▶ Q1

$$p_\mu = \begin{pmatrix} 200 \\ 30 \\ 100 \\ 150 \end{pmatrix} = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

(a) From the metric we have $m^2 = E^2 - p^2$

$$|p| = \sqrt{30^2 + 10^2 + 150^2}$$
$$m^2 = (200c)^2 - 30^2 - 100^2 - 150^2$$
$$m = 81.24$$

This is a bit more than the rest mass of a W boson 80.37 GeV.

In [40]:
```python
p2 = 30**2 + 100**2 + 150**2
print('momentum magnitude: ', p2**(1/2))

E = 200

m2 = E**2 - p2 #c=1
m = m2**(1/2)
print('mass: ', m)
```

```
momentum magnitude:  182.75666882497066
mass:  81.24038404635961
```

Based on the fact that a particle's mass is the same in every inertial frame, we'll assume that the mass is 81.24 GeV rather than the mass of $W^\pm$

(b) We can use $E/c = \gamma mc$ to find γ. This works using natural units $c = 1$

$$\gamma = \frac{E}{c}\frac{1}{mc} \overset{c=1}{=} \frac{E}{m}$$

This gives $\gamma = 2.46$ for our calculated m, and $\gamma = 2.49$ using a W boson's rest mass

$$\beta = \frac{v}{c} = \sqrt{1 - \frac{1}{\gamma^2}} = 0.914$$

In [30]:
```python
E = 200

gamma = E/m
gamma

beta = (1-(1/(gamma)**2))**(1/2) # beta formula

print(gamma, '\n', beta)
```

```
2.4618298195866544
 0.9137833441248533
```

In [31]:
```python
E = 200
gamma = E/m
print(gamma)
print(E/80.37)

beta = (1-(1/(gamma)**2))**(1/2) # beta formula
v = p2**(1/2) / (m* gamma) # p = gamma*mv -> v = p/(gamma m)

assert beta == v # sanity check
print(beta)
```

```
2.4618298195866544
2.488490730372029
0.9137833441248533
```

(c) Boost the particle into its own rest frame: $p = v = 0$ so $\beta = 0$ and $\gamma = 1$.

Using $m = 81.24$ and $E/c = \gamma mc$, our 4-vector is

$$p_\mu = \begin{pmatrix} 81.24 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Next the decay results.

$$m_e = 0.511 \text{ MeV } = 0.000511 \text{ GeV}$$
$$m_e^2 = 2.6 \times 10^{-7}$$

It turns out the electron mass is very small relative to the GeV scale. We should expect the energy distribution to be about even

$$|p_e| = |p_{\nu_e}|$$
$$E_W = E_e + E_{\nu_e}$$

Our solution is based primarily on the lorentz invariant

$$(E/c)^2 = p^2 + (m_0 c)^2$$
$$E/c = \sqrt{p^2 + (m_0 c)^2}$$

$$E_e = \sqrt{|p_e|^2 + m_e^2}$$
$$E_{\nu_e} = \sqrt{|p_{v_e}|^2} = |p_{v_e}| = |p_e|$$
$$E_W = \sqrt{|p_e|^2 + m_e^2} + |p_e|$$

Considering m_e^2 is well below our precision, we have

$$E_W \approx \sqrt{|p_e|^2} + |p_e| = 2|p_e|$$
$$p_e^z = |p_e| = E_W/2 = 40.62$$
$$p_{\nu_e}^z = -p_e^z = -40.62$$
$$p_e^\mu = \begin{pmatrix} 40.62 \\ 0 \\ 0 \\ 40.62 \end{pmatrix}$$
$$p_{\nu_e}^\mu = \begin{pmatrix} 40.62 \\ 0 \\ 0 \\ -40.62 \end{pmatrix}$$

(d) After we transfer from particle rest frame to lab rest frame, we should be able to get our min/max momentum easily

Max momentum: $\vec{p}_e = |\vec{p}_W| + |\vec{p'}_e|$ aligned vectors
Min momentum: $\vec{p}_e = |\vec{p}_W| - |\vec{p'}_e|$ anti-aligned vectors

The actual computation is more complicated than this sketck. Let's find the magnitude of momentum in the rest frame

$$\begin{pmatrix} E'/c \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

The above matrix is from a slide in-class for a boost along the x-axis. Our choice of x-axis / z-axis momentum is arbitrary, since we'll end up aligning our vectors with the direction of momentum in the end.

The inverse matrix is:

$$\begin{pmatrix} \frac{1}{\gamma-\gamma\beta^2} & \frac{\beta}{\gamma-\gamma\beta^2} & 0 & 0 \\ \frac{\beta}{\gamma-\gamma\beta^2} & \frac{1}{\gamma-\gamma\beta^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 40.62 \\ 40.62 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\gamma = 2.46$$
$$\beta = 0.901$$
$$|p_e| = p_x = \frac{40.62(1+\beta)}{\gamma - \gamma\beta^2} = 191.4$$

In [35]:
```
(m/2) * (1+beta) / (gamma - (gamma*beta**2))
```

Out[35]: `191.3783344124855`

Thus, the maximum momentum magnitude is $184 + 191 = 375$ GeV and the minimum is $|184 - 191| = 7$ GeV (natural units, GeV/c otherwise).

▶ Q2

The combined momentum and mass of the inital particles must equal the mass of each resultant particle.

$E_2 \ngtr E_1$
$E_1 \geq E_2$
$\min(E_1) = E_2$

$$E_A + E_B = \sum_{i=1}^{n} E_{C_i}$$
$$E^2 = m^2 + p^2$$
$$E = \sqrt{m^2 + p^2}$$
$$\sqrt{m_A^2 + p_A^2} + m_B = \sum_{i=1}^{n} m_{C_i}$$
$$p_A = \sqrt{\left(\sum_{i=1}^{n} m_{C_i} - m_B\right)^2 - m_A^2}$$

Below I calculate the momentum needed for each reaction

In [52]:
```python
# mass MeV, natural units c=1
m_pi = 140 #+/-
m_pi0 = 135 #just following pdg
m_p = 938
m_K0 = 497
m_S0 = 1193
m_n = 940

# i.
E = m_K0+m_S0-m_p
print(f'i. total energy: {E} MeV')
momentum = (E**2-m_pi**2)**(1/2)
print(f'i. pion momentum: {momentum:.0f} MeV \n')

# ii.
E = m_p + m_p + m_pi0 - m_p
print(f'ii. total energy: {E} MeV')
momentum = (E**2- m_p**2)**(1/2)
print(f'ii. proton momentum: {momentum:.0f} MeV \n')

# iii.
E = m_p+m_p+m_n-m_p
print(f'iii. total energy: {E} MeV')
momentum = (E**2-m_pi**2)**(1/2)
print(f'iii. pion momentum: {momentum:.0f} MeV \n')
```

```
i. total energy: 752 MeV
i. pion momentum: 739 MeV

ii. total energy: 1073 MeV
ii. proton momentum: 521 MeV

iii. total energy: 1878 MeV
iii. pion momentum: 1873 MeV
```

▶ Q3

Pions have odd-parity (-1), and the lack of spin (angular momentum) of the $\eta(549)$ meson means that the parity before and after decay is conserved/equivalent.

Thus, the parity of $\eta(549) = P_\pi^3 = (-1)^3 = -1$, and the forbidden decays have incorrect parity to occur

$$\eta \rightarrow \pi + \pi$$
$$-1 \neq (-1)(-1) = 1$$

Notice I do not specify types of pions, this is because they all have the same parity.

▶ Q4

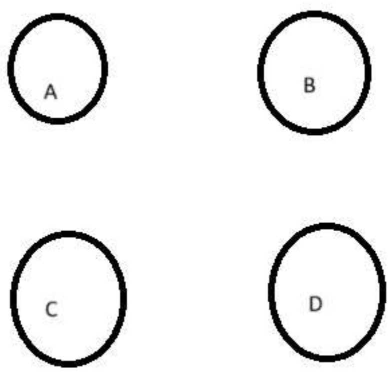
The Cayley table shows all elements and their products

	I	r	r^2	r^3	f_x	f_y	f_{AC}	f_{BD}
I	I	r	r^2	r^3	f_x	f_y	f_{AC}	f_{BD}
r	r	r^2	r^3	I	f_{AC}	f_{BD}	f_y	f_x
r^2	r^2	r^3	I	r	f_y	f_x	f_{BD}	f_{AC}
r^3	r^3	I	r	r^2	f_{BD}	f_{AC}	f_x	f_y
f_x	f_x	f_{BD}	f_{AC}	f_y	I	r^2	r^3	r
f_y	f_y	f_{AC}	f_x	f_{BD}	r^2	I	r	r^3
f_{AC}	f_{AC}	f_x	f_{BD}	f_y	r^3	r	I	r^2
f_{BD}	f_{BD}	f_y	f_{AC}	f_x	r	r^3	r^2	I

r represents a 90 degree rotation. Only 90, $90^2 = 180$, $90^3 = 270$, and $90^4 = I$ rotations are allowed in the square group. r can be clockwise or counterclockwise, but cw r is the same as ccw r^3, so it isn't really a new action. The same is true for "flipping" 180 degrees cw or ccw. The above table uses ccw r

Flips along the x-axis and y-axis through the center of the square also have closure (swap A,B with C,D or A,C with B,D), alongside flips on the diagonals (swap BC or AD).

Note that $f_x \cdot r = f_{AC}$, while $r \cdot f_x = f_{BC}$, so not all operators commute and the group is non-abelian



r -> f_x
B D -> A C
A C -> B D

f_x -> r
C D -> D B
A B -> C A