Harmonic Motion

Experiment Seven

Physics 191
Michigan State University

Before Lab

- Read the lab guide and attempt all theory questions
- Review error analysis concepts in the Experiment One and Experiment Two lab guides as needed
- Watch a short Crash Course video on Simple Harmonic Motion

Experiment Overview

This is a two-week experiment. You are expected to show all of your work for any calculations. It is best to perform any algebra symbolically. All tables and plots should include a caption, and each section needs a short introduction providing context for your work.

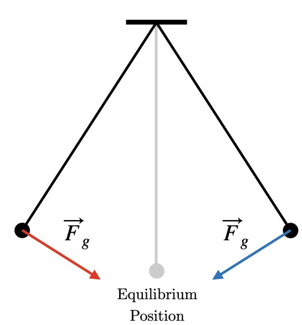
- Write a lab report that can be understood by a general scientific audience
- Study the properties of oscillations described by Hooke's law
- Characterize the stiffness of a spring by measuring its spring constant in different situations
- Observe how friction manifests in periodic motion

1 Theory

Oscillations are ubiquitous in nature. The action of a heartbeat or firing of a neuron, the business cycles in economics, the operation of electrical grids, and the seismic activity of the earth are just a few examples of systems which undergo oscillation. An oscillation is the periodic (repetitive) variation of some quantity. Physical oscillation is commonly referred to as **harmonic motion**, a name originating from the first investigations of musical harmony. Early experiments performed by Pythagoras and his students were the first to quantify the relationship between physical motion and sound[1]. In this lab, you will study the behavior of a weight attached to a spring. Despite the simplicity of the system, the corresponding analysis extends far beyond the realm of springs (and even mechanics).

1.1 Simple Harmonic Motion

In the pendulum motion lab, you were introduced to the concept of a restoring force. When stationary, the pendulum hangs in its equilibrium position. If displaced a small amount to the left, the gravitational force acts to pull the bob back to equilibrium. A displacement to the right does the same thing. In this example, the gravitational force behaves as a **restoring force** as it always acts to restore the bob to its equilibrium position. The defining characteristic of **simple harmonic motion** is that the force responsible for motion is a restoring force which depends *linearly* on the displacement of the weight from equilibrium. The spring force happens to meet both of these criteria in most physical situations.



[1]More information on Pythagoras's experiments can be found on the Pythagoreanism Wikipedia page.

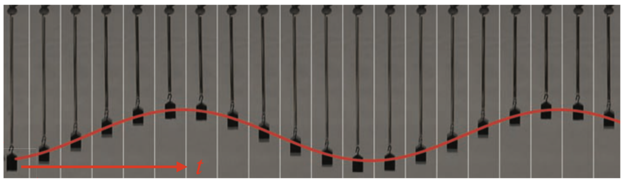


Figure 1: Successive snapshots of a weight on a spring show that the position of the weight varies sinusoidally in time (image source: *Bauer & Westfall*, 2014).

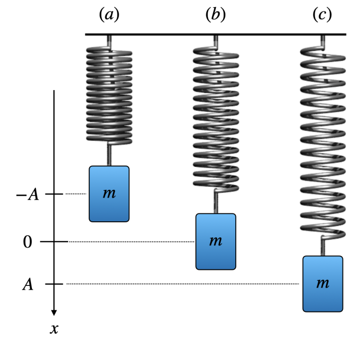


Figure 2: The position of the weight at the minimum, equilibrium, and maximum positions labeled on the x-axis.

Hooke's law is the empirical description of the force exerted by a spring. Consider the spring shown in Figure 1. The weight oscillates because the spring exerts a restoring force as it extends and compresses. This force is described by **Hooke's law** which, in one dimension, has the form

$$F_s = -kx \tag{1}$$

where k is a number characterizing the stiffness of the spring called the **spring constant** and x is the displacement from equilibrium. Figure 2 shows three important positions of the weight. The x-axis is oriented vertically and pointed downward[2].

(a) The spring is fully compressed with the weight at its minimum position, $x = -A$. Here, the weight experiences a force $F_s = -k(-A) = kA$. This amounts to a *push* towards equilibrium.

(b) The spring is in its equilibrium position with the weight at $x = 0$. According to Equation 1, the spring force at this point is $F_s = 0$. While the spring exerts no force at this point, the weight's inertia causes it to continue moving through the equilibrium position.

(c) The spring is fully extended with the weight at its maximum position, $x = A$. Here, the weight experiences a force $F_s = -kA$. This amounts to a *pull* towards equilibrium.

This process repeats periodically, with the weight continuously moving between $-A$ and A. The number A is called the **amplitude**, which is a measure of the distance traveled in one cycle. The **period** is the time of one complete cycle and is typically denoted as T. The **ordinary frequency** is the number of cycles per second and is directly related to the period by

$$f = \frac{1}{T}. \tag{2}$$

With a qualitative understanding of simple harmonic motion, we are in a position to develop a quantitative model of the displacement as a function of time, $x(t)$. The first step is to identify any involved forces and use our friend $F_{net} = ma$. Consider the simplest case in which the only force involved is the spring force, $F_s = -kx$. This gives the relationship

$$F_{net} = ma = -kx. \tag{3}$$

This equation can be used to determine the displacement as a function of time, but doing so requires some footwork. The first step is to recognize that the acceleration is the second derivative of the position, $a = d^2x/dt^2$. Substituting this into Equation 3 gives

$$m\frac{d^2x}{dt^2} = -kx. \tag{4}$$

[2]You might associate horizontal motion with x and vertical motion with y. This choice is arbitrary. For motion in one dimension, it is common to label the coordinate as x regardless of the orientation of the system.

Taking one final step makes the relationship more clear, writing the previous equation as

$$\frac{d^2x(t)}{dt^2} = -\frac{k}{m}x(t) \tag{5}$$

where x is written as $x(t)$ to explicitly show that the position is a function of time. This is an ordinary differential equation with a form identical to that for the angle $\theta(t)$ in the pendulum motion lab. The equation above says that $x(t)$ must be a function whose second derivative equals itself times a *negative* constant. The sine and cosine functions behave in this way (you should verify this for yourself).

With most differential equations, the best way to find a solution is to guess and check. We have a hunch that the solution is sinusoidal, which is further supported by the red curve traced in Figure 1. Let's try

$$x(t) = A\cos(\omega_0 t) \tag{6}$$

where ω_0 is the angular frequency. In order to check if our guess is valid, we can take derivatives of $x(t)$ and see if Equation 5 is satisfied:

$$\begin{aligned} \frac{dx(t)}{dt} &= \omega_0 A \sin(\omega_0 t) \Rightarrow \\ \frac{d^2x(t)}{dt^2} &= -\omega_0^2 A\cos(\omega_0 t) = -\omega_0^2 x(t). \end{aligned} \tag{7}$$

The trick is to compare Equations 5 and 7, as they are both equal to the second derivative of the position:

$$\frac{d^2x(t)}{dt^2} = -\omega_0^2 x(t) = -\frac{k}{m}x(t). \tag{8}$$

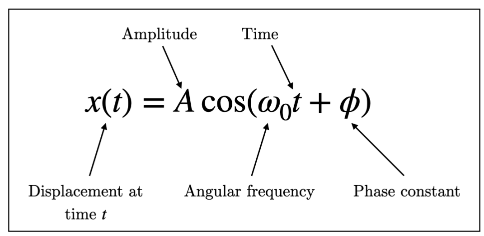
This shows that Equation 6 is a valid solution, with the added bonus of finding a relationship between the angular frequency, spring constant, and mass (shown in red). Thus, the **angular frequency** of our oscillating mass is

$$\omega_0 = \sqrt{\frac{k}{m}}. \tag{9}$$

There is *one* problem left to address. According to Equation 6, the position at $t = 0$ is fixed: $x(0) = A\cos(0) = A$. If we start measuring when the weight is not exactly at its maximum value, then $x(0) \neq A$. This is illustrated in Figure 3, where $t = 0$ corresponds to a value less than the maximum. To account for this shift in position, we introduce a number called a **phase constant** which is typically denoted by a lowercase Greek phi, ϕ. With this modification, we arrive at a general solution for the position as a function of time:

$$x(t) = A\cos(\omega_0 t + \phi). \tag{10}$$

With this equation, the position of the weight can be predicted at any point in time as long as long as A, ω_0, and ϕ are known.



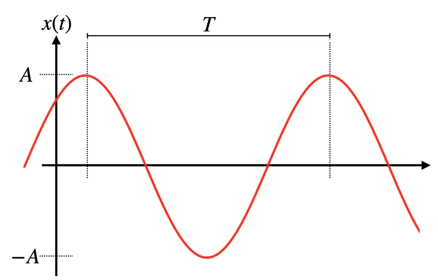


Figure 3: A graph of $x(t)$ with the amplitude A and period T indicated.

1.2 Damped Harmonic Motion

The motion described in the previous section is called "simple" because the behavior does not change for different masses and springs. Regardless of the situation, an object in simple harmonic motion will continue to oscillate indefinitely. While this is a good approximation for many systems, a more accurate model must consider the effects of dissipative processes. Without an external energy source, all macroscopic oscillations will eventually decay as stored energy is converted into heat. This decay reduces the oscillation amplitude in a process called **damping**.

The effects of damping can be introduced to the model by adding a damping force that resists motion. The damping force can be approximated as $F_d = -bv$ where b called the damping coefficient and v is the

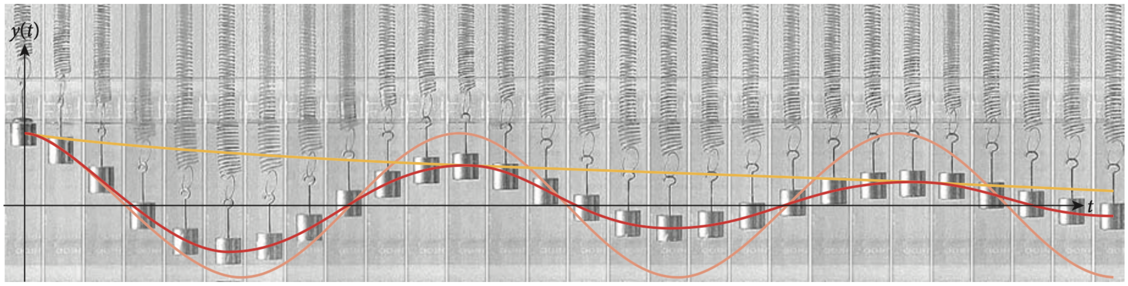


Figure 4: Successive snapshots of a weight on a spring oscillating in water, where the amplitude rapidly decays (image source: *Bauer & Westfall*, 2014).

velocity of the object. Circling back to $F_{net} = ma$, we can follow the same procedure as in the previous section, where the total force on the mass is now

$$\begin{aligned} F_{net} &= F_d + F_s \\ ma &= -bv - kx \\ m\frac{d^2x}{dt^2} &= -b\frac{dx}{dt} - kx \end{aligned}$$

The last line gives the relationship necessary to determine the position of a damped oscillator as a function of time. By defining $\gamma = b/2m$ and using the definition $\omega_0^2 = k/m$ from above, the resulting differential equation can be expressed compactly as

$$\frac{d^2x(t)}{dt^2} = -2\gamma\frac{dx(t)}{dt} - \omega_0^2 x(t). \tag{11}$$

In the last line, we have acknowledged that the velocity is the time derivative of position, $v = dx/dt$. This is the damped analog of Equation 5 for the simple harmonic oscillator. When damping is present, the properties of the spring, mass, and friction significantly affect the character of $x(t)$. Under the conditions of this experiment, you will study **underdamped** oscillations. The equation for the position as a function of time resembles Equation 10, but now the *amplitude* is also a function of time, giving $x(t) = A(t)\cos(\omega t + \phi)$. It turns out that the amplitude decays exponentially as $A(t) = A_0 e^{-\gamma t}$ where A_0 is the maximum amplitude. This gives the explicit form of the position for a damped oscillator,

$$x(t) = A_0 e^{-\gamma t}\cos(\omega t + \phi), \tag{12}$$

which is the red curve shown above in Figure 4. Let's take a closer look at the parameters in this equation:

- A_0 is the maximum amplitude. This value is reached only in the first oscillation, as all subsequent oscillations have a decaying amplitude.

- γ is the **damping rate**, defined above as

  $$\gamma = \frac{b}{2m} \tag{13}$$

  where γ is a lowercase Greek gamma. The damping rate is directly related to the oscillation decay time, with units of s^{-1}.

- ω is the *system* angular frequency. This is the measurable frequency of the oscillator. It is related to the undamped angular speed, ω_0, and the damping angular speed, γ, by

  $$\omega = \sqrt{\omega_0^2 - \gamma^2} = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \tag{14}$$

  This shows that the oscillation frequency is always reduced when compared with the undamped frequency, with the reduction dependent upon the damping coefficient (b) and the mass of the oscillator.

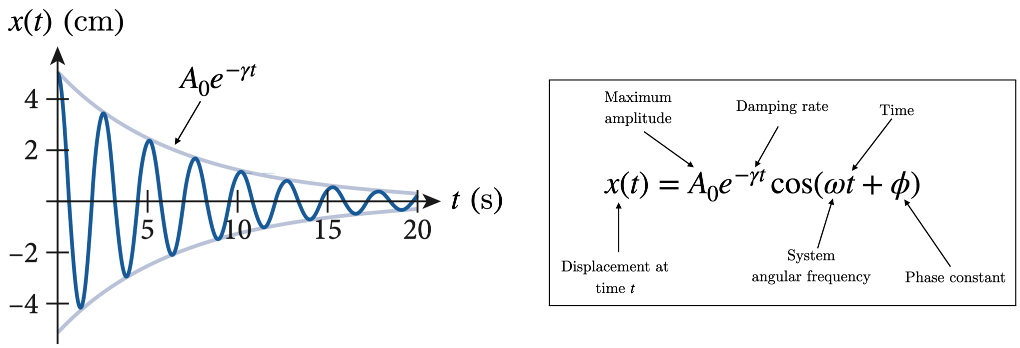


Figure 5: The position of a underdamped harmonic oscillator as a function of time, with the exponential damping curve shown in light blue (image source: *Bauer & Westfall*, 2014).

The graph in Figure 5 shows the position as a function of time with the effects of damping. The sinusoidal nature is still evident, but now the curve is "enveloped" by an exponential function which decays with time (light blue curve). The rate at which oscillations decay is extremely important to engineers in many fields. The development of car suspension, rocket engines, bridges and buildings, and audio systems are just a few examples which require careful damping rate tuning.

1.3 Damping Rate Estimation

There is a useful method for estimating the damping rate for a decaying oscillation that only requires a graph of the position as a function of time. Take a close look at decaying amplitude of the graph in Figure 5, represented by $A(t) = A_0 e^{-\gamma t}$. Dimensional analysis tells us that the damping rate, γ, must have units of s^{-1} (the argument of an exponential function *must* be unitless). It is common to define a corresponding characteristic **time constant**

$$\tau = \frac{1}{\gamma} \tag{15}$$

where τ is a lowercase Greek tau. This is analogous to the relationship between the period and the ordinary frequency, $T = 1/f$. With this definition, the amplitude can be expressed as

$$A(t) = A_0 e^{-t/\tau}. \tag{16}$$

This curve is shown to the right. Consider the value of $A(t)$ at two different times:

- At $t = 0$, $A(0) = A_0$. This is the maximum amplitude in this example.
- At $t = \tau$, $A(\tau) = A_0 e^{-\tau/\tau} = A_0 e^{-1}$. Recalling that the number $e \approx 2.718$, its inverse is $e^{-1} = 0.368$. This gives $A(\tau) = 0.368 A_0$. In other words, the time at which the amplitude is 36.8% of its initial value *is* the time constant.

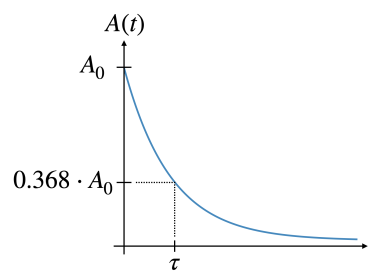


Figure 6

Using this process, the time constant can be used in conjunction with Equation 15 to determine the damping rate. A table summarizing important quantities related to oscillation and damping times is given below.

Quantity	Relationship
Ordinary Frequency	$f = 1/T$
Angular Frequency	$\omega = 2\pi f$
Damping Rate	$\gamma = b/2m$
Time Constant	$\tau = 1/\gamma$

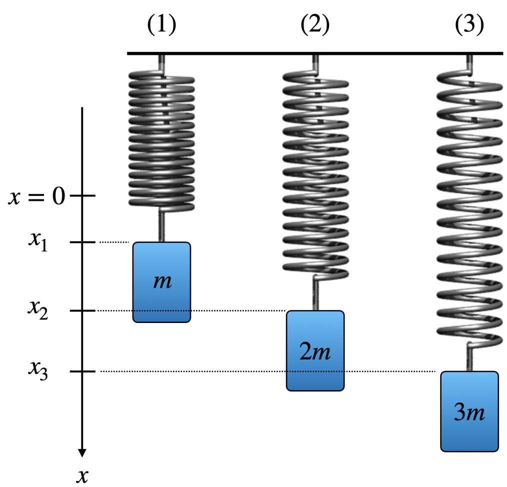
1.4 Theory Questions

1. It is important to recognize that the spring force depends on the displacement from equilibrium. A more general expression for Hooke's law is

$$F_s = -k\Delta x$$

where $\Delta x = x - x_0$ is the amount by which the spring is stretched. We typically define coordinates (the measurement system) such that the equilibrium position is $x_0 = 0$ to obtain the form $F_s = -kx$.

The diagram below shows three cases of an identical spring suspending different masses. The masses are *stationary* with displacements indicated on the x-axis. The position $x = 0$ corresponds to the end of the unstretched spring.



a. (1 pt) A free-body diagram is a tool to help you write an equation for F_{net}. Draw a free-body diagram showing the forces acting on an arbitrary mass, m.

b. (1 pt) Write down an equation for net force acting on a mass. With this, solve for the acceleration, a, in terms of the spring constant k, the displacement x, the gravitational acceleration g, and the mass.

c. (2 pt) What is the acceleration when the mass is at rest in the equilibrium position? Use this fact and your result from the previous question to obtain an equation that gives the displacement in terms of g, k, and m.

d. (1 pt) Use your equation to predict the displacement you would measure in a lab if you hung a 1 kg mass from a spring with $k = 7$ N/m. Use $g = 9.8$ m/s^2 as the gravitational acceleration.

e. (1 pt) Sketch a graph of the displacement vs. mass. Indicate the three points (m, x_1), $(2m, x_2)$, and $(3m, x_3)$ corresponding to the picture above. What can be said about the slope of the graph?

f. (1 pt) How would the graph change if the positions were measured from the top of the spring instead of the position labeled $x = 0$? Does changing the starting point of the measurement change the slope of the graph sketched in the previous question?

g. (1 pt) Does the force exerted by a spring depend on the choice of a coordinate system?

2. (1 pt) If a large weight is attached to a small spring, the spring will overstretch and deform. The deformation changes the behavior of the spring. With this observation, do you think Hooke's law is valid for any displacement?

3. (1 pt) Is gravity necessary for a mass on a spring to oscillate?

2 Experimental Setup

In this experiment, you will use electronic equipment to measure the mechanical properties of a system comprised of a spring and a set of masses. A digital oscilloscope is a tool that produces a real-time visualization of an input voltage. If a damped oscillating voltage is fed into an oscilloscope, the device can generate an image like that shown below, where the vertical axis is voltage and the horizontal is time. This allows for digital measurements of the amplitude or period (labeled as T Figure 7).

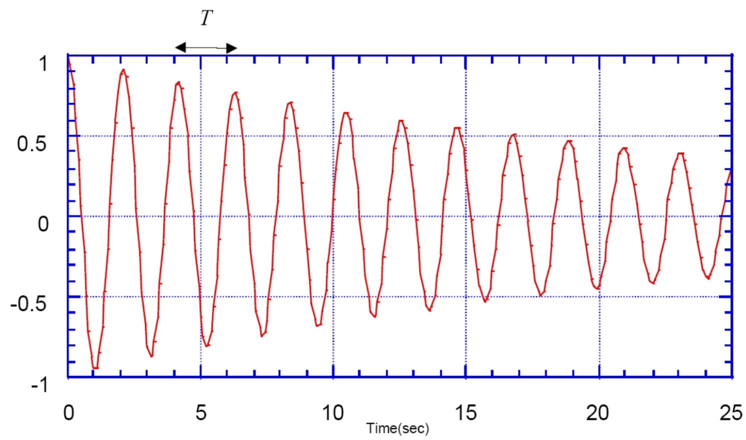


Figure 7

1. (1 pt) For the oscillation shown in Figure 7, determine the peak amplitude A_0 (no units are necessary), the period T, the ordinary frequency f, and the angular frequency ω.
2. (1 pt) Does the signal in Figure 7 look like a damped or an undamped oscillation?

A **dynamic force transducer** is an interface between the spring (a mechanical object) and the digital oscilloscope. The dynamic force transducer (DFT) has an arm that extends outward and attaches directly to the spring. When a force is exerted on the arm, such as the force from a spring, the DFT generates a voltage that is directly proportional to the applied force.

DFT Operation

- The side of the DFT has two knobs:
  - *Sensitivity* should always be turned to its maximum value (clockwise).
  - *Zero Adjust* allows the output voltage to be shifted to 0 V when a force is applied to the arm. This is akin to the "tare" setting on a digital scale.
- If you do not see a signal from the DFT, make sure the device is switched to ON. If this does not work, your TA can check the battery.

Oscilloscope Operation

The oscilloscope has a myriad of settings and features, but you will only need to change a few things throughout the experiment. Changing any settings on the device only effects what is observed on the screen, nothing about the input signal changes.

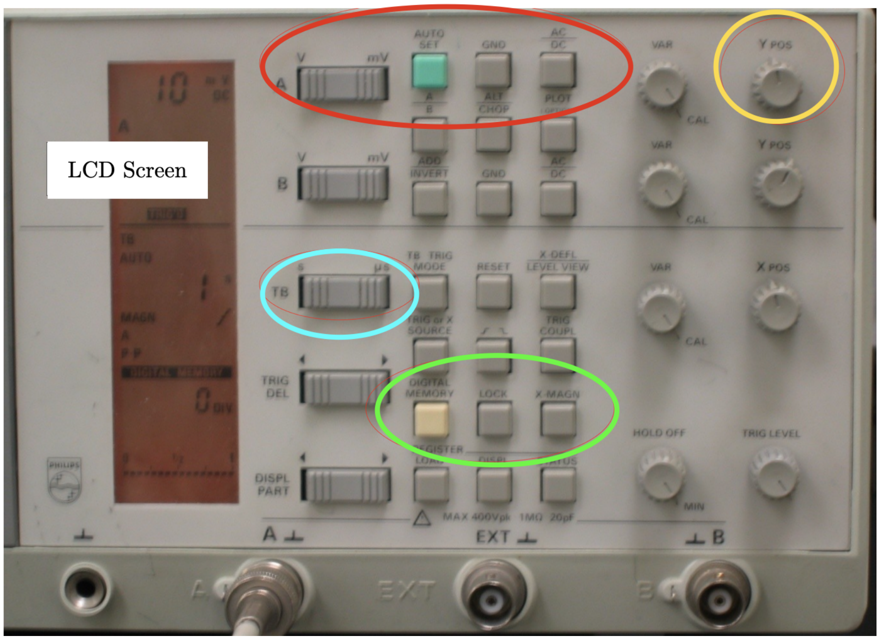


Figure 8: The oscilloscope control panel with the relevant controls circled.

Oscilloscope Setup & Adjustments

- A BNC cable connects the DFT to the oscilloscope. Ensure the the cable is attached to channel A.
- Make sure the input is set to channel A by pressing the "$\frac{\text{A}}{\text{B}}$" button.
- Ensure that the VAR knob is pointed at "CAL" and left alone.
- Turn on the oscilloscope with the green "POWER" button.
- Press the "$\frac{\text{AC}}{\text{DC}}$" button (circled in red) until the display shows "DC."
- Press the "DIGITAL MEMORY" button (circled in green).
- Use the "TB" control (circled in cyan) to adjust the time base. This is the scale for the horizontal axis which corresponds to time. The time on the display indicates the amount of time corresponding to the 1 cm grid on the output screen. Adjust this setting to "0.5 s."
- The voltage scale operates in the same way, but the changes are to the vertical axis. To adjust the voltage scale, use the leftmost control circled in red, labeled as "A V – mV." Set the value to "0.1 V."
- *Note:* you are encouraged to play with the time and voltage scales, choosing values which give the best presentation on the output screen.
- The "XMAGN" button is the rightmost button circled in green. Press this button until the display shows [- - - - - - - - -] so the device continuously collects new data. This may change the TB setting. Readjust to the value to "0.4 s."
- Press "GND" (circled in red) to ground the input, setting it to 0 V. Turn the "X-POS" and "Y-POS" knobs to center the output line on the screen. **Do not** touch these knobs for the rest of the experiment. This ensures consistency of values on the output screen.
- Press "GND" again so the signal from the DFT is displayed on the output screen. As the force applied to the DFT changes, the signal on the screen will move up or down. If the line moves beyond the output screen, adjust the DFT "Zero Adjust" knob.

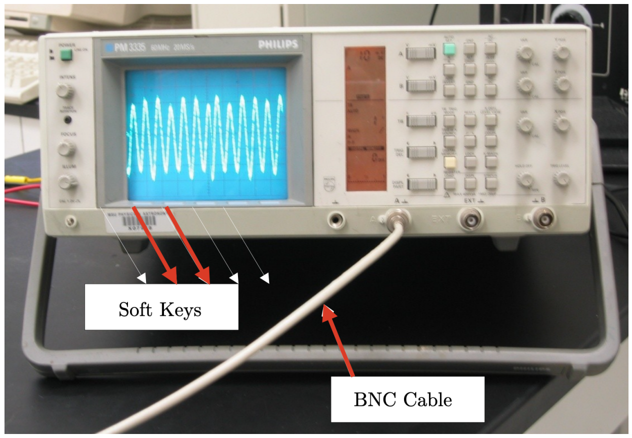


Figure 9

Oscilloscope Measurements

The oscilloscope has horizontal and vertical cursors (lines) that allow for measurements of voltage and time intervals. The cursors are setup and moved using the "soft-keys" which are shown in the Figure 9 (the surface of the buttons point toward the ceiling).

Inspect the main screen. If there is no text at the bottom of the screen, press one of the blue soft-keys below the screen to make the text appear. If one of the soft-keys has "RETURN" displayed above it, keep pressing it until it no longer says "RETURN" to get back to the top menu. You should see:

CURSORS SETTINGS TEXT OFF

- To display the measurement cursors, first press the "MODE" soft-key. Toggle "V-CURS" and "T-CURS" to show or hide the lines. The press "RETURN" to move to the previous options.
- In order to measure time intervals, you will use the T-cursors which display as two vertical lines. The horizontal distance between the two lines corresponds to an amount of time, which is displayed on the top of the screen.
- Use the soft-keys to move the two lines to the left or right. These cursors allow you to measure the period of oscillation. By aligning the cursors to adjacent peaks, the period can be determined.
- Instead of determining the frequency using the value displayed on the screen, calculate the value using $f = 1/T$. This will be more accurate than using the displayed value.

3 Static Measurement of the Spring Constant

The first set of measurements will be made without the electronic devices. The multipart theory question was written to help you prepare for this experiment. Your goal is to use Hooke's law to determine the spring constant, k, by making a series of measurements with the weight stationary. The spring will stretch by different amounts depending on the suspended mass. The displacements can be measured directly with a ruler or meter stick.

1. (1 pt) Write an equation that relates the mass, spring constant, and displacement (this was the subject of the first set of theory questions).
2. (1 pt) Does the spring constant depend on the mass suspended by the spring?
3. (3 pt) Describe the method you will use to make your measurements and analyze the results. Be sure to include the dependent and independent variables in your experiment. Think about a plot you can make to determine the value of k from your measurements of mass and displacement.

4. Organize a table and make your measurements. The mass of the spring can be neglected, but the hook *must* be included in all measurements of mass.

5. (3 pt) Generate a plot of your data using curve.fit and determine $k \pm \delta k$. There are multiple ways this can be done. You are free to choose a method. Be sure to include horizontal and vertical error bars corresponding to measurement uncertainties. **Show your error bar calculations.**

6. (1 pt) Predict the undamped angular frequencies, ω_0, using your determined value of k for each different mass.

4 Simple Harmonic Motion

In this part of the lab, you will experiment with simple harmonic motion. The objective is to determine the spring constant through different methods and compare the precision. While there is not a complete absence of damping, the effects are minimal, so Equation 9 can be used to relate the angular frequency to the spring constant. You will use two different methods to obtain data:

I. Manual measurements of the period using oscilloscope T-cursors

II. Automatic measurements of the DFT output voltage using National Instruments digital converter

4.1 Manual Measurements

The first set of measurements will be done manually using the oscilloscope. The T-cursors give you the ability to *directly* measure the period of one or more oscillations. **Tip**: use the "LOCK" button (circled in green in Figure 8) to freeze the screen once nice oscillations are observed (otherwise the signal actively updates).

1. (2 pt) Prove that the spring constant can be expressed in terms of the period as

$$k = \frac{4\pi^2}{T^2}m.$$

Show all of your steps.

2. (6 pt) Determine the spring constant for oscillations with three different masses (approximately 100 g, 150 g, and 200 g). Then calculate your best estimate, $\bar{k} \pm \delta\bar{k}$. Explain your process. *Hint:* the uncertainty in the mean is the standard error, $\delta\bar{k} = \sigma/\sqrt{N}$.

4.2 Automatic Measurement

The DFT output is sent both to the oscilloscope and an analog-to-digital converter box attached to your computer. The program "Force Transducer 2017.vi" samples the voltage from the DFT and generates an Excel spreadsheet containing the voltage as a function of time. The objective of this part of the experiment is to determine the spring constant by fitting a curve to the voltage data.

1. Login to the computer at your bench using password **Physics191**. Locate the "U" (user) drive and create a folder with a unique name to store your data.

2. Click the "Force Transducer 2017.vi" icon on the desktop. Ensure that "Total number of Individual Voltages read" is set to 1000, and the "Rate (V readings per Second" is set to 100. This amounts to a 10 second measurement.

3. (1 pt) Suspend a mass of around 200 g from your spring and set it into motion. Click the "Run" icon (arrow pointing to the right) to begin measuring. When the measurement is completed, you will be prompted to save the output file. Save it in your user folder. Open the spreadsheet and add a new column for time. The existing time column is not well-formatted for analysis. If 1000 measurements are made over 10 seconds, what is the time interval between voltages?

4. (2 pt) Plot your data in curve.fit and fit a damped oscillator function. Add a proper title and axes labels and include your plot in your report. From your angular frequency fit parameter, calculate the spring constant. *Note:* because very little damping is present, $\omega = \omega_0$ (remember, $\omega_0 = \sqrt{k/m}$).

5. (2 pt) The value of k obtained in this method depends on a calculation of two numbers with uncertainties: $\omega \pm \delta\omega$ and $m \pm \delta m$. Use uncertainty propagation to calculate

$$\delta k = \sqrt{\left(\frac{\partial k}{\partial \omega} \cdot \delta\omega\right)^2 + \left(\frac{\partial k}{\partial m} \cdot \delta m\right)^2}.$$

6. (6 pt) **Make a summary table** containing the spring constants obtained in the static measurement method, the manual measurement method, and the automatic measurement method. Include k, δk, and $\delta k/k$. Which method was the most precise? How can you tell?

5 Damped Harmonic Motion

For the final part of the lab, you will observe the effects of damping by attaching a styrofoam plate to your oscillating mass. The plate introduces a significant amount of drag, damping the oscillations. The objective of this part of the experiment is to determine the time constant τ and the damping rate γ. Once again, you will make use of the analog-to-digital converter to obtain the amplitude as a function of time.

1. (4 pt) You will analyze the motion of two different masses. Be sure to include the mass of the plate. Follow the same procedure from the previous section record and plot the amplitude as a function of time for each mass. Include the total mass in the plot titles.

2. (2 pt) Use your fit results to determine the time constant and its uncertainty for each mass. Explain what this number quantifies and describe its relationship to the mass.

3. (2 pt) Use a pen to label τ on each plot. Sketch the exponential decay curve enveloping the oscillation (see Figure 5).

4. (2 pt) Write a short summary of the lab, including the most important concepts you learned and any difficulties experienced in the process.