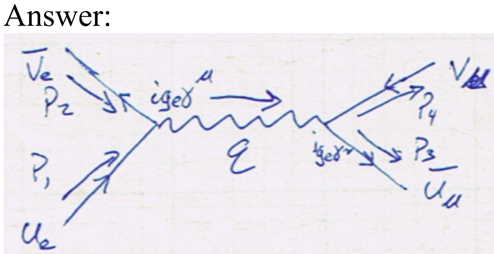
PHY493/803, Intro to Elementary Particle Physics

Homework 5

Please clearly state any assumptions, show all your work, number the equations, and indicate logical connections between the lines.

1. (10pts + 15pts + 15pts + 15pts + 15pts = 70 pts total)
   Calculate the cross section for the reaction $e^+e^- \rightarrow \mu^+\mu^-$. Assume that the incoming particles are not polarized and that the spin projections of the outgoing muons are not measured. Further assume that the energy of the incoming electrons in the center-of-momentum frame is much larger than the electron or muon masses.
   Hint: For this problem, it will be highly instructive to work through Griffiths' examples 7.1, 7.3, 7.5 and 7.7 first. Griffiths' problems 7.26 and 7.38 will also be of use to peruse first. Note the "crossing" symmetry with the $e^+\mu^- \rightarrow e^+\mu^-$ scattering process.

   a) Draw the relevant Feynman diagram(s) for the lowest order process
   Answer:



   b) Use the Feynman rules to determine the matrix element.
   Answer:

$$M = i \int [\bar{v}_e i g_e \gamma^\mu u_e] \frac{-i g_{\mu\nu}}{q^2} [\bar{u}_{muon} i g_e \gamma^\nu v_{muon}]$$
$$\times (2\pi)^4 \delta^4(p_1 + p_2 - q)(2\pi)^4 \delta^4(q - p_3 - p_4) \frac{d^4 q}{(2\pi)^4}$$

   The first line contains the two currents and the propagators and vertices, the second line the momentum conservation delta functions.
   Taking care of the delta function and the integral to get rid of q, this simplifies to

$$M = -i^4 g_e^2 [\bar{v}_e \gamma^\mu u_e] \frac{g_{\mu\nu}}{(p_1 + p_2)^2} [\bar{u}_{muon} (\gamma^\nu v_{muon}] (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4)$$

   and $g_{\mu\nu}$ lowers the index on the second gamma function, plus we drop the extra delta function and $(2\pi)^4$. Therefore the matrix element becomes

$$M = -\frac{g_e^2}{(p_1 + p_2)^2} [\bar{v}_e \gamma^\mu u_e][\bar{u}_{muon} \gamma_\mu v_{muon}]$$

   c) Use Casimir's trick to calculate the spin-averaged square of the matrix element $\langle|\mathcal{M}^2|\rangle$. You can use the approximation $E_e \gg m_\mu$.

Hint: You will need the following trace relationships:
$Tr[\gamma^\mu(\not{p}_1+m_e)\gamma^\nu(\not{p}_2-m_e)] = 4\{p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu}(p_1 \cdot p_2)\}$
$Tr[\gamma_\mu(\not{p}_3+m_\mu)\gamma_\nu(\not{p}_4-m_\mu)] = 4\{p_{3,\mu}p_{4,\nu} + p_{4,\mu}p_{3,\nu} - g_{\mu\nu}(p_3 \cdot p_4)\}$
And remember to average over initial state spins and sum over final state spins. And $g^{\mu\nu}g_{\mu\nu} = 4$.

Answer: We average over the initial state spins, which gives factor ¼ because it is two fermions, and we sum over the final state spins. Also write $s = (p_1 + p_2)^2$:

$$\langle |M|^2 \rangle = \frac{g_e^4}{4s^2} \sum_{e\ spins} \sum_{\substack{muon \\ spins}} [\bar{v}_e\, \gamma^\mu u_e][\bar{u}_{muon}\gamma_\mu v_{muon}]([\bar{v}_e\, \gamma^\nu u_e][\bar{u}_{muon}\gamma_\nu v_{muon}])^*$$

The sums can be separated into an electron one and a muon one:
$\sum_{e\ spins}[\bar{v}_e\, \gamma^\mu u_e][\bar{v}_e\, \gamma^\nu u_e]^*$ and $\sum_{muon\ spins}[\bar{u}_{muon}\gamma_\mu v_{muon}][\bar{u}_{muon}\gamma_\nu v_{muon}]^*$, to which we can apply the trace theorems, so these become:
$\mathrm{Tr}[\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu(\not{p}_2-m_e)]$ and $\mathrm{Tr}[\gamma_\mu(\not{p}_3+m_\mu)\gamma_\nu(\not{p}_4-m_\mu)]$
The minus signs arise because particles 2 and 4 are antiparticles.
Then with the hint, the traces become
$4\{p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu}(p_1 \cdot p_2)\}$ and
$4\{p_{3,\mu}p_{4,\nu} + p_{4,\mu}p_{3,\nu} - g_{\mu\nu}(p_3 \cdot p_4)\}$.
With these, the average matrix element is now

$$\langle |M|^2 \rangle = \frac{g_e^4}{4s^2} 4 \cdot 4\{p_1^\mu p_2^\nu + p_2^\mu p_1^\nu - g^{\mu\nu}(p_1 \cdot p_2)\}\{p_{3,\mu}p_{4,\nu} + p_{4,\mu}p_{3,\nu} - g_{\mu\nu}(p_3 \cdot p_4)\}$$

$$= \frac{4g_e^4}{s^2}[2(p_1 \cdot p_3)(p_2 \cdot p_4) + 2(p_1 \cdot p_4)(p_2 \cdot p_3) - 2(p_1 \cdot p_2)(p_3 \cdot p_4) - 2(p_3 \cdot p_4)(p_1 \cdot p_2) + 4(p_1 \cdot p_2)(p_3 \cdot p_4)]$$

$$= \frac{8g_e^4}{s^2}[(p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)]$$

The assumption $E_e \gg m_\mu$ means in the CM frame that for all four particles, $E \sim p$. We get for these dot products:

$$p_1 \cdot p_3 = p_2 \cdot p_4 = E^2 - \vec{p}_1 \cdot \vec{p}_3 = E^2(1 - cos\,\theta)$$

where the angle θ is between particle 1 and 3 and therefore also between 2 and 4.

$$p_1 \cdot p_4 = p_2 \cdot p_3 = E^2 - \vec{p}_1 \cdot \vec{p}_4 = E^2(1 - cos(\pi - \theta))$$

And in the CM system, $cos(\pi - \theta) = -cos\theta$, thus

$$p_1 \cdot p_4 = p_2 \cdot p_3 = E^2(1 + cos\,\theta)$$

Plugging this into the average matrix element gives

$$\langle |M|^2 \rangle = \frac{4g_e^4}{s^2}(E^2(1 - cos\,\theta))^2 + (E^2(1 + cos\,\theta))^2$$

Since under these assumptions, $s = (2E)^2$, the factors of E^2 cancel out and we are left with
$\langle |M|^2 \rangle = g_e^4(1 + cos^2\theta)$.

d) Calculate the cross section, starting from the average matrix element $\langle |M|^2 \rangle = \frac{8g_e^4}{s^2}((p_1 \cdot p_3)(p_2 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3)$, or from your result in part c if you already simplified that, and remember that the cross section is given by the golden rule, $\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2}\frac{\langle|M|^2\rangle}{(E_1+E_2)^2}\frac{|p_f|}{|p_i|}$. Give your result in the center-of-momentum frame. You can use the approximation $E_e \gg m_\mu$.

Answer:

$$\frac{d\sigma}{d\Omega} = \frac{1}{(8\pi)^2}\frac{g_e^4(1+cos^2\theta)}{(E_1+E_2)^2}\frac{|p_f|}{|p_i|}$$

and $|p_f| = |p_i|$ and $E_1 = E_2 = E$ in the CM system. Then

$$\frac{d\sigma}{d\Omega} = \frac{g_e^4}{256\pi^2E^2}(1+cos^2\theta)$$

The total cross section is $\sigma = \int \frac{d\sigma}{d\Omega}\, sin\theta d\theta d\phi$, or

$$\sigma = \frac{g_e^4}{256\pi^2 4E^2}\int(1+cos^2\theta)sin\theta d\theta d\phi$$

$$= \frac{g_e^4}{256\pi^2 4E^2}\left[4\pi + 2\pi\int cos^2\theta sin\theta d\theta\right]$$

We can get the integral over $cos^2\theta sin\theta$ through a change of variables, or by looking it up: $\int cos^2\theta sin\theta d\theta = -\frac{1}{3}x^3|_1^{-1} = \frac{2}{3}$. Then

$$\sigma = \frac{g_e^4}{256\pi^2 4E^2}\left(4\pi + \frac{4}{3}\pi\right) = \frac{g_e^4}{48\pi E^2}$$

e) How does the differential cross section for $e^+e^- \to \mu^+\mu^-$ compare with the cross section for $e^+\mu^- \to e^+\mu^-$ scattering at the same electron energy (see Griffiths problem 7.38)?

Answer:

Here, for $e^+e^- \to \mu^+\mu^-$, we have

$$\frac{d\sigma}{d\Omega} = \frac{g_e^4}{64\pi^2}\frac{1}{s}(1+cos^2\theta)$$

(where we use $s = 4E^2$).

For $e^+\mu^- \to e^+\mu^-$, we have

$$\frac{d\sigma}{d\Omega} = \frac{g_e^4}{64\pi^2}\frac{2}{s}\frac{(1+cos^4\theta/2)}{sin^4\theta/2}$$

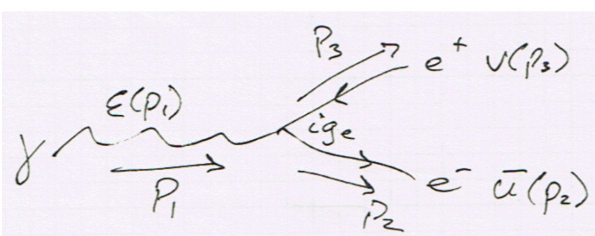
Observations:

- The dependence on the coupling is the same
- The dependence on s is the same
- The angular dependence is different:
  - The cross-section for $e^+\mu^-$ scattering goes to infinity for small angles, while the integral for $e^+e^- \to \mu^+\mu^-$ remains finite for all angles.
  - One diagram is the rotated version of the other, and this is also clear when expressing the cross-section in terms of s, t, u. This was studied in class.

2. (25+15 pts) {**Required for PHY803 students only. +40 pts extra credit for PHY493 students.**}
    a) Griffiths' problem 7.48(a), show that the result is $\Gamma = \frac{g^2(m_\gamma^2 - 4m_e^2)^{3/2}}{8\pi m_\gamma^2}$:

    Imagine that the photon, instead of being a massless vector (spin 1) particle, were a massive scalar (spin 0). Specifically, suppose the QED vertex factor were $ig_e 1$ (where 1 is the 4 by 4 unitary matrix), and the 'photon' propagator were $\frac{-i}{q^2 - (m_\gamma c)^2}$.
    There is no photon polarization vector now, and hence no factor for external photon lines. Apart from that, the Feynman rules for QED are unchanged. Assuming it is heavy enough, this 'photon' can decay. Calculate the decay rate for $\gamma \to e^+ + e^-$.
    Notes:
    a. The description of the problem begins on the previous page (pg 272 in my book).
    b. It is helpful to calculate the width in the COM frame.
    c. The trace to compute in this case is
    $$Tr\left(\gamma_\mu p_2^\mu + m_e\right)\left(\gamma_\mu p_2^\mu - m_e\right) = p_2 \cdot p_3 - 4m_e^2$$
    d. Use the golden rule for 2-particle decays (book Eq. 6.35) to determine the width.

Answer: The Feynman diagram for this process is



The matrix element is $M = [\bar{u}(p_2)(ig_e)v(p_3)](2\pi)^4\delta(p_1 - p_2 - p_3)$
$$= (i)^2 g_e[\bar{u}(p_2)v(p_3)]$$

The average matrix element squared is
$$\langle |M|^2 \rangle = g_e^2 Tr\left[\left(\not{p}_2 + m_e\right)\left(\not{p}_3 - m_e\right)\right]$$
$$= g_e^2 Tr\left[\not{p}_2 \not{p}_3\right] - 4m_e^2$$

$$= 4g_e^2(p_2 \cdot p_3 - m_e^2)$$

Because 4-momentum is conserved, we can evaluate $p_2 \cdot p_3$:

$$m_\gamma^2 = p_1^2 = p_2^2 + p_3^2 + 2p_2 \cdot p_3$$

Therefore, $\langle |M|^2 \rangle = 4g_e^2\left(\frac{m_\gamma^2}{2} - 2m_e^2\right) = 2g_e^2\left(m_\gamma^2 - 4m_e^2\right)$

We obtain the width from Fermi's golden rule,

$$\Gamma = \frac{|p_3|}{8\pi m_\gamma^2} 2g_e^2\left(m_\gamma^2 - 4m_e^2\right)$$

In the CM frame, $|p_3| = \frac{1}{2}\sqrt{m_\gamma^2 - 4m_e^2}$, therefore the width is

$$\Gamma = \frac{g_e^2(m_\gamma^2 - 4m_e^2)^{3/2}}{8\pi m_\gamma^2}$$

b) Griffiths' problem 7.48(b): If $m_\gamma = 300$ MeV, find the lifetime of the 'photon', in seconds.
Notes:
   a. You can use $\alpha = \frac{1}{137} = \frac{g^2}{4\pi}$.
   b. To get the correct units, you will have $\hbar = 6.58 \times 10^{-22}$ MeV·s to get the time in seconds.

Answer:

$$\tau = \frac{1}{\Gamma} = \frac{8\pi m_\gamma^2}{g_e^2(m_\gamma^2 - 4m_e^2)^{3/2}}$$

Since the electron mass is much smaller than the photon mass, we can simplify this to

$$\tau = \frac{2\hbar}{\alpha m_\gamma c^2}$$

Plug in all of the numbers gives 6E-22 seconds.