Chapter II. Basic Topology of sets

DEF A function OR MAP BETWEEN TWO (nonempty) sets A & B (denoted $A \to B$, $A \xrightarrow{f} B$, $f(A)$, etc) is a CORRESPONDENCE that set in correspondence to every element $a \in A$ a unique $b \in B$: $b = f(a)$.

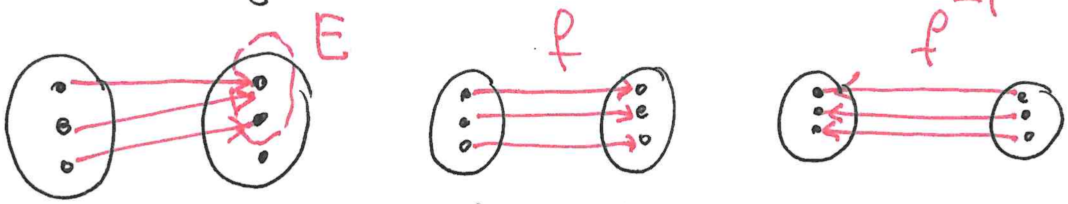
$f(A) = E \subseteq B$, A is the domain of f and E is the range of f.

DEF IF $E = B$ then the map is called surjective (OR mapping ONTO)

IF $f(a_1) \neq f(a_2)$ for $a_1 \neq a_2$, the map is injective.

DEF If a map is surjective and injective it is called a bijection, OR 1-1 correspondence

Euler-Venn diagrams:



A B A B

bijection

If f is a bijection, $\forall y \in B$ $\exists!$ $x \in A$: $f(x) = y$, then we have an inverse function $f^{-1}(y) = x$.

unique

SEQUENCE $f: \mathbb{N} \to \mathbb{R}$. $(x_1, x_2, x_3 \ldots)$

Cardinality : "size" of a set : $|A|$

$|A| = |B|$ iff $\exists f$ a bijection $A \leftrightarrow B$ $(A \sim B)$

"if and only if"

$|A| \leq |B|$ if $\exists$ an injection $A \to B$ $(A \leftrightarrow E \subseteq B)$

Theorem (hard) $|A| \leq |B| \wedge |B| \leq |A| \Rightarrow |A| = |B|$

[see Appendix]

Classification : let $I_n = 1, 2, \ldots, n \subset \mathbb{N}$ then

(a) A is finite if $A \leftrightarrow I_n$ for some n.
(then $|A| = n$)

(b) A is infinite if not finite

(c) A is countable if $A \sim \mathbb{N}$

(d) A is uncountable if it is not finite nor countable

(e) A is at most countable if A is finite or countable.

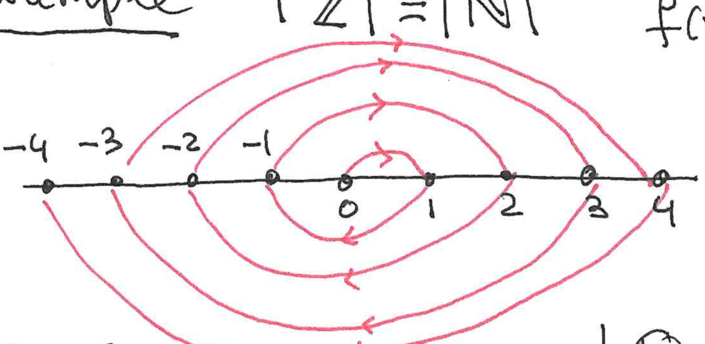
Theorem $|A| = |B|$ is an equivalence relation : $A \sim B$

(i) $A \sim A$

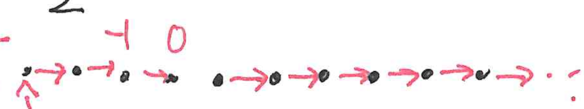
(ii) if $A \sim B$ then $B \sim A$: $B = f(A)$: $\exists f^{-1}$ so $A = f^{-1}(B)$
by injection

(iii) if $A \sim B$ and $B \sim C$ then $A \sim C$
(composite function) $B = f(A)$, $C = g(B)$ then
$C = g(f(A)) = g \circ f(A)$

Example $|\mathbb{Z}| = |\mathbb{N}|$ $\quad f(n) = \begin{cases} \frac{n}{2} & n \text{ even} \\ -\frac{n-1}{2} & n \text{ odd} \end{cases}$



NOT



because here we cannot assign a specific finite n to, say -1, or 0.

Example (Lab Work) $|\mathbb{Q}| = |\mathbb{N}|$.

The set of $\alpha \in (0,1]$ is uncountable

Assume α is countable; so $\exists\ f(n) \Leftrightarrow (0,1]$:
Arrange $x \in \alpha$ is an order of increasing n:

n				
1	$0.10110110\ldots$ x_1		0.00110110	
2	$0.01110010\ldots$ x_2		0.00110010	
3	$0.10100101\ldots$ x_3	$\rightarrow$	0.10000101	and take
4	$0.01100100\ldots$ x_4		0.01110100	
	$\vdots$			

diagonal

$x_0 = 0.0001011\ldots$ such that each n'th element is the changed (swapped) n^{th} digit of x_n

Then $x_0 \neq x_n\ \forall n \in \mathbb{N}$ (since they differ by n^{th} digit
So x_0 is NOT in the list of Dedekind cuts $\{x_1, \ldots x_n \ldots$
so this list is incomplete – contradiction.

[This is called Cantor's "diagonal process"]

$\boxed{\text{DEF}}$ Union of sets $\bigcup_{\alpha \in A} E_\alpha$: all x : $x \in E_\alpha$ for SOME $\alpha \in A$

Intersection of sets $\bigcap_{\alpha \in A} E_\alpha$: all x: $x \in E_\alpha$ for ALL $\alpha \in A$.

A can be finite or infinite.

interesting EXAMPLES: $\{ E_n = (0, \frac{1}{n}],\ n = 1, 2, \dots \}$

$\bigcup_{n=1}^{\infty} E_n = E_1 = (0, 1]$ $\qquad E_1 \supseteq E_2 \supset E_3 \supset E_4 \dots$

$\bigcap_{n=1}^{\infty} E_n = \{ \text{all } x : \ 0 < x \leq \frac{1}{n} \text{ for all } n \in \mathbb{N} \}$

but $\forall x > 0 \ \exists n : \ \frac{1}{n} < x \Rightarrow$ no such x exist

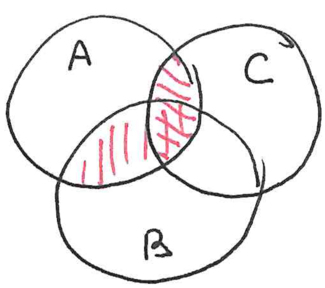
$\bigcap_{n=1}^{\infty} E_n = \emptyset$

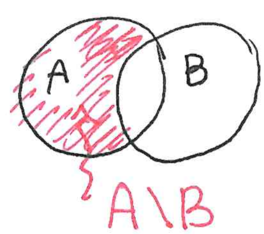
properties $\quad A \cup B = B \cup A \qquad A \cap B = B \cap A$

distributive law:

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

what about $A \cup (B \cap C)$?



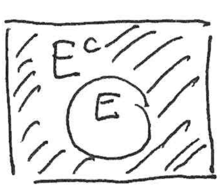


$A \setminus B = ?$

$\underline{A \setminus B = A \cap B^C}$

Complement $\quad E^C$: implies we HAVE some ambient space

[... let E be a nonempty subset of $\mathbb{R}$ ...]



$E^C = \{ x \in \mathbb{R} : \ x \notin E \}$.

$$[A \cup B]^C = A^C \cap B^C.$$

Metric spaces ($\mathbb{R}$)

$d: A \to \mathbb{R}_{+,0}$:

(i) $d(x,x) = 0$; $d(x,y) = 0 \Leftrightarrow x = y$

(ii) $d(x,y) = d(y,x)$

(iii) $d(x,z) \leq d(x,y) + d(y,z)$. Δ inequality

(on $\mathbb{R}$ $d(x,y) = |x-y|$

Already on $\mathbb{R}^2 := (x_1, x_2)$ - ordered pairs of real numbers we have many different "distances")

$\boxed{\text{DEF}}$ (a) A ball (a neighborhood) $B_r(p)$ centered at p:

$\forall x \in \mathbb{R}$: $|x-p| < r$

(b) A point p is a limit point of the set E if $\forall r > 0$ $\exists q \neq p$, $q \in E$: $q \in B_r(p)$

(c) if $p \in E$ and p is not a limit point, then p is an isolated point of E.

(d) A point p is an interior point of E if $\exists r > 0$: $B_r(p) \subseteq E$

$\boxed{\text{DEF}}$ A set E is OPEN iff every point of E is an ~~isolated~~ interior point.

EXAMPLE: $(0,1)$ is open : $\forall x \in (0,1)$, $0 < x < 1$; take $r = \min\{x, 1-x\}$ then $\forall y \in B_r(x)$

if $y \leq x$ then $x - y < \min\{x, 1-x\} \leq x$, so $y > 0$ hence $0 < y \leq x < 1$ and $y \in (0,1)$.

if $y \geq x$ then $y - x < \min\{x, 1-x\} \leq 1-x$, so $y < 1$ and $x \leq y < 1$ and again $y \in (0,1)$, so proven

Theorem every $B_r(x)$ is open:

proof Take $y \in B_r(x)$, then $|y-x| < r$, so

$r - |y-x| = \varepsilon > 0$. Prove that $B_\varepsilon(y) \subseteq B_r(x)$:

Take $\forall z \in B_\varepsilon(y)$, then $|x-z| \leq |x-y| + |y-z| <$

$<$ ~~[illegible]~~ $|y-x| + \varepsilon = r$ so $z \in B_r(x)$. $\Box$

Theorem (a) Any union of open sets is an open set.

$$\left(\bigcup_{\alpha \in A} E_\alpha = E \text{ - open} \right)$$

proof if $x \in \bigcup_{\alpha \in A} E_\alpha \Rightarrow x \in E_\alpha$ for some α, then

$\exists r: B_r(x) \subseteq E_\alpha$, so $B_r(x) \subseteq \bigcup_{\alpha \in A} E_\alpha = E$, so E is open

(b) Intersection of a finite number of open sets is open.

$\bigcap_{j=1}^{n} E_j = E$ is open:

proof. if $x \in \bigcap_{j=1}^{n} E_j$ then $\forall j \; \exists r_j > 0: B_{r_j}(x) \subseteq E_j$

Take $r = \min\{r_j, j=1,\dots,n\}$, then the ball ~~[illegible]~~

$B_r(x) \subseteq B_{r_j}(x) \;\; \forall j=1,\dots n$, so $B_r(x) \subseteq \bigcap_{j=1}^{n} E_j$

[why it does not work for infinite intersections? can be that no minimum exist, instedd $\inf\{r_\alpha\} = 0$]

Can be that $\bigcap E_j = \emptyset$, then $\emptyset$ is OPEN.

$\mathbb{R}$ is open; proper open sets are those not equal $\mathbb{R}$, $\emptyset$

__Theorem__ If p is a limit point, then every $B_r(p)$ contains ∞ many points of E.

__proof by contradiction__: assume $\exists r>0$:

$$\left(B_r(p)\setminus\{p\}\right)\cap E=\{q_1,\dots,q_n\}\subset E.$$

$q_i\neq p$, so take $r_0=\min\{|p-q_i|,\ i=1,\dots,n\}$; $r_0>0$

then $\{q_1,\dots,q_n\}\cap B_{r_0}(p)=\emptyset$, so $B_{r_0}(p)$ does not contain any point of E (besides, possibly, p itself)

So p is not a limit point – contradiction

__Corollary__ A finite point set has no limit points.

$\boxed{\text{DEF}}$ E is **CLOSED** if every limit point of E belongs to E

$\Updownarrow$

$\boxed{\text{DEF}}$ THM: E is closed if and only if its complement is open

$\Rightarrow$ Assume E^c is open. Then $\forall x\in E^c\ \exists B_r(x)\subseteq E^c$

so $B_r(x)\cap E=\emptyset$, so x is NOT a limit point of E and ALL limit points of E (if any exist) ARE in E.

$\Leftarrow$ Assume E contains ALL its limit points. Then no limit points of E ARE in E^c, so $\forall x\in E^c\ \exists r>0$ such that $B_r(x)\setminus\{x\}$ contains no points of E.

Since $x\notin E$, $B_r(x)\cap E=\emptyset$ so $B_r(x)\subseteq E^c$ so E^c is open.

Example (a) $\mathbb{N} \subset \mathbb{R}$ is a closed subset (no limit points)

(b) $\mathbb{Q} \subset \mathbb{R}$ is neither closed nor open.

(c) set $\{\frac{1}{n}, n \in \mathbb{N}\}$ is not closed as $B_\varepsilon(0)$ contain ∞ many points. But if we add this limit point then the set $\{\frac{1}{n}, n \in \mathbb{N}\} \cup \{0\}$ is closed.

Let E_{lim} be the set of all limit points of E. Then the set $E_{lim} \cup E$ called the closure of E and denoted by $\overline{E}$ is closed.

proof. Assume x is a limit point of $\overline{E}$ and $x \notin \overline{E}$ (therefore $x \notin E$) Then $\forall r > 0$ ~~such that~~ $\exists y \in B_r(x) \cap \overline{E}$. We show that $\exists y' \in E$ such that $y' \in B_r(x)$. If $y \in E$ we done. If y is a limit point of E then take $\varepsilon = r - |y - x| > 0$, $\exists y' \in E$: $y' \in B_\varepsilon(y)$ But then $|y' - x| \leq |y' - y| + |y - x| < \varepsilon + |y - x| = r$ so $y' \in B_r(x)$. Thus x is a limit point of E Contradiction. So $\overline{E}$ is closed.

EXAMPLE ... SOME SETS CAN BE WEIRD...

$$W = \bigcup_{\substack{p/q \in \mathbb{Q} \\ 0 \leq |p/q| \leq 1}} \left(\frac{p}{q} - \frac{1}{2q^3}, \frac{p}{q} + \frac{1}{2q^3} \right)$$

: W is OPEN AS A UNION of open sets.

Therefore W^c is closed TRY TO DESCRIBE IT.

<u>Theorem</u> (continuation)

(b) $E = \overline{E}$ iff E is closed

(c) $\overline{E} \subseteq F$ $\forall$ closed set $F \supseteq E$.

(c) Since F is closed, F^c is open and $\forall x \in F^c$ $\exists B_r(x) \subset F^c$, so F^c contains no limit points of E and no points of E. So $F^c \cap \overline{E} = \emptyset$ and $\overline{E} \subseteq F$.

<u>Theorem</u> Let $E \subset \mathbb{R}$ be nonempty, bounded above. Let $y = \sup E$. Then $y \in \overline{E}$. So $y \in E$ if E is closed.

proof If $y \in E$ then $y \in \overline{E}$. If $y \notin E$ then $\forall r > 0$ $\exists x \in E$: $|y - x| < r$ otherwise $y - r$ is upper boun. So y is a limit point of E. $\Rightarrow y \in \overline{E}$.

$\boxed{\text{DEF}}$ Interior of a set E denoted by E^o is the <u>largest</u> open subset of E.
[why exists?]

<u>Theorem</u> $E^o = \bigcup$ ALL OPEN subsets of E: $\left(\bigcup_{\alpha \in I} A_\alpha\right)$

Indeed, (i) $\forall$ $A \subseteq E$ A-open, $A \subseteq \bigcup_{\alpha \in I} A_\alpha$

(ii) Union of any number of open sets is open.

<u>Example</u> $[0,1]^o = (0,1)$; $\mathbb{Q}^o = \emptyset$.

Obviously $E^o = \emptyset$ for any countable set (because any ball is not countable.

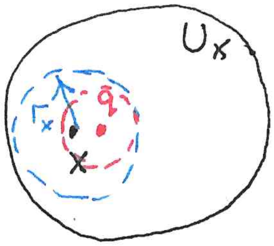
Lindelöf ~~Heine-Borel~~ theorem

Consider any subset $E \subseteq \mathbb{R}$.
(open)
A <u>covering</u> of E is a union of open $A_\alpha \subset \mathbb{R}$:

$E \subseteq \bigcup_{\alpha \in I} A_\alpha$. Prove that any open covering has a countable <u>subcovering</u> (i.e. $\exists\, \alpha_i \in I$, $i = 1, 2, \ldots$ such that $E \subseteq \bigcup_{i=1}^{\infty \text{ (or } n)} A_{\alpha_i}$.

proof (constructive)

$\forall x \in E \quad \exists\, U_x \in \{A_\alpha\}$ so $\exists\, r_x$: $B_{r_x}(x) \subseteq U_x$



Then INSIDE $B_{r_x}(x)$ we have a rational point q_x such that $|x - q_x| < \frac{1}{3} r_x$. Then take ANY rational number $\frac{1}{3} r_x < s_x < \frac{2}{3} r_x$ AND CONSIDER A BALL $B_{s_x}(q_x)$:

$B_{s_x}(q_x) \ni x$ and $B_{s_x}(q_x) \subseteq B_{r_x}(x)$, so

$\bigcup_x B_{s_x}(q_x) \supseteq E$. But we have at most countable set of $B_{s_x}(q_x)$ (parameterized by $\mathbb{Q} \times \mathbb{Q}_+$)
Taking exactly one U_x for each $B_{s_x}(q_x)$ we have an at most countable subcovering.

Compact sets

First, what about intersections / unions of closed sets?

Theorem (in fact, corollary of the theorem on OPEN sets)

(a) $\bigcap_{\alpha \in I} E_\alpha$ is closed (E_α are closed sets)

(b) $\bigcup_{i=1}^{n} E_i$ is closed (union of finite number of closed sets

proof: if E_α is closed, E_α^c is open. Then

$$\bigcap_{\alpha \in I} E_\alpha = \left(\bigcup_{\alpha \in I} E_\alpha^c\right)^c = (\text{open})^c = \text{closed}$$

$$\bigcup_{i=1}^{n} E_i = \left(\bigcap_{i=1}^{n} E_i^c\right)^c = (\text{open})^c = \text{closed}.$$

Boundary: Lemma $\partial E = \partial(E^c)$: Indeed: $\partial E = \overline{E} \setminus E^\circ$.

But $E^\circ = \left(\overline{E^c}\right)^c$, so $\partial E = \overline{E} \cap \overline{E^c} = \overline{E^c} \cap \overline{E} = \overline{E^c} \setminus (E^c$

$\boxed{\text{DEF}}$ A subset ~~of~~ K of a metric space X is COMPACT if every open covering contains a finite subcovering.

Theorem Compact subsets of metric spaces are closed

proof. by contradiction : Assume K is compact and $\exists$ limit point $x \notin K$. Then any $B_r(x)$ contains a point of K. Take all balls with $r = 1/n$ and consider $W_n(x) = \mathbb{R} \setminus \overline{B_{1/n}(x)}$ – open sets (sometimes associated with balls "centered" at infinity) Then $\bigcup_{n=1}^{\infty} W_n(x) = \mathbb{R} \setminus \{x\} \supset K$ is a covering. However, for any finite subcovering $\{W_{n_i}(x), i=1..,m\}$ we have $\max\{n_i, i=1...m\} = N$ and then $B_{1/N}(x) \cap \left(\bigcup_{i=1}^{m} W_{n_i}\right) = \emptyset$. But $\exists y \in K$: $y \in B_{1/N}(x)$, so it is NOT a subcovering, so K is not compa

<u>Theorem</u> Closed subsets of compact sets are compact.

~~Assume~~
<u>proof</u> Let F be closed and K be compact and $F \subset K$.

Assume we have a covering of F that has no finite subcovering. ($\bigcup_{n=1}^{\infty} E_n$). Take $\bigcup_{n=1}^{\infty} E_n \cup F^c$.
F^c is open and $\{E_n, n=1, \dots; F^c\}$ is a covering of K (since $\bigcup_{n=1}^{\infty} E_n \cup F^c = \mathbb{R}$). But this covering has NO finite subcovering; otherwise, since $F^c \cap F = \emptyset$ this covering ~~[illegible]~~ with F^c removed (if necessary) will be a finite subcovering of F by $\{E_{n_i}, i=1, \dots, m\}$ – contradiction. So F is compact

<u>Corollary</u> If F is closed and K is compact then $F \cap K$ is compact
$F \cap K$ is closed and $F \cap K \subset K$, so $F \cap K$ is compact.

<u>Theorem</u> If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every finite collection of $\{K_\alpha\}$ is nonempty, then $\cap K_\alpha$ is nonempty (NOTABLE EXAMPLE: $K_1 \supset K_2 \supset K_3 \supset \dots \supset K_n \supset \dots$)

<u>proof</u>. Fix a member K_1 of $\{K_\alpha\}$ and let $E_\alpha = K_\alpha^c$.
Assume NO point of K_1 belongs to <u>every</u> K_α. Then $\cup K_\alpha^c \supset K_1$ and K_α^c is an open covering of K_1. Since K_1 is compact, $\exists$ finite subcovering $\{K_{\alpha_i}^c \ i=1, \dots, m\}$
But then $K_1 \cap K_{\alpha_1} \cap \dots \cap K_{\alpha_m} = \emptyset$ – contradiction.

Corollary If $\{K_n\}$ are nonempty compact sets such that $K_1 \supset K_2 \supset .. \supset K_n \supset K_{n+1} \supset \cdots$ then $\bigcap_{n=1}^{\infty} K_n \neq \emptyset$.

Theorem If E is an infinite subset of a compact set K, then E has a limit point in K.

proof If no point of K were a limit point of E then $\forall x \in K$ $\exists B_r(x)$ containing either no points of E (if $X \notin E$) OR containing exactly one point of E (if $X = q \in E$). $B_r(x)$ is a covering of K but it has no finite subcovering $B_{r_i}(x_i)$, $i = 1 \ldots m$ (otherwise $|E| \leq m$). contradiction.

We are still short of examples. Some are obvious:
Any finite set is compact (can be covered by at most $|E|$ open sets)

Theorem. let $I_1 \supseteq I_2 \supset I_3 \supset \ldots \supset I_n \supset I_{n+1} \supset$ be a set of closed intervals in $\mathbb{R}$. Then $\bigcap_{n=1}^{\infty} I_n$ is not empty.

If $I_n = [a_n, b_n]$, ~~[illegible]~~ we have $a_i \leq b_j$ for all i and j so $\{a_i\}$ is bounded above. let $a = \sup\{a_i\}$. Correspondingly $b = \inf\{b_j\}$. Then since every a_i is a lower bound of $\{b_j\}$, $a_i \leq \inf(b_j)$ $\forall i$, so $\inf(b_j)$ is an upper bound for $\{a_i\}$ $\Rightarrow$ $\sup\{a_i\} \leq \inf\{b_j\}$ and, say $X = \sup\{a_i\}$: $\forall i$ $a_i \leq X \leq b_i$, so $x \in I_i$ and $x \in \bigcap_{n=1}^{\infty} I_i$ $\square$

Finally!

Theorem $[a,b]$ is compact!

proof. Assume $\exists$ covering E_n, $n=1,\dots\infty$ that has no finite subcovering. Split $[a,b]$ into $[a,\frac{a+b}{2}] \cup [\frac{a+b}{2}, b]$. For at least one of the new intervals the covering E_n has no finite subcovering. Choose this interval (if it is true for both intervals, choose any of them, say, the left one). Declare endpoints of this new interval to be $[a_1, b_1]$ and repeate splitting for this interval.

We then have the sequence of intervals

$[a,b] \supset [a_1,b_1] \supset [a_2,b_2] \supset \dots \supset [a_n,b_n] \supset \dots$

such that $|a_i - b_i| = \dfrac{|a-b|}{2^i}$. Take $\sup\{a_i\} = x$

Obviously $x = \inf b_i$ (since $a_i \le \sup_i(a_i) \le \inf(b_i) \le b_i \ \forall i$ so $|\sup(a_i) - \inf(b_i)| \le \dfrac{|a-b|}{2^i} \ \forall i$)

So $\{x\} = \bigcap_{i=1}^{\infty} [a_i, b_i]$. But $x \in E_\alpha$ for some α, so, since E_α is open, $\exists r>0$: $B_r(x) \subseteq E_\alpha$.

Take n: $\dfrac{|a-b|}{2^n} < r$, then, since $a_n \le x \le b_n$ and $|b_n - a_n| < r$, $|x - a_n|, |x - b_n| < r$ and $[a_n, b_n] \subset B_r(x) \subseteq E_\alpha$, so it admits a finite subcovering – contradiction. So $[a,b]$ is compact.

<u>Corollary</u> Every bounded closed subset of $\mathbb{R}$ is compact and vice versa, every compact in $\mathbb{R}$ is its closed and bounded subset.

Now we have plenty of compact sets! But we are puzzled WHY we need such an elaborated definition of compact sets? Why not to declare bounded and closed sets to be compact and live happily everafter? In fact, this property is true <u>only</u> in $\mathbb{R}$ and $\mathbb{R}^n$ for n <u>finite</u>. Already in an infinite-dimensional Euclidean space (e.g. in the space of functions $C^1[a,b]$) it is generally NOT true. In $\mathbb{R}^\infty$ NOT every closed and bounded set is compac

<u>Heine-Borel theorem</u> The three properties are equivalent in $\mathbb{R}^n$

(a) E is closed and bounded

(b) E is compact

(c) Every infinite subset of E has a limit point in E

(a) $\Rightarrow$ (b) : take $I = [\inf E, \sup E]$ then I is compact and $E \cap I$ is compact.

(b) $\Rightarrow$ (c) proven.

(c) $\Rightarrow$ (a) if E is not bounded, then $\exists$ infinite set $x_i \in E$: $|x_i| > |x_{i-1}| + 1$. Then $|x_i - x_{i-1}| \geq |x_i| - |x_{i-1}| > 1$ and $|x_{i+n} - x_i| > n$ so any $B_{1/2}(x)$ contains at most one point of E. If E is not closed, then $\exists$ limit point $x_0 \notin E$.

Construct an infinite subset of E:

$x_1 \in B_1(x_0)$; $x_2 \in B_{\frac{|x_1-x_0|}{2}}(x_0)$ etc.

$x_i \neq x_j$ for $i \neq j$ and $|x_i - x_0| < \frac{1}{2^{i-1}}$, so

x_0 is the only limit point, but $x_0 \notin E$ – contradiction.

Theorem (Weierstrass) Every bounded infinite subset of $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$. Take $I = [\inf E, \sup E]$. Then I is compact and thus E has a limit point in I.

PERFECT SETS

$\boxed{\text{DEF}}$ E is PERFECT if E is closed and every point of E is a limit point of E.

Theorem Let P be a nonempty perfect set in $\mathbb{R}$. Then P is uncountable.

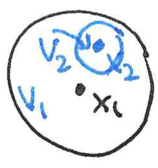
Proof. Since P has limit points, P is infinite.

Suppose P is countable: $\{x_1, x_2, x_3, \dots\}$

Take $V_1 = B_{r_1}(x_1)$ and consider $\overline{V_1} = \overline{B_{r_1}(x_1)}$: $\forall y \in \mathbb{R}: |y - x_1| \leq 1$

Suppose V_n was constructed: $V_n \cap P$ is nonempty.

Since every point of P is a limit point, $\exists V_{n+1}$ such that (i) $\overline{V_{n+1}} \subset V_n$; (ii) $x_n \notin \overline{V_{n+1}}$, (iii) $V_{n+1} \cap P$ is nonempty. We can continue indefinitely.



Let $K_n = \overline{V_n} \cap P$; $K_1 \supseteq K_2 \supseteq \dots \supseteq K_n \supseteq K_{n+1} \supseteq \dots$

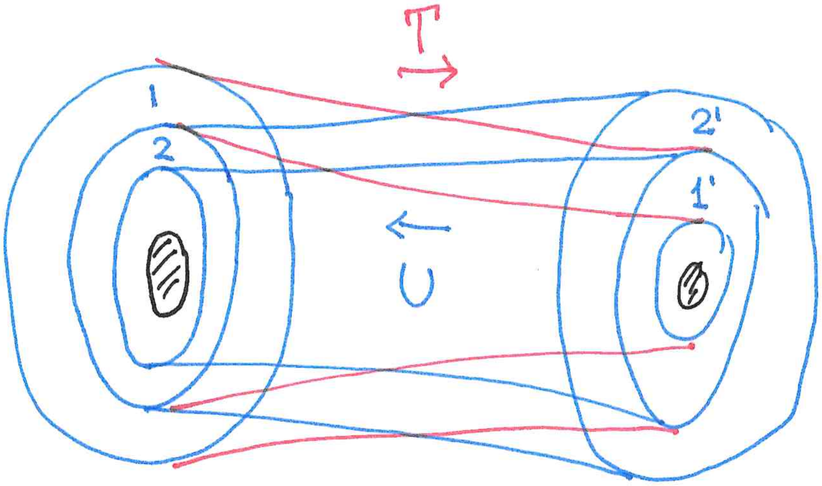
so $\bigcap_{n=1}^{\infty} K_n$ is nonempty. Since $x_n \notin K_{n+1}$ no point of P lies in $\bigcap_{n=1}^{\infty} K_n$. $\Rightarrow$ $\bigcap_{n=1}^{\infty} K_n$ is empty

Contradiction!

SUPPLEMENT - I

Theorem $|A| \leq |B| \wedge |B| \leq |A|$ impeies $|A| = |B|$

Proof by constructing a 1-1 map (Bijection) between A and B:



$B_1 \subseteq B$, $A_1 \subseteq A$

$T: A \to B_1$ Bijection

$U: B \to A_1$ Bijection

$B_i = T(A_{i-1})$

$(A_0 = A)$

$A_i = U(B_{i-1})$

$(B_0 = B)$

Split B: $B = (B \setminus B_1) \cup (B_1 \setminus B_2) \cup (B_2 \setminus B_3) \cup \dots$

$\cup \{\text{CORE } B\}$ $\quad \text{CORE } B = \bigcap_{n=1}^{\infty} B_i$

$A = (A \setminus A_1) \cup (A_1 \setminus A_2) \cup \dots \cup \{\text{CORE } A\}$

$\text{CORE } A = \bigcap_{n=1}^{\infty} A_n$

Then $T(A_i \setminus A_{i+1}) = B_{i+1} \setminus B_{i+2}$ and $(i = 0, 1, 2, \dots)$

$T^{-1}(B_{i+1} \setminus B_{i+2}) = A_i \setminus A_{i+1}$,

$U(B_i \setminus B_{i+1}) = A_{i+1} \setminus A_{i+2}$

$U^{-1}(A_{i+1} \setminus A_{i+2}) = B_i \setminus B_{i+1}$

$T\{\text{CORE } A\} = T\left(\bigcap_{n=1}^{\infty} A_n\right) = \bigcap_{n=1}^{\infty} T(A_n) = \bigcap_{n=1}^{\infty} B_{n+1} = \text{CORE}$

So we constract a one-to-one (Bijective) map:

$$\boxed{T(A \setminus A_1) + U^{-1}(A_1 \setminus A_2) + T(A_2 \setminus A_3) + U^{-1}(A_3 \setminus A_4) + \dots + \\ + U\{\text{CORE } A\}}$$

SUPPLEMENT II

continued fractions and approximations of algebraic numbers by rationals

Address the problem: given an irrational number x "how close" we may approach it by $\frac{p}{q} \in \mathbb{Q}$ provided p and q are not "too large".

Consider a "golden mean": the number X that is a solution of $X^2+X-1=0$ OR $X = \frac{\sqrt{5}-1}{2}$.

① Continuous fraction is the expression

$$a_1 + \cfrac{b_1}{b_1 + \cfrac{1}{a_2 + \cfrac{b_2}{b_2 + \cfrac{1}{a_3 + \dots}}}}$$

α). Every rational number can be uniquely written as a finite continuous fraction:

$$\frac{4}{7} = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3}}};$$

β). Every infinite continuous fraction with periodic structure: $a_{n+k} = a_k$, $b_{n+k} = b_k$ for n fixed and $\forall k > M$.

is a solution of quadratic equation

Set e.g. all $a_k = b_k = 1$: $\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots}}} = x$, then $\frac{1}{1+x} = x$,

so $X = \frac{\sqrt{5}-1}{2}$.

② Approximation of x:

$$\frac{p_n}{q_n} = \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\ddots 1+1}}} . \quad \text{Then} \quad \frac{p_{n+1}}{q_{n+1}} = \frac{1}{1+\frac{p_n}{q_n}} = \frac{q_n}{p_n+q_n} .$$

OR $(p_{n+1}, q_{n+1}) = (q_n, p_n+q_n)$

Note that p_{n+1}, q_{n+1} are <u>coprime</u> provided p_n and q_n are coprime, and the same is true for any continued fraction, so the continuous fraction representation of $p/q \in \mathbb{Q}$ is unique.

We now consider how $\frac{p_n}{q_n}$ approximate our x. Note that x simultaneously satisfies an infinite set of consistent equations:

$$x = \cfrac{1}{1+\cfrac{1}{\underbrace{1+\dots}_{n}\cfrac{}{1+\frac{1}{x}}}} = \frac{p_n + p_{n-1}x}{q_n + q_{n-1}x}$$

, so let us compare

$$\left|\frac{p_n}{q_n} - x\right| = \left|\frac{p_n}{q_n} - \frac{p_n+p_{n-1}x}{q_n+q_{n-1}x}\right| = \left|\frac{p_nq_n + p_nq_{n-1}x - p_nq_n - p_{n-1}q_nx}{q_n(q_n+q_{n-1}x)}\right| =$$

$$= |x| \frac{|p_nq_{n-1} - p_{n-1}q_n|}{|q_n(q_n+q_{n-1}x)|} .$$

Now use that:

$$p_nq_{n-1} - p_{n-1}q_n = q_{n-1}^2 - p_{n-1}q_{n-1} - p_{n-1}^2$$

But $q_n^2 - p_nq_n - p_n^2 = (p_{n-1}+q_{n-1})^2 - q_{n-1}(p_{n-1}+q_{n-1}) - q_{n-1}^2 = p_{n-1}^2 + p_{n-1}q_{n-1} - q$

and since $p_1 = q_1 = 1$, $q_n^2 - p_nq_n - p_n^2 = (-1)^n \ \forall n$.

So, finally, we obtain

$$\left|\frac{p_n}{q_n} - x\right| = \frac{|(-1)^n|\,|x|}{|q_n(\underbrace{q_n + q_{n-1}x}_{=p_n})|} \approx \frac{|x|}{|q_n||p_n||x + \frac{q_n}{p_n}|} \gtrsim \frac{1}{2|q_n|^2|x|}$$

For example, an interval

$$\left(\frac{p_n}{q_n} - \frac{1}{2|q_n|^3}, \frac{p_n}{q_n} + \frac{1}{2|q_n|^3}\right) \text{ will never contain } x!$$

In fact, algebraic numbers are <u>WORST</u> when we try to approximate them by rationals. An (almost) theorem claims that for (almost all) algebraic numbers x, $\inf_{p/q \in \mathbb{Q}} \left|\frac{p}{q} - x\right| \cdot |q|^2 > 0$.

In this respect, $x = \frac{1-\sqrt{5}}{2}$ is an absolute champion! As we see, this infimum is equal $\frac{1}{2|x|} = \frac{1}{1-\sqrt{5}}$ and it is the maximum of these quantities among all real numbers!