CLASS 0, DATE SEPTEMBER 6: INTRODUCTION

There are many reasons to study complex analysis.

1) **It is beautiful in its own right:** It is a common misconception that complex analysis would be more complicated than real analysis, due to the fact that $\mathbb{C}$ is more complicated and even contains $\mathbb{R}$. However there are many things that make it far simpler to study.

   One result we will prove in the first half of the course is the following:

   Theorem 0.1 (The Fundamental Theorem of Algebra)**.** *If $p(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ is a polynomial with complex coefficients, then $p(x)$ factors into linear terms.*

   Another question we will formalize within the first half of the class is the question of differentiability of complex functions. It turns out that something quite magical happens which greatly differs from the case of real analysis:

   Theorem 0.2 (Informal Version)**.** *If $f : \mathbb{C} \to \mathbb{C}$ is a complex valued function, then the first derivative $f'(z)$ exists if and only if the* **all** *derivatives $f^{(n)}(z)$ exist.*

   It is easy to conceive of real valued functions for which derivatives exist up to an arbitrary order, and then fail to exist for higher orders (via, for example, integrating $|x|$).

2) **Signal Processing:** Historically (since roughly the 17^{th} century), complex analysis was developed to study waves in physics. There is a particularly easy way to study a wave as a complex valued function; using the exponential! It simplifies a great deal of the difficult trigonometric identities and unifies other forces of nature.

   One of the central things we aim to study in this course is the Fourier Transform. Its definition is given as follows: if $f : \mathbb{R} \to \mathbb{C}$ is a complex valued function, its **Fourier Transform** is given by

   $$\hat{f}(y) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x y}dx$$

   For now (until next week) you can think of the exponential factor as making f into a wave.

   The more precise advantage of doing this is that it has the ability to transfer complex differential equations to easier to deal with algebraic (polynomial) equations. Moreover, there is an inverse Fourier Transform which can transfer all of the data back, making for a very simple transference of a solution back to the original space.

3) **Number Theory:** It may be surprising (it certainly was to me the first time I saw these types of results!) that complex analysis can relate to something arithmetic. It does so however via some of the deepest mathematical results and conjectures.

One of the most baffling problems in mathematics is measuring the growth rate of the prime numbers. Several estimates are easily generated: Let $\pi(x)$ denote the number of primes less than a given real number x. Then

$$\pi(x) \sim \frac{x}{\ln(x)} \sim \int_2^x \frac{dt}{\ln(t)}$$

This is the renowned **prime number theorem**, one of the most foundation results about the primes. Now the question becomes 'they are asymptotically the same, i.e. the limit of their quotients are 1 as $x \to \infty$, but how far off are we along the way?'

Denote by $Li(x)$ the integral on the far right. It is known that

$$\pi(x) = Li(x) + O\left(x \cdot e^{-a\sqrt{\ln(x)}}\right)$$

for some $a > 0$, where the O notation implies that they differ by at most a constant times the quantity within.

One of the most famous open problems in mathematics is the **Riemann Hypothesis**, which states that the non-trivial complex roots of (an analytic continuation of) $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ all lie on the line $s = \frac{1}{2} + iy$.

If (and only if) RH is true, this would improve our approximation of the above error to

$$\pi(x) = Li(x) + O\left(\sqrt{x}\log(x)\right)$$

The constant is even estimated to be $\frac{1}{8\pi}$!

We will study the ζ function later on and time permitting touch on number theory near the end of the course.

4) **Algebra, Topology, etc:** The list goes on :)

CLASS 1, SEPTEMBER 9TH: COMPLEX NUMBERS

We will begin our course by describing complex numbers and various representations of them based on our existent knowledge of the reals.

Definition 1.1. A **complex number** is a symbol of the form $a+ib$, where $a, b \in \mathbb{R}$, and $i = \sqrt{-1}$ is the imaginary unit. The collection of all complex numbers is denoted $\mathbb{C}$.

For referencing a complex number, $z = a + ib$, we write $Re(z) = a$ and $Im(z) = b$, the **real** and **imaginary** parts of z.

We also have two natural operations on $\mathbb{C}$:

$$(a+ib) + (c+id) := (a+c) + i(b+d)$$
$$(a+ib) \cdot (c+id) := (ac-bd) + i(ad+bc)$$

This yields a few clear identification of $\mathbb{C}$ with other familiar objects:

1) As real vector spaces, $\mathbb{C}$ is exactly $\mathbb{R}^2$.
2) $\mathbb{C}$ is a commutative ring; it is an Abelian group under addition, closed under multiplication (also commutative), and a quick check shows that multiplication distributes over addition.
3) As a more complicated (though often fruitful) representation, we can view complex numbers $a+ib$ as matrices of the form
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$
This is compatible with matrix multiplication!
$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -(ad+bc) \\ ad+bc & ac-bd \end{pmatrix}$$
We won't focus on this representation so much, but in complex analysis it is often beneficial to see how it fits into the broad framework of mathematics.

Focusing on the identification of $\mathbb{C}$ with $\mathbb{R}^2$, there is another useful method for viewing points: by their distance from the origin r and the angle they form with the x-axis θ. This is often referred to as **polar coordinates**, and written $z = re^{i\theta}$.

Note: θ is only determined up to 2π, so it is often convenient to restrict θ to $(-\pi, \pi]$ and choose only non-negative values for r.

As in Calculus, there is a natural way to transfer between the 2 coordinate systems: $z = a + ib = re^{i\theta}$, then

$$a = r\cos(\theta) \qquad b = r\sin(\theta) \qquad r = \sqrt{a^2+b^2}$$

Considerations for θ, as in trigonometry, are a little more bizarre. If for example $a > 0$, then it is concise:

$$\theta = \tan^{-1}\left(\frac{b}{a}\right).$$

Otherwise, some care should be applied (c.f. homework).

Definition 1.2. We call θ the **argument** of z and write $\theta = Arg(z)$. r is called the **absolute value** of z, written $r = |z|$.

The question is then why would we care about polar coordinates? Well, for one, they make multiplication quite a bit easier!

$$z \cdot w = re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)}.$$

This will be checked as one of your homework assignments. Note in particular that multiplication behaves exactly as you might expect based on your knowledge of the exponential function. This is no coincidence.

The absolute value is exactly the Euclidean norm, the standard measure of distance, on $\mathbb{R}^2$. A simple way to determine it is through the use of the complex conjugate.

Definition 1.3. If $z = a + ib$ is a complex number, then the **complex conjugate** of z, denoted $\bar{z}$, is $\bar{z} = a - ib$.

The complex conjugate can be viewed as the reflection of z across the real axis, making it so that if $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$. An immediate conclusion regarding it is

$$|z|^2 = r^2 = z \cdot \bar{z}.$$

Additionally, the following equalities are easily verified:

$$Re(z) = \frac{z + \bar{z}}{2} \qquad\qquad Im(z) = \frac{z - \bar{z}}{2}$$

The complex conjugate also immediately realizes the following:

Theorem 1.4. *Given, $z \neq 0$ a complex number, there exists $z^{-1} \in \mathbb{C}$ such that $z \cdot z^{-1} = 1$. This is to say that $\mathbb{C}$ is a field.*

Proof. Let $z^{-1} = \frac{1}{|z|^2}\bar{z}$. □

Example 1.5. Considering the complex number $z = 1 + 2i$ (it is a strange but convenient notation to list real numbers in front of i, and variables after i), it is clear that $Re(z) = 1$ and $Im(z) = 2$. Using our rules from above, we can further compute $|z| = \sqrt{5}$ and $Arg(z) = \tan^{-1}(2) \approx 1.10714872$ radians, or roughly 63.4 degrees. This should be confirmed by our intuition. So we could write

$$1 + 2i \approx \sqrt{5}e^{1.10714872i}$$

Finally, since $\mathbb{C}$ with the absolute value is nothing but $\mathbb{R}^2$ with the Euclidean norm, we have:

Theorem 1.6. $\mathbb{C}$ *is a complete metric space.*

To be a **complete metric space** means that every **Cauchy sequence** will **converge**. These words should be familiar to you from real analysis. Next time we will talk about sets in $\mathbb{C}$ and begin talking about complex functions.

CLASS 2, SEPTEMBER 11TH: COMPLEX FUNCTIONS

Last time we talk about the complex number system in various frames of reference. Today, we will study complex functions and how we can verify differentiability. First, we need to review a few notions from real analysis to produce a domain for our functions.

Definition 2.1. We define the **open ball** of radius r and centered at $z \in \mathbb{C}$ to be

$$B(z,r) = \{w \in \mathbb{C} \mid |w - z| < r\}$$

Similarly, we can define the **closed ball** with the same parameters:

$$\bar{B}(z,r) = \{w \in \mathbb{C} \mid |w - z| \leq r\}$$

Note that the book uses the notation $D_r(z)$ instead of $B(z,r)$ for a 'disc'. You may use either but my notation is used more commonly in analysis. Worthy of its own name and notation is the **unit disc** $\mathbb{D} = B(0,1)$.

This allows us to define the notion of an open and closed set in the complex plane:

Definition 2.2. If $\Omega \subseteq \mathbb{C}$ is a subset, $z \in \Omega$ is called an **interior point** if there exists $r > 0$ such that $B(z,r) \subseteq \Omega$.

Ω is **open** if every point is an interior point.

Ω is called **closed** if its complement $\Omega^c = \mathbb{C} \setminus \Omega$ is open.

An alternative characterization (agreeing with the situation of real analysis) of Ω being closed is as follows:

Proposition 2.3. *Ω is closed if and only if every every convergent sequence $z_n \to z$ in $\mathbb{C}$ with $z_n \in \Omega$ implies that $z \in \Omega$.*

Proof. $\Rightarrow$: If Ω is closed, Ω^c is open. If $z \in \Omega^c$, then $B(z,r) \subseteq \Omega^c$ for some $r > 0$. But then z_n cannot converge to z, since $z_n \notin B(z,r)$ (which implies $|z - z_n| \geq r > 0$).

$\Leftarrow$: Suppose Ω is not closed. Then there exists $z \in \Omega^c$ such that $B(z, \frac{1}{n}) \bigcap \Omega \neq \emptyset$ for each $n \in \mathbb{N}$. Choose $z_n \in B(z, \frac{1}{n}) \bigcap \Omega$. Then $z_n \to z$ but $z \notin \Omega$. $\square$

Finally, recall the definition of **bounded**:

Definition 2.4. $\Omega \subseteq \mathbb{C}$ is called **bounded** if there exists $z \in \mathbb{C}$ and $r > 0$ such that

$$\Omega \subseteq B(z,r).$$

As a result, we have the following equivalence from real analysis:

Theorem 2.5. *Let $\Omega \subseteq \mathbb{C}$. Then TFAE (The Following Are Equivalent):*

1) *If U_α is any collection of open sets* **covering** *Ω, i.e. $\Omega \subseteq \bigcup_\alpha U_\alpha$, then there exists $\alpha_1, \dots, \alpha_n$ such that $\Omega \subseteq U_{\alpha_1} \cup \dots \cup U_{\alpha_n}$ is a finite subcover.*
2) *If $z_n \subseteq \Omega$, then there exists a subsequence z_{n_i} which converges.*
3) *Ω is complete and* **totally bounded***.*
4) *Ω is closed and bounded.*

Definition 2.6. Ω satisfying any of the equivalent conditions of Theorem 2.5 is **compact**.

Note condition 2) is often called **sequential compactness**. The first 3 conditions are equivalent for any metric space. 4) is strictly weaker in the infinite dimensional cases.

We can now pivot to the notion of a continuous function. This is exactly as it was in real analysis:

Definition 2.7. If $\Omega \subseteq \mathbb{C}$, and $f : \Omega \to \mathbb{C}$ is a function, then f is said to be **continuous at** $z \in \Omega$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $w \in \Omega$ with $|w - z| < \delta$, necessarily $|f(w) - f(z)| < \epsilon$.

f is **continuous** if f is continuous at z for every $z \in \Omega$.

There are some immediate and convenient rephrasings of this statement in terms of preimages. Recall that if $\Gamma \subseteq \mathbb{C}$, then

$$f^{-1}(\Gamma) = \{z \in \Omega \mid f(z) \in \Gamma\}$$

This should NOT be confused as an inverse function!

Theorem 2.8. *Let $f : \Omega \to \mathbb{C}$ be a function. TFAE:*

0) If $x_n \to x$ in Ω, then $f(x_n) \to f(x)$.
1) f is continuous.
2) For every $z \in \Omega$ and $\epsilon > 0$, there exists $\delta > 0$ such that $f(B(z, \delta)) \subseteq B(f(z), \epsilon)$.
3) For every $z \in \Omega$ and $\epsilon > 0$, there exists $\delta > 0$ such that $B(z, \delta) \subseteq f^{-1}(B(f(z), \epsilon))$.

If Ω is itself open, these conditions are equivalent to the following:

5) If $U \subseteq \mathbb{C}$ is an open set, then $f^{-1}(U)$ is an open set.

Proof. Sketch. For 0) implies 1), it is convenient to prove the contrapositive. If f is not continuous at x, there exists ϵ such that no $\delta > 0$ has the property that $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Choose x_n to be y violating this property with $\delta = \frac{1}{n}$. 2) is exactly the statement of 1) but with set notation. Applying f^{-1} to the inclusion in 2) yields 3) once you notice that

$$\Gamma \subseteq f^{-1}(f(\Gamma))$$

is true for any set Γ. 3) implies 4) is a result of the fact that preimages and unions can be interchanged:

$$\bigcup_\alpha f^{-1}(\Gamma_\alpha) = f^{-1}(\bigcup_\alpha \Gamma_\alpha)$$

(here let $U = \bigcup_\alpha \Gamma_\alpha$ where each Γ_α is an open ball). 4) implies 0) is simply established by considering $f^{-1}(B(f(x), \epsilon)$ for any given ϵ! $\square$

For context, 4) is the definition of continuity given outside of the context of metric spaces. A useful thing to recover from real analysis is the **extreme value theorem**:

Theorem 2.9. *If Ω is a compact set and $f : \Omega \to \mathbb{C}$ is a continuous function, then Ω obtains its maximum and minimum with respect to absolute value.*

Proof. We can note that the image of a compact set is compact (this is easiest to see with the open cover definition of compactness). But this implies that $f(\Omega)$ is a closed and bounded set. Boundedness then yields an $R > 0$ such that $f(\Omega) \subseteq \bar{B}(0, R)$. Choose R minimally satisfying this condition. If $z_n \in \Omega$ is a convergent sequence such that $|z_n| \to R$, then closedness of Ω implies its limit z is in Ω. Thus maxima are achieved.

If $0 \in \Omega$, we are done. Otherwise, $\frac{1}{f}$ is a continuous function on Ω. Applying the previous result shows $\frac{1}{f}$ has a maximum, which in turn implies f has a minimum. $\square$

Next time we will get into the idea of when a complex function is differentiable.

CLASS 3, SEPTEMBER 13: HOLOMORPHIC FUNCTIONS

Today we will focus what it means for a complex function to be differentiable. Though the definition is exactly the one from real analysis, the repercussions are far deeper.

Definition 3.1. Let $\Omega \subseteq \mathbb{C}$ be an open subset of the complex plane, and $f : \Omega \to \mathbb{C}$ a function. If $z \in \Omega$, then f is said to be **holomorphic at** z if

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

exists. The value of the above limit is denoted $f'(z)$, the **derivative of** f **at** z.

f is called **holomorphic** if $f'(z)$ exists at each point $z \in \Omega$.

If Ω is not open, f is **holomorphic on** Ω if there exists an open set $\Gamma \supseteq \Omega$ for which f is defined and holomorphic on.

If $\Omega = \mathbb{C}$, we call f an **entire** function.

A key difference to note here is that $h \in \mathbb{C}$, which differs from the case of real analysis in the sense that the limit is not a 2-sided approach, but instead a circular/∞-sided one.

Just to show you that much of you intuition is correct about differentiation, we have the following examples:

Example 3.2. If $f(z) = a_0 + a_1 z + \ldots + a_n z^n$ is a polynomial, then f is holomorphic on any domain Ω, thus entire. As you may expect,

$$f'(z) = a_1 + \ldots + n a_n z^{n-1}.$$

This follows by the binomial theorem.

Example 3.3. $f(z) = \frac{1}{z^n}$ is holomorphic for any Ω not containing 0 (thus f is not entire). One can compute

$$f'(z) = \lim_{h\to 0} \frac{\frac{1}{(z+h)^n} - \frac{1}{z}}{h} = \lim_{h\to 0} \frac{z^n - (z+h)^n}{hz^n(z+h)^n} = -\frac{1}{z^n} \lim_{h\to 0} \frac{(z+h)^n - z^n}{h(z+h)^n}$$

$$= -\frac{1}{z^n} \lim_{h\to 0} \frac{\binom{n}{1} h z^{n-1} + \binom{n}{2} h^2 z^{n-2} + \ldots + \binom{n}{n} h^n}{h(z+h)^n} = -\frac{1}{z^n} \lim_{h\to 0} \frac{\binom{n}{1} z^{n-1} + \binom{n}{2} h z^{n-2} + \ldots + \binom{n}{n} h^{n-1}}{h(z+h)^n}$$

$$= \frac{-nz^{n-1}}{z^n \cdot z^n} = \frac{-n}{z^{n+1}}$$

Note here that my limits only allow values of h in $B(0, |z|)$, to avoid the possibility of $z+h = 0$. This is fine because the definition of a limit only requires some open neighborhood of the desired point $h = 0$.

Example 3.4. The assignment $f(z) = \bar{z}$ is a perfectly well defined continuous function. It is however nowhere holomorphic. Indeed, noting that conjugation commutes with sums ($\overline{z+h} = \bar{z} + \bar{h}$), we have

$$\frac{f(z+h) - f(z)}{h} = \frac{\bar{h}}{h}$$

This has no limit as $h \to 0$. For example, if we approach 0 along the real axis, we would get 1. On the other hand, approaching 0 along the imaginary axis yields -1. Distinct limits imply that the function cannot be differentiated.

The last example shows that real and complex derivatives are distinct. Indeed, viewed as a function $F : \mathbb{R}^2 \to \mathbb{R}^2$, complex conjugation is differentiable: There is a linear transformation (**the Jacobian**) $J : \mathbb{R}^2 \to \mathbb{R}^2$ such that

$$\frac{|F(P+H) - F(P) - J(H)|}{H} \to 0$$

as $H \to 0$. Equivalently, we can write

$$F(P+H) - F(P) = J(H) + |H|\Psi(H)$$

with $\Psi(H) \to 0$ with H (i.e. using little o notation, they are equal $o(|H|)$). Given coordinatewise, $F(x, y) = (x, -y)$. Thus the corresponding matrix J is

$$J(x,y) = \begin{pmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In the case of the complex derivative, the result is a complex number and not a matrix. This yields a reduction of 2 degrees of freedom from real differentiation!

This can be realized explicitly. Assume f is holomorphic at $z = x + iy$. Then we know $f'(z)$ exists. Let's let h approach along the real and imaginary axes:

$$f'(z) = \lim_{h \to 0} \frac{f(x+h, y) - f(x,y)}{h} = \frac{\partial f}{\partial x}(z)$$

$$f'(z) = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{ih} = \frac{1}{i}\frac{\partial f}{\partial y}(z)$$

Therefore, if f is holomorphic, these 2 limits agree:

$$\frac{\partial f}{\partial x}(z) = \frac{1}{i}\frac{\partial f}{\partial y}(z)$$

Viewing f in terms of 2 real valued functions $f = u(x, y) + iv(x, y)$, we produce

$$f'(z) = \frac{\partial u}{\partial x}(x,y) + i\frac{\partial v}{\partial x}(x,y) = -i\frac{\partial u}{\partial y}(x,y) + \frac{\partial v}{\partial y}(x,y)$$

Relating the real and imaginary parts of this equation gives us the **Cauchy-Riemann equations**:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad\qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

These equations fundamentally link real and complex analysis. We can clarify the situation further by defining two differential operators:

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \qquad\qquad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

These results can be summed up as follows:

Theorem 3.5. *If f is holomorphic at $z_0 \in \mathbb{C}$, then f satisfies the Cauchy-Riemann equations. Additionally,*

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \qquad\qquad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$$

Finally, f is differentiable in the sense of real analysis, and

$$|f'(z)|^2 = \det(J(x_0, y_0))$$

where J is the Jacobian as above.

CLASS 4, SEPTEMBER 16: CAUCHY-RIEMANN EQUATIONS

Last time we finished up with the following result:

Theorem. *If f is holomorphic at $z_0 \in \mathbb{C}$, then f satisfies the Cauchy-Riemann equations. Additionally,*

$$\frac{\partial f}{\partial \bar{z}}(z_0) = 0 \qquad\qquad f'(z_0) = \frac{\partial f}{\partial z}(z_0) = 2\frac{\partial u}{\partial z}(z_0)$$

Finally, f is differentiable in the sense of real analysis, and

$$|f'(z)|^2 = \det(J(x_0, y_0))$$

where J is the Jacobian as above.

I'll begin with a proof sketch:

Proof. f satisfies the CR equations was demonstrated last time. For the second equality;

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right]$$

The CR equations allow us to write

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left[\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial x}\right) - i\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial y}\right)\right] = 2\frac{\partial u}{\partial z}(z_0)$$

The case of $\frac{\partial f}{\partial \bar{z}}(z_0)$ is similar but easier. A final computation shows the following:

$$\det(J(x_0, y_0)) = \det\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2 = \left(2\frac{\partial u}{\partial z}\right)^2 = |f'(z_0)|^2$$

□

We have put a lot of effort into the Cauchy-Riemann equations and it has been stated that they are essential in complex analysis. The following partial converse is one of the main motivations to study these equations:

Theorem 4.1. *Let $\Omega \subseteq \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$ a function. Write $f = u + iv$ to denote its real and imaginary parts. If u and v are* **continuously** *differentiable and satisfy the Cauchy-Riemann equations in Ω, then f is holomorphic in Ω and $f'(z) = \frac{\partial f}{\partial z}$.*

This is a beautiful result yielding not only a path from real to complex analysis, but a simple way to check holomorphicity of functions!

Proof. Given u and v are differentiable, write

$$u(x + h_1, y + h_2) - u(x, y) = \frac{\partial u}{\partial x}h_1 + \frac{\partial u}{\partial y}h_2 + |h|\psi_1(h)$$

$$v(x + h_1, y + h_2) - v(x, y) = \frac{\partial v}{\partial x}h_1 + \frac{\partial v}{\partial y}h_2 + |h|\psi_2(h)$$

where $h = (h_1, h_2)$ and $\lim_{h \to 0} \psi_i(h) = 0$. Using CR, we have that

$$f(z + h) - f(z) = \left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)(h_1 + ih_2) + |h|\psi(h)$$

where $\psi = \psi_1 + \psi_2$. This implies f is differentiable by our equivalent characterizations. $\square$

Next up we will talk about power series. Many of the functions with power series from calculus are equivalently defined for complex analysis! They also provide canonical examples of holomorphic functions. Recall

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

For real values of z, this series converges absolutely! It turns out that this condition is enough to ensure convergence for any choice of complex number[1] and thus yields a definition for the exponential as a complex function.

Theorem 4.2. *Given a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists $0 \leq R \leq \infty$ such that the series converges absolutely for $|z| < R$ and diverges for $|z| > R$.*

Additionally, we can calculate R explicitly using **Hadamard's formula***:*

$$1/R = \limsup |a_n|^{\frac{1}{n}}$$

Definition 4.3. R as in Theorem 4.2 is called the **radius of convergence**, whereas $B(0, R)$ is called the **disc of convergence**.

Proof. Let $L = \frac{1}{R}$ be as in Hadamard's formula. Suppose $L \neq 0, \infty$ (which are easy special cases). If $|z| < R$, choose $\epsilon > 0$ (by openness) such that

$$(L + \epsilon)|z| < 1$$

Since $|a_n|^{\frac{1}{n}} \leq L + \epsilon$ for $n \gg 0$,

$$|a_n||z|^n \leq ((L + \epsilon)|z|)^n = r^n$$

By what we know about geometric series, the series converges absolutely. If $|z| > R$, the same argument shows the existence of a sequence of terms going to infinity. Since a series can only converge in any sense if the terms go to 0, we conclude that the sequence diverges. $\square$

The same ideas allow us to define our trig functions:

Definition 4.4. Define the complex functions

$$\sin(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$$

It is now easy to check $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ for real numbers θ by comparing power series. Additionally, we get some new **Euler relations**:

$$\sin(z) = \frac{e^{iz} - e^{-iz}}{2} \qquad \cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

Next time, we'll begin by proving the following result:

Theorem 4.5. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then f is holomorphic in its disc of convergence. Furthermore,*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

has the same radius of convergence R.

[1]This can be seen, for example, by considering real and imaginary parts.

CLASS 5, SEPTEMBER 18: POWER SERIES

Last time, we left off with the following result:

Theorem. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $R > 0$, then f is holomorphic in its disc of convergence. Furthermore,*

$$f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$$

has the same radius of convergence R.

Proof. First note that $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$. As a result, if we can prove the other assertions, Hadamard's formula will ensure that they share the same radius of convergence.

Let $|z_0| < r < R$ for some fixed r. Let $S_N(z)$ denote the N^{th} partial sum of the series $\sum_{n=0}^{N} a_n z^n$, and $E_n(z)$ denote their difference (i.e. the error in this estimation). Let $g(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$. If h is chosen such that $|z_0 + h| < r$, then

$$\frac{f(z_0+h)-f(z_0)}{h} - g(z_0) = \left(\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right) + (S_N'(z_0) - g(z_0)) + \left(\frac{E_N(z_0+h)-E_N(z_0)}{h}\right)$$

Now, using the equality $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + b^{n-1})$, letting $a = z_0 + h$ and $b = z_0$ yields

$$\left|\frac{E_N(z_0+h)-E_N(z_0)}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| \left|\frac{(z_0+h)^n - z_0^n}{h}\right| \leq \sum_{n=N+1}^{\infty} |a_n| n r^{n-1}$$

the last equality is due to our choices of $|z_0| < r$ and $|z_0 + h| < r$. On the right we have the end of a convergent series by the first part. Thus it can be made arbitrarily small: Given $\epsilon > 0$, we can choose N_1 such that $\frac{E_N(z_0+h)-E_N(z_0)}{h} < \epsilon$ for all $N > N_1$.

Additionally, since $\lim_N S_N'(z_0) = g(z_0)$, we can choose N_2 such that $|S_N'(z_0) - g(z_0)| < \epsilon$ for $N > N_2$.

If we choose N larger than both of them, and choose $|h| < \delta$ so that $\left|\frac{S_N(z_0+h)-S_N(z_0)}{h} - S_N'(z_0)\right| < \epsilon$ we get that the left hand side is bounded above by 3ϵ, which can be made arbitrarily small. Therefore $g(z_0) = f'(z_0)$. □

By induction, we achieve the following corollary:

Corollary 5.1. *Power series are infinitely complex differentiable in their disc of convergence, with derivatives obtained via term-wise differentiation.*

One should also note that we have been focusing on power series centered at the origin. In general, a **power series centered at** z_0 has the form

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

It's disc of convergence is centered at z_0. We can even transfer between the two notions with the transform $g(z) = f(z - z_0)$ assuming z_0 is in the disc of convergence of f.

Definition 5.2. A complex function $f : \Omega \to \mathbb{C}$ is **analytic at** $z_0 \in \Omega$ if there exists a power series

$$\sum_{n=0}^{\infty} a_n(z - z_0)^n$$

with positive radius of convergence R agreeing with f in $B(z_0, R) \cap \Omega$. f is **analytic** if it is analytic at all its points.

As a result of the theorem above, we note immediately that all analytic functions are holomorphic. Using Cauchy's Theorem from chapter 2, we will prove that the converse holds as well. This demonstrates something much stronger than the result stated in Class 0.

We now switch gears to the question of integration. Unlike with $\mathbb{R}$, where we can usually simply specify the bounds of integration, in $\mathbb{C}$ we need to specify a curve with which to integrate along. Note we are implicitly doing this in Calculus I & II.

Definition 5.3. A **path** is a function $\gamma : [a, b] \to \mathbb{C}$ for $a < b$ real numbers. A path is **smooth** if $\gamma'(t)$ exists and is a non-zero continuous function. At the boundaries, we make special notations

$$\gamma'(a) = \lim_{t \to a^+} \frac{\gamma(t) - \gamma(a)}{t - a} \qquad \gamma'(b) = \lim_{t \to b^-} \frac{\gamma(t) - \gamma(b)}{t - a}$$

We also call a path **piecewise-smooth** if there exist $a = a_0 < a_1 < \ldots < a_n = b$ such that $\gamma|_{[a_i, a_{i+1}]}$ is a smooth curve for each $i = 0, \ldots, n - 1$.

Intuitively, we can break up a piecewise smooth curve into several smooth curves. We call $\gamma_1 \simeq \gamma_2$ **equivalent**, where $\gamma_1 : [a, b] \to \mathbb{C}$ and $\gamma_2 : [c, d] \to \mathbb{C}$ if there exists a continuously differentiable bijection $\sigma : [a, b] \to [c, d]$ with $\sigma'(t) > 0$[1] such that $\gamma_1(t) = \gamma_2(\sigma(t))$.

Definition 5.4. We will call a piecewise smooth path simply a **curve**.

All equivalent paths yield $C \subseteq \mathbb{C}$ given by the image of γ with an orientation. We also have C traversed in the opposite direction, denote $\bar{C}$, which is determined by the path $\bar{\gamma}(t) = (b + a - t)$.

Definition 5.5. A path is **closed** (sometimes called a **loop**) if $\gamma(a) = \gamma(b)$. It is **simple** if γ is an injective map, i.e. it doesn't self intersect except perhaps at its endpoints.

Example 5.6. We can easily produce a curve using a circle:

$$C_r(z_0) = \{z \in \mathbb{C} \mid |z - z_0| = r\}$$

To give this a parameterization, we can use the exponential:

$$\gamma : [0, 2\pi] \to \mathbb{C} : t \mapsto z_0 + re^{it}$$

γ is said to have **positive orientation** (counterclockwise). To produce its negative counterpart (clockwise), we have naturally

$$\bar{\gamma} : [0, 2\pi] \to \mathbb{C} : t \mapsto z_0 + re^{-it}$$

You can also easily produce piecewise smooth paths by considering polygons (i.e. triangles, squares, etc). Next time, we will setup the familiar idea of a path-integral using these notions.

[1]This condition preserves orientation of γ_1, so that you don't traverse it backwards and yield negative signs.

CLASS 6, SEPTEMBER 20: PATH INTEGRALS

Last time, we established some conditions for what a good definition of a path should be. These are important for a definition of an integral in the complex plane. It turns out that the story is quite similar to that of calculus.

Definition 6.1. Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth curve, and f be a continuous function on C (the image curve of γ). Then we define

$$\int_C f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt$$

Something funny has happened here; we used a generic parameterization of C. Therefore, we must establish that it is independent of the choice of parameterization to ensure that our asserted definition is well-defined.

Suppose $\gamma_1 \simeq \gamma_2$ are two equivalent curves. Recall that this means that there exists a continuously differentiable bijection $\sigma : [a, b] \to [c, d]$ with positive derivative such that $\gamma_1(t) = \gamma_2(\sigma(t))$. The chain rule then implies

$$\int_a^b f(\gamma_1(t))\gamma_1'(t)dt = \int_a^b f(\gamma_1(t))(\gamma_2(\sigma(t))'dt = \int_a^b f(\gamma_2(\sigma(t)))\gamma_2'(\sigma(t))\sigma'(t)dt$$

The last term is simply $\int_c^d f(\gamma_2(t))\gamma_2'(t)dt$ by the change of base formula (from calculus). This is precisely the reason we call 2 such curves equivalent.

We can naturally generalize this to piecewise smooth curves by dividing into smooth components:

$$\int_\gamma f(z)dz = \sum_{i=0}^{n-1} \int_{a_i}^{a_{i+1}} f(\gamma(t))\gamma'(t)$$

One additional extremely important quantity is the length of a curve. This is given by taking the function f to be 1 and removing the notion of positive/negative orientation:

$$\int_a^b |\gamma'(t)|dt$$

Example 6.2. Consider the curve parameterized by $\gamma(t) = e^{it}$. As we know, this traces a circle counterclockwise. This gives us some interesting things to consider: First, if $f(z) = z^n$ where $n \neq -1$, then we get

$$\int_C z^n dz = \int_a^b ie^{i(n+1)t}dt = \left[\frac{e^{i(n+1)t}}{n+1}\right]_{t=a}^b$$

Now we can choose our bounds. If we aim for a half circle, i.e. $a = 0$ and $b = \pi$, we yield an integrand of $\frac{2}{n+1}$.

Similarly, if we do the whole circle we yield 0! This is an example inside of a much broader result.

Example 6.3. If we continue with the previous example, but instead consider the function $f(z) = \frac{1}{z}$, then we will setup the integral

$$\int_C \frac{1}{z} dz = \int_a^b e^{-it} i e^{it} dt = \int_a^b i dt = i(b-a)$$

So in particular, if we let C be the whole circle, in this case we get $2\pi i$. Again, this is part of a much grander theorem that we will tackle in chapter 2.

Next, we get to a relation similar to the fundamental theorem of calculus.

Definition 6.4. If f is a function on an open set Ω, then f has a **primative** if there exists F a holomorphic function on Ω such that $F'(z) = f(z)$ for every $z \in \Omega$.

Theorem 6.5. *Let F be a primative for f in Ω, and $\gamma : [a, b] \to \Omega$ be a curve in Ω. Then*

$$\int_\gamma f(z) dz = F(\gamma(b)) - F(\gamma(a))$$

Proof. This follows from the standard fundamental theorem of calculus, since

$$\int_\gamma f(z) dz = \int_a^b f(\gamma(t))\gamma'(t) dt = \int_a^b \frac{\partial}{\partial t} F(\gamma(t)) dt$$

□

This yields two expected yet useful corollaries:

Corollary 6.6. *If γ is a loop in an open set Ω, and f has a primative in Ω, then*

$$\int_\gamma f(z) dz = 0$$

Proof. $F(a) = F(a)$! □

Given our example of $\int_{S^1} \frac{dz}{z} = 2\pi i$, where S^1 is the circle in $\mathbb{C}$, we know by this result that $\frac{1}{z}$ has no primative in $\mathbb{C} \setminus 0$. This has to do with the fact that we can't define the logarithm on all of $\mathbb{C}$ in a coherent way!

If we try to follow the homework's assertion that $\log(z) = \log(r) + i\theta$ for $r \geq 0$ and $\theta \in (-\pi, \pi)$, then as θ varies towards $-\pi$ and π we would expect different answers!

However, if we consider $\frac{1}{z^n}$ for $n > 1$, we do have the expected primative $\frac{-1}{(n-1)z^{n-1}}$. This immediately confirms the result

$$\int_{S^1} \frac{dz}{z^n} = 0$$

This can be verified by a straightforward computation, but we can avoid such work!

Corollary 6.7. *If f is holomorphic in Ω a connected and open set in $\mathbb{C}$, and $f' = 0$, then f is necessarily constant.*

Proof. f is certainly a primative for f'. As a result, we know

$$f(b) - f(a) = \int_\gamma f'(z) dz = \int_\gamma 0 = 0$$

But there exists a curve connecting any two points b, a by connectedness. As a result, $f(a) = f(b)$ for any $a, b \in \Omega$. □

CLASS 7, SEPTEMBER 23: GOURSAT'S THEOREM

Chapter 2 of the book is filled with many theorems which are at the heart of complex analysis. We will begin with one of the assertions made previously; Goursat's Theorem. This provides a similar result to the case of primatives from last class. It should be noted that this result derives many of the other important upcoming theorems.

Theorem 7.1 (Goursat's' Theorem)**.** *Let $\Omega \subseteq \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $T \subseteq \Omega$ is a triangle, then*

$$\int_T f(z)dz = 0$$

The theorem seems very restrictive, i.e. why study triangles? They are able to closely represent other shapes as well through use of a triangulation of a given region. We will formalize this idea later.

Proof. The proof is divided into some steps due to its length.

1) **Barycentric Subdivision:** Write $T = T^{(0)}$. Here we represent a natural way to subdivide a triangle into 4 triangular pieces. Given a triangle with sides ABC, choose the midpoints a, b, c of each side. Connecting them with lines naturally subdivides ABC into 4 pieces; $T_1^{(1)}, \dots, T_4^{(1)}$.

   The important aspect with regards to integrals is the orientation of the triangle. We already know that if we reverse the orientation, we negate the corresponding integral.

   To do this, we maintain the natural positive/counterclockwise orientation of the volume. That is to say, we give each of the new triangles the same counterclockwise orientation. This allows us to cancel equal sides and produce the following subdivision of the integral:

$$\int_{T^{(0)}} f(z)dz = \int_{T_1^{(1)}} f(z)dz + \int_{T_2^{(1)}} f(z)dz + \int_{T_3^{(1)}} f(z)dz + \int_{T_4^{(1)}} f(z)dz$$

   As a result, we can obtain that for some j (maximal say), we have

$$\int_{T^{(0)}} f(z)dz \leq 4 \left| \int_{T_j^{(1)}} f(z)dz \right|$$

2) **Proceed by induction:** We can continue the barycentric subdivision of the chosen triangle $T_j^{(1)}$ satisfying the previous inequality. This would yield the inequality

$$\int_{T^{(0)}} f(z)dz \leq 4 \left| \int_{T^{(1)}} f(z)dz \right| \leq 16 \left| \int_{T_j^{(2)}} f(z)dz \right|$$

   for some j. Inductively, we can produce the smaller and smaller triangles $T^{(n)}$ satisfying

$$\int_{T^{(0)}} f(z)dz \leq 4 \left| \int_{T^{(1)}} f(z)dz \right| \leq 4^n \left| \int_{T^{(n)}} f(z)dz \right|$$

3) **Notes about the triangles:** If we let d_n and p_n denote the diameter and perimeter respectively of $T^{(n)}$, then

$$d_n = 2^{-n} d_0 \qquad\qquad p_n = 2^{-n} p_0$$

If we denote by $\mathcal{T}^{(n)}$ the solid closed triangle enclosed by $T^{(n)}$, then we have constructed

$$\mathcal{T}^{(0)} \supseteq \mathcal{T}^{(1)} \supseteq \mathcal{T}^{(2)} \supseteq \dots$$

a nested sequence of triangles whose diameter goes to 0. Since each triangle was compact, this implies that there is a unique point z_0 belonging to $\bigcap_{n=0}^{\infty} \mathcal{T}^{(n)}$.

4) **Holomorphicity:** Given f is holomorphic at z_0, we can write

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$$

where $\psi(z) \to 0$ as $z \to z_0$. Now notice that $F(z) = f(z_0)z + \frac{f'(z_0)}{2}(z - z_0)^2$ is a primative for $f(z_0) + f'(z_0)(z - z_0)$, so their integrals are 0. As a result,

$$\int_{T^{(n)}} f(z)dz = \int_{T^{(n)}} \psi(z)(z - z_0)dz$$

Now, since $z_0 \in \mathcal{T}^{(n)}$ and $z \in T^{(n)}$, we have $|z - z_0| < d_n = 2^{-n} d_0$. As a result, we can estimate

$$\left| \int_{T^{(n)}} \psi(z)(z - z_0)dz \right| \leq \int_{T^{(n)}} |\psi(z)| \cdot |(z - z_0)| dz \leq \sup_{z \in T^{(n)}} \{\psi(z)\} \cdot p_n \cdot d_n$$

If we let $\epsilon_n = \sup_{z \in T^{(n)}} \{\psi(z)\}$, we know $\lim_{n \to \infty}(\epsilon_n) = 0$.

5) **Combining previous estimates:** We have

$$\left| \int_{T^{(n)}} f(z)dz \right| \leq \epsilon_n 4^{-n} d_0 p_0$$

Furthermore,

$$\left| \int_{T^{(0)}} f(z)dz \right| \leq 4^n \left| \int_{T^{(0)}} f(z)dz \right| \leq \epsilon_n d_0 p_0$$

Since the right-hand side goes to 0 as $n \to \infty$, we conclude that the independent of n left hand side must also be 0. This proves the theorem.

□

Corollary 7.2. *Let $\Omega \subseteq \mathbb{C}$ be an open set, and $f : \Omega \to \mathbb{C}$ be a holomorphic function. If $P \subseteq \Omega$ is a n-gon, then*

$$\int_P f(z)dz = 0$$

Proof. There is a very natural way to triangulate an n-gon. Each triangle satisfies the previous theorem and is therefore 0 in integral. □

CLASS 8, SEPTEMBER 25: CAUCHY'S THEOREM IN A DISC

As a corollary to Goursat's theorem, we can acquire the following result in a disc. Triangles turn out to be quite powerful objects.

Theorem 8.1. *A holomorphic function $f : B(z_0, r) \to \mathbb{C}$ in an open disc has a primative in that disc.*

Proof. Using the change of variables $z \mapsto z - z_0$, we may assume $z_0 = 0$. Let $z \in B(0, r)$. Consider the piecewise smooth path γ_z going from 0 to $Re(z)$, then to z itself. Orient the curve from 0 to z.

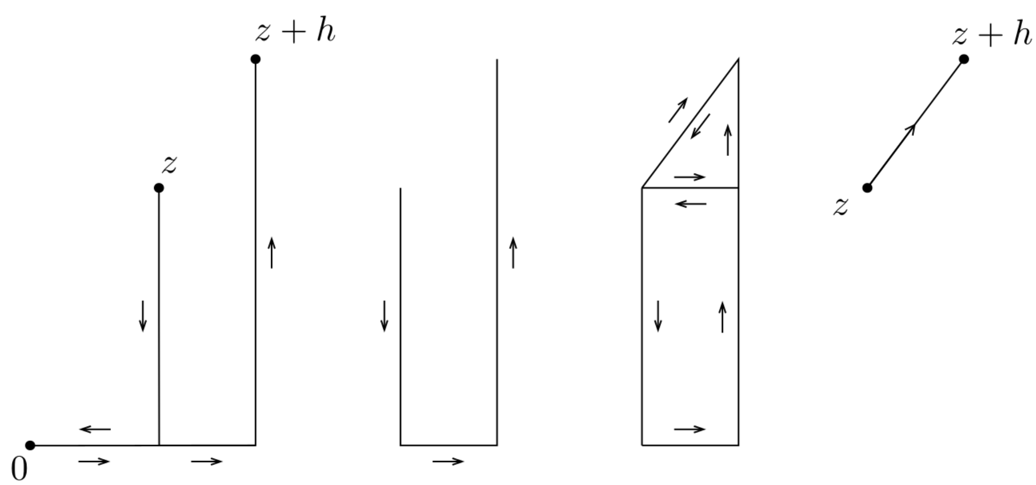
Define

$$F(z) = \int_{\gamma_z} f(w)dw.$$

We assert that $F(z)$ is holomorphic in $B(0, r)$, with $F'(z) = f(z)$. Note that

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(w)dw - \int_{\gamma_z} f(w)dw$$

It is best to think of this equation geometrically; we can replace the second integral with an integral with the same curve but in the opposite direction. Thus we are able to cancel the trip from 0 to z. This yields (b) in the picture (which is courtesy of pg 38 of Stein/Shakarchi). To obtain (c), we add a curve in both its forwards and backwards orientation, which doesn't change the integral. Finally, notice that all except the curve connecting z to h are parts of a triangular region and a rectangular region. From Goursat we may conclude those integrals are 0, leaving only (d).



(a) (b) (c) (d)

All this geometry yields us

$$F(z+h) - F(z) = \int_{\gamma} f(w)dw$$

where γ is simply the straight line connecting z to $z+h$. Since f is continuous, we have that $f(w) - f(z) = \psi(w) \to 0$ as $w \to z$. So as a result,

$$F(z+h) - F(z) = \int_\gamma f(z)dw + \int_\gamma \psi(w)dw = hf(z) + \int_\gamma \psi(w)dw$$

Finally,

$$\left| \int_\gamma \psi(w)dw \right| \leq \sup_{w \in \gamma} |\psi(w)| \cdot |h|.$$

But the supremum goes to 0 as $|h|$ does, so dividing the equality by h and taking the limit yields

$$\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = F'(z) = f(z).$$

□

Theorem 8.2 (Cauchy's Theorem in a Disc)**.** *If f is holomorphic in a disc D, then*

$$\int_\gamma f(z)dz = 0$$

for any loop $\gamma : [a, b] \to D$.

We can also generalize this statement slightly:

Corollary 8.3. *If f is holomorphic in an open set Ω containing a circle C and its interior, then*

$$\int_C f(z)dz = 0$$

Proof. If C is the boundary of $D = \bar{B}(z_0, r)$, there exists $\epsilon > 0$ such that $B(z_0, r + \epsilon) \subseteq \Omega$. This follows by compactness of the disc. As a result, the previous result yields the corollary. □

This corollary actually extends to any loop with a notion of an interior. Fortunately, there is a beautiful result called the "Jordan Curve Theorem" that tells us this is always the case when γ is simple and piecewise smooth[1]. I refer the interested reader to Appendix B of the book. More general versions also exist.

But since we won't have such a result in this class, we will call loops with an obvious interior **toy contours**. These include polygons. One very useful one in complex analysis is the **keyhole contour**. This is the curve that is designed to exclude a certain arc in the complex plane (such as the negative real axis).

The main idea is that when γ is a toy countour and f is holomorphic in an open region containing the interior of γ and γ itself, the

$$\int_\gamma f(z)dz = 0.$$

We will use the keyhole contour and toy contours to great effect later when trying to integrate more interesting functions.

[1]If you're willing to head to the topological world, piecewise smooth is also not needed.

Example 8.4. A difficult to evaluate integral using classical methods is the following:

$$\int_0^\infty \frac{1-\cos(x)}{x^2}dx.$$

Using our new theory, we can show that this is exactly $\frac{\pi}{2}$. To make such an integral appear in the complex world, we consider the function $\frac{1-e^{iz}}{z^2}$. We integrate over a large and small semicircle in the upper halfplane (call their radii R and ϵ respectively), as well as their connecting line segments. Since $f(z)$ is holomorphic everywhere except 0, we have that the total integral of this path is 0. This yields:

$$0 = \int_{-R}^{-\epsilon} \frac{1-e^ix}{x^2}dx + \int_\epsilon^R \frac{1-e^ix}{x^2}dx - \int_{C_\epsilon} f(z)dz + \int_{C_R} f(z)dz$$

where C_r is the circle of radius r centered at 0 with counterclockwise orientation. Letting $R \to \infty$, we have that $|\frac{1-e^{iz}}{z^2}| \leq \frac{2}{|z|^2}$. As a result, $\int_{C_R} f(z)dz \to 0$ and therefore

$$\int_{|x|\geq\epsilon} \frac{1-e^{ix}}{x^2}dx = \int_{C_\epsilon} f(z)dz$$

We also have a nice power series expansion for e^{iz}:

$$e^{iz} = \sum_{n=0}^\infty \frac{(iz)^n}{n!} = 1 + iz - \frac{z^2}{2} - \frac{iz^3}{6} + \dots$$

This produces $f(z) = \frac{-iz}{z^2} + E(z)$ where $E(z)$ is bounded as $z \to 0$. Therefore as $\epsilon \to 0$,

$$\int_{C_\epsilon} f(z)dz = \int_{C_\epsilon} \frac{-iz}{z^2} = \int_0^\pi \frac{-i\epsilon e^{i\theta}}{\epsilon^2 e^{2i\theta}} i\epsilon e^{i\theta} d\theta = \int_0^\pi d\theta = \pi$$

Using the fact that we have an even function, we are done!

CLASS 9, SEPTEMBER 27: CAUCHY'S INTEGRAL FORMULAS

Today we will begin by showing one more example following from local Cauchy and then move into some extremely useful integral formulas which also follow.

Example 9.1. If $\xi \in \mathbb{R}$, then I claim

$$e^{-\pi\xi^2} = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi} dx$$

If $\xi = 0$, then $1 = \int_{-\infty}^{\infty} e^{-\pi x^2} dx$ can be calculated directly. If $\xi > 0$, then $f(z) = e^{-\pi z^2}$is an entire function. Consider the contour γ which is a rectangle with vertices $-R, R, R+i\xi, -R-i\xi$ given counterclockwise orientation. By Cauchy's theorem,

$$\int_\gamma f(z)dz = 0$$

The integral over the bottom is exactly 1 as $R \to \infty$. The right side's integral is

$$\int_0^\xi f(R+iy)idy = i\int_0^\xi e^{-\pi(R^2+2iRy-y^2)}dy$$

As $R \to \infty$, this integral goes to 0 since it has absolute value bounded above by $Ce^{-\pi R^2}$ (since ξ is fixed). The same is true on the left. Finally, for the top, we have

$$\int_{-R}^{R} e^{-\pi(x+i\xi)^2}dx = -e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2}e^{-2\pi i x\xi}dx$$

Sending $R \to \infty$ yields

$$0 = 1 - e^{\pi\xi^2}\int_{-R}^{R} e^{-\pi x^2}e^{-2\pi i x\xi}dx$$

This shows that the Fourier Transform of $e^{-\pi z^2}$ is itself! We now shift to one of the most central theorems in complex analysis; Cauchy's integral theorem:

Theorem 9.2 (Cauchy's Integral Theorem)**.** *If $f : \Omega \to \mathbb{C}$ is a holomorphic function, and $\bar{B}(z_0, r) \subseteq \Omega$, then*

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw \qquad\qquad \forall z \in B(z_0, r)$$

Here C is the positively oriented circle bounding $\bar{B}(z_0, r)$.

Proof. Consider the keyhole K with outer circle $\bar{B}(z_0, r)$ and inner circle of radius ϵ centered at z. Let $\delta > 0$ be the width of the corridor. Since f is holomorphic, we know that $\frac{f(w)}{w-z}$ is holomorphic on the interior of this region. As a result, Local Cauchy tells us

$$\int_K \frac{f(w)}{w-z}dw = 0$$

Now, we can send $\delta \to 0$ since $\frac{f(w)}{w-z}$ is a continuous function away from z. This makes it so that the path join and cancel each other, leaving only the 2 circles. Orienting both positively as we have been, we have

$$\int_C \frac{f(w)}{w-z}dw = \int_{C_\epsilon} \frac{f(w)}{w-z}dw$$

It goes to compute the left hand side. We can break up the equation as

$$\int_{C_\epsilon} \frac{f(w)}{w-z}dw = \int_{C_\epsilon} \frac{f(w)-f(z)}{w-z}dw + \int_{C_\epsilon} \frac{f(z)}{w-z}dw$$

The first integral on the right hand side is bounded as $\epsilon \to 0$, since the inner portion approaches the derivative. Thus the integral is 0. Therefore, we are left with

$$\int_{C_\epsilon} \frac{f(w)}{w-z}dw = f(z)\int_{C_\epsilon} \frac{1}{w-z}dw$$

We can change variables using $w \mapsto w+z$ to recenter the integral at 0. This shows

$$\int_{C_\epsilon} \frac{f(w)}{w-z}dw = f(z)\int_{C_\epsilon} \frac{1}{w}dw = f(z)2\pi i.$$

Dividing precisely yields the desired result. $\square$

Just like with our previous Cauchy-style results, we can replace the circle with any toy contour that has z in its interior and yield identical results. Note that if z isn't in the interior we are holomorphic and thus the integral vanishes. This gives us an easy route to solving the homework problem:

Example 9.3. If $C = \partial B(0,r)$ is positively oriented, where $|a| < r < |b|$, then

$$\int_C \frac{dz}{(z-a)(z-b)} = \int_C \frac{\frac{1}{z-b}}{z-a}dz = 2\pi i\frac{1}{a-b}.$$

Here we are taking $f(z) = \frac{1}{z-b}$.

As a corollary to this theorem, which may seem a bit ambiguous as to why we would care about such a specific integral, is the following result I asserted in Class 0:

Corollary 9.4. *If $f : \Omega \to \mathbb{C}$ is a holomorphic function, then f has infinitely many complex derivatives in Ω. Furthermore, if $\bar{B}(z_0, r) \subseteq \Omega$ and $C = \partial\bar{B}(z_0, r)$, then*

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw \qquad \forall z \in B(z_0, r)$$

Proof. The proof follows by induction. $n = 0$ is Cauchy's integral theorem. So suppose it is true up to n-derivatives:

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z)^{n+1}}dw \qquad \forall z \in B(z_0, r)$$

If we write the difference quotient, we yield

$$\frac{f^{(n)}(z+h) - f^{(n)}(z)}{h} = \frac{n!}{2\pi i}\int_C \frac{f(w)}{h} \cdot \left[\frac{1}{(w-z-h)^{n+1}} - \frac{1}{(w-z)^{n+1}}\right] dw$$

Again we use the difference of 2 powers rule for $a^{n+1} - b^{n+1}$:

$$\frac{1}{(w-z-h)^{n+1}} - \frac{1}{(w-z)^{n+1}} = \frac{h}{(w-z-h)(w-z)}\left[\frac{1}{(w-z-h)^n} + \ldots + \frac{1}{(w-z)^n}\right]$$

Notice that if h is sufficiently small we stay within C. As a result

$$= \frac{n!}{2\pi i}\int_C f(w) \cdot \frac{1}{(w-z)^2}\frac{n+1}{(w-z)^n}dw$$

$\square$

CLASS 10, SEPTEMBER 30: CAUCHY'S COROLLARIES

Last time to great effect we used Cauchy's integral theorem to prove that if f is holomorphic, then it has infinitely many complex derivatives. Today we will improve this to showing it is in fact analytic! But first, we can show the following inequality.

Corollary 10.1 (Cauchy's inequality)**.** *If f is holomorphic in an open set containing a $\bar{B}(z_0, r)$, and $C = \partial\bar{B}(z_0, r)$, then*

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{r^n}$$

where $\|f\|_C = \sup_{z \in C} |f(z)|$.

Proof. This follows by our usual inequality for an integral via its sup:

$$|f^{(n)}(z_0)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(w)}{(w-z_0)^{n+1}}dw\right| = \frac{n!}{2\pi}\left|\int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{r^{n+1}e^{i(n+1)\theta}}re^{i\theta}d\theta\right| \leq \frac{n!}{2\pi}\frac{\|f\|_C}{r^n}2\pi$$

□

This yields an easy numeric method to test the size of a given integral. Now we can get to the amazing result about holomorphic functions being analytic (thus it is an if and only if statement). This is especially surprising since we are assuming a single derivative exists. We have seen an example of a real valued function having infinitely many derivatives but not being analytic on homework 2.

Theorem 10.2. *If f is holomorphic in Ω an open set, then if $\bar{B}(z_0, R) \subseteq \Omega$, then f is analytic at z_0. Furthermore,*

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$$

Proof. The last equality follows from our previous analysis of derivatives of power series. Fix $z \in \bar{B}(z_0, r)$. Cauchy then yields

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w - z_0}dw$$

We can write

$$\frac{1}{w-z} = \frac{1}{w - z_0 - (z - z_0)} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \frac{z-z_0}{w-z_0}}$$

There exists $0 < r < 1$ such that

$$\left|\frac{z - z_0}{w - z_0}\right| < r$$

Thus the geometric series yields

$$\frac{1}{1 - \frac{z-z_0}{w-z_0}} = \sum_{n=0}^{\infty}\left(\frac{z - z_0}{w - z_0}\right)^n$$

Here the series converges uniformly for $w \in C$. As a result, we can interchange the sum and the integral in the previous equations:

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\left(\int_C \frac{f(w)}{(w - z_0)^{n+1}}dw\right) \cdot (z - z_0)^n$$

This proves the result given Cauchy's integral theorem. ∎

These major results also yields quite a great deal about the structure of holomorphic functions. For example, in the complex world, everywhere holomorphic functions which are bounded are necessarily constant. This differs drastically with the real case, where functions like $\tan^{-1}(x)$ exist.

Corollary 10.3 (Louiville's Theorem). *If $f : \mathbb{C} \to B(0, R)$ is an entire function, then f is constant.*

Proof. Given *Corollary* 10.1, we have that

$$|f^{(n)}(z_0)| \leq \frac{n!\|f\|_C}{r^n}$$

for any $r > 0$. Furthermore, since f is bounded, $\|f\|_C < R$ by our assumption. Therefore, sending r to infinity produces $|f^{(n)}(z_0)| = 0$. Now, given Theorem 10.2, we can conclude f is necessarily constant (equal to $f(z_0)$). ∎

As a final major corollary, we have something that you likely know quite well; the fundamental theorem of algebra. It states that every polynomial factors into linear polynomials over the complex numbers.

Theorem 10.4 (FTOA). *If $p(z) = a_n z^n + \ldots + a_1 z + a_0$ with $n > 0$ and $a_n \neq 0$, then $p(z)$ has a root: there exists z_0 such that $p(z_0) = 0$.*

As a result, $p(z)$ factors as

$$p(z) = a_n(z - z_1) \cdots (z - z_n)$$

for some $z_i \in \mathbb{C}$ potentially repeating.

Proof. Suppose p has no roots. I assert that $\frac{1}{p(z)}$ is a bounded function. To see this, consider

$$\frac{p(z)}{z^n} = a_n + \frac{a_{n-1}}{z} + \ldots + \frac{a_0}{z^n}$$

As $|z|$ becomes large, say bigger than R, this function from below by $c = |\frac{a_n}{2}|$, thus

$$|p(z)| \geq c|z|^n$$

or equivalent $\frac{1}{|p(z)|} \leq C|z|^{-n} \leq CR^{-n}$. Now, for $|z| < R$, we have a closed disc for which $p(z)$ is never 0. Therefore, $p(z)$ is bounded away from 0, say by r. Immediately, we can conclude $\frac{1}{p(z)}$ is bounded above by $\max\{\frac{1}{r}, CR^{-n}\}$. But by Corollary 10.3, we have that $\frac{1}{p(z)}$ is constant, and therefore so is p. This contradicts our assumption that $n > 0$.

To show the final statement, we could use the division algorithm. But a natural way to realize this is the following: start by factoring out a_n. If z_0 is a root of p, then using the binomial theorem, we can write

$$p(z) = (z - z_0)^n + b_{n-1}(z - z_0)^{n-1} + \ldots + b_1(z - z_0) + b_0$$

Plugging in z_0 yields 0 on the left, so $b_0 = 0$. Therefore, we can write

$$p(z) = p_1(z)(z - z_0) = (z - z_0)^{n-1} + b_{n-1}(z - z_0)^{n-2} + \ldots + b_1$$

Now induction will yield the desired result.

∎

CLASS 11, OCTOBER 2: ANALYTIC CONTINUATION

Cauchy's theory is yielding the following general principle: Holomorphicity is an extremely strong condition. We have seen it yield strong bounds on functions, the analytic property, and only constant bounded functions! Today, we will study the idea of extending a function outside of its domain.

We intend to show that such an extension must be unique. To do this, we need a lemma about accumulation of zeroes of holomorphic functions.

Lemma 11.1. *Suppose f is holomorphic in Ω a connected set. If $z_n \in \Omega$ is a sequence of distinct points which converges to some $z_\infty \in \Omega$, and $f(z_n) = 0$ for all $n \in \mathbb{N}$, then f is the zero function on Ω.*

Proof. We begin by showing f is 0 in a neighborhood of z_∞. We can choose a disc for which f has a power series:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_\infty)^n$$

Choosing m minimal such that $a_m \neq 0$, we can write

$$f(z) = a_m (z - z_\infty)^m (1 + (z - z_0) g(z))$$

where $(z - z_\infty) g(z) \to 0$ as $z \to z_\infty$. Choosing z_k sufficiently close to z_∞, we can ensure $|(z - z_0) g(z)| < \frac{1}{2}$. But as a result

$$0 = |f(z_k)| = |a_m (z_k - z_\infty)^m (1 + (z - z_0) g(z))| \geq \frac{1}{2} |a_m| |z_k - z_\infty|^m > 0$$

This is a contradiction to our assumptions. So f is 0 in a neighborhood of z_∞.

Now we finish by use of connectedness. First, let U be the interior of $f^{-1}(0)$. We just showed U is non-empty. Furthermore, since f is continuous, if $z_n \to z$ in U, $f(z) = 0$. Thus U is closed. But to be connected implies that the only open and closed sets are empty or Ω. Thus $U = \Omega$. □

We can use this to prove the desired statement about function's extensions by considering their difference:

Corollary 11.2. *If f and g are holomorphic in a connected region Ω, and $f(z) = g(z)$ in an open subset of Ω, then $f = g$ throughout Ω.*

Here is a way to reinterpret this statement. If $\Omega \subseteq \Omega'$ are two open sets, and $f : \Omega \to \mathbb{C}$ is a holomorphic function, there exists **at most** one extension of f to Ω'. Here an extension is a holomorphic function $g : \Omega' \to \mathbb{C}$ such that $f(z) = g(z)$ for all $z \in \Omega$. Of course, an extension needn't exist. If they do, we call g an **analytic continuation** of f to Ω'.

Example 11.3. The function $f(z) = \frac{1}{z}$ is holomorphic on $\mathbb{C} \setminus 0$. However, there cannot exist even a continuous function on $\mathbb{C}$ agreeing with f.

In the case of real valued functions, there can exist as many distinct extensions as one may with.

Example 11.4. Going back to our function $f(x) = e^{-\frac{1}{x^2}}$ on $(0, \infty)$, the homework exercise shows that $f(x)$ can be extended to $\mathbb{R}$ by 0. But you could extend it also by $e^{-\frac{c}{x^2}}$ for any $c > 0$, or more generally any C^∞ function with all derivatives vanishing at the origin.

Example 11.5. We know that the power series $\sum_{n=0}^{\infty} z^n$ has a radius of convergence of $R = 1$. So our earlier result of Theorem 4.5 tells us explicitly that the series diverges for $|z| > 1$. On the other hand, in its radius of convergence this function agress with $\frac{1}{1-z}$. This function can be defined anywhere where $z \neq 1$, and is therefore the analytic continuation of the power series to $\mathbb{C} \setminus \{1\}$.

Example 11.6. The zeta function as discussed on day 1 is $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. This function makes sense whenever $Re(s) > 1$. However, in this domain one can show that

$$\zeta(s) = \frac{\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx}{\Gamma(s)} = \frac{\int_0^\infty \frac{x^{s-1}}{e^x - 1} dx}{\int_0^\infty x^{s-1} e^{-x} dx}$$

We will examine this equality later, but suffice it to say that the RHS makes sense whenever $s \neq 1$. Therefore, we can extend $\sum_{n=1}^{\infty} \frac{1}{n^s}$ to $\mathbb{C} \setminus \{1\}$.

It should be noted that $\Gamma(s)$ is itself a very important function. First of all, if s is an integer, then one can immediately conclude by integration by parts that

$$\Gamma(s) = \int_0^\infty x^{s-1} e^{-x} dx = \left[-x^{n-1} e^{-x}\right]_{x=0}^{\infty} - \int_0^\infty (s-1) x^{s-2} e^{-x} dx$$

By induction, with base case $\int_0^\infty e^{-x} dx = 1$, shows that $\Gamma(s) = (s-1)!$, the factorial of $s - 1$. Thus as a very cool analogy, $\Gamma(n+1)$ is continuation of the factorial from the non-negative integers to $\mathbb{C} \setminus (\mathbb{Z}_{n<0})$! The removal of the negative integers is due to the fact that the function isn't defined at thus values (there are **simple poles** there).

To finish up, I want to discuss an important converse to Cauchy's theorem.

Theorem 11.7 (Morera's Theorem)**.** *If f is continuous in an open disc $B(z, r)$ and such that for any triangle T, we have*

$$\int_T f(z) dz = 0$$

then f is holomorphic in $B(z, r)$.

It is actually quite a simple proof:

Proof. By the proof of Cauchy's integral theorem, f has a primative in $B(z, r)$, namely $F(z) = \int_{\gamma_z} f(z) dz$ where γ_z is a path from a chosen point to z. Since $F'(z) = f(z)$, and F is holomorphic, F has infinitely many derivatives. But this implies $F''(z) = f'(z)$, i.e. f is holomorphic. $\square$

Example 11.8. Returning to our previous example, Morera's Theorem shows that $\zeta(s)$ is a holomorphic function. Indeed, given a triangle T inside of the region of complex numbers with real part bigger than 1, then one can show

$$\int_T \zeta(s) = \int_T \sum_{n=1}^{\infty} \frac{1}{n^s} ds = \sum_{n=1}^{\infty} \int_T \frac{1}{n^s} ds = \sum_{n=1}^{\infty} 0 = 0$$

Here, interchanging the $\sum$ and $\int_T$ is a delicate process. One can cite Fubini or Tonelli's theorem, but at the very least one should note the importance of the uniform absolute convergence of the series on the triangle T.

Regardless, this shows that $\zeta(s)$ is a holomorphic function. The same sort of trick can be applied to the gamma function $\Gamma(s)$.

CLASS 12, OCTOBER 4: FURTHER APPLICATIONS

Last time we saw a few very important corollaries of Cuachy's integral theorem, such as the idea of analytic continuation and Morera's theorem. Today we will study sequences of Holomorphic functions and the Schwarz reflection principle.

The first thing I want to tackle is when the limit of holomorphic functions is holomorphic. It turns out it is sufficient to assume uniform convergence on compact subsets:

Theorem 12.1. *If $f_n : \Omega \to \mathbb{C}$ is a sequence of holomorphic functions such that $f_n \to f$ uniformly for every compact subset $K \subseteq \Omega$, then f is holomorphic on Ω.*

Note that finite collections of points are always compact. So this is a much stronger type of convergence than pointwise convergence.

Proof. Let $T \subseteq B(z_0, r) \subseteq \Omega$ be a triangle inside an open disc with z_0 in its interior. Since f_n is holomorphic, we have

$$\int_T f_n(z)dz = 0$$

by Goursat's theorem. But by uniform convergence, we can choose $N \gg 0$ such that $|f(z) - f_n(z)| < \frac{\epsilon}{l(T)}$ for any $\epsilon > 0$, where $l(T)$ is the length of the triangle. This shows

$$\int_T f_n(z)dz \to \int_T f(z)dz = 0$$

Now as a result of the fact that a uniform limit of continuous functions is continuous, Morera's theorem implies f is holomorphic in $B(z_0, r)$. But these cover Ω, so the same is true for Ω. $\square$

There are many examples of real valued functions where this property is false. For example,

$$\sum_{n=0}^{\infty} \frac{1}{2^n} \sin(2^n x)$$

is uniformly approximatable by partial sums, but isn't differentiable at 0.

In the complex setting, we can do one better:

Theorem 12.2. *With the assumptions of Theorem 12.1, $f_n'(z)$ converges uniformly to $f'(z)$ on every compact subset $K \subseteq \Omega$.*

Proof. Let $\Omega_\delta = \{z \in \Omega \ : \ \bar{B}(z, \delta) \subseteq \Omega\}$. I claim that $f_n'(z) \to f(z)$ uniformly for each $\delta > 0$, which would prove the result. We claim for any holomorphic function F on Ω,

$$\sup_{z \in \Omega_\delta} |F'(z)| \leq \frac{1}{\delta} \sup_{w \in \Omega} |F(w)|$$

To see this, consider CIT:

$$F'(z) = \frac{1}{2\pi i} \int_C \frac{F(w)}{(w - z)^2} dw$$

where C is the circle of radius δ about z. Now our standard approximation yields

$$|F'(z)| \leq \frac{1}{2\pi} \int_C \frac{|F(w)|}{|w - z|^2} |dw| = \frac{1}{2\pi\delta^2} \int_C |F(w)||dw| \leq \frac{2\pi\delta}{2\pi\delta^2} \sup_{w \in C} |F(w)|$$

This demonstrates the claim. Now if we consider the known holomorphic function $F = f - f_n$, we have bounded $|f' - f_n'|$ by a constant times something which approaches 0 as $n \gg 0$. Therefore they must agree. $\square$

Corollary 12.3. *If $f_n : \Omega \to \mathbb{C}$ is a sequence of holomorphic functions such that $f_n \to f$ uniformly for every compact subset $K \subseteq \Omega$, then $f_n^{(m)} \to f^{(m)}$ uniformly on every compact set.*

Proof. By induction. $\square$

Now we turn to Schwarz reflection principle. This gives us an idea for extending a holomorphic function to a larger domain. This is easier in real analysis, since differentiability is such a weaker condition.

Let Ω be an open set which is symmetric over the real line: $z \in \Omega \iff \bar{z} \in \Omega$. Call Ω^+ and Ω^- the open subsets with positive and negative imaginary parts respectively. Additionally, let $I = \Omega \cap \mathbb{R}$.

Theorem 12.4 (Symmetry Principle)**.** *If f^+ and f^- are holomorphic functions on Ω^+ and Ω^- respectively that can be continuously extended to I and agree with one another, then f defined piecewise by these functions is holomorphic.*

Proof. Let $B(iy, r) \subseteq \Omega$ be a disc centered along I, and $T \subseteq B(iy, r)$ be a triangle with a vertex or side on I. In the case of a side, we can consider an $\epsilon > 0$ shift of the triangle upward or downward to place it entirely in Ω^+ or Ω^-. Doing so ensures

$$\int_{T_\epsilon} f(z)dz = 0$$

but since f is continuous in Ω, this converges to $\int_T f(z)dz$. In the case of a vertex, we can subdivide the triangle to reduce to the previous case. The same holds for the general case! Thus again by Morera's theorem we can conclude f is holomorphic. $\square$

We can rephrase Theorem 12.4 slightly to yield a similar result with only the task of defining a function in the lower half-plane.

Theorem 12.5 (Schwarz Reflection Principle)**.** *Suppose f is holomorphic on Ω^+ and can be extended continuously to a real valued function on I. Then f can be extended to a holomorphic function on the whole region Ω.*

Proof. It goes to define the function holomorphically on Ω^- and use Theorem 12.4. A good choice is $f^-(z) = \overline{f(\bar{z})}$. This makes it immediate that f^- agrees with f on I. Now it only goes to show that it is holomorphic.

Given $z_0 \in \Omega^-$, we have $\bar{z_0} \in \Omega^+$. Therefore, there is a power series expansion for f near $\bar{z_0}$:

$$f(\bar{z}) = \sum_{n=0}^{\infty} a_n(\bar{z} - \bar{z_0})^n$$

Applying the complex conjugate, we get that

$$f^-(z) = \sum_{n=0}^{\infty} \bar{a_n}(z - z_0)^n$$

This power series has the same radius of convergence as the original, so f^- is holomorphic in this radius. This holds for every point, so we are done. $\square$

It is fun to think about the importance of the conditions in Theorem 12.4 & Theorem 12.5 .

CLASS 13, OCTOBER 7: ZEROES AND POLES

Now that we have many of the underpinnings of what a holomorphic function is, we will ask how far we can deviate without losing everything. A deviation from holomorphic readily arrives at the singularity.

Definition 13.1. If f is a complex valued function defined in a neighborhood of a point z_0, but not defined at z_0, then z_0 is called an **isolated singularity for** f.

A canonical example of this is $f(z) = \frac{1}{z}$. It is necessarily singular at $z = 0$ because it can't be continuously extended to $z = 0$. An even 'worse' singularity would occur if you study the function $g(z) = e^{\frac{1}{z}}$. $f(z)$ at the very least can be made holomorphic by multiplying by z, where as $\lim_{z\to 0} |z^n e^{\frac{1}{z}}| = \infty$.

Definition 13.2. If $z_0 \in \mathbb{C}$ is such that $f(z_0) = 0$, we call z_0 a **zero** of f. The **order** of a zero of f is a positive integer m, if it exists, such that

$$f(z) = (z - z_0)^m \cdot g(z)$$

in a neighborhood of z_0, and $g(z_0) \neq 0$. We label it $\mathrm{ord}_{z_0}(f) = m$.

If f is holomorphic, then our analytic continuation result shows the zeroes of f are isolated (they can have no accumulation point). There is something to show here (namely that the order makes sense!):

Proposition 13.3. *If f is a holomorphic function with an isolated zero at z_0, then the order exists uniquely.*

Proof. We know that $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$ in some disc $B(z_0, r)$ around z_0. Since f is not identically 0 near z_0, by the analytic continuation results we have that $\exists m$ minimal such that $a_m \neq 0$:

$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \ldots = (z - z_0)^m(a_m + (z - z_0)h(z))$$

As a result, $g(z) = a_m + (z - z_0)h(z)$ demonstrates the result. Now, assume that there is another integer n such that $f(z) = (z - z_0)^m l(z)$ where $l(z_0) \neq 0$ in a neighborhood of z_0. We may assume by intersecting that they are equal and defined on a given set. Then we can divide by $(z - z_0)^m$:

$$(z - z_0)^{-m} f(z) = g(z) = (z - z_0)^{n-m} l(z)$$

If $n - m > 0$, then $z \to z_0$ would yield that $g(z_0) = 0$, a contradiction. Similarly, if $n - m < 0$, then $l(z_0) = 0$, another contradiction. So they must agree. □

The key point here is that we can now define the idea of a pole. If $f(z)$ is holomorphic in a neighborhood U of z_0 with an isolated zero (in U) at z_0, then we can consider $\frac{1}{f(z)}$ in $U \setminus \{z_0\}$.

Definition 13.4. If $\mathrm{ord}_{z_0}(f) = m$, then we say $\frac{1}{f}$ has a **pole of order** m at z_0. If $m = 1$, then it is sometimes referred to as a **simple pole**. We write $\mathrm{ord}_{z_0}(\frac{1}{f}) = -m$.

Example 13.5. If we let C be the counterclockwise oriented unit circle, then we know

$$\int_C \frac{dz}{z^m} = \begin{cases} 2\pi i & m = 1 \\ 0 & \text{otherwise} \end{cases}$$

This already shows the importance of the simple pole.

A different way to classify this, identical to Proposition 13.3 is the following result:

Proposition 13.6. *If f has a pole at $z_0 \in \Omega$ of order m, then in a neighborhood of z_0, there is a holomorphic function h such that*

$$f(z) = (z - z_0)^{-m} h(z)$$

and such that $h(z_0) \neq 0$.

I leave it to you to prove this result. The following corollary arises immediately:

Corollary 13.7. *If f is a function with a pole of order m at z_0, then*

$$f(z) = \frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \ldots + \frac{a_{-1}}{z - z_0} + g(z)$$

where $g(z)$ is a holomorphic function at z_0.

Proof. Consider the function $h(z)$ in Proposition 13.6. Since h is holomorphic, non-vanishing, we have

$$h(z) = b_0 + b_1(z - z_0) + \ldots + b_{m-1}(z - z_0)^{m-1} + b_m(z - z_0)^m + \ldots$$
$$(z - z_0)^{-m} h(z) = (z - z_0)^{-m} b_0 + b_1(z - z_0)^{-m+1} + \ldots + b_{m-1}(z - z_0)^{-1} + b_m + \ldots$$

As a result, if we let $a_{-n} = b_{m-n}$ for $n = 1, \ldots, m$ and $g(z) = b_m + b_{m+1}(z - z_0) + \ldots$ then we acquire the desired result. □

Definition 13.8. We call $\frac{a_{-m}}{(z-z_0)^m} + \frac{a_{-m+1}}{(z-z_0)^{m-1}} + \ldots + \frac{a_{-1}}{z-z_0}$ the **principal part of** f at z_0. a_1 is called the **residue** and we write $a_1 = res_{z_0} f$.

As a result of the example above and Corollary 13.7, given any function $f(z)$ with a pole of order m at z_0, in a neighborhood of z_0 where z_0 is the only pole, we have

$$\int_C f(z) dz = \int_C \left(\frac{a_{-m}}{(z - z_0)^m} + \frac{a_{-m+1}}{(z - z_0)^{m-1}} + \ldots + \frac{a_{-1}}{z - z_0} + g(z) \right) dz = 2\pi i a_{-1}$$

where C is a circle centered at z_0 in the neighborhood. So the residue is quite an important concept for our purposes. It can be computed quite easily for **simple poles**:

$$\text{res}_{z_0} f = \lim_{z \to z_0} (z - z_0) f(z)$$

If f has a pole of order m, then we can use the following more general result:

Theorem 13.9. *If f has a pole of order m at z_0, then*

$$res_{z_0} f = \lim_{z \to z_0} \frac{1}{(m-1)!} \left(\frac{\partial}{\partial z} \right)^{m-1} ((z - z_0)^m f(z))$$

Proof. Notice that due to Corollary 13.7,

$$(z - z_0)^m f(z) = a_{-m} + a_{-m+1}(z - z_0) + \ldots + a_{-1}(z - z_0)^{m-1} + (z - z_0)^m g(z)$$

A simple computation now proves the result. □

CLASS 14, OCTOBER 9: THE RESIDUE THEOREM

Last time we studied how zeroes become poles when you invert the function, and how the order of a pole is an important quantity telling you how you can multiply away the pole. Today we will use this machinery to study integrals of such functions.

Lemma 14.1. *Suppose f is a function holomorphic on $\bar{B}(w,r)$ except at some point $z_0 \in B(w,r)$ where it has a pole or order m. Then*

$$\int_C f(z)dz = 2\pi i \cdot res_{z_0} f$$

This result is only a slight deviation from what we've already proved.

Proof. Consider the keyhole contour connecting C to a circle C_ϵ of radius ϵ about z_0. By our assumptions, f is holomorphic here so the integral along the keyhole is 0. Sending the width of the corridor to 0 produces

$$\int_C f(z)dz = \int_{C_\epsilon} f(z)dz$$

where C_ϵ is oriented clockwise. Expressing f here, we see

$$\int_{C_\epsilon} f(z)dz = \sum_{n=2}^{m} \int_{C_\epsilon} \frac{a_n}{(z-z_0)^n}dz + \int_{C_\epsilon} \frac{\mathrm{res}_{z_0} f}{z-z_0}dz + \int_{C_\epsilon} g(z)dz$$

where g is holomorphic. Now using Cauchy's integral theorem, we have

$$\int_{C_\epsilon} \frac{a_n}{(z-z_0)^n}dz = \left(\frac{\partial}{\partial z}\right)^{n-1} (a_n)_{z_0} = 0$$

$$\int_{C_\epsilon} f(z)dz = \int_{C_\epsilon} \frac{\mathrm{res}_{z_0} f}{z-z_0}dz = 2\pi i \cdot \mathrm{res}_{z_0} f$$

□

The same result, as usual, holds for any toy contour. This is because CIT does as well. We can use this to produce a more general result using this observation:

Theorem 14.2 (The Residue Formula)**.** *Suppose f is a function holomorphic on Ω except at some points $z_1, \ldots, z_n \in B(w,r)$ where it has a poles. If C is a toy contour enclosing $z_1, \ldots, z_n$, then*

$$\int_C f(z)dz = 2\pi i \cdot \sum_{i=1}^{n} res_{z_i} f$$

Proof. We can proceed by induction. The base case is the toy contour version of Lemma 14.1. By a finiteness argument, we can produce a keyhole contour around z_n with the corridor paths and circle avoiding $z_1, \ldots, z_{n-1}$. Our inductive hypothesis ensures that

$$\int_K f(z)dz = 2\pi i \cdot \sum_{i=1}^{n-1} \mathrm{res}_{z_i} f$$

Examining the left hand side, we see

$$\int_K f(z)dz = \int_C f(z)dz - \int_{C_\epsilon} f(z)dz$$

So again applying Lemma 14.1 yields the desired result:

$$\int_C f(z)dz = \int_{C_\epsilon} f(z)dz + 2\pi i \cdot \sum_{i=1}^{n-1} \mathrm{res}_{z_i} f = 2\pi i \cdot \sum_{i=1}^{n} \mathrm{res}_{z_i} f.$$

□

So far in evaluating several of the real integrals we needed to invoke Cauchy's theorem and ensure holomorphicity. This improves this technique dramatically by allowing finitely many poles!

Example 14.3. Consider the integral

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx$$

This can be evaluated by methods of trigonometric integrals. But we can simplify this quite a bit! Consider the upper half semi-circle C. The integral along the bottom portion is exactly what we want. Now noticing

$$\int_C \frac{dz}{z^2+1} = \int_C \frac{dz}{(z+i)(z-i)}$$

So this function has a pole at i and $-i$. However, our semi-circle doesn't enclose $-i$, so we have

$$\int_C \frac{dz}{z^2+1} = 2\pi i \mathrm{res}_i f = 2\pi i \frac{1}{i+i} = \pi$$

Thus it only goes to show that the upper portion C_R doesn't contribute anything. But this is easy!

$$|\int_{C_R} \frac{dz}{z^2+1}| \leq \pi R \cdot \sup|f(z)| = \pi R \frac{1}{R^2-1} \to 0$$

It should be noted that the choice is yours to integrate over the upper or lower half semicircles in the previous case. Note $\mathrm{res}_{-i} f = \frac{1}{z-i}|_{z=-i} = -\frac{1}{2i}$. But this is accounted for because we have gone around clockwise instead of counterclockwise. The rest of the story goes through as expected.

This naturally generalizes to any rational function $\frac{p(z)}{q(z)}$ without poles on the real line. Assuming $\deg(p(z)) + 1 < \deg(q(z)$, then we know the integral converges (this is actually an if and only if statement. We would simply choose either the upper or lower semicircles (note that if p and q have real coefficients, then both will have the same number of enclosed poles), and apply the residue theorem. The same result will show C_R contributes nothing to the integral, so we would derive

$$\int_{-\infty}^{\infty} \frac{p(z)}{q(z)} dz = \sum_{\substack{z_i \\ q(z_i)=0}} \mathrm{res}_{z_i} \left(\frac{p(z)}{q(z)}\right)$$

This is amazing! You never need to evaluate integrals of rational functions with simple poles again. Instead in suffices to simply plug in values.

CLASS 15, OCTOBER 16: APPLICATIONS OF RESIDUES

Today we will consider a few other important examples where the residue theorem can be directly applied.

Example 15.1. Consider the following integral with $a \in (0,1)$:

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$$

To evaluate this, we will use the rectangle with sides the real axis and $Im(z) = 2\pi$. Now, notice that the only singular point occurs when $e^z = 1$, which occurs exactly at πi. Therefore, to compute this integral, we only need to compute its residue here. Consider $(z - \pi i)f(z)$. Given the fact that $e^{i\pi} = -1$, we can rewrite this as

$$(z - \pi i)f(z) = \frac{z - \pi i}{1 + e^z} \cdot e^{az} = \frac{z - \pi i}{e^z - e^{i\pi}} \cdot e^{az} = \frac{1}{\frac{e^z - e^{i\pi}}{z - \pi i}} \cdot e^{az}$$

Now, as we send $z \to \pi i$, we acquire that this is the inverted derivative of e^z:

$$\text{res}_{\pi i}((z - \pi i)f(z)) = \lim_{z \to \pi i} (z - \pi i)f(z) = \frac{e^{a\pi}}{e^{\pi i}}$$

As a result, the residue formula yields

$$\int_\gamma \frac{e^{az}}{1+e^z} dz = 2\pi i \frac{e^{a\pi i}}{e^{\pi i}} = -2\pi i e^{a\pi i}$$

Now, it goes to consider the parts individually. For the right side of the integral, we achieve

$$\int_0^{2\pi} f(R + it) i dt = \int_0^{2\pi} \frac{e^{a(R+it)}}{1 + e^{R+it}} i dt$$

In absolute value, the integrand is bounded by

$$2\pi \frac{e^{aR}}{e^R - 1} \to 0$$

as $R \to \infty$. As a result, the integral is bounded by 2π times something going to 0. The same is true for the left hand side of the rectangle with slight modification of the bound. Therefore, it only goes to compute the remaining portions. The bottom is the quantity we are interested in. The top is

$$-\int_{-R}^{R} f(t + 2\pi i) dt \to -\int_{-\infty}^{\infty} \frac{e^{at} e^{2\pi i a}}{1 + e^t} dt = -e^{2\pi i a} \int_{-\infty}^{\infty} \frac{e^{at}}{1+e^t} dt$$

So in total, we get

$$(1 - e^{2\pi i a}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = -2\pi i e^{a\pi i}$$

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx = -\frac{2\pi i e^{a\pi i}}{1 - e^{2\pi i a}} = -2\pi i \frac{1}{e^{-\pi i a} - e^{\pi i a}} = \frac{\pi}{\sin(\pi a)}$$

We have already seen that $e^{-\pi x^2}$ is its own Fourier transform. Here we will find another function with this property.

Example 15.2. I will demonstrate that

$$\frac{1}{\cosh(\pi\xi)} = \int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)} dx$$

We will again use the rectangle, but this time let the height be $2i$. Since $\cosh(z) = \frac{e^z+e^{-z}}{2}$, we have $\cosh(\pi z) = 0$ precisely when $e^{\pi z} = -e^{-\pi z}$, or when $e^{2\pi z} = -1$. This occurs exactly when $z = \alpha := i\frac{1+2n}{2}$ for $n \in \mathbb{Z}$. Again, one can check that these are simple poles:

$$(z-\alpha)f(z) = 2e^{-2\pi i z\xi}e^{\pi z}\frac{z-\alpha}{e^{2\pi z}-e^{2\pi\alpha}} \to 2e^{-2\pi i\frac{i}{2}\xi}e^{2\pi\alpha}\frac{1}{2\pi e^{2\pi\alpha}} = \frac{e^{\pi\xi}}{\pi i}$$

as $z \to \alpha$. Again, the vertical sides go to 0 as $R \to \infty$, since

$$|e^{2\pi i z\xi}| \leq e^{4\pi|\xi|}$$

and ξ is fixed, where as

$$\cosh(z)| \geq \frac{1}{2}\left|\left(|e^{\pi z}| - |e^{-\pi z}|\right)\right| = \frac{1}{2}(e^{\pi R} - e^{-\pi R}) \to \infty$$

Thus again we have

$$(1-e^{4\pi\xi})\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)} dx = 2\pi i\left(\frac{e^{\pi\xi}}{\pi i} + \frac{-e^{3\pi\xi}}{\pi i}\right) = 2(e^{\pi\xi} - e^{3\pi\xi})$$

Finally, rearranging cosh yields the desired result:

$$\frac{1}{\cosh(\pi\xi)} = \frac{2}{e^{\pi\xi}+e^{-\pi\xi}} = \frac{2}{e^{\pi\xi}+e^{-\pi\xi}}\frac{e^{\pi\xi}-e^{-\pi\xi}}{e^{\pi\xi}-e^{-\pi\xi}} = 2\frac{e^{\pi\xi}-e^{-\pi\xi}}{e^{2\pi\xi}-e^{-2\pi\xi}}$$

The last quantity is exactly the desired integral.

The importance of this integral is again to do with the Fourier transform. What we have just proven is that the Fourier transform of $\frac{1}{\cosh(\pi x)}$ is itself. We will go into more depth in chapter 4 regarding these transforms.

Next time we will talk singularities more in general. We will see that not all of them are simply poles. There is an idea of an **essential singularity**, which is something like an infinite order pole!

Example 15.3. Consider the function $e^{\frac{1}{z}}$. At 0, it is easy to check that the following limit never exists:

$$\lim_{z\to 0} z^n e^{\frac{1}{z}}$$

One can think about this as when we plug $\frac{1}{z}$ into the taylor expansion for e^z, we will never remove all of the powers of $\frac{1}{z}$.

So we will attempt to tackle functions like this and create a notion of a function that avoids such difficult properties.

CLASS 16, OCTOBER 18: SINGULARITIES

So far we have only talked about poles in the class of isolated singularities. Today we will study the remaining classes.

Definition 16.1. Let $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ be a holomorphic function, with z_0 internal to Ω. A singularity $z_0 \in \Omega$ of f is called **removable** if there exists w such that defining $f(z_0) = w$ makes $f : \Omega \to \mathbb{C}$ a holomorphic function.

Thus a singularity is removable if it can be removed from the list of singularities. The following theorem makes this more rigorous:

Theorem 16.2 (Riemann's theorem on removable singularities)**.** *Suppose f is holomorphic on Ω except possibly at a point $z_0 \in \Omega$. If f is bounded near z_0, then z_0 is a removable singularity.*

Proof. We can focus on $\bar{B}(z_0, r) \subseteq \Omega$. Let C be the boundary circle oriented counterclockwise. We want to show that Cauchy's Integral theorem holds:

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)}{w - z} dw$$

for $z \neq z_0$ internal to $\bar{B}(z_0, r)$. We will show that the RHS is a holomorphic function on all of $\bar{B}(z_0, r)$, and it agrees with $f(z)$ whenever $z \neq z_0$. As a result, analytic continuation will yield that the RHS is the desired extension of f to z_0. To show holomorphicity, we use the following lemma:

Lemma 16.3. *Let $F : \Omega \times [0, 1]$ be a function where Ω is open. If*

1) $F(z, t)$ is holomorphic in z for each fixed t.
2) F is continuous.

Then the function $f(z) = \int_0^1 F(z, t) dt$ is holomorphic.

The idea of this lemma is to allow a function to be deformed with respect to a parameter.

Proof. For $n \geq 1$, we can consider the Riemann sum

$$f_n(z) = \frac{1}{n} \sum_{m=1}^{n} F(z, \frac{m}{n})$$

Then $f_n(z)$ is holomorphic by assumption 1). Now we want to show that for any given disc $\bar{B}(z_0, r) \subseteq \Omega$, the sequence f_n converges uniformly to f. Since F is continuous on the compact set $\bar{B}(z_0, r) \times [0, 1]$, we have that it is uniformly continuous: $\forall \epsilon > 0$, there exists $\delta > 0$ such that

$$|s - t| \leq \delta \qquad \Longrightarrow \qquad \sup_{z \in \bar{B}(z_0, r)} |F(z, s) - F(z, t)| < \epsilon$$

Now if $\frac{1}{n} < \delta$, i.e. $n \gg 0$, then

$$\begin{aligned} |f_n(z) - f(z)| &= \left| \sum_{m=1}^{n} \int_{\frac{(m-1)}{n}}^{\frac{m}{n}} F(z, \frac{k}{n}) - F(z,s) ds \right| \\ &\leq \sum_{m=1}^{n} \int_{\frac{(m-1)}{n}}^{\frac{m}{n}} \left| F(z, \frac{k}{n}) - F(z,s) \right| ds \\ &= \sum_{m=1}^{n} \frac{\epsilon}{n} = \epsilon \end{aligned}$$

This shows uniformity of convergence. Finally, since f_n are themselves holomorphic, we have that f is as well by Theorem 12.1 in the notes. □

Returning to the proof of the original result, Lemma 16.3 yields the fact that $\frac{1}{2\pi i} \int_C \frac{f(w)}{w-z} dw$ is holomorphic everywhere on $B(z_0, r)$.

It now suffices to check that the equality holds. To do this, we will use the 'double keyhole' contour which avoids both z_0 and our point of interest z. Since we are holomorphic on the interior, we yield that

$$\int_C \frac{f(w)}{w-z} dz - \int_{C_z} \frac{f(w)}{w-z} dz - \int_{C_{z_0}} \frac{f(w)}{w-z} dz = 0$$

where C_w is a circle of small radius $\epsilon > 0$ about w oriented clockwise. The residue theorem yields

$$\int_{C_z} \frac{f(w)}{w-z} dz = 2\pi i f(z)$$

Additionally, using the fact that $f(z)$ is bounded near z_0, as in the homework exercise, we may conclude $\int_{C_{z_0}} \frac{f(w)}{w-z} dz = 0$. This shows the desired result. □

A very nice corollary of Theorem 16.2 is the following perhaps expected result is something that you may have initially suspected.

Corollary 16.4. *If f has an isolated singularity at z_0, then z_0 is a pole if and only if $\lim_{z \to z_0} |f(z)| = \infty$.*

Proof. ($\Rightarrow$): If z_0 is a pole of order m, then $\frac{1}{f}$ has a zero of order m at z_0. Thus $|f(z)| \to \infty$ as $z \to z_0$.

($\Leftarrow$): If $\lim_{z \to z_0} |f(z)| = \infty$, then $\frac{1}{f}$ is bounded near z_0 (in fact close to 0). Therefore, $\frac{1}{f}$ has a removable singularity at z_0 necessarily with limit 0. Therefore, writing

$$\frac{1}{f} = a_m (z-z_0)^m + a_{m+1}(z-z_0)^{m+1} + \ldots = a_m (z-z_0)(1 + (z-z_0)g(z))$$

we have that f has a pole of order m at z_0. □

Example 16.5. We said $e^{\frac{1}{z}}$ does not have a pole at 0. Corollary 16.4 now can ensure this: Approach 0 from the direction $z = iy$ as $y \to 0$:

$$e^{\frac{1}{iy}} = e^{-i\frac{1}{y}}.$$

This is bounded in absolute value by 1. Thus it cannot have limit ∞ (in fact, it doesn't exist).

CLASS 17, OCTOBER 23: ESSENTIAL SINGULARITIES

We now have produced a way to subclassify all types of singularities. They must fall into one of the following buckets:

1) **Removable singularities**: Fixable without modification
2) **Poles**: Fixable by multiplication by $(z - z_0)^m$
3) **Essential Singularities**: Not fixable $(z - z_0)^m$.

These words are quite loose, but Corollary 16.4 from last class really firms up this understanding. As a result, one may ask what can we say about essential singularities. One interesting observation is the following:

Theorem 17.1 (Casorati-Weierstass)**.** *If f is holomorphic near z_0 and has an essential singularity at z_0, then $f(B_*(z_0, r)) \subseteq \mathbb{C}$ is dense, where the $*$ indicates without its center.*

$B_*(z_0, r)$ is often called the **punctured disc**. This shows just how wild these essential singularities are: any neighborhood however small with fill up the entire complex plane up to closure!

Proof. Suppose the assertion is false. This is equivalent to saying there exists w and δ such that

$$B(w, \delta) \subseteq \mathbb{C} \setminus f(B(z_0, r))$$

This allows us to consider a new function on $B_*(z_0, r)$:

$$g(z) = \frac{1}{f(z) - w}$$

Note that $g(z)$ is bounded above by $\frac{1}{\delta}$ and is furthermore holomorphic on its domain. As a result of Riemann's theorem (Theorem 16.2, we get that $g(z)$ has a removable singularity at z_0. If $g(z_0) \neq 0$, then $f(z)-w$ is holomorphic at z_0. This is impossible since f has an essential singularity there and w is just a constant. If $g(z_0) = 0$, then $\lim_{z\to z_0}(|f(z)-w|) = \infty$, which implies it is a pole by Corollary 16.4. We have reach a contradiction. $\square$

It should be noted that Picard proved in fact that f takes on each complex value infinitely often with the exception of a single point. So at the very least, the image of the punctured disc misses a single point!

Example 17.2. Examining again our essential singularity $e^{\frac{1}{z}}$, we claim it hits every point except 0. Suppose $w = re^{i\theta} \in \mathbb{C}$ with $r > 0$. Then we note

$$re^{i\theta} = e^{\frac{1}{z}} = e^{\frac{1}{R}e^{-i\phi}} = e^{\frac{1}{R}\cos(\phi)}e^{-i\frac{1}{R}\sin(\phi)}$$

This gives us a set of 2 real valued equations:

$$r' = \ln(r) = \frac{1}{R}\cos(\phi)$$

$$\theta = \frac{1}{R}\sin(\phi) \pmod{2\pi}$$

For any fixed choice of $m \in \mathbb{N}$, and r', we can find a unique $R > 0$ and $\phi \in (-\pi, \pi]$ such that the equations above hold with $\theta + 2\pi m = \frac{1}{R}\sin(\phi)$ (think of them as points on a circle centered at the origin). Therefore there are infinitely many elements in the preimage of any non-zero complex number, as Picard expects.

We can now turn to the function which I would deem best without being holomorphic.

Definition 17.3. $f : \Omega \to \mathbb{C}$ is called **meromorphic** if there exist at most countably many points $z_1, z_2, \dots$ without a limit point such that f is holomorphic for $z \neq z_i$, and f has a pole at z_i.

There is also a natural way to view the idea of being meromorphic on the **extended complex plane**. This is similar to the case where we adjoin ∞ to $\mathbb{R}$ and make it into a circle.

Definition 17.4. We define the extended complex plane, or the **Riemann sphere** to be $\mathbb{C} \cup \{\infty\}$. It is denoted $\mathbb{C}_\infty$.

It 'looks like' a sphere since we can do the stereographic projection to the complex plane for all values $\neq \infty$, which allows us to identify $\mathbb{C} \subseteq \mathbb{C}_\infty$ in a nice geometric way.

If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic for all large values of z, then we can study $F(z) = f(\frac{1}{z})$. This function is now holomorphic in a neighborhood of 0 with an isolated singularity at 0. Therefore we can say that f has a pole, or essential singularity, or a removable singularity (thus is holomorphic) at ∞ if F has those properties at 0.

The following is an excellent classification of the meromorphic functions on $\mathbb{C}_\infty$:

Theorem 17.5. *f is meromorphic on $\mathbb{C}_\infty$ if and only if f is a rational function.*

Proof. It is clear that rational functions are meromorphic (by clearing denominators). So suppose f is meromorphic. Then f must be either holomorphic or have a pole at ∞. In either case, it is holomorphic in a neighborhood of ∞. Therefore, f can only have finitely many poles in the plane since removing a neighborhood of ∞ yields a compact set. Call them $z_1, \dots, z_n$.

For each z_i, we can write

$$f(z) = p_i(z) + f_i(z)$$

where p_i is the principal part of f at z_i and f_i is holomorphic at z_i. Similarly, we can write

$$f\left(\frac{1}{z}\right) = \tilde{p}_\infty(z) + f_\infty(z)$$

Additionally, let $p_\infty(z) = \tilde{p}(\frac{1}{z})$. Combining this information, we assert that

$$H(z) = f(z) - p_\infty(z) - \sum_{i=1}^{n} p_i(z)$$

is a entire and bounded, thus constant by Liouville. Note first that subtracting off the principal part ensures that H has removable singularities at each z_i, so in particular is holomorphic there.

Additionally, subtracting off the principal part in each neighborhood yields that f is bounded in those neighborhoods, since f is continuous on a compact set. Finally, $\mathbb{C}$ without all these neighborhoods is a closed and bounded set, thus compact. As a result, f is everywhere bounded as claimed. $\square$

CLASS 18, OCTOBER 25: THE ARGUMENT PRINCIPLE

Today we will discuss in more depth the idea of the logarithm in the complex world. We call log a 'multivalued function', since it can't be defined unambiguously on all of $\mathbb{C} \setminus \{0\}$ unless we allow ourselves to work $\pmod{2\pi}$ in the imaginary axis. But if it does exist, then it would need to behave as we have mentioned previously:

$$\log(f(z)) = \log|f(z)| + i\arg(f(z))$$

for $f(z)$ living in some particular range. In such a case, we still have

$$\frac{\partial}{\partial z}\log(f(z)) = \frac{f'(z)}{f(z)}$$

As such, the integral of $\frac{f'(z)}{f(z)}$ represents the rate of change of the argument of $f(z)$.

Example 18.1. If $\gamma(t) = e^{it}$ for $t \in [a, b]$, then

$$\int_\gamma \frac{dz}{z} = i(b - a)$$

One can also observe that

$$\log(f_1 f_2) = \log(f_1) + \log(f_2)$$

where the equation makes sense due to the fact that

$$\frac{(f_1 f_2)'}{f_1 f_2} = \frac{f_1' f_2 + f_1 f_2'}{f_1 f_2} = \frac{f_1'}{f_1} + \frac{f_2'}{f_2}$$

Suppose f has a zero of order m at z_0. Then $f(z) = (z - z_0)^m g(z)$ is such that $g(z)$ is non-vanishing near z_0. As a result,

$$\frac{f'(z)}{f(z)} = \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}$$

As a result, if f has a zero of order m, then $\frac{f'(z)}{f(z)}$ also has a simple pole at z_0 with residue m. We can conclude the same but with a negative sign if f has a pole of order m, so we derive the formula

$$\operatorname{ord}_{z_0}(f) = \operatorname{res}_{z_0}\left(\frac{f'}{f}\right)$$

This reasoning yields the following result:

Theorem 18.2 (The Argument Principle)**.** *Suppose that f is a meromorphic function in Ω, and C is a simple positively oriented loop in its interior. If f has no poles nor zeroes on C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = (\#\text{ of zeroes in } C) \;-\; (\#\text{ of poles in } C)$$

where the # is counted with its order (sometimes called multiplicity).

This result is very impressive in the sense that it is very easy to compute integrals of this form. This yields even more impressive corollaries. The first we will tackle is Rouché's theorem, which gives a method to perturb a given holomorphic function without changing the number of zeroes inside a region.

Theorem 18.3 (Rouché's theorem)**.** *Suppose f and g are holomorphic in an open set Ω, and C is a simple positively oriented curve with interior inside Ω. If*

$$|f(z)| > |g(z)| \qquad \forall z \in \mathbb{C}$$

then both f and $f+g$ have the same number of zeroes (counted with multiplicity) in C.

This may not be terribly surprising, but one should notice that we are only asking for the inequality **on** C, not within it!

Proof. We will consider this as a *deformation*. Let

$$f_t(z) = f(z) + tg(z)$$

for $t \in [0,1]$. Let n_t denote the number of zeroes of f_t in C. Our condition also yields that f_t has no zeroes on C, so Theorem 18.2 produces

$$n_t = \frac{1}{2\pi i}\int_C \frac{f_t'(z)}{f_t(z)}dz$$

To prove n_t is constant, it suffices to check that the RHS is a continuous function of t. Note that $\frac{f_t'(z)}{f_t(z)}$ is a jointly continuous function for t and $z \in C$, since the same is true for the numerator and denominator and the denominator is never 0.

As a result, the same is true for the integral since C is compact. Therefore, n_t is a continuous integer valued function of t, but the only way that is possible is if n_t is constant, which shows that $n_0 = n_1$. ☐

Example 18.4. Let's determine the number of zeroes of the polynomial $p(z) = z^7 - 2z^3 + 7$ inside $B(0,2)$. Notice that we can break up $p(z)$ as

$$p(z) = (z^7) + (-2z^3 + 7) = f(z) + g(z)$$

Now notice that $|f(z)| = |z|^7 = 2^7 = 128$, whereas $|g(z)| = |2z^3 - 7| \leq 2|z|^3 + 7 = 2^4 + 7 = 23$. So these two functions satisfy the conditions of Theorem 18.3, and therefore p shares the same number of zeroes as $f = z^7$. This is clearly just a single zero at 0 of order 7.

But by the fundamental theorem of algebra, we know that p has **exactly** 7 zeroes in $\mathbb{C}$, and thus we just showed that all of them are in $B(0,2)$. Further refinements can be made: for example, if we choose $r = 1.5$, then $r^7 > 17$ and $2r^3 + 7 = 13.75$, so really they exist within $B(0,1.5)$.

If you've ever studied Galois theory or more specifically generalizations of the quadratic formula, then you know that we have formulae to find the zeroes up degree 4 polynomials, and beyond that no formula *can* exist. As a result, the actual zeroes of $z^7 - 2z^3 + 7$ are difficult to find. Computer estimates are

$$z \approx -1.443, \quad -.71 - .98i, \quad -.71 + .98i, \quad .213 - 1.39i, \quad .213 + 1.39i$$

The fact that there are 5 distinct roots is not detectable by this method of course, as we count *with* multiplicity.

CLASS 19, OCTOBER 28: ARGUMENTATIVE COROLLARIES

Last time we proved the argument principal using the idea of the logarithm. It is stated as follows:

Theorem. *Suppose that f is a meromorphic function in Ω, and C is a simple positively oriented loop in its interior. If f has no poles nor zeroes on C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = (\#\ \textit{of zeroes in } C)\ -\ (\#\ \textit{of poles in } C)$$

where the # is counted with its order (sometimes called multiplicity).

This allowed us to prove Rouche's theorem, which allows one to perturb holomorphic functions without changing the number of zeroes in a given region. Today we will examine some other very important corollaries of this result.

The first is the open mapping theorem. Thus it would be helpful to know what an open mapping is:

Definition 19.1. Let $\Omega \subseteq \mathbb{C}$ be an open set. Then $f : \Omega \to \mathbb{C}$ is said to be an **open mapping** if whenever $\Omega' \subseteq \Omega$ is an open set, then so is $f(\Omega')$ open.

Notice that this is very different from continuity!

Example 19.2. The constant map $f : \mathbb{C} \to \mathbb{C} : z \to z_0$ is continuous but not open.

Example 19.3. If $f : \Omega \to \mathbb{C}$ is a bijective function, then in fact f is continuous if and only if f^{-1} is an open map.

Constructing maps which are open but not continuous in the complex world (as opposed to the topological world) is a bit more difficult, but a non-constructive example can be constructed as follows:

Example 19.4. Consider an equivalence relation $\sim$ on $\mathbb{C}$ defined by $z \sim z'$ if and only if $Re(z-z'),\ Im(z-z') \in \mathbb{Q}$. Then there are uncountably many equivalences classes in $\mathbb{C}/\sim$. So there exists a bijection p between $\mathbb{C}/\sim$ and $\mathbb{C}$. Now consider the map which sends z to its equivalence class $[z]$, then to $p([z])$. Call it f:

$$f : \mathbb{C} \to \mathbb{C} : z \mapsto p([z])$$

This map has an astounding property:

$$f(B(z, r)) = \mathbb{C}$$

for any $z \in \mathbb{C}$ and $r > 0$. This is because $\mathbb{Q}^2 \subseteq \mathbb{C}$ is dense! As a result it is necessarily open. However, for the same reason it can't be continuous; the preimage of $B(f(z), \epsilon)$ can't contain any neighborhood of z.

This example is pathological, but does demonstrate the assertion that open and continuous do not imply one another. Now we can move onto the open mapping theorem:

Theorem 19.5 (Open mapping theorem)**.** *If f is non-constant and holomorphic on Ω, then f is an open map on Ω.*

Proof. Consider $\Lambda = f(B(z_0, r))$ and suppose $f(z_0) = w_0$. It goes to demonstrate the existence of $\epsilon > 0$ such that $B(w_0, \epsilon) \subseteq \Lambda$.

Consider $g(z) = g_w(z) = f(z) - w$ and write

$$g(z) = (f(z) - w_0) + (w_0 - w) = F(z) + G(z)$$

Choose $0 < \delta < r$ such that $f(z) \neq w_0$ for $|z - z_0| = \delta$. Choose $\epsilon > 0$ such that $|f(z) - w_0| > \epsilon$ on this circle (by compactness). If $|w - w_0| < \epsilon$, we get that $|F(z)| > |G(z)|$ on this circle. Rouche's theorem ensures that $F(z)$ has the same number of zeroes as $g(z)$. But this implies $g(z)$ has a zero, since we assumed $F(z)$ has one. This shows the assertion. □

Another wonderful corollary is the maximum modulus principle, which states that maxima for holomorphic functions can only exist on the boundary.

Theorem 19.6 (Maximum Modulus Principle)**.** *If f is non-constant and holomorphic in Ω an open set, then f cannot attain its maximum in Ω.*

Proof. Suppose $z_0 \in \Omega$ was a maximum for f. Then let $B(z_0, r) \subseteq \Omega$. In this case, we know $f(B(z_0, r))$ is an open set containing $f(z_0)$. But this implies that larger values for f exist! A direct contradiction to our assumptions on z_0. □

Rephrasing this realization a bit, we get the following:

Corollary 19.7. *If Ω is an open set with compact closure, and if f is holomorphic on Ω and continuous on $\bar{\Omega}$, then*

$$\sup_{z \in \Omega}(f(z)) \leq \sup_{z \in \bar{\Omega} \setminus \Omega}(f(z))$$

Proof. Since $\bar{\Omega}$ is compact and f is continuous, f attains its maximum on $\bar{\Omega}$. But Theorem 19.6 informs us that no point in the interior can be maximal. □

Example 19.8. It should be noted here that the compactness assumption is necessary to our claim. Otherwise, you could consider something like the upper half plane and the function $\sin(z)$. On the boundary, the real line, the function is bounded. However, in the upper half plane the values of $\sin(z)$ can be chosen arbitrarily large.

Example 19.9. A special case of Theorem 19.6 is the story for analytic functions at the origin with radius of convergence $R > 1$. If we consider

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

we know that on $\bar{B}(0, 1)$, f must attain its maximum for some $|z| = 1$. This can be interpreted as saying we can solve a system of linear equations to make the first N terms have near the same argument (or be 0) and thus to make f as large as possible, the absolute value of z with such argument should also be maximized.

To specialize even further, we can consider $f(z) = \cos(\frac{\pi}{2} z)$. If we consider only real values, we know f is maximized at $z = 0$ where it have value 1. at ± 1, the function is 0. So Theorem 19.6 implies there is $|z| = 1$ such that $|f(z)| > 1$. It is easy to check that $z = \pm i$ works:

$$|\cos(\pm \frac{\pi}{2} i)| = \frac{1}{2}\left(e^{\frac{\pi}{2}} - e^{-\frac{\pi}{2}}\right) > 2.3$$

CLASS 20, OCTOBER 30: HOMOTOPY

Today we will study the idea of a homotopy, a feature that permeates geometry, topology, and even abstract algebra at the highest level. We already saw an example of the idea of homotopy in our proof of Riemann's theorem on removable singularities (Lemma 16.3), when we study how an integral *deforms*.

Definition 20.1. Let $\gamma_0, \gamma_1 : [a, b] \to \Omega$ be two curves in Ω an open set that share common endpoints α, β (representing the start and finish). Then $\gamma_0 \simeq \gamma_1$ are said to be **homotopic** in Ω if there exists a continuous function $F : [a, b] \times [0, 1] \to \Omega$ such that

$$F(t, 0) = \gamma_1(t) \qquad\qquad F(t, 1) = \gamma_1(t)$$

$$F(a, s) = \alpha \qquad\qquad F(b, s) = \beta$$

F is called a **homotopy** connecting γ_0 to γ_1.

It is convenient to think of this geometrically: for a fixed value of s, $\gamma_s(t) = F(t, s)$ is a curve with endpoints α, β. Therefore, as s varies from 0 to 1, we deform γ_0 to γ_1. A nice feature of homotopic curves is that their integrals of holomorphic functions agree!

Theorem 20.2. *If f is holomorphic in Ω, and $\gamma_1 \simeq \gamma_2$, then*

$$\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$$

Proof. Our proof will rely on the fact that the statement is true locally, i.e. if the curves are sufficiently close to one another then the theorem holds.

Let K be the image of F. Since the image of a compact set is compact, K is compact. Therefore, there exists $\epsilon > 0$ such that $B(z, 3\epsilon) \subseteq \Omega$ for each $z \in K$. Now, since F is continuous on a compact set, it is uniformly continuous. Thus we can select δ with

$$\sup_{t\in[a,b]} |\gamma_{s_1}(t) - \gamma_{s_2}(t)| < \epsilon \qquad\qquad \text{if } |s_1 - s_2| < \delta$$

Fixing such a choice of s_1, s_2, we can choose discs $D_0, \dots, D_n$ of radius 2ϵ and consecutive points z_i, w_i on $\gamma_{s_1}, \gamma_{s_2}$ respectively for $i = 0, \dots, n + 1$ with

$$z_i, z_{i+1}, w_i, w_{i+1} \in D_i$$

Further assume $z_0 = w_0 = \alpha$ and $z_{n+1}, w_{n+1} = \beta$ so our points go from start to finish. By Cauchy's theorem, we have that on each disc D_i we have a primative for f, say F_i. On adjacent overlaps, we get that $F_{i+1} - F_i = C_i$ is a constant (since both are primatives of the same function). This implies centrally that

$$F_{i+1}(z_{i+1}) - F_i(z_{i+1}) = F_{i+1}(w_{i+1}) - F_i(w_{i+1})$$

$$F_{i+1}(z_{i+1}) - F_{i+1}(w_{i+1}) = F_i(z_{i+1}) - F_i(w_{i+1})$$

As a result, we have that

$$\begin{aligned}\int_{\gamma_{s_1}} f - \int_{\gamma_{s_2}} f &= \sum_{i=0}^{n} [F_{i+1}(z_{i+1}) - F_i(z_i)] - \sum_{i=0}^{n} [F_{i+1}(w_{i+1}) - F_i(w_i))] \\ &= \sum_{i=0}^{n} [F_i(z_{i+1}) - F_i(w_{i+1}) - (F_i(z_i) - F_i(w_i)))]\end{aligned}$$

Canceling nearby signed terms leaves us with

$$\int_{\gamma_{s_1}} f - \int_{\gamma_{s_2}} f = F_{n+1}(z_{n+1}) - F_{n+1}(w_{n+1}) - (F_0(z_0) - F_i(w_0)) = 0$$

Since $z_{n+1} = w_{n+1}$ and $z_0 = w_0$. This shows the equality. Subdividing the whole interval $[0,1]$ into $\frac{\delta}{2}$-sized pieces, we can prove that

$$\int_{\gamma_0} f = \int_{\gamma_1} f$$

□

An important type of domain is one in which all curves starting and ending at the same point are homotopic. This notion is called a **simply connected** domain, and should be thought of as the next order of (path) connectedness. In a path connected space, all points can be connected by a path. In a simply connected domain, all paths (with the same starting and ending points) can be connected by a path of paths.

Example 20.3. A disc $\bar{B}(z_0, r)$ is an example of a simply connected domain. Indeed, we can transform one curve to another by the straight line homotopy:

$$F : [a,b] \times [0,1] \to \bar{B}(z_0, r) : (s,t) \mapsto t\gamma_0(s) + (1-t)\gamma_1(s)$$

This in fact works for any convex domain!

Example 20.4. A non-convex domain that is still simply connected is $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Remember that this is where we were for example able to define $\log(z) = \log(re^{i\theta}) = \log(r) + i\theta$.

This can be seen by writing the curves in polar coordinates: $\gamma_i(s) = r_i(s)e^{i\theta_i(s)}$. Now again, we can apply the straight line contours to r_i and θ_i again:

$$F : [a,b] \times [0,1] \to \bar{B}(z_0, r) : (s,t) \mapsto (tr_0(s) + (1-t)r_1(s))\, e^{i(t\theta_0(s) + (1-t)\theta_1(s))}$$

Noticing that since $r_i > 0$ and $\theta_i \in (-\pi, \pi)$, so are they for each $t \in [0,1]$.

Example 20.5. For an example of a non-simply connected domain, consider $\bar{B}(0,1) \setminus \{0\}$. If we consider, for example, the constant path at 1, and the path $\gamma(s) = e^{2\pi i s}$ for $s \in (0,1)$. Intuitively, there is certainly no way to deform one loop to the other, as looping is always undone. Formally, this is provable by virtue of the fact that

$$\int_1 \frac{1}{z} dz = 0 \neq \int_C \frac{1}{z} dz = 2\pi i$$

Corollary 20.6. *Any holomorphic function $f : \Omega \to \mathbb{C}$ in a simply connected domain Ω has a primitive given by*

$$F(w) = \int_{\gamma_w} f(z) dz$$

where γ_w is (any) path from some fixed point z_0 to w.

CLASS 21, NOVEMBER 1: THE COMPLEX LOGARITHM

We now return to a discussion about what a complex logarithm could look like. It is still defined to be an inverse to the exponential function (on some domain) and thus must satisfy

$$\log(z) = \log(re^{i\theta}) = \log(r) + i\theta$$

for θ in a fixed range. As we discovered on the homework, this is holomorphic if $\theta \in (-\pi, \pi)$ and $r > 0$, but can't be even continuous if we attempt to extend it any further.

We can somewhat remedy this; as θ varies only a little, we can define the logarithm around θ coherently. So we could define log locally, patch the local definitions together, and yield useful information.

Example 21.1. Let C be a curve winding m times counterclockwise around the origin. Then we can write

$$\int_C \frac{dz}{z} = \sum_{j=1}^{2m} \int_{C_j} \frac{dz}{z}$$

where C_j is the j^{th} piece of the curve with angle change π. Thus we can rewrite

$$\sum_{j=1}^{2m} [\log_j(z)]_{r_{j-1}e^{i\pi j}}^{r_j e^{i\pi j}} = \sum_{j=1}^{2m} (\log(r_j) - \log(r_{j-1})) + i(\pi j - \pi(j-1)) = \sum_{j=1}^{2m} i\pi = 2\pi i m$$

where $\log_j$ has a branch on $\theta = \frac{3\pi}{2}$ if j is odd, and $\theta = \frac{\pi}{2}$ if j is even. The same formula holds if $m < 0$

This yields the following ubiquitous definition:

Definition 21.2. If C is a curve with $z_0 \notin C$, then the **winding number** of C about z_0 is

$$m_C = \frac{1}{2\pi i} \int_C \frac{dz}{z - z_0}$$

An additional phrase also came up, the idea of a **branch** for log. This is precisely a choice of $\phi \in \mathbb{R}$ such that

$$\log(re^{i\theta}) = \log(r) + i\theta$$

for all $\theta \in (\phi - 2\pi, \phi)$ and $r > 0$. Thus the log we discussed previously has a branch about $\phi = \pi$. This is called **the principal branch** of the logarithm. We will call the logarithm with such a branch $\log_\phi$ as a shorthand.

Note not every logarithm has a branch. You can produce open sets for which no such value is possible. In general however, we have the following theorem:

Theorem 21.3. *If Ω is a simply connected open set, with $1 \in \Omega$ and $0 \notin \Omega$, then there exists F such that F is holomorphic in Ω, $e^F = z$, and $F(r) = \log(r)$ for every $r \in \mathbb{R} \cap \Omega$ nearby 1.*

Rephrased geometrically, this is saying simply connected domains must have a ray pointed out from the origin within their complement.

Proof. We will construct our logarithm according to the fact that it should be an antiderivative for $f(z) = \frac{1}{z}$. Since $0 \notin \Omega$, f is holomorphic in Ω. So we may define

$$F(z) = \int_\gamma \frac{dw}{w}$$

where γ is a path connecting 1 to z. By simple connectedness, this is well defined (in the sense that any γ will yield the same result since they are automatically homotopic). This yields that F is a primative, and thus is necessarily holomorphic.

Now we need to show $e^{F(z)} = z$. This is equivalent to $1 = ze^{-F(z)}$. We do this by differentiating:

$$\frac{\partial}{\partial z} ze^{-F(z)} = e^{-F(z)} - ze^{-F(z)}F'(z) = (1 - F'(z)z)e^{-F(z)} = 0$$

Therefore $ze^{-F(z)}$ is constant. So it is enough to plug in a single value: $z = 1$. This yields $C = 1 \cdot e^{-F(1)} = 1 \cdot e^0 = 1$. This was our intention.

Finally, for z close enough to 1, we can ensure that γ be chosen on the real line. In this case, it is just the standard log from calculus! $\square$

Example 21.4. One needs to be quite careful with some expected results about log (taking for example $\log_0$). For example, it is not true in general that

$$\log(z \cdot w) = \log(z) + \log(w)$$

for two complex numbers z, w avoiding some particular branch. Indeed, if we consider

$$\log(e^{\frac{2\pi}{3}} \cdot e^{\frac{2\pi}{3}})) = \log(e^{\frac{4\pi}{3}}) = \log(e^{\frac{-2\pi}{3}}) = -\frac{2\pi}{3} \neq \frac{4\pi}{3}$$

The same goes for any $\log_\phi$ and even F as in Theorem 21.3.

Note that if we do the taylor series expansion of the principal branch of the logarithm shifted about 1, we get the expected formula:

$$\log(z+1) = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{z^n}{n}$$

which naturally has radius of convergence 1.

Additionally, we can expand our usual definitions of z^n to z^α for any $\alpha \in \mathbb{R}$! If Ω is as in Theorem 21.3, we can take $\log = F$ in the theorem and write

$$z^\alpha = e^{\alpha \log(z)}$$

We get automatically that $1^\alpha = 1$ and if $\alpha = \frac{1}{n}$, then

$$(z^{\frac{1}{n}})^n = (e^{\frac{1}{n}\log(z)})^n = z$$

This extends even more broadly by the following theorem we will prove next time:

Theorem 21.5. *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists another holomorphic function $g(z)$ such that*

$$f(z) = e^{g(z)}$$

CLASS 22, NOVEMBER 4TH: FOURIER SERIES

Today we will finish our discussion of the logarithm and move onto a study of Fourier Series (preceding the chapter on the Fourier transform).

Recall that last time we ended with the following unproven theorem:

Theorem. *If f is a nowhere vanishing holomorphic function in a simply connected region Ω, then there exists another holomorphic function $g(z)$ such that*

$$f(z) = e^{g(z)}$$

Proof. Fix z_0 in Ω, and define

$$g(z) = \int_\gamma \frac{f'(w)}{f(w)} dw + c_0$$

where γ is a path connecting z_0 to z, and $c_0 \in \mathbb{C}$ is such that $e^{c_0} = f(z_0)$. Immediately it should be stated that this is well defined due to simple connectedness. As expected, g is a primative for $\frac{f'(z)}{f(z)}$. Moreover,

$$\frac{\partial}{\partial z}\left[f(z)e^{-g(z)}\right] = f'(z)e^{-g(z)} - f(z)e^{-g(z)}g'(z) = 0$$

Thus the function itself is constant. Checking the equation at z_0 shows the desired result. $\square$

This gives an interesting presentation of f for any f satisfying the assumptions above. We now switch gears to Fourier Series.

Let f be holomorphic on $B(z_0, R)$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be its power series expansion.

Theorem 22.1. *The coefficients of f are defined by*

$$a_n = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta$$

for all $n \geq 0$ and any $0 < r < R$. Additionally,

$$\frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta = 0$$

for any $n < 0$.

Proof. We already have that $a_n = \frac{f^{(n)}(z_0)}{n!}$ for $n \geq 0$. Applying Cauchy's integral theorem now yields

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{1}{2\pi i} \int_C \frac{f(z_0 + re^{i\theta})}{r^{n+1} e^{i(n+1)\theta}} re^{i\theta} d\theta = \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta$$

Now, it goes to show the statement for $n < 0$. But this is an integral of a holomorphic function! So it is 0 by Goursat. $\square$

The last part of Theorem 22.1 gives the idea that this calculation could also work for meromorphic functions with mild modification. In particular, if g has a pole of order m at z_0, then

$$g(z) = \frac{b_{-m}}{(z-z_0)^m} + \ldots + \frac{b_{-1}}{(z-z_0)} + h(z)$$

where $h(z)$ is a holomorphic function. Therefore, we can consider $f(z) = (z-z_0)^m g(z)$, which is a nice holomorphic function. Writing $f(z) = \sum_{n=0} a_n (z-z_0)^n$ and applying Theorem 22.1, we would produce

$$\begin{aligned} b_{n-m} = a_n &= \frac{1}{2\pi r^n} \int_0^{2\pi} f(z_0 + re^{i\theta}) e^{-in\theta} d\theta \\ &= \frac{1}{2\pi r^n} \int_0^{2\pi} g(z_0 + re^{i\theta}) r^m e^{im\theta} e^{-in\theta} d\theta \\ &= \frac{1}{2\pi r^{n-m}} \int_0^{2\pi} g(z_0 + re^{i\theta}) e^{-i(n-m)\theta} d\theta \end{aligned}$$

for $n \geq 0$. Substituting $n-m$ with n yields that

$$b_n = \frac{1}{2\pi r^n} \int_0^{2\pi} g(z_0 + re^{i\theta}) e^{-in\theta} d\theta$$

for all $n \geq -m$, which naturally generalizes the previous result.

2 other neat corollaries of the result which might go under the radar are the following:

Theorem 22.2 (Mean Value Property)**.** *If f is holomorphic on $B(z_0, R)$, then*

$$f(z_0) = \frac{1}{2\pi} \int_C f(z) dz = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) d\theta$$

If we consider this statement for its real and imaginary parts, we conclude the following:

Corollary 22.3. *If $f = u + iv$ is holomorphic on $B(z_0, R)$, then*

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) d\theta$$

This property holds for any harmonic function u.[1] This can be deduced from the fact that every harmonic function is the real part of some holomorphic function. This is exercise 12 in chapter 2, and goes as follows: consider $2\frac{\partial u(w)}{\partial z}$. Then consider $f(z) = \int_\gamma 2\frac{\partial u(w)}{\partial w} dw$, where γ is a path connecting 0 to z. Then $f'(z) = 2\frac{\partial u(w)}{\partial w}$. This is always true for a holomorphic function by our analysis of the CR equations.

[1] Recall from an ancient homework that u is harmonic if and only if $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

CLASS 23, NOVEMBER 6TH: FOURIER TRANSFORMS INTRO

Recall previously that we have discussed the idea of the Fourier transform of a function f:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi} dx$$

An important variation on this formula is the **Fourier Inversion Formula**:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{-2\pi i x\xi} d\xi$$

Since we are performing an integral over the whole real line, there is a question of convergence. We will work to establish a sufficient condition, called moderate descent. But since we work in the complex world, we will attempt to analytically continue a real valued function with this property to a strip around the real line.

Definition 23.1. A real valued function $f : \mathbb{R} \to \mathbb{R}$ has **moderate descent** if

$$|f(x)|, |\hat{f}(x)| \leq \frac{A}{1+x^2}$$

for some constant A.

This makes it so that the integrals in question in the above formulas are well define, since

$$\left|\hat{f}(\xi)\right| = \left|\int_{-\infty}^{\infty} f(x)e^{-2\pi i x\xi} dx\right| \leq \int_{-\infty}^{\infty} \left|f(x)e^{-2\pi i x\xi}\right| dx \leq \int_{-\infty}^{\infty} \frac{A}{1+x^2} dx = A\pi$$

Using the fact that the last integral has primative $A\tan^{-1}(x)$. The same goes for the inversion formula.

Now, since we work in the complex plane, we will work to define a class of functions $\mathcal{F}$ for which the same results hold (inversion formula, Poisson summation, etc). We intend to define it analogously to that of moderate descent. But our class will be larger, opening a wider swath of applications.

For each $a > 0$, define $\mathcal{F}_a \subseteq \{f : \mathbb{C} \to \mathbb{C}\}$ satisfying the following properties:

(1) f is holomorphic in the horizontal strip $S_a = \{z = x + iy \mid |y| < a\}$.
(2) There exists $A > 0$ such that for $z = x + iy \in S_a$,

$$f(x+iy) \leq \frac{A}{1+x^2}$$

This is to say $\mathcal{F}_a$ is the set of functions satisfying moderate descent on fixed real lines of S_a uniformly.

Example 23.2. $f(z) = e^{-\pi z^2}$ has $f \in S_a$ for each $a > 0$. This is because

$$|f(z)| = e^{-\pi x^2 + \pi y^2}$$

and since y is bounded, f decays exponentially with f. Note that previously we showed that this function is its own Fourier Transform!

Similarly, $f(z) = \frac{1}{\pi}\frac{c}{c^2+z^2}$ is in $\mathcal{F}_a$ for each $a < c$ (to avoid the poles at $z = \pm ic$). Away from these poles, we get

$$|f(z)| = \left|\frac{1}{\pi}\frac{c}{c^2+z^2}\right| = \frac{c}{\pi}\left|\frac{1}{c^2+z^2}\right|$$

For $x > 2c$, this function experiences the desired rate of decay.

Lastly, we have also shown $\frac{1}{\cosh(\pi z)}$ is its own Fourier transform. It can be shown to be in $\mathcal{F}_a$ for each $a < \frac{1}{2}$.

A nice application of Cauchy's Integral theorem is that if $f \in \mathcal{F}_a$, then so is $f^{(n)}$ for each $n \geq 0$. Indeed, consider

$$\left|f^{(n)}(z)\right| \leq \frac{n!\|f\|_C}{R^n} \leq \frac{n!}{R^n} \sup_{w=x+iy\in C}(\frac{A}{1+x^2})$$

$f^{(n)}(z)$ will be bounded near $x = 0$, and for the rest we can choose a constant uniformly.

NOTE: We can allow more functions into $\mathcal{F}_a$ without change if we define moderate decrease by $|f(z)| \leq \frac{C}{1+|x|^{1+\epsilon}}$ for some $\epsilon > 0$.

Definition 23.3. We define $\mathcal{F}$ to be the set of functions f that are in some $\mathcal{F}_a$:

$$\mathcal{F} = \bigcup_{a>0} \mathcal{F}_a$$

Theorem 23.4. *If $f \in \mathcal{F}_a \subseteq \mathcal{F}$, then $|\hat{f}(\xi)| \leq Be^{-2\pi b|\xi|}|$ for all $0 \leq b < a$.*

This result says that if f has moderate decay, then $\hat{f}$ has exponential decay. At the very least, this allows us to say that $\hat{f} \in \mathcal{F}$ given $f \in \mathcal{F}$. But it is a far stronger result than this.

Proof. For $b = 0$, we have that $\hat{f}$ is bounded. This is immediate given its definition. So suppose $0 < b < a$. Begin with the case $\xi > 0$. The idea is to shift the integral down to the line where the imaginary part is $-b$ using contour integration along the rectangle $\mathcal{R} = [-R, R, R - ib, -R - ib]$. Note that the vertical sides of this rectangle have the property that

$$\left|\int_{R-ib}^{R} f(z)e^{-2\pi iz\xi}dz\right| \leq \int_0^b \left|f(R-it)e^{-2\pi i(R-it)}\right| dt \leq \int_0^b \left|\frac{A}{R^2}e^{-2\pi i(R-it)}\right| dt \leq \frac{C}{R^2} \to 0$$

as $R \to \infty$. Therefore, by Cauchy/Goursat, we have that

$$\tilde{f}(\xi) = \int_{-\infty}^{\infty} f(x-ib)e^{-2\pi i(x-ib)\xi}dx$$

and thus

$$|\tilde{f}(\xi)| \leq \left|\int_{-\infty}^{\infty} \frac{A}{1+x^2}e^{-2\pi b\xi}dx\right| \leq A\pi e^{-2\pi b\xi}$$

A very similar argument works for $\xi < 0$, but instead you need to consider the rectangle $\mathcal{R} = [-R, R, R + ib, -R + ib]$. □

This at the very least ensures that Fourier Inversion makes sense. Next time we will show that it *is* an inversion formula.

CLASS 24, NOVEMBER 8TH: FOURIER INVERSION

Last time we proved that the Fourier transform of a function of moderate descent has quite steep descent. This allows us to apply an inversion formula to calculate the inversion formula:

Theorem 24.1 (Fourier Inversion Formula)**.** *If $f \in \mathcal{F}$, then*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi$$

The proof will use Contour integration, much like Theorem 23.4 from last class. In addition, we need the following Lemma:

Lemma 24.2. *If $a, b \in \mathbb{R}$ and $a > 0$, then*

$$\int_0^{\infty} e^{-(a+ib)\xi}d\xi = \frac{1}{a+ib}$$

Proof. Indeed, this is calculus!

$$\int_0^{\infty} e^{-(a+ib)\xi}d\xi = \frac{1}{a+ib}\int_{-\infty}^{b} e^u du = \frac{1}{a+ib}\left[e^u\right]_{u=-\infty}^{0} = \frac{1}{a+ib}$$

□

Now we can proceed to the proof of Theorem 24.1.

Proof. Let $f \in \mathcal{F}_a$ and write

$$\int_{-\infty}^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi = \int_{-\infty}^{0} \hat{f}(\xi)e^{2\pi i x\xi}d\xi + \int_0^{\infty} \hat{f}(\xi)e^{2\pi i x\xi}d\xi =: (-) + (+)$$

to separate into cases based on sign. For $(+)$, let $0 < b < a$. As in the proof from last time, we may conclude that

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(u - ib)e^{-2\pi i(u-ib)\xi}du$$

This yields a computation:

$$\begin{aligned}
(+) &= \int_0^{\infty}\int_{-\infty}^{\infty} f(u-ib)e^{-2\pi i(u-ib)\xi}e^{2\pi i x\xi}du d\xi \\
&= \int_{-\infty}^{\infty} f(u-ib)\int_0^{\infty} e^{-2\pi i(u-ib)\xi}e^{2\pi i x\xi}d\xi du \\
&= \int_{-\infty}^{\infty} f(u-ib)\frac{1}{2\pi b + 2\pi i(u-x)}du \\
&= \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(u-ib)}{u-ib-x}du \\
&= \frac{1}{2\pi i}\int_{-\infty-ib}^{\infty-ib} \frac{f(\zeta)}{\zeta - x}d\zeta
\end{aligned}$$

Going through this set of equations requires some explanation. The first line is the definition. The second equality is through a process of flipping the 2 integrals. In general, this is a delicate process. But since in this case we have that the integrals of the absolute values converge, it is easy to ensure using only the finite case. One should see Fubini's theorem for any further clarification.

The 3rd equality is Lemma 24.2. The forth is simply rearranging terms. The final is using the substitution $\zeta = u - ib$, $d\zeta = du$. The integral is stated more precisely as the integral along the straight line path with imaginary part b.

A similar computation works for $(-)$ and shows

$$(-) = -\frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{f(\zeta)}{\zeta - x}d\zeta$$

Now we can consider the rectangle $\mathcal{R}$ of height 2b centered at the origin. Cauchy's integral theorem tells us that

$$\frac{1}{2\pi i}\int_{\mathcal{R}}\frac{f(\zeta)}{\zeta - x}d\zeta = f(x)$$

Just like in the proof of the Theorem 23.4, the integrals over the vertical sides of this rectangle approach 0. Thus we are left with

$$\begin{aligned} f(x) &= -\frac{1}{2\pi i}\int_{-\infty+ib}^{\infty+ib}\frac{f(\zeta)}{\zeta - x}d\zeta + \frac{1}{2\pi i}\int_{-\infty-ib}^{\infty-ib}\frac{f(\zeta)}{\zeta - x}d\zeta \\ &= \int_{-\infty}^{0}\hat{f}(\xi)e^{2\pi i x\xi}d\xi + \int_{0}^{\infty}\hat{f}(\xi)e^{2\pi i x\xi}d\xi = \int_{-\infty}^{\infty}\hat{f}(\xi)e^{2\pi i x\xi}d\xi \end{aligned}$$

As an application of this analysis, we get some neat formulas involving all of the Fourier $\square$ transforms we have computed so far. An additional application which is very common in signal processing is the following example:

Example 24.3. Define the square function by

$$sq(x) = \begin{cases} 1 & |x| \leq \frac{1}{2} \\ 0 & |x| > \frac{1}{2} \end{cases}$$

This function looks like a flat line seperated by 2 big jumps. Let's compute its Fourier transform:

$$\begin{aligned} \hat{sq}(\xi) &= \int_{-\infty}^{\infty} sq(x)e^{-2\pi i\xi x}dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i\xi x}dx \\ &= -\frac{1}{2\pi i\xi}[e^{-\pi i\xi} - e^{\pi i\xi}] = \frac{1}{\pi\xi}\frac{e^{-\pi i\xi} - e^{\pi i\xi}}{2i} = \frac{\sin(\pi\xi)}{\pi\xi} \end{aligned}$$

So the Fourier transform of the square function is a sin-wave.

An interesting application used for the past century is then the fact that a sin-wave has an inverse Fourier transform of one of these square functions. If we went through this analysis with a larger magnitude, that would pull out from the integral. Similarly, if we change the periodicity, that would be an adjusted square function (a rectangle). In particular, if $\hat{f} = M\frac{\sin(f\xi)}{x}$, then

$$f(x) = \begin{cases} \pi M & |x| \leq \frac{f}{2\pi} \\ 0 & |x| > \frac{f}{2\pi} \end{cases}$$

CLASS 25, NOVEMBER 11TH: POISSON SUMMATION FORMULA

Last time we proved that the Fourier inversion holds for functions in the class $\mathcal{F}$. This was done through the method of contour integration, which as we know is a powerful technique that we've built up for the entire semester. Today we will again institute these methods to prove the wonderful Poisson summation formula.

Theorem 0.1 (Poisson Summation Formula)**.** *If $f \in \mathcal{F}$, then*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

Proof. First note that by our previous estimates and assumptions, both sums converge! Suppose $f \in \mathcal{F}_a$. We can consider the function $g(z)\frac{f(z)}{e^{2\pi iz}-1}$, which has simple poles at the integers with residues $\frac{f(n)}{2\pi i}$. If $0 < b < a$, we can consider the rectangle R_N of height $2b$ and of length $2N+1$ centered at the origin. Note that this encompasses the poles $-N, -N+1, \ldots, N$. Thus we have

$$\sum_{n=-N}^{N} f(n) = \int_{R_N} g(z)dz$$

Sending N off to infinity, we get

$$\sum_{n\in\mathbb{Z}} f(n) = \int_{-\infty-ib}^{\infty-ib} g(z)dz - \int_{-\infty+ib}^{\infty+ib} g(z)dz$$

Now we will use the identity that if $w > 1$, we have

$$\frac{1}{w-1} = \frac{\frac{1}{w}}{1-\frac{1}{w}} = w^{-1}\sum_{n=0}^{\infty} w^{-n}$$

Applying this to $\frac{1}{e^{2\pi iz}-1}$ in the first integral, we produce

$$\int_{-\infty-ib}^{\infty-ib} g(z)dz = \int_{-\infty-ib}^{\infty-ib} f(z)e^{-2\pi iz}\sum_{n=0}^{\infty} e^{-2\pi inz}dz$$

Similarly, using the more standard equality for the second integral produces:

$$\int_{-\infty+ib}^{\infty+ib} g(z)dz = -\int_{-\infty+ib}^{\infty+ib} f(z)\sum_{n=0}^{\infty} e^{2\pi inz}dz$$

So in total, we have

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} f(n) &= \int_{-\infty-ib}^{\infty-ib} f(z)e^{-2\pi iz}\sum_{n=0}^{\infty} e^{-2\pi inz}dz + \int_{-\infty+ib}^{\infty+ib} f(z)\sum_{n=0}^{\infty} e^{2\pi inz}dz \\
&= \sum_{n=0}^{\infty}\int_{-\infty-ib}^{\infty-ib} f(z)e^{-2\pi iz}e^{-2\pi inz}dz + \sum_{n=0}^{\infty}\int_{-\infty+ib}^{\infty+ib} f(z)e^{2\pi inz}dz \\
&= \sum_{n=0}^{\infty}\int_{-\infty-ib}^{\infty-ib} f(z)e^{-2\pi iz}e^{-2\pi inz}dz + \sum_{n=0}^{\infty}\int_{-\infty+ib}^{\infty+ib} f(z)e^{2\pi inz}dz \\
&= \sum_{n=0}^{\infty}\int_{-\infty-ib}^{\infty-ib} f(z)e^{-2\pi i(n+1)z}dz + \sum_{n=0}^{\infty}\int_{-\infty+ib}^{\infty+ib} f(z)e^{2\pi inz}dz \\
&= \sum_{n=-\infty}^{-1}\int_{-\infty-ib}^{\infty-ib} f(z)e^{2\pi inz}dz + \sum_{n=0}^{\infty}\int_{-\infty+ib}^{\infty+ib} f(z)e^{2\pi inz}dz \\
&= \sum_{n=-\infty}^{\infty} \hat{f}(n)
\end{aligned}$$

□