

## Problem 1

Kittel & Kroemer, Chapter 3, problem 9 [Partition function for two systems]: 2 points

Show that the partition function  $Z(\mathbf{1} + \mathbf{2})$  of two independent systems  $\mathbf{1}$  and  $\mathbf{2}$  in thermal contact at a common temperature  $\tau$  is equal to the product of the partition functions of the separate systems:

$$Z(\mathbf{1} + \mathbf{2}) = Z(\mathbf{1})Z(\mathbf{2})$$

Let's see:

$$\begin{aligned} Z(\mathbf{1})Z(\mathbf{2}) &= \left( \sum_{s_1} e^{-\epsilon_{s_1}/\tau} \right) \left( \sum_{s_2} e^{-\epsilon_{s_2}/\tau} \right) \\ &= \sum_{s_1} \sum_{s_2} e^{-(\epsilon_{s_1} + \epsilon_{s_2})/\tau}. \end{aligned}$$

Since system 1 and system 2 are independent, a state of the combined system is given by a pair of states, one for system 1 and one for system 2; so a sum over all states of the combined system should appear as two sums, one over the states of system 1 and one over the states of system 2. For each pair, the energy of the combined system should be  $\epsilon_{s_1} + \epsilon_{s_2}$ . So we expect

$$Z(\mathbf{1} + \mathbf{2}) = \sum_{s_1} \sum_{s_2} e^{-(\epsilon_{s_1} + \epsilon_{s_2})/\tau}.$$

This is just what we've written above, so we can see that

$$Z(\mathbf{1} + \mathbf{2}) = Z(\mathbf{1})Z(\mathbf{2}).$$

## Problem 2

Adapted from: Kittel & Kroemer, Chapter 3, problem 2 [Magnetic susceptibility]: 6 points

### (a): 2 points

Find the partition function first for a single spin, then for  $N$  independent spins, using the result from the first problem on this homework set.

For a single spin with magnetic moment  $m$  in a magnetic field  $B$ , there are two possible energies:  $\epsilon_{\pm} = \pm mB$ . Thus the partition function for a single spin is

$$Z_1 = e^{mB/\tau} + e^{-mB/\tau} = 2 \cosh(mB/\tau).$$

For  $N$  independent spins, we use the result from Problem 1 to write

$$Z_N = (Z_1)^N = 2^N (\cosh(mB/\tau))^N.$$

(a): 2 points

### (b): 2 points

Find the magnetization  $M$  directly from the partition function using the Boltzmann factor. (You don't need to find  $F$  as the book suggests). You can work either with the energy  $U$  or the spin excess  $s$ , which are related as  $U = -2mB\langle s \rangle$ .  $M$  is then a function of  $U$  or  $s$ .

The magnetization is defined in Eq. 46 in the text as the magnetic moment per unit volume:

$$M = \langle 2s \rangle m / V.$$

A natural quantity to work with here is the number of spins per unit volume  $n \equiv N/V$ , in which case the magnetization is written as

$$M = \langle 2s \rangle mn / N.$$

Let's work with the energy  $U = -2mB\langle s \rangle$ , in terms of which the magnetization is

$$M = -\frac{Un}{NB}.$$

Since we have the partition function from part (a), we can use the fact that  $U = \tau^2 \frac{\partial \ln Z}{\partial \tau}$  to calculate:

$$\begin{aligned} M &= -\frac{\tau^2 n}{NB} \frac{\partial}{\partial \tau} (\ln(2^N (\cosh(mB/\tau))^N)) \\ &= -\frac{\tau^2 n}{NB} \frac{\partial}{\partial \tau} (\ln(2^N) + \ln(\cosh(mB/\tau))^N) \\ &= -\frac{\tau^2 n}{B} \frac{\partial}{\partial \tau} \ln(\cosh(mB/\tau)) \\ &= -\frac{\tau^2 n}{B} \frac{1}{\cosh(mB/\tau)} \frac{\partial}{\partial \tau} \cosh(mB/\tau) \\ &= -\frac{\tau^2 n}{B} \frac{\sinh(mB/\tau)}{\cosh(mB/\tau)} (-mB/\tau^2) \\ &= mn \tanh(mB/\tau). \end{aligned}$$

(b): 2 points

**(c): 1 point**

Show that the magnetic susceptibility is  $\chi = nm^2/\tau$  in the high-temperature limit.

Magnetic susceptibility is defined as  $\chi \equiv dM/dB$ . Using our result from part (b), we can calculate:

$$\begin{aligned}\chi &= \frac{d}{dB}(mn \tanh(mB/\tau)) \\ &= mn(1 - \tanh^2(mB/\tau))(m/\tau) \\ &= \frac{m^2 n}{\tau}(1 - \tanh^2(mB/\tau)).\end{aligned}$$

(c): 1 point

In the high-temperature limit,  $\lim_{\tau \rightarrow \infty}(\tanh^2(mB/\tau)) = 0$ , so we see that

$$\chi \rightarrow \frac{m^2 n}{\tau}.$$

**(d): 1 point**

Compare your answer with the one you obtained on Problem 2 of the previous homework assignment. There you calculated the dimensionless magnetization per atom,  $M/Nm$ , for a spin system. (That answer was only valid at high temperatures where  $s \ll N$ .) Calculate the magnetic susceptibility from that problem, and compare it with your answer to part (c) above. The inverse temperature dependence you found for the magnetic susceptibility is called the “Curie Law”, and is very famous.

On the previous homework assignment, we found that the net magnetization is given by

$$M_{\text{net}} = \frac{Nm^2 B}{\tau}.$$

Dividing by the volume gives us the net magnetization per unit volume, which is what we’re calling  $M$  here:

$$M = \frac{nm^2 B}{\tau}.$$

(d): 1 point

$$\chi = \frac{dM}{dB} = \frac{nm^2}{\tau},$$

in accordance with what we found here. We’ve done a bit better here and found a result that works a larger range of temperatures.

## Problem 3

### Kittel & Kroemer, Chapter 3, problem 4 [Energy fluctuations]: 3 points

Consider a system of fixed volume in thermal contact with a reservoir. Show that the mean square fluctuation in the energy of the system is

$$\langle(\epsilon - \langle\epsilon\rangle)^2\rangle = \tau^2(\partial U/\partial\tau)_V.$$

Here  $U$  is the conventional symbol for  $\langle\epsilon\rangle$ .

Hint: Use the partition function  $Z$  to relate  $\partial U/\partial\tau$  to the mean square fluctuation. Also, multiply out the term  $(\dots)^2$ .

Note: The temperature  $\tau$  of a system is a quantity that by definition does not fluctuate in value when the system is in thermal contact with a reservoir. Any other attitude would be inconsistent with our definition of the temperature of a system. The energy of such a system may fluctuate, but the temperature does not. Some workers do not adhere to a rigorous definition of temperature. Thus Landau and Lifshitz give the result

$$\langle(\Delta\tau)^2\rangle = \tau^2/C_V,$$

but this should be viewed as just another form of the result of this problem with  $\Delta\tau$  set at  $\Delta U/C_V$ . We know that  $\Delta U = C_V\Delta\tau$ , whence  $\langle(\Delta U)^2\rangle = \tau^2 C_V$ , which is our result.

The left-hand side looks like:

$$\langle(\epsilon - \langle\epsilon\rangle)^2\rangle = \langle\epsilon^2\rangle - \langle\epsilon\rangle^2.$$

Here  $\langle\epsilon\rangle \equiv U$ . In class we have shown that

$$\langle\epsilon\rangle \equiv U = \tau^2 \frac{1}{Z} \frac{\partial Z}{\partial\tau};$$

we will use this result below. Let's take a look at the partial derivative:

$$\begin{aligned} \left(\frac{\partial U}{\partial\tau}\right)_V &= \frac{\partial}{\partial\tau} \left( \frac{\sum_s \epsilon_s e^{-\epsilon_s/\tau}}{Z} \right) \\ &= \frac{\frac{\partial}{\partial\tau} \sum_s \epsilon_s e^{-\epsilon_s/\tau}}{Z} - \frac{\sum_s \epsilon_s e^{-\epsilon_s/\tau}}{Z^2} \frac{\partial Z}{\partial\tau} \\ &= \frac{\frac{\partial}{\partial\tau} \sum_s \epsilon_s e^{-\epsilon_s/\tau}}{Z} - \frac{\langle\epsilon\rangle}{\tau^2} \frac{\sum_s \epsilon_s e^{-\epsilon_s/\tau}}{Z} \\ &= \frac{\frac{\partial}{\partial\tau} \sum_s \epsilon_s e^{-\epsilon_s/\tau}}{Z} - \frac{\langle\epsilon\rangle^2}{\tau^2} \\ &= \frac{\frac{1}{\tau^2} \sum_s \epsilon_s^2 e^{-\epsilon_s/\tau}}{Z} - \frac{\langle\epsilon\rangle^2}{\tau^2} \\ &= \frac{\langle\epsilon^2\rangle}{\tau^2} - \frac{\langle\epsilon\rangle^2}{\tau^2}. \end{aligned}$$

Multiplying both sides by  $\tau^2$  gives us our result.