Problem 1

Kittel & Kroemer, Chapter 3, problem 6 [Rotation of diatomic molecules]

In our first look at the ideal gas we considered only the translational energy of the particles. But molecules can rotate, with kinetic energy. The rotational motion is quantized; and the energy levels of a diatomic molecule are of the form

$$\epsilon(j) = j(j+1)\epsilon_0$$

where j is any positive integer including zero: $j = 0, I, 2, \ldots$. The multiplicity of each rotational level is $g(j) = 2j + 1$.

(a): 1 point

Find the partition function $Z_R(\tau)$ for the rotational states of one molecule. Remember that Z is a sum over all states, not over all levels—this makes a difference.

(b): 1 point

Evaluate $Z_R(\tau)$ approximately for $\tau \gg \epsilon_0$, by converting the sum to an integral.

(c): 1 point

Do the same for $\tau \ll \epsilon_0$, by truncating the sum after the second term.

(d): 1 point

Give expressions for the energy U and the heat capacity C, as functions of τ, in both limits. Observe that the rotational contribution to the heat capacity of a diatomic molecule approaches 1 (or, in conventional units, k_B) when $\tau \gg \epsilon_0$.

(e): 1 point

Sketch the behavior of $U(\tau)$ and $C(\tau)$, showing the limiting behaviors for $\tau \to \infty$ and $\tau \to 0$.

Problem 2

Adapted from Kittel & Kroemer, Chapter 3, problem 11 [One-dimensional gas]: 3 points

Consider an ideal gas of N particles, each of mass M, confined to a one-dimensional line of length L. Find the entropy at temperature τ. The particles have spin zero. [Note: treat the ideal gas as being in thermal contact with a reservoir at temperature τ. This is a quantum mechanics problem—try to follow the 3-dimensional version that we treated in class and in the book.]

Problem 3

3 points

This problem will give you a better understanding of the "quantum concentration" used repeatedly in your textbook. The de Broglie wavelength λ for a particle with momentum p is defined as $\lambda = h/p$, where $h = 2\pi\hbar$ is Planck's constant. If we don't know the momentum, but we know that the particle is part of a gas at temperature τ , then we can define the "thermal de Broglie wavelength" λ_{th} by assuming that the kinetic energy of the particle is equal to $\frac{3}{2}\tau$, as we derived in class. Derive an expression for the thermal de Broglie wavelength in terms of $\hbar$, τ, the particle mass m, and some numerical constants. What is the concentration n of the gas if the average spacing between particles equals λ_{th}? (Hint: n has units of inverse volume.) Compare your answer with the definition of the quantum concentration n_Q given in the book. The two formulas should differ only by a numerical constant. Evaluate the ratio n/n_Q.

Problem 4

Before it was understood that particles, even in the classical regime, are indistinguishable (a revelation of quantum physics), the statistical treatment of the ideal gas of N non-interacting particles gives a partition function $Z_N = (Z_1)^N$, where Z_1 is the partition function of a single particle (as opposed to $Z_N = (Z_1)^N/N!$, which takes the particles' indistinguishability into account in the classical regime). Let's feign ignorance for a moment, and see what the differences are between the predictions of these two models. For both, let's use the partition function for a single particle that we have derived in class, $Z_1 = n_Q V$, where $n_Q \equiv \left(\frac{m\tau}{2\pi\hbar^2}\right)^{3/2}$.

(a): 1 point

Calculate the entropy for an ideal gas of N particles, *ignoring the indistinguishability of particles*, i.e. start with the partition function $Z_N = (Z_1)^N$.

(b): 1 point

Show that the entropy derived in part (a) is not extensive. This is the "Gibbs Paradox". (Reminder: if you double N and V for the system, an extensive quantity should also double).

(c): 1 point

Show that the Sackur-Tetrode equation for the ideal gas entropy, which takes into account the indistinguishability of the particles, gives an entropy that is extensive.