CLASS 6, SEPTEMBER 20: PATH INTEGRALS

Last time, we established some conditions for what a good definition of a path should be. These are important for a definition of an integral in the complex plane. It turns out that the story is quite similar to that of calculus.

Definition 6.1. Let $\gamma : [a,b] \to \mathbb{C}$ be a smooth curve, and f be a continuous function on C (the image curve of γ). Then we define

$$\int_C f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Something funny has happened here; we used a generic parameterization of C. Therefore, we must establish that it is independent of the choice of parameterization to ensure that our asserted definition is well-defined.

Suppose $\gamma_1 \simeq \gamma_2$ are two equivalent curves. Recall that this means that there exists a continuously differentiable bijection $\sigma : [a,b] \to [c,d]$ with positive derivative such that $\gamma_1(t) = \gamma_2(\sigma(t))$. The chain rule then implies

$$\int_a^b f(\gamma_1(t))\gamma_1'(t)dt = \int_a^b f(\gamma_1(t))(\gamma_2(\sigma(t))'dt = \int_a^b f(\gamma_2(\sigma(t)))\gamma_2'(\sigma(t))\sigma'(t)dt$$

The last term is simply $\int_c^d f(\gamma_2(t))\gamma_2'(t)dt$ by the change of base formula (from calculus). This is precisely the reason we call 2 such curves equivalent.

We can naturally generalize this to piecewise smooth curves by dividing into smooth components:

$$\int_\gamma f(z)dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)$$

One additional extremely important quantity is the length of a curve. This is given by taking the function f to be 1 and removing the notion of positive/negative orientation:

$$\int_a^b |\gamma'(t)|dt$$

Example 6.2. Consider the curve parameterized by $\gamma(t) = e^{it}$. As we know, this traces a circle counterclockwise. This gives us some interesting things to consider: First, if $f(z) = z^n$ where $n \neq -1$, then we get

$$\int_C z^n dz = \int_a^b ie^{i(n+1)t}dt = \left[\frac{e^{i(n+1)t}}{n+1}\right]_{t=a}^b$$

Now we can choose our bounds. If we aim for a half circle, i.e. $a = 0$ and $b = \pi$, we yield an integrand of $\frac{2}{n+1}$.

Similarly, if we do the whole circle we yield 0! This is an example inside of a much broader result.

Example 6.3. If we continue with the previous example, but instead consider the function $f(z) = \frac{1}{z}$, then we will setup the integral

$$\int_C \frac{1}{z} dz = \int_a^b e^{-it} i e^{it} dt = \int_a^b i dt = i(b-a)$$

So in particular, if we let C be the whole circle, in this case we get $2\pi i$. Again, this is part of a much grander theorem that we will tackle in chapter 2.

Next, we get to a relation similar to the fundamental theorem of calculus.

Definition 6.4. If f is a function on an open set Ω, then f has a **primative** if there exists F a holomorphic function on Ω such that $F'(z) = f(z)$ for every $z \in \Omega$.

Theorem 6.5. *Let F be a primative for f in Ω, and $\gamma : [a,b] \to \Omega$ be a curve in Ω. Then*

$$\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$$

Proof. This follows from the standard fundamental theorem of calculus, since

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt = \int_a^b \frac{\partial}{\partial t} F(\gamma(t))dt$$

□

This yields two expected yet useful corollaries:

Corollary 6.6. *If γ is a loop in an open set Ω, and f has a primative in Ω, then*

$$\int_\gamma f(z)dz = 0$$

Proof. $F(a) = F(a)$! □

Given our example of $\int_{S^1} \frac{dz}{z} = 2\pi i$, where S^1 is the circle in $\mathbb{C}$, we know by this result that $\frac{1}{z}$ has no primative in $\mathbb{C} \setminus 0$. This has to do with the fact that we can't define the logarithm on all of $\mathbb{C}$ in a coherent way!

If we try to follow the homework's assertion that $\log(z) = \log(r) + i\theta$ for $r \geq 0$ and $\theta \in (-\pi, \pi)$, then as θ varies towards $-\pi$ and π we would expect different answers!

However, if we consider $\frac{1}{z^n}$ for $n > 1$, we do have the expected primative $\frac{-1}{(n-1)z^{n-1}}$. This immediately confirms the result

$$\int_{S^1} \frac{dz}{z^n} = 0$$

This can be verified by a straightforward computation, but we can avoid such work!

Corollary 6.7. *If f is holomorphic in Ω a connected and open set in $\mathbb{C}$, and $f' = 0$, then f is necessarily constant.*

Proof. f is certainly a primative for f'. As a result, we know

$$f(b) - f(a) = \int_\gamma f'(z)dz = \int_\gamma 0 = 0$$

But there exists a curve connecting any two points b, a by connectedness. As a result, $f(a) = f(b)$ for any $a, b \in \Omega$. □