Physics 410 Quiz #3 – Thursday, February 27, 2025

Name: Solutions

1. [12] The partition function for a photon gas in a cubical box of side length L is $Z = \prod_n \left[1 - e^{-\frac{n\pi\hbar c}{L\tau}}\right]^{-1}$, where $\prod_n$ indicates the product over modes $n = \sqrt{n_x^2 + n_y^2 + n_z^2}$, $n_x, n_y, n_z = 1,2,3, \ldots$

a) [3] Write an expression for the free energy F; leave the result as a sum.

$$F = -\tau \ln Z = -\tau \ln\left(\prod_n (1-e^{-x})^{-1}\right) = -\tau \sum_{n_x n_y n_z} \ln(1-e^{-x})^{-1}$$

$$= +\tau \sum_{n_x n_y n_z} \ln(1-e^{-x})$$

where $x = \frac{n\pi\hbar c}{L\tau}$

b) [5] Now attempt to write your sum from part a) as an integral over n. Change the integration variable to the dimensionless variable $x = n\pi\hbar c/L\tau$. Do not attempt to evaluate the integral.

$$\sum_{n_x, n_y, n_z} \rightarrow \underbrace{2}_{\text{polarizations}} \times \int_0^\infty dn_x \int_0^\infty dn_y \int_0^\infty dn_z = 2 \times \underbrace{\frac{1}{8} \int_0^\infty 4\pi n^2 dn}_{\text{spherical coordinates}}$$

$$F = +\tau \cdot \pi \int dn\, n^2 \ln\left(1 - e^{-\frac{n\pi\hbar c}{\tau L}}\right) = +\tau\pi \left(\frac{L\tau}{\pi\hbar c}\right)^3 \int_0^\infty dx \cdot x^2 \ln(1-e^{-x})$$

c) [4] From your result in part b), find the pressure p and entropy σ of this photon gas (still do not attempt to evaluate the integral from your result in b)).

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right)_{V,N} = 4\pi \left(\frac{L\tau}{\pi\hbar c}\right)^3 \int_0^\infty dx\, x^2 \ln(1-e^{-x})$$

$$p = -\left(\frac{\partial F}{\underbrace{\partial V}_{V = L^3}}\right)_{\tau, N} = +\tau\pi \left(\frac{\tau}{\pi\hbar c}\right)^3 \int_0^\infty dx\, x^2 \ln(1-e^{-x})$$

2. [8] In the homework you showed that the heat capacity of N phonons in the high-temperature limit is approximately $C_V = 3N\left(1 - \frac{1}{20}\left(\frac{k_B\theta}{\tau}\right)^2\right)$ where $\theta \equiv \frac{\hbar v}{k_B}\left(\frac{6\pi^2 N}{V}\right)^{\frac{1}{3}}$ is the Debye temperature, v is the speed of waves in the medium, and V is the system volume. Recall that the definition of heat capacity is $C_V = \left(\frac{\partial U}{\partial \tau}\right)_V$.

a) [4] From this result, find the energy U of this system.

$$C_V = \left(\frac{\partial U}{\partial \tau}\right)_V \rightarrow \quad U = 3N\left(\tau + \frac{1}{20}\tau\left(\frac{k_B\theta}{\tau}\right)^2\right) + \underbrace{f(V,N)}_{\text{some function independent of } \tau.}$$

It's ok if you set this to zero for the quiz.

some function independent of τ. We need more info to determine this

b) [4] A result from Chapter 3 of our textbook is that the energy fluctuations in the canonical ensemble are given by $\langle(\epsilon - \langle\epsilon\rangle)^2\rangle = \tau^2\left(\frac{\partial U}{\partial \tau}\right)_V$. Use this, and your result from part a), to evaluate the fractional energy fluctuations $\langle(\epsilon - \langle\epsilon\rangle)^2\rangle/\langle\epsilon\rangle^2$.

$$\frac{\langle(\epsilon - \langle\epsilon\rangle)^2\rangle}{\langle\epsilon\rangle^2} = \frac{\tau^2 C_V}{U^2} = \frac{\tau^2 \cdot 3N\left(1 - \frac{1}{20}\left(\frac{k_B\theta}{\tau}\right)^2\right)}{\left[3N\tau\left(1 + \frac{1}{20}\left(\frac{k_B\theta}{\tau}\right)^2\right) + f(V,N)\right]^2}$$