COMPLEX ANALYSIS: MIDTERM

(1) (10 points) Define the integral of a continuous function $f : \Omega \to \mathbb{C}$ along a piecewise-smooth path $\gamma : [a, b] \to \Omega$.

Solution: Assuming γ is piecewise smooth, we can break it into smooth components on intervals $[a_i, a_{i+1}]$ for $a = a_0 < \ldots < a_n = b$. Then

$$\int_\gamma f(z)dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)dt$$

(2) (10 points) Define what it means for a function to be analytic at $z_0 \in \mathbb{C}$.

Solution: There exists $a_i \in \mathbb{C}$ and $r > 0$ such that

$$f(z) = \sum_{i=0}^{\infty} a_i(z - z_0)^i$$

for all $z \in B(z_0, r)$.

(3) (15 points) State the Cauchy-Riemann equations. When do they ensure holomorphicity?

Solution: The CR equations are given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

The function $f = u + iv$ is holomorphic if the CR equations hold and all of the partials are continuous.

(4) (15 points) State Cauchy's Integral Theorem.

Solution: If f is holomorphic on Ω, and $C \subseteq \Omega$ is a positively oriented simple curve, then

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}} dw$$

for each z in the interior of C.

(5) (25 points) Let $z = x + iy$ and $f(z) = u(z) + iv(z)$. Suppose $u(z) = 4xy^3 - 4x^3y$. Find a function $v(z)$ that makes $f(z)$ an entire function.

Solution: Following the CR equations, we see that

$$\frac{\partial v}{\partial y} = 4y^3 - 12x^2y$$

$$\frac{\partial v}{\partial x} = -(12xy^2 - 4x^3) = 4x^3 - 12xy^2$$

Integrating both of the equations with respect to the desired variable yields

$$v = y^4 - 6x^2y^2 + C(x) = x^4 - 6x^2y^2 + D(y)$$

Thus we see that any v of the form

$$v = x^4 + y^4 - 6x^2y^2 + E$$

for $E \in \mathbb{C}$ will do.

(6) (25 points) Define for each $\alpha \in \mathbb{R}$ the quantity

$$I(\alpha) = \int_{-\infty}^{\infty} e^{-(x+i\alpha)^2} dx$$

Show that in fact $I(\alpha)$ is independent of α, and thus equal to $I(0) = \sqrt{\pi}$.

Solution: Consider the rectangle Λ with vertices $R, -R, R+i\alpha, -R+i\alpha$. Orient it clockwise. Since $f(z) = e^{z^2}$ is entire, by Goursat's Theorem we have that

$$0 = \int_{\Lambda} f(z)dz = \int_{-R}^{R} f(x)dx - \int_{-R}^{R} f(x+it)dx + \int_{0}^{\alpha} f(R+it)dt - \int_{0}^{\alpha} f(-R+it)dt$$

So it suffices to show the latter 2 integrals are 0. This follows from our typical bound:

$$\left| \int_{0}^{\alpha} f(\pm R+it)dt \right| \leq \alpha \cdot e^{-R^2+\alpha^2} \to 0$$

(7) (20 points) Prove Liouville's theorem assuming Cauchy's inequality.

Solution: Cauchy's inequality produces

$$|f^{(n)}(z_0)| \leq \frac{n! \cdot \|f\|_C}{r^n}$$

where C is a circle centered at z_0 of radius r. If f is bounded and entire, we also have $\|f\|_C \leq B$ for some $B > 0$. Thus

$$|f^{(n)}(z_0)| \leq \frac{n! \cdot \|f\|_C}{r^n} \leq \frac{n! \cdot B}{r^n} \to 0$$

as $r \to \infty$. Thus $f^{(n)}(z_0) = 0$ for $n > 0$. That is to say

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0}{n!}(z - z_0)^n = f(z_0)$$

(8) (30 points) Compute the following integral:

$$\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{(x^2+4)^2} dx$$

Solution: Consider the upper semicircle S of radius R oriented positively. The function above has poles of order 2 at $\pm 2i$, with only $2i$ being enclosed. Therefore the residue theorem yields

$$\int_S f(z)dz = 2\pi i \cdot \mathrm{res}_{2i}(f(z))$$

where $f(z) = \frac{e^{\pi i z}}{(z^2+4)^2}$. Notice that the upper portion of the semicircle is sent to 0:

$$\left| \int_{\gamma_R} f(z)dz \right| \leq \pi R \frac{1}{(R^2-4)^2} \to 0$$

as $R \to \infty$. So in fact

$$\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{(x^2+4)^2} dx = 2\pi i \cdot \mathrm{res}_{2i}(f(z))$$

since $i \sin(\pi x)$ is odd. Finally

$$\mathrm{res}_{2i}(f(z)) = \lim_{z \to 2i} \frac{\partial}{\partial z}(z-2i)^2 f(z) = \lim_{z \to 2i} \frac{\partial}{\partial z} \frac{e^{\pi i z}}{(z+2i)^2}$$

$$= \lim_{z \to 2i} \left(\pi i \frac{e^{\pi i z}}{(z+2i)^2} - 2 \frac{e^{\pi i z}}{(z+2i)^3} \right) = \pi i \frac{e^{-2\pi}}{(4i)^2} - 2 \frac{e^{-2\pi}}{(4i)^3} = e^{-2\pi} \frac{\pi}{16i} - e^{-2\pi} \frac{1}{32i^3}$$

Thus

$$\int_{-\infty}^{\infty} \frac{\cos(\pi x)}{(x^2+4)^2} dx = 2\pi i e^{-2\pi} \cdot \left(\frac{\pi}{16i} - \frac{1}{32i^3} \right) = e^{-2\pi} \left(\frac{\pi^2}{8} + \frac{\pi}{16} \right)$$