# CLASS 32, NOVEMBER 30: (BABY) ASCOLI'S THEOREM

Ascoli's Theorem produces an excellent peek into functional analysis, where you study the space of operators from a metric space to $\mathbb{R}$ subject to various (continuity, differentiability, integrability) conditions. To state and prove the theorem, we need some prerequisites from real.

Recall the following theorem from real analysis:

**Theorem 32.1.** *If $X$ is a metric space, then TFAE:*

*1) $X$ is compact.*
*2) $X$ is sequentially compact (e.g. every sequence has a convergent subsequence).*
*3) $X$ is complete and totally bounded (e.g. $\forall \epsilon > 0$, $\exists x_1, \ldots, x_n$ such that $X = \bigcup_{i=1}^n B(x_i, \epsilon)$).*

There is a proof in Munkres of 1) $\Leftrightarrow$ 3), but this is a foundational result in Real analysis so I omit the proof here.

Next, I add some definition which are very important in functional analysis. Equicontinuous gives a kind of uniform continuity of a *collection* of functions, and pointwise-bounded means almost exactly what it says.

**Definition 32.2.** Let $Y$ be a metric space, and $X$ a topological space. Let $C(X, Y)$ be the set of continuous functions $X \to Y$. A subset $D \subseteq C(X, Y)$ is said to be **equicontinuous** at $x$ if for each $\epsilon > 0$, there exists a neighborhood $U$ of $x$ such that $\forall y \in U$,

$$d(f(x), f(y)) < \epsilon$$

$D$ is **equicontinuous** if it is for every point $x \in X$.

$D$ is said to be **pointwise bounded** if for each *point* $x \in X$, the set

$$D_x = \{f(x) \mid f \in D\}$$

is bounded in $Y$.

**Lemma 32.3.** *If $X$ is a space and $Y$ is a metric space, then if $\mathcal{Z} \subseteq C(X, Y)$ is totally bounded, then $\mathcal{Z}$ is equicontinuous.*

*Proof.* Choose $\epsilon > 0$. Given $\mathcal{Z}$ is totally bounded, choose finitely-many $f_1, \ldots, f_n$ such that balls of radius $\frac{\epsilon}{3}$ cover the space. This implies that every function $f \in \mathcal{Z}$ has

$$d(f_i(x), f(x)) < \frac{\epsilon}{3}$$

for some $i$ and *all* $x \in X$. Choose $U$ a neighborhood of $x$ such that $d(f_i(y), f_i(x)) < \frac{\epsilon}{3}$ for $y \in U$. Again by the triangle inequality, we see that for $y \in U$, $d(f(y), f(x)) < \epsilon$. $\square$

In the compact case, a partial converse exists.

**Lemma 32.4.** *If $X$ is compact and $Y$ is a compact metric space, and $\mathcal{Z} \subseteq C(X, Y)$ is equicontinuous, then $\mathcal{Z}$ is totally bounded in the uniform topology.*

*Proof.* Note that the uniform metric $\rho(f, g) = \sup_x\{d(f(x), g(x))\}$ is well-defined since $X$ is compact and therefore so is it's image. Thus its image is totally bounded.

For a given $\epsilon > 0$, it suffices to cover $\mathcal{Z}$ by finitely many $\epsilon$-balls.

Choose neighborhoods $U_x \subseteq X$ of $x$ such that $d(f(x), f(y)) < \frac{\epsilon}{3}$ for each $y \in U_x$ and $f \in \mathcal{Z}$. Since $X$ is compact, finitely many will do, say $U_{x_1}, \ldots, U_{x_n}$. This is possible by equicontinuity. We can similarly cover $Y$ by finitely many $\frac{\epsilon}{3}$-balls, say $C_1, \ldots, C_m$.

For a given pair $(i,j) \in [n] \times [m]$, choose a function $f_{i,j} : U_{x_i} \to C_j$ with $f_{i,j} \in \mathcal{Z}$ if it exists. This is a finite collection of functions. Then I claim $B(f_{i,j}, \epsilon)$ cover $\mathcal{Z}$, proving total boundedness.

This follows again by the triangle inequality. Suppose $f \in \mathcal{Z}$ and $f(x_i) \in C_j$. Then

$$d(f(x), f_{i,j}(x)) \leq d(f(x), f(x_i)) + d(f(x_i), f_{i,j}(x_i)) + d(f_{i,j}(x_i), f_{i,j}(x)) < \epsilon$$

But $x \in X$ was arbitrary, showing $\rho(f, f_{i,j}) < \epsilon$, as the supremum is a maximum given $X$ is compact and $Y$ is Hausdorff. □

**Theorem 32.5** (Ascoli's Theorem). *Let $X$ be a compact topological space. Let $\mathcal{C} = C(X, \mathbb{R}^n)$ be the set of continuous functions from $X$ to $\mathbb{R}^n$, with the uniform topology. Then $\mathcal{Z} \subseteq \mathcal{C}$ has compact closure if and only if $\mathcal{Z}$ is equicontinuous and pointwise bounded.*

*Proof.* $(\Rightarrow)$ : Suppose $\overline{\mathcal{Z}}$ is compact. It suffices to check $\overline{\mathcal{Z}}$ is equicontinuous and pointwise bounded.

By virtue of Lemma 32.3, we can can conclude that since $\overline{\mathcal{Z}}$ is compact, it is totally bounded, and therefore equicontinuous. Totally bounded implies bounded. Indeed, choose

$$R = \max_{i=2,\ldots,n} \{d(x_1, x_i) + \epsilon \mid X = \bigcup_{i=1}^{n} B(x_i, \epsilon)\}$$

then $\overline{\mathcal{Z}} \subseteq \bar{B}(x_1, R)$. But this implies $\overline{\mathcal{Z}}$ is pointwise-bounded since $\rho(f,g) = \sup\{f(x), g(x)\}$.

$(\Leftarrow)$ : First, I want to show that if $\mathcal{Z}$ is equicontinuous and pointwise bounded, then so is $\overline{\mathcal{Z}}$. Given $x \in X$, $\epsilon > 0$, there is a neighborhood of $x$, say $U$, such that $d(f(x), f(x_0)) < \frac{\epsilon}{3}$ for each $x \in U$ and $f \in \mathcal{Z}$. We can furthermore choose $f \in \mathcal{Z} \cap B(g, \frac{\epsilon}{3})$ for some $g \in \overline{\mathcal{Z}}$. As a result, for $x \in U$,

$$d(g(x_0), g(x)) \leq d(g(x_0), f(x_0)) + d(f(x_0), f(x)) + d(f(x), g(x)) < \epsilon$$

A similar argument shows pointwise boundedness.

Next, I show that if $\overline{\mathcal{Z}}$ is equicontinuous and pointwise bounded, then there exists a compact space $Y \subseteq \mathbb{R}^n$ such that $ev(X \times \overline{\mathcal{Z}}) \subseteq Y$.

For each $x \in X$, choose $U_x$ a neighborhood such that for each $y \in U$, $d(g(x), g(y)) < 1$ for every $g \in \overline{\mathcal{Z}}$. Since $X$ is compact, there exists a finite covering $U_{x_1}, \ldots, U_{x_n}$. Since each $\overline{\mathcal{Z}}_{x_i} = ev(U_{x_i} \times \bar{Z})$ is bounded by pointwise boundedness, so is their union. Thus $\exists R > 0$ s.t.

$$\overline{\mathcal{Z}}_{x_i} \subseteq \bar{B}(0, R)$$

But this implies $\overline{\mathcal{Z}}_x \subseteq B(0, R+1) \subseteq \bar{B}(0, R+1) = Y$.

Finally, I prove the Theorem. It suffices to check that $\overline{\mathcal{Z}}$ is complete & totally bounded in $\mathcal{C}$.

Since $\overline{\mathcal{Z}} \subseteq \mathcal{C}$ is closed, and $\mathcal{C}$ is complete, $\overline{\mathcal{Z}}$ is automatically complete. Additionally, the previous parts show that $\overline{\mathcal{Z}} \subseteq C(X, Y)$ is equicontinuous when $Y$ is compact. Thus Lemma 32.4 shows that it is totally bounded in $\mathcal{C}$. This completes the proof. □

Rephrasing a few things, we note the following:

**Corollary 32.6.** *If $X$ is compact, and $C(X, \mathbb{R}^n)$ has the uniform topology, then $\mathcal{Z} \subseteq C(X, \mathbb{R}^n)$ is compact if and only if it is closed, bounded, and equicontinuous.*