HOMEWORK 8: FINITENESS & NAGATA-SMIRNOFF
DUE: NOVEMBER 2

1) An open covering is **point-finite** if any point is in at most finitely many elements. Find an example of a point finite open covering which is not locally finite.

   Solution: Let $X = [0, 1]$, and consider $[0, 1]$ and $U_n = (0, \frac{1}{n})$. If $x = 0$, $x \notin U_n$ for any n. Any point $x \neq 0$ has the property that $\frac{1}{n} < x$ for all $n \geq N$, so in particular the cover is point-finite. On the other hand it isn't locally finite since a neighborhood of 0 will intersect all of them.

2) Show that if X is second-countable, the $\mathcal{A}$ is a countably locally finite set if and only if $\mathcal{A}$ is countable.

   Solution: If $\mathcal{A}$ is countable, then it is trivially countably locally finite.

   On the other hand, suppose that X is countably locally finite. Let $\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n$ be a decomposition into locally finite components. Since X is second countable, there exist countably many neighborhoods to check say $U_1, U_2, \ldots$. Therefore, for each $x \in X$, there is some U_i containing it which intersects $\mathcal{A}_n$ finitely many times. Call the finite collection of such things $\mathcal{A}_{n,i} \subseteq \mathcal{A}_n$. Since every element of $\mathcal{A}_n$ intersects at least one neighborhood, we see that $\mathcal{A}_n = \bigcup_{i \text{ possible}} \mathcal{A}_{n,i}$. But this implies

$$\mathcal{A} = \bigcup_{n\in\mathbb{N}} \mathcal{A}_n = \bigcup_{i\in\mathbb{N}} \bigcup_{i \text{ possible}} \mathcal{A}_{n,i}$$

   which is a countable union of finite sets, thus countable.

3) In the uniform topology, let $\mathfrak{B}_n$ is the collection of all subsets $\prod_{i\in\mathbb{N}} X_i \subseteq \mathbb{R}^{\mathbb{N}}$ with $X_1 = \ldots = X_n = \mathbb{R}$, and $U_m = \{0\}$ or $U_m = \{1\}$ for $m > n$. Show $\mathfrak{B} = \cup_{i=1}^{\infty} \mathfrak{B}_i$ is countably locally finite but not countable or locally finite.

   Solution: First note that each $\mathfrak{B}_n$ is in bijection with $2^{\mathbb{N}}$, which is uncountable (in bijection with $\mathbb{R}$, or more obviously the Cantor set). Therefore $\mathfrak{B}$ is of course uncountable.

   Additionally, the total set is not locally finite. Consider $0 = (0, 0, \ldots)$. This point is in $\mathbb{R}^n \times \prod_{m=n+1}^{\infty} \{0\}$ for each n, which is an element of $\mathfrak{B}_n$.

   Finally, it goes to check that each $\mathfrak{B}_n$ is locally finite. Suppose $(x_1, x_2, \ldots) \in \mathbb{R}^{\mathbb{N}}$. If some $x_m \neq 0, 1$ for $m > n$, choose $B(x, \min\{|x_m|, |1 - x_m|\})$. This doesn't intersect ANY element of $\mathfrak{B}_n$. Otherwise, every x_m is either 0 or 1 for $m > n$. In this case, choose $U = B(x_m, \frac{1}{2})$. This intersects only $\mathbb{R}^n \times \prod_{m=n+1}^{\infty} \{x_m\}$, as every other set has at least one element differing by 1. This proves the claim.

4) Show that a T1 space has a locally finite basis if and only if it is discrete.

   Solution: If X is discrete, it has a basis of 1-point sets. Therefore it has a locally finite basis.

On the other hand, suppose that X has a locally finite basis. Let $x \in X$ and U an open neighborhood be such that $B_1, \ldots, B_n$ are the only basis elements intersecting U. At least one contains x, so we can form a set

$$B_x = \bigcap_{x \in B_i} B_i$$

This is an open set containing x. Furthermore it is the smallest open set containing x. Suppose $y \neq x$ is in B_x. Then there exists no open set containing x which doesn't contain y. Therefore, X isn't T1. A contradiction. Therefore, $B_x = \{x\}$, so X is discrete.

5) Find a non-discrete space which has a countably locally finite basis, but is not second-countable.

Solution: Consider $\mathbb{R}^{\mathbb{N}}$ with the uniform topology. We already know this space is not second-countable, so it suffices to check that it is countably locally finite. For a fixed n, let

$$\mathfrak{B}_n = \left\{ B\left(x, \frac{1}{n}\right) \mid x \in X \right\}$$

Since every metric space is paracompact, we know that there exists a locally finite open covering refinement of $\mathfrak{B}_n$. Call it $\mathfrak{B}'_n$. Then I claim $\mathfrak{B} = \bigcup_{i=1}^{\infty} \mathfrak{B}'_n$ is the desired basis. Since $\mathfrak{B}'_n$ covers X, we know that for any given x there exists an open set U_n containing x in $\mathfrak{B}'_n$, and thus there exists a ball of a given radius $B(x, \epsilon)$. But once $\frac{1}{m} < \epsilon$, the same can be said about $\mathfrak{B}'_m$. This shows it is a basis, which by construction is countably locally finite.

6) Find an example to show that a paracompact space can have an open cover $X = \bigcup_{\alpha \in \Lambda} U_\alpha$ which doesn't have a locally finite *subcover* $X = \bigcup_{\alpha \subseteq \Lambda'} U_\alpha$ with $\Lambda' \subseteq \Lambda$.

Solution: Choose $B(0, n) \subseteq \mathbb{R}^m$. This is clearly a covering of $\mathbb{R}^m$, and every element contains 0. Therefore no such subcover can exist.

However, $\mathbb{R}^n$ is paracompact.

7) Find an example of a locally compact Hausdorff space which is not normal.[1] You may use any asserted results from Class 16.

Solution: Note that every paracompact Hausdorff space is normal. Therefore, it suffices to find a locally compact Hausdorff space which isn't normal.

An example of such a space results from Class 16; the tangent-disc topology. This is a non-normal space, so it only goes to show that it is locally compact. Given a point not on the y-axis, we have that there exists a neighborhood whose closure doesn't intersect the y-axis. Therefore it has the Euclidean topology in the subspace.

Now take $(0, y) \subseteq U$. The basis ensures that there is some $r > 0$ such that

$$\{(0, y)\} \cup B((-2r, y), r) \cup B((2r, y), r) \subseteq U$$

[1]Therefore locally compact and Hausdorff do not imply paracompact.

Note that for $x \in \bar{B}((r,y),r)$, we have (using the standard Euclidean metric)

$$d(x,(2r,y)) \leq d(x,(r,y)) + d((r,y),(2r,y)) \leq r + r = 2r$$

with equality if and only if $x = (0,y)$. This implies

$$Y = \bar{B}((-r,y),r) \cup \bar{B}((r,y),r) \subseteq \{(0,y)\} \cup B((-2r,y),r) \cup B((2r,y),r) \subseteq U$$

It is closed in the Euclidean topology thus also in the tangent-disc. Now, I claim it is a compact space. Suppose $Y = \bigcup_\alpha U_\alpha$. Choose $(0,y) \in U_{\alpha_0}$. Then $B \backslash U$ contains no points on the y-axis, and therefore has the Euclidean subspace topology. Since it is a closed and bounded set of $\mathbb{R}^2$, it is compact!