CLASS 34, DECEMBER 5: THE FUNDAMENTAL GROUP

Today we introduce the fundamental group, which provides a tool to easily distinguish some non-homotopic spaces via abstract algebra. It is a surprising achievement of the 20th century that relates distinct regions of math.

Proposition 34.1. *Let $\Omega(X,x)$ be the set of loops based at x. Define*

$$\pi_1(X,x) := (\Omega(X,x)/\simeq \ \textit{rel}\ \{0,1\}\ ,\ *)$$

That is to say our elements are equivalence classes of homotopic loops relative to the basepoint x with the operation of composition of paths. Then $\pi_1(X,x)$ is a group.

Proof. ∘ **Identity:** The identity element for $*$ is the constant path $e : I \to X : t \to x$. Indeed,

$$F : I \times I \to X : (s,t) \mapsto \begin{cases} x & s \leq \frac{t}{2} \\ \gamma\left(\frac{s-\frac{t}{2}}{1-\frac{t}{2}}\right) & s \geq \frac{t}{2} \end{cases}$$

This is an explicit homotopy $\gamma \simeq e * \gamma$ (since $\frac{s-\frac{1}{2}}{1-\frac{1}{2}} = 2s-1$). Additionally, $F(s,0) = x$ and $F(s,1) = \gamma(1) = x$ for all s. So in fact $\gamma \simeq e * \gamma$ rel $\{0,1\}$, as desired.

A similar homotopy show $\gamma \simeq \gamma * e$ rel $\{0,1\}$

Existence of inverses: I claim that for a given loop γ, the inverse is given by

$$\bar{\gamma}(s) = \gamma(1-s)$$

(note the bar here is used to avoid confusion with the inverse image under γ). Again, we demonstrate this with an explicit homotopy operator:

$$F : I \times I \to X : (s,t) \mapsto \begin{cases} \gamma(2s) & s \leq \frac{t}{2} \\ \gamma(t) & \frac{t}{2} \leq s \leq 1-\frac{t}{2} \\ \bar{\gamma}(2s-1) & s \geq 1-\frac{t}{2} \end{cases}$$

The idea here is quite simple; run through γ for a shorter and shorter period of time, stop at whatever point you get to, and then go back. Note that clearly $F(s,0) = e(x) = x$, and additionally that $F(s,1) = \gamma * \bar{\gamma}(s)$. Finally, note that this is a continuous map by the pasting lemma: $\gamma(2\frac{t}{2}) = \gamma(t) = \bar{\gamma}(2(1-\frac{t}{2})-1) = \bar{\gamma}(1-t)$.

Associative: One can compute

$$\gamma_1 * (\gamma_2 * \gamma_3) : s \mapsto \begin{cases} \gamma_1(2s) & s \in [0,\frac{1}{2}] \\ \gamma_2(4s-2) & s \in [\frac{1}{2},\frac{3}{4}] \\ \gamma_3(4s-3) & s \in [\frac{3}{4},1] \end{cases} \qquad (\gamma_1 * \gamma_2) * \gamma_3 : s \mapsto \begin{cases} \gamma_1(4s) & s \in [0,\frac{1}{4}] \\ \gamma_2(4s-1) & s \in [\frac{1}{4},\frac{1}{2}] \\ \gamma_3(2s-1) & s \in [\frac{1}{2},1] \end{cases}$$

The homotopy can be explicitly constructed as follows:

$$\Gamma : (s,t) \mapsto \begin{cases} \gamma_1\left(\frac{4}{1+t}s\right) & s \leq \frac{1}{4}+\frac{t}{4} \\ \gamma_2(4s-1-t) & \frac{1}{4}+\frac{t}{4} \leq s \leq \frac{1}{2}+\frac{t}{4} \\ \gamma_3\left(\frac{4}{2-t}s - \frac{t+2}{2-t}\right) & \frac{1}{2}+\frac{t}{4} \leq s \end{cases}$$

I leave it to you to check this is the desired homotopy. □

Therefore, $\pi_1(X, x)$ is a group! This is a very exciting result. Now we can proceed to discover some neat features of it. Recall the following definition:

Definition 34.2. A space X is called **path connected** if for every 2 points x, y, there exists a path γ such that $\gamma(0) = x$ and $\gamma(1) = y$. In such a case we write $\pi_1(X)$ instead of $\pi_1(X, x)$.

The reason we can do this is the following proposition:

Proposition 34.3. *If X is path connected, then for any two points x, y,*

$$\pi_1(X, x) \cong \pi_1(X, y)$$

Proof. Let γ be as in the definition of path connected. Then we can construct an explicit group isomorphism:

$$\Gamma : \pi_1(X, x) \to \pi_1(X, y) : \sigma \mapsto \bar{\gamma} * \sigma * \gamma$$

Note that this is a loop based at y, so it as at least a function. It is furthermore a group homomorphism:

$$\Gamma(\sigma * \sigma') = \bar{\gamma} * \sigma * \sigma' * \gamma \simeq \bar{\gamma} * \sigma * \gamma * \bar{\gamma} * \sigma' * \gamma = \Gamma(\sigma) * \Gamma(\sigma')$$

Of course, the inverse of this is given by a map of the same type:

$$\Gamma^{-1} : \pi_1(X, y) \to \pi_1(X, x) : \sigma \mapsto \gamma * \sigma * \bar{\gamma}$$

□

Now onto some examples (with nice definitions):

Example 34.4.
- $\pi_1(\mathbb{R}^n) = 0$. This is because $\mathbb{R}^n$ is a **contractible space**: $\mathbb{R}^n \simeq pt$. Indeed, we can use the simple homotopy $F(x, t) = (1 - t) \cdot x$ to show this. Now, given a loop γ based at 0 (WLOG by Proposition 34.3), then we can consider

$$G(s, t) = F(\gamma(s), t)$$

  This shows that $e \simeq \gamma$ rel $\{0, 1\}$, and thus $\pi_1(\mathbb{R}^n) = 0$. A path-connected space with trivial fundamental group is called **simply-connected** (c.f. the Poincaré conjecture). Note that since $\mathbb{R}^n \ncong \mathbb{R}^m$ for $n \neq m$, we have that the fundamental group doesn't distinguish non-homeomorphic spaces.
- Additionally, not all spaces with trivial fundamental group are contractible. This is demonstrated by S^n for $n \geq 2$. All of these spaces bound an n-dimensional space, so are non-contractible.

  If $\gamma : I \to S^n$ is a path, then either γ is surjective (space filling) or it isn't. If it isn't surjective, then $\gamma : I \to S^n \setminus \{pt\} \cong \mathbb{R}^n$, and since $\mathbb{R}^n$ is contractible, so is the curve. If γ is surjective, we can take a small open neighborhood of a non-basepoint, and modify γ homotopically by taking the curve through the disc instead along the boundary, making it non-surjective. This shows $pi_1(S^n) = 0$.
- $\pi_1(S^1) = \mathbb{Z}$. This takes some work to show, but intuitively is quite easy to visualize. We can count the number of times a curve loops around the circle counterclockwise (the winding number of γ). This is a homotopy invariant (cf Example 33.3), and therefore establishes a well defined map $\pi_1(S^1) \to \mathbb{Z}$, treating clockwise rotations as negative. The difficult part is showing this is injective.