Chapter 10 Solutions

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Wasserman: All of Statistics

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Problem 10.1. Let us consider the power of the Wald test when the null hypothesis is false.

Suppose the true value of θ is $\theta_\star \neq \theta_0$. Prove that the power $\beta(\theta_\star)$, the probability of correctly rejecting the null hypothesis, is given (approximately) by

$$1 - \Phi\left(\frac{\theta_0 - \theta_\star}{\widehat{\text{se}}} + z_{\alpha/2}\right) + \Phi\left(\frac{\theta_0 - \theta_\star}{\widehat{\text{se}}} - z_{\alpha/2}\right).$$

Solution. The Wald test rejects the null hypothesis when $\frac{|\widehat{\theta} - \theta_0|}{\widehat{\text{se}}} > z_{\alpha/2}$.

Note that under the assumption that $\widehat{\theta}$ is asymptotically normal, we have

$$\frac{\widehat{\theta} - \theta_\star}{\widehat{\text{se}}} \xrightarrow{\text{d}} N(0, 1).$$

Then, we have

$$\begin{aligned}
\mathbb{P}\left(\frac{|\widehat{\theta} - \theta_0|}{\widehat{\text{se}}} > z_{\alpha/2}\right) &= \mathbb{P}\left(\frac{\widehat{\theta} - \theta_0}{\widehat{\text{se}}} > z_{\alpha/2}\right) + \mathbb{P}\left(\frac{\theta_0 - \widehat{\theta}}{\widehat{\text{se}}} > z_{\alpha/2}\right) \\
&= \mathbb{P}\left(\frac{\widehat{\theta} - \theta_\star}{\widehat{\text{se}}} > z_{\alpha/2} + \frac{\theta_0 - \theta_\star}{\widehat{\text{se}}}\right) + \mathbb{P}\left(\frac{\theta_\star - \widehat{\theta}}{\widehat{\text{se}}} > z_{\alpha/2} - \frac{\theta_0 - \theta_\star}{\widehat{\text{se}}}\right) \\
&= 1 - \Phi\left(z_{\alpha/2} + \frac{\theta_0 - \theta_\star}{\widehat{\text{se}}}\right) + \mathbb{P}\left(\frac{\widehat{\theta} - \theta_\star}{\widehat{\text{se}}} < \frac{\theta_0 - \theta_\star}{\widehat{\text{se}}} - z_{\alpha/2}\right) \\
&= 1 - \Phi\left(\frac{\theta_0 - \theta_\star}{\widehat{\text{se}}} + z_{\alpha/2}\right) + \Phi\left(\frac{\theta_0 - \theta_\star}{\widehat{\text{se}}} - z_{\alpha/2}\right)
\end{aligned}$$

as desired. □

Problem 10.2. Prove that if the test statistic has a continuous distribution, then under $H_0 : \theta = \theta_0$, the p-value has a Uniform$(0, 1)$ distribution. Therefore, if we reject H_0 when the p-value is less than α, the probability of a Type I error is α.

Solution. We note first the probability integral transform: if a random variable X has a continuous distribution function F, then $F(X) \sim$ Uniform$(0, 1)$.

A proof for this lemma: for $0 \leq x \leq 1$ we have $\mathbb{P}(F(X) < x) = \mathbb{P}(X < F^{-1}(x)) = F(F^{-1}(x)) = x$. Thus for $0 \leq x \leq 1$ we have $\mathbb{P}(F(X) < x) = x$, which implies that $F(X) \sim$ Uniform$(0, 1)$.

Let our test statistic be T, such that $T(X^n)$ is a random variable. Let its distribution function be F. Then $F(k) = \mathbb{P}(T(X^n) \leq k)$. Therefore

$$\begin{aligned}
F(T(x^n)) &= \mathbb{P}(T(X^n) \leq T(x^n)) \\
&= 1 - \text{p-value}
\end{aligned}$$

for any given x^n. But the left-hand-side is just the probability integral transform, so therefore $1 -$ p-value is distributed according to $1 -$ Uniform$(0, 1)$, so the p-value has a Uniform$(0, 1)$ distribution.

Now, a type I error is the rejection of H_0 when H_0 is true. So rejecting H_0 when the p-value is less than α does give us that the probability of a Type I error is α. □

Problem 10.3. Prove that the size α Wald test rejects $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ if and only if $\theta_0 \notin C$ where

$$C = (\widehat{\theta} - \widehat{\text{se}} \cdot z_{\alpha/2}, \widehat{\theta} + \widehat{\text{se}} \cdot z_{\alpha/2})$$

Solution. Suppose first that the size α Wald test rejects $H_0 : \theta = \theta_0$. This occurs when

$$\frac{|\widehat{\theta} - \theta_0|}{\widehat{\text{se}}} > z_{\alpha/2}$$

which is equivalent to

$$|\widehat{\theta} - \theta_0| > z_{\alpha/2} \cdot \widehat{\text{se}}$$

or

$$\theta_0 \notin (\widehat{\theta} - \widehat{\text{se}} \cdot z_{\alpha/2}, \widehat{\theta} + \widehat{\text{se}} \cdot z_{\alpha/2}).$$

Now suppose that $\theta_0 \notin C$. Then $|\widehat{\theta} - \theta_0| > z_{\alpha/2} \cdot \widehat{\text{se}}$ and thus $\frac{|\widehat{\theta} - \theta_0|}{\widehat{\text{se}}} > z_{\alpha/2}$, so the size α Wald test rejects $H_0 : \theta = \theta_0$. □

Problem 10.4. Suppose that the size α test is of the form

$$\text{reject } H_0 \text{ if and only if } T(X^n) \geq c_\alpha.$$

Prove that

$$\text{p-value} = \sup_{\theta \in \Theta_0} \mathbb{P}_\theta(T(X^n) \geq T(x^n))$$

where x^n is the observed value of X^n. If $\Theta_0 = \{\theta_0\}$ then

$$\text{p-value} = \mathbb{P}_{\theta_0}(T(X^n) \geq T(x^n)).$$

Solution. Our size α test is of the form "reject the null hypothesis if and only if $T(X^n) \geq c_\alpha$." Note that the function $\sup_{\theta \in \Theta_0} \mathbb{P}(T(X^n) \geq c_\alpha)$ is a decreasing function of c_α; this function returns the maximum probability that the test statistic is greater than or equal to c_α, so as c_α increases, this function decreases.

Upon observing $T(x^n)$, given how our test works, we will reject the null hypothesis if $c_\alpha \leq T(x^n)$. Thus the maximum c_α for which we reject the null hypothesis is exactly $T(x^n)$.

But now as $\sup_{\theta \in \Theta_0} \mathbb{P}(T(X^n) \geq c_\alpha)$ is a decreasing function of c_α, it follows that the minimum value of $\sup_{\theta \in \Theta_0} \mathbb{P}(T(X^n) \geq c_\alpha)$ for which we reject the null hypothesis occurs when $c_\alpha = T(x^n)$. But then $\sup_{\theta \in \Theta_0} \mathbb{P}(T(X^n) \geq T(x_n))$ is exactly the p-value, as desired. The specific case follows immediately, as then there is only one value in Θ_0. □

Problem 10.5. Let $X_1, \ldots, X_n \sim \text{Uniform}(0, \theta)$ and let $Y = \max(X_1, \ldots, X_n)$. We want to test

$$H_0 : \theta = \frac{1}{2} \text{ versus } H_1 : \theta > \frac{1}{2}.$$

The Wald test is not appropriate since Y does not converge to a Normal. Suppose we decide to test this hypothesis by rejecting H_0 when $Y > c$.

(a) Find the power function.

(b) What choice of c will make the size of the test .05?

(c) In a sample of size $n = 20$ with $Y = 0.48$ what is the p-value? What conclusion about H_0 would you make?

(d) In a sample of size $n = 20$ with $Y = 0.52$ what is the p-value? What conclusion about H_0 would you make?

Solution. Here we are given that the rejection region should be $R = \{x_1, \ldots, x_n : \max(x_1, \ldots, x_n) > c\}$.

(a) With $X = (x_1, \ldots, x_n)$ the power function is $\beta(\theta) = \mathbb{P}_\theta(X \in R)$; that is, the probability that given that θ is the true parameter, that we reject the null hypothesis.

So we have

$$\begin{aligned}\beta(\theta) &= \mathbb{P}_\theta(X \in R)\\ &= \mathbb{P}_\theta(\max(x_1, \ldots, x_n) > c)\\ &= 1 - [\mathbb{P}_\theta(x_i < c)]^n\\ &= 1 - (c/\theta)^n.\end{aligned}$$

with the assumption, of course, that $c < \theta$. If $c > \theta$ the power is evidently 0, because there is no chance of us rejecting the null hypothesis.

(b) The size of a test is $\alpha = \sup_{\theta \in \Theta_0} \beta(\theta)$, and as $\Theta_0 = \{\frac{1}{2}\}$, we need to set $\beta(\frac{1}{2}) = 0.05$.

We have

$$\begin{aligned}0.05 &= \beta\left(\frac{1}{2}\right)\\ &= 1 - (2c)^n\end{aligned}$$

so $c = \left(\frac{0.95}{2^n}\right)^{1/n}$.

(c) The p-value is $\mathbb{P}_{\theta=\frac{1}{2}}(Y > 0.48)$. So we need to find $\mathbb{P}_{\theta=\frac{1}{2}}(\max(x_1, \ldots, x_{20}) > 0.48)$.

That comes out to

$$\begin{aligned}\text{p-value} &= 1 - \mathbb{P}(x_i < 0.48)^{20}\\ &= 1 - (0.96)^{20}\\ &\approx 0.558\end{aligned}$$

so we would not reject the null.

(d) We would certainly reject the null, because $Y = 0.52$ implies at least one $X_i = 0.52$, and it is impossible to draw 0.52 from Uniform$(0, 0.5)$.

□

Problem 10.6. There is a theory that people can postpone their death until after an important event. To test the theory, Phillips and King (1988) collected data on deaths around the Jewish holiday Passover. Of 1919 deaths, 922 died the week before the holiday and 997 died the week after. Think of this as a binomial and test the null hypothesis that $\theta = \frac{1}{2}$. Report and interpret the p-value. Also construct a confidence interval for θ.

Solution. Suppose $X_1, \ldots, X_{1919} \sim \text{Bernoulli}(\theta)$, where $X_i = 0$ if person i died the week before the holiday and $X_i = 1$ if person i died the week after the holiday. We will use the Wald test. The null hypothesis and alternative hypothesis would be $H_0 : \theta = \frac{1}{2}$ and $H_1 : \theta \neq \frac{1}{2}$, respectively.

The MLE for θ would be $\frac{\sum_{i=1}^{1919} X_i}{1919} = \frac{997}{1919}$, and the estimated standard error is $\widehat{\text{se}} = \sqrt{\frac{\widehat{\theta}(1-\widehat{\theta})}{1919}} \approx 0.0114$. Thus the Wald test statistic is

$$W = \frac{\frac{997}{1919} - \frac{1}{2}}{0.0114} \approx 1.7134.$$

and the p-value is given by $\mathbb{P}(|Z| > 1.7134) \approx 0.0866$. So the p-value is 0.0866.

A confidence interval for θ would be

$$\begin{aligned}C &= (\widehat{\theta} - \widehat{\text{se}} \cdot z_{\alpha/2}, \widehat{\theta} + \widehat{\text{se}} \cdot z_{\alpha/2})\\ &= (0.48046 - 0.0114 z_{\alpha/2}, 0.48046 + 0.0114 z_{\alpha/2}).\end{aligned}$$

Note that as expected, when we take $\alpha = 0.0866$, we obtain that the confidence interval contains 0.5. □

Problem 10.7. In 1861, 10 essays appeared in the New Orleans Daily Crescent. They were signed "Quintus Curtius Snodgrass" and some people suspected they were actually written by Mark Twain. To investigate this, we will consider the proportion of three letter words found in an author's work. From eight Twain essays we have:

$.225, .262, .217, .240, .230, .229, .235, .217$

From 10 Snodgrass essays we have:

$.209, .205, .196, .210, .202, .207, .224, .223, .220, .201$

(a) Perform a Wald test for equality of the means. Use the nonparametric plug-in estimator. Report the p-value and a 95 percent confidence interval for the difference of means. What do you conclude?

(b) Now use a permutation test to avoid the use of large sample methods. What is your conclusion?

Solution. Let us regard this data $X_1, \dots, X_{n_1}$ and $Y_1, \dots, Y_{n_2}$ as being sampled from two distributions F_1 and F_2 for Twain and Snodgrass, respectively, with true means μ_1 and μ_2, and $n_1 = 8, n_2 = 10$. Let $\theta = \mu_1 - \mu_2$, so that $H_0 : \theta = 0$ and $H_1 : \theta \neq 0$.

The nonparametric plug-in estimators for the means and the differences in means are just the sample means and the differences in the sample means. That is, $\widehat{\mu}_1 = 0.231875$ and $\widehat{\mu}_2 = 0.2097$, and $\widehat{\theta} = \widehat{\mu}_1 - \widehat{\mu}_2 = 0.022175$.

Now, we have

$$\begin{aligned}\widehat{\mathrm{se}}(\widehat{\theta}) &= \widehat{\mathrm{se}}(\widehat{\mu}_1 - \widehat{\mu}_2) \\ &= \sqrt{\mathbb{V}(\widehat{\mu}_1 - \widehat{\mu}_2)} \\ &= \sqrt{\mathbb{V}(\widehat{\mu}_1) + \mathbb{V}(\widehat{\mu}_2)}.\end{aligned}$$

But $\mathbb{V}(\widehat{\mu}_1) = \mathbb{V}(\frac{1}{n_1}\sum_{i=1}^{n_1} X_i) = \frac{1}{n_1}\mathbb{V}(X_1)$. We can estimate the variance by $\mathbb{V}(X_1)$ is $\frac{1}{n_1-1}\sum_{i=1}^{n_1}(X_i - \overline{X}_n)^2 = 0.000212125$. Thus we obtain $\mathbb{V}(\widehat{\mu}_1) = 0.000026515625$.

In a similar fashion we obtain $\mathbb{V}(\widehat{\mu}_2) = 0.00000933444$. It follows that the standard error $\widehat{\mathrm{se}}(\widehat{\theta})$ is

$$\widehat{\mathrm{se}}(\widehat{\theta}) = 0.005987492.$$

Thus the observed value of the Wald test statistic is

$$\begin{aligned}w &= \frac{\widehat{\theta} - \theta_0}{\widehat{\mathrm{se}}} \\ &= \frac{0.022175}{0.005987492} \approx 3.7.\end{aligned}$$

So the p-value is given by $\mathbb{P}(|Z| > |w|) \approx 0.00022$. A 95 percent confidence interval is given by

$$\begin{aligned}C &= (0.022175 - 0.005987492 \cdot z_{\alpha/2}, 0.022175 + 0.005987492 \cdot z_{\alpha/2}) \\ &\approx (0.0104, 0.0339).\end{aligned}$$

Regarding part (b), see the RMD file. □

Problem 10.8. Let $X_1, \dots, X_n \sim N(\theta, 1)$. Consider testing

$$H_0 : \theta = 0 \text{ versus } \theta = 1.$$

Let the rejection region be $R = \{x^n : T(x^n) > c\}$ where $T(x^n) = n^{-1}\sum_{i=1}^{n} X_i$.

(a) Find c so that the test has size α.

(b) Find the power under H_1; that is, find $\beta(1)$.

(c) Show that $\beta(1) \to 1$ as $n \to \infty$.

Solution. As usual, let Z be a standard Normal.

(a) We know that the test size is $\sup_{\theta\in\Theta_0}\beta(\theta)$, if the test must have size α, it follows that $\beta(0)=\alpha$. But $\beta(0)=\mathbb{P}_0(x^n\in R)=\mathbb{P}_0(\overline{X}_n>c)$, the probability that $\overline{X}_n>c$ given that $\theta=0$ is the true parameter.

Then we know that $\overline{X}_n\sim N(0,1/n)$, so $\mathbb{P}_0(\overline{X}_n>c)=\mathbb{P}(Z>c\sqrt{n})=1-\Phi(c\sqrt{n})$. It follows that $\alpha=1-\Phi(c\sqrt{n})$, so $c=\frac{1}{\sqrt{n}}\Phi^{-1}(1-\alpha)$.

(b) We have $\beta(1)=\mathbb{P}_1(x^n\in R)=\mathbb{P}_1(\overline{X}_n>c)$, the probability that $\overline{X}_n>c$ given that $\theta=1$ is the true parameter.

If $\theta=1$ is the true parameter, then $\overline{X}_n\sim N(1,1/n)$. It follows that

$$\begin{aligned}\mathbb{P}_1(\overline{X}_n>c)&=\mathbb{P}_1(\sqrt{n}(\overline{X}_n-1)>\sqrt{n}(c-1))\\&=\mathbb{P}(Z>\sqrt{n}(c-1))\\&=1-\Phi(\sqrt{n}(c-1)).\end{aligned}$$

(c) We have

$$\begin{aligned}\beta(1)&=1-\Phi(\sqrt{n}(c-1))\\&=1-\Phi\left(\sqrt{n}\left(\frac{1}{\sqrt{n}}\Phi^{-1}(1-\alpha)-1\right)\right)\\&=1-\Phi(\Phi^{-1}(1-\alpha)-\sqrt{n})\end{aligned}$$

which means that as $\Phi^{-1}(1-\alpha)$ is a constant, then $\Phi^{-1}(1-\alpha)-\sqrt{n}\to-\infty$. Therefore $\Phi(\Phi^{-1}(1-\alpha)-\sqrt{n})\to 0$ as $n\to\infty$, and the result follows.

□

Problem 10.9. Let $\widehat{\theta}$ be the MLE of a parameter θ and let $\widehat{\text{se}}=\frac{1}{\sqrt{nI(\widehat{\theta})}}$, where $I(\theta)$ is the Fisher information. Consider testing

$$H_0:\theta=\theta_0 \text{ versus } \theta\neq\theta_0.$$

Consider the Wald test with rejection region $R=\{x^n:|Z|>z_{\alpha/2}\}$ where $Z=(\widehat{\theta}-\theta_0)/\widehat{\text{se}}$. Let $\theta_1>\theta_0$ be some alternative. Show that $\beta(\theta_1)\to 1$.

Solution. We want to show that

$$\beta(\theta_1)=\mathbb{P}_{\theta_1}\left(\left|\frac{\widehat{\theta}-\theta_0}{\widehat{\text{se}}}\right|>z_{\alpha/2}\right)\to 1.$$

Note that by the asymptotic normality of the MLE, we know that

$$\frac{\widehat{\theta}_n-\theta_1}{\widehat{\text{se}}}\xrightarrow{\text{d}}N(0,1).$$

But

$$\left|\frac{\widehat{\theta}-\theta_0}{\widehat{\text{se}}}\right|=\left|\frac{\widehat{\theta}_n-\theta_1}{\widehat{\text{se}}}+\frac{\theta_1-\theta_0}{\widehat{\text{se}}}\right|$$

so as $\widehat{\text{se}}\to 0$ as $n\to\infty$, and as θ_1 and θ_0 are distinct constants, it follows that $\frac{\widehat{\theta}-\theta_0}{\widehat{\text{se}}}\to\infty$. Thus $\beta(\theta_1)\to 1$. □

Problem 10.10. Here are the number of elderly Jewish and Chinese women who died just before and after the Chinese Harvest Moon Festival. Compare the two mortality patterns.

Week	Chinese	Jewish
-2	55	141
-1	33	145
1	70	139
2	49	161

Solution. Assume $X \sim \text{Multinomial}(207, p_{-2}, p_{-1}, p_1, p_2)$ where $X = (55, 33, 70, 49)$ and p_i refers to the probability of death occurring in week i, given that it occurs in one of the four given weeks.

The null hypothesis is $H_0 : p = p_0 = (0.25, 0.25, 0.25, 0.25)$, and the alternative hypothesis is $H_1 : p \neq p_0$. We will test both the Chinese and the Jewish women under this hypothesis. We will reject the null hypothesis if the p-value is less than 0.05.

For the Chinese women: with $(X_1, X_2, X_3, X_4) = (55, 33, 70, 49)$ we have

$$\begin{aligned} T_C &= \sum_{j=1}^{4} \frac{(X_j - E_j)^2}{E_j} \\ &= \sum_{j=1}^{4} \frac{(X_j - 51.75)^2}{51.75} \\ &\approx 13.58. \end{aligned}$$

Similarly, for the Jewish women we obtain $T_J \approx 2.04$.

The p-values are approximately $\mathbb{P}(\chi_3^2 > t)$ for $t = 13.58, 2.04$. This gives us p-values of $0.00354, 0.56$ respectively, so we reject the null for the Chinese women and do not reject the null for the Jewish women. □

Problem 10.11. A randomized, double-blind experiment was conducted to assess the effectiveness of several drugs for reducing postoperative nausea. The data are as follows.

	Number of Patients	Incidences of Nausea
Placebo	80	45
Chlorpromazine	75	26
Dimenhydrinate	85	52
Pentobarbital (100 mg)	67	35
Pentobarbital (150 mg)	85	37

(a) Test each drug versus the placebo at the 5 percent level. Also, report the estimated odds-ratios. Summarize your findings.

(b) Use the Bonferroni and FDR method to adjust for multiple testing.

Solution. Let p_1, p_2, p_3, p_4, p_5 be the proportion of nausea for Placebo, Chlorpromazine, Dimenhydrinate, Pentobarbital (100 mg), and Pentobarbital (150 mg).

(a) We will test the hypothesis $H_0 : \delta_i = 0$ versus $H_1 : \delta_i \neq 0$ for $\delta_i = p_i - p_1$ for $i = 2, 3, 4, 5$.

We can model these as binomial random variables: $\text{Binomial}(n_i, p_i)$. In this case $n_1 = 80, n_2 = 75, \ldots, n_5 = 85$.

In this case, from the textbook, the size α Wald test is to reject H_0 when $|W_i| > z_{\alpha/2}$, where

$$W_i = \frac{\widehat{p}_1 - \widehat{p}_i}{\sqrt{\frac{\widehat{p}_1(1-\widehat{p}_1)}{n_1} + \frac{\widehat{p}_i(1-\widehat{p}_i)}{n_i}}}$$

with $\widehat{p}_i = \frac{X_i}{n_i}$ and x_i as the observed value from the binomial draw.

So we compute as follows: $\widehat{p}_1 = .5625, \widehat{p}_2 = .3467, \widehat{p}_3 = .6118, \widehat{p}_4 = .5224, \widehat{p}_5 = .4353$. Thus we have $w_2 = 2.7639, w_3 = -0.6435, w_4 = 0.4863, w_5 = 1.6465$. Given $z_{\alpha/2} \approx 1.96$, that means we reject the null hypothesis precisely for Chlorpromazine.

The odds ratios are approximately, respectively, $0.4127, 1.2256, 0.8507, 0.5995$.

(b) The Bonferroni method would have us reject a null hypothesis if a p-value satisfies $P_i < \frac{\alpha}{m}$. Here, we have $m = 4$ p-values and $\alpha = 0.05$. So $\alpha/m = 0.0125$. Our p-values are $P_1 = 0.0058, P_2 = 0.5198, P_3 = 0.6268, P_4 = 0.0996$, so we still reject the null hypothesis for Chlorpromazine and do not reject for anything else.

As for the FDR method: the ordered p-values are

$$P_{(1)} = 0.0058, P_{(2)} = 0.0996, P_{(3)} = 0.5198, P_{(4)} = 0.6268.$$

Then

$$\ell_1 = 0.0125, \ell_2 = 0.025, \ell_3 = 0.0375, \ell_4 = 0.05.$$

It follows that if $R = \max\{i : P_{(i)} < \ell_i\}$ then $R = 1$. Then the BH rejection threshold sets $T = 0.0058$. We thus reject only the null hypothesis for Chlorpromazine.

□

Problem 10.12. Let $X_1, \ldots, X_n \sim \text{Pois}(\lambda)$. Let $\lambda_0 > 0$. Find the size α Wald test for

$$H_0 : \lambda = \lambda_0 \text{ versus } H_1 : \lambda \neq \lambda_0.$$

Solution. The size α Wald test is to reject H_0 when $|W| > z_{\alpha/2}$, where

$$W = \frac{\widehat{\lambda} - \lambda_0}{\widehat{\text{se}}}.$$

We use the standard estimate $\overline{X}_n$ for $\widehat{\lambda}$. Then

$$\begin{aligned} \widehat{\text{se}} &= \sqrt{\mathbb{V}\left(\frac{\sum_{i=1}^n X_i}{n}\right)} \\ &= \sqrt{\frac{1}{n} \cdot \mathbb{V}(X_i)} \\ &= \sqrt{\lambda_0/n}. \end{aligned}$$

Thus we reject H_0 when

$$\left| \frac{\overline{X}_n - \lambda_0}{\sqrt{\lambda_0/n}} \right| > z_{\alpha/2}.$$

□

Problem 10.13. Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. Construct the likelihood ratio test for

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu \neq \mu_0.$$

Compare to the Wald test.

Solution. First, we will compute the MLE under H_0.

Denote $\theta = (\mu, \sigma), \theta_0 = (\mu_0, \sigma)$. The likelihood function is

$$\begin{aligned} \mathcal{L}_{\mu=\mu_0}(\theta) &= \prod_{i=1}^n f(X_i; \theta_0) \\ &= \prod_{i=1}^n \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x_i - \mu_0)^2} \end{aligned}$$

so the log-likelihood is

$$\ell_{\mu=\mu_0}(\theta) = \sum_{i=1}^{n} \left[\log\left(\frac{1}{\sigma\sqrt{2\pi}} \right) - \frac{1}{2\sigma^2}(x_i - \mu_0)^2 \right]$$

and discarding constant terms and simplifying some more yields

$$\ell_{\mu=\mu_0}(\theta) = n \log \frac{1}{\sigma} - \sum_{i=1}^{n} \frac{1}{2\sigma^2}(x_i - \mu_0)^2.$$

Then

$$\frac{d}{d\sigma} \ell_{\mu=\mu_0}(\theta) = -\frac{n}{\sigma} + \sum_{i=1}^{n} \frac{1}{\sigma^3}(x_i - \mu_0)^2$$

and setting this equal to 0 yields

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu_0)^2.$$

Thus the MLE under H_0 is $\widehat{\theta}_0 = (\mu_0, \widehat{\sigma}_0)$, where $\widehat{\sigma}_0 = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2}$.

The MLE in general is well-known to be $\widehat{\theta} = (\widehat{\mu}, \widehat{\sigma})$, where $\widehat{\mu} = \overline{X}_n$ and $\widehat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - \widehat{\mu})^2}$.

That allows us to compute the likelihood ratio statistic: we have

$$\lambda = 2 \log \left(\frac{\mathcal{L}(\widehat{\theta})}{\mathcal{L}(\widehat{\theta}_0)} \right).$$

So

$$\begin{aligned} \mathcal{L}(\widehat{\theta}) &= \prod_{i=1}^{n} f(X_i; \widehat{\theta}) \\ &= \prod_{i=1}^{n} \frac{1}{\widehat{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2\widehat{\sigma}^2}(x_i - \widehat{\mu})^2} \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}(\widehat{\theta}_0) &= \prod_{i=1}^{n} f(X_i; \widehat{\theta}_0) \\ &= \prod_{i=1}^{n} \frac{1}{\widehat{\sigma}_0\sqrt{2\pi}} e^{-\frac{1}{2\widehat{\sigma}_0^2}(x_i - \mu_0)^2}. \end{aligned}$$

Thus

$$\frac{\mathcal{L}(\widehat{\theta})}{\mathcal{L}(\widehat{\theta}_0)} = \prod_{i=1}^{n} \frac{\widehat{\sigma}_0}{\widehat{\sigma}} \cdot e^{-\frac{1}{2\widehat{\sigma}^2}(x_i - \widehat{\mu})^2 + \frac{1}{2\widehat{\sigma}_0^2}(x_i - \mu_0)^2}$$

and

$$\begin{aligned} \lambda &= 2 \log \left(\frac{\mathcal{L}(\widehat{\theta})}{\mathcal{L}(\widehat{\theta}_0)} \right) \\ &= 2 \cdot \left[n \log \frac{\widehat{\sigma}_0}{\widehat{\sigma}} + \sum_{i=1}^{n} \left(-\frac{1}{2\widehat{\sigma}^2}(x_i - \widehat{\mu})^2 + \frac{1}{2\widehat{\sigma}_0^2}(x_i - \mu_0)^2 \right) \right] \end{aligned}$$

and the limiting distribution of λ under H_0 is χ_1^2. The p-value for this test is $\mathbb{P}(\chi_1^2 > \lambda)$.

Now, for the Wald test, we are testing $H_0 : \mu = \mu_0$ versus $\mu \neq \mu_0$. That means that the Wald test statistic is

$$W = \frac{\widehat{\mu} - \mu_0}{\widehat{\text{se}}}$$

where we have $\widehat{\mu} = \overline{X}_n$.

Next, we have

$$\begin{aligned}
\widehat{\text{se}}(\widehat{\mu}) &= \sqrt{\mathbb{V}(\overline{X}_n)} \\
&= \sqrt{\frac{1}{n}\widehat{\sigma}^2} = \frac{\widehat{\sigma}}{\sqrt{n}}.
\end{aligned}$$

so therefore, with w as the observed value of the Wald test statistic, the p-value is approximately

$$\begin{aligned}
\mathbb{P}(|Z| > |w|) &= \mathbb{P}(Z^2 > w^2) \\
&= \mathbb{P}\left(\chi_1^2 > \left(\frac{\sqrt{n}(\widehat{\mu} - \mu_0)}{\widehat{\sigma}}\right)^2\right).
\end{aligned}$$

$\square$

Problem 10.14. Let $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$. Construct the likelihood ratio test for

$$H_0 : \sigma = \sigma_0 \text{ versus } H_1 : \sigma \neq \sigma_0.$$

Compare to the Wald test.

Solution. Denote $\theta = (\mu, \sigma)$ and $\theta_0 = (\mu, \sigma_0)$. The MLE under H_0 is $\widehat{\theta}_0 = (\widehat{\mu}, \sigma_0)$, where $\widehat{\mu} = \overline{X}_n$. The MLE under H_1 is $\widehat{\theta} = (\widehat{\mu}, \widehat{\sigma})$, where $\widehat{\sigma} = \sqrt{\frac{1}{n}\sum_{i=1}^n (X_i - \widehat{\mu})^2}$.

So

$$\begin{aligned}
\mathcal{L}(\widehat{\theta}) &= \prod_{i=1}^n f(X_i; \widehat{\theta}) \\
&= \prod_{i=1}^n \frac{1}{\widehat{\sigma}\sqrt{2\pi}} e^{-\frac{1}{2\widehat{\sigma}^2}(x_i - \widehat{\mu})^2}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{L}(\widehat{\theta}_0) &= \prod_{i=1}^n f(X_i; \widehat{\theta}_0) \\
&= \prod_{i=1}^n \frac{1}{\sigma_0\sqrt{2\pi}} e^{-\frac{1}{2\sigma_0^2}(x_i - \widehat{\mu})^2}.
\end{aligned}$$

Thus we compute the likelihood ratio statistic as

$$\begin{aligned}
\lambda &= 2\log\left(\frac{\mathcal{L}(\widehat{\theta})}{\mathcal{L}(\widehat{\theta}_0)}\right) \\
&= 2\log\left(\left(\frac{\sigma_0}{\widehat{\sigma}}\right)^n \prod_{i=1}^n e^{\frac{1}{2\sigma_0^2}(x_i - \widehat{\mu})^2 - \frac{1}{2\widehat{\sigma}^2}(x_i - \widehat{\mu})^2}\right) \\
&= 2\left[n\log\frac{\sigma_0}{\widehat{\sigma}} + \sum_{i=1}^n \left(\frac{1}{2\sigma_0^2}(x_i - \widehat{\mu})^2 - \frac{1}{2\widehat{\sigma}^2}(x_i - \widehat{\mu})^2\right)\right]
\end{aligned}$$

and the limiting distribution of λ under H_0 is χ_1^2. The p-value for this test is $\mathbb{P}(\chi_1^2 > \lambda)$.

Now, for the Wald test, we are testing $H_0 : \sigma = \sigma_0$ versus $\sigma \neq \sigma_0$. That means that the Wald test statistic is

$$W = \frac{\widehat{\sigma} - \sigma_0}{\widehat{\text{se}}}$$

where, to reiterate, $\sigma = \sqrt{\frac{1}{n}\sum_{i=1}^n (X_i - \overline{X}_n)^2}$.

To get $\widehat{\text{se}}$, we'll note that

$$\begin{aligned}
I(\sigma) &= -\mathbb{E}_\sigma \left(\frac{\partial^2 \log f(X;\sigma)}{\partial \sigma^2} \right) \\
&= -\mathbb{E}_\sigma \left(\frac{\partial^2}{\partial \sigma^2} \left(\log \frac{1}{\sigma\sqrt{2\pi}} - \frac{1}{2\sigma^2}(X-\mu)^2 \right) \right) \\
&= -\mathbb{E}_\sigma \left(\frac{\partial}{\partial \sigma} \left(-\frac{1}{\sigma} + \frac{1}{\sigma^3}(X-\mu)^2 \right) \right) \\
&= -\mathbb{E}_\sigma \left(\frac{1}{\sigma^2} - \frac{3}{\sigma^4}(X-\mu)^2 \right) \\
&= -\frac{1}{\sigma^2} + \frac{3}{\sigma^4}\mathbb{E}_\sigma[(X-\mu)^2] \\
&= \frac{2}{\sigma^2}.
\end{aligned}$$

Thus

$$\begin{aligned}
\widehat{\text{se}} &= \sqrt{1/I_n(\widehat{\sigma})} \\
&= \sqrt{\frac{\widehat{\sigma}^2}{2n}}
\end{aligned}$$

and the observed value of the Wald test statistic becomes

$$W = \frac{\widehat{\sigma} - \sigma_0}{\sqrt{\frac{\widehat{\sigma}^2}{2n}}}.$$

Thus the p-value is approximately

$$\begin{aligned}
\mathbb{P}(|Z| > |w|) &= \mathbb{P}(Z^2 > w^2) \\
&= \mathbb{P}\left(\chi_1^2 > \frac{2n(\widehat{\sigma} - \sigma_0)^2}{\widehat{\sigma}^2} \right).
\end{aligned}$$

□

Problem 10.15. Let $X \sim \text{Binomial}(n, p)$. Construct the likelihood ratio test for

$$H_0 : p = p_0 \text{ versus } H_1 : p \neq p_0.$$

Compare to the Wald test.

Solution. The MLE for p is $\widehat{p} = \frac{X}{n}$. The MLE under H_0 is just $p = p_0$. Therefore, the likelihood ratio statistic is

$$\begin{aligned}
\lambda &= 2\log\left(\frac{\mathcal{L}(\widehat{p})}{\mathcal{L}(p_0)} \right) \\
&= 2\log\left(\frac{\binom{n}{x}\widehat{p}^x(1-\widehat{p})^{n-x}}{\binom{n}{x}p_0^x(1-p_0)^{n-x}} \right) \\
&= 2\log\left(\left(\frac{\widehat{p}}{p_0}\right)^x \left(\frac{1-\widehat{p}}{1-p_0}\right)^{n-x} \right) \\
&= 2\left[x\log\frac{\widehat{p}}{p_0} + (n-x)\log\frac{1-\widehat{p}}{1-p_0} \right]
\end{aligned}$$

and the limiting distribution of λ under H_0 is χ_1^2. The p-value for this test is $\mathbb{P}(\chi_1^2 > \lambda)$.

The Wald test statistic is

$$W = \frac{\widehat{p} - p_0}{\widehat{\text{se}}}$$

where $\widehat{p} = \frac{X}{n}$ and p_0 is as given in the problem statement.

Also, we have

$$\begin{aligned}
\text{se} &= \sqrt{\mathbb{V}(\widehat{p})} \\
&= \sqrt{\mathbb{V}(X/n)} \\
&= \frac{\sqrt{\mathbb{V}(X)}}{n} \\
&= \frac{\sqrt{np(1-p)}}{n}
\end{aligned}$$

so the observed value of the Wald test statistic is

$$w = \frac{n(\widehat{p} - p_0)}{\sqrt{n\widehat{p}(1-\widehat{p})}}$$

and the p-value is approximately

$$\begin{aligned}
\mathbb{P}(|Z| > |w|) &= \mathbb{P}(Z^2 > w^2) \\
&= \mathbb{P}\left(\chi_1^2 > \frac{n(\widehat{p} - p_0)^2}{\widehat{p}(1-\widehat{p})}\right).
\end{aligned}$$

□

Problem 10.16. Let θ be a scalar parameter and suppose we test

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta \neq \theta_0.$$

Let W be the Wald test statistic and let λ be the likelihood ratio test statistic. Show that these tests are equivalent in the sense that

$$\frac{W^2}{\lambda} \xrightarrow{P} 1$$

as $n \to \infty$.

Solution. We know that

$$\lambda = 2\log\left(\frac{\mathcal{L}(\widehat{\theta})}{\mathcal{L}(\theta_0)}\right)$$

and that

$$W = \frac{\widehat{\theta} - \theta_0}{\widehat{\text{se}}}.$$

Note that $2\log\left(\frac{\mathcal{L}(\widehat{\theta})}{\mathcal{L}(\theta_0)}\right) = 2(\ell(\widehat{\theta}) - \ell(\theta_0))$. But a Taylor expansion on the log-likelihood yields

$$\ell(\widehat{\theta}) \approx \ell(\theta_0) + \ell'(\theta_0)(\widehat{\theta} - \theta_0) + \frac{1}{2}\ell''(\theta_0)(\widehat{\theta} - \theta_0)^2$$

so

$$\begin{aligned}
\lambda &= 2(\ell(\widehat{\theta}) - \ell(\theta_0)) \\
&\approx 2\left(\ell'(\theta_0)(\widehat{\theta} - \theta_0) + \frac{1}{2}\ell''(\theta_0)(\widehat{\theta} - \theta_0)^2\right).
\end{aligned}$$

Next, note that $\ell'(\widehat{\theta}) = 0$ as $\widehat{\theta}$ is the MLE. But moreover, we have

$$0 = \ell'(\widehat{\theta}) \approx \ell'(\theta_0) + \ell''(\theta_0)(\widehat{\theta} - \theta_0)$$

so we obtain

$$\begin{aligned}
\lambda &\approx 2\left(\ell'(\theta_0)(\widehat{\theta} - \theta_0) + \frac{1}{2}\ell''(\theta_0)(\widehat{\theta} - \theta_0)^2\right) \\
&\approx -\ell''(\theta_0)(\widehat{\theta} - \theta_0)^2.
\end{aligned}$$

Next, we have

$$W = \frac{\widehat{\theta} - \theta_0}{\widehat{\mathrm{se}}}$$

so

$$\begin{aligned}
\frac{W^2}{\lambda} &\approx -\frac{1}{\widehat{\mathrm{se}}^2 \ell''(\theta_0)} \\
&= -\frac{I_n(\widehat{\theta})}{\ell''(\theta_0)} \\
&= -\frac{nI(\widehat{\theta})}{\ell''(\theta_0)}.
\end{aligned}$$

Now we note that $I(\widehat{\theta}) \xrightarrow{P} I(\theta_0)$, using the consistency of the MLE and the continuity of I. Furthermore, we have

$$\frac{1}{n}\ell''(\theta_0) = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial^2}{\partial\theta^2} \log f(X;\theta_0)$$

and thus, as the right-hand-side is an average of n IID terms, we can use the Weak Law of Large Numbers to conclude that

$$\begin{aligned}
\frac{1}{n}\ell''(\theta_0) &\xrightarrow{P} -\mathbb{E}_{\theta_0}\left(\frac{\partial^2}{\partial\theta^2} \log f(X;\theta_0)\right) \\
&= -I(\theta_0).
\end{aligned}$$

It thus follows that $\frac{W^2}{\lambda} \xrightarrow{P} 1$, as desired. We are done. $\square$