Attractor reconstruction, Takens theorem and causality detection

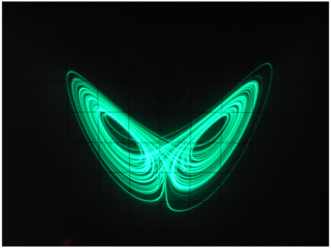
Marco Vitturi

School of Mathematics

PG Colloquium - 25th February 2016

Attractors

Attractors are a feature of the solutions of systems of ODEs that was first noticed in numerical experiments by Lorenz in 1963.



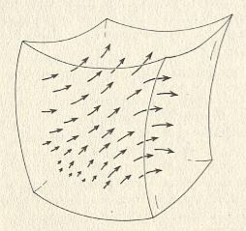
$$\begin{cases} x' = \sigma(y - x) \\ y' = x(\rho - z) - y \\ z' = xy - \beta z \\ \sigma = 10, \;\; \rho = 28, \;\; \beta = 8/3. \end{cases}$$

Some definitions

A **dynamical system** on a manifold M (e.g. $\mathbb{R}^n$) is specified by a **flow**

$$\Phi_{(\cdot)}(\cdot) : \mathbb{R} \times M \to M$$

, where $\Phi_0(x) = x$, Φ_t is a diffeomorphism for all t and $\Phi_t \circ \Phi_s = \Phi_{s+t}$ (e.g. the solution to a system of ODEs).



An **attracting set** A for flow Φ_t is a closed set s.t.

- the basin of attraction[a] $B(A)$ has positive measure;
- for $A' \subsetneq A$ the difference $B(A) \backslash B(A')$ has positive measure.

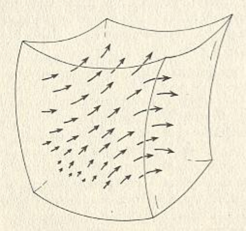
[a] i.e. $B(A) := \bigcap_{\varepsilon>0} \bigcup_{t>0} \bigcap_{s>t} \Phi_s^{-1}(A_\varepsilon)$.

Some definitions

A **dynamical system** on a manifold M (e.g. $\mathbb{R}^n$) is specified by a **flow**

$$\Phi_{(\cdot)}(\cdot) : \mathbb{R} \times M \to M$$

, where $\Phi_0(x) = x$, Φ_t is a diffeomorphism for all t and $\Phi_t \circ \Phi_s = \Phi_{s+t}$ (e.g. the solution to a system of ODEs).



An **attracting set** A for flow Φ_t is a closed set s.t.

- the basin of attraction[a] $B(A)$ has positive measure;
- for $A' \subsetneq A$ the difference $B(A) \backslash B(A')$ has positive measure.

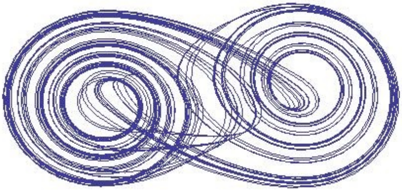
[a] i.e. $B(A) := \bigcap_{\varepsilon>0} \bigcup_{t>0} \bigcap_{s>t} \Phi_s^{-1}(A_\varepsilon)$.

Some definitions

A set $A \subset M$ is an **attractor** if it is an attracting set that contains a dense orbit[a] of the flow.

[a] $\exists x_0$ s.t. $\overline{\bigcup_{t>0} \Phi_t(x_0)} = A$.

An attractor A is a **strange attractor** if its (box-counting) dimension d is non-integer.

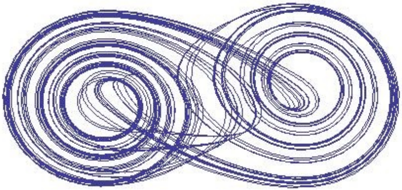


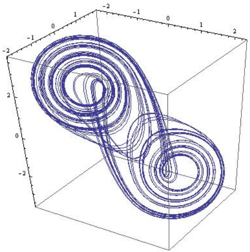
Some definitions

A set $A \subset M$ is an **attractor** if it is an attracting set that contains a dense orbit[a] of the flow.

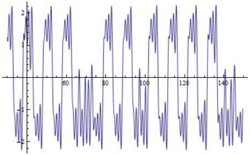
[a] $\exists x_0$ s.t. $\overline{\bigcup_{t>0} \Phi_t(x_0)} = A$.

An attractor A is a **strange attractor** if its (box-counting) dimension d is non-integer.

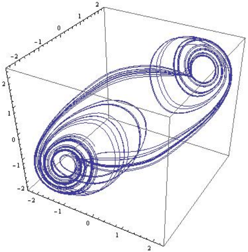




(a) Chua's attractor in X, Y, Z space



(b) time series of $X(t)$

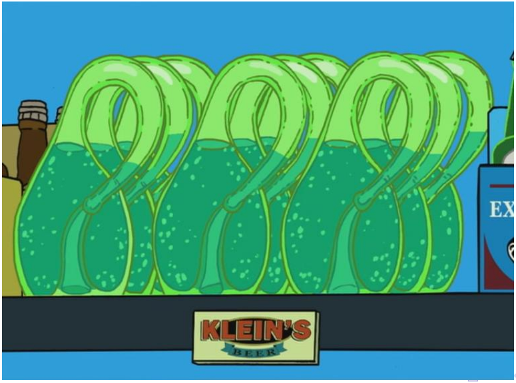


(c) Reconstructed attractor from trajectory $(X(t), X(t-\tau), X(t-2\tau))$

Entrée

Whitney Embedding Theorem

Let $\mathscr{M}$ be a smooth compact manifold of (integer) dimension n. Then $\mathscr{M}$ can be embedded smoothly in $\mathbb{R}^{2n}$ (in particular, no self-intersections).



Main course

Takens' theorem (clean statement) [1981]

Let $\mathscr{M}$ be a compact manifold of (integer) dimension d. Then for generic pairs (ϕ, y), where

- $\phi : \mathscr{M} \to \mathscr{M}$ is a C^2-diffeomorphism of $\mathscr{M}$ in itself,
- $y : \mathscr{M} \to \mathbb{R}$ is a C^2-differentiable function,

the map $\Phi_{(\phi,y)} : \mathscr{M} \to \mathbb{R}^{2d+1}$ given by

$$\Phi_{(\phi,y)}(x) := \left(y(x), y(\phi(x)), y(\phi^2(x)), \cdots, y(\phi^{2d}(x))\right)$$

is an embedding[a] of $\mathscr{M}$ in $\mathbb{R}^{2d+1}$.

[a] i.e. an injective and immersive map.

Main course

Takens' theorem (clean statement) [1981]

Let $\mathscr{M}$ be a compact manifold of (integer) dimension d. Then for generic pairs (ϕ, y), where

- $\phi : \mathscr{M} \to \mathscr{M}$ is a C^2-diffeomorphism of $\mathscr{M}$ in itself,
- $y : \mathscr{M} \to \mathbb{R}$ is a C^2-differentiable function,

the map $\mathbf{\Phi}_{(\phi,y)} : \mathscr{M} \to \mathbb{R}^{2d+1}$ given by

$$\mathbf{\Phi}_{(\phi,y)}(x) := \left(y(x), y(\phi(x)), y(\phi^2(x)), \cdots, y(\phi^{2d}(x))\right)$$

is an embedding[a] of $\mathscr{M}$ in $\mathbb{R}^{2d+1}$.

[a] i.e. an injective and immersive map.

A restatement with some dirt on...

Takens' theorem (beta version)

Let $\mathscr{M}$ be a compact manifold of (integer) dimension d. Let $\phi : \mathscr{M} \to \mathscr{M}$ be a C^2-diffeomorphism s.t.

- the periodic points with period $T \leqslant 2d$ are finite in number,
- if x has period $T \leqslant 2d$ then $D(\phi^T)$ has all distinct eigenvalues;

then for generic $y \in C^2(\mathscr{M}, \mathbb{R})$ the map $\Phi_{(\phi,y)} : \mathscr{M} \to \mathbb{R}^{2d+1}$ given by

$$\Phi_{(\phi,y)}(x) := \left(y(x), y(\phi(x)), y(\phi^2(x)), \cdots, y(\phi^{2d}(x))\right)$$

is an embedding of $\mathscr{M}$ in $\mathbb{R}^{2d+1}$.

A restatement with some dirt on...

Takens' theorem (beta version)

Let $\mathscr{M}$ be a compact manifold of (integer) dimension d. Let $\phi : \mathscr{M} \to \mathscr{M}$ be a C^2-diffeomorphism s.t.

- the periodic points with period $T \leqslant 2d$ are finite in number,
- if x has period $T \leqslant 2d$ then $D(\phi^T)$ has all distinct eigenvalues;

then for generic $y \in C^2(\mathscr{M}, \mathbb{R})$ the map $\Phi_{(\phi,y)} : \mathscr{M} \to \mathbb{R}^{2d+1}$ given by

$$\Phi_{(\phi,y)}(x) := \left(y(x), y(\phi(x)), y(\phi^2(x)), \cdots, y(\phi^{2d}(x))\right)$$

is an embedding of $\mathscr{M}$ in $\mathbb{R}^{2d+1}$.

A restatement with some dirt on...

Takens' theorem (beta version)

Let $\mathscr{M}$ be a compact manifold of (integer) dimension d. Let $\phi : \mathscr{M} \to \mathscr{M}$ be a C^2-diffeomorphism s.t.

- the periodic points with period $T \leqslant 2d$ are finite in number,
- if x has period $T \leqslant 2d$ then $D(\phi^T)$ has all distinct eigenvalues;

then for generic $y \in C^2(\mathscr{M}, \mathbb{R})$ the map $\Phi_{(\phi,y)} : \mathscr{M} \to \mathbb{R}^{2d+1}$ given by

$$\Phi_{(\phi,y)}(x) := (y(x), y(\phi(x)), y(\phi^2(x)), \cdots, y(\phi^{2d}(x)))$$

is an embedding of $\mathscr{M}$ in $\mathbb{R}^{2d+1}$.

A restatement with some dirt on...

Takens' theorem (beta version)

Let $\mathscr{M}$ be a compact manifold of (integer) dimension d. Let $\phi : \mathscr{M} \to \mathscr{M}$ be a C^2-diffeomorphism s.t.

- the periodic points with period $T \leqslant 2d$ are finite in number,
- if x has period $T \leqslant 2d$ then $D(\phi^T)$ has all distinct eigenvalues;

then for generic $y \in C^2(\mathscr{M}, \mathbb{R})$ the map $\mathbf{\Phi}_{(\phi,y)} : \mathscr{M} \to \mathbb{R}^{2d+1}$ given by

$$\mathbf{\Phi}_{(\phi,y)}(x) := \left(y(x), y(\phi(x)), y(\phi^2(x)), \cdots, y(\phi^{2d}(x))\right)$$

is an embedding of $\mathscr{M}$ in $\mathbb{R}^{2d+1}$.

Why periodic points are bad

- fixed points of ϕ are mapped to the diagonal of $\mathbb{R}^{2d+1}$,

$$x \mapsto (y(x), y(x), \cdots, y(x)).$$

  Imagine there is a continuum of fixed points that contains a circle. Thus $\Phi_{(\phi,y)}$ sends a circle to a line, which can't be done injectively.

- consider the set of points P_2 of period 2, and suppose they form a continuum. Define on P_2

$$y'(x) := y(x) - y(\phi^{-1}(x)),$$

  then $y'(x_0) = -y'(\phi(x_0))$, and therefore there exists $z_0 \in P_2$ s.t. $y'(z_0) = 0$, or equivalently $y(z_0) = y(\phi^{-1}(z_0))$. But then

$$\Phi_{(\phi,y)}(z_0) = \Phi_{(\phi,y)}(\phi(z_0)),$$

  so injectivity fails again.

Why periodic points are bad

- fixed points of ϕ are mapped to the diagonal of $\mathbb{R}^{2d+1}$,

$$x \mapsto (y(x), y(x), \cdots, y(x)).$$

  Imagine there is a continuum of fixed points that contains a circle. Thus $\Phi_{(\phi,y)}$ sends a circle to a line, which can't be done injectively.

- consider the set of points P_2 of period 2, and suppose they form a continuum. Define on P_2

$$y'(x) := y(x) - y(\phi^{-1}(x)),$$

  then $y'(x_0) = -y'(\phi(x_0))$, and therefore there exists $z_0 \in P_2$ s.t. $y'(z_0) = 0$, or equivalently $y(z_0) = y(\phi^{-1}(z_0))$. But then

$$\Phi_{(\phi,y)}(z_0) = \Phi_{(\phi,y)}(\phi(z_0)),$$

  so injectivity fails again.

Why periodic points are bad

- fixed points of ϕ are mapped to the diagonal of $\mathbb{R}^{2d+1}$,

$$x \mapsto (y(x), y(x), \cdots, y(x)).$$

  Imagine there is a continuum of fixed points that contains a circle. Thus $\Phi_{(\phi,y)}$ sends a circle to a line, which can't be done injectively.

- consider the set of points P_2 of period 2, and suppose they form a continuum. Define on P_2

$$y'(x) := y(x) - y(\phi^{-1}(x)),$$

  then $y'(x_0) = -y'(\phi(x_0))$, and therefore there exists $z_0 \in P_2$ s.t. $y'(z_0) = 0$, or equivalently $y(z_0) = y(\phi^{-1}(z_0))$. But then

$$\Phi_{(\phi,y)}(z_0) = \Phi_{(\phi,y)}(\phi(z_0)),$$

  so injectivity fails again.

delay embeddings

- If Φ_t is a flow on $\mathbb{R}^n$ and $\tau > 0$ is a fixed time, then we can define the **delay map**

$$\mathbf{\Phi}(x) := \big(y(x), y(\Phi_{-\tau}(x)), y(\Phi_{-2\tau}(x)), \cdots, y(\Phi_{-k\tau}(x))\big).$$

- If $\mathscr{M}$ is a manifold that is an attractor for the flow Φ_t then $\Phi_{-\tau}$ is a diffeomorphism of $\mathscr{M}$ into itself, so if $k \geqslant 2\dim(\mathscr{M}) + 1$ for generic y, τ the theorem says $\mathbf{\Phi}$ is actually an embedding!

If the flow is given by a system of ODEs and $X(t)$ is one of the $m+1$ coordinates, then the **delayed coordinate map**

$$t \mapsto \big(X(t), X(t-\tau), X(t-2\tau), \cdots, X(t-m\tau)\big)$$

is an embedding of an orbit. One variable contains ALL the information!

delay embeddings

- If Φ_t is a flow on $\mathbb{R}^n$ and $\tau > 0$ is a fixed time, then we can define the **delay map**

$$\mathbf{\Phi}(x) := \big(y(x), y(\Phi_{-\tau}(x)), y(\Phi_{-2\tau}(x)), \cdots, y(\Phi_{-k\tau}(x))\big).$$

- If $\mathscr{M}$ is a manifold that is an attractor for the flow Φ_t then $\Phi_{-\tau}$ is a diffeomorphism of $\mathscr{M}$ into itself, so if $k \geqslant 2\dim(\mathscr{M}) + 1$ for generic y, τ the theorem says $\mathbf{\Phi}$ is actually an embedding!

If the flow is given by a system of ODEs and $X(t)$ is one of the $m+1$ coordinates, then the **delayed coordinate map**

$$t \mapsto \big(X(t), X(t-\tau), X(t-2\tau), \cdots, X(t-m\tau)\big)$$

is an embedding of an orbit. One variable contains ALL the information!

delay embeddings

- If Φ_t is a flow on $\mathbb{R}^n$ and $\tau > 0$ is a fixed time, then we can define the **delay map**

$$\mathbf{\Phi}(x) := \big(y(x), y(\Phi_{-\tau}(x)), y(\Phi_{-2\tau}(x)), \cdots, y(\Phi_{-k\tau}(x))\big).$$

- If $\mathscr{M}$ is a manifold that is an attractor for the flow Φ_t then $\Phi_{-\tau}$ is a diffeomorphism of $\mathscr{M}$ into itself, so if $k \geqslant 2\dim(\mathscr{M}) + 1$ for generic y, τ the theorem says $\mathbf{\Phi}$ is actually an embedding!

If the flow is given by a system of ODEs and $X(t)$ is one of the $m + 1$ coordinates, then the **delayed coordinate map**

$$t \mapsto \big(X(t), X(t-\tau), X(t-2\tau), \cdots, X(t-m\tau)\big)$$

is an embedding of an orbit. One variable contains ALL the information!

Genericity

In the statement of Takens theorem the word *generic* is used. What does it mean?

Genericity

Let P be a property of functions in $C^k(\mathscr{M}, \mathscr{N})$ they might or might not have. Then we say that $P(f)$ is true for **generic** $f \in C^k(\mathscr{M}, \mathscr{N})$ if the set of function for which it holds is open and dense[a].

[a] in the C^k-topology

So arbitrary small perturbations turn bad choices in good choices.
$\Rightarrow$ Still somewhat unsatisfactory: would like an almost-certain sort of statement.

Genericity

In the statement of Takens theorem the word *generic* is used. What does it mean?

Genericity

Let P be a property of functions in $C^k(\mathscr{M}, \mathscr{N})$ they might or might not have. Then we say that $P(f)$ is true for **generic** $f \in C^k(\mathscr{M}, \mathscr{N})$ if the set of function for which it holds is open and dense[a].

[a] in the C^k-topology

So arbitrary small perturbations turn bad choices in good choices.
$\Rightarrow$ Still somewhat unsatisfactory: would like an almost-certain sort of statement.

Genericity

In the statement of Takens theorem the word *generic* is used. What does it mean?

Genericity

Let P be a property of functions in $C^k(\mathscr{M}, \mathscr{N})$ they might or might not have. Then we say that $P(f)$ is true for **generic** $f \in C^k(\mathscr{M}, \mathscr{N})$ if the set of function for which it holds is open and dense[a].

[a] in the C^k-topology

So arbitrary small perturbations turn bad choices in good choices.

$\Rightarrow$ Still somewhat unsatisfactory: would like an almost-certain sort of statement.

Genericity

In the statement of Takens theorem the word *generic* is used. What does it mean?

Genericity

Let P be a property of functions in $\mathrm{C}^k(\mathscr{M}, \mathscr{N})$ they might or might not have. Then we say that $\mathrm{P}(\mathrm{f})$ is true for **generic** $\mathrm{f} \in \mathrm{C}^k(\mathscr{M}, \mathscr{N})$ if the set of function for which it holds is open and dense[a].

[a] in the C^k-topology

So arbitrary small perturbations turn bad choices in good choices.
$\Rightarrow$ Still somewhat unsatisfactory: would like an almost-certain sort of statement.

Prevalence

No obvious probability on infinite dimensional spaces of functions. But...

Prevalence

We say property P is **prevalent** in (infinite dimensional) V vector space if there exists a finite dimensional subspace $E < V$ (the probe space) s.t. for any $v \in V$ one has that $P(v + e)$ holds (Lebesgue)-a.e. in E.

E.g. property $P(f) = \{\int_{\mathbb{T}^1} f \neq 0\}$ is prevalent amongst $L^1(\mathbb{T}^1)$ periodic functions. Indeed, take $E = \{\text{constants}\}$, then $\int_{\mathbb{T}^1} f + c \neq 0$ for a.e. $c \in \mathbb{R}$.

Prevalence

No obvious probability on infinite dimensional spaces of functions. But...

Prevalence

We say property P is **prevalent** in (infinite dimensional) V vector space if there exists a finite dimensional subspace $E < V$ (the probe space) s.t. for any $v \in V$ one has that $P(v + e)$ holds (Lebesgue)-a.e. in E.

E.g. property $P(f) = \{\int_{\mathbb{T}^1} f \neq 0\}$ is prevalent amongst $L^1(\mathbb{T}^1)$ periodic functions. Indeed, take $E = \{\text{constants}\}$, then $\int_{\mathbb{T}^1} f + c \neq 0$ for a.e. $c \in \mathbb{R}$.

Prevalence

No obvious probability on infinite dimensional spaces of functions. But...

Prevalence

We say property P is **prevalent** in (infinite dimensional) V vector space if there exists a finite dimensional subspace $E < V$ (the probe space) s.t. for any $v \in V$ one has that $P(v + e)$ holds (Lebesgue)-a.e. in E.

E.g. property $P(f) = \{\int_{\mathbb{T}^1} f \neq 0\}$ is prevalent amongst $L^1(\mathbb{T}^1)$ periodic functions. Indeed, take $E = \{\text{constants}\}$, then $\int_{\mathbb{T}^1} f + c \neq 0$ for a.e. $c \in \mathbb{R}$.

Dessert

Generalization by Sauer, Yorke, Casdagli [1991]

Let Φ_t be a flow on $\mathcal{U}$ open in $\mathbb{R}^k$ and let $\mathscr{A}$ be a compact subset of $\mathcal{U}$ with $\mathrm{boxdim}(\mathscr{A}) = d$ (possibly non-integer). Take $n \geqslant \lceil 2d \rceil$. Then, under similar hypotheses[a] as in Takens', for prevalent y, τ, the delay map

$$\Phi(x) := (y(x), y(\Phi_{-\tau}(x)), y(\Phi_{-2\tau}(x)), \cdots, y(\Phi_{-n\tau}(x)))$$

is

i) injective on $\mathscr{A}$;

ii) an embedding on (compact) smooth manifolds contained in $\mathscr{A}$ (e.g. orbits).

[a] finitely many periodic points of period less than $n\tau$, no periods τ or 2τ, $D(\phi^T)$ has all distinct eigenvalues at T-periodic points.

Dessert

Generalization by Sauer, Yorke, Casdagli [1991]

Let Φ_t be a flow on $\mathcal{U}$ open in $\mathbb{R}^k$ and let $\mathscr{A}$ be a compact subset of $\mathcal{U}$ with $\mathrm{boxdim}(\mathscr{A}) = d$ (possibly non-integer). Take $n \geqslant \lceil 2d \rceil$. Then,under similar hypotheses[a] as in Takens', for prevalent y, τ, the delay map

$$\mathbf{\Phi}(x) := \big(y(x), y(\Phi_{-\tau}(x)), y(\Phi_{-2\tau}(x)), \cdots, y(\Phi_{-n\tau}(x))\big)$$

is

i) injective on $\mathscr{A}$;

ii) an embedding on (compact) smooth manifolds contained in $\mathscr{A}$ (e.g. orbits).

[a] finitely many periodic points of period less than $n\tau$, no periods τ or 2τ, $D(\phi^T)$ has all distinct eigenvalues at T-periodic points.

What's attractor reconstruction good for?

- allows to recover the dynamics from the observation of just ONE variable;
- preserves the topology;
- it's diffeomorphic $\rightarrow$ can be used for system identification;
- allows to calculate the box-counting dimension of the attractor;
- allows to calculate the (positive) Lyapunov exponents;
- ...

What's attractor reconstruction good for?

- allows to recover the dynamics from the observation of just ONE variable;
- preserves the topology;
- it's diffeomorphic $\rightarrow$ can be used for system identification;
- allows to calculate the box-counting dimension of the attractor;
- allows to calculate the (positive) Lyapunov exponents;
- ...

What's attractor reconstruction good for?

- allows to recover the dynamics from the observation of just ONE variable;
- preserves the topology;
- it's diffeomorphic $\rightarrow$ can be used for system identification;
- allows to calculate the box-counting dimension of the attractor;
- allows to calculate the (positive) Lyapunov exponents;
- ...

What's attractor reconstruction good for?

- allows to recover the dynamics from the observation of just ONE variable;
- preserves the topology;
- it's diffeomorphic $\rightarrow$ can be used for system identification;
- allows to calculate the box-counting dimension of the attractor;
- allows to calculate the (positive) Lyapunov exponents;
- ...

What's attractor reconstruction good for?

- allows to recover the dynamics from the observation of just ONE variable;
- preserves the topology;
- it's diffeomorphic $\rightarrow$ can be used for system identification;
- allows to calculate the box-counting dimension of the attractor;
- allows to calculate the (positive) Lyapunov exponents;
- ...

What's attractor reconstruction good for?

- allows to recover the dynamics from the observation of just ONE variable;
- preserves the topology;
- it's diffeomorphic $\rightarrow$ can be used for system identification;
- allows to calculate the box-counting dimension of the attractor;
- allows to calculate the (positive) Lyapunov exponents;
- ...

Fisheries

Fish population dynamics is modeled by equations like

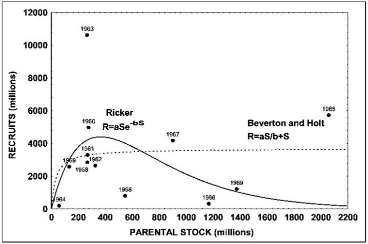
$$N_{t+1} = N_t e^{r(1-N_t/k)}, \qquad \text{[Ricker, '54]}$$

$$N_{t+1} = \frac{R_0 N_t}{1 + N_t/M}, \qquad \text{[Beverton-Holt, '57]}$$

$$C = \frac{F}{F+M}(1 - e^{-(F+M)T})N_0. \qquad \text{[Baranov, '18]}$$

These perform very bad:

- unknown parameters;
- parameter values might vary in time;
- dependence on other variables / interaction with other species;
- principles behind dynamics not fully understood.



Fisheries

Fish population dynamics is modeled by equations like

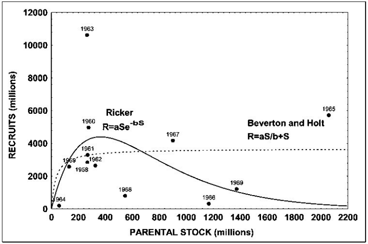
$$N_{t+1} = N_t e^{r(1-N_t/k)}, \qquad \text{[Ricker, '54]}$$

$$N_{t+1} = \frac{R_0 N_t}{1 + N_t/M}, \qquad \text{[Beverton-Holt, '57]}$$

$$C = \frac{F}{F+M}(1 - e^{-(F+M)T})N_0. \qquad \text{[Baranov, '18]}$$

These perform very bad:

- unknown parameters;
- parameter values might vary in time;
- dependence on other variables / interaction with other species;
- principles behind dynamics not fully understood.



Empirical Dynamic Modeling

We could let the variable estimate itself: from the time series of fish abundance, reconstruct the attractor, then use the attractor to make short term predictions (sort of an empirical interpolation).

No equations needed!

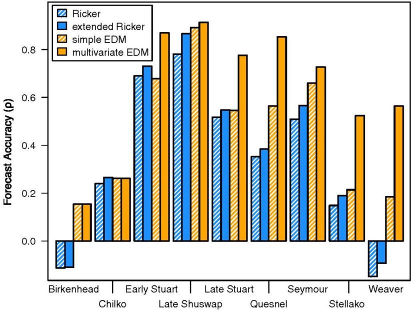


Figure: Performance of Ricker vs. EDM in the case of canadian salmon stocks

Empirical Dynamic Modeling

We could let the variable estimate itself: from the time series of fish abundance, reconstruct the attractor, then use the attractor to make short term predictions (sort of an empirical interpolation).

No equations needed!

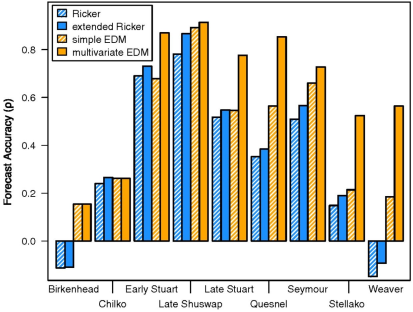


Figure: Performance of Ricker vs. EDM in the case of canadian salmon stocks

Empirical Dynamic Modeling

We could let the variable estimate itself: from the time series of fish abundance, reconstruct the attractor, then use the attractor to make short term predictions (sort of an empirical interpolation).

No equations needed!

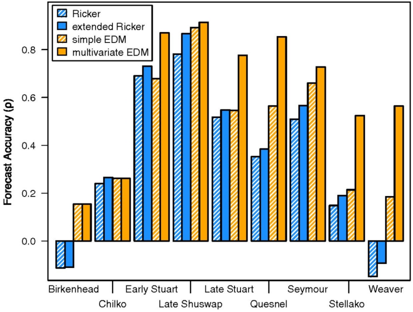
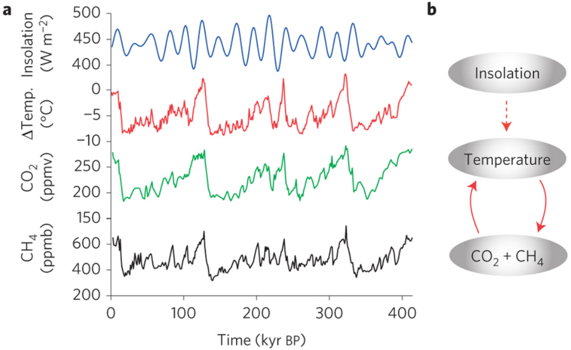


Figure: Performance of Ricker vs. EDM in the case of canadian salmon stocks

CO2 vs. Temperature

In the Vostok ice core data, CO2 rises are sometimes delayed with respect to temperature rises. Does this mean it's higher temperatures that cause higher CO2 in the atmosphere? or the other way round? or both?



Convergent Cross Mapping

Let $\mathscr{A}$ be the attractor of a dynamical system in variables $(X, Y, Z, \ldots)$. Remember the delayed coordinate map

$$t \mapsto \big(X(t), X(t-\tau), \ldots, X(t-n\tau)\big)$$

produces a diffeomorphic image of the attractor $\mathscr{A}$, denoted $\mathscr{A}_X$.

If X and Y belong to the same dynamical system, then $\mathscr{A}_X$ **must be diffeomorphic to** $\mathscr{A}_Y$!

In particular, points that are close in $\mathscr{A}_X$ must be close in $\mathscr{A}_Y$
$\rightarrow$ can predict one from the other (for nearby points). How well depends on the strength of the causal relations

Convergent Cross Mapping

Let $\mathscr{A}$ be the attractor of a dynamical system in variables $(X, Y, Z, \ldots)$. Remember the delayed coordinate map

$$t \mapsto \big(X(t), X(t-\tau), \ldots, X(t-n\tau)\big)$$

produces a diffeomorphic image of the attractor $\mathscr{A}$, denoted $\mathscr{A}_X$.

If X and Y belong to the same dynamical system, then $\mathscr{A}_X$ **must be diffeomorphic to $\mathscr{A}_Y$** !

In particular, points that are close in $\mathscr{A}_X$ must be close in $\mathscr{A}_Y$
$\rightarrow$ can predict one from the other (for nearby points). How well depends on the strength of the causal relations

Convergent Cross Mapping

Let $\mathscr{A}$ be the attractor of a dynamical system in variables $(X, Y, Z, \ldots)$. Remember the delayed coordinate map

$$t \mapsto \big(X(t), X(t-\tau), \ldots, X(t-n\tau)\big)$$

produces a diffeomorphic image of the attractor $\mathscr{A}$, denoted $\mathscr{A}_X$.

If X and Y belong to the same dynamical system, then $\mathscr{A}_X$ **must be diffeomorphic to** $\mathscr{A}_Y$!

In particular, points that are close in $\mathscr{A}_X$ must be close in $\mathscr{A}_Y$
$\rightarrow$ can predict one from the other (for nearby points). How well depends on the strength of the causal relations

Convergent Cross Mapping

Let $\mathscr{A}$ be the attractor of a dynamical system in variables $(X, Y, Z, \ldots)$. Remember the delayed coordinate map

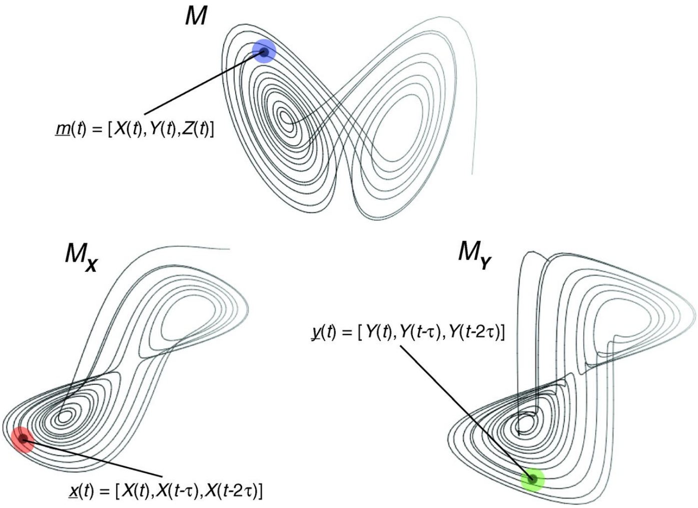
$$t \mapsto \big(X(t), X(t-\tau), \ldots, X(t-n\tau)\big)$$

produces a diffeomorphic image of the attractor $\mathscr{A}$, denoted $\mathscr{A}_X$.

If X and Y belong to the same dynamical system, then $\mathscr{A}_X$ **must be diffeomorphic to** $\mathscr{A}_Y$!

In particular, points that are close in $\mathscr{A}_X$ must be close in $\mathscr{A}_Y$
$\rightarrow$ can predict one from the other (for nearby points). How well depends on the strength of the causal relations

Convergent Cross Mapping



back to CO2 vs. Temperature