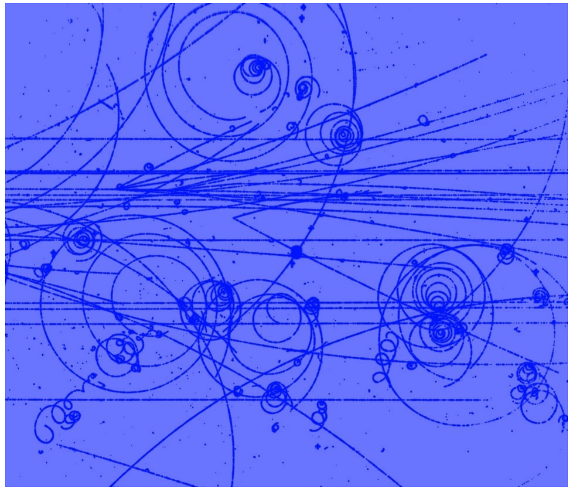
Announcements



Quiz:
- Pick up quiz after class. **No quiz this week** – midterm instead.

Homework:
Homework 1 grades posted on gradescope. Solutions posted on D2L.
Homework 2 + solutions will be posted before next class.
Third HW posted on D2L : Q1/Q2 can be done now

Midterm: Friday, Feb 21 in class
Will cover through "bound states".
Equation sheet: 1 letter-sized (8 ½ by 11 inches) page front and back, handwritten

Paper:
Topic due Monday Feb 17 at 3pm. Fill out this google form before then:
https://forms.gle/MmCk8NtrMm7RdfLC7

Office hours: on **Wednesday 4-5pm this week**, not on Friday

The Golden Rule Reminder

Fermi's Golden Rule states:

The transition rate for a system depends on two fundamental quantities

1) The "Matrix Element" or "Transition Amplitude"
2) The final state phase space

$$W = \frac{2\pi}{\hbar} \left| \mathcal{M}_{if} \right|^2 \rho(E)$$

The transition rate (decays or interactions)

The Matrix Element. We may not actually know this.

Phase Space or Density of States

W is proportional to cross section or decay rate

See derivation from last class

Feynman Rules

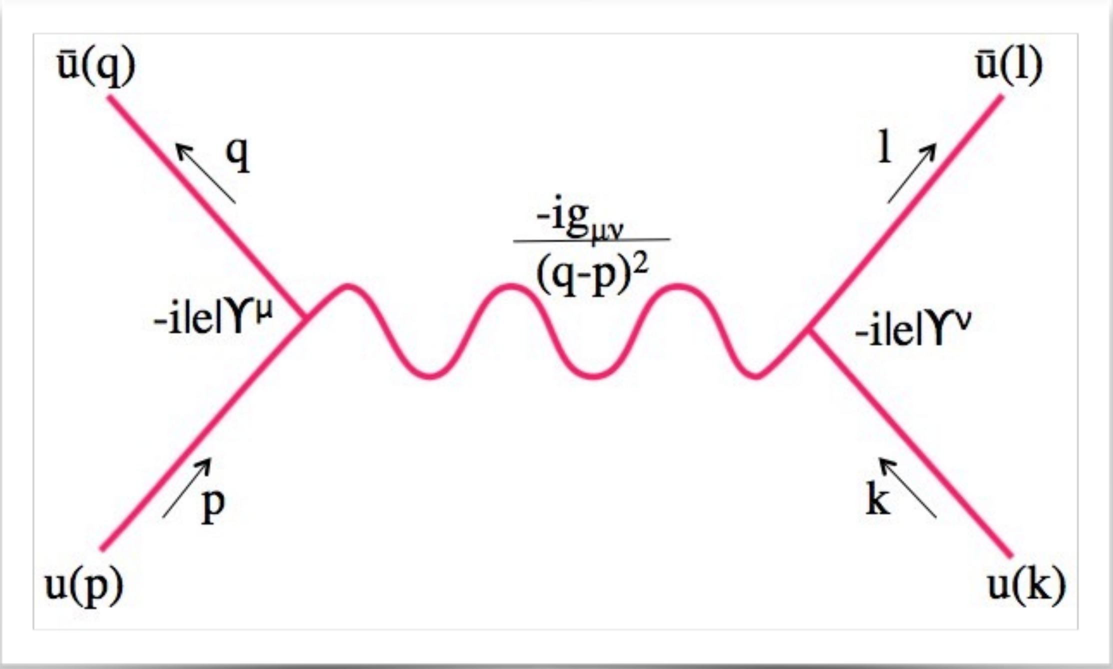
We've been completely ignoring the dynamical part of the equation

All of the specific interaction dynamics are bound up in the **Matrix Element**

We need to now build up the rules for calculating it!

The Feynman diagram will be our guide in most cases

We will assign kinematical features to each part of the diagram, depending on what type of particle it represents.



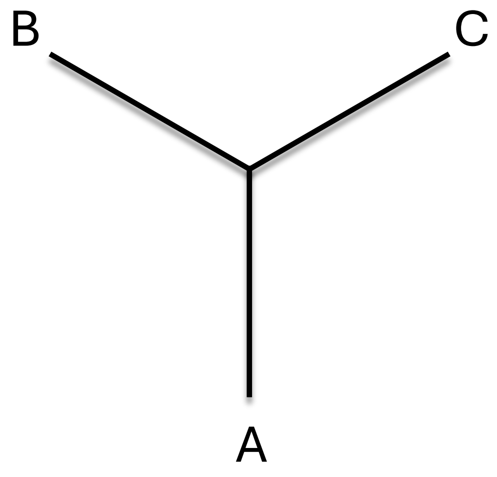
The Feynman diagram will give rise to the matrix element.

ABC Theory

To start with the Feynman rules, use a toy theory: ABC theory

- There are 3 spinless particles: A, B, C
- Each particle is its own antiparticle (No arrows!)
- Only 1 vertex exists and includes all three particles (ABC). Eg, (AAA) is forbidden.
- Masses not assumed, different cases lead to different physics, i.e.
  - If $m_A > m_B + m_C$, then A can decay to B and C.

Basic vertex



Last time: decay of the A into B + C

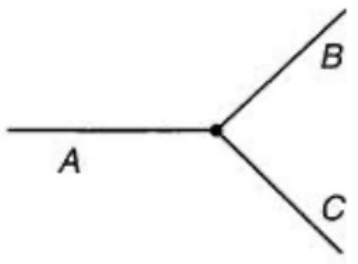
Today: Scattering

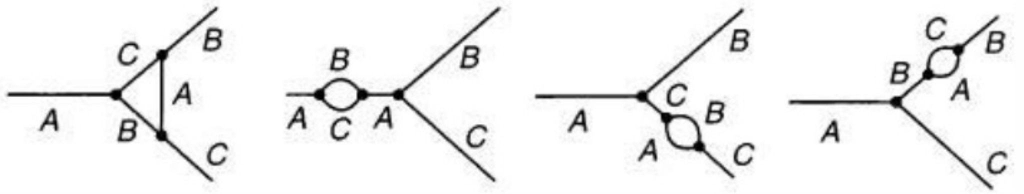
Feynman Rules for ABC Theory

The Feynman rules provide the recipe for constructing an amplitude (Matrix Element $\mathcal{M}$) from a Feynman diagram.

<u>Rule 1:</u>
Draw the Feynman diagram with the minimum number of vertices.
There may be more than 1.

Eg. for A -> B+C decay:





Feynman Rules for ABC Theory

Rule 2:

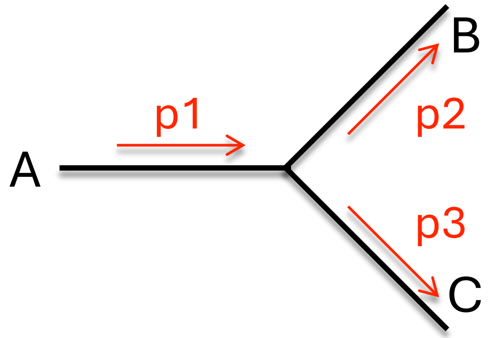
Label the four-momentum of each line (with arrows), enforcing four-momentum conservation at each vertex.

p1, p2, ... external momenta: need arrows

q1, q2, ... internal momenta: arrow is arbitrary

We'll keep track of arrows into/out of vertices.

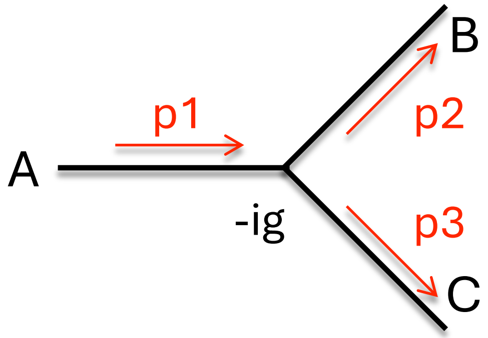
Eg. for A -> B+C decay:



Feynman Rules for ABC Theory

Rule 3:
Each vertex contributes a factor of $(-ig)$, where g is referred to as the coupling constant. It specifies the strength of the ABC interaction.

Eg. for A -> B+C decay:



Feynman Rules for ABC Theory

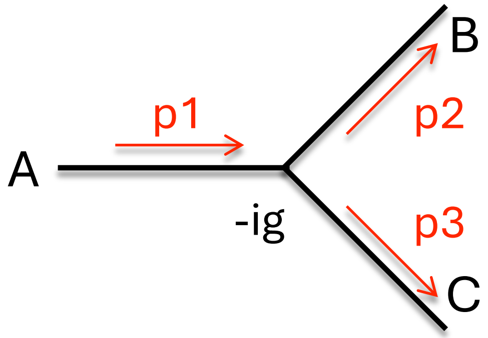
<u>Rule 4:</u>
Each internal line, or propagator, with mass **m** and four-momentum **q** gets a factor of:

$$\frac{i}{q^2 - m^2c^2}$$

q^2 = 4 momentum of internal line

Note: q^2 doesn't have to equal m^2. These are virtual particles!

Eg. for A -> B+C decay:



-> There are no propagator lines here

Feynman Rules for ABC Theory

Rule 5:

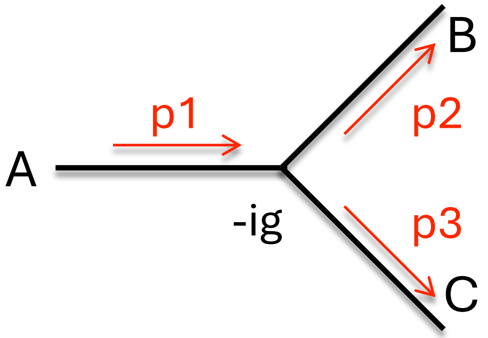
Each vertex contributes a delta function to conserve energy and momentum. The **k_i** are the momenta coming into the vertex:

$$(2\pi)^4\ \delta^4(k_1 + k_2 + k_3)$$

k= external (p) or internal (q) momentum

k is negative if it is outgoing

Eg. for A -> B+C decay:



$$(2\pi)^4\ \delta^4(p_A - p_B - p_C)$$

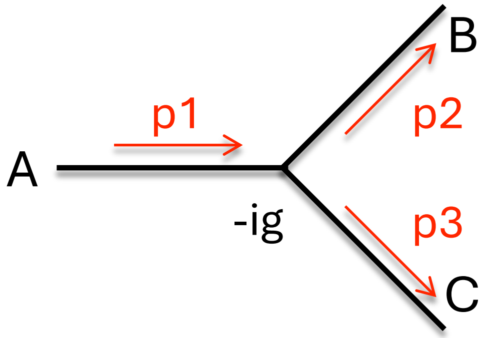
Feynman Rules for ABC Theory

Rule 6:

Build up the proto-Matrix Element from the previous factors and add an **i**:

$$\mathcal{M} = i \text{ (vertex factors) (propagator factors) (momentum conservation)}$$

Eg. for A -> B+C decay:



$$\mathcal{M} = i\,(-ig)\,(2\pi)^4\,\delta^4(p_A - p_B - p_C)$$

Feynman Rules for ABC Theory

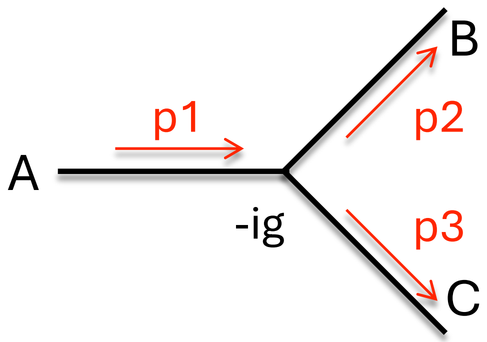
<u>Rule 7:</u>

Integrate over the internal momenta:

For each internal line write down a factor: $$\frac{1}{(2\pi)^4}\, d^4 q_i$$

This connects initial/final state momenta via the delta functions.

Eg. for A -> B+C decay:



$$\mathcal{M} = i\,(-ig)\;(2\pi)^4\,\delta^4(p_A - p_B - p_C)$$

$$= g\;(2\pi)^4\,\delta^4(p_A - p_B - p_C)$$

-> No internal lines to integrate over

Feynman Rules for ABC Theory

Rule 8:
The result contains a delta function that is a statement of total momentum conservation
Drop this delta function. This statement of total momentum conservation will be included in the Golden Rule. (Not included here because the matrix element is squared)

The result of step 8 is the matrix element!

$$\mathcal{M} = g \; \cancel{(2\pi)^4 \, \delta^4(p_A - p_B - p_C)}$$

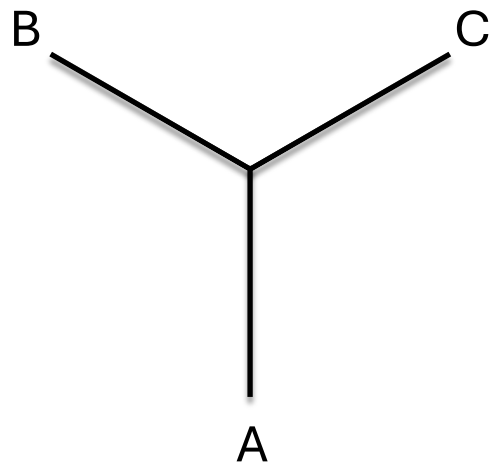
$$= g$$

➔ In this simple case the delta function was unnecessary
➔ Once we have the matrix element, plug in to Golden rule to calculate decay rate

Scattering examples

- AA to BB scattering

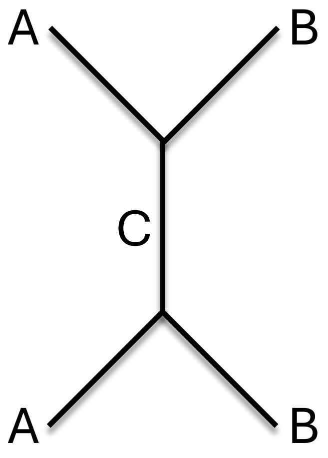
- AB to AB scattering

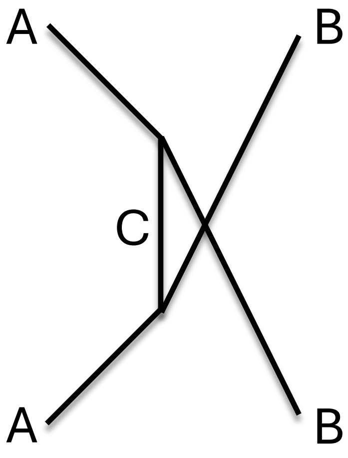


1. How many Feynman diagrams for each process
2. Draw each of them

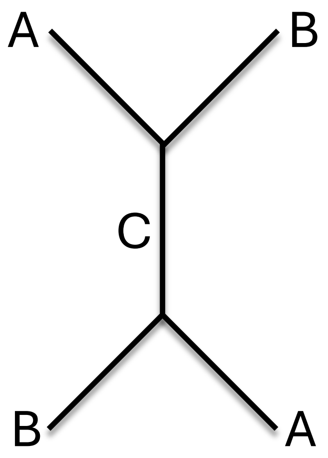
Scattering example

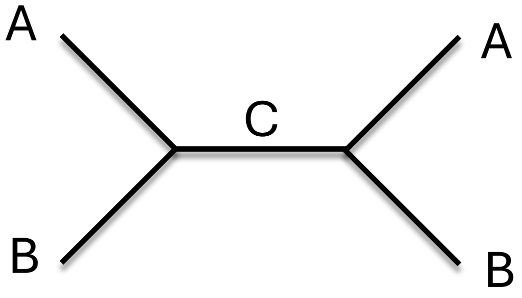
- AA to BB scattering





- AB to AB scattering

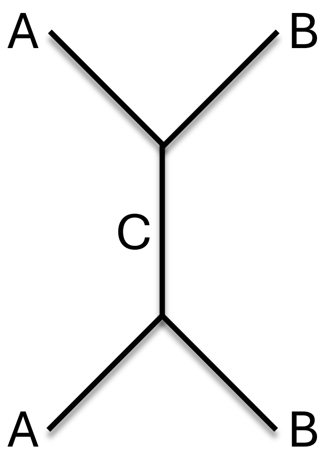




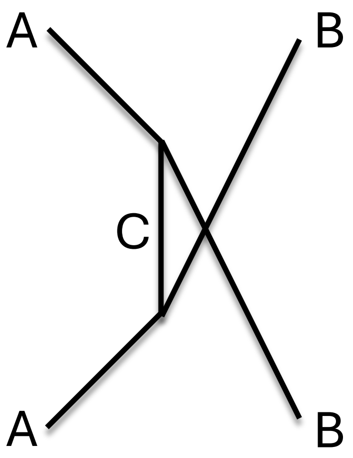
A+A→B+B Scattering

In ABC theory A+A into B+B occurs via the exchange of particle C.

We have the t-channel diagram



And also the u-channel diagram!



Why two diagrams?

- The u-channel diagram is necessary because we cannot tell which outgoing particles connect to which vertex. So we have to include both possibilities in our matrix element and, thus, in the integrations.
- Our S-factor will now be S=1/2!=1/2. We effectively average over the two diagrams -> No double counting

Golden Rule for Scattering Cross Section

Last class, 2-body scattering (1,2→3,4):

$$\frac{d\sigma}{d\Omega} = \left(\frac{\hbar c}{8\pi}\right)^2 \frac{S|\mathcal{M}|^2}{(E_1 + E_2)^2} \frac{|p_f|}{|p_i|}$$

Set S= ½ for A+A -> B+B scattering:

$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{2(E_1 + E_2)^2} \frac{|p_f|}{|p_i|}$$

Also:

$$\hbar = c = 1$$

A+A→B+B Scattering: Calculate M

The scattering of A+A into B+B is highly relevant
In ABC theory, this occurs via the exchange of particle C.

Rule 1: draw all lowest order diagrams

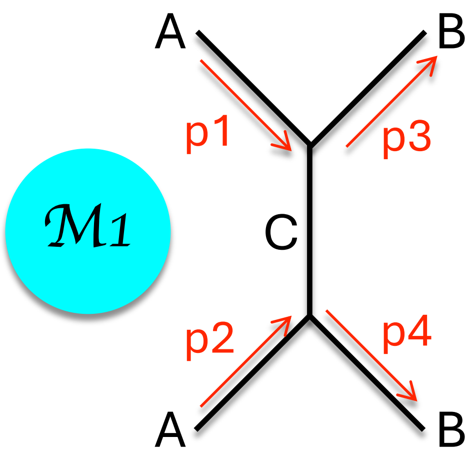
A+A→B+B Scattering: Calculate M

The scattering of A+A into B+B is highly relevant
In ABC theory, this occurs via the exchange of particle C.

Rule 1: draw all lowest order diagrams

Rule 2: label incoming and outgoing 4-momentum

We have the t-channel diagram



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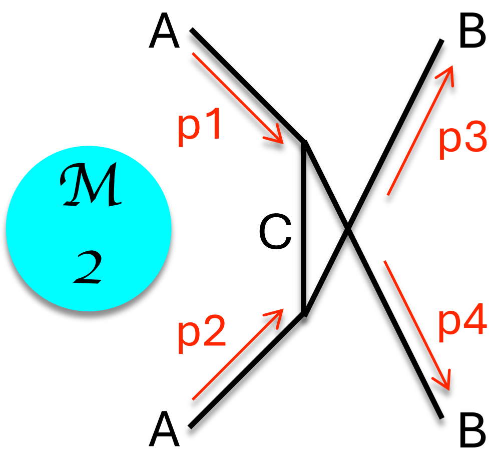
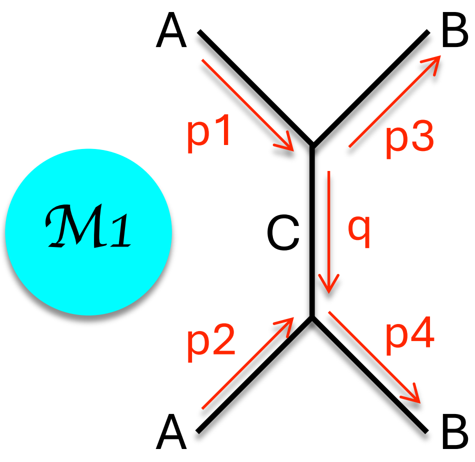


A+A→B+B Scattering: t-channel M

Rule 3: Each vertex contributes a factor of (-ig)

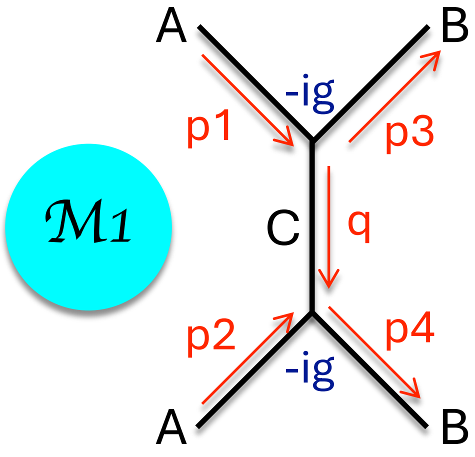
How many are there?



A+A→B+B Scattering: t-channel M

Rule 3: Each vertex contributes a factor of (-ig)

Two vertices -> Two factors of -ig



A+A→B+B Scattering: t-channel M

Rule 3: Each vertex contributes a factor of (-ig)

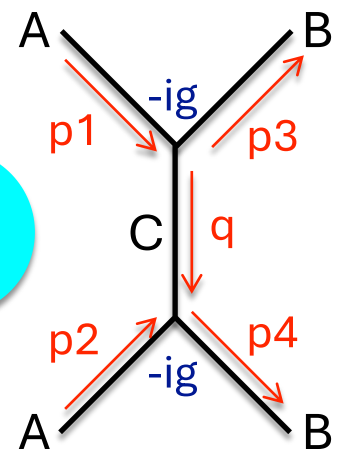
Two vertices -> Two factors of -ig

Rule 4: Each internal line, or propagator, with mass **m** and four momentum **q** gets a factor of:

$$\frac{i}{q^2 - m_C^2}$$

So, how many do we have?

$\mathcal{M}_1$



A+A→B+B Scattering: t-channel M

Rule 3: Each vertex contributes a factor of (-ig)

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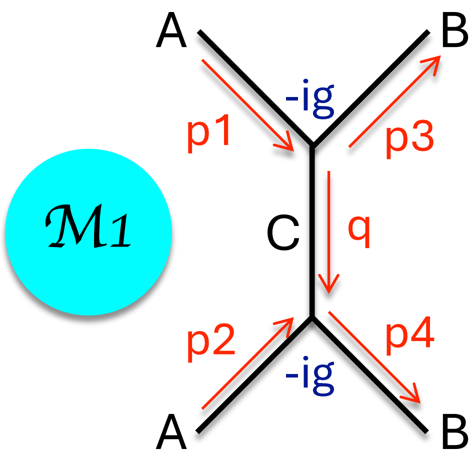
Rule 4: Each internal line, or propagator, with mass **m** and four momentum **q** gets a factor of:

One propagator, one factor: $$\frac{i}{q^2 - m_C^2}$$



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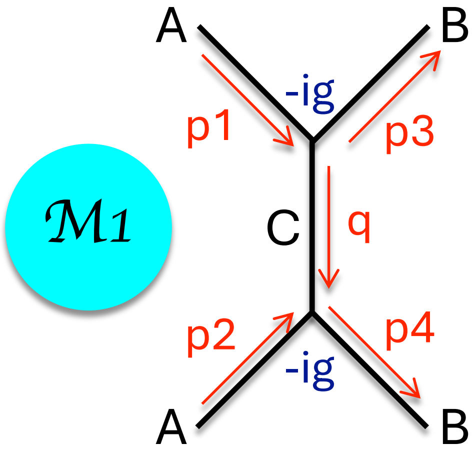
Rule 5: Each vertex contributes a delta function to conserve energy and momentum

What do the delta functions look like?



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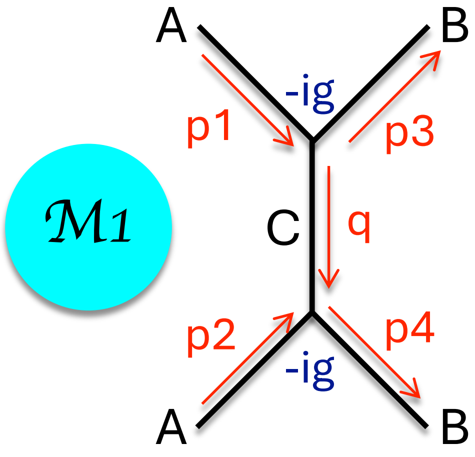

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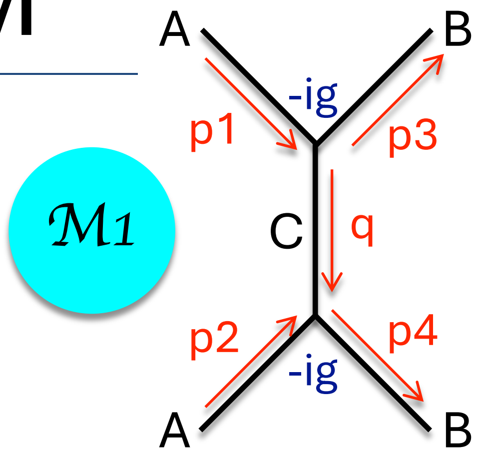
Rule 5: Each vertex contributes a delta function to conserve energy and momentum

Two delta functions, one for momentum in and out at each vertex:

$$(2\pi)^4\ \delta^4(p_1 - q - p_3) \qquad\qquad (2\pi)^4\ \delta^4(p_2 + q - p_4)$$

A+A→B+B Scattering: t-channel M



Rule 6:

Build up the Matrix Element from the previous factors & add an **i**:

$$\mathcal{M} = i \text{ (vertex factors) (propagator factors) (momentum conservation)}$$

Rule 7:

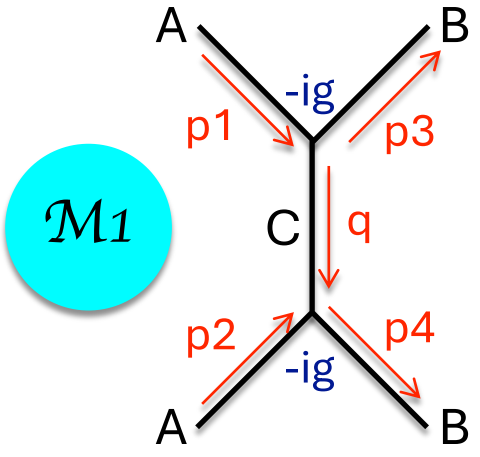
Integrate over the internal momenta:

For each internal line write down a factor:

$$\frac{1}{(2\pi)^4}\, d^4 q_i$$

This connects initial/final state momenta via the delta functions.

A+A→B+B Scattering: t-channel M



Rule 3: Vertex Factors

Rule 5: Vertex E/p conservation

Rule 6 and 7:

$$\mathcal{M}_1 = i\int (-ig)^2 \left(\frac{i}{q^2 - m_C^2}\right) (2\pi)^4\delta^4(p_1 - q - p_3)\ (2\pi)^4\delta^4(p_2 + q - p_4)\ \frac{d^4q}{(2\pi)^4}$$

Rule 4: Propagator Factor

Rule 7: Vertex E/p conservation

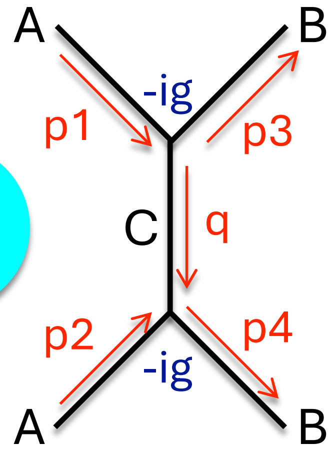
A+A→B+B Scattering: t-channel M

The propagator integral sends either:

$$\boxed{q \to p_4 - p_2}$$

or:

$$q \to p_3 - p_1$$



Rule 6 and 7:

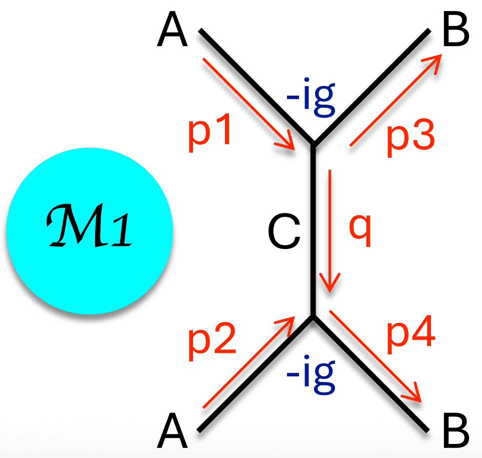
$$\mathcal{M}_1 = i\int(-ig)^2\left(\frac{i}{q^2-m_C^2}\right)(2\pi)^4\delta^4(p_1-q-p_3)\,(2\pi)^4\delta^4(p_2+q-p_4)\,\frac{d^4q}{(2\pi)^4}$$

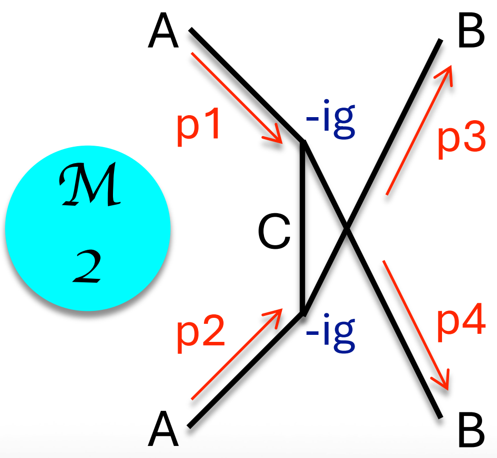
$$\mathcal{M}_1 = \frac{g^2}{(p_4-p_2)^2-m_C^2}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)$$

A+A→B+B Scattering

Now we can consider the u-channel matrix element.

No work required, just swap out the relevant momenta





$$\mathcal{M}_1 = \frac{g^2}{(p_4 - p_2)^2 - m_C^2} \qquad \mathcal{M}_2 = \frac{g^2}{(p_3 - p_2)^2 - m_C^2}$$

$$\mathcal{M}_{\text{tot}} = \quad ?$$

A+A→B+B Scattering

Now we can consider the u-channel matrix element.

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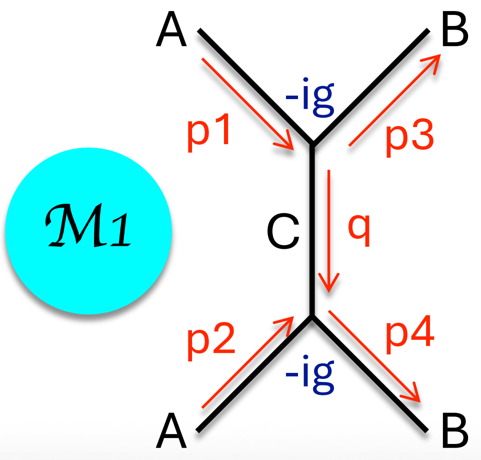
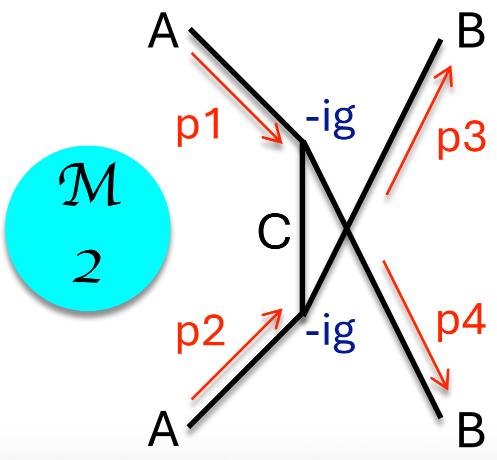




$$\mathcal{M}_1 = \frac{g^2}{(p_4 - p_2)^2 - m_C^2} \qquad \mathcal{M}_2 = \frac{g^2}{(p_3 - p_2)^2 - m_C^2}$$

$$\mathcal{M}_{\rm tot} = \mathcal{M}_1 + \mathcal{M}_2$$

A+A→B+B Scattering

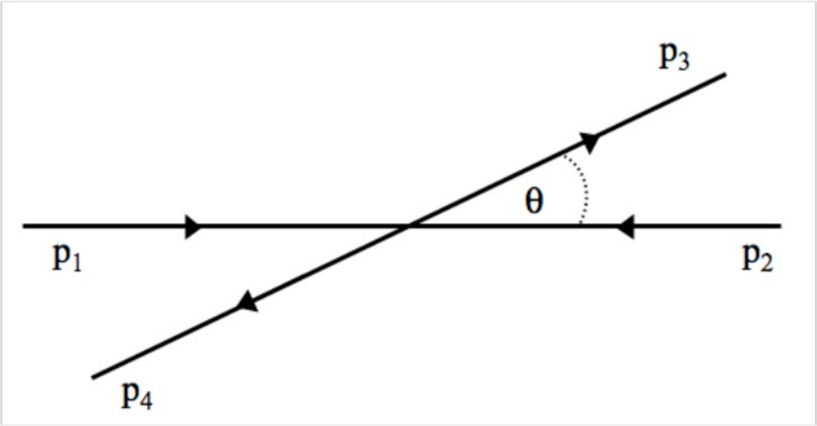
The A+A→B+B scattering matrix element is Lorentz Invariant

But the evaluated quantities may depend on the inertial frame

Let's consider the CoM frame and let $M_A=M_B=m$ and $M_C=0$

$$\mathcal{M}_1 = \frac{g^2}{(p_4 - p_2)^2 - m_C^2}$$

$$\mathcal{M}_2 = \frac{g^2}{(p_3 - p_2)^2 - m_C^2}$$



$$t = (p_4 - p_2)^2 = p_4^2 + p_2^2 - 2p_2 \cdot p_4 = -2p^2(1 - \cos\theta)$$

$$u = (p_3 - p_2)^2 = p_3^2 + p_2^2 - 2p_3 \cdot p_2 = -2p^2(1 + \cos\theta)$$

A+A→B+B Scattering

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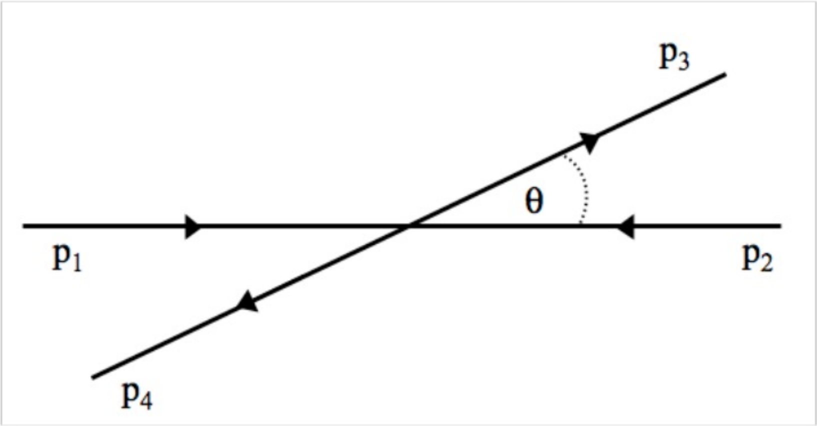
Let's consider the CoM frame and let $M_A=M_A=m$ and $M_C=0$

$$\mathcal{M}_1 = \frac{g^2}{(p_4 - p_2)^2 - m_C^2}$$

$$\mathcal{M}_2 = \frac{g^2}{(p_3 - p_2)^2 - m_C^2}$$

$$\mathcal{M} = -\frac{g^2}{p^2 \sin^2\theta}$$



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$$u = (p_3 - p_2)^2 = p_3^2 + p_2^2 - 2p_3 \cdot p_2 = -2p^2(1 + \cos\theta)$$

A+A→B+B Scattering

Finally, we can evaluate the differential cross section!

We worked out the 2-body kinematics earlier

We just calculated the matrix element

Using Golden rule:

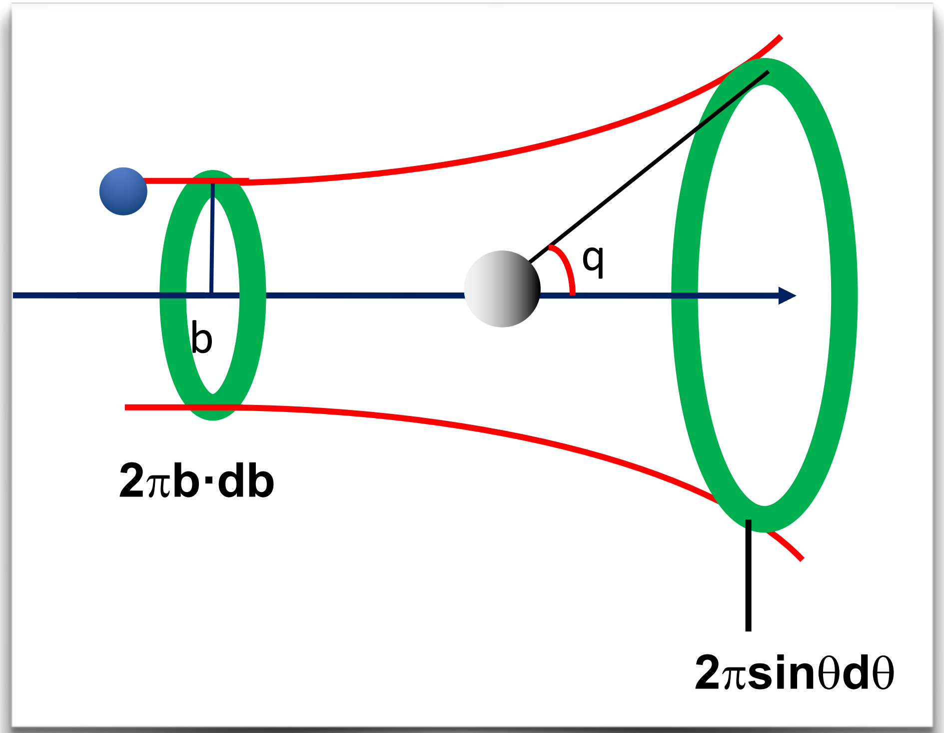
$$\frac{d\sigma}{d\Omega} = \left(\frac{1}{8\pi}\right)^2 \frac{|\mathcal{M}|^2}{2(E_1+E_2)^2} \frac{|p_f|}{|p_i|} \quad \mathcal{M} = -\frac{g^2}{p^2 \sin^2\theta}$$

From the CoM condition we have:

$$E_1 = E_2$$
$$|p_f| = |p_i|$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{g^2}{16\pi\ E\ |p|^2 \sin^2\theta}\right)^2$$

Rutherford Scattering Again



AA→BB Scattering

$$\frac{d\sigma}{d\Omega} = \frac{1}{2}\left(\frac{g^2}{16\pi\ E\ |p|^2 \sin^2\theta}\right)^2$$

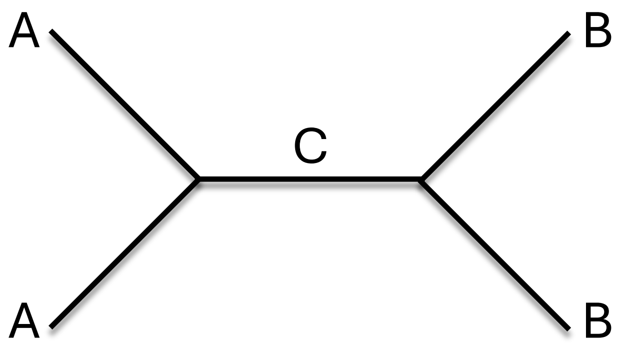
Rutherford Scattering

$$\left|\frac{d\sigma}{d\Omega}\right| = \left(\frac{zZe^2}{16\pi\epsilon_0}\frac{1}{E_{\text{kin}}}\right)^2 \operatorname{cosec}^4\frac{\theta}{2}$$

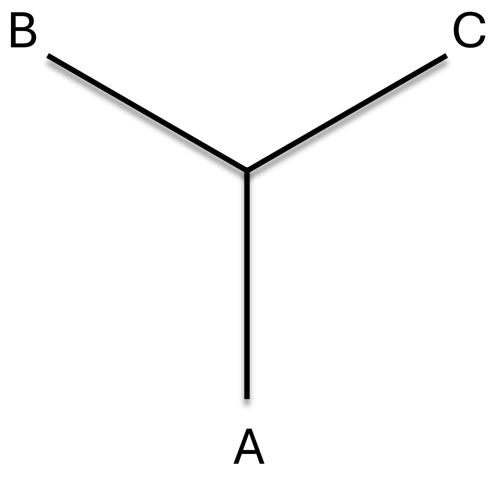
A+A→B+B Scattering

Why didn't we consider this diagram?

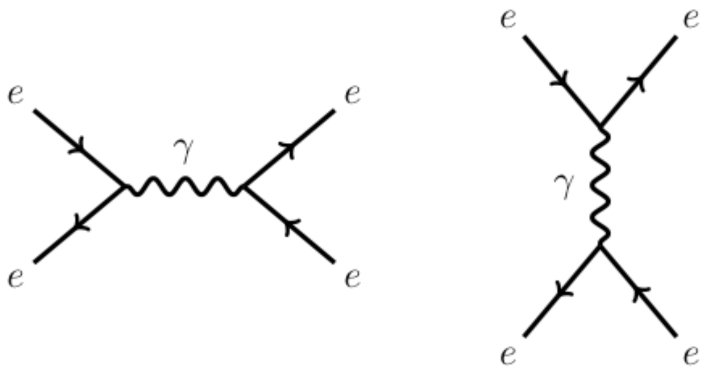
Be careful to not read too deeply into ABC theory!



Primitive vertex



Answer: There are no AAC or CBB vertices in ABC theory. But this diagram does exist in QED, for example.



Another question

Earlier we said cross sections sum:

$$\sigma_{\text{tot}} = \sum \sigma_i$$

And we just said matrix elements sum:

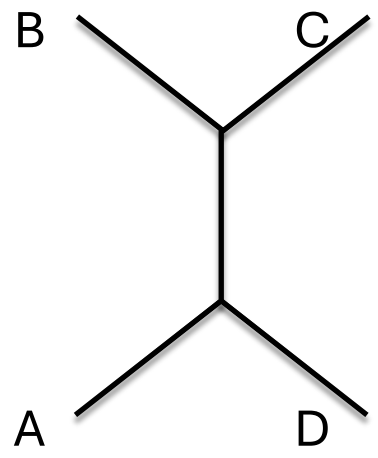
$$\mathcal{M}_{\text{tot}} = \mathcal{M}_1 + \mathcal{M}_2$$

The first statement is only true if there is no interference between the matrix elements!

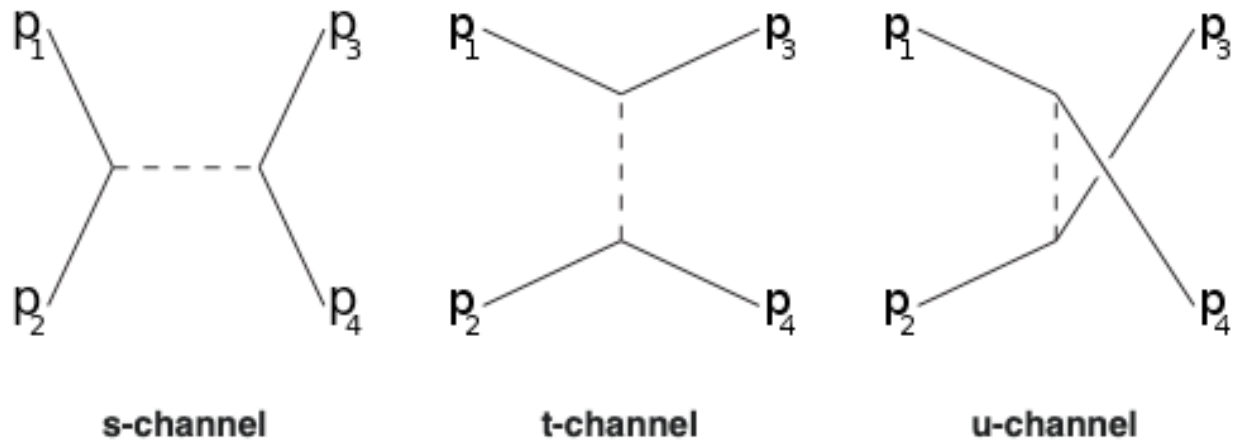
$$|\mathcal{M}|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2|\mathcal{M}_1 \cdot \mathcal{M}_2|^2$$

Mandlestam Variables

- Relate 4-momentum of incoming/outgoing particles



Sometimes classify Feynman diagrams in similar terms:



$$s = (p_A + p_B)^2$$
$$t = (p_A - p_c)^2$$
$$u = (p_A - p_D)^2$$

Higher Order Diagrams

These interference effects frequently (but not always) arise due to higher order diagrams

- ABC theory limits the number of such diagrams, so continue to be wary
- We're not going to dig deep into this right now, but be aware they exist