Problem 1

(a): 2 pts

A coin is tossed 5 times. Write down all possible configurations with 3 heads. A useful notation is to represent each configuration by a row of 5 arrows, with heads and tails denoted by up and down arrows, respectively. What is the probability of getting 3 heads when tossing the coin 5 times?

We can see explicitly that there are 10 possible ways to have 3 heads in 5 coin tosses:

$$\begin{array}{ccccc} \uparrow\uparrow\uparrow\downarrow\downarrow & \uparrow\uparrow\downarrow\uparrow\downarrow & \uparrow\uparrow\downarrow\downarrow\uparrow & \uparrow\downarrow\uparrow\downarrow\uparrow & \uparrow\downarrow\downarrow\uparrow\uparrow \\ \uparrow\downarrow\uparrow\uparrow\downarrow & \downarrow\downarrow\uparrow\uparrow\uparrow & \downarrow\uparrow\downarrow\uparrow\uparrow & \downarrow\uparrow\uparrow\downarrow\uparrow & \downarrow\uparrow\uparrow\uparrow\downarrow \end{array}$$

When tossing the coin 5 times, there are $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32$ total possibilities. So the probability of getting 3 heads is

$$P = \frac{10}{32} = 0.3125.$$

(a): 2 pts

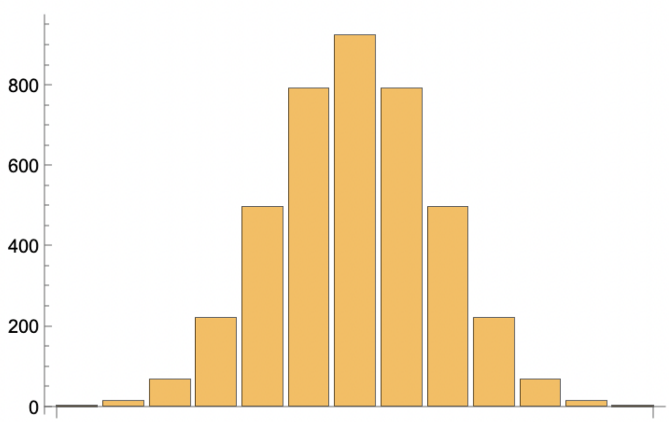
(b): 3 pts

A coin is tossed 12 times. Draw a histogram showing the number of states as a function of the number of heads. (See Figure 1.7 in the text as an example.) Use the binomial expansion coefficients to calculate the exact multiplicity of each state.

Shown below in curly brackets is a table of the multiplicities from 1 to 12 (e.g. 66 is the multiplicity of having 3 heads with 12 coin flips). Below that is a visual representation of the coefficients. I used Mathematica to produce this (more on Mathematica in the next problem).

```
BinomCoeff[N_] := Table[Binomial[N, k], {k, 0, N}]
BinomCoeff[12]
BarChart[BinomCoeff[12]]
```

```
{1, 12, 66, 220, 495, 792, 924, 792, 495, 220, 66, 12, 1}
```



(b): 3 pts

Problem 2

On pages 18-20 of your textbook, the author shows that the binomial distribution takes on a Gaussian shape for large N. You will use Mathematica to simulate the binomial distribution by generating random coin flips. If you prefer to write your own program using C++, Python, or some other programming language, that is fine, too. But Mathematica is a powerful tool that I urge you to learn both for this course and for your own benefit. Mathematica is available for free to MSU students for use on your own computers: go to the MSU Tech Store website, search "Mathematica", and click on the Student Software version.

With Mathematica running, just type in commands. Mathematica executes the commands when you hit SHIFT-ENTER. If you make a mistake you can go back to the command, edit it or re-type it, then hit SHIFT-ENTER again. Mathematica will erase the old result and execute the revised command. Use the HELP menu if you have trouble figuring out what these commands do.

(a): 2 pts

To get started, here are some commands you might try:

Random[]
Random[Integer]
Table[Random[Integer],{10}]

Here is a way to toss m coins, find the number of heads, repeat the whole trial N times, and then histogram the result. In the example below I used $m = 10$ and $N = 200$.

NumberOfHeads[m_]:=Sum[Random[Integer], {m}]
ManyTrials[m_,N_]:=Table[NumberOfHeads[m], {N}]
Histogram[ManyTrials[10, 200]]

Notice that when you define a function, the variable name on the left-hand-side of the definition is followed by the underscore, hence m_ instead of m.

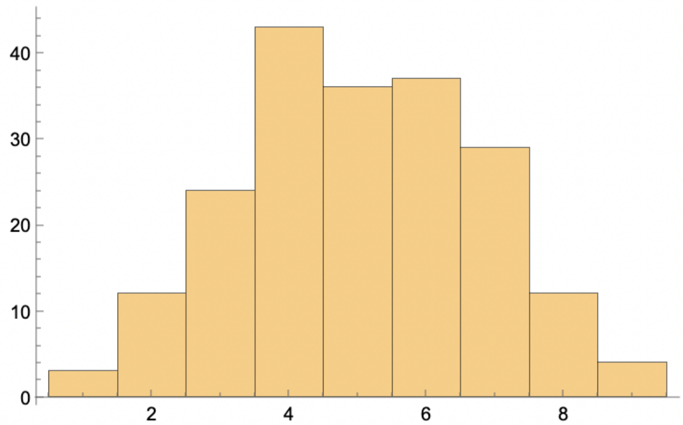
Here are the results of running the shown code:

```
In[51]:= Random[]
         Random[Integer]
         Table[Random[Integer], {10}]

Out[51]= 0.0367606

Out[52]= 1

Out[53]= {1, 1, 1, 1, 1, 0, 0, 1, 1, 0}
```

(a): 2 pts

```
In[7]:= NumberOfHeads[m_] := Sum[Random[Integer], {m}]
        ManyTrials[m_, N_] := Table[NumberOfHeads[m], {N}]
        Histogram[ManyTrials[10, 200]]
```

Out[9]=



(a): 2 pts

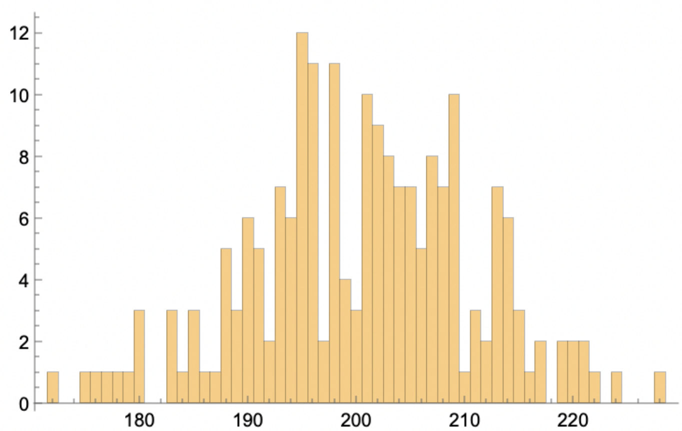
(b): 2 pts

Have Mathematica toss 400 coins, find the number of heads, and repeat the experiment 200 times. The resulting histogram will be very bumpy. In some versions of Mathematica the default bin width will be larger than 1. To set the bin width to 1, you can specify the number of bins this way:

Histogram[ManyTrials[400, 200], 60]

```
In[13]:= Histogram[ManyTrials[400, 200], 60]
```

Out[13]=



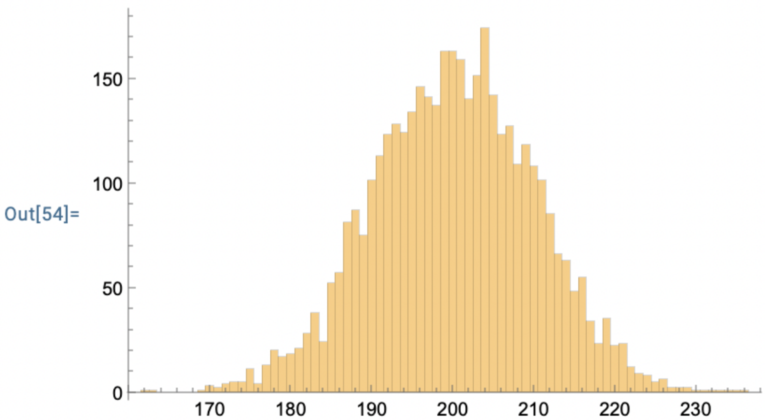
(b): 2 pts

(c): 2 pts

Now have Mathematica repeat the 400-coin toss experiment 4000 times. (This may take your computer a few seconds.) The resulting histogram should be pretty smooth. Give the histogram a name, such as **P1=Histogram ...** so you can show it again later (see below).

```
In[54]:= P1 = Histogram[ManyTrials[400, 4000], 60]
```

Out[54]=



(c): 2 pts

(d): 2 pts

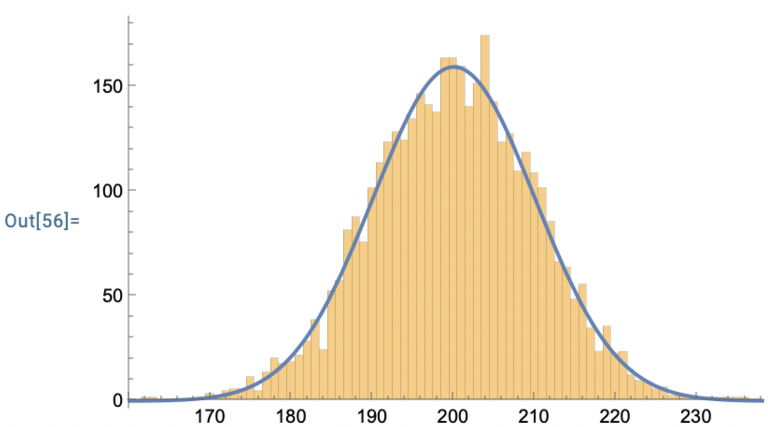
Now graph a Gaussian function and see how well it matches your histogram. Note that the normalized Gaussian must be multiplied by the number of trials (4000) to match the histogram, assuming your bin width = 1. I chose x-axis limits of 160 to 240, since the function is peaked at x=200.

P2=Plot[4000*Sqrt[2/(400*Pi)]*Exp[-2*(x-200)^2/400],{x,160,240}]
Show[P1,P2]

You are welcome to do more than I have outlined here. Your Mathematica program and output should both be included in your homework.

```
In[55]:= P2 = Plot[4000 * Sqrt[2 / (400 * Pi)] * Exp[-2 * (x - 200) ^2 / 400], {x, 160, 240}] ;
         Show[P1, P2]
```

Out[56]=



(d): 2 pts

(e): 2 pts

Define n to be the number of heads obtained from the N coin tosses. The mean value of n is called $\overline{n}$ or $\langle n \rangle$. The standard deviation of n is defined as $\delta n \equiv \sqrt{\langle (n - \langle n \rangle)^2 \rangle}$. What is δn for $N = 12$ and $N = 400$? (You may use results we derived in class.) What is the relative uncertainty in n, defined as $\delta n / \langle n \rangle$, for the two cases? Is your result for $N = 400$ consistent with the graphs you made in parts (c) and (d)? Discuss how likely (or unlikely) it is to obtain fewer than 30% heads in the two cases.

We have learned that for a binomial distribution, $\delta n = \sqrt{Np(1-p)}$. For an unbiased coin, $p = 1/2$, so that for the unbiased coin $\delta n = \sqrt{N/4}$. So,

$$\delta n = \begin{cases} 1.73, & N = 12 \\ 10.0, & N = 400. \end{cases}$$

We have also learned that for a binomial distribution, $\langle n \rangle = Np$, so that for the unbiased coin, $\langle n \rangle = N/2$, which gives $\delta n / \langle n \rangle = 1/\sqrt{N}$. So the relative uncertainty for these cases is

(e): 2 pts

$$\delta n / \langle n \rangle = \begin{cases} 0.289, & N = 12 \\ 0.0500, & N = 400. \end{cases}$$

The relative uncertainty is very small for the case of $N = 400$, and indeed, looking at our histograms, obtaining fewer than 30% heads (fewer than 120 heads for the case of $N = 400$) is extremely small. For $N = 12$, obtaining fewer than 30% heads means obtaining fewer than 4 heads; inspecting our histogram, this is relatively likely.

Problem 3

Consider a gas of N_0 noninteracting molecules enclosed in a container of volume V_0. Focus attention on a subvolume V of this container and denote by N the number of molecules located within V. Each molecule is equally likely to be located anywhere within V_0, hence the probability of finding a given molecule in V is $p = V/V_0$.

(a): 1 pt

What is the mean number $\langle N \rangle$ of molecules located within V? Express your answer in terms of N_0, V_0, and V. (Hint: This problem is just like the biased coin flip problem we discussed in class, where p is the probability to get "heads".)

In class, we showed that

$$\langle N \rangle = pN_0 = \frac{V}{V_0} N_0.$$

(a): 1 pt

(b): 1 pt

Find the relative dispersion $\langle (N - \langle N \rangle)^2 \rangle / \langle N \rangle^2$ in the number of molecules located within V. (This is the square of the quantity we discussed in class.) Express your answer in terms of $\langle N \rangle$, V and V_0.

In class, we showed that $\langle (\Delta N)^2 \rangle = N_0 p(1-p)$. So

$$\frac{\langle (\Delta N)^2 \rangle}{\langle N \rangle^2} = \frac{N_0 p(1-p)}{(N_0 p)^2} = \frac{1-p}{p} \frac{1}{N_0} = \left(1 - \frac{V}{V_0}\right) \frac{1}{\langle N \rangle}$$

(b): 1 pt

(c): 1 pt

What does the answer to part (b) become when $V \ll V_0$? (The answer is not zero.)

When $V \ll V_0$, $V/V_0 \to 0$, so

$$\frac{\langle (\Delta N)^2 \rangle}{\langle N \rangle^2} \to \frac{1}{\langle N \rangle}.$$

Note that $\langle N \rangle$ also tends to zero in this limit; this doesn't change the fact that the term we keep dominates the result.

(c): 1 pt

(d): 1 pt

What value should the dispersion $\langle (N - \langle N \rangle)^2 \rangle$ assume when $V \to V_0$? Does the answer in part (b) agree with this?

When $V \to V_0$, the number of particles N in V will *always* be equal to the total number of particles N_0, so there should be no flucuations: we expect that the dispersion $\langle (N - \langle N \rangle)^2 \rangle \to 0$. We can get the dispersion from part (b) by multiplying both sides by $\langle N \rangle^2$: $\langle (\Delta N)^2 \rangle = \left(1 - \frac{V}{V_0}\right) \langle N \rangle$.

(d): 1 pt

Problem 3 continued on next page. . .

(e): 1 pt

Consider the case $N_0 = 5$, and $V = V_0/4$. At any given time, what is the probability of finding all 5 molecules in V? What is the probability of finding exactly one molecule in V?

For this case, $p = V/V_0 = \frac{1}{4}$. The probability of finding all five molecules in V is thus

$$P_5 = p^5 = (1/4)^5 = 1/1024 = 9.8 \times 10^{-4}.$$

(e): 1 pt

The probability of finding exactly one molecule in V is

$$P_1 = 5p(1-p)^4 = 0.396.$$