CLASS 20, OCTOBER 26: THE TIETZE EXTENSION THEOREM

Finally, we come to the Tietze Extension Theorem. This provides a tool which is applicable to things such as metric spaces, but also far more broadly. The formal statement is as follows:

Theorem 20.1 (Tietze Extension Theorem)**.** *Let A be a closed subspace of a normal topological space X. Then if $f : A \to \mathbb{R}$ is a continuous map, then there exists a continuous map $f' : X \to \mathbb{R}$ such that $f(a) = f'(a)$ for all $a \in A$. Moreover, if f is bounded in $[a, b]$, we may assume f' is as well.*

Note that this produces Urysohn as a corollary. Take the subspace $A \cup B$ and the continuous function f to be constantly 0 on A and 1 on B (it is continuous since A, B are open and closed in the subspace topology by virtue of their separation). Teitze allows us to extend f to a continuous function on all of X! Since we use Urysohn's Lemma in the proof of Tietze, we actually show that they are equivalent theorems.

We will use one tool from real analysis here, since our image is a metric space.

Lemma 20.2. *If $f_n : X \to \mathbb{R}$ are a sequence of continuous functions converging uniformly to $f : X \to \mathbb{R}$ ($\forall \epsilon > 0$, $\exists N > 0$ such that $|f(x) - f_n(x)| < \epsilon \ \forall n \geq N$). Then f is a continuous function.*

Proof. This follows by the triangle inequality. For a given $\epsilon > 0$, consider

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f_n(x)|$$

Therefore, we can intersect the 3 open neighborhoods of x resulting from $\frac{\epsilon}{3}$ in the definition of continuity of f_n, and uniform convergence of $f_n \to f$. $\square$

Proof. (of Theorem 20.1): I will begin by proving the bounded version: $f : A \to [-m, m]$ where $m = \sup\{|f(a)| \mid a \in A\}$. Consider the subsets $Z_0^- = f^{-1}(\big(-\infty, -\frac{m}{3}\big])$ and $Z_0^+ = f^{-1}(\big(\frac{m}{3}, \infty\big))$. They are closed and disjoint subsets of X (since A is closed). Therefore, we can apply Urysohn's Lemma to find a continuous function $g_0 : X \to [-\frac{m}{3}, \frac{m}{3}]$ which is constantly $-\frac{m}{3}$ on Z_0^- and $\frac{m}{3}$ on Z_0^+.

Now we may consider the function $f_1 = f - g_0 : A \to \mathbb{R}$ which is still bounded, since it is a difference of bounded functions. Note that since $|g_0(a)| \leq \frac{m}{3}$, we have $|f_1(a)| \leq \frac{2m}{3}$ by the triangle inequality. Let $Z_1^- = f_1^{-1}(\big(-\infty, -\frac{2m}{9}\big))$ and $Z_1^+ = f^{-1}(\big(\frac{2m}{9}, \infty\big))$. Repeating the procedure, we retrieve a function $g_1 : X \to [-\frac{2m}{9}, \frac{2m}{9}]$ constantly $-\frac{2m}{9}$ on Z_1^- and $\frac{2m}{9}$ on Z_1^+. Therefore, we obtain $|(f - g_0 - g_1)(a)| < \frac{4m}{9}$ for all $a \in A$.

Continuing in this manner, we can produce g_n with

$$|(f - g_0 - \ldots - g_n)(a)| < \frac{2^{n+1}m}{3^{n+1}}$$

Therefore, the functions $f_n = g_0 + g_1 + \ldots + g_n$ converges uniformly to f on A. Letting $f' = \lim_{n\to\infty} f_n$ produces a function on X agreeing with f on A.

Now suppose that $f : A \to \mathbb{R}$ isn't necessarily bounded. We note that $\mathbb{R} \cong (-\frac{\pi}{2}, \frac{\pi}{2})$ by virtue of the continuous function $h = \tan^{-1}$. Therefore, note that

$$h \circ f : A \to (-\frac{\pi}{2}, \frac{\pi}{2})$$

So we can apply the previous result to obtain an extended function $f' : X \to (-\frac{\pi}{2}, \frac{\pi}{2})$. Finally, note that $h^{-1} \circ f' : X \to \mathbb{R}$ has the property that $h^{-1} \circ f'(x) = f(x)$ for all x. Indeed,

$$f' = h \circ f \quad \Rightarrow h^{-1} \circ f' = h^{-1} \circ h \circ f = f$$

This completes the proof. □

As an immediate consequence, we get the following result from analysis;

Corollary 20.3. *A metric space (X, d) is compact if and only if every continuous function $f : X \to \mathbb{R}$ is bounded.*

Proof. ($\Rightarrow$): If X is compact, we get that $f^{-1}((n, n+1))$ forms a cover as n varies. Therefore, a finite subcover will do, and the maximum $|n| + 1$ from this sub will do.

($\Leftarrow$): Suppose every continuous function is bounded. Note that in the world of metric spaces, X is compact if and only if it is sequentially compact. Assume (aiming for a contradiction) $(x_1, x_2,)$ is a sequence in X with no convergent subsequence. Then the set $A = \{x_1, x_2, \ldots\}$ is closed (since it contains all of its limit points). Therefore, we can define the function $f : A \to \mathbb{R}$ by $x_n \mapsto n$. This is continuous since A has the discrete topology as a subspace. Therefore, we can extend f to an unbounded function $f : X \to \mathbb{R}$ by Tietze. This contradicts are assumption. □

It should be noted that the proof does not extend to arbitrary normal spaces, as we only have compactness implies sequential compactness. An example to show that the reverse is not true is the normal subspace of ordinals $< \Omega := \omega^\omega$, denoted in the book by S_Ω.

We can also extend Tietze's Extension Theorem to more general ranges. Here is a formal statement:[1]

Theorem 20.4 (Tietze Extension V2)**.** *Suppose $A \subseteq X$ is a closed subspace of a normal space X and Z is a retraction of the space $\mathbb{R}^\Lambda$ for some indexing set Λ. Then if $f : A \to Z$ is a continuous map, there exists $f' : X \to Z$ extending f.*

Proof. Since Z is a retraction of $\mathbb{R}^\Lambda$, there exists a continuous function $r : \mathbb{R}^\Lambda \to Z$ which is the identity when restricted to $Z \subseteq \mathbb{R}^\Lambda$. Therefore, given f as in the theorem, it suffices to construct $f' : X \to \mathbb{R}^\Lambda$ agreeing with f on A and apply r: letting $f'(a) = z$, note that $r \circ f : X \to Z$ has the property that

$$r(f(a)) = r(f'(a)) = r(z) = z$$

Therefore, it suffices to check the statement when $Z = \mathbb{R}^\Lambda$. But $f : A \to \mathbb{R}^\Lambda$ is continuous if and only if each of its coordinate functions $f_\alpha = \pi_\alpha \circ f : A \to \mathbb{R}$ are continuous. Therefore, we can apply the original Tietze Extension Theorem to produce $f'_\alpha : X \to \mathbb{R}$ agreeing with f_α. Then let $f' = (f_\alpha) : X \to \mathbb{R}^\Lambda$ be the desired extension. □

[1]In Munkres, this is phrased as the **Universal Extension Property**, introduced in $\oint$ 35, problem 5.