CLASS 9, SEPTEMBER 27: CAUCHY'S INTEGRAL FORMULAS

Today we will begin by showing one more example following from local Cauchy and then move into some extremely useful integral formulas which also follow.

Example 9.1. If $\xi \in \mathbb{R}$, then I claim

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx$$

If $\xi = 0$, then $1 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx$ can be calculated directly. If $\xi > 0$, then $f(z) = e^{-\pi z^2}$is an entire function. Consider the contour γ which is a rectangle with vertices $-R, R, R+i\xi, -R-i\xi$ given counterclockwise orientation. By Cauchy's theorem,

$$\int_{\gamma} f(z)dz = 0$$

The integral over the bottom is exactly 1 as $R \to \infty$. The right side's integral is

$$\int_0^{\xi} f(R+iy)idy = i\int_0^{\xi} e^{-\pi(R^2+2iRy-y^2)}dy$$

As $R \to \infty$, this integral goes to 0 since it has absolute value bounded above by $Ce^{-\pi R^2}$ (since ξ is fixed). The same is true on the left. Finally, for the top, we have

$$\int_{-R}^{R} e^{-\pi(x+i\xi)^2}dx = -e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2}e^{-2\pi i x\xi}dx$$

Sending $R \to \infty$ yields

$$0 = 1 - e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2}e^{-2\pi i x\xi}dx$$

This shows that the Fourier Transform of $e^{-\pi z^2}$ is itself! We now shift to one of the most central theorems in complex analysis; Cauchy's integral theorem:

Theorem 9.2 (Cauchy's Integral Theorem)**.** *If $f : \Omega \to \mathbb{C}$ is a holomorphic function, and $\bar{B}(z_0, r) \subseteq \Omega$, then*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw \qquad\qquad \forall z \in B(z_0, r)$$

Here C is the positively oriented circle bounding $\bar{B}(z_0, r)$.

Proof. Consider the keyhole K with outer circle $\bar{B}(z_0, r)$ and inner circle of radius ϵ centered at z. Let $\delta > 0$ be the width of the corridor. Since f is holomorphic, we know that $\frac{f(w)}{w-z}$ is holomorphic on the interior of this region. As a result, Local Cauchy tells us

$$\int_K \frac{f(w)}{w-z}dw = 0$$

Now, we can send $\delta \to 0$ since $\frac{f(w)}{w-z}$ is a continuous function away from z. This makes it so that the path join and cancel each other, leaving only the 2 circles. Orienting both positively as we have been, we have

$$\int_C \frac{f(w)}{w-z}dw = \int_{C_\epsilon} \frac{f(w)}{w-z}dw$$

It goes to compute the left hand side. We can break up the equation as

$$\int_{C_\epsilon} \frac{f(w)}{w-z}dw = \int_{C_\epsilon} \frac{f(w)-f(z)}{w-z}dw + \int_{C_\epsilon} \frac{f(z)}{w-z}dw$$

The first integral on the right hand side is bounded as $\epsilon \to 0$, since the inner portion approaches the derivative. Thus the integral is 0. Therefore, we are left with

$$\int_{C_\epsilon} \frac{f(w)}{w-z}dw = f(z)\int_{C_\epsilon} \frac{1}{w-z}dw$$

We can change variables using $w \mapsto w+z$ to recenter the integral at 0. This shows

$$\int_{C_\epsilon} \frac{f(w)}{w-z}dw = f(z)\int_{C_\epsilon} \frac{1}{w}dw = f(z)2\pi i.$$

Dividing precisely yields the desired result. $\square$

Just like with our previous Cauchy-style results, we can replace the circle with any toy contour that has z in its interior and yield identical results. Note that if z isn't in the interior we are holomorphic and thus the integral vanishes. This gives us an easy route to solving the homework problem:

Example 9.3. If $C = \partial B(0,r)$ is positively oriented, where $|a| < r < |b|$, then

$$\int_C \frac{dz}{(z-a)(z-b)} = \int_C \frac{\frac{1}{z-b}}{z-a}dz = 2\pi i \frac{1}{a-b}.$$

Here we are taking $f(z) = \frac{1}{z-b}$.

As a corollary to this theorem, which may seem a bit ambiguous as to why we would care about such a specific integral, is the following result I asserted in Class 0:

Corollary 9.4. *If $f : \Omega \to \mathbb{C}$ is a holomorphic function, then f has infinitely many complex derivatives in Ω. Furthermore, if $\bar{B}(z_0,r) \subseteq \Omega$ and $C = \partial\bar{B}(z_0,r)$, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw \qquad \forall z \in B(z_0,r)$$

Proof. The proof follows by induction. $n = 0$ is Cauchy's integral theorem. So suppose it is true up to n-derivatives:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw \qquad \forall z \in B(z_0,r)$$

If we write the difference quotient, we yield

$$\frac{f^{(n)}(z+h) - f^{(n)}(z)}{h} = \frac{n!}{2\pi i}\int_C \frac{f(w)}{h}\cdot\left[\frac{1}{(w-z-h)^{n+1}} - \frac{1}{(w-z)^{n+1}}\right]dw$$

Again we use the difference of 2 powers rule for $a^{n+1} - b^{n+1}$:

$$\frac{1}{(w-z-h)^{n+1}} - \frac{1}{(w-z)^{n+1}} = \frac{h}{(w-z-h)(w-z)}\left[\frac{1}{(w-z-h)^n} + \ldots + \frac{1}{(w-z)^n}\right]$$

Notice that if h is sufficiently small we stay within C. As a result

$$= \frac{n!}{2\pi i}\int_C f(w)\cdot\frac{1}{(w-z)^2}\frac{n+1}{(w-z)^n}dw$$

$\square$