Problem 1

Adapted from Kittel & Kroemer, Chapter 7, problem 1 [Density of orbitals in 1D and 2D.]

(a): 1 point

Find the density of states for a free electron in a one-dimensional box of length L.

In 1D, the energy levels are $\epsilon = \frac{\hbar^2\pi^2}{2mL^2}n^2$, with $n = 1, 2, 3, \ldots$. The total number of particles N is

$$N = \sum_{\text{orbitals}} f(\epsilon)$$
$$= 2\int_0^\infty f(\epsilon)dn.$$

From the energy level equation, we see $d\epsilon = \frac{\hbar^2\pi^2}{2mL^2}2n\,dn$. Also, $n = \sqrt{\frac{2mL^2}{\hbar^2\pi^2}\epsilon}$, so $dn = \frac{1}{2}\sqrt{\frac{2mL^2}{\hbar^2\pi^2}}\frac{1}{\sqrt{\epsilon}}\,d\epsilon$, so that

(a): 1 point

$$N = \int_0^\infty \sqrt{\frac{2mL^2}{\hbar^2\pi^2}}\frac{1}{\sqrt{\epsilon}}f(\epsilon)d\epsilon.$$

Comparing this with $N = \int_0^\infty \mathcal{D}(\epsilon)f(\epsilon)d\epsilon$, we see that

$$\mathcal{D}_1(\epsilon) = L\sqrt{\frac{2m}{\hbar^2\pi^2}}\frac{1}{\sqrt{\epsilon}}$$

(b): 1 point

Find the total ground-state energy of a gas of non-interacting electrons in 1D.

Using our answer in part (a), we can write

$$U_0 = \int_0^\infty \epsilon\mathcal{D}_1(\epsilon)f(\epsilon, \tau = 0)d\epsilon$$
$$= L\sqrt{\frac{2m}{\hbar^2\pi^2}}\int_0^{\epsilon_F}\sqrt{\epsilon}d\epsilon$$
$$= \frac{2L}{3}\sqrt{\frac{2m}{\hbar^2\pi^2}}(\epsilon_F)^{3/2}.$$

We need to figure out what ϵ_F is in 3d. We can get this by writing down the equation for N at $\tau = 0$:

(b): 1 point

$$N = \int_0^{\epsilon_F}\mathcal{D}_1(\epsilon)d\epsilon$$
$$= L\sqrt{\frac{2m}{\hbar^2\pi^2}}\int_0^{\epsilon_F}\frac{1}{\sqrt{\epsilon}}d\epsilon$$
$$= 2L\sqrt{\frac{2m}{\hbar^2\pi^2}}\sqrt{\epsilon_F}.$$

Solving for ϵ_F, we find

$$\epsilon_F = \frac{1}{4}\frac{\hbar^2\pi^2N^2}{2mL^2}$$

Thus

$$U_0 = \frac{2L}{3}\sqrt{\frac{2m}{\hbar^2\pi^2}}\left(\frac{1}{4}\frac{\hbar^2\pi^2N^2}{2mL^2}\right)^{3/2}$$
$$= \frac{L}{12}\frac{\hbar^2\pi^2}{2m}(N/L)^3$$
$$= \frac{1}{3}N\epsilon_F$$

(b): 1 point

(c): 1 point

Find the density of states for a free electron in a two-dimensional square of side length L.

Let's repeat the process from (a). In 2D, the energy levels are $\epsilon = \frac{\hbar^2\pi^2}{2mL^2}(n_x^2 + n_y^2)$, with $n_x, n_y = 1, 2, 3, \ldots$. The total number of particles N is

$$N = \sum_{\text{orbitals}} f(\epsilon)$$
$$= 2\int_0^\infty dn_x \int_0^\infty dn_y f(\epsilon_n)$$
$$= 2\frac{1}{4}(2\pi)\int_0^\infty dn\, n f(\epsilon_n),$$

where in the last line we switch to the variable $n = \sqrt{n_x^2 + n_y^2}$; the 2π comes from the angular integral and the factor of $1/2$ comes from the fact that we are only integrating over the positive quadrant. In terms of n, $\epsilon = \frac{\hbar^2\pi^2}{2mL^2}n^2$. Then, like before, $d\epsilon = \frac{\hbar^2\pi^2}{2mL^2}2n\,dn$, so that

(c): 1 point

$$N = \frac{mL^2}{\hbar^2\pi}\int_0^\infty f(\epsilon)d\epsilon.$$

Comparing this with $N = \int_0^\infty \mathcal{D}(\epsilon)f(\epsilon)d\epsilon$, we see that

$$\mathcal{D}_2(\epsilon) = \frac{mL^2}{\hbar^2\pi}.$$

This is independent of energy!

(d): 1 point

Find the total ground-state energy of a gas of non-interacting electrons in 2D.

Let's compute:

$$U_0 = \int_0^{\epsilon_F} \epsilon\mathcal{D}_2(\epsilon)d\epsilon$$
$$= \frac{mL^2}{\hbar^2\pi}\int_0^{\epsilon_F}\epsilon d\epsilon$$
$$= \frac{mL^2}{2\hbar^2\pi}\epsilon_F^2.$$

(d): 1 point

We should compute ϵ_F. We can do this by noting that

$$\begin{aligned}
N &= \int_0^{\epsilon_F} \mathcal{D}_2(\epsilon) d\epsilon \\
&= \frac{mL^2}{\hbar^2\pi} \int_0^{\epsilon_F} d\epsilon \\
&= \frac{mL^2}{\hbar^2\pi} \epsilon_F,
\end{aligned}$$

which gives

(d): 1 point

$$\epsilon_F = \frac{\hbar^2 \pi N}{mL^2}.$$

Then

$$\begin{aligned}
U_0 &= \frac{1}{2} \frac{\hbar^2 \pi N^2}{mL^2} \\
&= \frac{1}{2} N \epsilon_F
\end{aligned}$$

Problem 2

Kittel & Kroemer, Chapter 7, problem 2 [Energy of relativistic Fermi gas.]

For electrons with an energy $\epsilon \gg mc^2$, where m is the rest mass of the electron, the energy is given by $\epsilon \simeq pc$, where p is the momentum. For electrons in a cube of volume $V = L^3$ the momentum is of the form $(\pi\hbar/L)$, multiplied by $(n_x^2 + n_y^2 + n_z^2)^{1/2}$, exactly as for the nonrelativistic limit.

(a): 1 point

Show that in this extreme relativistic limit the Fermi energy of a gas of N electrons is given by

$$\epsilon_F = \hbar\pi c(3n/\pi)^{1/3},$$

where $n = N/V$.

If we assume that the energy of *all* particles takes the relativistic form, we can write

$$\begin{aligned} N &= \sum_{\text{orbitals}} f(\epsilon) \\ &= 2\frac{1}{8}(4\pi)\int_0^{n_F} n^2 dn \\ &= \frac{\pi}{3}n_F^3 \end{aligned}$$

Thus

$$n_F = \left(\frac{3N}{\pi}\right)^{1/3}.$$

(a): 1 point

The Fermi energy should be related to this quantity as $\epsilon_F = \pi\hbar c n_F/L$, so that

$$\begin{aligned} \epsilon_F &= \pi\hbar c\left(\frac{3N}{\pi}\right)^{1/3}/L \\ &= \pi\hbar c\left(\frac{3N}{L^3\pi}\right)^{1/3} \\ &= \pi\hbar c\left(\frac{3n}{\pi}\right)^{1/3} \end{aligned}$$

(b): 1 point

Show that the total energy of the ground state of the gas is

$$U_0 = \frac{3}{4} N \epsilon_F.$$

Above, we saw that

$$N = \pi \int_0^{n_F} n^2 dn.$$

Using $\epsilon = \pi\hbar cn/L$, we can convert this to an integral over energy, since $d\epsilon = \pi\hbar c dn/L$:

$$N = \frac{L^3}{\pi^2 \hbar^3 c^3} \int_0^{\epsilon_F} \epsilon^2 d\epsilon.$$

From this, we can see that the density of states is

$$\begin{aligned} \mathcal{D}(\epsilon) &= \frac{L^3}{\pi^2 \hbar^3 c^3} \epsilon^2 \\ &= 3N \frac{\epsilon^2}{\epsilon_F^3}. \end{aligned}$$

(b): 1 point

We can use this to calculate the ground-state energy:

$$\begin{aligned} U_0 &= \int_0^{\epsilon_F} \epsilon \mathcal{D}(\epsilon) d\epsilon \\ &= \frac{3N}{\epsilon_F^3} \int_0^{\epsilon_F} \epsilon^3 d\epsilon \\ &= \frac{3}{4} N \epsilon_F. \end{aligned}$$

Problem 3

Kittel & Kroemer, Chapter 7, problem 3 [Pressure and entropy of degenerate Fermi gas.]

(a): 1 point

Show that a Fermi electron gas in the ground state exerts a pressure

$$p = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{5/3}.$$

In a uniform decrease of the volume of a cube every orbital has its energy raised: The energy of an orbital is proportional to $1/L^2$ or to $1/V^{2/3}$.

We showed in the book that the ground state energy of a Fermi gas is

$$\begin{aligned} U_0 = \frac{3}{5}N\epsilon_F &= \frac{3}{5}N\frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \\ &= \frac{3(3\pi^2)^{2/3}}{10}N^{5/2}\frac{\hbar^2}{m}\left(\frac{1}{V}\right)^{2/3} \end{aligned}$$

We can calculate the pressure:

(a): 1 point

$$\begin{aligned} p &= -\frac{\partial U_0}{\partial V} \\ &= \frac{(3\pi^2)^{2/3}}{5}N^{5/2}\frac{\hbar^2}{m}\left(\frac{1}{V}\right)^{5/3} \\ &= \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m}\left(\frac{N}{V}\right)^{5/3} \end{aligned}$$

(b): 1 point

Find an expression for the entropy of a Fermi electron gas in the region $\tau \ll \epsilon_F$. Notice that $\sigma \to 0$ as $\tau \to 0$.

The entropy is related to the heat capacity as $C_V = \tau(\partial\sigma/\partial\tau)_V$. We showed in class that the low-temperature heat capacity for the Fermi gas is $C_V = N\tau/\tau_F$, where $\tau_F = \epsilon_F$. Then $C_V/\tau = N/\tau_F$, and we can integrate this to find

(b): 1 point

$$\sigma(\tau) = \int_0^\tau (C_V/\tau')d\tau' = N\tau/\tau_F.$$

Problem 4

Kittel & Kroemer, Chapter 7, problem 5 [Liquid ^{3}He as a Fermi gas]: 4 points

(a)

Since ^{3}He has spin-1/2, the form of ϵ_F will be the exact same as for electrons:

$$\epsilon_F = \frac{\hbar^2}{2M}\left(\frac{3\pi^2 N}{V}\right)^{2/3}.$$

The problem gives the mass density of the gas as $\rho = 0.081$ g/cm^3, so we should put this quantity in terms of mass density $\rho = MN/V$:

$$\epsilon_F = \frac{\hbar^2}{2M^{5/3}}\left(\frac{3\pi^2 MN}{V}\right)^{2/3} = \frac{\hbar^2}{2M^{5/3}}\left(3\pi^2\rho\right)^{2/3}.$$

The molar mass of ^{3}He is 3.02 g/mol, so that $M = (3.02)/(6.02 \times 10^{23}) = 0.5 \times 10^{-23}$ g. Plugging this all in gives (a)

$$\epsilon_F = 4.25 \times 10^{-4} \text{ eV}.$$

The velocity of particles that have this energy is found using $\epsilon_F = \frac{1}{2}Mv_F^2$, so that

$$v_F = 1.65 \times 10^4 \text{ cm/s}.$$

The Fermi energy corresponds to a temperature via $T_F = \epsilon_F/k_B$, which gives

$$4.93 \text{ K}.$$

This system needs to be quite cold to treat it as a non-interacting degenerate Fermi gas.

(b)

We saw that for the degenerate Fermi gas the heat capacity at low temperatures is given by (in conventional units) $C = \frac{1}{2}\pi^2 N k_B T/T_F$. Plugging in our $T_F = 4.93$ K, we see that (b)

$$C_V = 1.00 N k_B T,$$

in certain units. This is significantly lower than the given experimental value at low temperatures.

Problem 5

Kittel & Kroemer, Chapter 7, problem 11 [Fluctuations in a Fermi gas.]: 2 points

Show for a single orbital of a fermion system that

$$\langle(\Delta N)^2\rangle = \langle N\rangle(1 - \langle N\rangle),$$

if $\langle N\rangle$ is the average number of fermions in that orbital. Notice that the fluctuation vanishes for orbitals with energies deep enough below the Fermi energy so that $\langle N\rangle = 1$. By definition, $\Delta N \equiv N - \langle N\rangle$.

Let's first see what $1 - \langle N\rangle$ looks like for fermions so we recognize it when we see it:

$$\begin{aligned}
1 - \langle N\rangle &= 1 - \frac{1}{e^{(\epsilon-\mu)/\tau} + 1} \\
&= \frac{e^{(\epsilon-\mu)/\tau} + 1}{e^{(\epsilon-\mu)/\tau} + 1} - \frac{1}{e^{(\epsilon-\mu)/\tau} + 1} \\
&= \frac{e^{(\epsilon-\mu)/\tau}}{e^{(\epsilon-\mu)/\tau} + 1}.
\end{aligned}$$

We can use a result from Chapter 6 to compute the fluctuations:

$$\langle(\Delta N)^2\rangle = \tau\frac{\partial\langle N\rangle}{\partial\mu}.$$

For fermions, we know $\langle N\rangle = \frac{1}{e^{(\epsilon-\mu)/\tau}+1}$, so that

$$\begin{aligned}
\langle N^2\rangle &= \tau\frac{\partial}{\partial\mu}\frac{1}{e^{(\epsilon-\mu)/\tau} + 1} \\
&= \frac{e^{(\epsilon-\mu)/\tau}}{(e^{(\epsilon-\mu)/\tau} + 1)^2} \\
&= \langle N\rangle(1 - \langle N\rangle).
\end{aligned}$$

A more elegant way of approaching this problem recognizes that for fermions, there are only two options for N, $N = 0$ or $N = 1$, and for both of these values, $N^2 = N$, so $\langle N^2\rangle = \langle N\rangle$, and then our result follows from the definition of ΔN.

Problem 6

Adapted from: Kittel & Kroemer, Chapter 7, problem 13 [Chemical potential versus concentration.]: 2 points

Sketch carefully the chemical potential versus the number of particles for a Fermi gas in volume V at temperature τ. Include both classical and quantum regimes.

It's easiest to plot this as μ vs $N/n_Q V$. In the classical regime, we know that the chemical potential is given by $\mu = \tau \ln(n/n_Q) = \tau \ln((N/V)/n_Q)$, and that in that regime $n \ll n_Q$ so that $\mu < 0$. In the quantum regime, the chemical potential should approach the Fermi energy $(\hbar^2/2m)(3\pi^2(N/V))^{2/3}$, which is positive. The chemical potential must cross from negative to positive at some point, which likely occurs near $N/V \approx n_Q$, or $N/(n_Q V) \approx 1$. We can guess about the behavior in the intermediate regime: