Nov ~~10~~ 13

There is a mild problem w/ Simplicial Homology: Not all spaces are Δ-complexes.

In fact: It was proved ~~in~~ that E8 (dim=4) is not all manifolds are Δ-complexes!

Manolescu

Therefore, we relax the conditions of a Δ-complex, and allow singular n-chains:

Continuous maps $\sigma_\alpha^n : \Delta^n \longrightarrow X$

The collection of these is $\widetilde{\Delta}_n(X)$. The boundary map still makes sense, satisfies $\partial_{n-1} \circ \partial_n = 0$:

$$\cdots \longrightarrow \widetilde{\Delta}_{n+1} \xrightarrow{\partial_{n+1}} \widetilde{\Delta}_{n} \xrightarrow{\partial_n} \widetilde{\Delta}_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$

$$H_n(X) := \ker(\partial_n) / \operatorname{Im}(\partial_{n-1})$$

Prop: If $X = \coprod_\alpha X_\alpha$ is a decomposition into path/ connected components, then

$$H_n(X) = \bigoplus_{\alpha} H_n(X_\alpha)$$

Pf: If $\Delta^n \xrightarrow{\sigma^n} X$ is a continuous map, Δ^n is connected $\Rightarrow \sigma^n(\Delta^n)$ is connected.

$\Rightarrow \sigma^n(\Delta^n) \subseteq X_\alpha$ for some α

$$
\begin{array}{ccccc}
\tilde{\Delta}_{n+1}(X) & \xrightarrow{d_{n+1}} & \tilde{\Delta}_n(X) & \xrightarrow{d_n} & \tilde{\Delta}_{n-1}(X) \\
\| & & \| & & \| \\
\bigoplus_\alpha \Delta_{n+1} & \xrightarrow{d_{n+1}} & \bigoplus_\alpha \Delta_n(X_\alpha) & \xrightarrow{d_n} & \bigoplus_\alpha \tilde{\Delta}_{n-1}(X_\alpha)
\end{array}
$$

Prop: If $X \neq \phi$ is path connected, $H_0(X) = \mathbb{Z}$

Pf: $H_0(X) = C_0(X)/\mathrm{Im}(\partial_1)$. Define

$$\varepsilon : C_0(X) \twoheadrightarrow \mathbb{Z} : \sum_{i=0}^{m} n_i \sigma_i = \sum_{i=0}^{m} n_i$$

$\mathrm{Im}(\partial_1) \subseteq \mathrm{Ker}(\varepsilon)$ since $\varepsilon(\sigma^1) = \varepsilon(\sigma^1(v_1) - \sigma^1(v_0))$
$= 1 - 1 = 0$.

$\mathrm{Ker}(\varepsilon) \subseteq \mathrm{Im}(\partial_1)$: Let $\varepsilon(\sum n_i \sigma_i) = 0$
then $\sum n_i = 0$.

$\cancel{n_{i_1} + \dots + n_{i_k} = n_{i_{k+1}} + \dots + n_{i_m}}$

$\tau_i : I \to X$, $\tau_i(0) = x_0$, $\tau_i(1) = \sigma_i$

$$\tau = \sum n_i \tau_i \xmapsto{\partial_1} \sum n_i \partial_1 \tau_i = \sum n_i (\sigma_i - x_0)$$

Prop: if X is a point, $H_i(X) = 0$ if $i > 0$.

Pf: $\partial_n : \Delta_n(X) \cong \mathbb{Z} \to \Delta_{n-1}(X) \cong \mathbb{Z}$

$$\partial_n(\Delta^n) = \sum_{i=0}^{n} (-1)^i \Delta^{n-1} = \begin{cases} \Delta^{n-1} & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

$$\cdots \xrightarrow{\cong} \underset{C_3}{\mathbb{Z}} \xrightarrow{0} \underset{C_2}{\mathbb{Z}} \xrightarrow{\cong} \underset{C_1}{\mathbb{Z}} \xrightarrow{0} \underset{C_0}{\mathbb{Z}} \to 0$$

- - -

Ker = 0 Ker = $\mathbb{Z}$

Im = 0 Im = $\mathbb{Z}$

Nov 15: Homotopy Invariance

Reduced Homology.

Similar to $\pi_1(X)$, given a continuous map $f: X \to Y$, we can define

$$f_\#: C_n(X) \longrightarrow C_n(Y)$$
$$[\sigma: \Delta^n \to X] \longmapsto [f \circ \sigma: \Delta^n \to Y]$$

This has a nice property:

$$\begin{array}{ccccccc}
\to C_{n+1}(X) & \xrightarrow{\partial^X_{n+1}} & C_n(X) & \xrightarrow{\partial^X_n} & C_{n-1}(X) \to \\
\downarrow f_\# & \circlearrowleft & \downarrow f_\# & & \downarrow f_\# \\
\to C_{n+1}(Y) & \xrightarrow{\partial^Y_{n+1}} & C_n(Y) & \xrightarrow{\partial^Y_n} & C_{n-1}(Y) \to
\end{array}$$

$$\partial^Y_n \circ f_\# = f_\# \circ \partial^X_n$$

$\Rightarrow f_\#$ is a chain map.

Prop: $f_\#$ induces a map on Homology:

$$f_*: H_n(X) \to H_n(Y)$$

Pf: $\mathrm{Ker}(\partial^X_n) \xrightarrow{f_\#} \mathrm{Ker}(\partial^Y_n)$ since

$$0 = f_\#(0) = f_\#(\partial_X(\sigma)) = \partial_Y(f_\#(\sigma))$$

$$\mathrm{Im}(\partial^X_{n+1}) \rightarrow \mathrm{Im}(\partial^Y_{n+1})$$
$$f_\#(\partial_X(\sigma)) = \partial_Y(f_\#(\sigma))$$

$\leadsto$ f_* is well defined

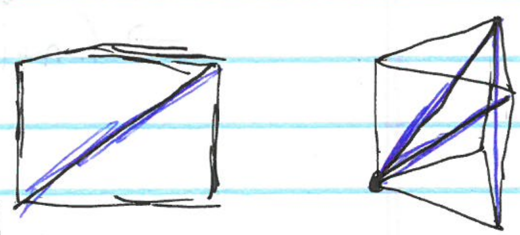
~~Prop~~ Thm [Homotopy Invariance]: If $f \simeq g : X \to Y$, then $f_* = g_*$. As a result, Homotopic eq spaces have $\cong$ homology.

Note $(f \circ g)_* = f_* \circ g_*$, so since $f \circ g \simeq Id_X$
$$f_* \circ g_* = Id_{X*} = Id_{H^n(X)}$$

Pf: The proof idea is to subdivide $\Delta^n \times I$ into $(n+1)$-many Δ^{n+1}, giving

$$F : X \times I \to Y$$

a nice singular structure.



Let $\Delta^n \times \{0\} = [v_0, \dots, v_n]$
$\Delta^n \times \{1\} = [w_0, \dots, w_n]$

For each $i = 0, \dots, n$, make a Δ^n by $[v_0, \dots, v_i, w_i, \dots, w_n]$

Consider the prism operator $P : C_n(X) \to C_{n+1}(Y)$
$$: \sigma \mapsto \sum_i (-1)^i F \circ (\sigma \times \mathbb{1})|_{[v_0, \dots, v_i, w_i, \dots, w_n]}$$

We will show $\partial_Y \circ P = g_* - f_* - P\partial^X$

$$\partial \circ P(\sigma) = \sum_{j \le i} (-1)^i (-1)^j F \circ (\sigma \times \mathbb{1})|_{[v_0, \ldots, \hat{v}_j, \ldots v_i, w_i, \ldots w_n]}$$

$$+ \sum_{j \ge i} (-1)(-1)^{j+1} F \circ (\sigma \times \mathbb{1})|_{[v_0, \ldots, v_i, w_i, \ldots \hat{w}_{j(+1)}, \ldots w_n]}$$

$i = j$ cancel except $F \circ (\sigma \times \mathbb{I})|_{[\hat{v}_0, w_0, \ldots, w_n]}$ and

$-F \circ (\sigma \times \mathbb{1})|_{[v_0, \ldots, v_n, \hat{w}_n]}$. These are

$g_\#(\sigma)$, $-f_\#(\sigma)$ respectively. Left w/ $i<j, i>j$

$\Rightarrow g_\# - f_\# = \partial P + P\partial$ (P is a chain homotopy operator)

If $\alpha \in Z_n(X)$, then

$$g_\#(\alpha) - f_\#(\alpha) = \underbrace{\partial \circ P(\alpha)}_{\in B_n(Y)} - \underbrace{P \circ \partial(\alpha)}_{=0}$$

$\Rightarrow g_* = f_*$!

Cor: Contractible Spaces have $\tilde{H}_n(X) = 0$ $\forall n \ge 0$.

Nov 17: Excision & Exactness

A sequence of group homs

$$\cdots \to G_{n+1} \xrightarrow{\partial_{n+1}} G_n \xrightarrow{\partial_n} G_{n-1} \xrightarrow{\partial_{n-1}} \cdots$$ is called exact

If $\ker(\partial_i) = \operatorname{Im}(\partial_{i+1})$ for all $i \in \mathbb{Z}$.

This is saying that Homology groups are all 0.
Often called an "acyclic" chain complex.

Nice statements: The following are exact

- $0 \to A \xrightarrow{i} B \iff i$ in an injection
- $B \xrightarrow{p} A \to 0 \iff p$ is surjective
- $0 \to A \to B \to C \to 0 \iff C \cong B/A$ (SES)

We can use this language to get good information about H_n of $A \subseteq X$, and X/A.

If $A \subseteq X$ is s.t. $\exists\, U \supseteq A$ w/ U def ret to A, we say (X, A) is a good pair.

$$A \xhookrightarrow{i} X \xrightarrow{q} X/A$$

Thm: If (X,A) is a good pair, the following sequence is exact:

$$\cdots \xrightarrow{\partial} \tilde{H}_n(A) \xrightarrow{i_*} \tilde{H}_n(X) \longrightarrow \tilde{H}_n(X/A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

∂ will be constructed in the proof, using the "snake lemma."

Nice thing (X, A) a CW pair $\Rightarrow$ Good pair
Ex/ $(\mathbb{D}^n, S^{n-1})$: $\mathbb{D}^n / S^{n-1} \cong S^n$. So

$$\cdots$$

$$\xrightarrow{\delta} \tilde{H}_{i}(S^{n-1}) \to \underset{}{\overset{0}{\overset{\parallel}{\tilde{H}_i(\mathbb{D}^n)}}} \to \tilde{H}_i(S^n) \to$$

$$\xrightarrow{\delta} \tilde{H}_{i-1}(S^{n-1}) \to \underset{\overset{\parallel}{0}}{\tilde{H}_{i-1}(\mathbb{D}^n)} \to \tilde{H}_{i-1}(S^n) \xrightarrow{\delta} \cdots$$

$$\tilde{H}_i(S^n) \cong \tilde{H}_{i-1}(S^{n-1})$$

$$\Rightarrow \tilde{H}_i(S^n) = \begin{cases} \mathbb{Z} & i=n \\ 0 & \text{otherwise} \end{cases}$$

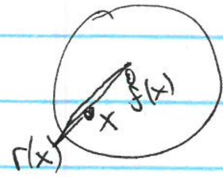
Cor: [Brower fixed pt] Every map $\mathbb{D}^n \xrightarrow{f} \mathbb{D}^n$ has a fixed point.

Pf: Note: $\partial\mathbb{D}^n$ is not a retract of $\mathbb{D}^n$; otherwise $r_* \underset{=0}{\tilde{H}_{n-1}(\mathbb{D}^n)} \twoheadrightarrow \underset{=\mathbb{Z}}{\tilde{H}_{n-1}(\partial\mathbb{D}^n)}$

Suppose not. Then $f(x) \neq x$ $\forall x \in \mathbb{D}^n$.

Thus

$r(x) =$ unique point from $f(x)$ to x on $\partial\mathbb{D}^n$ is a continuous retraction



$r(x)$ is a retraction $\lightning$

Nov 20

Relative Homology

$A \subseteq X$. Let $C_n(X,A) := C_n(X)/C_n(A)$

Let $C_n(X,A) \xrightarrow{d_n} C_{n-1}(X,A)$ be the usual boundary.

Let $H_n(X,A)$ be the homology of such a cx.

$$\rightsquigarrow \quad \begin{array}{ccccccccc} 0 & \to & C_n(A) & \xrightarrow{i_\#} & C_n(X) & \xrightarrow{p_\#} & C_n(X,A) & \to & 0 \\ & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \\ 0 & \to & C_{n-1}(A) & \to & C_{n-1}(X) & \to & C_{n-1}(X,A) & \to & 0 \end{array}$$

$\rightsquigarrow$ LES on Homology:

$$\to H_{n+1}(X,A) \xrightarrow{\delta} H_n(A) \to H_n(X) \to H_n(X,A) \xrightarrow{\delta} H_{n-1}(A) \to$$

Construction of δ. More generally, if we assume

$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$ is an exact seq of chain complexes:

$$\begin{array}{ccccccccc} & & K_A & \to & K_B & \to & K_C & & \\ 0 & \to & A_{n+1} & \to & B_{n+1} & \to & C_{n+1} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A_n & \to & B_n & \to & C_n & \to & 0 \\ & & \cancel{A_{n-1}}\ \text{Coker}_A & \to & C_B & & C_C & & \end{array}$$

$\cancel{C_n(X)} \to \cancel{C_n(X,A)}$

$\cancel{a'}$ $\quad$ $\cancel{a}$

Let $c \in Z_n(X,A)$. Then since $C_n(X) \xrightarrow{p_\#} C_n(X,A)$, $\exists\, c' \in C_n(X)$: $c' \mapsto c$. Then $\partial_X(c') \in C_{n-1}(X)$ has the property that $p_\# : \partial_X(c') \mapsto 0$, since $p_\# \partial_X(c') = \partial_{X,A}(c) = 0$. So $\partial_X(c') \in \ker(p_\#) = \mathrm{Im}(i_\#) \Rightarrow \exists\, a \in C_{n-1}(A)$ $a \mapsto \partial_X(c')$. Let

$$\delta(c) := a \in Z_{n-1}(A) \subseteq C_{n-1}(A)$$

Note $\partial_A(a) = 0$ since $\overset{\text{inj}}{i_\#}\partial_A(a) = \partial_X(\partial_X(c')) = 0$

Additionally, this is well defined:

- a is uniquely determined by ∂, since i is injective
- If c', c'' are 2 lifts of c, then
  $c' - c'' \in \ker(j_\#) = \mathrm{im}(i_\#)$. Thus $c' - c'' = i_\#(a)$
  So $\partial(c') - \partial(c') = \partial\, i_\#(a) = i_\# \partial(a)$ boundary
- Similarly, a boundary in $C_n(X,A) \mapsto$ boundary $C_{n-1}(A)$

$\rightsquigarrow$ LES on Homology

Ex/ If (X, x_0) is considered, then $\tilde{H}_n(x_0) = 0$ $\forall n > 0$. So for $n > 0$

$$\underbrace{\tilde{H}_n(x_0)}_{=0} = 0 \to \tilde{H}_n(X) \xrightarrow{\cong} H_n(X, x_0) \to \underbrace{\tilde{H}_{n-1}(x_0)}_{=0}$$

$$\Rightarrow H_n(X) \cong H_n(X, x_0)$$

Excision: Let $Z \subseteq A \subseteq X$ s.t. $\overline{Z} \subseteq \mathring{A}$. Then $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ yields an isomorphism

$$H_n(X \setminus Z, A \setminus Z) \xrightarrow{i_*} H_n(X, A)$$

We can form a chain complex for $\mathcal{U} = \{U_i\}$, $X = \bigcup U_i$

$$C_n^{\mathcal{U}}(X) \subseteq C_n(X)$$
$$\| $$
$$\{\sigma \in C_n(X) : \mathrm{Im}(\sigma) \subseteq U_i \text{ for some } i\}$$

Lemma: This induces an isomorphism on homology:

$$H_n^{\mathcal{U}}(X) \cong H_n(X) \quad \forall n \geq 0$$

Pf (Available on pg 119–124), using Barycentric subdivision).

Pf: Let $X = A \cup B$. Note that $C_n(\mathcal{U}) \ni \sigma$ if
$$\sigma = \sum m_i \sigma_i^A + \sum n_i \sigma_i^B$$

w/ $C_n^{\mathcal{U}}(X)/C_n(A) \hookrightarrow C_n(X)/C_n(A)$

w/ $H_n^{\mathcal{U}}(X, A) \cong H_n(X, A)$

Similarly

$C/H_n^{\mathcal{U}}(B, A \cap B) \cong H_n(X, A)$

~~Therefore~~ To convert to the version above,

~~Replace A B w/ A w/ B~~

set $B = X \setminus Z$