HOMEWORK 5: COHEN-MACAULAY AND CHARACTERISTIC $p > 0$
DUE: MONDAY, APRIL 23

1) Show that the ring $R = K[x, y, u, v]/\langle x, y\rangle \cap \langle u, v\rangle$ from class 18 is non-CM (localized at $\langle x, y, z, w\rangle$ if desired). In particular, show that $\dim(R) = 2$ (what are its prime ideals?) and that modding out by any NZD leaves a ring with only units and zero-divisors.

2) Show that $\mathrm{Tor}_i^R(M \oplus M', N) \cong \mathrm{Tor}_i^R(M, N) \oplus \mathrm{Tor}_i^R(M', N)$.

3) Compute $\mathrm{Tor}_i^R(M, M)$ where $R = \mathbb{Z}/6\mathbb{Z}$ and $M = \mathbb{Z}/3\mathbb{Z}$.

4) Prove the following Proposition from class:

   Proposition 0.1. *A ring R is equal characteristic 0 if and only if $\mathbb{Q} \subseteq R$. A ring is characteristic $p > 0$, where p is a prime number, if and only if $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z} \subseteq R$. All other rings are mixed characteristic, which holds if and only if $\mathrm{char}(R) \neq \mathrm{char}(R/\mathfrak{m})$ for some maximal ideal $\mathfrak{m}$.*

5) Prove (or recall) the following proposition from class:

   Proposition 0.2. *If R is a ring of characteristic $p > 0$, then $F : R \to R$ is injective if and only if R is reduced (e.g. the nilradical $\mathcal{N} = 0$).*

6) Show that if $R \subseteq S$, M is an S-module, and $Hom_R(S, R) \cong S$ as S-modules, then the map given by composition is surjective:
$$\mathrm{Hom}_S(M, S) \times \mathrm{Hom}_R(S, R) \to \mathrm{Hom}_R(M, R)$$

7) Show that localization commutes with F_*. That is to say $F_*W^{-1}R \cong W^{-1}F_*R$.

8) Find an example of a field which is not F-finite.

9) Find an example of an F-finite field which isn't perfect.

10) Let $R = \mathbb{F}_q[x_1, \ldots, x_n]$ for some $q = p^e$. Show that F_*R is a free R-module and calculate its rank.