Physics 410 Quiz #1 – Thursday, January 30, 2025

Name: Solutions

I. Probability and statistics, and other mathematical formulas:

mean value and variance: $\bar{X} \equiv \langle X \rangle = \sum_s X(s)P(s)$, $\langle(\Delta X)^2\rangle = \langle X^2 \rangle - \langle X \rangle^2$

where P(s) is a normalized probability distribution: $\sum_s P(s) = 1$

binomial distribution: $(p+q)^N = \sum_{n=0}^{N} \frac{N!}{n!(N-n)!} p^n q^{N-n}$, so $g(N,n) = \frac{N!}{(n)!(N-n)!}$

Stirling's approximation: $ln(n!) = \frac{1}{2} ln(2\pi n) + n \cdot ln(n) - n$

binomial multiplicity for large N: $g(N,s) = \frac{N!}{(\frac{N}{2}+s)!(\frac{N}{2}-s)!} \approx \left(\frac{2}{\pi N}\right)^{1/2} 2^N e^{-2s^2/N}$

II. Microcanonical ensemble: independent variables *U, V, N*: multiplicity function: *g(U,V,N)*

entropy: $\sigma(U,V,N) = \ln g(U,V,N)$ temperature: $\frac{1}{\tau} = \left(\frac{\partial \sigma}{\partial U}\right)_{V,N}$

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1. [4] Consider a biased coin, where the probability of heads is 1/3 and the probability of tails is 2/3. You flip the coin 10 times. (You don't need to simplify your answers.)

a) [1] What is the probability of obtaining 10 heads, 0 tails?

$$P(10) = g(10,10)\left(\tfrac{1}{3}\right)^{10} = \left(\tfrac{1}{3}\right)^{10}$$

b) [1] What is the probability of obtaining 0 heads, 10 tails?

$$P(0) = g(10,0)\left(\tfrac{2}{3}\right)^{10} = \left(\tfrac{2}{3}\right)^{10}$$

c) [2] What is the probability of obtaining 4 heads, 6 tails?

$$P(4) = g(10,4)\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^6 = \frac{10!}{6!4!}\left(\tfrac{1}{3}\right)^4\left(\tfrac{2}{3}\right)^6$$

2. [6] Consider two small systems of spins in a uniform magnetic field. System 1 has $N_1 = 8$ spins, while system 2 has $N_2 = 24$ spins. Initially, system 1 has all 8 spins up, and system 2 has all 24 spins down, and the systems are not in thermal contact.

a) [2] What is the entropy of the initial state of the combined system?

$\ln(1) = 0$

$$\sigma_0 = \ln g_0 = \ln\left((g_1)_0 (g_2)_0\right) = \ln\left(\frac{8!}{8!0!} \cdot \frac{24!}{0!24!}\right) = 0$$

b) [4] Now bring the two systems into thermal contact with each other. The most probable state is the one where six spins from system 1 flip from up to down, and six spins from system 2 flip from down to up. What is the entropy g of that state? (You do not need to simplify your result).

$$\sigma \cong \ln\left((g_1 g_2)_{max}\right) = \ln\left(\frac{8!}{2!\,6!} \cdot \frac{24!}{6!\,18!}\right)$$

3. [10] Now consider two larger systems of spins in a uniform magnetic field. System 1 has $N_1 = 1200$ spins, while system 2 has $N_2 = 3600$ spins. The energy of a state with spin excess s is $U = -2smB$, where m is the magnetic moment of a spin and B is the magnetic field. Initially, system 1 has half its spins up and half down, so its spin excess is $s_1 = 0$. Initially, system 2 has a spin excess $s_2 = 240$. <u>Show all your calculations for this problem, but you do not need to plug results into a calculator.</u>

a) [1] What is the initial energy of each system?

$$(U_1)_0 = -2s_1 mB = 0$$

$$(U_2)_0 = -2s_2 mB = -480\,mB$$

b) [3] From the homework, you know that the entropy of the system is given by: $\sigma(s) = ln(g(0)) - \frac{2s^2}{N}$. Find an expression for the temperature $\tau(U, N)$. Use your result to calculate the initial temperature of each system.

$$\sigma(U) = \ln g(0) - \frac{U^2}{2Nm^2B^2}$$

$$\frac{1}{\tau} = \left(\frac{\partial\sigma}{\partial U}\right)_{N,V} = \frac{-U}{Nm^2B^2}$$

$$\boxed{\tau = \frac{-Nm^2B^2}{U}}$$

$$(\tau_1)_0 = \frac{-N_1 m^2B^2}{(U_1)_0} = \infty$$

$$(\tau_2)_0 = \frac{-N_2 m^2B^2}{(U_2)_0} = \frac{3600}{480}\,mB$$

c) [3] Now bring the two spin systems into thermal contact with each other. In thermal equilibrium, what is the average spin excess for each system, $\hat{s}_1$ and $\hat{s}_2$? [Hint: use what you know about temperature in thermal equilibrium, along with your result from part (b).]

$$\tau_{sys} = \frac{-N_{tot}\cdot m^2B^2}{U_{sys}} = \frac{4800}{480}\,mB = 10\,mB \quad \because \text{ in equilibrium, } \tau_1 = \tau_2 = \tau_{sys} = 10\,mB$$

System 1: $10\,mB = \dfrac{1200\,m^2B^2}{2\hat{s}_1 mB} \longrightarrow \boxed{\hat{s}_1 = 60}$

System 2: $10\,mB = \dfrac{3600\,m^2B^2}{2\hat{s}_2 mB} \longrightarrow \boxed{\hat{s}_2 = 180}$

d) [3] By how much did the total entropy increase during the process of thermal equilibration? (Hints: You may use the relation $\sigma = \sigma_1 + \sigma_2$ both before and after the systems are in thermal equilibrium. You do <u>not</u> need to calculate any sums or integrals!)

$$\sigma_f - \sigma_i = \ln\left(\frac{\sqrt{\frac{2}{\pi N_1}}\sqrt{\frac{2}{\pi N_2}}\cdot 2^{N_1}\cdot 2^{N_2}\cdot e^{-2\hat{s}_1^2/N_1 - 2\hat{s}_2^2/N_2}}{\sqrt{\frac{2}{\pi N_1}}\sqrt{\frac{2}{\pi N_2}}\cdot 2^{N_1}\cdot 2^{N_2}\cdot e^{-2s_1^2/N_1 - 2s_2^2/N_2}}\right)$$

$$= \ln\left(e^{-\frac{2}{N_1}(\hat{s}_1^2 - s_1^2) - \frac{2}{N_2}(\hat{s}_2^2 - s_2^2)}\right) = \frac{-2}{N_1}(60^2 - 0^2) - \frac{2}{N_2}(180^2 - 240^2)$$

$$= \boxed{8}$$