CHAPTER III
SEQUENCES AND SERIES

Convergent sequences

Sequence $(a_n) \subset \mathbb{R}$: any function $\mathbb{N} \to \mathbb{R}$.

$\boxed{\text{DEF}}$ (a_n) converges to $p \in \mathbb{R}$ if

$\forall \varepsilon > 0\ \exists N:\ \forall n > N\ \ \|a_n - p\| < \varepsilon$. p is called the limit of (a_n). ($\lim\limits_{n\to\infty} a_n = p$ OR $a_n \xrightarrow[n\to\infty]{} p$ OR $a_n \to p$)

if no such p exists, then (a_n) diverges.

A sequence is bounded if the set $\{a_n\}$ is bounded.

Examples
(a) $a_n = 1/n$: $\lim\limits_{n\to\infty} a_n = 0$
(b) $a_n = 1 + \frac{(-1)^n}{n}$: $\lim\limits_{n\to\infty} a_n = 1$
(c) $a_n = 1\ \forall n$: $\lim\limits_{n\to\infty} a_n = 1$
(d) $a_n = (-1)^n$: bounded, but $\lim\limits_{n\to\infty} a_n$ DNE.

Theorem (a_n) is a sequence. Then $a_n \to p$ iff $\forall r > 0$ $B_r(p)$ contains (a_n) for all but finitely many terms.

$\Rightarrow$ assume $\forall r > 0$ $(a_n) \cap (B_r(p))^c$ is a finite set $a_{n_1} \ldots a_{n_k}$. Take $N = \max\{n_1 \ldots n_k\}$. Then $\forall n > N$ $a_n \in B_r(p)$ so $a_n \to p$.

$\Leftarrow$ assume $\forall r > 0\ \exists N:\ \forall n > N\ a_n \in B_r(p)$. Then $(a_n) \cap (B_r(p))^c \subset \{a_1, \ldots, a_N\}$ and is therefore finite.

Theorem The limit is unique: if $\lim\limits_{n\to\infty} a_n = p$ and $\lim\limits_{n\to\infty} a_n = q$ and $p \neq q$ then take $r = \frac{1}{2}|p-q|$, $[B_r(p)]^c \cap (a_n) =$ finite set. But $B_r(q) \subset [B_r(p)]^c$, so $B_r(q) \cap (a_n)$ is a finite set – contradiction.

Theorem if (a_n) converges, (a_n) is bounded.

Take $B_1(p)$, then $[B_1(p)]^c \cap \{a_n\}$ is finite and therefore has a maximum a_s. Then $\forall n$ $|a_n| \le \max\{|a_s|, |p|+1\}$

MAIN THEOREMS Suppose $a_n \to A$, $b_n \to B$.

Then: (a) $\lim_{n\to\infty} (a_n + b_n) = A + B$

(b) $\lim_{n\to\infty} c\, a_n = cA$

(c) $\lim_{n\to\infty} a_n \cdot b_n = AB$

(d) Let $a_n \neq 0$ and $A \neq 0$, then $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{A}$

(and thus $\lim_{n\to\infty} b_n/a_n = B/A$)

Proof (a) [$\varepsilon/2$] $\forall \varepsilon > 0$ $\exists N_1, N_2$: $\forall n > N_1$ $|a_n - A| < \varepsilon/2$ and $\forall n > N_2$ $|b_n - B| < \varepsilon/2$. Then $\forall n > \max\{N_1, N_2\}$

$$|A + B - a_n - b_n| \leq |A - a_n| + |B - b_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \square$$

(b) $\forall \varepsilon > 0$ $\exists N$: $\forall n > N$ $|a_n - A| < \varepsilon/|c|$ then

$$|c a_n - cA| = |c|\,|a_n - A| < |c| \cdot \varepsilon/|c| = \varepsilon \quad \square$$

(c) since $a_n \to A$, $b_n \to B$, a_n, b_n are bounded, so $|a_n| < M$, $|b_n| < R$ $\forall n$;

$$\|AB - a_n b_n\| = \|(A - a_n)B + a_n(B - b_n)\| \leq$$

$$\leq |A - a_n||B| + |a_n||B - b_n| < |A - a_n||B| + M|B - b_n|$$

$\forall \varepsilon > 0$ $\exists N_1$: $|A - a_n| < \frac{\varepsilon}{2|B|}$ (if $B \neq 0$ otherwise this term is merely zero) and $|B - b_n| < \frac{\varepsilon}{2M}$, so $|A - a_n||B| + M|B - b_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\square$

(d) $\left|\frac{1}{A} - \frac{1}{a_n}\right| = \frac{1}{|A \cdot a_n|}|A - a_n|$. Since $A \neq 0$, $\exists N_0$: $\forall n > N_0$ $|A - a_n| < \frac{|A|}{2}$. $\forall \varepsilon > 0$ $\exists N$: $\forall n > N$ $|A - a_n| < \varepsilon \frac{|A|^2}{2}$. Then $\forall n > \max\{N_0, N\}$ $|a_n| > |A|/2$

$$\frac{1}{|A \cdot a_n|}|A - a_n| < \frac{1}{|A| \cdot |A|/2}|A - a_n| = \frac{2}{|A|^2}|A - a_n| < \varepsilon. \square$$

Corollary Let $\lim_{n\to\infty} b_n = B = 0$ and let (a_n) be a bounded sequence (not necessarily convergent).
Then $\lim_{n\to\infty} a_n b_n = 0$.

Theorem "squeezing" Let $a_n \le b_n \le c_n$ $\forall n \in \mathbb{N}$
OR for "ALMOST ALL" n, that is, for all but finite number of n.
Let $\lim_{n\to\infty} a_n = A = \lim_{n\to\infty} c_n$. Then $\lim_{n\to\infty} b_n = A$.

Indeed, $\forall \varepsilon > 0$ $\exists N$: $\forall n > N$ $|a_n - A| < \varepsilon \vee |c_n - A| < \varepsilon$
then $|b_n - A| \le \max\{|c_n - A|, |a_n - A|\} < \varepsilon$ so $b_n \to A$.

SPECIAL SEQUENCES

Theorem
(a) if $p > 0$ then $\lim_{n\to\infty} \frac{1}{n^p} = 0$
(b) if $p > 0$ then $\lim_{n\to\infty} \sqrt[n]{p} = 1$
(c) $\lim \sqrt[n]{n} = 1$
(d) for $x > 0$: $\lim_{n\to\infty} x^n = \begin{cases} 0, & x<1 \\ 1, & x=1 \\ \infty, & x>1 \end{cases}$

$\boxed{\text{DEF}}$ $\lim_{n\to\infty} x_n = +\infty$ (OR $-$) if $\forall M > 0$ $\exists N$: $\forall n > N$ $x_n > M$.

Say, $(a_n) = 2^{(-1)^n \cdot n}$ is not bounded but $\lim_{n\to\infty} a_n \neq \infty$

(a) Take $N = (1/\varepsilon)^{1/p}$
(b) if $p > 1$ put $x_n = \sqrt[n]{p} - 1$ then $(1 + nx_n) \le (1 + x_n)^n =$
why? Prove by induction: $n = 1$ $1 + x_n \le 1 + x_n$
$(1 + (n+1)x_n) = 1 + nx_n + x_n \le$
$\le (1 + nx_n)(1 + x_n) \le (1 + x_n)^{n+1}$

so $0 < x_n < \frac{p-1}{n} \Rightarrow x_n \to 0$.

(c) [perhaps there will be other proofs of this statement].

(d): Prove that $(1+x_n)^n \geq \frac{n(n-1)}{2}x_n^2$ for $x_n \geq 0$.

[~~we already proved~~ let us prove $(1+x_n)^n \geq 1+nx_n+\frac{n(n-1)}{2}x_n^2$]

Assumption: $(1+x_n)^n \geq \frac{n(n-1)}{2}x_n^2+nx_n+1$

then $(1+x_n)^{n+1} = (1+x_n)\cdot(1+x_n)^n \geq$

$\geq \cancel{(1+x_n)\cdot\frac{1}{2}\left(1+nx_n+\frac{n(n-1)}{2}x_n^2\right)}$

$$\geq (1+x_n)\left(1+nx_n+\frac{n(n-1)}{2}x_n^2\right) =$$

$$= 1+(n+1)x_n+nx_n^2+\frac{n(n-1)}{2}x_n^2+\frac{n(n-1)}{2}x_n^3 =$$

$$= 1+(1+n)x_n+\frac{n(n+1)}{2}x_n^2+\frac{n(n-1)}{2}x_n^3 \geq$$

$$\geq 1+(1+n)x_n+\frac{n(n+1)}{2}x_n^2 .$$

So $(1+x_n)^n > \frac{n(n-1)}{2}x_n^2$. Take $x_n = \sqrt[n]{n}-1 > 0$

Then $(1+x_n)^n = n > \frac{n(n-1)}{2}x_n^2$ so $x_n^2 < \frac{2}{n-1}$

$0 < x_n < \sqrt{\frac{2}{n-1}} \xrightarrow[n\to\infty]{} 0$.

IF WE DO NOT APPLY THEOREMS ON SEQUENCES, PROOFS CAN BE NOTORIOUSLY LONG.

Prove, using only definition of the limit, that

$$\lim_{n\to\infty} \frac{3n^2+n}{n^2-6} = 3.$$

first, we need an estimate:

$$\left|\frac{3n^2+n}{n^2-6} - 3\right| = \left|\frac{3n^2+n-3n^2+18}{n^2-6}\right| = \left|\frac{n+18}{n^2-6}\right|$$

we "intuitively" feel that $|n+18| << |n^2-6|$ for BIG n, but imagine that we only know Archimedean property (and therefore that $\forall \varepsilon>0\ \exists n: \frac{1}{n}<\varepsilon$. First, restrict to sufficiently large n: $|n+18| < 2n$ for $n \geq 18$, $|n^2-6| > \frac{1}{2}n^2$ for $n \geq 4$, So, for $n \geq 18$, $\left|\frac{n+18}{n^2-6}\right| < \frac{2n}{\frac{1}{2}n^2} = \frac{4}{n}$

and ~~we can conclude since 4/n~~

so $\forall \varepsilon > 0\ \exists N = \max\{18, 4/\varepsilon\}$ such that

$$\forall n > N \quad \left|\frac{3n^2+n}{n^2-6} - 3\right| < \frac{4}{n} < \frac{4}{N} = \varepsilon. \quad \square.$$

Whereas, with theorems:

$$\lim_{n\to\infty} \frac{3n^2+n/n^2}{n^2-6/n^2} = \lim_{n\to\infty} \frac{3+\frac{1}{n}}{1-6/n^2} = \frac{\lim_{n\to\infty}(3+\frac{1}{n})}{\lim_{n\to\infty}(1-6/n^2)} =$$

$$= \frac{3+\lim 1/n}{1-6\lim 1/n^2} = \frac{3+0}{1-0} = 3.$$

NOTE $(a_n) \neq \{a_n\}$! We may have ∞ many repeating terms in (a_n). So, be careful...

UPPER AND LOWER LIMITS

HERE WE USE OUR KNOWLEDGE OF SETS:

HAVING A SEQUENCE (a_n), DEFINING A SET

$A_K = \{a_{K+1}, a_{K+2}, \dots\}$ that is, we remove first K terms. As a set, A_K has inf and sup. (if not, we declare $\sup A_K = +\infty$ AND/OR $\inf A_K = -\infty$)

Then obviously $A_0 \supset A_1 \supset A_2 \supset \dots \supset A_K \supset A_{K+1} \supset \dots$

so, say $\sup A_K \leq \sup A_{K-1} \leq \dots \leq \sup A_0$. (OR ALL OF THEM ARE INFINITE!) We call

$$\limsup (a_n) = \inf (\sup A_K, K = 0, 1, 2, \dots)$$
$$\liminf (a_n) = \sup (\inf A_K, K = 0, 1, 2, \dots)$$

NOTE THAT $\limsup (a_n)$ can be ANY number, OR $+\infty$ OR $-\infty$! [Take $a_k = -k$]

Theorem (a_n) is convergent if and only if $\limsup(a_n) = \liminf(a_n) \left(= \lim\limits_{n\to\infty} a_n\right)$.

Indeed, (i) $\liminf(a_n) \leq \limsup(a_n)$

(ii) if $a_n \to A$, then $\forall \varepsilon > 0 \; \exists N : \forall n > N$

$A - \varepsilon < a_n < A + \varepsilon$, so ~~$\sup A_n \leq A+\varepsilon$~~

$A - \varepsilon \leq \inf A_n \leq \sup A_n \leq A + \varepsilon \Rightarrow \sup A_n - \inf A_n \leq 2\varepsilon$

so $\limsup(a_n) - \liminf(a_n) \leq 2\varepsilon \;\; \forall \varepsilon > 0 \Rightarrow$

$\limsup(a_n) = \liminf(a_n)$.

(iii) $\liminf(a_n) = A = \limsup(A_n) \Rightarrow \exists N : \forall n > N \;\; A - \varepsilon < \inf A_n < \sup A_n < A$

so $\forall a_n \in A_n \;\; |a_n - A| < \varepsilon \Rightarrow A = \lim a_n$

Theorem Let $s_n \to s > 0$ and (t_n) be any sequence. Then $\limsup (s_n t_n) = s \cdot \limsup t_n$

Proof. $\forall \varepsilon > 0 \ \exists N : \forall n > N \ |s_n - s| < |s| \cdot \varepsilon$. Then $s(1-\varepsilon) < s_n < s(1+\varepsilon)$ so $s(1-\varepsilon) t_n < s_n t_n < s t_n (1+\varepsilon)$ and $s(1-\varepsilon) \sup\limits_n t_n \leq \sup(s_n t_n) \leq s(1+\varepsilon) \sup t_n$ OR $(\sup(s_n t_n) - s \cdot \sup t_n) \leq$

$\leq \varepsilon \cdot s \cdot \sup t_n$. If $\limsup t_n$ is finite, we have

$|\limsup (s_n t_n) - s \limsup t_n| \leq \varepsilon \cdot s \cdot \limsup t_n \ \forall \varepsilon$, so

$\limsup (s_n t_n) = s \cdot \limsup t_n$.

If $\limsup t_n = +\infty$, then $\sup t_n = +\infty$ for each n and $\sup(s_n t_n) = +\infty$ for $n > N$, so $\limsup(s_n t_n) = +\infty$

If $\limsup (t_n) = -\infty$, take $\varepsilon = 1$, then $\sup(s_n t_n) \leq 2s \sup t_n$ and $\limsup(s_n t_n) \leq 2s \limsup t_n = -\infty$.

Theorem Let (s_n) be any sequence of nonzero real numbers.

Then $\liminf \left|\frac{s_{n+1}}{s_n}\right| \leq \liminf |s_n|^{1/n} \leq \limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$

prove that $\limsup |s_n|^{1/n} \leq \limsup \left|\frac{s_{n+1}}{s_n}\right|$. Assume $\limsup \left|\frac{s_{n+1}}{s_n}\right| = L$

$L < \infty$. Show that $a \leq L_1$ for ANY $L_1 > L$. Since

$L = \limsup \left|\frac{s_{n+1}}{s_n}\right| = \lim\limits_{N \to \infty} \sup\left\{ \left|\frac{s_{n+1}}{s_n}\right| : n > N \right\} < L_1$,

$\exists N : \left|\frac{s_{n+1}}{s_n}\right| < L_1$ for $n \geq N$. NOTE THAT $|s_n| = \left|\frac{s_n}{s_{n-1}}\right| \cdot \left|\frac{s_{n-1}}{s_{n-2}}\right| \cdot \cdots |s$

for $n > N$. Then $|s_n| < L_1^{n-N} |s_N|$, $\varkappa = |s_N| \cdot L_1^{-N}$ is fixed and positive. Then $|s_n|^{1/n} < L_1 \cdot \varkappa^{1/n}$ and

$\limsup |s_n|^{1/n} \leq L_1 \cdot \limsup \varkappa^{1/n} = L_1 \cdot \lim\limits_{n \to \infty} \varkappa^{1/n} = L_1$. $\square$

Monotonous sequences

DEF A sequence (a_n) increases monotonically (OR just increases) (OR STRICtly increases) if

$a_n \leqslant a_{n+1}$ ($a_n < a_{n+1}$) $\forall n \in \mathbb{N}$.

Analogously (a_n) decreases monotonically if $a_n \geqslant a_{n+1}$ $\forall n$. If a sequence either increases OR decreases, it is called monotonous sequence.

<u>Theorem</u> monotonous bounded sequences have limits in $\mathbb{R}$. (OR just "are convergent")

<u>proof</u>: Consider only the case of increasing (a_n). "$(a_n \nearrow)$"

Then, first $\inf\{a_n\} = \min a_n = a_1$

Assume (a_n) is bounded above. Then, by the completeness axiom, the set $\{a_n\}$ has LUB $\sup(a_n)$

We now show that $\sup\{a_n\}$ IS the limit!

Indeed, $\forall \varepsilon > 0$ $\exists a_N$: $|\sup(a_n) - a_N| < \varepsilon$ (otherwise it is not L.U.B.) Then, $\forall n > N$: $\sup(a_n) - \varepsilon < a_N \leqslant a_n \leqslant \sup(a_n)$

so $|\sup(a_n) - a_n| < \varepsilon$, so $\sup(a_n) = \lim\limits_{n\to\infty} a_n$.

Examples ARE abundant; ~~e.g.~~ $a_n = 1/n$;

- $a_n = n$ [~~it~~ has no limit, but since $\forall M$ $\exists N$: $N > M$, and $\forall n > N$ $a_n > M$, then $a_n \nearrow \infty$.]
- $a_n = \prod\limits_{k=1}^{n} (1 - q^k)$, $0 < q < 1$: decreasing; what is $\inf(a_n)$?

[ANSWER IN CHAPTER 6].

Cauchy sequences

(answering a question about convergence if we do not know the limit).

$\boxed{\text{DEF}}$ A sequence (a_n) is called Cauchy if

$$\forall \varepsilon>0\ \exists N : \forall n,m>N,\ |a_n-a_m|<\varepsilon.$$

Theorem (a) convergent sequences are Cauchy (in any metric space);
(b) Cauchy sequences converge (in a complete space).

(a) If $\lim_{n\to\infty} a_n = p$ then $\exists N: \forall n>N\ |a_n-p|<\varepsilon/2$, then $\forall n,m>N$, $|a_n-a_m| = |(a_n-p)+(p-a_m)| \le |a_n-p|+|a_m-p| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon$

(b) Let (a_n) be Cauchy. ~~Since~~ Take $\varepsilon>0$. Then
$\exists N: |a_n-a_m|<\varepsilon/2\ \forall n,m>N$ Fix n. Then $\forall m>N$
$a_n-\varepsilon/2 < a_m < a_n+\varepsilon/2$, so $a_n-\varepsilon/2 \le \inf\{a_m, m>N\} \le$
$\le \sup\{a_m, m>N\} \le a_n+\varepsilon/2$ and therefore
~~$\inf\{a_m, m>N\}$~~ $\sup\{a_m, m>N\} - \inf\{a_m, m>N\} \le \varepsilon$

Then use that $\inf A_N \le \liminf a_n \le \limsup a_n \le \sup A_N$,
so $\limsup a_n - \liminf a_n \le \varepsilon$ for any $\varepsilon>0 \Rightarrow$
$\Rightarrow \limsup a_n = \liminf a_n = \lim_{n\to\infty} a_n$.

[NB: to prove this part we need an axiom of completeness, otherwise we may not have limsup & liminf.

Subsequences

$a_1\ a_2\ a_3\ a_4\ a_5\ a_6 \dots$

$\left.\begin{array}{l}\{a_1, a_3, a_5, a_7, \dots\} \\ \{a_2, a_4, a_6, a_8, \dots\}\end{array}\right\}$ two subsequences of (a_n).

DEF ~~...~~ ORDERED INFINITE subset (a_{n_k}) $k=1,2,\dots$ such that $n_k < n_{k+1}\ \forall k \in \mathbb{N}$ is called a <u>subsequence</u> of (a_n).

<u>Theorem</u> If (a_n) is convergent, so is any its subsequence $\lim\limits_{k\to\infty} a_{n_k} = \lim\limits_{n\to\infty} a_k$: NOTE FIRST THAT $n_k \geq k$

Since $\forall \varepsilon > 0\ \exists N : \forall k > N\ |a_k - p| < \varepsilon$, then for the same N, $n_k \geq k$, so $|a_{n_k} - p| < \varepsilon$ and $a_{n_k} \xrightarrow[k\to\infty]{} p$.

It often happens that a sequence is NOT convergent, but its subsequent does converge...

Say $a_n = (-1)^n$: $a_{2n} = 1$ – constant (sub) sequence.

DEF If subsequence (a_{n_k}) converges, its limit is called a subsequential limit of (a_n)

Say; $(-1)^n$ has exactly two subsequential limits: $+1$ and -1

<u>Theorem</u> If (a_n) is a sequence on a compact set K $(\subset \mathbb{R})$ then it has a subsequence convergent to a point of K.

<u>proof</u>: Consider a set $\{a_n\} \subset K$. If this set is finite, then at least one of points $a_i \in \mathbb{R}$ is repeated ∞ many times in (a_n) and we have a constant subsequence $a_{n_k} = a_i$

Otherwise $\{a_n\}$ is infinite and has a limit point $p \in K$

Construct a subsequence: $a_{n_1} \in B_1(p)$, $a_{n_1} \neq p$; then $a_{n_2} \in B_{r_2}(p)$ with $r_2 = \frac{1}{2}|a_{n_1} - p|$ AND $n_2 > n_1$. Such point always exists because $\forall r$ $B_r(p)$ contains ∞ many points of (a_n)

Continuing this process we, at every step, has $a_{n_{i+1}}$: $a_{n_{i+1}} \neq p$, $n_{i+1} > n_i$, and $|a_{n_{i+1}} - p| < \frac{1}{2}|a_{n_i} - p|$, so $|a_{n_{i+1}} - p| < \frac{1}{2^i}$.

Obviously subsequence $(a_{n_i}) \xrightarrow[i\to\infty]{} p$.

[Why not just to take $a_{n_i} \in B_{1/i}(p)$ with $n_i > n_{i-1}$? It ~~is~~ also gives the required subsequence. But with this procedure we constructed a little more than just convergent subsequence].

<u>Corollary</u> Any bounded sequence in $\mathbb{R}$ contains a monotonous convergent subsequence.

<u>proof</u>. First, since (a_n) is bounded, $(a_n) \subset [\inf(a_n), \sup(a_n)]$ — a compact subset of $\mathbb{R}$, so, by the above construction $\exists (a_{n_i}) \to p \in [\inf(a_n), \sup(a_n)]$. Second, we have that $|a_{n_i} - p| < \frac{1}{2}|a_{n_{i-1}} - p|$ $\forall i$. At least one of subsets of a_{n_i}:

$$\{a_{n_i} : a_{n_i} < p\} \text{ and } \{a_{n_i} : a_{n_i} > p\}$$

is infinite. Consider elements of this subset a subsequence of (a_{n_i}), say $(a_{n_i^*})$. Then this subsequence $(a_{n_i^*})$ is monotonous. (and strictly increasing or decreasing). Otherwise, if $\{a_n\}$ is finite, this subsequence is a constant sequence.

Theorem For any sequence (a_n) the set of its subsequential limits is closed in $\mathbb{R}$ (can be empty; $+\infty$ and $-\infty$ ARE NOT POINTS!)

Proof. Let E be ~~closed~~ the set of subsequential limits of (a_n). Let q be a limit point of E. We show then that $\forall \varepsilon > 0$ $B_\varepsilon(q)$ contains ∞ many points of (a_n) that are not equal q: indeed $\exists$ a limit point $p \in B_\varepsilon(q)$, $p \neq q$. Then, since p is a limit point, take $r = \min\{\varepsilon - |p-q|, |p-q|\}$, $r > 0$ and $B_r(p)$ contains ∞ many points of (a_n). Since $q \notin B_r(p)$ neither of these points is equal q.

So let us construct a subsequence like in the previous proof: take $a_{n_1} \in B_1(q)$, $a_{n_1} \neq q$, then $a_{n_2} \in B_{r_2}(q)$ with $n_2 > n_1$ and $r_2 = \frac{1}{2}|a_{n_1} - q|$ and continue. We then obtain a subsequence (a_{n_k}) converging to q, so q is a subsequential limit.

Theorem $\limsup a_n$ and $\liminf a_n$ (if finite) are subsequential limits; moreover $\liminf a_n = \inf E$ and $\limsup a_n = \sup E$ where E is the set of subsequential limits.

$= \min E$ $\qquad$ $= \max E$

Examples
- (a_n): $\{a_n\} = \mathbb{Q}$; $E = \mathbb{R}$.
- (a_n): $a_n = \sin(n)$ $\quad E = [-1, 1]$
- (a_n): $a_n = \sin\left(\frac{2\pi n}{N}\right)$: $\quad E = \{ \sin \frac{2\pi k}{N}, k = 0, .., N-1 \}$