CLASS 16, MARCH 13TH: NN & INTEGRAL FIELD EXTENSIONS

Recall that last time we proved Noether Normalization using the following result:

Lemma 1. *Given the set up of Theorem 15.6, if $z_1, \ldots, z_n \in A = K[z_1, \ldots, z_n]$ are not algebraically independent, then there exists $z_1^*, \ldots, z_{n-1}^*$ such that z_n is integral over $A^* = K[z_1^*, \ldots, z_{n-1}^*]$. Moreover, $A = A^*[z_n]$.*

Today, we will prove this statement (in the case that K is an infinite field) and talk about integrality with respect to fields. The statement for non-infinite fields K is more technical and is due to Nagata. It is available in section 4.7 of the book for those interested.

Proof. (of Lemma 1) We will pick elements of the field $\alpha_1, \ldots, \alpha_{n-1} \in K$ such that

$$z_i^* = z_i - \alpha_i z_n$$

play the desired role. Define

$$G(z_1^*, \ldots, z_{n-1}^*, z_n) = F(z_1^* + \alpha_1 z_n, \ldots, z_{n-1}^* + \alpha_{n-1} z_n, z_n) = 0$$

achieved simply by substituting for z_i using our new equation. Let

$$F = \sum_m a_m z^m = \sum_m a_m z_1^{m_1} \cdots z_n^{m_n}$$

Then

$$G = \sum_m a_m (z_1^* + \alpha_1 z_n)^{m_1} \cdots (z_1^* + \alpha_1 z_n)^{m_{n-1}} z_n^{m_n}$$

Let $d = \deg(F)$ be the largest number such that there exists m such that $|m| = m_1 + \ldots + m_n = d$ with $a_m \neq 0$. Then notice that the coefficient in $K[z_1^*, \ldots, z_{n-1}^*]$ of z_n^d of G is given by

$$F_d(\alpha_1, \ldots, \alpha_{n-1}, 1) = \sum_{|m|=d} a_m \alpha_1^{m_1} \cdots \alpha_{n-1}^{m_{n-1}}$$

which is simply an element of K! So it only goes to ensure we can choose α_i such that $F_d(\alpha_1, \ldots, \alpha_{n-1}, 1)$ is a unit, or equivalently non-zero.

This can rephrased as follows; $f \in K[x_1, \ldots, x_n]$ where K is an infinite field is zero if and only if $f(\alpha_1, \ldots, \alpha_n) = 0$ for any choice of $\alpha_i \in K$. When $n = 1$, this is clear (f has only finitely many roots in $\bar{K}$, thus also in K). Assume we have proved this for up to n variables. But

$$f \in K[x_1, \ldots, x_n] \subseteq K(x_1, \ldots, x_{n-1})[x_n]$$

So there are only finitely many roots $x_n = \alpha$ for which $f(x_1, \ldots, x_{n-1}, \alpha) = 0$. Choose β not one of these roots, and notes that

$$0 \neq f(x_1, \ldots, x_{n-1}, \beta) \in K[x_1, \ldots, x_{n-1}]$$

As a result, we can conclude by induction that there exist $\alpha_1, \ldots, \alpha_{n-1} \in K$ such that

$$0 \neq f(\alpha_1, \ldots, \alpha_{n-1}, \beta)$$

as desired. □

We will finish up with one neat consideration for integral extensions:

Proposition 16.1. *Let $A \subseteq B$ be an integral extension of integral domains. Then*
$$A \text{ is a field} \quad \Longleftrightarrow \quad B \text{ is a field}$$

Of course, the same is not true if we weaken our assumptions:

Example 16.2. $K \subseteq K[x]/\langle x^n \rangle$ is an integral extension, but $K[x]/\langle x^n \rangle$ is not even a domain! If we try to drop the integral assumption, examples such as $\mathbb{Z} \subseteq \mathbb{Q}$ and $K \subseteq K[x]$ provide natural counterexamples.

Proof. (of Proposition 16.1) $\Rightarrow$: Suppose $x \in B$ and A is a field. Then
$$x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n = 0$$
implies
$$x \cdot (-a_n^{-1})(x^n + a_1 x^{n-1} + \ldots + a_{n-1}) = 1$$
So x is a unit. Note here we can assume $a_n \neq 0$, as otherwise we could simply factor out by a large enough power of x, which is non-zero since $x \in B$ can be assumed not in A.

$\Leftarrow$: Suppose $x \in A$ and B is a field. Then $x^{-1} \in B$ and
$$x^{-n} + a_1 x^{-(n-1)} + \ldots + a_{n-1} x^{-1} + a_n = 0$$
implies (by multiplying by x^{n-1})
$$x^{-1} = -(a_1 + a_2 x + \ldots + a_n x^{n-1})$$
So $x^{-1} \in A$. $\square$

Corollary 16.3 (Weak Nullstellensatz)**.** *If K/k is a field extension, and K is a finitely generated k-algebra, then K/k is algebraic/integral, and thus is a finite field extension.*

Proof. By Noether Normalization, there exist $z_1, \ldots, z_n$ algebraically independent elements such that $k[z_1, \ldots, z_n] \subseteq K$ is a finite extension of rings. By Proposition 16.1, we know that $k[z_1, \ldots, z_n]$ is a field. This is only possible if $n = 0$ by algebraic independence. And thus K/k is finite. $\square$

Remember, the exam is on Friday!