CLASS 22, FRIDAY APRIL 20TH: KUNZ THEOREM I

We have already spoken about how nice regular rings are in commutative algebra, with every module having finite projective dimension. In fact, they are the commutative algebraic analog of smooth manifolds from differential geometry. Today, we give a classical yet extremely powerful way to detect regularity in positive characteristic.

Theorem 0.1 (Kunz's Theorem)**.** *Let R be an F-finite ring. Then R is regular if and only if F_*R is a flat module.*

We can use some of the previous results to apply the following characterization:

Theorem 0.2 (Local Kunz's Theorem)**.** *If R is a local F-finite ring, then R is an RLR if and only if F_*R is a free R-module.*

We have already indicated the following implications for R-modules:

$$\text{Free} \Rightarrow \text{Projective} \Rightarrow \text{Flat} \Rightarrow \text{Torsion-free}$$

We additionally have the following under additional conditions (R Noetherian):

$$\text{Free} \overset{R \text{ local}}{\Longleftarrow} \text{Projective} \overset{\text{f.g.}}{\Longleftarrow} \text{Flat} \overset{\text{Dedekind}}{\Longleftarrow} \text{Torsion-free}$$

To make Local Kunz sufficient for Kunz, we need to prove Projective $\overset{\text{f.g.}}{\Longleftarrow}$ Flat. Suppose M is flat. We may assume $(R, \mathfrak{m})$ is a local ring. Choose $\langle m_1, \ldots, m_n \rangle$ a minimal generating set for M. Then we get a surjection $R^n \to M$. We can expand to a SES and tensor with $R/\mathfrak{m}$ to yield a SES

$$0 \to K \otimes_R R/\mathfrak{m} \to R^n \otimes_R R/\mathfrak{m} \to M \otimes_R R/\mathfrak{m} \to 0$$

The zero on the left comes from the fact that M is flat (which implies $\mathrm{Tor}_1^R(M, R/\mathfrak{m}) = 0$). Since the generating set was chosen minimal, we see that $R^n \otimes R/\mathfrak{m} \to M \otimes R/\mathfrak{m}$ is an isomorphism of vector spaces, implying $K \otimes_R R/\mathfrak{m} = K/\mathfrak{m}K = 0$. By Nakayama's lemma, we see that $K = 0$. Therefore, M is free.

Therefore, it suffices to prove Theorem 0.2. First, we need some information about the completion of a ring.

Definition 0.3. Let $(R, \mathfrak{m})$ be a local ring. We denote $\hat{R}$, called the **completion** of the ring R at $\mathfrak{m}$, to be the inverse limit of the sequence

$$\hat{R} \to \ldots R/\mathfrak{m}^n \to \ldots \to R/\mathfrak{m}^2 \to R/\mathfrak{m} \to 0$$

Alternatively, it is the completion of the metric space R with the $\mathfrak{m}$-adic metric:

$$d(x, y) = 2^{-\max\{n \mid x - y \in \mathfrak{m}^n\}}$$

This is a metric by Krull's intersection theorem.

This can be thought of as an even more local version of localization. $\hat{R}$ is always a faithfully (taking non-zero modules to non-zero modules under tensor) flat R-module. The reason this is important is because regular rings themselves can be quite strange looking:

Example 0.4. The ring $R = K[x,y]/\langle y^2 - x(x-1)(x+1)\rangle$ with $\mathrm{char}(K) > 2$ is a regular ring. This can be seen via the Jacobian criterion.

However, by the following theorem of Cohen, we get a much nicer characterization for complete rings:

Theorem 0.5 (Cohen's Structure Theorem: Regular case)**.** *Suppose that R is a ring containing a field K. Then $\hat{R} = K[\![x_1, \dots, x_n]\!]/I$. Furthermore, if R is regular, then $\hat{R}$ is a power series ring:*

$$\hat{R} \cong K[\![x_1, \dots, x_{\dim(R)}]\!]$$

This is a very nice result, and will be used to simplify the proof of Kunz dramatically.

Easy direction. Let R be a regular F-finite ring. As a result of Cohen's structure theorem, we note that $\hat{R} \cong K[\![x_1, \dots, x_d]\!]$. Now, we can apply F_* to get the following commutative diagram:

$$\begin{tikzcd} \hat{R} \arrow[r, "F"] & F_*\hat{R} \\ R \arrow[u] \arrow[r, "F"] & F_*R \arrow[u] \end{tikzcd}$$

As mentioned, the extension $R \subseteq \hat{R}$ is flat, and given $F_*\hat{R} \cong \widehat{F_*R}$, we see the the right arrow is flat as well. I now demonstrate that $F_*\hat{R}$ is flat as a $\hat{R}$-module. Notice that

$$\hat{R} = K[\![x_1, \dots, x_n]\!] \subseteq (F_*K)[\![x_1, \dots, x_n]\!] \subseteq F_*K[\![x_1, \dots, x_n]\!] = F_*\hat{R}$$

The first extension is free: given R is F-finite, we see K is as well. Therefore, there is a basis of F_*K over K consisting of $F_*k_1 = F_*1, \dots, F_*k_m$. This implies directly that

$$(F_*K)[\![x_1, \dots, x_n]\!] = \bigoplus_{i=1}^{m} F_*k_i \cdot \hat{R} \cong \hat{R}^n$$

Now, if we apply the same analysis of Homework 5, problem 7, we see that $F_*\hat{R} = F_*K[\![x_1, \dots, x_n]\!]$ is a free $(F_*K)[\![x_1, \dots, x_n]\!]$-module of rank N. Combining these 2 pieces of data, we see that

$$F_*\hat{R} \cong ((F_*K)[\![x_1, \dots, x_n]\!])^N \cong (\hat{R}^m)^N \cong \hat{R}^{mN}$$

In particular, we note that $F_*\hat{R}$ is a flat $\hat{R}$-module. Now, consider an injection $M \subseteq N$ of R-modules. Suppose that $F_*R \otimes_R M \not\subseteq F_*R \otimes_R N$. Let K be its kernel. We can now apply the exact functor $- \otimes_{F_*R} F_*\hat{R}$:

$$\begin{tikzcd} 0 \arrow[r] & K \otimes_{F_*R} F_*\hat{R} \arrow[r] \arrow[d, leftrightarrow, "\cong"] & M' \otimes_R F_*R \otimes_{F_*R} F_*\hat{R} \arrow[r] \arrow[d, leftrightarrow, "\cong"] & M' \otimes_R F_*R \otimes_{F_*R} F_*\hat{R} \arrow[d, leftrightarrow, "\cong"] \\ 0 \arrow[r] & K \otimes_{F_*R} F_*\hat{R} \arrow[r] & M' \otimes_R F_*\hat{R} \arrow[r] & M' \otimes_R F_*\hat{R} \end{tikzcd}$$

Since $K \neq 0$, $K \otimes_{F_*R} F_*\hat{R} \neq 0$ by faithful flatness. However, $M' \otimes_R F_*\hat{R} \hookrightarrow M' \otimes_R F_*\hat{R}$ since $F_*\hat{R}$ is $\hat{R}$-flat, and $\hat{R}$ is R-flat. Thus, a contradiction is reached. $\square$