CLASS 1, SEPTEMBER 7: COMPARISON WITH METRIC SPACES

From real analysis, one of the most prominent objects is that of a metric space. This vastly generalizes many of the spaces you have seen in Calculus and even elementary geometry, and gives a way to measure 'how far apart' 2 points are in your space. Just to recall, here is a definition:

Definition 1.1. Let S be a set. A function $d : S \times S \to \mathbb{R}_{\geq 0}$ is called a **metric** if it satisfies the following conditions:

1) **Separation Axiom:** $d(x,y) = 0$ if and only if $x = y$
2) **Symmetry:** $d(x,y) = d(y,x)$
3) **Triangle Inequality:** $d(x,y) + d(y,z) \leq d(x,z)$

A pair (S,d) as above is called a **metric space**.

Example 1.2.
- $\mathbb{R}$ equipped with the metric $d(x,y) = |y - x|$ is a metric space.
- Let V be a finite dimensional vector space. Then $d_1(u,v) = |v_1 - u_1| + \ldots |v_n - u_n|$, $d_2(u,v) = \sqrt{(v_1 - u_1)^2 + \ldots + (v_n - u_n)^2)}$, and $d_\infty(u,v) = \max\{|v_i - u_i|\}$ all produce (equivalent!) metric spaces. These metrics are called the Manhattan, the Euclidean, and the Chebyshev metrics respectively.
- On a sphere S^2 (or S^n for any $n \geq 0$) is a metric space. This can be seen since it sits within $\mathbb{R}^3$ (or $\mathbb{R}^{n+1}$) which are metric spaces with a (or many) choices of d.
- Vertices on connected graphs have a metric, defined by how many edges one needs to travel to get between two vertices.

So many of the object you hold close have a notion of distance. This brings about the notion of an open or closed set in a natural way:

Definition 1.3. If (S,d) is a metric space, a subset $U \subseteq S$ is called **open** if for every point $x \in U$, there exists $\epsilon > 0$ (depending on x) such that $B(x,\epsilon) \subset U$, where

$$B(x,\epsilon) := \{y \in S \mid d(x,y) < \epsilon\}$$

This is commonly called an ϵ-ball around x or ϵ-neighborhood of x.

A subset $Z \subseteq S$ is called **closed** if its complement $Z^c = S \setminus Z$ is open.

Thus objects such as *open* intervals $(a,b) \subseteq \mathbb{R}$ are also open in a metric sense. Phrased differently, a set is called open if it is a union of ϵ-neighborhoods:

$$U = \bigcup_{x \in U} B(x, \epsilon_x)$$

Here are some nice properties of open sets (which you can transfer to corresponding statements for closed sets):

Proposition 1.4. *Let (S,d) be a metric space.*

1) S and $\emptyset$ are open sets.
2) If $U_\alpha \subseteq S$ are any collection of open sets indexed by $\alpha \in \Lambda$, then so is

$$U = \bigcup_{\alpha \in \Lambda} U_\alpha$$

3) If $U_1, \ldots, U_n$ are open sets, then so is $V = U_1 \cap \ldots \cap U_n$

Proof. 1) Obvious. Take any ϵ your heart desires.

2) If $x \in U$, then $x \in U_\alpha$ for some $\alpha \in \Lambda$, and therefore, $B(x, \epsilon_x) \subseteq U_\alpha \subseteq U$.

3) If $x \in V$, then $x \in U_i$ for each $i = 1, \ldots, n$. Therefore, there is $\epsilon_i > 0$ such that $B(x, \epsilon_i) \subseteq U_i$. Let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\} > 0$. Then $B(x, \epsilon) \subseteq B(x, \epsilon_i) \subseteq U_i$, and thus $B(x, \epsilon) \subseteq V$.

$\square$

Note that the collection of open sets is *heavily* dependent on the choice of metric:

Example 1.5. Let (V, d_2) be a finite dimensional vector space with the Euclidean metric as above. We can also define $d_0 : V \times V \to \mathbb{R}$ by

$$d_0(x, y) = \begin{cases} 1 & x \neq y \\ 0 & x = y \end{cases}$$

With some thought, you can check that this is a metric on V (or any set) and every subset is open. Of course this is not the case for d_2; c.f. $[a, b]$. d_0 is called the **discrete** metric.

This brings about the following question: How can we tell when the collections of open sets induced by two metrics are the same?

Definition 1.6. Two metrics $d_1, d_2 : V \times V \to \mathbb{R}_{\geq 0}$ are said to be **strongly equivalent** if there exists $a, b > 0$ such that for every pair of points $x, y \in V$, we have

$$a \cdot d_1(x, y) \leq d_2(x, y) \leq b \cdot d_1(x, y)$$

That is to say the ratio $\frac{d_2}{d_1}$ is uniformly bounded above and below by positive numbers. The following proposition realizes the importance of this definition.

Proposition 1.7. *If d_1 and d_2 are strongly equivalent, then they share the exact same collection of open sets.*

Proof. Suppose U is d_1-open. Then if $B_1(x, \epsilon) \subseteq U$ (shorthand for ball in the d_1-metric), then $B_2(x, b \cdot \epsilon) \subseteq U$. This follows, as by definition $d_2(x, y) \leq b \cdot d_1(x, y) < b \cdot \epsilon$. Similarly, if V is d_2-open, and $B_2(x, \epsilon) \subseteq V$, then $B_1(x, \frac{\epsilon}{a}) \subseteq V$. Thus being open in one metric is *equivalent* to being open in the other. $\square$

There is also a notion of (non-strongly) equivalent metrics, which give not only a sufficient, but also a necessary condition! It simply takes away the uniformity of a and b in the above definition. In particular, it says that for a fixed x and $r > 0$, we can find r', r'' such that

$$B_1(x, r') \leq B_2(x, r) \leq B_1(x, r'')$$

Now, the collection of open sets determine most of the important data about metric spaces, e.g. continuity of functions, differentiability of functions, completions, etc. Topology peals away the rigidity of a metric and dealing all of the numerics, and instead focuses simply on the collection of open sets. This yields a broadened and rich field of study which we will embark on next class!