Problem 1

Kittel & Kroemer, Chapter 3, problem 7 [Zipper problem]

A zipper has N links; each link has a state in which it is closed with energy 0 and a state in which it is open with energy ϵ. We require, however, that the zipper can only unzip from the left end, and that the link numbers can only open if all links to the left $(1, 2, \ldots, s-1)$ are already open.

(a): 1 point

Show that the partition function can be summed in the form

$$Z = \frac{1 - \exp[-(N+1)\epsilon/\tau]}{1 - \exp(-\epsilon/\tau)}$$

(b): 1 point

In the limit $\epsilon \gg \tau$, find the average number of open links. The model is a very simplified model of the unwinding of two-stranded DNA molecules.

[Hint: It is easier to work with the inverse temperature $\beta \equiv 1/\tau$.]

[Note: your answer to this problem shouldn't simply be 0. That result is obvious! We are interested in the *limiting behavior* of this quantity at low temperatures: is it polynomial? exponential? etc]

Problem 2

Kittel & Kroemer, Chapter 3, problem 10 [Elasticity of polymers]

The thermodynamic identity for a one-dimensional system is

$$\tau d\sigma = dU - f dl$$

when f is the external force exerted on the line and dl is the extension of the line. By analogy with (32) we form the derivative to find

$$-\frac{f}{\tau} = \left(\frac{\partial \sigma}{\partial l}\right)_U .$$

The direction of the force is opposite to the conventional direction of the pressure.

We consider a polymeric chain of N links each of length ρ, with each link equally likely to be directed to the right and to the left.

(a): 1 point

Show that the number of arrangements that give a head-to-tail length of $l = 2|s|\rho$ is

$$g(N, -s) + g(N, s) = \frac{2N!}{(\frac{1}{2}N + s)!(\frac{1}{2}N - s)!}$$

Problem 2 continued on next page. . .

(b): 1 point

For $|s| \ll N$ show that

$$\sigma(l) = \ln[2g(N,0)] - l^2/2n\rho^2.$$

(c): 1 point

Show that the force at extension l is

$$f = l\tau/N\rho^2.$$

The force is proportional to the temperature. The force arises because the polymer wants to curl up: the entropy is higher in a random coil than in an uncoiled configuration. Warming a rubber band makes it contract; warming a steel wire makes it expand.

Problem 3

Kittel & Kroemer, Chapter 4, problem 1 [Number of thermal photons]: 2 points

Show that the number of photons $\sum\langle s_n\rangle$ in equilibrium at temperature τ in a cavity of volume V is

$$N = 2.40\pi^{-2}V(\tau/\hbar c)^3.$$

From (23) the entropy is $\sigma = (4\pi^2 V/45)(\tau/\hbar c)^3$, whence $\sigma/N \sim 3.602$. It is believed that the total number of photons in the universe is 10^8 larger than the total number of nucleons (protons, neutrons). Because both entropies are of the order of the respective number of particles (see Eq. 3.76), the photons provide the dominant contribution to the entropy of the universe, although the particles dominate the total energy. We believe that the entropy of the photons is essentially constant, so that the entropy of the universe is approximately constant with time.

Problem 4

Kittel & Kroemer, Chapter 4, problem 7 [Free energy of a photon gas]: 3 points

(a): 1 point

Show that the partition function of a photon gas is given by

$$Z = \prod_n [1 - e^{-\hbar\omega_n/\tau}]^{-1},$$

where the product is over the modes n.

Problem 4 continued on next page. . .

(b): 1 point

The Helmholtz free energy is found directly from your result from part (a) as

$$F = \tau \sum_n \ln[1 - e^{-\hbar\omega_n/\tau}].$$

Transform the sum to an integral; integrate by parts to find

$$F = -\pi^2 V \tau^4 / 45\hbar^3 c^3.$$