Problem 1

Kittel & Kroemer, Chapter 8, problem 1 [Heat Pump.]

(a): 1 point

Show that for a reversible heat pump the energy required per unit of heat delivered inside the building is given by the Carnot efficiency (6):

$$\frac{W}{Q_h} = \eta_C = \frac{\tau_h - \tau_l}{\tau_h}.$$

What happens if the heat pump is not reversible?

For a heat pump, work W is expended to pull heat from the cold reservoir (Q_l) and dump it into the hot reservoir (Q_h). This is the same concept as the operation of a refrigerator. For these processes, the book derives the equations $W = Q_h - Q_l$ and $Q_h = (\tau_h/\tau_l)Q_l$. Plugging the second into the first, we see that

$$W = Q_h(1 - \tau_l/\tau_h),$$

or

$$\frac{W}{Q_h} = \frac{\tau_h - \tau_l}{\tau_h}.$$

This is identical to the Carnot efficiency. If the heat pump is not reversible, entropy is generated during the operation of the pump, so that $Q_h > (\tau_h/\tau_l)Q_l$. In order to conserve energy, it then takes correspondingly more work W to move this heat, and W is increased by the same amount that Q_h is, which leaves W/Q_h unchanged.

(a): 1 point

(b): 1 point

Assume that the electricity consumed by a reversible heat pump must itself be generated by a Carnot engine operating between the temperatures τ_{hh} and τ_l. What is the ratio Q_{hh}/Q_h, of the heat consumed at τ_{hh}, to the heat delivered at τ_h? Give numerical values for $T_{hh} = 600$ K; $T_h = 300$ K; $T_l = 270$ K.

For the Carnot engine we have $W = Q_{hh} - Q_l = [1 - (\tau_l/\tau_{hh})]Q_{hh}$. That work is then consumed by the heat pump, for which $W = [1 - (\tau_l/\tau_h)]Q_h$. Setting these equal, we have

$$[1 - (\tau_l/\tau_h)]Q_h = [1 - (\tau_l/\tau_{hh})]Q_{hh},$$

or

$$Q_{hh}/Q_h = \frac{1 - \tau_l/\tau_h}{1 - \tau_l/\tau_{hh}}$$

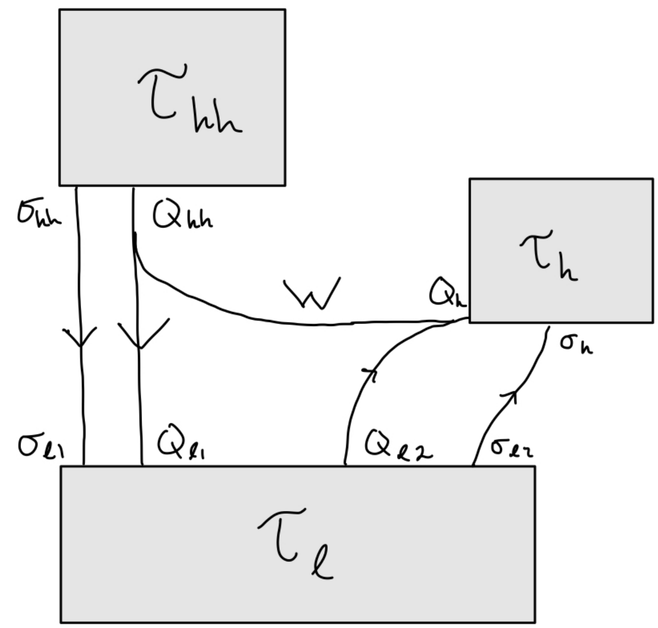
For the given numerical values of temperature, this yields

$$Q_{hh}/Q_h = 0.18.$$

(b): 1 point

(c): 1 point

Draw an energy-entropy flow diagram for the combination heat engine-heat pump, similar to Figures 8.1, 8.2 and 8.4, but involving no external work at all, only energy and entropy flows at three temperatures.



(c): 1 point

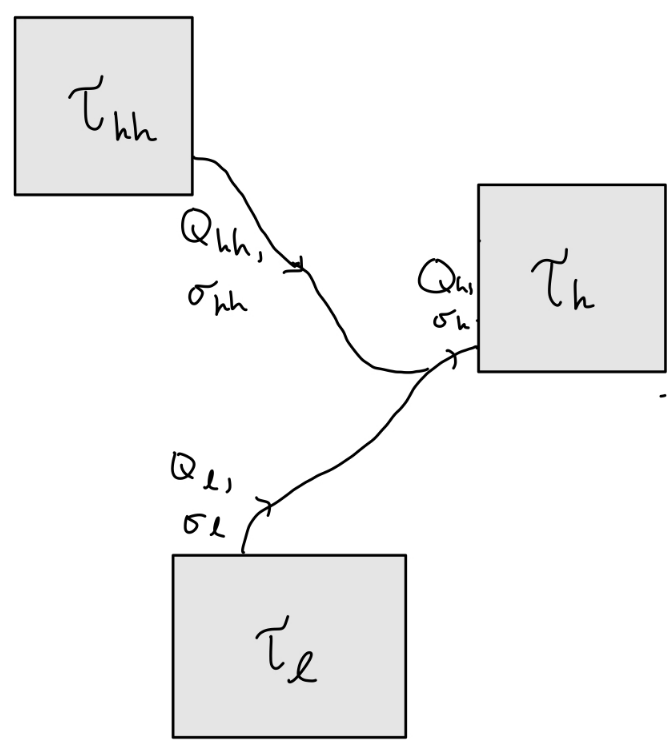
Problem 2

Kittel & Kroemer, Chapter 8, problem 2 [Absorption Refrigerator.]

In absorption refrigerators the energy driving the process is supplied not as work, but as heat from a gas flame at a temperature $\tau_{hh} > \tau_h$. Mobile home and cabin refrigerators may be of this type, with propane fuel.

(a): 1 point

Give an energy-entropy flow diagram similar to Figures 8.2 and 8.4 for such a refrigerator, involving no work at all, but with energy and entropy flows at the three temperatures $\tau_{hh} > \tau_h > \tau_l$.



(a): 1 point

(b): 1 point

Calculate the ratio Q_l/Q_{hh}, for the heat extracted at $\tau = \tau_l$, where Q_{hh} is the heat input at $\tau = \tau_{hh}$. Assume reversible operation.

Based on our diagram, we can write down energy and entropy conservation equations for this combination:

$$Q_{hh} + Q_l = Q_h$$
$$\sigma_{hh} + \sigma_l = \sigma_h$$

(b): 1 point

Using $\sigma = Q/\tau$, these become

$$Q_{hh} + Q_l = Q_h$$
$$Q_{hh}/\tau_{hh} + Q_l/\tau_l = Q_h/\tau_h.$$

Solving the second equation for Q_h and plugging into the first equation, we see

(b): 1 point

$$Q_{hh} + Q_l = Q_{hh}\tau_h/\tau_{hh} + Q_l\tau_h/\tau_l.$$

Rearranging we find

$$Q_l/Q_{hh} = \frac{\tau_h/\tau_{hh} - 1}{1 - \tau_h/\tau_l}$$

Problem 3

Kittel & Kroemer, Chapter 8, problem 3 [Photon Carnot engine.]

Consider a Carnot engine that uses as the working substance a photon gas.

(a): 1 point

Given τ_h and τ_l as well as V_1 and V_2, determine V_3 and V_4.

The first step of the Carnot cycle is the isothermal expansion at τ_h from V_1 to V_2. The next will be an isentropic expansion, taking τ_h to τ_c and V_2 to V_3. We showed in a previous homework that for an isentropic transformation of a photon gas, VT^3 is constant. Thus

$$V_3 = V_2 \left(\frac{\tau_h}{\tau_l}\right)^3.$$

(a): 1 point

Similarly,

$$V_4 = V_1 \left(\frac{\tau_h}{\tau_l}\right)^3.$$

(b): 1 point

What is the heat Q_h taken up and the work done by the gas during the first isothermal expansion? Are they equal to each other, as for the ideal gas?

The pressure-volume relationship for a photon gas is

$$p = \frac{\pi^2}{45c^3\hbar^3}\tau^4.$$

That is, the pressure is independent of volume! It is thus simple for us to calculate the work done in the first isothermal expansion at τ_h:

$$\begin{aligned} W_1 = \int_{V_1}^{V_2} p\,dV &= \int_{V_1}^{V_2} \frac{\pi^2}{45c^3\hbar^3}\tau_h^4 dV \\ &= \frac{\pi^2}{45c^3\hbar^3}\tau_h^4(V_2 - V_1). \end{aligned}$$

(b): 1 point

The entropy of the photon gas is $\sigma = \frac{4\pi^2}{45c^3\hbar^3}V\tau^3$, and during an isothermal process $d\sigma = \frac{4\pi^2}{45c^3\hbar^3}dV\tau^3$ so we can also calculate the heat:

$$\begin{aligned} Q_1 = \int_{\sigma_1}^{\sigma_2} \tau_h\,d\sigma &= \int_{V_1}^{V_2} \frac{4\pi^2}{45c^3\hbar^3}\tau_h^4\,dV \\ &= \frac{4\pi^2}{45c^3\hbar^3}\tau_h^4(V_2 - V_1). \end{aligned}$$

We see that they are not equal to each other, but that $Q_1 = 4W_1$. This is fine, since the internal energy of the photon gas is not independent of volume, and also changes during this process.

Problem 3 continued on next page. . .

(c): 1 point

Do the two isentropic stages cancel each other, as for the ideal gas?

Let's calculate. For the isentropic stages, we know the heat exchanged is zero, so the work done must equal the change in internal energy. The internal energy of the photon gas is given by $U = \frac{\pi^2}{15c^3\hbar^3} V\tau^4$. So, during the adiabatic expansion (from V_2, τ_h to V_3, τ_l), the work done is

$$W_2 = -\Delta U_2 = -\frac{\pi^2}{15c^3\hbar^3}(V_3\tau_l^4 - V_2\tau_h^4).$$

Similarly, the work done during the adiabatic compression (from V_4, τ_l to V_1, τ_h) is

$$W_4 = -\Delta U_4 = -\frac{\pi^2}{15c^3\hbar^3}(V_1\tau_h^4 - V_4\tau_l^4).$$

From part (a), we can eliminate V_3, V_4 from these results:

$$\begin{aligned} W_2 &= -\frac{\pi^2}{15c^3\hbar^3} V_2\tau_h^3(\tau_l - \tau_h) \\ W_4 &= -\frac{\pi^2}{15c^3\hbar^3} V_1\tau_h^3(\tau_h - \tau_l). \end{aligned}$$

We see that these steps no longer cancel each other, but that

$$W_2 + W_4 = \frac{\pi^2}{15c^3\hbar^3}(V_2 - V_1)\tau_h^3(\tau_h - \tau_l)$$

(c): 1 point

(d): 1 point

Calculate the total work done by the gas during one cycle. Compare it with the heat taken up at τ_h and show that the energy conversion efficiency is the Carnot efficiency.

We have already done most of the work, and all we need is to calculate W_3, the work done during the isothermal compression at τ_h from V_3 to V_4:

$$\begin{aligned} W_3 = \int_{V_3}^{V_4} p\,dV &= \int_{V_3}^{V_4} \frac{\pi^2}{45c^3\hbar^3}\tau_l^4 dV \\ &= \frac{\pi^2}{45c^3\hbar^3}\tau_l^4(V_4 - V_3), \end{aligned}$$

so that

$$W = W_1 + W_2 + W_3 + W_4 = \frac{\pi^2}{45c^3\hbar^3}\left(3(V_2 - V_1)\tau_h^3(\tau_h - \tau_l) + \tau_h^4(V_2 - V_1) + \tau_l^4(V_4 - V_3)\right).$$

(d): 1 point

From part (a), $V_4 - V_3 = (\tau_h/\tau_l)^3(V_1 - V_2)$, so this becomes

$$\begin{aligned} W &= \frac{\pi^2}{45c^3\hbar^3}\left(3(V_2 - V_1)\tau_h^3(\tau_h - \tau_l) + \tau_h^4(V_2 - V_1) + \tau_h^3\tau_l(V_1 - V_2)\right) \\ &= \frac{4\pi^2}{45c^3\hbar^3}\tau_h^3(V_2 - V_1)(\tau_h - \tau_l) \end{aligned}$$

We shwed in (b) that the heat taken up at τ_h is $Q_1 = \frac{4\pi^2}{45c^3\hbar^3}\tau_h^4(V_2 - V_1)$, so that $W/Q_1 = \frac{\tau_h - \tau_l}{\tau_h}$, which is the Carnot efficiency, as expected.

Problem 4

Kittel & Kroemer, Chapter 8, problem 4 [Heat engine-refrigerator cascade]: 2 points

The efficiency of a heat engine is to be improved by lowering the temperature of its low-temperature reservoir to a value τ_r, below the environmental temperature τ_l, by means of a refrigerator. The refrigerator consumes part of the work produced by the heat engine. Assume that both the heat engine and the refrigerator operate reversibly. Calculate the ratio of the net (available) work to the heat Q_h supplied to the heat engine at temperature τ_h. ls it possible to obtain a higher net energy conversion efficiency in this way?

Let's split the output work of the heat engine into two parts, $W = W_{net} + W_r$, where W_r is the work that the refrigerator will consume, and W_{net} is the remainder. Here we have a heat engine operating between τ_h and τ_r, and a refrigerator operating between τ_r and τ_l. Energy conservation gives us

$$Q_h = Q_{r,in} + W_{net} + W_r$$
$$Q_{r,out} + W_r = Q_l.$$

We can eliminate W_r in these equations to write

$$Q_h = (Q_{r,in} - Q_{r,out}) + W_{net} + Q_l.$$

We will assume that our refrigerator is not an infinite reservoir, so that for reversible operation the net heat and entropy exchanged with the refrigerator should be zero (otherwise it would ultimately accumulate entropy and heat up), so that $(Q_{r,in} - Q_{r,out})$, and we have

$$Q_h = W_{net} + Q_l.$$

With reversible operation, entropy is also conserved, and we have the equations

$$\sigma_h = \sigma_{r,in}$$
$$\sigma_{r,out} = \sigma_l,$$

and with our assumption $\sigma_{r,in} = \sigma_{r,out}$, so that these equations combine to give

$$\sigma_h = \sigma_l,$$

or

$$Q_h/\tau_h = Q_l/\tau_l.$$

Using this to eliminate Q_l above, we have

$$Q_h = W_{net} + Q_h(\tau_l/\tau_h),$$

which yields

$$W_{net}/Q_h = \frac{\tau_h - \tau_l}{\tau_h},$$

from which we can see that the efficiency of this cycle is still the Carnot efficiency between τ_h and τ_l! Adding the refrigerator did not make our heat engine more efficient, but neither did it make it *less* efficient. We see that the ultimate efficiency is independent of how cold we make our refrigerator.

Problem 5

Kittel & Kroemer, Chapter 8, problem 5 [Thermal pollution]: 2 points

A river with a water temperature $T_l = 20°$ C is to be used as the low temperature reservoir of a large power plant, with a steam temperature of $T_h = 500°$ C. If ecological considerations limit the amount of heat that can be dumped into the river to 1500 MW, what is the largest electrical output that the plant can deliver? If improvements in hot-steam technology would permit raising T_h by 100° C, what effect would this have on the plant capacity?

The power output is limited by the Carnot efficiency. To use this we need to put the temperatures in kelvin, $\tau_l = 293.15K$ and $\tau_h = 773.15$ K. Then the Carnot efficiency between these temperatures is $\eta_C = \frac{773.15-293.15}{773.15} = 0.62$. This is the ratio of the output to the heat *absorbed* by the hot reservoir, $W/Q_h = 0.62$. We know that for a reversible heat engine, $Q_h = (T_h/T_l)Q_l$, so we have

$$W = 0.62Q_h = 0.62(773.15/293.15)(1500 \text{ MW}) = 2450 \text{ MW}.$$

Note that technically here we are using power rather than energy; the same arguments flow through when we take the rates instead of the absolute energies. □

If the hot reservoir is bumped up by 100 K, this changes the Carnot efficiency to 0.665, so that

$$W_{new} = 0.665(873.15/293.15)(1500 \text{ MW}) = 2970 \text{ MW}.$$

This is a substantial ($\sim 25\%$) increase in the plant's output.

Problem 6

Kittel & Kroemer, Chapter 8, problem 6 [Room air conditioner.]

A room air conditioner operates as a Carnot cycle refrigerator between an outside temperature T_h and a room at a lower temperature T_l. The room gains heat from the outdoors at a rate $A(T_h - T_l)$: this heat is removed by the air conditioner. The power supplied to the cooling unit is P.

(a): 1 point

Show that the steady state temperature of the room is

$$T_l = (T_h + P/2A) - [(T_h + P/2A)^2 - T_h^2]^{1/2}.$$

The steady state temperature of the room will occur when the rate of heat removal from the air conditioner equals the rate of heat gain from the outdoors. From our discussion of refrigerators, we know that

$$Q_l = W\frac{T_l}{T_h - T_l},$$

where Q_l is the amount of heat removed from the cold reservoir, and W is the amount of work input to the refrigerator. This can be written as a rate of energy flow by taking the time derivative of both sides; here $dW/dt = P$ is the power supplied to the unit. So

$$dQ_l/dt = P\frac{T_l}{T_h - T_l}.$$

In steady state, this will equal the rate of heat gain from the outdoors, so that

(a): 1 point

$$A(T_h - T_l) = P\frac{T_l}{T_h - T_l}.$$

Multiplying both sides by $T_h - T_l$, this becomes a quadratic equation for T_l:

$$\begin{aligned} 0 &= A(T_h - T_l)^2 - PT_l \\ &= T_l^2 + (-2T_h - P/A)T_l + T_h^2 \end{aligned}$$

Solving this for T_l using the quadratic formula, we find

$$T_l = (T_h + P/2A) - [(T_h + P/2A)^2 - T_h^2]^{1/2},$$

as expected. We know to take the negative root because T_l must be less than T_h.

(b): 1 point

If the outdoors is at 37° C and the room is maintained at 17° C by a cooling power of 2 kW, find the heat loss coefficient A of the room in W K^{-1}.

In part (a) we showed that the steady-state equation for this system is $A(T_h - T_l) = P\frac{T_l}{T_h - T_l}$, which we can solve for A to find

(b): 1 point

$$A = P\frac{T_l}{(T_h - T_l)^2} = (2\text{ kW})\frac{290.15\text{ K}}{(20\text{ K})^2} = 1450\text{ W/K}.$$