Physics 410 Quiz #5 – Thursday, April 3, 2025

Name: Solutions

1.

a. [5 points] Calculate the density of states $D(\epsilon)$ for a 2-dimensional degenerate electron gas consisting of N particles in a square of side length L. The orbital energies are given by $\frac{\hbar^2}{2ML^2}(n_x^2 + n_y^2)$, where $n_x, n_y = 1,2,3, \ldots$.

$$N = \underbrace{2}_{\text{spin}} \times \frac{1}{4} \cdot 2\pi \int_0^\infty n \cdot f(\varepsilon)\, dn \qquad d\varepsilon = \frac{\hbar^2 \pi^2}{2ML^2} \cdot 2n\, dn \longrightarrow n\, dn = \frac{ML^2}{\hbar^2 \pi^2} d\varepsilon$$

$$= \int_0^\infty \underbrace{\frac{ML^2}{\pi \hbar^2}} f(\varepsilon)\, d\varepsilon$$

$$\boxed{D(\varepsilon) = \frac{ML^2}{\pi \hbar^2}}$$

b. [5 points] Calculate the Fermi energy ϵ_F of this gas.

$$N = \int_0^\infty D(\varepsilon) f(\varepsilon)\, d\varepsilon \overset{\tau = 0}{=\!=} \int_0^{\varepsilon_F} D(\varepsilon)\, d\varepsilon$$

$$= \frac{ML^2}{\pi \hbar^2} \int_0^{\varepsilon_F} d\varepsilon = \frac{ML^2}{\pi \hbar^2} \varepsilon_F$$

$$\Downarrow$$

$$\boxed{\varepsilon_F = \frac{\pi \hbar^2 N}{ML^2}}$$

c. [5 points] Calculate the ground-state energy U_0 of this gas. You may leave your result in terms of ϵ_F.

$$U_0 = \langle \varepsilon \rangle_0 = \int_0^\infty \varepsilon \, D(\varepsilon) \, f(\varepsilon, \tau{:}0) \, d\varepsilon$$

$$= \int_0^{\varepsilon_F} \varepsilon \frac{ML^2}{\pi \hbar^2} \, d\varepsilon$$

$$\boxed{= \frac{ML^2}{\pi \hbar^2} \cdot \frac{1}{2} \varepsilon_F^2 \qquad = \frac{1}{2} N \varepsilon_F}$$

2. [5] Find the chemical potential of an *ideal* (classical regime), spinless, monatomic gas in two dimensions, with *N* particles confined to a square of side length *L* at temperature τ (work in the Canonical Ensemble). *You must show your work.* You may use the result that $\int_0^\infty x \, e^{-x^2} dx = \frac{1}{2}$. The orbital energies are the same as in Problem 1.

$$N = \sum_{\text{orbitals}} f_{CL}(\varepsilon) = \sum_{\text{orbitals}} e^{\frac{(\mu - \varepsilon)}{\tau}} = e^{\mu/\tau} \sum_{\text{orbitals}} e^{-\varepsilon/\tau}$$

$$= e^{\mu/\tau} \underline{Z_1}$$

$$Z_1 = 2 \cdot \frac{1}{4} \cdot 2\pi \cdot \int_0^\infty e^{-\frac{\hbar^2 \pi^2 n^2}{2mL^2\tau}} n \cdot dn$$

$$x = \sqrt{\frac{\hbar^2 \pi^2}{2ML^2\tau}}\, n$$

$$\downarrow$$

$$Z_1 = \frac{2ML^2\tau}{\hbar^2 \pi} \int_0^\infty e^{-x^2} \cdot x \, dx$$

$$= \frac{ML^2}{\hbar^2 \pi}$$

$$\mu = \tau \ln(N/Z_1)$$

$$\boxed{\mu = \tau \ln\left(\frac{N\pi\hbar^2}{ML^2\tau}\right)}$$