Problem Set #4

✔ **5/5** points earned (100%)

Quiz passed!

✔ 1 / 1 points

1.

Consider a directed graph with real-valued edge lengths and no negative-cost cycles. Let s be a source vertex. Assume that there is a unique shortest path from s to every other vertex. What can you say about the subgraph of G that you get by taking the union of these shortest paths? [Pick the strongest statement that is guaranteed to be true.]

○ It is a path, directed away from s.

◉ It is a tree, with all edges directed away from s.

▲

Correct Response
Subpaths of shortest paths must themselves be shortest paths. Combining this with uniqueness, the union of shortest paths cannot include two different paths between any source and destination.

○ It is a directed acyclic subgraph in which s has no incoming arcs.

○ It has no strongly connected component with more than one vertex.

✔ 1 / 1 points

2.

Consider the following optimization to the Bellman-Ford algorithm. Given a graph $G = (V, E)$ with real-valued edge lengths, we label the vertices $V = \{1, 2, 3, \ldots, n\}$. The source vertex s should be labeled "1", but the rest of the labeling can be arbitrary. Call an edge $(u, v) \in E$ *forward* if $u < v$ and *backward* if $u > v$. In every odd iteration of the outer loop (i.e., when $i = 1, 3, 5, \ldots$), we visit the vertices in the order from 1 to n. In every even iteration of the outer loop (when $i = 2, 4, 6, \ldots$), we visit the vertices in the order from n to 1. In every odd iteration, we update the value of $A[i, v]$ using only the forward edges of the form (w, v), using the *most recent* subproblem value for w (that from the current iteration rather than the previous one). That is, we compute $A[i, v] = \min\{A[i - 1, v], \min_{(w,v)} A[i, w] + c_{wv}\}$, where the inner minimum ranges only over forward edges sticking into v (i.e., with $w < v$). Note that all relevant subproblems from the current round ($A[i, w]$ for all $w < v$ with $(w, v) \in E$) are available for constant-time lookup. In even iterations, we compute this same recurrence using only the backward edges (again, all relevant subproblems from the current round are available for constant-time lookup). Which of the following is true about this modified Bellman-Ford algorithm?

○ This algorithm has an asymptotically superior running time to the original Bellman-Ford algorithm.

◉ It correctly computes shortest paths if and only if the input graph has no negative-cost cycle.

▲

Correct Response
Indeed. Can you prove it? As a preliminary step, prove that with a directed acyclic graph, considering destinations in topological order allows one to compute correct shortest paths in one pass (and thus, in linear time). Roughly, pass i of this optimized Bellman-Ford algorithm computes shortest paths amongst those comprising at most i "alternations" between forward and backward edges.

○ It correctly computes shortest paths if and only if the input graph has no negative edges.

○ It correctly computes shortest paths if and only if the input graph is a directed acyclic graph.

✓ 1 / 1 points

3.

Consider a directed graph in which every edge has length 1. Suppose we run the Floyd-Warshall algorithm with the following modification: instead of using the recurrence A[i,j,k] = min{A[i,j,k-1], A[i,k,k-1] + A[k,j,k-1]}, we use the recurrence A[i,j,k] = A[i,j,k-1] + A[i,k,k-1] * A[k,j,k-1]. For the base case, set A[i,j,0] = 1 if (i,j) is an edge and 0 otherwise. What does this modified algorithm compute -- specificially, what is A[i,j,n] at the conclusion of the algorithm?

○ None of the other answers are correct.

Correct Response
Indeed. How would you describe what the recurrence is in fact computing?

○ The length of a longest path from i to j.

○ The number of shortest paths from i to j.

○ The number of simple (i.e., cycle-free) paths from i to j.

✔ 1 / 1 points

4.

Suppose we run the Floyd-Warshall algorithm on a directed graph $G = (V, E)$ in which every edge's length is either -1, 0, or 1. Suppose further that G is strongly connected, with at least one u-v path for every pair u, v of vertices. The graph G may or may not have a negative-cost cycle. How large can the final entries A[i,j,n] be, in absolute value? Choose the smallest number that is guaranteed to be a valid upper bound. (As usual, n denotes $|V|$.) [WARNING: for this question, make sure you refer to the implementation of the Floyd-Wardshall algorithm given in lecture, rather than to some alternative source.]

○ $n - 1$

○ n^2

○ $+\infty$

● 2^n

Correct Response
By induction. Can you prove a sharper (exponential) bound, or is this tight?

✓ 1 / 1
points

5.

Which of the following events cannot possibly occur during the reweighting step of Johnson's algorithm for the all-pairs shortest-paths problem? (Assume that the input graph has no negative-cost cycles.)

○ Reweighting strictly increases the length of some s-t path, while strictly decreasing the length of some t-s path.

○ In a directed acyclic graph, reweighting causes the length of every path to strictly increase.

○ The length of some edge strictly decreases after the reweighting.

◉ In a directed graph with at least one cycle, reweighting causes the length of every path to strictly increase.

▲

Correct Response

Consider two "halves" of a cycle. The increase in length of one of these paths equals the decrease in length of the other path.