CLASS 21, NOVEMBER 1: THE COMPLEX LOGARITHM

We now return to a discussion about what a complex logarithm could look like. It is still defined to be an inverse to the exponential function (on some domain) and thus must satisfy

$$\log(z) = \log(re^{i\theta}) = \log(r) + i\theta$$

for θ in a fixed range. As we discovered on the homework, this is holomorphic if $\theta \in (-\pi, \pi)$ and $r > 0$, but can't be even continuous if we attempt to extend it any further.

We can somewhat remedy this; as θ varies only a little, we can define the logarithm around θ coherently. So we could define log locally, patch the local definitions together, and yield useful information.

Example 21.1. Let C be a curve winding m times counterclockwise around the origin. Then we can write

$$\int_C \frac{dz}{z} = \sum_{j=1}^{2m} \int_{C_j} \frac{dz}{z}$$

where C_j is the j^{th} piece of the curve with angle change π. Thus we can rewrite

$$\sum_{j=1}^{2m} [\log_j(z)]_{r_{j-1}e^{i\pi j}}^{r_j e^{i\pi j}} = \sum_{j=1}^{2m} (\log(r_j) - \log(r_{j-1})) + i(\pi j - \pi(j-1)) = \sum_{j=1}^{2m} i\pi = 2\pi i m$$

where $\log_j$ has a branch on $\theta = \frac{3\pi}{2}$ if j is odd, and $\theta = \frac{\pi}{2}$ if j is even. The same formula holds if $m < 0$

This yields the following ubiquitous definition:

Definition 21.2. If C is a curve with $z_0 \notin C$, then the **winding number** of C about z_0 is

$$m_C = \frac{1}{2\pi i} \int_C \frac{dz}{z - z_0}$$

An additional phrase also came up, the idea of a **branch** for log. This is precisely a choice of $\phi \in \mathbb{R}$ such that

$$\log(re^{i\theta}) = \log(r) + i\theta$$

for all $\theta \in (\phi - 2\pi, \phi)$ and $r > 0$. Thus the log we discussed previously has a branch about $\phi = \pi$. This is called **the principal branch** of the logarithm. We will call the logarithm with such a branch $\log^\phi$ as a shorthand.

Note not every logarithm has a branch. You can produce open sets for which no such value is possible. In general however, we have the following theorem:

Theorem 21.3. *If Ω is a simply connected open set, with $1 \in \Omega$ and $0 \notin \Omega$, then there exists F such that F is holomorphic in Ω, $e^F = z$, and $F(r) = \log(r)$ for every $r \in \mathbb{R} \cap \Omega$ near 1.*

Thus we may define a log on any simply connected set.

Proof. We will construct our logarithm according to the fact that it should be an antiderivative for $f(z) = \frac{1}{z}$. Since $0 \notin \Omega$, f is holomorphic in Ω. So we may define

$$F(z) = \int_\gamma \frac{dw}{w}$$

where γ is a path connecting 1 to z. By simple connectedness, this is well defined (in the sense that any γ will yield the same result since they are automatically homotopic). This yields that F is a primative, and thus is necessarily holomorphic.

Now we need to show $e^{F(z)} = z$. This is equivalent to $1 = ze^{-F(z)}$. We do this by differentiating:

$$\frac{\partial}{\partial z} ze^{-F(z)} = e^{-F(z)} - ze^{-F(z)}F'(z) = (1 - F'(z)z)e^{-F(z)} = 0$$

Therefore $ze^{-F(z)}$ is constant. So it is enough to plug in a single value: $z = 1$. This yields $C = 1 \cdot e^{-F(1)} = 1 \cdot e^0 = 1$. This was our intention.

Finally, for z close enough to 1, we can ensure that γ be chosen on the real line. In this case, it is just the standard log from calculus! $\square$

Example 21.4. One needs to be quite careful with some expected results about log (taking for example $\log_0$). For example, it is not true in general that

$$\log(z \cdot w) = \log(z) + \log(w)$$

for two complex numbers z, w avoiding some particular branch. Indeed, if we consider

$$\log(e^{\frac{2\pi i}{3}} \cdot e^{\frac{2\pi i}{3}})) = \log(e^{\frac{4\pi i}{3}}) = \log(e^{\frac{-2\pi i}{3}}) = -\frac{2\pi i}{3} \neq \frac{4\pi i}{3}$$

The same goes for any $\log_\phi$ and even F as in Theorem 21.3.

Note that if we do the taylor series expansion of the principal branch of the logarithm shifted about 1, we get the expected formula:

$$\log(z+1) = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}$$

which naturally has radius of convergence 1.

Additionally, we can expand our usual definitions of z^n to z^α for any $\alpha \in \mathbb{R}$! If Ω is as in Theorem 21.3, we can take $\log = F$ in the theorem and write

$$z^\alpha = e^{\alpha \log(z)}$$

We get automatically that $1^\alpha = 1$ and if $\alpha = \frac{1}{n}$, then

$$(z^{\frac{1}{n}})^n = (e^{\frac{1}{n}\log(z)})^n = z$$

This extends even more broadly by the following theorem we will prove next time:

Theorem 21.5. *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists another holomorphic function $g(z)$ such that*

$$f(z) = e^{g(z)}$$