Physics 410 Quiz #4 – Thursday, March 20, 2025

Name:________________________________

1. [5] Consider a cylinder of length L and radius R, containing an ideal gas of N monatomic, spinless particles each of mass M. The cylinder rotates around its long axis at angular frequency ω. The box is in thermal contact with a reservoir at temperature τ. In the reference frame of the particles, their situation is equivalent to having a potential energy $U(r) = \frac{1}{2}M\omega^2(L^2 - r^2)$ for a particle a distance r from the axis of rotation. Suppose the concentration of particles on the axis is $n(0)$; find an expression for the concentration of particles a distance r from the axis of rotation.

$\mu_{tot}(0) = \mu_{tot}(r)$ in diffusive equilibrium

$$\tau \ln\left(\frac{n(0)}{n_Q}\right) + U(0) = \tau \ln\left(\frac{n(r)}{n_Q}\right) + U(r)$$

$$+\tau \ln\left(\frac{n(r)}{n(0)}\right) = \frac{1}{2}M\omega^2(r^2\,\text{[redacted]}$$

$$\boxed{n(r) = n(0)\, e^{\frac{M\omega^2(r^2\,\text{[redacted]})}{2\tau}}}$$

2. [5] Consider an ideal gas of N spinless particles, each of mass M, confined to a two-dimensional square of side length L. The energy levels of the particles are $\epsilon_{n_x,n_y} = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2)$, where $n_x, n_y = 1,2,3, \ldots$. Calculate the partition function Z_1 for a single particle. If you are confronted with a sum, turn it into an integral; it is OK to leave a *dimensionless* integral in your answer.

$$Z_1 = \sum_{n_x n_y} e^{-\frac{\hbar^2\pi^2(n_x^2+n_y^2)}{2mL^2\tau}} \Rightarrow \boxed{\int_0^\infty dn_x \int_0^\infty dn_y\, e^{-\frac{\hbar^2\pi^2(n_x^2+n_y^2)}{2mL^2\tau}}} \quad : \quad n^2 = n_x^2 + n_y^2$$

$$\int_0^\infty dn_x \int_0^\infty dn_y \to \frac{1}{4}\int_0^\infty (2\pi n)\,dn$$

$$Z_1 = \frac{\pi}{2}\int n\, e^{\hbar^2\pi^2 n^2/2mL^2\tau}\, dn \quad : \quad x = \sqrt{\frac{\hbar^2\pi^2}{2mL^2\tau}}\cdot n$$

$$= \boxed{\frac{\pi}{2}\cdot\frac{2mL^2\tau}{\hbar^2\pi^2}\int x\, e^{-x^2}\, dx}$$

OR

$$x = \sqrt{\frac{\hbar^2\pi^2}{2mL^2\tau}}\cdot n_x, \quad y = \sqrt{\frac{\hbar^2\pi^2}{2mL^2\tau}}\cdot n_y$$

$$\boxed{Z_1 = \frac{2mL^2\tau}{\hbar^2\pi^2}\left(\int_0^\infty dx\, e^{-x^2}\right)^2}$$

3. [10] Consider a system with only two orbitals, one at energy 0 and one at energy ϵ. Each orbital can be occupied by either zero, one, or two particles at the same time, i.e. $N_0 = 0,1,2$ and $N_\epsilon = 0,1,2$. The system is in diffusive and thermal contact with a reservoir at temperature τ and chemical potential μ.

a) [4] Write down the Gibbs sum for this system.

There should be 9 total terms in the sum, corresponding to $(N_0, N_\epsilon) = (0,0), (0,1), (1,0), \ldots$

Easiest to write by considering this as two independent systems:

$$\mathcal{Z} = \mathcal{Z}_0 \cdot \mathcal{Z}_\epsilon = \left(e^0 + e^{\mu/\tau} + e^{2\mu/\tau}\right)\left(e^0 + e^{\frac{\mu-\epsilon}{\tau}} + e^{\frac{2\mu-2\epsilon}{\tau}}\right)$$

$$= \underbrace{1 + e^{\mu/\tau} + e^{2\mu/\tau}}_{N_\epsilon=0} + \underbrace{e^{\frac{\mu-\epsilon}{\tau}} + e^{\frac{2\mu-\epsilon}{\tau}} + e^{\frac{3\mu-\epsilon}{\tau}}}_{N_\epsilon=1} + \underbrace{e^{\frac{2\mu-2\epsilon}{\tau}} + e^{\frac{3\mu-2\epsilon}{\tau}} + e^{\frac{4\mu-2\epsilon}{\tau}}}_{N_\epsilon=2}$$

b) [3] Find the thermal average number of particles in each orbital, $\langle N_0 \rangle$ and $\langle N_\epsilon \rangle$.

$$\langle N_0 \rangle = \frac{\sum N_0 e^{(\mu N - \epsilon)/\tau}}{\mathcal{Z}} = \left[(0)\sum_{N_\epsilon} e^{(\mu(0+N_\epsilon)-\epsilon)/\tau} + (1)\sum_{N_\epsilon} e^{(\mu(1+N_\epsilon)-\epsilon)/\tau} + (2)\sum_{N_\epsilon} e^{(\mu(2+N_\epsilon)-\epsilon)/\tau}\right]\Big/\mathcal{Z}$$

$$= \left[e^{\mu/\tau} + 2e^{2\mu/\tau} + e^{\frac{2\mu-\epsilon}{\tau}} + 2e^{\frac{3\mu-\epsilon}{\tau}} + e^{\frac{3\mu-2\epsilon}{\tau}} + 2e^{\frac{4\mu-2\epsilon}{\tau}}\right]\Big/\mathcal{Z}$$

$$\boxed{\langle N_\epsilon \rangle = \left[e^{\frac{\mu-\epsilon}{\tau}} + e^{\frac{2\mu-\epsilon}{\tau}} + e^{\frac{3\mu-\epsilon}{\tau}} + 2\left(e^{\frac{2\mu-2\epsilon}{\tau}} + e^{\frac{3\mu-2\epsilon}{\tau}} + e^{\frac{4\mu-2\epsilon}{\tau}}\right)\right]\Big/\mathcal{Z}}$$

c) [3] Find the average energy $\langle \epsilon \rangle$ of the system.

$$\langle \epsilon \rangle = \left[0\left(1 + e^{\mu/\tau} + e^{2\mu/\tau}\right) + (\epsilon)\left(e^{\frac{\mu-\epsilon}{\tau}} + e^{\frac{2\mu-\epsilon}{\tau}} + e^{\frac{3\mu-\epsilon}{\tau}}\right) + (2\epsilon)\left(e^{\frac{2\mu-2\epsilon}{\tau}} + e^{\frac{3\mu-2\epsilon}{\tau}} + e^{\frac{4\mu-2\epsilon}{\tau}}\right)\right]\Big/\mathcal{Z}$$

$$= \frac{\epsilon\left[e^{\frac{\mu-\epsilon}{\tau}} + e^{\frac{2\mu-\epsilon}{\tau}} + e^{\frac{3\mu-\epsilon}{\tau}} + 2\left(e^{\frac{2\mu-2\epsilon}{\tau}} + e^{\frac{3\mu-2\epsilon}{\tau}} + e^{\frac{4\mu-2\epsilon}{\tau}}\right)\right]}{\mathcal{Z}}$$