Physics 410 Quiz #2 – Thursday, February 20, 2025

Name: Solutions

1. [8] Consider a system with infinite evenly spaced energy levels $\epsilon_s = s\hbar\omega$, labeled by index $s = 0,1,2, \ldots$, with $\omega > 0$. The system is in thermal equilibrium with a reservoir at temperature τ.

a) [4] Write down the partition function Z for the system in terms of ω and τ. *Do not leave your answer as a sum. If you find yourself confronted with an infinite sum, attempt to evaluate it.*

$$Z = \sum_s e^{-\epsilon_s/\tau} = \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau} = 1 + \sum_{s=1}^{\infty} e^{-s\hbar\omega/\tau}$$

$$= 1 + e^{-\hbar\omega/\tau} \sum_{s=0}^{\infty} e^{-s\hbar\omega/\tau}$$

$$Z = 1 + e^{-\hbar\omega/\tau} \cdot Z$$

$$\longrightarrow \boxed{Z = \frac{1}{1 - e^{-\hbar\omega/\tau}}}$$

b) [4] Write an expression for the *relative* probability of finding the system in the $s = 1$ state compared to the $s = 0$ state, i.e. $P_{s=1}/P_{s=0}$. Then, evaluate this result in the following two cases. (Give a numerical answer for each case.)

$$P_{s=1}/P_{s=0} = \frac{e^{-\hbar\omega/\tau}}{e^{0}} = e^{-\hbar\omega/\tau}$$

i) $\tau = 0$:

$$e^{-\infty} \to 0 \text{, so } P_{s=1}/P_{s=0} \to 0$$

ii) $\tau \gg \hbar\omega$ (think of the limit $\tau \to \infty$):

$$e^{0} \to 1 \text{, so } \frac{P_{s=1}}{P_{s=0}} = 1 \quad \text{: equally likely!}$$

2. [8] Consider a system with two states, one of energy ϵ and one of energy $-\epsilon$, in contact with a reservoir at temperature τ.

a) [4] Find an expression for the Helmholtz free energy F as a function of τ. Your answer must *only* include the symbols ϵ and τ.

$$F = -\tau \ln Z = -\tau \ln\left(e^{\varepsilon/\tau} + e^{-\varepsilon/\tau}\right)$$

b) [4] From your result from part (a), find an expression for the entropy as a function of τ. Your answer must *only* include the symbols ϵ and τ.

$$\sigma = -\left(\frac{\partial F}{\partial \tau}\right) = \ln\left(e^{\varepsilon/\tau} + e^{-\varepsilon/\tau}\right) + \tau \frac{\partial}{\partial \tau} \ln\left(e^{\varepsilon/\tau} + e^{-\varepsilon/\tau}\right)$$

$$= \ln\left(e^{\varepsilon/\tau} + e^{-\varepsilon/\tau}\right) + \tau \cdot \frac{1}{e^{\varepsilon/\tau} + e^{-\varepsilon/\tau}} \cdot \left(-\frac{\varepsilon}{\tau^2} e^{\varepsilon/\tau} + \frac{\varepsilon}{\tau^2} e^{-\varepsilon/\tau}\right)$$

$$= \ln\left(e^{\varepsilon/\tau} + e^{-\varepsilon/\tau}\right) + \frac{\varepsilon}{\tau} \cdot \frac{e^{-\varepsilon/\tau} - e^{\varepsilon/\tau}}{e^{-\varepsilon/\tau} + e^{\varepsilon/\tau}}$$

3. [4]

a) [2] True or false: for an isolated system S, the entropy of S tends to increase or remain constant over time. Explain your reasoning.

This is true. Just because the system is isolated, does not mean nothing is happening! The system continuously samples different possible microstates, as long as the total energy remains constant. It tends to find itself in the microstates with the highest multiplicity g, simply because there are more of those states. (e.g. gas particles evenly spaced across the room, rather than bunched up on one side). Thus since entropy is ln(g), and g tends not to decrease, entropy behaves the same way. This is the Second Law of Thermodynamics.
Note that it is not impossible for entropy to briefly decrease by the system momentarily entering a state of lower multiplicity; it is simply exceedingly unlikely that this will continue to happen over extended periods of time.

b) [2] True or false: for a system S in thermal equilibrium with a reservoir R at temperature τ, the entropy of S tends to increase or remain constant over time. Explain your reasoning.

False. In thermal contact, energy can be exchanged with the reservoir, which may increase <u>or</u> decrease the entropy of S. It is the free energy F that is minimized in this case.

Note: the problem statement was intended to read "thermal contact" rather than "thermal equilibrium". Responses will be graded loosely depending on students' interpretation.