HOMEWORK 6: SINGULARITIES
DUE: WEDNESDAY, OCTOBER 30TH

(1) Show that if $u \in \mathbb{R} \setminus \mathbb{Z}$, then
$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{\sin(\pi u)^2}.$$
This can be done by integrating $f(z) = \frac{\pi \cot(\pi z)}{(u+z)^2}$ on the circle of radius $N + \frac{1}{2}$ with $N \in \mathbb{Z}$, and sending $N \to \infty$. Show why.[1]

(2) Suppose f is holomorphic in $B_*(0,1)$ and that
$$|f(z)| \leq A|z|^{-1+\epsilon}$$
for some $\epsilon > 0$ and all z_0 near 0. Show that f has a removable singularity at 0.

(3) Show that all entire functions which are also injective ($f(z) = f(w)$ if and only if $z = w$) are linear:
$$f(z) = az + b \qquad\qquad a \neq 0$$
(**hint:** Use Casorati-Weirstrass on $f(\frac{1}{z})$, and apply the open mapping theorem).

(4) Suppose f and g are holomorphic on $\bar{B}(0,1)$, and that f has only a simple zero at $z = 0$. Show that
$$f_\epsilon(z) = f(z) + \epsilon g(z)$$
has exactly one zero on $\bar{B}(0,1)$, and if we call it z_ϵ, then z_ϵ varies continuously in ϵ.

(5) Let f be non-constant holomorphic in $\Omega \supseteq \bar{B}(0,1)$. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then $\bar{B}(0,1) \subseteq f(\Omega)$.

If instead $|f(z)| \geq 1$ whenever $|z| = 1$ and there is some $z_0 \in \bar{B}(0,1)$ with $|f(z_0)| < 1$, then $\bar{B}(0,1) \subseteq f(\Omega)$.

(**hint:** for the first part, show that it suffices to check that $f(z)$ has a root. Then apply the maximum modulus principle).

[1] This is a sort of shifted ζ-function at $s = 2$.