CLASS 25, APRIL 19TH: MACAULAY2 & SUPPORT OF MODULES

Today's class will be divided into two parts; first, I will do a small tutorial on Macaulay2, a computer algebra system which can be used to verify examples and thus hypotheses for general theorems. Then we will move on to the support of a module (chapter 7 of Reid). Here are some of the important websites to associate with Macaulay2 (M2):

- `http://www2.macaulay2.com/Macaulay2/`: This is the official webpage for M2. It contains the installable binaries for many popular distributions, such as OSX and Ubuntu, under the appended /Downloads/ url. There are also a few guides under /GettingStarted/ written by some prolific M2 programmers and mathematicians.
- `https://github.com/Macaulay2/M2`: The GitHub repository for Macaulay2, containing the most up-to-date code. There may be some compiler optimizations available if you compile the packages on your own, which may increase speed of code execution.
- `http://habanero.math.cornell.edu:3690/`: Cornell (in particular Mike Stillman) provide a remote web client version of Maculay2. It is currently at version 1.12 (as of April 9^{th}, 2019) which is nearly the newest version (1.13). This can be helpful if there isn't a pre-compiled version for your distribution.

The code executed in class will be made available as Class25.M2.pdf on GLOW.

Now we will begin to focus our attention on the structure of ideals. We have seen in the last section that radical ideals in Noetherian rings can be decomposed into finite intersections of prime ideals. Of course, such a thing can't be expected to hold for more general ideals (such as powers of a single prime ideal). We will begin this story by studying the support of a module.

Definition 25.1. Given an R-module M, the **support of** M is the set

$$\operatorname{Supp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_\mathfrak{p} \neq 0\}$$

Recall the **annihilator of** M is the ideal

$$\operatorname{Ann}_R(M) = \{r \in R \mid rM = 0\}$$

If the context is clear, sometimes the R subscript is omitted in these notations. Finally, $r \in R$ is called a **zero divisor** for M if $rm = 0$ for some $m \neq 0$.

This has a clear geometric significance. $M_\mathfrak{p}$ should be viewed as the structure of M near a point $\mathfrak{p} \in \operatorname{Spec}(R)$. The support tells us where the module has local significance (i.e. is non-zero). Thus, by the dicussion of Class 19, $\operatorname{Supp}(M) = \emptyset$ if and only if $M = 0$. The following theorem tells us that in most cases of interest, modules are interesting most of the time.

Proposition 25.2. (a) *If* $\mathfrak{p} \in \operatorname{Supp}(M)$, *then* $V(\mathfrak{p}) \subseteq \operatorname{Supp}(M)$.
(b) *If* $M = \langle x \rangle$ *is principal, then* $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$.
(c) *If* $M = \sum_i M_i$, *then* $\operatorname{Supp}(M) = \bigcup_i \operatorname{Supp}(M_i)$.
(d) *If* M *is finitely generated, then* $\operatorname{Supp}(M) = V(\operatorname{Ann}(M))$.

(e) *If* $0 \to M' \to M \to M'' \to 0$ *is a SES, then* $\mathrm{Supp}(M) = \mathrm{Supp}(M') \cup \mathrm{Supp}(M'')$.

Proof. (a) If $\mathfrak{p}' \supseteq \mathfrak{p}$ is a prime ideal, then

$$M_{\mathfrak{p}} = (M_{\mathfrak{p}'})_{\mathfrak{p}}$$

As a result, we have that if $M_{\mathfrak{p}'} = 0$, then $M_{\mathfrak{p}} = 0$.

(b) If $Ann(x) \not\subseteq \mathfrak{p}$, then let y be in their difference. We have that $(w, rx) \sim (1, 0)$:

$$y(rx - 0w) = r(yx) = r0 = 0$$

Thus $M_{\mathfrak{p}} = 0$. If $\mathrm{Ann}(x) \subseteq \mathfrak{p}$, for the same reason we can conclude $(1, x) \not\sim (1, 0)$.

(c) Localization is exact, so $(M_i)_{\mathfrak{p}} \subseteq M_{\mathfrak{p}}$. This shows the $\supseteq$ direction. For $\subseteq$, if

$$m = m_{i_1} + \cdots + m_{i_n}$$

where $m_{i_j} \in M_{i_j}$, and $x_j \notin \mathfrak{p}$ is such that $x_j \cdot m_{i_j} = 0$, then $x_1 \cdots x_n \cdot m = 0$.

(d) $M = \langle m_1, \ldots, m_n \rangle = \langle m_1 \rangle + \ldots + \langle m_n \rangle$, then

$$\mathrm{Supp}(M) = \bigcup_{i=1}^{n} \mathrm{Supp}(\langle m_i \rangle) = \bigcup_{i=1}^{n} V(\mathrm{Ann}(m_i)) = V\left(\bigcap_{i=1}^{n} \mathrm{Ann}(m_i)\right) = V(\mathrm{Ann}(M))$$

(e) Since localization is exact, we see

$$0 \to M'_{\mathfrak{p}} \to M_{\mathfrak{p}} \to M''_{\mathfrak{p}} \to 0$$

is an exact sequence of $R_{\mathfrak{p}}$-modules. As a result, $M'_{\mathfrak{p}} = 0$ and $M''_{\mathfrak{p}} = 0$ if and only if $M_{\mathfrak{p}} = 0$. This shows the desired statement.

□

Example 25.3. We have essentially handled the case of finitely generated modules in part (d) of Proposition 25.2. Consider the $\mathbb{Z}$ module $M = \oplus_{n \in \mathbb{N}} \mathbb{Z}/2^n\mathbb{Z}$. The annihilator of each component of the sum is precisely $\langle 2^n \rangle \subseteq \mathbb{Z}$. Therefore, if we tried to generalize the result of part (d), we would note that

$$\mathrm{Ann}(M) = \bigcap_{n \in \mathbb{N}} \mathrm{Ann}(Z/2^n\mathbb{Z}) = 0$$

and thus we might expect the support to be $V(0) = \mathrm{Spec}(\mathbb{Z})$. However, if we localize at any prime different than $\langle 2 \rangle$, we invert 2 and thus 2^n. As a result, we have that for all $a \in \mathbb{Z}/2^n\mathbb{Z}$,

$$2^n a = 0 \qquad \Longrightarrow \qquad (1, m) \sim (1, 0) = 0$$

So in fact $\mathrm{Supp}(M) = \langle 2 \rangle$.

There is a similarly startling realization when considering the module $M = \oplus_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z}$. Note that $\mathrm{Ann}(M) = 0$, thus $V(\mathrm{Ann}(M)) = \mathrm{Spec}(\mathbb{Z})$. However, $0 \notin \mathrm{Supp}(M)$.

One can note however that $\mathrm{Supp}(M) \subseteq V(\mathrm{Ann}(M))$ in full generality.