CLASS 17, FRIDAY APRIL 6TH: REGULAR LOCAL RINGS

Next up, we will study the most idealized version of a ring from the perspective of Algebraic Geometry and commutative algebra. It corresponds closely with the idea of a manifold being smooth, or without nodes, cusps, or similar phenomena.

Definition 0.1. The dimension of a ring R is the longest length n of a chain of prime ideals of R

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n$$

Therefore, the dimension of a field K is 0, the dimension of $\mathbb{Z}$ and $K[x]$ is 1 (because they are PIDs), and the dimension of $K[x_1, \ldots, x_n]$ can be shown to be n. This is one of the most important invariants of a ring. We will restrict our focus to finite dimensional rings for the remainder of today (and possibly the course).

Definition 0.2. A local ring $(R, \mathfrak{m})$ is **regular** (or a **regular local ring**, or **RLR**) if $\mathfrak{m}$ is generated minimally by exactly $\dim(R)$ many elements.

We note that in general, we can compute the number of generators in this setting quite naturally:

Proposition 0.3. *If $(R, \mathfrak{m})$ is a local ring,*

$$\beta(\mathfrak{m}) = min\{n \mid \mathfrak{m} = \langle f_1, \ldots, f_n\rangle\} = \dim_{R/\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2)$$

Proof. By Nakayama's Lemma++, we know that generators of $\mathfrak{m}$ correspond directly with those of $\mathfrak{m}/\mathfrak{m}\cdot\mathfrak{m} = \mathfrak{m}/\mathfrak{m}^2$. Therefore, the minimal number of generators of one corresponds precisely to those of the other. $\square$

We think of $\mathfrak{m}/\mathfrak{m}^2$ as the cotangent space of R at $\mathfrak{m}$. Another very important thing to note is that $dim(R) \leq \beta(\mathfrak{m})$, so being regular is stating a minimality of the generating set of $\mathfrak{m}$. This follows from the following theorem:

Theorem 0.4 (Principal Ideal Theorem)**.** *If $x_1, \ldots, x_c$ are elements of a ring R, then if $I = \langle x_1, \ldots, x_c\rangle$, the* **codimension** *or* **height** *of I is*

$$\operatorname{codim}(I) = ht(I) := \min\{\dim(R_\mathfrak{p}) \mid I \subseteq \mathfrak{p} \textit{ is prime}\} \leq c$$

I will black box this proof for now, as it takes some work and machinery. However, we note that for a local ring $(R, \mathfrak{m})$, $\operatorname{codim}(\mathfrak{m}) = \dim(R)$. So this equality gives us directly that $dim(R) \leq \beta(\mathfrak{m})$. Even moreso, it gives that $R/\langle x_1, \ldots, x_i\rangle$ has dimension $\dim(R) - i$. We will now explore some nice properties of regular local rings, by use of Krull's Intersection Theorem.

Theorem 0.5 (Krull's Intersection Theorem)**.** *Let R be a Noetherian ring and let $I \subsetneq R$ be an ideal. If M is a finitely generated R-module, then $\cap_{n\geq 0} I^n M = 0$*

I will prove this momentarily, but first some nice semi-corollaries.

Proposition 0.6. *If $(R, \mathfrak{m})$ is a regular local ring of dimension d, then considering the* **graded Algebra** *of $\mathfrak{m}$, we have*

$$gr_\mathfrak{m}(R) := R/\mathfrak{m} \oplus \mathfrak{m}/\mathfrak{m}^2 \oplus \mathfrak{m}^2/\mathfrak{m}^3 \oplus \ldots \cong (R/\mathfrak{m})[X_1, \ldots, X_d]$$

Proof. Given $\mathfrak{m} = \langle x_1, \ldots, x_d \rangle$, we know that $\mathfrak{m}^n$ is generated by elements of the form $x_1^{n_1} \cdots x_n^{n_d}$ where $n = n_1 + \ldots + n_d$. Therefore, there is a natural surjection

$$(R/\mathfrak{m})[X_1, \ldots, X_d] \to gr_{\mathfrak{m}}(R)$$

Moreover, this map is injective by Krull's Intersection Theorem. This completes the proof. □

Proposition 0.7. *Every regular local ring R is an integral domain.*

Proof. We will use crucially that $\cap_{n \geq 0} \mathfrak{m}^n = 0$, which holds as a result of the Krull Intersection Theorem, to be seen momentarily. Suppose that $f \cdot g = 0$. Then there exists $a, b \geq 0$, we have $f \in \mathfrak{m}^a, g \in \mathfrak{m}^b$, but not in any larger power. Proposition 0.6 then implies that since $0 = f \cdot g \in \mathfrak{m}^{a+b+1}$, that $f \in \mathfrak{a}^{a+1}$ or $g \in \mathfrak{m}^{b+1}$. This contradicts our assumptions. □

I will now prove Krull's Intersection Theorem.

Lemma 0.8 (Artin-Rees Lemma)**.** *Suppose R is a Noetherian ring, and $I \subseteq R$ is an ideal. Let $N \subseteq M$ be two finitely generated R-modules. Then there exists $c > 0$ such that*

$$I^n M \cap N = I^{n-c}(I^c M \cap N)$$

for every $n \geq c$.

Proof. Consider the **blowup algebra** of I:

$$bl_I(R) = R \oplus I \oplus I^2 \oplus I^3 \oplus \ldots = \oplus_{n \geq 0} I^n$$

where we multiply elements of I^n and I^m into I^{n+m}. If $I = \langle f_1, \ldots, f_t \rangle$, then $bl_I(R)$ is Noetherian, since it is a quotient of the Noetherian ring $R[X_1, \ldots, X_t]$ by $X_i \mapsto f_i$. Similarly, we may consider

$$bl_I(M) = M \oplus IM \oplus I^2 M \oplus I^3 M \oplus \ldots = \oplus_{n \geq 0} I^n M$$

This is certainly a finitely generated $bl_I(R)$-module. This yields the following submodule:

$$\oplus_{n \geq 0} I^n M \cap N$$

This is a finitely generated S-module as well. Take $\alpha_i = \alpha_i^0 + \ldots + \alpha_i^{n_i}$ generators (decomposed into their direct sums). Set $c = \max_i\{n_i\}$. Then for $n \geq c$. Then for every $n \geq c$, every element has the form

$$\sum_{i,j} h_j \alpha_i^j \text{ where } h_j \in I^{n-j} \subseteq I^{n-c}$$

This implies that $I^n M \cap N \subseteq I^{n-c}(I^c M \cap N)$. The other direction is left as an exercise. □

Finally, we conclude Krull:

Proof. Let $N = \cap_{n \geq 0} I^n M$. Then $N = I^n M \cap N$ for all $n \geq 0$. By Artin-Rees, we get

$$N = I^n M \cap N \subseteq IN$$

for some suitably large power of n. But Nakayama's Lemma then implies $N = 0$. □

Next time we will talk briefly about global dimension and it's relation to regularity.