Chapter 6 Solutions

Andrew Wu
Wasserman: All of Statistics

March 1, 2025

Problem 6.1. Let $X_1, \dots, X_n \sim \text{Pois}(\lambda)$ and let $\widehat{\lambda} = n^{-1}\sum_{i=1}^n X_i$. Find the bias, SE, and MSE of this estimator.

Solution. We have $\text{bias}(\widehat{\lambda}) = \mathbb{E}(\widehat{\lambda}) - \lambda$. Note that

$$\begin{aligned}
\mathbb{E}(\widehat{\lambda}) &= \mathbb{E}\left(n^{-1}\sum_{i=1}^n X_i\right) \\
&= \frac{1}{n} \cdot n \cdot \mathbb{E}(X_i) \\
&= \lambda
\end{aligned}$$

so $\text{bias}(\widehat{\lambda}) = 0$.

Next, we have $\text{se} = \sqrt{\mathbb{V}(\widehat{\lambda})}$. So

$$\begin{aligned}
\mathbb{V}(\widehat{\lambda}) &= \mathbb{V}\left(\frac{1}{n}\sum_{i=1}^n X_i\right) \\
&= \frac{1}{n^2} \cdot n \cdot \mathbb{V}(X_i) \\
&= \frac{\lambda}{n}
\end{aligned}$$

so $\text{se} = \sqrt{\lambda/n}$.

Finally, we have $\text{mse} = \mathbb{V}(\widehat{\lambda}) + \text{bias}^2(\widehat{\lambda}) = \frac{\lambda}{n}$. □

Problem 6.2. Let $X_1, \dots, X_n \sim \text{Uniform}(0, \theta)$ and let $\widehat{\theta} = \max\{X_1, \dots, X_n\}$. Find the bias, SE, and MSE of this estimator.

Solution. We have $\text{bias}(\widehat{\theta}) = \mathbb{E}(\widehat{\theta}) - \theta$, with $\mathbb{E}(\widehat{\theta}) = \mathbb{E}(\max(X_1, \dots, X_n\}))$.

Let F and f be the CDF and PDF of random variable $\widehat{\theta}$. We have $\mathbb{P}(\widehat{\theta} \le y) = F(y) = \left(\frac{y}{\theta}\right)^n$. Thus $f(y) = \frac{ny^{n-1}}{\theta^n}$, so

$$\begin{aligned}
\mathbb{E}(\widehat{\theta}) &= \int_0^\theta y \cdot f(y)dy \\
&= \frac{n}{\theta^n}\int_0^\theta y^n dy \\
&= \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1} \\
&= \theta \cdot \frac{n}{n+1}.
\end{aligned}$$

Thus $\text{bias}(\widehat{\theta}) = \mathbb{E}(\widehat{\theta}) - \theta = \theta \cdot \frac{n}{n+1} - \theta = -\frac{\theta}{n+1}$.

We have $\text{se} = \sqrt{\mathbb{V}(\widehat{\theta})}$, and

$$\begin{aligned}
\mathbb{V}(\widehat{\theta}) &= \mathbb{E}(\widehat{\theta}^2) - \mathbb{E}(\widehat{\theta})^2 \\
&= \int_0^\theta y^2 f(y)dy - \left(\frac{n\theta}{n+1}\right)^2 \\
&= \frac{n}{n+2} \cdot \theta^2 - \left(\frac{n}{n+1}\right)^2 \cdot \theta^2 \\
&= \frac{n}{(n+2)(n+1)^2} \cdot \theta^2.
\end{aligned}$$

Thus $\text{se} = \frac{\theta}{n+1}\sqrt{\frac{n}{n+2}}$.

Finally, we have $\text{mse} = \mathbb{V}(\widehat{\theta}) + \text{bias}^2(\widehat{\theta}) = \frac{2\theta^2}{(n+1)(n+2)}$. □

Problem 6.3. Let $X_1, \ldots, X_n \sim \text{Uniform}(0, \theta)$ and let $\widehat{\theta} = 2\overline{X}_n$. Find the bias, SE, and MSE of this estimator.

Solution. We have $\text{bias}(\widehat{\theta}) = \mathbb{E}(\widehat{\theta}) - \theta$, with $\mathbb{E}(\widehat{\theta}) = \mathbb{E}(2\overline{X}_n)$. Note that

$$\begin{aligned}
\mathbb{E}(2\overline{X}_n) &= 2\mathbb{E}(\overline{X}_n) \\
&= \frac{2}{n} \cdot n \cdot \mathbb{E}(X_i) \\
&= 2 \cdot \frac{\theta}{2} = \theta
\end{aligned}$$

so $\text{bias}(\widehat{\theta}) = 0$.

We have $\text{se} = \sqrt{\mathbb{V}(\widehat{\theta})} = \sqrt{\mathbb{V}(2\overline{X}_n)}$. So as

$$\begin{aligned}
\mathbb{V}(2\overline{X}_n) &= 4\mathbb{V}(\overline{X}_n) \\
&= \frac{4}{n^2} \cdot n \cdot \mathbb{V}(X_i) \\
&= \frac{\theta^2}{3n},
\end{aligned}$$

we have $\text{se} = \frac{\theta}{\sqrt{3n}}$.

Finally, we have $\text{mse} = \mathbb{V}(\widehat{\theta}) + \text{bias}^2(\widehat{\theta}) = \frac{\theta^2}{3n}$. □