327H - Honor Analysis

I-1 Natural numbers

$\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

ALL positive integers.

Every n has a successor $n+1$

Peano Axioms

N1. 1 Belongs to $\mathbb{N}$

N2. If n belongs to $\mathbb{N}$, then its successor $n+1$ belongs to $\mathbb{N}$

N~~3~~4. If n and m has the same successor, then $n=m$.

[not quite canonical order]

N~~4~~3 1 is not the successor of any element in $\mathbb{N}$

N5. A subset of $\mathbb{N}$ which contains 1 and which contains $n+1$ whenever it contains n must be equal $\mathbb{N}$

Axioms : A set of statements sufficient for determining the object uniquely, but ~~not nufer~~ free of redundancy".

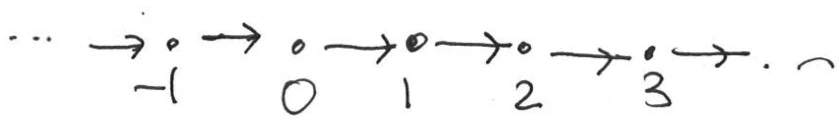
That is, if we relax ANY of N1—N5 we may construct a set that does not look like $\mathbb{N}$:

$$1 \to 2 \to 3 \to \cdots \to n \to \cdots$$

HERE $A \to B$ indicates "B is successor of A"

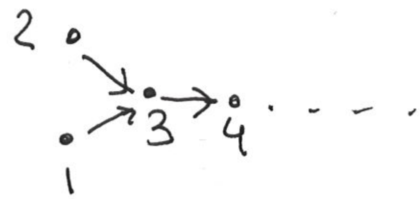
<u>Discussion</u> : without N2 we can hardly define the set, so assume N2 holds:

If N1, N2, N~~3~~4, and N5 hold, we may have



set of all integers.

N1, N2, N~~4~~3, N5



... Most famous example: five postulates of Euclid

MATHEMATICAL INDUCTION

Base (I_1) P_1 is true

implication (I_2) P_{n+1} is true whenever P_n is true

Then P_n is true for all $n \in \mathbb{N}$.

Innumerable applications in proofs. (Dozens in this course alone).

Ex. 1 Arithmetic progression

$$1+2+\dots+n = \frac{1}{2}n(n+1):$$

Base: P_1: $1 = \frac{1}{2}1(1+1)$

Assertion P_n: $1+2+\dots+n = \frac{1}{2}n(n+1)$

Implication $\underbrace{1+2+\dots+n}_{P_{n+1}}+(n+1) \underset{P_n}{=} \frac{1}{2}n(n+1)+(n+1)$

$= \frac{1}{2}(n+2)(n+1)$, whis IS P_{n+1}. so proven

By the principle of math. induction, P_n is true $\forall n$

Ex. 2 $7^n - 6n - 1$ divisible by 36.

Base $n=1$: $7-6-1=0 \,\vdots\, 36$ (OR $36 \mid 7-6-1=0$)

Assumption (Assertion) $7^n - 6n - 1 \,\vdots\, 36$: $7^n = 36k + 6n + 1$

Implication $7^{n+1} - 6(n+1) - 1 = 7.(7^n) - 6(n+1) - 1$

$= 7(36k + 6n + 1) - 6(n+1) - 1 = 7 \cdot 36k + 36n =$

$= 36(7k+n) \,\vdots\, 36$ thus proven. By m.i. P_n is true for any n.

Ex 3 $\quad |\sin(nx)| \leq n|\sin x| \quad \forall n, \ \forall x \in \mathbb{R}$

$P_1: \ |\sin(x)| \leq |\sin x|$

$P_n: \ |\sin(nx)| \leq n \cdot |\sin(x)|$

$P_{n+1} \quad |\sin((n+1)x)| = |\sin(nx) \cdot \cos x + \cos(nx)\sin x|$

$$\leq |\sin(nx)| \cdot \underbrace{|\cos x|}_{\leq 1} + \underbrace{|\cos(nx)|}_{\leq 1} \cdot |\sin x|$$

$$\leq |\sin(nx)| + |\sin x| \leq (n+1)|\sin(x)|$$

Integers $\quad \mathbb{Z} \ : \quad \ldots -n, \ldots, -1, 0, 1, 2, 3, \ldots$

"amend" axioms N1–N5 to define $\mathbb{Z}$

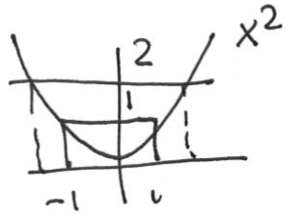
Rational numbers

$p \in \mathbb{Z}$

$q \in \mathbb{N}$

then $\frac{p}{q} \in \mathbb{Q}$ a unique choice is p & q are coprime

Not every equation can be solved:



$x^2 = 2$: Euclid:

if $x = \frac{p}{q}$ assume p & q are coprime

(not having a common divisor)

since $\frac{p^2}{q^2} = 2 \Rightarrow p^2 = 2q^2$, so $p^2 \in 2\mathbb{Z}$ (even)

and $p = 2k$, then $p^2 = 4k^2 = 2q^2$, so $q^2 = 2k^2$

and q is even itself, so p & q are NOT coprime

– contradiction [proof by contradiction].

sums, products, quotients of $x \in \mathbb{Q}$ are in $\mathbb{Q}$.

what about infinite sums (series)?

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$ (prove by induction!)

so $$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} + \dots = 1 - \frac{1}{2^{\infty}} ? = \lim_{n \to \infty} \left(1 - \frac{1}{2^n}\right) = 1$$

I-V

But $1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\dots+\frac{1}{n!}+\dots$ is NOT in $\mathbb{Q}$:

Indeed, let $1+\frac{1}{2!}+\dots+\frac{1}{q!}+\frac{1}{(q+1)!}+\dots=\frac{p}{q}$

multiply by $q!$:

$$\underbrace{q!\left(1+\frac{1}{2!}+\dots+\frac{1}{q!}\right)}_{\in\mathbb{Z}}+\frac{q!}{(q+1)!}+\frac{q!}{(q+2)!}+\dots=\underbrace{p(q-1)!}_{\in\mathbb{Z}}$$

so $\frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\dots+$ must be integer

but $\forall q>1$ $\quad 0<\frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\dots<\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots$

$=1$, so $\quad 0<\frac{1}{q+1}+\frac{1}{(q+1)(q+2)}+\dots<1$ so

Contradiction

Algebraic numbers

I-VI

Solutions of equations (algebraic equations)

$$c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0 = 0, \quad c_i \in \mathbb{Z}$$

All rational numbers <u>are</u> algebraic p/q: $qx - p = 0$

All radicals <u>are</u> algebraic numbers

$$\sqrt{2+\sqrt[3]{4}} = x \qquad x^2 = 2+\sqrt[3]{4} : \quad x^2 - 2 = \sqrt[3]{4} :$$

$$(x^2-2)^3 = 4.$$

rational zeros

Let r be a rational number satisfying a polynomial equation

$$c_n x^n + \dots + c_1 x + c_0 = 0 \quad n \geq 1, \; c_n \neq 0, \; c_0 \neq 0$$

if $r = p/q$ is a root then $p \mid c_0$ and $q \mid c_n$:

$$c_n \left(\frac{p}{q}\right)^n + \dots + c_1 \frac{p}{q} + c_0 = 0$$

Multiply by q^n:

$$c_n p^n + c_{n-1} p^{n-1} q + \dots + c_1 p q^{n-1} + c_0 q^n = 0$$

$q \mid c_n p^n$ but $q \nmid p$ so $q \mid c_n$

$p \mid c_0 q^n$ but $p \nmid q$ so $p \mid c_0$

<u>Corollary</u>: $x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0 = 0$: any rational solution is an integer dividing c_0.

$\mathbb{Q}$-axioms of number <u>fields</u>.

<u>OPERATIONS</u>:

$\mathbb{Q} = \frac{p}{q}$, p, q coprime

ADDITION AND MULTIPLICATION:

A0 : $\forall a, b \in \mathbb{Q} \ \exists \ c = a + b \in \mathbb{Q}$ ("sum")
M0 : $\forall a, b \in \mathbb{Q} \ \exists \ c = a \cdot b \in \mathbb{Q}$ ("product")

So, operations are <u>defined</u> on the whole $\mathbb{Q}$.

A1 $a+(b+c) = (a+b)+c$ associativity $\forall a, b, c$
A2 $a+b = b+a$ commutativity $\forall a, b$
[NOT for all fields, but required for all number fields]
A3 $\exists 0 \in \mathbb{Q} :\ a+0 = a \ \forall a \in \mathbb{Q}$
A4 $\forall a \in \mathbb{Q} \ \exists$ an element $-a$: $a+(-a) = 0$

This means that $\mathbb{Q}$ is a <u>GROUP</u> w.r.t. addition

M1 $a \cdot (bc) = (ab)c$
M2 $ab = ba$
M3 $\exists 1 : a \cdot 1 = a \ \forall a \in \mathbb{Q}$
M4 for each $a \neq 0$ there is an element a^{-1} : $a \cdot a^{-1} = 1$

So $\mathbb{Q}$ is "almost" a group w.r.t. multiplication
[and $\mathbb{Q} \setminus \{0\}$ is a group]

<u>Distributive law</u> – the only axiom that relates "$\cdot$" and "$+$"

$$\forall a,b,c \in \mathbb{Q}: \quad a\cdot(b+c) = (a\cdot b) + (a\cdot c)$$

OTHER NUMBER FIELDS: $\mathbb{R}, \mathbb{C}, \mathbb{Z}_p$

[example: $\mathbb{Z}_2$: $\{0,1\}$ $\begin{cases} 0+1=1, \ 1+1=0, \ 0+0=0 \\ 0\cdot 0=0, \ 0\cdot 1=0, \ 1\cdot 1=1 \end{cases}$

<u>Theorem 1</u>

(o) 0 is unique

(a) If $x+y=x+z$ then $y=z$

(b) If $x+y=x$ then $y=0$

(c) If $x+y=0$ then $y=-x$

(d) $-(-x)=x$

(o): Let 0_1 and 0_2 be such that $x+0_1=x$ and $x+0_2=x$ $\forall x\in\mathbb{Q}$

Then $0_2 = 0_1+0_2 = 0_1$ so zero is a unique element.

(a) $\quad y = y+0 = y+(x+(-x)) = (y+x)+(-x)$

$= (z+x)+(-x) = z+(x+(-x)) = z+0 = z$

so $y=z$

(b) $\quad x+y = x = x+0$, so by (a), $y=0$

(c) $\quad x+y=0=x+(-x)$, so by (a) $y=-x$

(d) $\quad x+(-x)=0$, so by (c) $x=-(-x)$

Analogous statements for multiplication:

Theorem 1' (o) 1 is unique

(a) If $x \neq 0$ and $xy = xz$ then $y = z$

(b) If $x \neq 0$ and $xy = x$ then $y = 1$

(c) If $x \neq 0$ and $xy = 1$ then $y = 1/x$

(d) If $x \neq 0$ then $1/(1/x) = x$.

NOW LET US SEE "INTERRELATIONS" BETWEEN OPERATIONS

<u>Theorem 2</u>

(a) $0 \cdot x = 0$

(b) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$

(c) $(-x) \cdot y = -(xy) = x \cdot (-y)$

(d) $(-x) \cdot (-y) = x \cdot y$

<u>Proof</u>: (a) $(0+0) \cdot x = 0 \cdot x + 0 \cdot x = 0 \cdot x + 0$

hence, by Th. 1 (b) $0 \cdot x = 0$

(b) Assume $x \neq 0$, $y \neq 0$ but $(xy) = 0$.

Then $\frac{1}{x} \cdot \frac{1}{y} \cdot y \cdot x = \frac{1}{x}\left(\frac{1}{y} \cdot y\right) \cdot x = \frac{1}{x} \cdot x = 1$

but, on the other hand $\frac{1}{x} \cdot \frac{1}{y} \cdot 0 = 0$

so $0 = 1$ - contradiction ?

[Discussion: Can it be that $0 = 1$?]

(c) $0 = (x + (-x)) \cdot y = x \cdot y + (-x) \cdot y$ so

$(-x) \cdot y = -(xy)$ $= x(-y)$ - by yourself

(d) $(-x) \cdot (-y) = -(x \cdot (-y)) = -(-(xy)) = xy$

ORDERED FIELDS

$\mathbb{Q}$ has an ORDER STRUSTURE: $\leq$

O1. Given a, b either $a \leq b$ OR $b \leq a$ (OR, we often say, $a \lneq b$)

O2. If $a \leq b$ and $b \leq a$ then $a = b$

O3. If $a \leq b$ and $b \leq c$ then $a \leq c$

O4. If $a \leq b$ then $a + c \leq b + c$ $\forall c \in \mathbb{Q}$

O5. If $a \leq b$ and $0 \leq c$ then $ac \leq bc$

PROPERTIES OF ORDERED FIELDS

$\wedge$ = "and"

Theorem (i) $a \leq b \Rightarrow -b \leq -a$

(ii) $a \leq b \wedge c \leq 0 \Rightarrow bc \leq ac$

(iii) $0 \leq a \wedge 0 \leq b \Rightarrow 0 \leq ab$

(iv) $0 \leq a^2$ $\forall a \in \mathbb{Q}$

(v) $0 < 1$

(vi) $0 < a \Rightarrow 0 < a^{-1}$

(vii) $0 < a < b \Rightarrow 0 < b^{-1} < a^{-1}$

Proof. (i) $a \leq b$, take $c = (-a) + (-b)$

$a + c = a + (-a) + (-b) = 0 + (-b) = -b \leq b + (-a) + (-b) = -a$

(ii) $a \leq b$, $c \leq 0$ then $c + (-c) \leq 0 + (-c)$, so $0 \leq (-c)$

so $a(-c) \leq b(-c)$; $-(ac) \leq -(bc)$, so

$-(-ac) = ac \geq -(-bc) = bc$ (same as $bc \leq ac$)

(iii) $0 \leq a \Rightarrow 0 \cdot b \leq a \cdot b$ OR $0 \leq a \cdot b$

(iv) if $0 \leq a$ then $0 \leq a \cdot a = a^2$; if $a \leq 0$ then

$0 \leq -a$ and $0 \leq (-a) \cdot (-a) = -(-a^2) = a^2$

(v) $0<1$. First $0 \neq 1$ Otherwise

$0 = 0 \cdot a = 1 \cdot a = a \quad \forall a$, so
we have only one element 0 in the field.

$1^2 = 1$, so $0 \leq 1^2 = 1$ and $0 \neq 1$, so $0<1$.

[$a<b$ iff $a \leq b$ and $a \neq b$]

(vi) $a \cdot 0 = 0 < 1 = a \cdot a^{-1}$ assume $a^{-1} < 0$ then

$a^{-1} \cdot a < 0 \cdot a$ and $1 < 0$ contradiction.

(vii) $\rightarrow$ HW exercise 3.4

ABSOLUTE VALUE $|a| = a$ if $0 \leq a$ and $|a| = -a$ if $a \leq 0$

DEFINE THE DISTANCE USING $|a|$: $dist(a,b) = |a-b|$

Thm 3.5 (i) $|a| \geq 0 \quad \forall a$ and $|a| = 0$ only for $a=0$
(ii) $|ab| = |a| \cdot |b| \quad \forall a,b$
(iii) $|a+b| \leq |a| + |b| \quad \forall a,b \in \mathbb{R}$

$a \leq |a|$, so $-|a| \leq a$

proof. $-|a| - |b| \leq a - |b| \leq a+b \leq |a| + b \leq |a| + |b|$

DISTANCE (OR METRIC) $dist(a,b) = |a-b|$

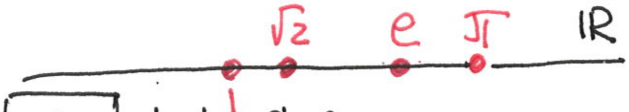
A1. $dist(a,b) \geq 0$ and $dist(a,b) = 0 \iff a=b$

A2. $dist(a,b) = dist(b,a)$

A3. $dist(a,b) + dist(b,c) \geq dist(a,c)$

UPPER AND LOWER BOUNDS; the completeness axiom

we "miraculously" passed to real numbers $\mathbb{R}$
(so far, represented by points on a line of real numbers)



(filling "gaps" in the set $\mathbb{Q}$)

DEF Let S be a nonempty subset of $\mathbb{R}$

(a) if S contains a LARGEST element s_0, that is $s_0 \in S$ and $s \leq s_0 \ \forall s \in S$, then s_0 is the maximum of S, $s_0 = \max S$

(b) if S contains a smallest element s_1 then $s_1 = \min S$

Example 1 (a) every finite subset has max & min

$S = \{3, 3.141592, 2, 2.71828\}$ $\max S = \pi, \min S = 2$

[proof by induction: problem in HW]

(b) $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ $\max [a, b] = b$
$\min [a, b] = a$

$[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ NO $\max [a, b)$
$\min [a, b) = a$

$(a, b) = \{x \in \mathbb{R} : a < x < b\}$ NO max & min

(c) $\mathbb{Z}$ and $\mathbb{Q}$ has no min OR max; $\min \mathbb{N} = 1$

(d) $\{x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2}\}$ $\min = 0$, but no maximum

DEF Let S be a nonempty subset of $\mathbb{R}$
If $M \in \mathbb{R}$ satisfies $s \leq M \ \forall s \in S$ then M is called UPPER BOUND of S (S is bounded above)
If $m \in \mathbb{R}$: $m \leq s \ \forall s \in S$ then m is LOWER BOUND (S is bounded below)
S is bounded if it is bounded above and below

$\max S$ (if it exists) in an upper bound

$\boxed{\text{DEF}}$ The Least upper bound (LUB) is the SMALLEST upper bound (if exists) min[UPPER BOUNDS]

The Greatest lower bound is max[lower bounds]

$\boxed{\text{DEF}}$ (a) If S is bounded above and S has LUB a then $a = \sup S$ (supremum)

(b) If S is bounded below and S has a GLB b then $b = \inf S$ (infimum)

$\boxed{\text{NOTE}}$ that a and b may NOT belong to S

$\boxed{\text{Exercise}}$ If $\max S$ exists then $\sup S = \max S$

proof: (a) (i) $\max S$ is an upper bound because $\forall s \in S$ $s \leq \max S$

(ii) take any upper bound M: since $\max S = s \in S$, $s \leq M$, so $\max S$ is LUB.

(b) $\sup[a,b] = \sup[a,b) = b$. Indeed, b is UB, take any upper bound M. Assume $a \leq M < b$. Then $\exists\ s = \frac{M+b}{2}$ such that (i) $a \leq s < b$, so $s \in [a,b)$; (ii) $M < s$, so M is NOT UB contradiction.

(c) Let $A = \{r \in \mathbb{Q} : r \leq \sqrt{2}\}$.

(i) A has no maximum: $\forall r \in \mathbb{Q}$, $r < \sqrt{2}$ we have r': $r < r' < \sqrt{2}$. We can construct r' explicitly: see HW

Here just mention that since $r < \sqrt{2}$, $\varepsilon = \sqrt{2} - r > 0$ and $\exists n \in \mathbb{N}$: $0 < \frac{1}{n} < \varepsilon$, then $r' = r + \frac{1}{n}$ is such that $r' \in \mathbb{Q}$ and $r < r' < \sqrt{2}$ so r is not a maximum of A; $\sqrt{2} \notin \mathbb{Q}$ so all $s \in A$ are such that $s < \sqrt{2}$.

(d) $A = \{\frac{1}{n^2}, n \in \mathbb{N}, n \geq 5\}$: $\max A = \frac{1}{25}$ $\sup A = \frac{1}{25}$

$\min A$ DNE $\inf A = 0$

Completeness Axiom

Every nonempty subset of $\mathbb{R}$ bounded above has a L.U.B.
[DOES NOT WORK for $\mathbb{Q}$!]

Corollary Every nonempty subset of $\mathbb{R}$ bounded below has a G.L.B.

Proof Take $-S: \{-s : s \in S\}$ since S bounded below, $\exists m: \; m \le s \; \forall s \in S$. Then $-s \le -m \; \forall s \in S$, so $-S$ is bounded above, hence, has LUB $m_0 = \sup(-S)$

Next, if m_1 is LB of S, then $-m_1$ is UB of $-S$ so $m_0 \le -m_1$ OR $m_1 \le -m_0$. Moreover, $-m_0$ is LB of S: $-s \le m_0 \; \forall s \in S$ so $-m_0 \le s \; \forall s \in S$

Hence $-m_0$ is G.L.B. of S.

Archimedean property

If $a > 0$ and $b > 0$ then, for some $n \in \mathbb{N}$, $na > b$

Proof Take the set $A = \{na, \; n \in \mathbb{N}\}$. If $\exists b > 0$ s.t. $na \le b$ for all n, then the set A is bounded above and has $\sup A \in \mathbb{R}$

$\forall n \quad na \le x = \sup A$, then $na - a \le x - a < x$

so $(n-1)a \le x - a$. But $\forall m \in \mathbb{N} \; \exists n = m+1$, so

$(n-1)a = ma \le x - a < x$, so $x - a$ is an UB of A and x is NOT LUB. Contradiction.

Denseness of $\mathbb{Q}$ $\quad \forall a, b \in \mathbb{R}$ with $a < b \; \exists r \in \mathbb{Q} : a < r < b$

Proof $\exists r = \frac{m}{n}$, $a < \frac{m}{n} < b \Rightarrow na < m < nb \Rightarrow \exists n, m \in \mathbb{N}$

Take $b - a = x > 0$ Then $\exists n: \; xn > 1$. Take

$\sup = \max \{m \in \mathbb{N}: \; m < nb\}$ If $m \le na$ then

$m + 1 \le na + 1 < na + nx = nb$, so $m+1$ is in the set – contradiction, so $\quad na < m < nb$ ☺

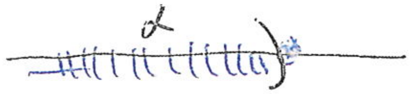
DEDEKIND CUTS

$\forall a \in \mathbb{R} \quad a = \sup\{r \in \mathbb{Q} : r < a\}$

ASSUME WE DO NOT KNOW ABOUT EXISTENCE OF IRRAT. #

SETS α $\quad \alpha \subset \mathbb{Q}$ $\quad$ (i) $\alpha \neq \mathbb{Q}$ and nonempty

(ii) if $r \in \alpha$, $s \in \mathbb{Q}$, $s < r$ then $s \in \alpha$



(iii) α contains no largest rational

α – DEDEKIND CUT ! $\quad$ SET OF $\alpha \cong \mathbb{R}$

COROLLARY EVERY DEDEKIND CUT α HAS AN UPPER BOUND

PROOF $\exists b \in \mathbb{Q}$ $\quad b \notin \alpha$ IF $\exists a \in \alpha$, $b \leq a$, then $b \in \alpha$ CONTRADICTION. SO $\forall a \in \alpha$, $a < b$ and b is UB

$\mathbb{Q}$ - ordered set. INTRODUCE A (NATURAL) ORDERING FOR α: IF α_1 AND α_2 ARE DEDEKIND CUTS, THEN EITHER $\alpha_1 \subseteq \alpha_2$ OR $\alpha_2 \subseteq \alpha_1$. IN THE FIRST CASE $\alpha_1 \leq \alpha_2$, OTHERWISE $\alpha_2 \leq \alpha_1$

PROVE THE COMPLETENESS

LET $\{\alpha\}$ BE ANY SET OF DEDEKIND CUTS. TAKE THE UNION $\mathcal{U} = \bigcup_{\alpha \in \{\alpha\}} \alpha$. IF $a \in \mathcal{U}$ then $\forall b < a$, $b \in \mathcal{U}$ because $a \in \alpha_1$ and $b \in \alpha_1$ (SOME SET FROM THE UNION)

EITHER $\mathcal{U} = \mathbb{Q}$ then not a cut (and NOT Bounded) OR $\mathcal{U} \neq \mathbb{Q}$ and is a DEDEKIND CUT. PROVE THAT $\mathcal{U}$ is $\sup\{\alpha\}$ TAKE ANY D.C. ω s.t. $\alpha \leq \omega \ \forall \alpha \in \mathcal{U}$ then, since for any $x \in \mathcal{U}$ ~~the~~ we have $x \in \alpha$ for some $\alpha \in \{\alpha\}$, $x \in \omega$, therefore $\mathcal{U} \leq \omega$ for ANY upper bound ω, so $\mathcal{U}$ is the LUB

$\boxed{\Phi\text{-}XVI}$

WE CAN <u>PROVE</u> THAT DEDEKIND CUTS SATISFY <u>ALL</u> AXIOMS OF ORDERED NUMBER FIELDS. [TEDIOUS AND NOT QUITE PLEASANT ORDEAL]
BUT, AT LEAST, LET US SHOW SEVERAL THINGS:

[i] FOR α, β − D. cuts, $\alpha+\beta$ IS A DEDEKIND CUT

(i) if $x \in \mathbb{Q}$ and $x \in \alpha+\beta$ TAKE $y \in \mathbb{Q}$, $y \le x$

SINCE $x \in \alpha+\beta$ $\exists\, y_1 \in \alpha$, $y_2 \in \beta$ SUCH THAT $y_1+y_2=x$

TAKE $y_1 \in \alpha$ AND $y_2+y-x \leq y_2$ SO $y_2+y-x \in \beta$,

so $y_1+(y_2+y-x) \in \alpha+\beta$, but it is just y!

(ii) $\alpha \neq \mathbb{Q}$, so $\exists$ UB $A \in \mathbb{Q}$ s.t. $\forall y_1 \in \alpha$ $y_1 < A$

$\beta \neq \mathbb{Q}$, so $\exists$ UB $B \in \mathbb{Q}$ s.t. $\forall y_2 \in \beta$ $y_2 < B$

Then $\forall$ $x = y_1+y_2$ $x < A+B$, so $A+B \notin \alpha+\beta$, so $\alpha+\beta \neq \mathbb{Q}$

(iii) IF $\exists \max(\alpha+\beta) = C \in \mathbb{Q}$ then $C = y_1+y_2$ for some $y_1 \in \alpha$, $y_2 \in \beta$. BUT $\forall y_1 \in \alpha$ $\exists\, y_1' \in \alpha$ SUCH THAT $y_1 < y_1'$ (No maximum element) and $\forall y_2 \in \beta$ $\exists\, y_2' \in \beta$ SUCH THAT $y_2 < y_2'$, then $y_1'+y_2' \in \alpha+\beta$ AND $y_1+y_2 < y_1'+y_2'$ SO y_1+y_2 CANNOT BE A MAXIMUM ELEMENT. $\square$

So, $\alpha+\beta$ IS A DEDEKIND CUT