Problem 1

Adapted from Kittel & Kroemer, Chapter 4, problem 6 [Pressure of thermal radiation]

(a): 1 point

Show for a photon gas that:

$$p = -(\partial U/\partial V)_\sigma = -\sum_j s_j \hbar (d\omega_j/dV),$$

where s_j is the number of photons in the mode j.

The total energy is given by

$$U = \sum_j \hbar \omega_j s_j$$

Holding the entropy of the gas constant is equivalent to holding the number of photons in each state constant (this is stated in the textbook, and you will prove it later on this homework). Thus

$$p = -(\partial U/\partial V)_\sigma = -\sum_j s_j \hbar (d\omega_j/dV).$$

(a): 1 point

(b): 1 point

Show for a photon gas that:

$$d\omega_j/dV = -\omega_j/3V.$$

For the photon, $\omega_j = j\pi c/L = j\pi c V^{-1/3}$. Then

$$\begin{aligned} \frac{d\omega_j}{dV} &= \frac{d}{dV}(j\pi c/V^{-1/3}) \\ &= -\frac{1}{3} j\pi c/V^{-4/3} \\ &= -\omega_j/3V. \end{aligned}$$

(b): 1 point

Problem 1 continued on next page...

(c): 1 point

Show for a photon gas that:

$$p = U/3V.$$

Thus the radiation pressure is equal to $\frac{1}{3}\times$(energy density).

Combining the first and second results, we have

$$\begin{aligned} p &= -\sum_j s_j \hbar(-\omega_j/3V) \\ &= \frac{1}{3V}\sum_j s_j \hbar\omega_j \\ &= U/3V \end{aligned}$$

(c): 1 point

Problem 2

Kittel & Kroemer, Chapter 4, problem 11 [Heat capacity of solids in high temperature limit]: 4 points

Show that in the limit $T \gg \theta$ the heat capacity of a solid goes towards the limit $C_V \to 3Nk_B$, in conventional units. To obtain higher accuracy when T is only moderately larger than θ, the heat capacity can be expanded as a power series in $1/T$, of the form

$$C_V = 3Nk_B \times \left[1 - \sum_n a_n/T^n\right].$$

Determine the first nonvanishing term in the sum. Check your result by inserting $T = \theta$ and comparing with Table 4.2.
(Hint: you will want to Taylor expand the integrand $x^3/(e^x - 1)$. Also, in conventional units, $C_V = (\partial U/\partial T)_V$).

We showed in class the Debye result for the energy of a solid:

$$\begin{aligned} U &= \frac{3V\tau^4}{2\pi^2\hbar^3 v^3} \int_0^{\theta/T} dx \frac{x^3}{e^x - 1} \\ &= \frac{3Vk_B^4 T^4}{2\pi^2\hbar^3 v^3} \int_0^{\theta/T} dx \frac{x^3}{e^x - 1} \end{aligned}$$

It is easiest to solve this when we write the temperature dependence in terms of θ/T. The Debye temperature $\theta = \frac{\hbar v}{k_B}\left(\frac{6\pi^2 N}{V}\right)^{1/3}$, or $\theta^3 = \frac{\hbar^3 v^3}{k_B^3}\frac{6\pi^2 N}{V}$. We can then write

$$U = 9Nk_B T(T/\theta)^3 \int_0^{\theta/T} dx \frac{x^3}{e^x - 1}$$

In the limit $T \gg \theta$, the upper bound of the integral goes to zero. To evaluate this, we can try to Taylor expand the integrand around $x = 0$, since we are restricted to a small region there anyway:

$$\begin{aligned} U &= 9Nk_B T(T/\theta)^3 \int_0^{\theta/T} dx \frac{x^3}{x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \cdots} \\ &= 9Nk_B T(T/\theta)^3 \int_0^{\theta/T} dx \frac{x^2}{1 + \frac{1}{2}x + \frac{1}{6}x^2 + \cdots} \\ &= 9Nk_B T(T/\theta)^3 \int_0^{\theta/T} dx\, x^2 \left(1 - x/2 + x^2/12 + \cdots\right) \end{aligned}$$

If we only keep the leading term, this becomes

$$\begin{aligned} \lim_{\theta/T \to 0} U &= 9Nk_B T(T/\theta)^3 \int_0^{\theta/T} dx\, x^2 \\ &= 3Nk_B T(T/\theta)^3(\theta/T)^3 \\ &= 3Nk_B T. \end{aligned}$$

In conventional units, $C_V = (\partial U/\partial T)_V$, so in this limit

$$\lim_{\theta/T \to 0} C_V = 3Nk_B.$$

This is the classical result.

We can also evaluate the next leading term in our expansion of the energy above; let's call it U_2:

$$\begin{aligned} U_2 &= 9Nk_BT(T/\theta)^3 \int_0^{\theta/T} dx\, x^2\, (-x/2) \\ &= -\frac{9}{2}Nk_BT(T/\theta)^3 \int_0^{\theta/T} dx\, x^3 \\ &= -\frac{9}{8}Nk_BT(T/\theta)^3(\theta/T)^4 \\ &= -\frac{9}{8}Nk_B\theta. \end{aligned}$$

This is independent of T, so C_V vanishes to this order. We have to look at the next term, which we call U_3:

$$\begin{aligned} U_3 &= 9Nk_BT(T/\theta)^3 \int_0^{\theta/T} dx\, x^2 \left(x^2/12\right) \\ &= \frac{9}{12}Nk_BT(T/\theta)^3 \int_0^{\theta/T} dx\, x^4 \\ &= \frac{3}{20}Nk_BT(T/\theta)^3(\theta/T)^5 \\ &= \frac{3}{20}Nk_B\theta^2/T. \end{aligned}$$

So we have

$$(C_V)_3 = (\partial U_3/\partial T)_V = -\frac{3}{20}Nk_B\theta^2/T^2,$$

or

$$C_V = 3Nk_B(1 - \frac{1}{20}(\theta/T)^2 + \cdots)$$

For 1 mole of solid, $Nk_B = 8.314$ J/mol·K (the ideal gas constant!) Plugging this in to our result with $T = \theta$, we have $C_V = 23.7$ J/mol·K. The value given in Table 4.2 is 23.74 J/mol·K at this temperature. Our approximation has done quite well, even at a value of θ/T that is not very small.

Problem 3

Kittel & Kroemer, Chapter 4, problem 13 [Energy fluctuations in a solid at low temperatures]: 4 points

Consider a solid of N atoms in the temperature region in which the Debye T^3 model is valid. The solid is in thermal contact with a heat reservoir. Use the results on energy fluctuations from Chapter 3 to show that the root mean square fractional energy fluctuation $\mathcal{F}$ is given by

$$\mathcal{F}^2 = \langle(\epsilon - \langle\epsilon\rangle)^2\rangle/\langle\epsilon\rangle^2 \approx \frac{0.07}{N}\left(\frac{\theta}{T}\right)^3.$$

Suppose that $T = 10^{-2}$ K, $\theta = 200$ K, and $N \approx 10^{15}$ for a particle 0.01 cm on a side; then $\mathcal{F} \approx 0.02$. At 10^{-5} K the fractional fluctuation in energy is of the order of unity for a dielectric particle of volume 1 cm^3.

From Chapter 3 (problem 4), we know that mean square fluctuation of the energy in a system at fixed volume in thermal contact with a reservoir at temperature τ is given by

$$\langle(\epsilon - \langle\epsilon\rangle)^2\rangle = \tau^2(\partial U/\partial\tau)_V.$$

The term in parentheses is simply the heat capacity C_V, so we can write the fractional energy fluctuation as

$$\mathcal{F}^2 = \tau^2 C_V/\langle\epsilon\rangle^2.$$

In the region in which the Debye T^3 model is valid, we have the results $C_V = \frac{12\pi^4 N}{5}(\tau/k_B\theta)^3$ and $U \equiv \langle\epsilon\rangle = \frac{3\pi^4 N\tau^4}{5(k_B\theta)^3}$. Thus

$$\begin{aligned}
\mathcal{F}^2 &= \tau^2 \frac{\frac{12\pi^4 N}{5}(\tau/k_B\theta)^3}{\left(\frac{3\pi^4 N\tau^4}{5(k_B\theta)^3}\right)^2} \\
&= \frac{20(k_B\theta/\tau)^3}{3\pi^4 N} \\
&= \frac{20}{3\pi^4 N}\left(\frac{\theta}{T}\right)^3 \\
&\approx \frac{0.07}{N}\left(\frac{\theta}{T}\right)^3.
\end{aligned}$$

Problem 4

Kittel & Kroemer, Chapter 4, problem 17 [Entropy and occupancy]: 4 points

We argued in this chapter that the entropy of the cosmic black body radiation has not changed with time because the number of photons in each mode has not changed with time, although the frequency of each mode has decreased as the wavelength has increased with the expansion of the universe. Establish the implied connection between entropy and occupancy of the modes, by showing that for one mode of frequency ω the entropy is a function of the photon occupancy $\langle s \rangle$ only:

$$\sigma = \langle s+1 \rangle \ln\langle s+1 \rangle - \langle s \rangle \ln\langle s \rangle.$$

It is convenient to start from the partition function.

Taking the problem statement at its word, let's start with the partition function (of a single mode):

$$Z = \sum_s e^{-s\hbar\omega/\tau} = \frac{1}{1 - e^{-\hbar\omega/\tau}},$$

and recall that the average photon occupancy is defined as

$$\langle s \rangle \equiv \frac{\sum_s s e^{-s\hbar\omega/\tau}}{Z} = \frac{1}{e^{\hbar\omega/\tau} - 1}.$$

We might as well also write

$$\begin{aligned}
\langle s+1 \rangle = \langle s \rangle + 1 &= \frac{1}{e^{\hbar\omega/\tau} - 1} + 1 \\
&= \frac{1}{e^{\hbar\omega/\tau} - 1} + \frac{e^{\hbar\omega/\tau} - 1}{e^{\hbar\omega/\tau} - 1} \\
&= \frac{e^{\hbar\omega/\tau}}{e^{\hbar\omega/\tau} - 1} \\
&= \frac{1}{1 - e^{-\hbar\omega/\tau}} \\
&= Z.
\end{aligned}$$

Interesting! We should also see what the natural logarithms of these quantities look like:

$$\begin{aligned}
\ln\langle s \rangle &= \ln \frac{1}{e^{\hbar\omega/\tau} - 1} \\
\ln\langle s+1 \rangle = \ln Z &= \ln \frac{e^{\hbar\omega/\tau}}{e^{\hbar\omega/\tau} - 1} \\
&= \ln\langle s \rangle + \hbar\omega/\tau.
\end{aligned}$$

This last equality will allow us to write $\hbar\omega/\tau = \ln\langle s+1 \rangle - \ln\langle s \rangle$ when we need to. With these in hand, we can move to calculate the entropy.

Problem 4 continued on next page. . .

$$\begin{aligned}
\sigma &= \frac{\partial}{\partial \tau}(\tau \ln Z) \\
&= \ln Z + \frac{\tau}{Z}\frac{\partial Z}{\partial \tau} \\
&= \ln\langle s+1\rangle + \frac{\tau}{Z}\frac{\partial}{\partial \tau}\sum_s e^{-s\hbar\omega/\tau} \\
&= \ln\langle s+1\rangle + \frac{1}{\tau Z}\sum_s (s\hbar\omega)e^{-s\hbar\omega/\tau} \\
&= \ln\langle s+1\rangle + \frac{\hbar\omega}{\tau}\langle s\rangle \\
&= \ln\langle s+1\rangle + (\ln\langle s+1\rangle - \ln\langle s\rangle)\langle s\rangle \\
&= (\langle s\rangle + 1)\ln\langle s+1\rangle - \langle s\rangle \ln\langle s\rangle \\
&= \langle s+1\rangle \ln\langle s+1\rangle - \langle s\rangle \ln\langle s\rangle.
\end{aligned}$$

This establishes that the entropy depends only on the occupancy of the modes, rather than on ω and τ independently.

Problem 5

Kittel & Kroemer, Chapter 4, problem 18 [Isentropic expansion of photon gas]

Consider the gas of photons of the thermal equilibrium radiation in a cube of volume V at temperature τ. Let the cavity volume increase; the radiation pressure performs work during the expansion, and the temperature of the radiation will drop. From the result for the entropy we know that $\tau V^{1/3}$ is constant in such an expansion.

(a): 2 points

Assume that the temperature of the cosmic black-body radiation was decoupled from the temperature of the matter when both were at 3000 K. What was the radius of the universe at that time, compared to now? If the radius has increased linearly with time, at what fraction of the present age of the universe did the decoupling take place?

We know that now, the cosmic black-body radiation has a temperature of $T \approx 2.725$ K and the observable universe has a radius of 46.5×10^9 light-years. If the expansion has occurred at constant photon entropy since decoupling, we know $T_i V_i^{1/3} = T_f V_f^{1/3}$; since volume goes like the cube of the radius this is equivalent to saying that $T_i R_i = T_f R_f$. Then we can calculate:

$$R_i = R_f \frac{T_f}{T_i} = (46.5 \times 10^9 \text{ light-years}) \frac{2.725}{3000} = 42 \times 10^6 \text{ light-years}.$$

If the radius has increased linearly with time, this means that $R_i / R_f = t_i / t_f$, so

$$\frac{t_i}{t_f} = \frac{42 \times 10^6 \text{ light-years}}{46.5 \times 10^9 \text{ light-years}} \approx 10^{-3}.$$

Thus decoupling took place at around 1/1000 the present age of the universe, based on this simple model.

(a): 2 points

(b): 2 points

Show that the work done by the photons during the expansion is

$$W = (\pi^2 / 15\hbar^3 c^3) V_i \tau_i^3 (\tau_i - \tau_f)$$

The subscripts i and f refer to the initial and final states.

Recall that for a photon gas, the total energy is $U = \frac{\pi^2}{15\hbar^3 c^3} V \tau^4$. For this isentropic process, we know that $\tau^3 V$ is a constant; let's say $V\tau^3 = V_i \tau_i^3$. Then throughout this process, $dU = \frac{\pi^2}{15\hbar^3 c^3} V_i \tau_i^3 \, d\tau$. We can integrate this from initial to final state to find the change in energy during the expansion:

$$\Delta U = \int_i^f dU = \frac{\pi^2}{15\hbar^3 c^3} V_i \tau_i^3 \int_{\tau_i}^{\tau_f} d\tau = \frac{\pi^2}{15\hbar^3 c^3} V_i \tau_i^3 (\tau_f - \tau_i).$$

Since there was no change in entropy during this process, the energy changed entirely due to the work done by the photon gas: $\Delta U = -W$. This shows the result.

(b): 2 points