CLASS 24, FRIDAY APRIL 27: FROBENIUS SPLITTINGS

Last time we finished the proof of the fact that a local ring R is regular in characteristic $p > 0$ if and only if F_*R (and thus F^e_*R) is a flat R-module. The Frobenius being able to detect local properties of a ring is extremely valuable in general in characteristic $p > 0$, and has no (or only using very modern methods) analog in characteristic 0 or mixed characteristic. Throughout I assume all rings are F-finite.

Writing this equationally, we have that R is regular local if and only if

$$F_*R \cong R^n \quad \text{for some } n \geq 1$$

So the quickly question becomes how can we allow ourselves some singularities (irregularities) without letting in massively bad ones? An immediate and profitable conclusion would be to weaken the free hypothesis. Maybe F_*R is not entirely composed of Rs, but simply contains one copy of R:

$$F_*R \cong R \oplus M$$

This gives us the following definition:

Definition 0.1. An F-finite ring R of positive characteristic is called F**-split** if and only if there exists an R-module M such that

$$F_*R \cong R \oplus M$$

As we will see a little later, a ring is F-split implies some very nice geometric statements about the ring. First, as with projective and injective modules, I would like to give some equivalent characterizations.

Proposition 0.2. *Let R be a ring of positive characteristic. Then TFAE:*

(1) R is F-split.
(2) $F^e_*R \cong R \oplus M_e$ for **all** *$e > 0$.*
(3) $F^e_*R \cong R \oplus M_e$ for **some** *$e > 0$.*
(4) There exists $\phi : F^e_*R \to R : F_*1 \mapsto 1$ for some (all) $e > 0$.
(5) There exists a surjective R-homomorphism $F^e_*R \to R$ for some (or all) $e > 0$.

Proof. (1) implies (2): Suppose that R is F split. Then note that we can proceed by induction on e. The base case, $e = 1$, is exactly given by (1). For the induction step:

$$F^{e+1}_*R \cong F_*(F^e_*R) \cong F_*(R \oplus M_e) \cong F_*R \oplus F_*M_e \cong R \oplus M \oplus F_*M_e$$

(2) implies (3): Obvious.

(3) implies (4): For the some portion, this is exactly the equivalent definitions of split exact sequences. To convince yourself of the **all** given **some** assumption, if it holds for $F^e_*R \to R$, then for $e' < e$, we have $F^{e'}_*R \to F^e_*R$ induced by the Frobenius, for which clearly $1 \mapsto 1$, and then we can compose with the given map. If $e' > e$, then there exists $m > 1$ such that $me > e'$, then we can use $F^{me}_*\phi : F^{me}_*R \to F^{(m-1)e}_*R$ and induction to conclude that there exists a splitting for $F^{e'}_*R \to R$.

(4) implies (5): Use ϕ.

(5) implies (1): Suppose that there is a surjective map $\phi : F_*R \to R$. I claim we can modify this map to send 1 to 1. Indeed, suppose $\phi(F_*r) = 1$. Then We can consider $\varphi(F_*s) = \phi(F_*(sr))$, obtained by premultiplying by F_*r. This map clearly has the desired property. □

Next up, like regularity, we can show that F-split is a local property:

Proposition 0.3. *R is an F-split ring if and only if $R_{\mathfrak{m}}$ is F-split for all maximal ideals $\mathfrak{m}$.*

Proof. See homework 6. □

As a corollary of this fact, we can say that regular rings are in fact F-split, because we can check the condition locally and use the above observation.

Proposition 0.4. *A ring R that is F-split is also reduced.*

Proof. If $r^n = 0$, then $r^{p^e} = 0$ for some $e > 0$ (for example, $e = n$). But by characterization (4) of Proposition 0.2, we have that there is a a map $\phi : F_*R \to R$ sending 1 to 1. But this implies

$$R \to F_*R \to R : r \mapsto rF^e_*1 = F_*r^{p^e} \mapsto r$$

But the middle term is 0, so $r = 0$. Thus R is reduced. □

Example 0.5. Consider the ring $R = K[x_1, \dots, x_n]/\langle x_1 \cdots x_n\rangle$ where K is a perfect field. According to our analysis so far, this has a chance to be F-split (since it is reduced). Let's see if we can prove this is the case.

It is natural to check that a generating set for F_*R as an R-module is

$$\langle F_*x_1^{i_1} \cdots \hat{x_j} \cdots x_n^{i_n} \mid 0 \le i_k < p\rangle_R$$

where the 'hat' indicates remove this variable from the product (to avoid producing 0). This is not a basis, since R is not regular: $\dim(R) = n-1$, but $\langle x_1, \dots, x_n\rangle$ is a minimal generating set for the maximal ideal. In particular, some generators have torsion (e.g. $x_1F_*x_2 \cdots x_n = 0$). However, there is a way to represent elements of F_*R uniquely in this basis (dividing everything into it's $x_1, \dots, x_n$ degrees). Therefore, we can create a map $\phi : F_*R \to R : F_*1 \mapsto 1$ and all the other generators to 0. This is well defined, surjective, and thus represents an F-splitting of R.

This seemed involved to conclude F-splittings exist. Next time we will simplify this via Fedder's Criterion. Here is one other classical result which yields further examples:

Proposition 0.6. *If $R \subseteq S$ are rings, and there exists $\psi \in \mathrm{Hom}_R(S, R)$ a surjection, then if S is F-split, so is R.*

Proof. $F_*R \hookrightarrow F_*S \xrightarrow{s} S \xrightarrow{\psi} R$ is surjective. □

Example 0.7. We can consider the ring $R = K[x^2, xy, y^2] \subseteq K[x, y] = S$. We can decompose elements of S into even and odd components, and surject the even components onto R (sending odd to 0 for example). S is a regular ring, so it is F-split. Therefore, Proposition 0.6 implies R is F-split.

This generalizes naturally to any number of variables and R a polynomial ring in the degree n monomials; the so-called **Veronese subrings**.