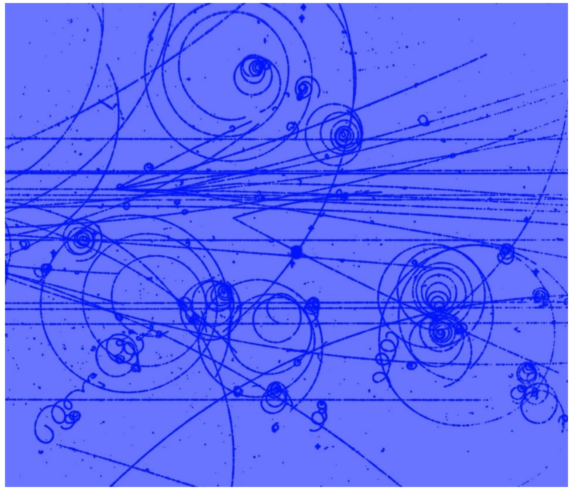
Announcements

Quiz:
- Pick up quiz after class. **No quiz next week** – midterm instead.



Homework:
Homework 1 grades posted on gradescope. Solutions posted on D2L.
Third HW posted on D2L : Q1/Q2 can be done now

Midterm: Friday, Feb 21 in class
Will cover through “bound states”.
Equation sheet: 1 letter-sized (8 ½ by 11 inches) page front and back, handwritten

Paper:
Topic due Monday Feb 17 at 3pm. Fill out this google form before then:
https://forms.gle/MmCk8NtrMm7RdfLC7

Office hours: next week on Wednesday 4-5pm, not on Friday

Observables

To learn anything about particle interactions, we have to observe something.

There are three basic categories:

A) Bound states & their spectra (this was Monday)

B) **When particles are "Left Alone", they can:**

1) Do nothing
2) Decay
3) Eventually find another particle and go to category C.

C) **When particles encounter another particle, they can:**

1) Do nothing
2) Scatter off the other particle.
3) Annihilate on the other particle

Use Fermi's golden rule to calculate these:

1. Decay - particle on its own
2. Cross section – interactions with another particle

The Golden Rule Reminder

Fermi's Golden Rule states:

The transition rate for a system depends on two fundamental quantities

1) The "Matrix Element" or "Transition Amplitude"
2) The final state phase space

$$W = \frac{2\pi}{\hbar} \left| \mathcal{M}_{if} \right|^2 \rho(E)$$

The transition rate (decays or interactions)

The Matrix Element. We may not actually know this.

Phase Space or Density of States

W is proportional to cross section or decay rate

Density of States

Density of states*

Also known as the available phase space or just phase space

Describes how many equivalent ways a final state can be configured

1) Assume we have a particle with quantized momentum confined to volume V

$$p = \hbar k = h/\lambda$$

Smallest element of phase space in any coordinate is **h**!

2) The number of equivalent states, N_i, can be calculated by dividing the total phase space volume by the elemental phase space volume:

$$N_i = \frac{1}{(2\pi\hbar)^3} \int dx\, dy\, dz\, dp_x\, dp_y\, dp_z = \frac{V}{(2\pi\hbar)^3} \int d^3p$$

Density of States/ Phase Space

Density of states

Also known as the available phase space or just phase space

Describes how many equivalent ways a final state can be configured

3) Simplify life by calculating over a unit volume (V=1 of your favorite unit). Also consider the option to have **n** particles in the final state, take product over all particles in final state:

$$N_n = \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n} d^3 p_i$$

4) Impose conservation of momentum in the final state, which means there is one less degree of freedom in the distribution of momenta using delta function:

$$N_n = \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n} d^3 p_i \; \delta\left[p_n - (p_0 - \sum_{i=1}^{n-1} p_i)\right]$$

$$= \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n-1} d^3 p_i$$

Impose that difference between initial and final momentum is 0

Dirac Delta Function

The Dirac delta function δ(x) is an infinitesimally small peak at zero

$$\delta(x) = \begin{cases} 0, & (x \neq 0) \\ \infty, & (x = 0) \end{cases}$$

It is the derivative of the Heaviside, or step function, θ(x)

$$\theta(x) = \begin{cases} 0, & (x < 0) \\ 1, & (x > 0) \end{cases}$$

The integral of the delta function has a valuable property

The delta function selects out the zeros of its argument

$$\int_{-\infty}^{+\infty} \delta(x) = 1 \longrightarrow \int_{-\infty}^{+\infty} f(x)\delta(x-a) = f(a)$$

Impose in this way that difference between initial and final momentum is 0

Density of States

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$$= \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n-1} d^3 p_i$$

Phase Space Problem!

What we did is just fine, but it's not Lorentz Invariant

Express our density of states as a 4-momentum instead of 3 momentum

3) Simplify life by calculating over a unit volume (V=1 of your favorite unit). Also consider the option to have **n** particles in the final state:

$$N_n = \frac{1}{(2\pi\hbar)^{3n}} \int \prod_{i=1}^{n} d^3 p_i \longrightarrow N_n = \frac{1}{(2\pi\hbar)^{4n}} \int \prod_{i=1}^{n} d^4 p_i$$

4) Impose Lorentz invariance by fixing the Lorentz invariant inner product for each final state particle:

$$N_n = \frac{1}{(2\pi\hbar)^{4n}} \int \prod_{i=1}^{n} d^4 p_i \, \delta\left[(p^\mu p_\mu)_i - m_i^2 c^2\right]$$

Delta function imposes the requirement that final-state particles are on-shell

Lorentz Invariant Phase Space

While we're at it, we may as well introduce "sanity checks"

1) Energy & momentum are conserved
2) There can be no negative energy states

3) Introduce a delta function for 4-momenta conservation (energy & momentum):

$$N_n = \frac{1}{(2\pi\hbar)^{4n}} \int \underline{\delta^4(p_i - \sum p_f)} \prod_{j=1}^{n} d^4p_j \, \delta\left[(p^\mu p_\mu)_j - m_j^2 c^2\right]$$

4) Introduce a Heaviside function to remove negative final state energies:

$$N_n = \frac{1}{(2\pi\hbar)^{4n}} \int \delta^4(p_i - \sum p_f) \prod_{j=1}^{n} d^4p_j \, \delta\left[(p^\mu p_\mu)_j - m_j^2 c^2\right] \, \underline{\theta(E_j)}$$

Lorentz Invariant Phase Space

We're almost there! One more thing to do...

Recall, our integrals were not truly over continuous momentum space.

Momentum is quantized, so we're actually integrating integer wave number k space!

5) Fourier transform from p-space (momentum) back to k-space (wave number) (or vice-versa):

$$\phi(p) = \frac{1}{2\pi}\int \psi(k)e^{ipk}dk$$

Every delta function gets a 2π in the integral.

6) We are done! We call the energy derivative the "differential Lorentz Invariant phase Space" or "
"dLIPS".

$$\rho(E) = \frac{\partial N_n}{\partial E} = \frac{(2\pi)^{n+4}}{(2\pi\hbar)^{4n}}\,\delta^4\left(p_i - \sum p_f\right)\prod_{j=1}^{n} d^4p_j\;\delta\left[(p^\mu p_\mu)_j - m_j^2c^2\right]\;\theta(E_j)$$

Solve this once for 2-particle final states (n=2), then derive the rest recursively

dLIPS

- dLIPS depends only on the number of particles in the final state
- Calculate once, re-use result
- 2-particle final states:

$$\rho(E) = \frac{\pi}{(2\pi)^6} \frac{p_1}{E}$$

where p_1 is the magnitude of the momentum of one of the outgoing particles ($= p_2$ in the CM frame) and E is the total energy of the decaying particle ($= M$ in the CM frame, i.e. the rest frame of the decaying particle

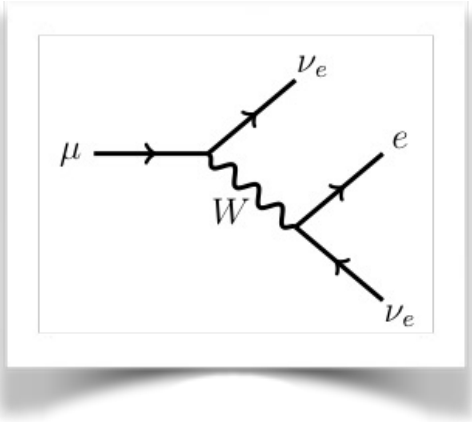
Golden Rule for Decays

We can now build the form of the decay rate

Assume we have a particle at rest decaying to n particles: 1→2,3,4…n

Here we assume the form for the decay rate and insert elements as needed

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 \, \rho(E)$$



The factor *S* factor takes into account final state particle interchange

If the final state has N identical particles, we include a factor of N!

Thus S = 1/N!

The integral is over the phase space $\rho(E)$, which corresponds to the allowed momenta of the outgoing particles

The matrix element *M* depends on momenta of incoming and outgoing particles

Golden Rule for Decays

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Here we assume the form for the decay rate and insert elements as needed

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 \; \rho(E)$$

Plug in density of states calculation:

$$\Gamma = \frac{S}{2\hbar m_1} \int |\mathcal{M}|^2 \; (2\pi)^4 \; \delta^4(p_1 - p_2 - p_3 \cdots - p_n)$$
$$\times \prod_{j=2}^{n} 2\pi \; \delta(p_j^2 - m_j^2 c^2) \; \theta(E_j) \; \frac{d^4 p_i}{(2\pi)^4}$$

Following the book, for 2-particle decays (1→2,3):

$$\Gamma = \frac{S \; |p|}{8\pi\hbar \, m_1^2 c} |\mathcal{M}|^2$$

S = for identical particles
|p| = momentum of outgoing particle
m1 = mass of initial particle

The Golden Rule Reminder

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The transition rate (decays or interactions)

The Matrix Element. We may not actually know this.

Phase Space or Density of States

W is proportional to cross section or decay rate

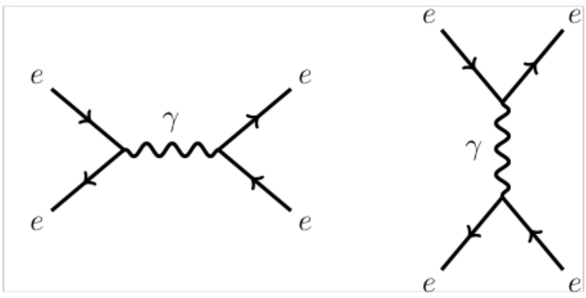
Golden Rule for Scattering Cross Section

We can also build the form for particle scattering

Suppose we have two particle scattering: 1,2 → 3,4,…,n

Don't worry about the inertial frame just yet.

$$\sigma = \frac{S\,\hbar^2}{4\sqrt{(p_1 \cdot p_2)^2 - (m_1 m_2 c^2)^2}} \int |\mathcal{M}|^2 \rho(E)$$



Scattering, so need initial and final state momentum/mass

Following the book's derivation for 2-body scattering (1,2→3,4):

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|p_f|}{|p_i|}$$

Feynman Rules

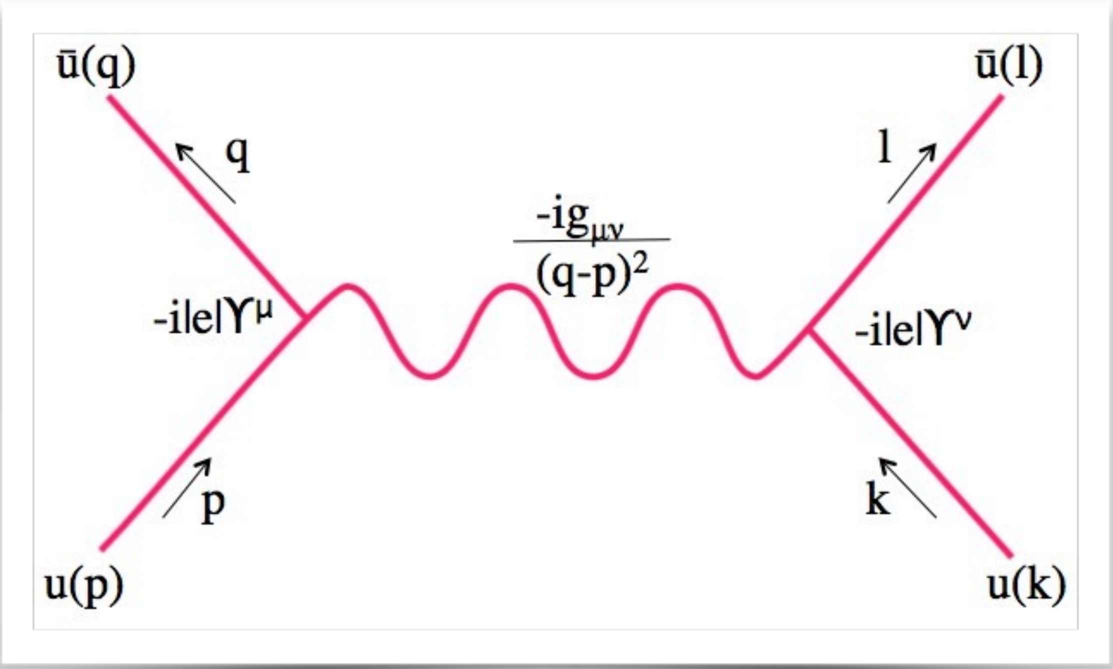
We've been completely ignoring the dynamical part of the equation

All of the specific interaction dynamics are bound up in the **Matrix Element**

We need to now build up the rules for calculating it!

The Feynman diagram will be our guide in most cases

We will assign kinematical features to each part of the diagram, depending on what type of particle it represents.

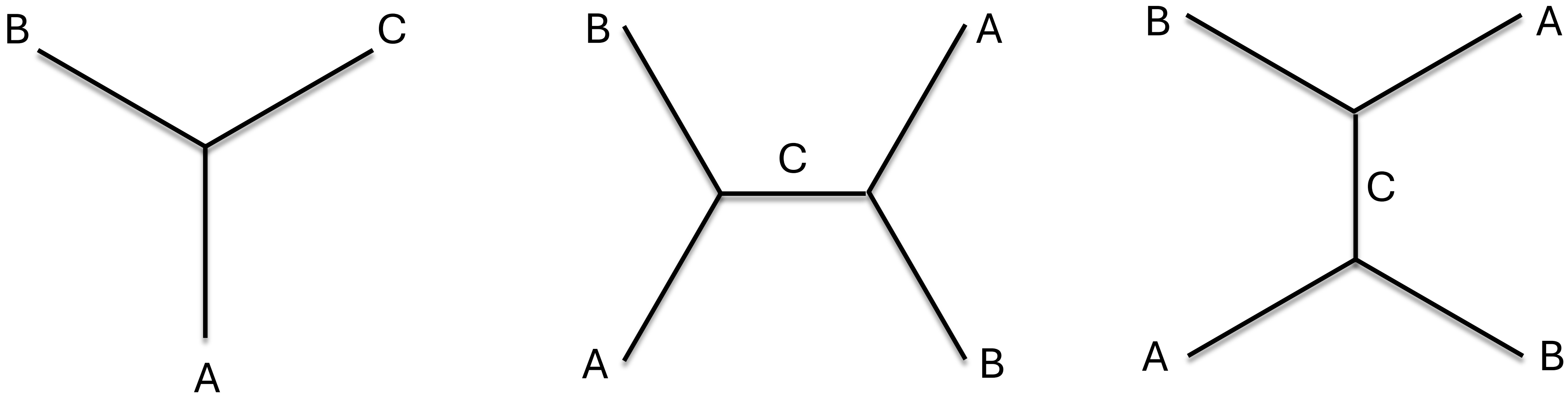


The Feynman diagram will give rise to the matrix element.

ABC Theory

To start with the Feynman rules, use a toy theory: ABC theory

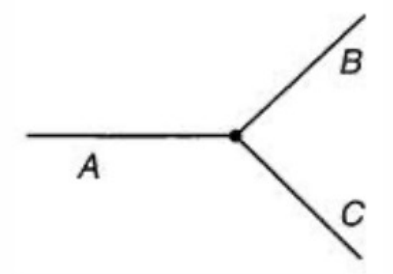
- There are 3 spinless particles: A, B, C
- Each particle is its own antiparticle (No arrows!)
- Only 1 vertex exists and includes all three particles (ABC). Eg, (AAA) is forbidden.
- Masses not assumed, different cases lead to different physics, i.e.
    - If $m_A > m_B + m_C$, then A can decay to B and C.

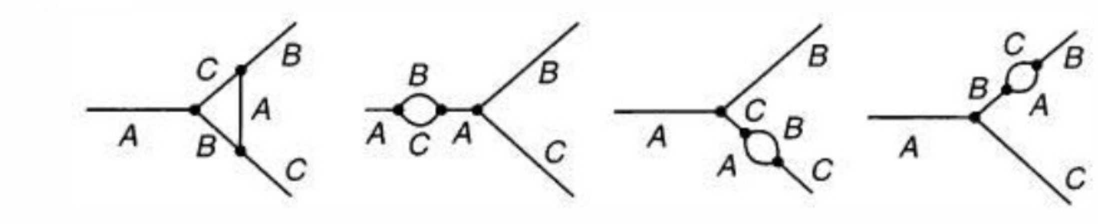


Feynman Rules for ABC Theory

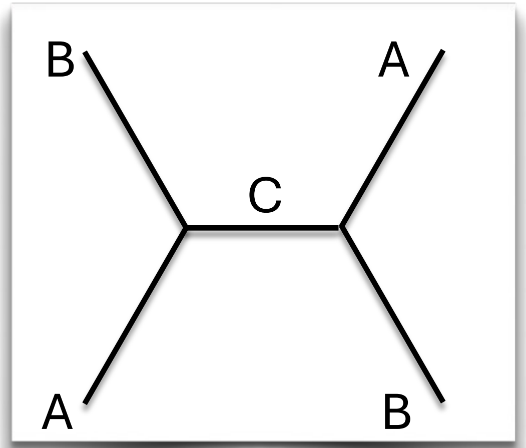
The Feynman rules provide the recipe for constructing an amplitude (Matrix Element $\mathcal{M}$) from a Feynman diagram.

Rule 1:
Draw the Feynman diagram with the minimum number of vertices.
There may be more than 1.





For A + B -> A+ B scattering:



Feynman Rules for ABC Theory

The Feynman rules provide the recipe for constructing an amplitude (Matrix Element $\mathcal{M}$) from a Feynman diagram.

Rule 1:

Draw the Feynman diagram with the minimum number of vertices. There may be more than 1.

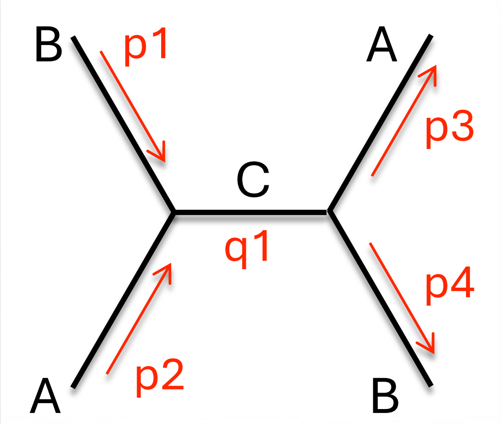
Rule 2:

Label the four-momentum of each line (with arrows), enforcing four-momentum conservation at each vertex.

p1, p2, … external momenta: need arrows

q1, q2, … internal momenta: arrow is arbitrary

We'll keep track of arrows into/out of vertices.



Feynman Rules for ABC Theory

<u>Rule 3:</u>
Each vertex contributes a factor of ($-ig$), where g is referred to as the coupling constant. It specifies the strength of the ABC interaction.

<u>Rule 4:</u>
Each internal line, or propagator, with mass **m** and four-momentum **q** gets a factor of:

$$\frac{i}{q^2 - m^2c^2}$$

q^2 = 4 momentum of internal line

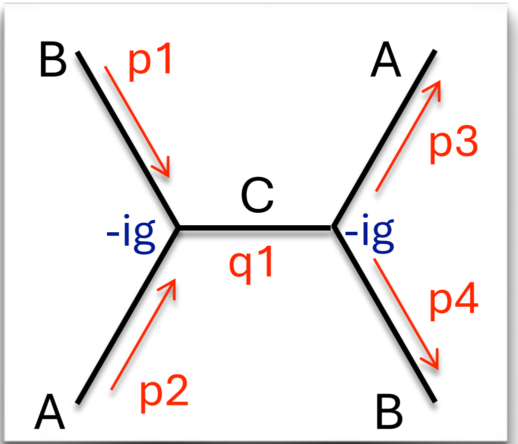
Note: q^2 doesn't have to equal m^2. These are virtual particles!

<u>Rule 5:</u>
Each vertex contributes a delta function to conserve energy and momentum. The **k**$_i$ are the momenta coming into the vertex:

$$(2\pi)^4\ \delta^4(k_1 + k_2 + k_3)$$

k= p or q

k is negative if it is outgoing



Feynman Rules for ABC Theory

Rule 6:

Build up the proto-Matrix Element from the previous factors & add an **i**:

$$\mathcal{M} = i\ \text{(vertex factors) (propagator factors) (momentum conservation)}$$

Rule 7:

Integrate over the internal momenta:

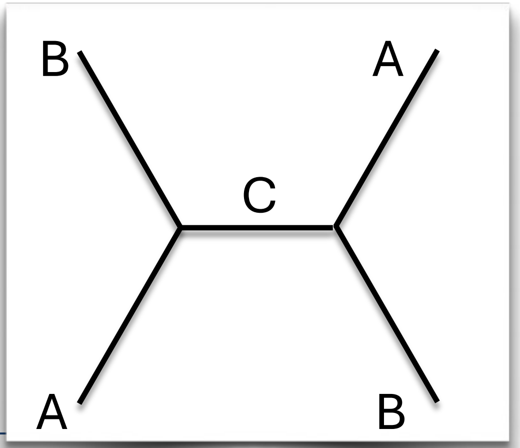
For each internal line write down a factor: $$\frac{1}{(2\pi)^4}\, d^4 q_i$$

This connects initial/final state momenta via the delta functions.

Rule 8:

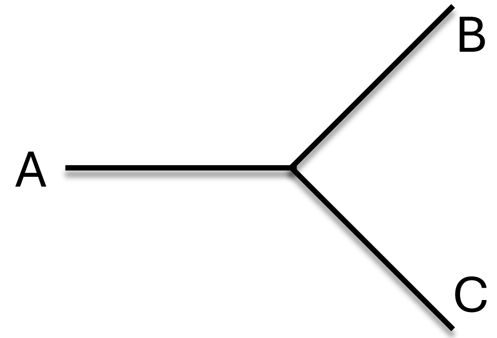
Drop the extra delta function. We do this because the Matrix Element gets squared, and that would double-count the delta function. Don't worry, it gets put back in the Golden Rule Equation!

The result of step 8 is the matrix element!

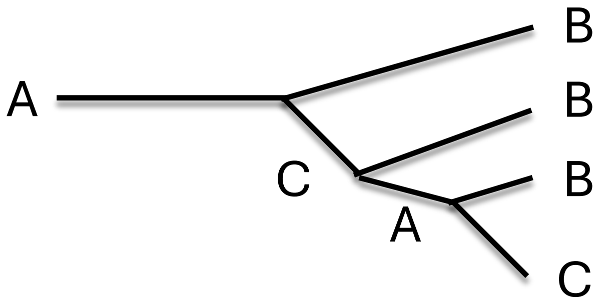


Example: Decay of the A

- Assume A is heavy and B, C are light, $m_A > m_B + m_C$
- Possible Feynman diagrams:



- And if $m_A \gg m_B + m_C$ then also multi-particle decays like:

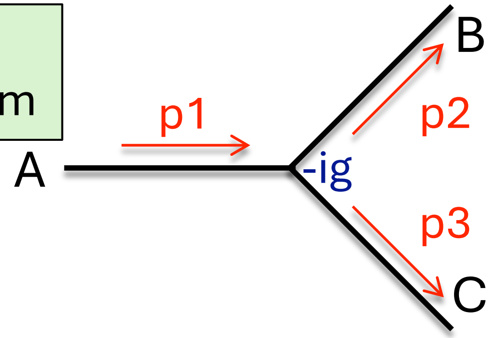


  - Diagrams of this type contain 3 vertices and a virtual A and are thus strongly suppressed

Decay of the A

We are now in a position to calculate something! The simplest is the lifetime and/or decay rate of A→BC.

Rule 1: draw the diagram
Rule 2: label the momentum



We've already done the kinematics!

$$\Gamma = \frac{S\,|p|}{8\pi\, m_A^2}|\mathcal{M}|^2$$

$\hbar = c = 1$

S Factor: The final state particles are not the same, so S=1

Rule 3: Only one vertex, so we get a factor of (-*ig*)

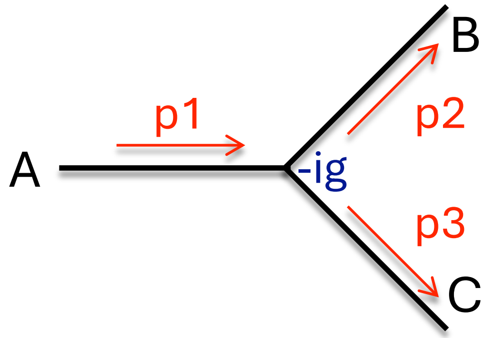
Rule 4: No propagators to worry about

Rule 5: We pick up a delta function over the vertex in/out momenta

$$(2\pi)^4\,\delta^4(p_A - p_B - p_C)$$

Decay of the A

We are now in a position to calculate something! The easiest is the lifetime and/or decay rate of A→BC.



We've already done the kinematics!

$$\Gamma = \frac{S\,|p|}{8\pi\, m_A^2}|\mathcal{M}|^2 \qquad \hbar = c = 1$$

Rule 6: Our matrix element is : $\mathcal{M} = i(-ig) = g$

Rule 7: No internal lines to integrate over

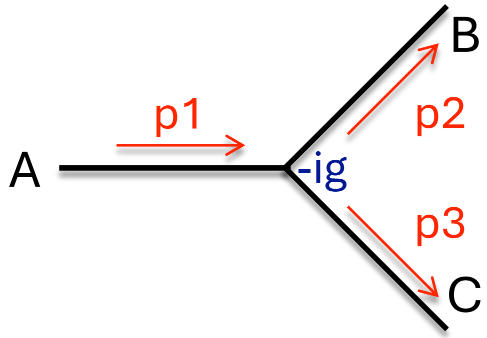
Rule 8: Now we drop the delta function we got from Rule 5. There was no integral, so it was unnecessary.

Momentum: $|\mathbf{p}_B|=|\mathbf{p}_C|= p$ is uniquely determined by M_A, M_B, & M_C.

See problem 3.19: $|p| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2m_B^2 - 2m_A^2m_C^2 - 2m_B^2m_C^2}$

Decay of the A

We are now in a position to calculate something! The easiest is the lifetime and/or decay rate of A→BC.



We've already done the kinematics!

$$\Gamma = \frac{S\,|p|}{8\pi\, m_A^2} |\mathcal{M}|^2$$

$\hbar = c = 1$

$$\Gamma(A \to BC) = \frac{g^2\,|p|}{8\pi\, m_A^2} \qquad \tau(A \to BC) = \frac{8\pi\, m_A^2}{g^2\,|p|}$$

See problem 3.19: $|p| = \frac{1}{2m_A}\sqrt{m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2}$