HOMEWORK 7: MANIFOLDS & TYCHONOFF
DUE: NOVEMBER 2

1) Recall that for $Y \subseteq Z$, a continuous map $r : Z \to Y$ is called a **retraction** if $r(y) = y$ for all $y \in Y$. Show that if Z is Hausdorff, then Y is a closed subset of Z. Also, show that there exists no retraction $r : \mathbb{R}^2 \to \{x, y\}$, but there is a retraction $R : \mathbb{R}^2 \setminus \{0\} \to S^1$.

2) Suppose $X_1 \subseteq X_2 \subseteq \ldots$ are a sequence of subspaces such that $X_i \subseteq X_{i+1}$ is a closed subset. We can put a topology on $X = \bigcup_{i=1}^{\infty} X_i$, called the **direct limit topology**, by requiring that $U \subseteq X$ is open if and only if $U \cap X_i \subseteq X_i$ is open. Show this is a topology upon which $X_i \subseteq X$ is a subspace. Additionally, show that $f : X \to Y$ is continuous if $f|_{X_i}$ is continuous for all i.

3) With the set up of the previous problem, show that X_i is normal for all i, then so is X with the direct limit topology. (**Hint**: Use Tietze iteratively.)

4) Show that every manifold is T3, and therefore is also metrizable. Is the Hausdorff condition used?

5) Show that if X is a connected, second-countable, locally Euclidean Hausdorff space, then the dimension m in the definition of locally Euclidean is the same on any chart.[1] You may assume the following beautiful result from algebraic topology[2]: If $U \subseteq \mathbb{R}^n$ and $V \subseteq \mathbb{R}^m$ are open subsets, and $U \cong V$, then $n = m$.

6) Show that if X is a locally Euclidean Hausdorff space, then if X is compact, each connected component of X is an m-manifold for some $m \in \mathbb{N}$.

7) Let $\mathcal{M}$ be a maximal collection of subsets of X with the finite intersection property, as in the proof of Tychonoff's Theorem.
   - Show $x \in \bar{A}$ for each $A \in \mathcal{M}$ if and only if every neighborhood U of x is in $\mathcal{M}$. Which direction(s) requires maximality?
   - Show that if $A \subseteq B$ for some $A \in \mathcal{M}$, then $B \in \mathcal{M}$.
   - Show that if X is T1, then $\bigcap_{A \in \mathcal{M}} \bar{A}$ is either empty or a single point.

[1]This demonstrates that calling X an m-manifold is well-defined.
[2]Usually referred to as **Invariance of Dimension**.