a$. Furthermore, since $\mathcal{B}$ is a basis, we note that there is some $U_{i(a)} \subseteq U_a'$ lying in $\mathcal{B}$. This allows us to choose a countable cover of A in $\mathcal{B}$ such that the closure of each set is disjoint from B.

Symmetrically, choose $V_{j(a)}$ covering B whose closures are disjoint from A.

$$A \subseteq \bigcup_{i(a)} U_{i(a)} = U \qquad\qquad B \subseteq \bigcup_{j(b)} V_{j(b)} = V$$

These are countable covers of their respective sets that need not be disjoint. Enumerate the $i(a)$ with $i_1, i_2, \ldots$ and the $j(b)$ by $j_1, j_2, \ldots$. By subtracting closed sets (e.g. intersecting with their open complements), we can form the desired open sets:

$$U_{i_k}' = U_{i_k} \setminus \bigcup_{l=1}^{i_k} \bar{V}_{j_l} \qquad\qquad V_{j_k}' = V_{j_k} \setminus \bigcup_{l=1}^{j_k} \bar{U}_{i_l}$$

Let U' and V' be the unions of the U_{i_k}' and V_{j_k}' respectively. Note these new sets still cover their respective spaces, since $a \notin \bar{V_{i_k}}$ and $b \notin \bar{U_{i_k}}$ for any $k \in \mathbb{N}$, $a \in A$, and $b \in B$. Furthermore, they are disjoint. If $x \in U' \cap V'$, then $x \in U_{i_k}' \cap V_{j_{k'}}'$. But one of these sets was removed from the other! This completes the proof. □

Next up, we can also upgrade T2/Hausdorff to normal if we assume the space is compact.

Theorem 17.2. *Every compact Hausdorff space X is normal.*

Proof. You've actually proved a more general version of this on the midterm. Indeed, if A, B are closed subsets of a compact space, they are themselves compact. Therefore, by problem 9 on the midterm, you can separate A, B by open sets. □

Finally, I add one statement about normality of the order topology:

Theorem 17.3. *If X is totally ordered set, then X with the order topology is normal*[1].

[1] In fact, it is T5.

This can be viewed as a generalization of the fact that $\mathbb{R}$ is normal.

Proof. Let A and B be closed subsets of X. We may assume WLOG that no element of A or B is an endpoint of X, i.e. A and B don't contain a largest or smallest element of X (If it does, add ∞ and $-\infty$ to X to enlarge the set). For $a \in A$, choose (invoking the axiom of choice) p_a, q_a satisfying the following conditions:

1) $p_a < a < q_a$.
2) $(p_a, q_a) \cap B \neq \emptyset$.
3) $(a, q_a) = \emptyset$ **or** $q_a \in A$ **or** ($q_a \notin B$ and $(a, q_a) \cap A = \emptyset$).
4) $(p_a, a) = \emptyset$ **or** $p_a \in A$ **or** ($p_a \notin B$ and $(p_a, a) \cap A = \emptyset$).

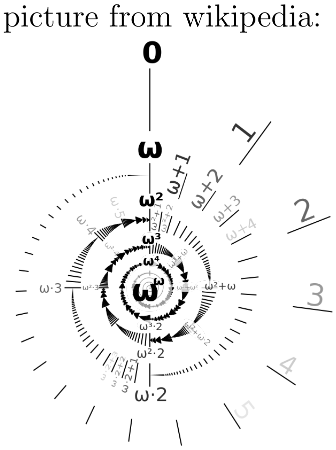
It goes to verify such points exist. 1) is satisfied by our assumption of non-max/minimality of A. 2) is by virtue of the fact that B^c is open and $a \in B^c$. For 3, we proceed as follows. Let $q > a$ satisfy the 2 previous properties. If $(a, q) = \emptyset$, let $q_a = q$. If $(a, q) \cap A \neq \emptyset$, choose $q_a \in (a, q) \cap A$. Lastly, if $(a, q) \neq \emptyset$ but is disjoint from A, choose $q_a \in (a, q)$. A similar argument shows p_a exists.

Now, we may consider $U = \bigcup_{a \in A}(p_a, q_a)$. This open set necessarily contains A. We can furthermore construct an open set $V = \bigcup_{b \in B}(p_b, q_b)$ containing B. Consider the intersection:

$$\begin{aligned}
U \cap V &= \left(\bigcup_{a \in A}(p_a, q_a)\right) \bigcap \left(\bigcup_{b \in B}(p_b, q_b)\right) \\
&= \bigcup_{a \in A}\bigcup_{b \in B}(p_a, q_a) \cap (p_b, q_b) \\
&= \bigcup_{a \in A}\bigcup_{b \in B}((p_a, a) \cup \{a\} \cup (a, q_a)) \cap ((p_b, b) \cup \{b\} \cup (b, q_b)) \\
&= \bigcup_{a \in A}\bigcup_{b \in B}((p_a, a) \cap (p_b, b)) \cup ((p_a, a) \cap (b, q_b)) \cup ((a, q_a) \cap (p_b, b)) \cup ((a, q_a) \cap (b, p_b))
\end{aligned}$$

Conditions 3/4 imply that each pairwise intersection must be empty. In particular, the last of the **or** conditions is the only one that isn't completely obvious. ☐

Example 17.4. The space of *ordinal numbers* is naturally ordered by size. We have notions of $0, 1, 2, 3, \ldots$, but then we reach countable infinity: $\omega, \omega+1, \omega+2, \ldots$. Next we reach $2\omega, 2\omega+1, \ldots, n\omega$ for all integers n, ω^2 as their limit, etc. This is excellently illustrated by a picture from wikipedia:



The set of such things, even up to ω^2, is in bijection with $\mathbb{R}$, thus the whole space is horribly uncountable. However, there is some interesting topology/geometry here. Endowing it with the order topology, we get a set which is normal while not being second countable or compact (even if we restrict to $[0, \omega]$).