Announcements

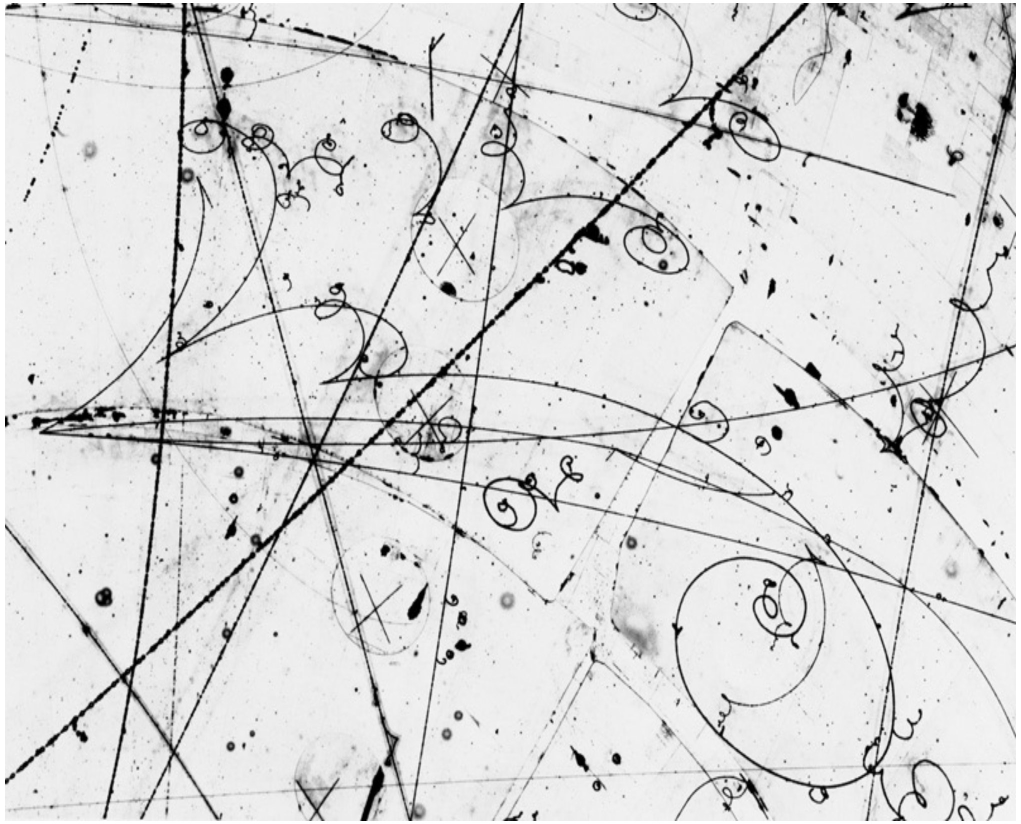
Homework:

Second assignment has been posted on D2L
To be submitted on gradescope, like HW1

Quizzes:

Graded quizzes if not picked up last time
Note: Grades on D2L do NOT include scaling mentioned in the syllabus; will be adjusted in final grades

Next quiz on Friday



Reminder from last time

Relativity has three main rules:

<u>Length Contraction</u> $L' = L_0/\gamma$ "Moving rulers are short"

<u>Time Dilation</u> $t' = \gamma\, t_0$ "Moving clocks are slow"

<u>Velocity Addition</u> $v_{AB} = (v_A + v_B)/(1 + v_A v_B/c^2)$

-> note, denominator is close to 1 if velocities are much smaller than c

$$\beta = \frac{v}{c}$$

$$\gamma^2 = \frac{1}{1-\beta^2}$$

Remember:

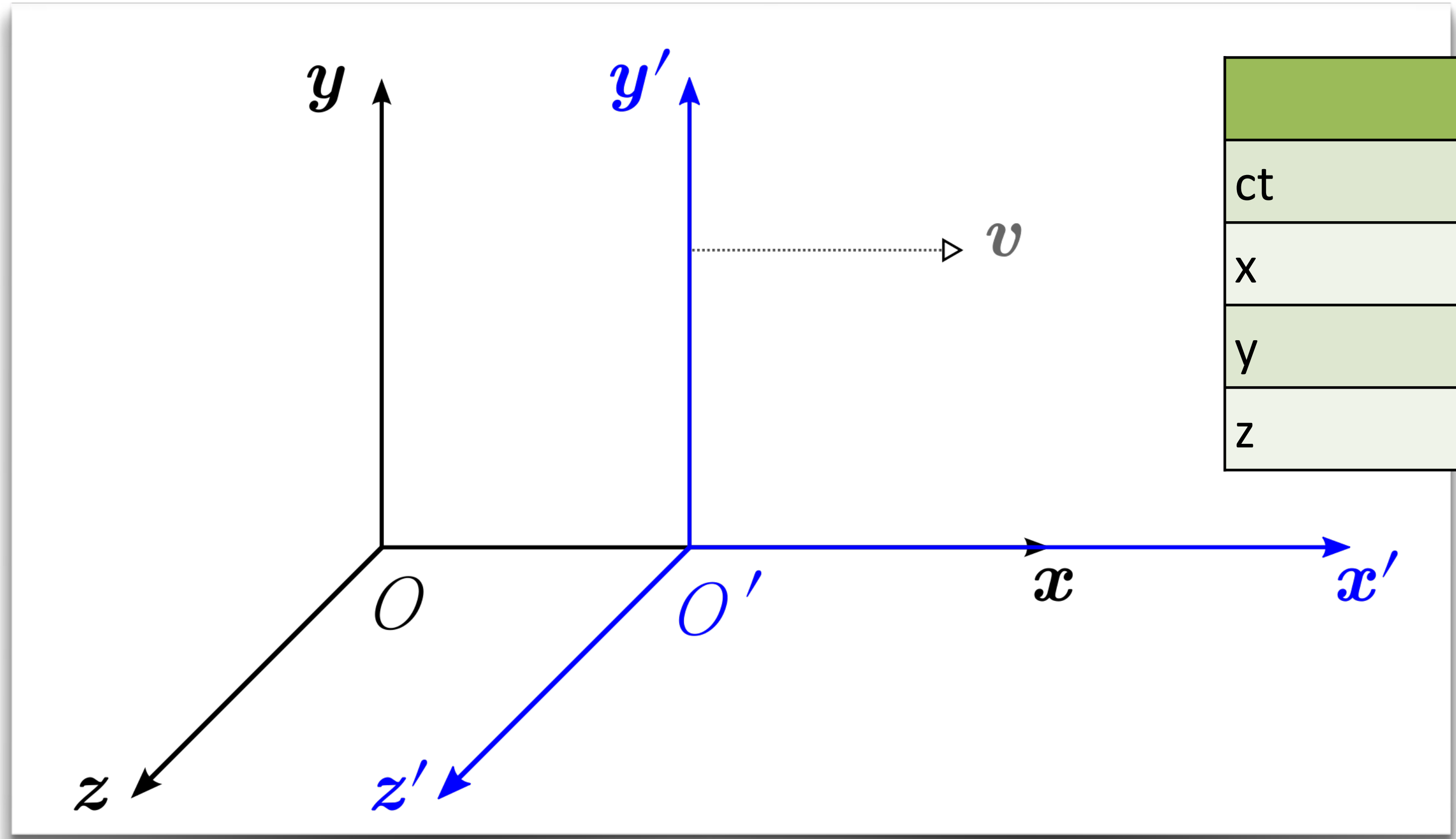
$\beta \leq 1$

$\gamma \geq 1$

Inertial Frame

<u>Inertial frame:</u> Any system that satisfies Newton's 1st law

Conservation of momentum implies that physics must be the same in all inertial frames



	Non-Relativistic	Relativistic
ct	ct	$\gamma(ct - \beta x)$
x	x	$\gamma(x - \beta ct)$
y	y	y
z	z	z

*For a velocity along the x-axis

4-Vectors

We can write these equations as:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

4-Vector

Lorentz transformation Matrix
In this example for a boost along the x axis.

4-Vector

Lorentz Transformation

We can write these equations as:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

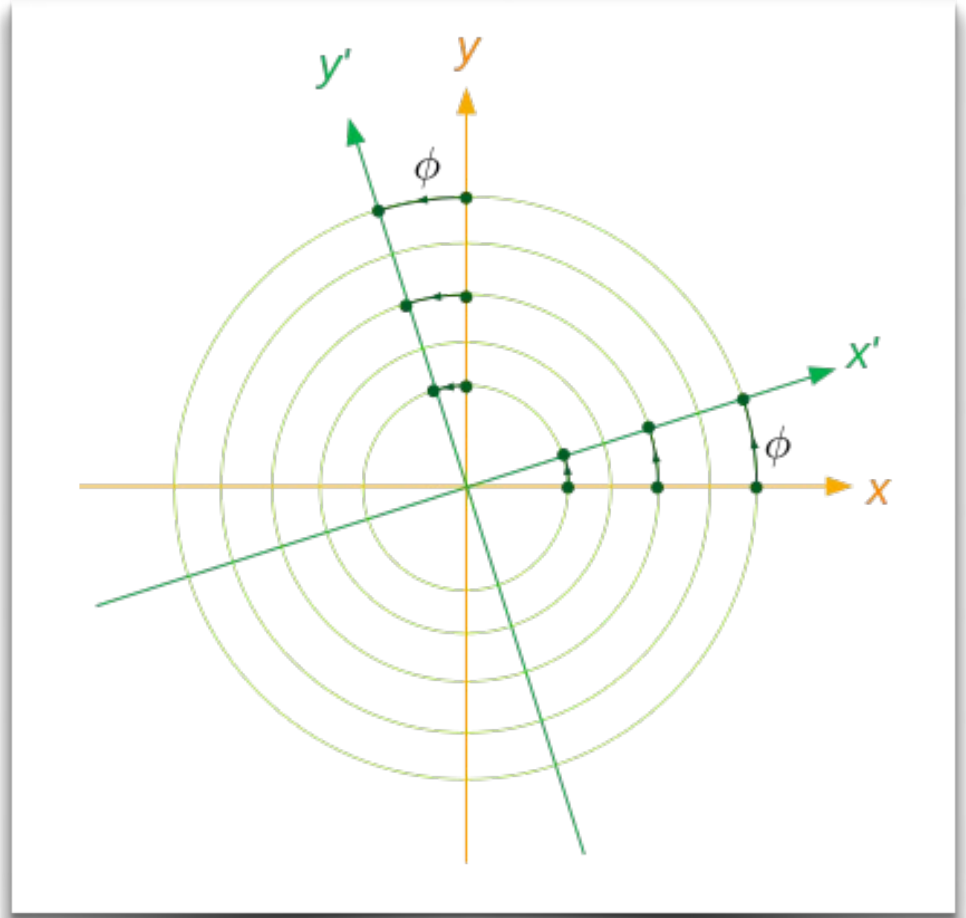
Or, more compactly:

$$x^{\mu'} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu} \qquad (\mu = 0, 1, 2, 3)$$

Coordinate Rotations

<u>Euclidean Rotations:</u> First, the ones you're used to!

Rotations of 3-D coordinates reveal an invariant quantity!



A given point has a constant displacement from the origin as the coordinate system rotates:

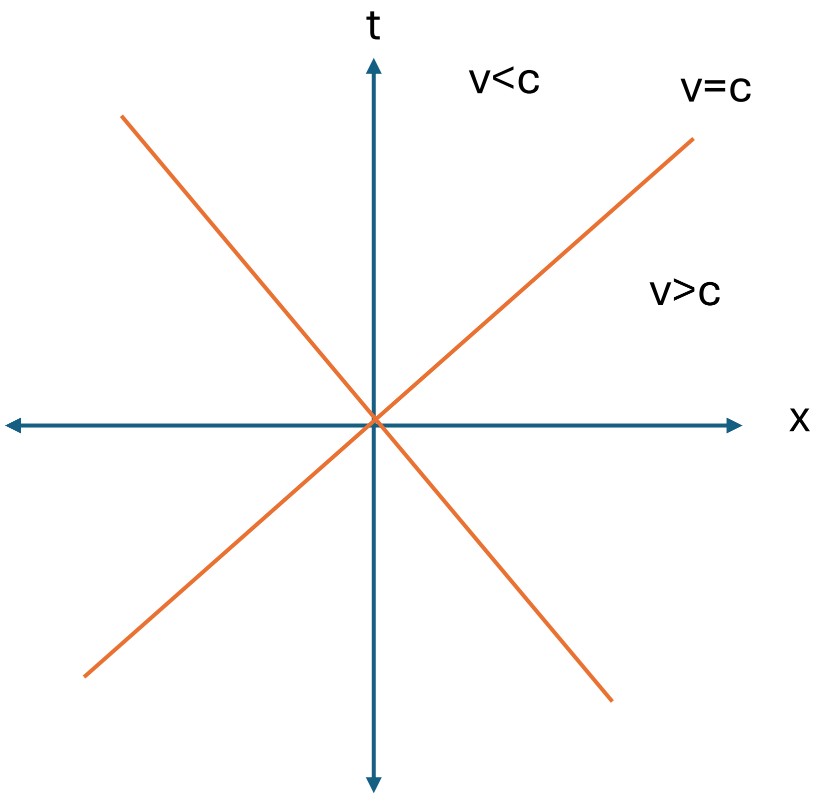
$$r^2 = x^2 + y^2$$

Space-time Coordinate System

Lorentz Rotations:

Now let's consider space-time, not just space as the fabric of our motion

Rotations of 4-D coordinates also reveal an invariant quantity!



x=vt

A given point transforms according to a Lorentz-invariant (ie, constant) value.

$$s^2 = (ct)^2 - |\vec{x}|^2$$

Coordinate System Rotations

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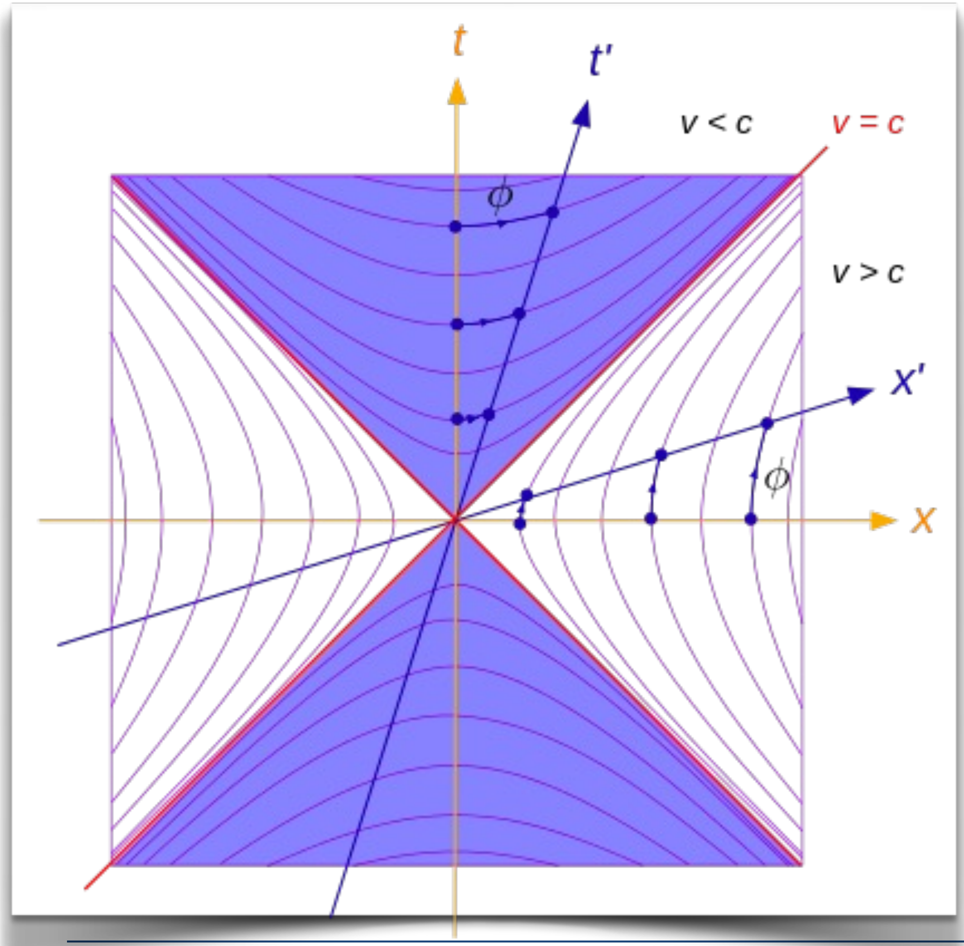

x=vt

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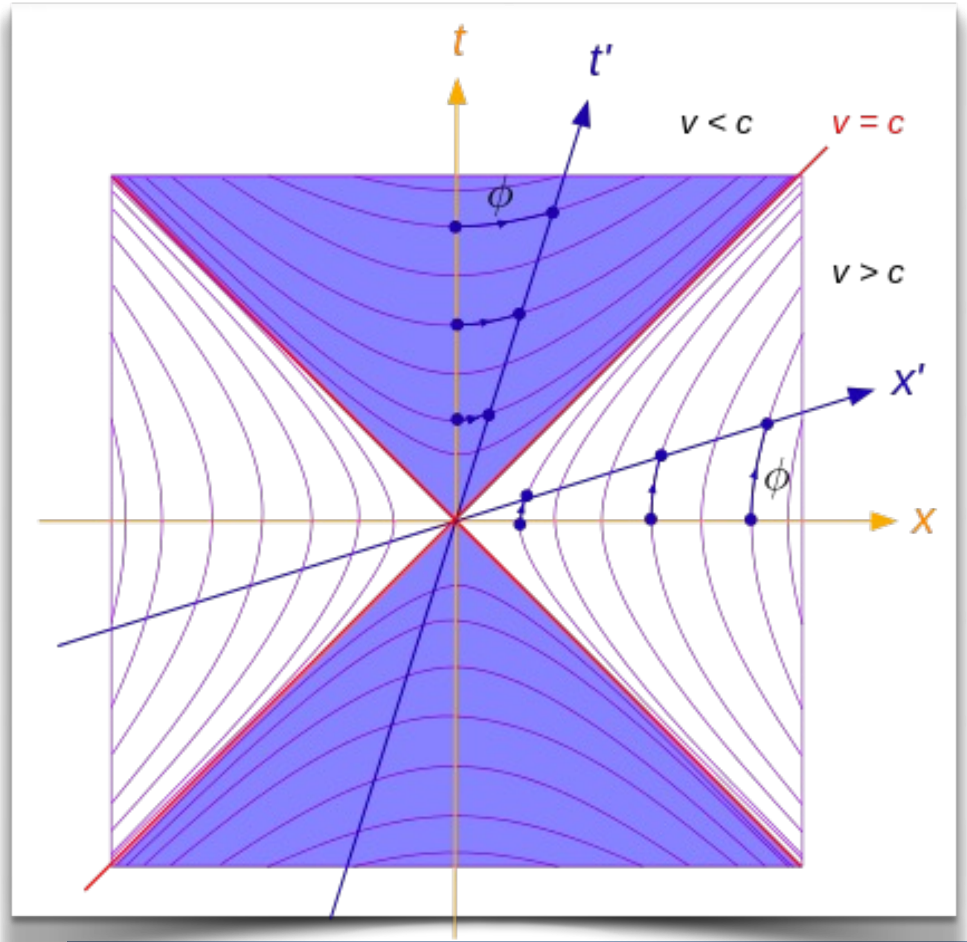
Definitions

$$s^2 = (ct)^2 - |\vec{x}|^2$$



$s^2 = 0$:

Spatial displacement is equal to the distance light travels in the time period.

Call this vector “light-like”.

$s^2 < 0$:

Vector describes 2 events that occurred at different spatial locations. They cannot be causally connected.

Call this vector “space-like”.

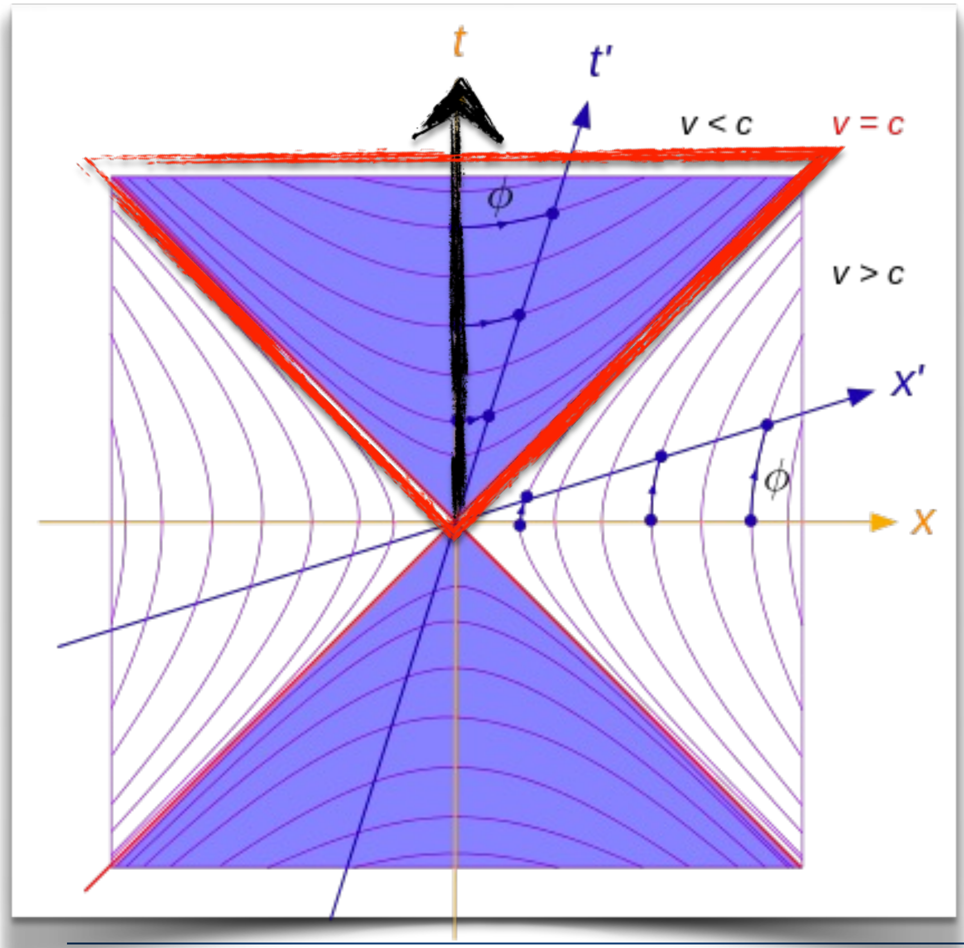
$s^2 > 0$:

Vector describes 2 events that occurred at different times. They can be causally connected.

Call this vector “time-like”.

Definitions

$$s^2 = (ct)^2 - |\vec{x}|^2$$



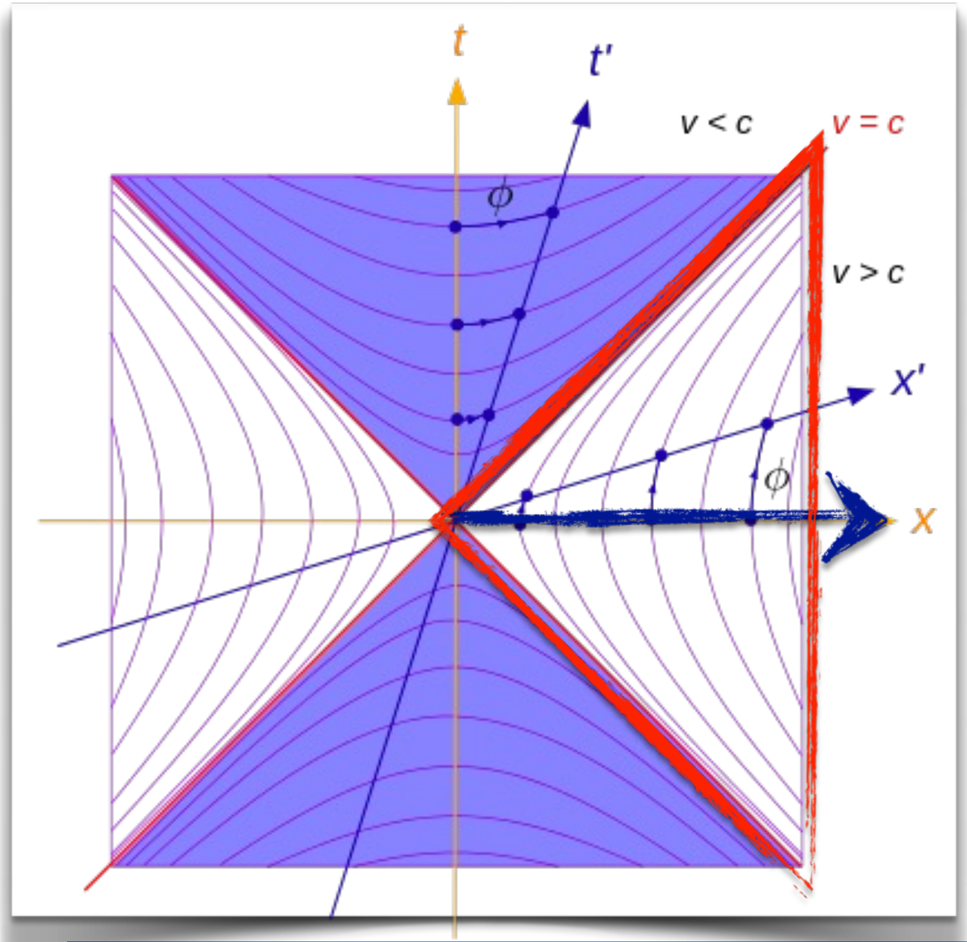
$s^2 > 0$:

Vector describes 2 events that occurred at different times. They can be causally connected.
Call this vector “time-like”.

There exists a transformation that allows the events along this vector to occur AT THE SAME PLACE and at different times.
Events occur in each other’s future/past.

Definitions

$$\boxed{s^2 = (ct)^2 - |\vec{x}|^2}$$



$s^2 < 0$:

Vector describes 2 events that occurred at different spatial locations. They cannot be causally connected. Call this vector “space-like”.

There exists a transformation that allows the events along this vector to occur AT THE SAME TIME and at different places. Events would then happen simultaneously.

Lorentz Transformation

We can write these equations as:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Or, more compactly:

$$x^{\mu'} = \sum_{\nu=0}^{3} \Lambda^{\mu}_{\nu} x^{\nu} \qquad (\mu = 0, 1, 2, 3)$$

Basis Transformations

- There are many ways to define basis vectors for n-D space.
- Transformations between different basis vectors are useful.
- The basis transformation can be represented by a transformation matrix.
- Here, essentially changing to new coordinate system

Basis Transformation

In new basis:

$$\rho = \Lambda \cdot \beta$$

Return where you started:

$$\beta = \Lambda^{-1} \cdot \rho$$

Rotation Matrix

$$\Lambda = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

counterclockwise

$$\Lambda^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

clockwise

Summation Notation

We typically use “Einstein Notation” to simplify the implied summation over indices:

$$A^\mu = \sum_{\nu=0}^{4} g^{\mu\nu} A_\nu$$

$$A_\mu = \sum_{\nu=0}^{4} g_{\mu\nu} A^\nu$$

Vector transformation using summation formalism

$$A^\mu = g^{\mu\nu} A_\nu$$

$$A_\mu = g_{\mu\nu} A^\nu$$

Vector transformation using Einstein Notation

Summation Notation

We typically use "Einstein Notation" to simplify the implied summation over indices:

	Quantity	Summation Formulation	Einstein Notation
Vector Dot Product	$c = \boldsymbol{a} \cdot \boldsymbol{b}$	$c = \sum_{i=1}^{3} a_i b_i$	$c = a_i b^i$
Vector Cross Product	$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$	$c_i = \sum_{j=1}^{3} \sum_{k=1}^{3} \epsilon_{ijk} a_j b_k$	$c^i = \epsilon^i{}_{jk} a^j b^k$
Matrix Multiplication	$\boldsymbol{C} = \boldsymbol{AB}$	$C_{ik} = \sum_{j=1}^{3} A_{ij} B_{jk}$	$C^i{}_k = A^i{}_j B^j{}_k$
Trace	$a = Tr(\boldsymbol{A})$	$a = \sum_{i=1}^{3} A_{ii}$	$a = A^i{}_i$

Co-/Contravariant Vectors

Vector: also called contravariant vector. These are the time/distance 4-vectors we have dealt with so far (contra-variant: if axis is scaled up, vector coordinates are scaled down).
Einstein Notation: upper indices

Contravariant Vectors:

$$\pi = \alpha \cdot \Lambda^{-1}$$

$$x^{\mu}$$

Covariant vector: a vector whose coordinates scale with the axes, for example where components are 1/distance.
Einstein Notation: lower indices

Covariant Vectors:

$$\rho = \Lambda \cdot \beta$$

$$x_{\nu}$$

Metrics

Metric: rule for measuring vector magnitudes, depends on the nature of the space-time fabric. Just a set of basis vectors!

$g_{\mu\nu}$ is the metric

The Euclidean Metric: Used to measure distances in flat, 3-D space.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Minkowski Metric: Used to measure distances in flat, 4-D space-time.

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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The Minkowski Metric: Used to measure distances in flat, 4-D space-time.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Both sign conventions are ok!

Metric used in particle physics

Particle physicists prefer the (+ - - -) metric because it gives us the spacetime interval

$$s^2 = (ct)^2 - x^2$$

It also gives an energy-momentum relationship with positive mass

$$m^2 = E^2 - p^2$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Special relativists prefer the Minkowski metric (- + + +) because it gives a + sign to space coordinates, as in Euclidian geometry

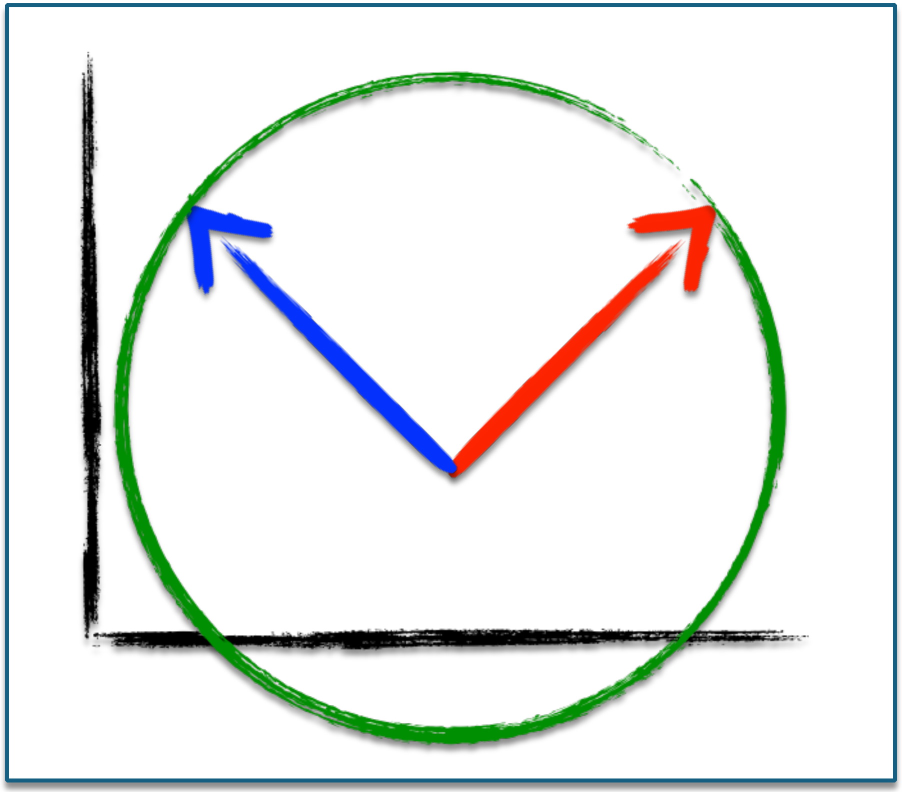
$$\eta = \pm \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Both are valid, just be consistent in using one or the other

Invariant Quantities

The inner product of a vector is a quantity that has the same magnitude regardless of rotations or displacements.

- The inner product, or vector magnitude, is invariant under all transformations that preserves the metric.



Inner Products

Inner products can be succinctly written using Einstein notation:

We write the Euclidean metric in matrix form, and it's equal to its inverse:

$$E^{\mu\nu} = E_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Vectors and co-vectors are written in the following way:

$$x^{\nu} = (1\ 2\ 3) \qquad x_{\nu} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

The inner product: $\vec{x} \cdot \vec{x} = E_{\mu\nu} x^{\nu} x^{\mu} = x_{\mu} x^{\mu} = x^{\mu} x_{\mu}$

Example:

$$E_{\mu\nu} x^{\nu} x^{\mu} = (1\ 2\ 3) \begin{pmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 1 \end{pmatrix} (1\ 2\ 3) = 1^2 + 2^2 + 3^2$$

Example: Inner product

$g_{\mu\nu}$ is the metric:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Inner product of a vector with itself in Einstein notation:

$$s^2 = x^\mu x_\mu = \sum_{\mu=0}^{3} x^\mu x_\mu = \sum_{\mu=0}^{3} x^\mu \sum_{\nu=0}^{3} g_{\mu\nu} x^\nu$$

And remembering that $g_{\mu\nu}$ is only non-zero along the diagonal,

$$s^2 = x^\mu x_\mu = \sum_{\mu=0}^{3} x^\mu g_{\mu\mu} x^\mu$$

Which is in components:

$$s^2 = (x^0)^2 - (x^1)^2 - (x^2)^2 - (x^3)^2$$

Or:

$$s^2 = (ct)^2 - |\vec{x}|^2$$

Contravariant Vectors

$$x^\mu = \left(x^0 \; x^1 \; x^2 \; x^3\right)$$

Covariant Vectors

$$x_\mu = \eta_{\mu\nu} x^\nu = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x^0 \\ -x^1 \\ -x^2 \\ -x^3 \end{pmatrix}$$

From Velocity to Momentum

$$\beta = \frac{v}{c}$$

$$\gamma^2 = \frac{1}{1-\beta^2}$$

The velocity of an inertial reference frame defines the transformation, so it should be a reversible measurement:

(Can boost into frame of moving particle)

$$v = dx/dt$$

$$v' = dx'/dt'$$

It's sometimes convenient to cast in terms of the "proper time" (τ), which is an expression of the time elapsed in the moving particle frame:

$$\begin{aligned} d\tau &= dx/dt' \\ &= dt/\gamma \end{aligned}$$

This yields a quantity some call "proper velocity", or just the 4-velocity:

$$\nu = dx/d\tau = \gamma v$$

4-Velocity & 4-Momentum

$$\beta = \frac{v}{c}$$

$$\gamma^2 = \frac{1}{1-\beta^2}$$

The 4-velocity is defined based on "gamma":

$$\gamma v^\mu = \gamma \frac{d}{dt} x^\mu = \gamma \begin{pmatrix} c \\ dx/dt \\ dy/dt \\ dz/dt \end{pmatrix}$$

And then can be extended to momentum:

$$\gamma m v^\mu = \gamma m \frac{d}{dt} x^\mu = \gamma m \begin{pmatrix} c \\ v_x \\ v_y \\ v_z \end{pmatrix} = \gamma \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Energy-Momentum 4-Vector

$$\begin{pmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Invariant Quantities

The inner product of a vector is a quantity that has the same magnitude regardless of rotations or displacements.

- The inner product, or vector magnitude, is invariant under all transformations that preserves the metric.

$$x_\mu x^\mu = \rho^2 = c^2t^2 - x^2 - y^2 - z^2$$

$$p_\mu p^\mu = s^2 = E^2/c^2 - p_x^2 - p_y^2 - p_z^2$$

These invariant quantities have the same value in EVERY inertial frame.

- This tool will be one of the most important in your analysis tool kit!

Invariant Quantities

The energy-momentum invariant quantity is very special!

- It's easy to show that the invariant quantity is the square of the mass
- This is a statement that a particle's mass is the same in every inertial frame.

$$E = \gamma m c^2$$
$$|p| = \gamma \beta m c$$

$$E^2/c^2 = \gamma^2 m^2 c^2$$
$$|p|^2 = \gamma^2 \beta^2 m^2 c^2$$

$$\begin{aligned} E^2/c^2 - |p|^2 &= \gamma^2 m^2 c^2 (1 - \beta^2) \\ &= \gamma^2 (1/\gamma^2) m^2 c^2 = m^2 c^2 \end{aligned}$$

Invariant Quantities

Be careful with units!

- The invariant quantity is invariant regardless of your choice of units
- But always compare apples to apples!

Momentum Units $p_\mu p^\mu = \mathbf{s^2} = E^2/c^2 - p_x^2 - p_y^2 - p_z^2$

Energy Units $p_\mu p^\mu = \mathbf{s_E^2} = E^2 - (p_x^2 - p_y^2 - p_z^2)c^2$

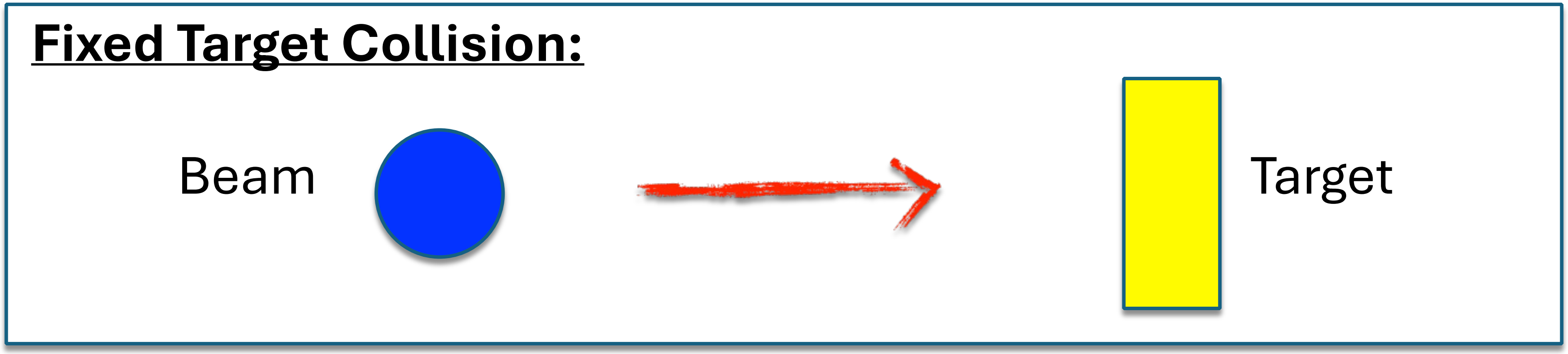
Natural Units $p_\mu p^\mu = \mathbf{s_N^2} = E^2 - p_x^2 - p_y^2 - p_z^2$

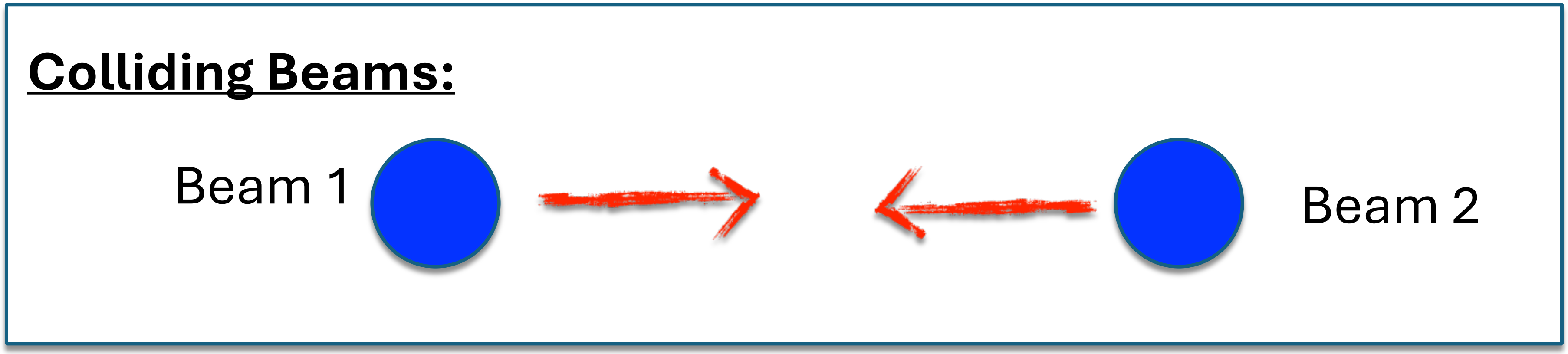
We use energy units of eV to easily reconcile natural units (h=c=1)

- Momentum: eV/c
- Mass: eV/c^2

Collisions

Particle physics frequently relies on high-energy collisions of particles to learn about nature.





Kinematics of Collisions

Some basic rules about relativistic collisions:

$$E = mc^2$$

- Total energy is always conserved

Einstein has already showed us how to reconcile the issue of energy & mass conservation

- Momentum is conserved
- Kinetic energy might not be conserved *This is also true of classical collisions*
- Mass may not be conserved *This never happens in classical collisions*