CLASS 6, FEBRUARY 18TH: LOCAL RINGS

Today we will look at a certain classification of rings determined by the shape of Spec(R). Local rings determine the structure of rings near a given prime ideal. We will see this comparison in more detail later on.

Definition 6.1. A ring is said to be **local** if there exists only a single maximal ideal $\mathfrak{m} \subsetneq R$. In this case, we often write $(R, \mathfrak{m})$ to indicate the ring, of even $(R, \mathfrak{m}, k)$, where $k = R/\mathfrak{m}$ is the residue field at $\mathfrak{m}$ of R.

There are a few methods to detect when a ring is local:

Proposition 6.2. *Let R be a ring. Then TFAE:*

1) R is a local ring with maximal ideal $\mathfrak{m}$.
2) $R \setminus R^\times$, the complement of the set of units of R, forms a proper ideal (namely $\mathfrak{m}$).
3) Everything of the form $1 + \alpha$, with $\alpha \in \mathfrak{m}$, is a unit.

Proof. 1) $\leftrightarrow$ 2) : Note that the units form a multiplicative set, so there exists a maximal ideal containing the ideal generated by every non-unit. As a result, since there is only 1 maximal ideal, $\mathfrak{m}$ contains every non-unit! Of course $\mathfrak{m}$ contains no units, so $\mathfrak{m} = R \setminus R^\times$ is an ideal.

2) $\to$ 3) : If $\alpha \in \mathfrak{m}$, then $1 + \alpha \notin \mathfrak{m}$. Otherwise, $1 \in \mathfrak{m}$ by additive closure. Therefore, by 2), $1 + \alpha$ is a unit.

3) $\to$ 2) : Suppose the condition of 2) is not satisfied. Namely, suppose there exist r, r' non-units such that $r + r'$ is a unit. Let u be its multiplicative inverse. Then

$$u(r + r') = ur + ur' = 1$$

But as a result, we have $1 - ur = ur'$, and $ur \in \mathfrak{m}$ since r is. As a result, since r' was not itself a unit, we note neither is ur' and thus $1 - ur$ is a non-unit. This implies 3) is false as claimed.

□

Now we turn to the idea of localization, at least in the case of an integral domain.

Definition 6.3. If R is an integral domain, $\mathfrak{p}$ is a prime ideal of R, then we define

$$R_\mathfrak{p} = \left\{ \frac{f}{g} \in Frac(R) \mid f, g \in R, g \notin \mathfrak{p} \right\}$$

with the standard method of adding and multiplying such fractions.

Note that these group structures are well defined, c.f. Homework 3 #1. In addition, everything outside of $\mathfrak{p}$ is made invertible! Therefore, by Proposition 6.2, $\mathfrak{p} \cdot R_\mathfrak{p}$ is the unique unique maximal ideal of $R_\mathfrak{p}$!

First, the integers.

Example 6.4. Consider the ring $R = \mathbb{Z}$, and let $\mathfrak{p}$ be a prime ideal. If $\mathfrak{p} = 0$, then we invert every non-zero element. As a result,

$$\mathbb{Z}_{\langle 0 \rangle} = \mathbb{Q}$$

If $\mathfrak{p} = \langle p \rangle$, where p is a prime number, then we invert everything not divisible by p. The effect is

$$\mathbb{Z}_{\langle p \rangle} = \mathbb{Z}\left[\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \ldots, \widehat{\frac{1}{p}}, \ldots\right]$$

Here the *hat*-symbol $\widehat{\cdot}$ denotes 'omit this element from the list'. By Homework 3 #2,$\mathbb{Z}_{\langle p \rangle}$ is an integral domain with exactly 2 prime ideals:

$$\mathrm{Spec}(\mathbb{Z}_{\langle p \rangle}) \cong \{0, \langle p \rangle\}$$

Next polynomial rings:

Example 6.5. Consider $R = K[x, y]$ and $\mathfrak{m} = \langle x, y \rangle$. Then

$$R_{\mathfrak{m}} = \left\{ \frac{p(x,y)}{q(x,y)} \mid q(x,y) \notin \mathfrak{m} \right\}$$

It is easy to check that $q(x, y) \notin \mathfrak{m}$ if and only if $q(0, 0) \neq 0$. Again utilizing Homework 3 #2, we see that

$$\mathrm{Spec}(R_{\mathfrak{m}}) \cong \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} \subseteq \mathfrak{m}\}$$

So again $R_{\mathfrak{m}}$ has one minimal prime ideal and one maximal ideal. Here however, there are many primes in between. If we reduce our attention to $\mathbb{C}$, then an example of such polynomials are $x^2 - y^3$ and $y^2 - x(x + 1)(x - 1) = y^2 - x^3 - x$. I leave it to you as Homework 3 #3 to verify this statement.

The final example I would like to study is that of a power series ring.

Example 6.6. Recall that $R = K[\![x_1, x_2, \ldots, x_n]\!]$ is the ring of formal power series in K. That is to say, it has elements of the form

$$f = \sum_{\boldsymbol{\alpha} \geq \mathbf{0}} c_{\boldsymbol{\alpha}} \boldsymbol{x}^{\boldsymbol{\alpha}}$$

Here a quick word about multi-indices. $\boldsymbol{x}$ denotes $(x_1, x_2, \ldots, x_n)$, and $\boldsymbol{\alpha}$ denotes $(\alpha_1, \alpha_2, \ldots, \alpha_n)$, a set of non-negative integers. $c_{\boldsymbol{\alpha}}$ is simply an element of K, and $\boldsymbol{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. In particular, power series allow terms to have non-zero coefficients indefinitely, unlike the case of polynomials.

I claim that R is already a local ring, with maximal ideal $\mathfrak{m} = \langle x_1, \ldots, x_n \rangle$. This is proved, using Proposition 6.2, by demonstrating that $f \in R$ with non-zero constant coefficient is a unit. This is the content of Homework 3#4.

The procedure of creating R from $K[x_1, x_2, \ldots, x_n]$ is called completion at the maximal ideal $\langle x_1, \ldots, x_n \rangle$. This should be thought of as an even more local version of localization.

Next time, we will begin to study modules.