▶ Problem 1

(a) $\sigma(U) = \ln(CU^{3N/2}) = \ln(C) + \frac{3N}{2}\ln(U)$

$$\begin{gathered}\frac{1}{\tau} = \frac{\partial\sigma}{\partial U} = \frac{d}{dU}\ln\left(CU^{3N/2}\right)\\ \frac{1}{\tau} = \frac{3N}{2U}\\ U = \frac{3N}{2}\tau\end{gathered}$$

(b) Just take the derivative of $\frac{3N}{2U}$

$$\frac{\partial}{\partial U}\frac{3N}{2U} = -\frac{3N}{2U^2}$$

Since N is positive, this must be negative

▶ Problem 2

In this case $U = -MB$. This gives us

We can find τ pretty easily using

$$\frac{1}{\tau} = \frac{\partial\sigma}{\partial U}$$

once we've worked in the relationship between U and s.

$$\frac{M}{m} = 2\langle s\rangle$$

Since we're working out τ we can replace $\langle s\rangle$ with s, since $\langle s\rangle$ is the equilibrium, which depends on τ

$$\begin{gathered}M = -\frac{U}{B}\\ s = \frac{M}{2m} = -\frac{U}{2mB}\\ \sigma(U) = \sigma_0 - 2\left(-\frac{U}{2mB}\right)^2/N\\ = \sigma_0 - \frac{U^2}{2m^2B^2N}\\ \frac{1}{\tau} = \frac{\partial\sigma}{\partial U} = -\frac{U}{m^2B^2N}\\ \tau = -\frac{m^2B^2N}{U}\end{gathered}$$

▶ Problem 3

The entropy is just $\sigma = \ln(g)$. From 1.55 we have

$$g(N,n) = \frac{(N+n-1)!}{n!(N-1)!}$$

which we adjust to $\frac{(N+n-1)!}{n!(N)!}$

$$\begin{gathered}\sigma(N,n) = \ln(g(N,n)) = \ln\left(\frac{(N+n-1)!}{n!(N)!}\right)\\ = \ln((N+n-1)!) - \ln(n!) - \ln(N!)\\ = (N+n-1)\ln(N+n-1) - (N+n-1) - n\ln(n) + n - N\ln(N) + N\\ = (N+n-1)\ln(N+n-1) - n\ln(n) - N\ln(N) - 1\end{gathered}$$

Not sure quite how far we're going with this. $\ln(e) = 1$

$$= \ln\left(\frac{(N+n-1)^{N+n-1}}{n^nN^Ne}\right)$$

Energy $U = n\hbar\omega$. From this we can relate energy and total quantum number. Does $\sigma(U,N)$ and $\sigma(N,U)$ mean different things? I hope not

$$n = \frac{U}{\hbar\omega}$$

$$\sigma(N,U) = \left(N + \frac{U}{\hbar\omega} - 1\right)\ln\left(N + \frac{U}{\hbar\omega} - 1\right) - \frac{U}{\hbar\omega}\ln\left(\frac{U}{\hbar\omega}\right) - N\ln(N) - 1$$

How to get U out of the logarithm? We can take the derivative and get τ first.

$$\frac{1}{\tau} = \frac{\partial\sigma}{\partial U} = \frac{1}{\hbar\omega}\ln\left(N + \frac{U}{\hbar\omega} - 1\right) + \frac{1}{\hbar\omega} - \frac{1}{\hbar\omega}\log\left(\frac{U}{\hbar\omega}\right) - \frac{1}{\hbar\omega}$$

$$\frac{\hbar\omega}{\tau} = \ln\left(\frac{N + \frac{U}{\hbar\omega}}{\frac{U}{\hbar\omega}}\right)$$

Alright this I can work with

$$e^{\hbar\omega/\tau} = \frac{N}{\frac{U}{\hbar\omega}} + 1$$

$$U = \frac{N\hbar\omega}{e^{\hbar\omega/\tau} - 1}$$

▶ Problem 4

(a) Just use the multiplicity formula. $g_{tot} = g_1 g_2$

$$g(N,s) = \frac{N!}{(N/2+s)!(N/2-s)!} = \frac{N!}{N_\uparrow! N_\downarrow!}$$

$$g(10,5) = 1$$

$$g(10,-3) = \frac{10!}{2!8!} = 45$$

$$g_{tot} = 45$$

(b) No better way than using code! Looks like entropy more than doubles compared to the initial state

In [1]:

```
import pandas as pd
import numpy as np
from scipy.special import factorial

s1 = np.arange(-3, 6)
s2 = 2 - s1
g1 = factorial(10)/(factorial(5+s1)*factorial(5-s1))
g2 = factorial(10)/(factorial(5+s2)*factorial(5-s2))
gtot = g1*g2
entropy = np.log(gtot)

table = pd.DataFrame({'s1':s1, 's2':s2, 'g1':g1, 'g2':g2, 'total multiplicity':gtot, 'combined entropy':entropy})
table
```

Out[1]:

	s1	s2	g1	g2	total multiplicity	combined entropy
0	-3	5	45.0	1.0	45.0	3.806662
1	-2	4	120.0	10.0	1200.0	7.090077
2	-1	3	210.0	45.0	9450.0	9.153770
3	0	2	252.0	120.0	30240.0	10.316921
4	1	1	210.0	210.0	44100.0	10.694215
5	2	0	120.0	252.0	30240.0	10.316921
6	3	-1	45.0	210.0	9450.0	9.153770
7	4	-2	10.0	120.0	1200.0	7.090077
8	5	-3	1.0	45.0	45.0	3.806662

(c) Probability is just # microstates/all possible states. It's more likely to be in the adjacent states rather than most probable state (2*24=48% vs 35%). The initial state is unlikely ~.04%

In [2]:

```
table['probability'] = table['total multiplicity']/table['total multiplicity'].sum()
table
```

Out[2]:

	s1	s2	g1	g2	total multiplicity	combined entropy	probability
0	-3	5	45.0	1.0	45.0	3.806662	0.000357
1	-2	4	120.0	10.0	1200.0	7.090077	0.009526
2	-1	3	210.0	45.0	9450.0	9.153770	0.075018
3	0	2	252.0	120.0	30240.0	10.316921	0.240057
4	1	1	210.0	210.0	44100.0	10.694215	0.350083
5	2	0	120.0	252.0	30240.0	10.316921	0.240057
6	3	-1	45.0	210.0	9450.0	9.153770	0.075018
7	4	-2	10.0	120.0	1200.0	7.090077	0.009526
8	5	-3	1.0	45.0	45.0	3.806662	0.000357

▶ Problem 5

(a)

Energy: $U_1 + U_2 = -2(0)mB - 2(1500)mB = -3000mB$

Entropy: Let's set up the gaussian approximation. Getting rid of $\sqrt{\frac{2}{\pi N}}$ (still has the 2^N term)

$$g(N,s) \approx 2^N e^{-2s^2/N}$$
$$\sigma(N,s) = \ln(g(N,s)) = \ln(2^N) + \ln(e^{-2s^2/N})$$
$$= N\ln(2) - \frac{2s^2}{N}$$

Entropy adds, so our entropy is

$$\sigma_{tot} = \sigma(10^4, 0) + \sigma(2\times 10^4, 1500) = 10^4\ln(2) + 2\times 10^4 \ln(2) - \frac{(1500)^2}{10^4}$$
$$\approx 20569$$

Just realized that we're asked for each system, not combined.

$U_1 = 0,\ U_2 = -3000mB$
$\sigma_1 \approx 6931,\ \sigma_2 \approx 13638$

2×10 4 ln2−112.5.

In [7]:
```
from numpy import log
print(10**4 * log(2), 2*10**4 * log(2) - (1500)**2/(10**4))
```

6931.471805599453 13637.943611198905

Out[7]: 225.0

(b)

Initial temp of each system depends on each $\sigma(N,s)$, which is converted to $\sigma(N, -\frac{U}{2mB})$. We can drop the $N\ln(2)$ term thanks to the derivative.

$$\frac{1}{\tau} = \frac{\partial\sigma}{\partial U} = \frac{d}{dU}\frac{-2\left(-\frac{U}{2mB}\right)^2}{N} = \frac{-2}{4m^2B^2N}\frac{d}{dU}U^2$$
$$= \frac{-U}{m^2B^2N}$$
$$\tau = -\frac{m^2B^2N}{U} = -\frac{m^2B^2N}{-2smB} = \frac{mBN}{2s}$$
$$\tau_1 = mB10^4/(2*0)???$$
$$\tau_2 = mB(2*10^4)/(2*1500) \approx 6.7mB$$

I guess the temperature τ_1 is infinite

(c) Shouldn't spin excess be constant? Oh, you mean spins in each system. I'm guessing $\hat{s}_2 = 2\hat{s}_1$ since $N_2 = 2N_1$, but we'll see.

Equilibrium is when $\tau_1 = \tau_2$. N and S_{tot} are constant, only s changes

$$\tau_1 = \tau_2$$
$$\frac{mB(10^4)}{2s_1} = \frac{mB(2\times 10^4)}{2s_2}$$
$$2s_2 = 2 * 2s_1$$
$$s_2 = 2s_1 \checkmark$$
$$s_1 + s_2 = 1500 \rightarrow s_1 = 500,\ s_2 = 1000$$
$$\tau = \frac{mB(10^4)}{2(500)} = 10mB$$

(d) I'll keep in the $N\ln(2)$ term so I can use my results from part (a)

$$\sigma_1 = 10^4\ln(2) - 2(500)^2/10^4 \approx 6881$$
$$\sigma_2 = 2\times 10^4\ln(2) - 2(1000)^2/(2\times 10^4) \approx 13763$$

Here's how entropy changed:

$$\sigma_1 : 6931 \rightarrow 6881,\ \Delta\sigma_1 = -50$$
$$\sigma_2 : 13637 \rightarrow 13762,\ \Delta\sigma_2 = 125$$
$$\Delta\sigma_{tot} = 75$$

In [16]:
```
print(10**4*log(2) - 2*(500)**2/10**4, 2*10**4*log(2) - 2*(1000)**2/(2*10**4))
```

```
6881.471805599453 13762.943611198905
```

In reflection, it looks like energy doesn't necessarily flow from high to low. If m or B have their sign flipped, the sign of energy is flipped, even if the sign of σ isn't. It merely flows from low-likelyhood configuration to high-likelyhood configuration.