Problem 1

Kittel & Kroemer, Chapter 2, problem 1 [Entropy and temperature]: 3 points

Suppose $g(U) = CU^{3N/2}$, where C is a constant and N is the number of particles. (This form of $g(U)$ actually applies to an ideal gas.)

(a)

Show that $U = \frac{3}{2} N \tau$.

(b)

Show that $(\partial^2 \sigma / \partial U^2)_N$ is negative.

Problem 2

Kittel & Kroemer, Chapter 2, problem 2 [paramagnetism]: 3 points

Find the equilibrium value at temperature τ of the fractional magnetization

$$M/Nm = 2\langle s \rangle / N$$

of the system of N spins each of magnetic moment m in a magnetic field B. The spin excess is $2s$. Take the entropy as the logarithm of the multiplicity $g(N, s)$ as given in (1.35):

$$\sigma(s) \simeq \ln g(N, 0) - 2s^2/N,$$

for $|s| \ll N$.

Hint 1: Show that in this approximation,

$$\sigma(U) = \sigma_0 - U^2/2m^2B^2N,$$

with $\sigma_0 = \ln g(N, 0)$. Further, show that $1/\tau = -U/m^2B^2N$, where U denotes $\langle U \rangle$, the thermal average energy.

Hint 2: In order to calculate the temperature, you need first to express the entropy in terms of the energy U, rather than the spin excess s. So write s in terms of U, and substitute into the formula for σ.

Problem 3

Kittel & Kroemer, Chapter 2, problem 3 [quantum harmonic oscillator]: 4 points

(a)

Find the entropy of a set of N oscillators of frequency ω as a function of the total quantum number n. Use the multiplicity function (1.55) and make the Stirling approximation $\ln N! \simeq N \ln N - N$. Replace $N - 1$ by N.

(b)

Let U denote the total energy $n\hbar\omega$ of the oscillators. Express the entropy as $\sigma(U, N)$. Show that the total energy at temperature τ is

$$U = \frac{N\hbar\omega}{e^{\hbar\omega/\tau} - 1}.$$

This is the Planck result; it is derived again in Chapter 4 by a powerful method that does not require us to find the multiplicity function.

Problem 4

Consider two systems of 10 spins each, i.e. $N_1 = N_2 = 10$. The systems are placed in a magnetic field, so configurations with different total spin have different energy. The systems are isolated from their surroundings, so their total energy (hence their total spin) is constant. Calculate all quantities in this problem exactly.

(a): 1 point

Initially, system 1 has all spins up, and system 2 has 2 up and 8 down. Using the notation of K&K, where $2s$ stands for the net excess of up spins, we have $s_1 = 5$, and $s_2 = -3$. What is the multiplicity of the combined system in its initial configuration?

(b): 2 points

Put the two systems in thermal contact with each other, so they can exchange energy with each other, but not with the outside world. So $s = s_1 + s_2 = 2$. Make a table showing all allowed combinations of s_1 and s_2, the individual multiplicities g_1 and g_2, and the combined multiplicity g_1g_2. (Note: If you prefer, you can choose to label the configurations by the number of up spins, $N_\uparrow$, rather than by the spin excess s.) What is the total possible number of configurations available to the combined system? How much did the total entropy increase as a result of putting the two systems in thermal contact?

(c): 1 points

What is the total probability that the combined system will be found in the most probable state $(s_1 = \hat{s}_1)$, or in one of the nearby states with $s_1 = \hat{s}_1 \pm 1$ What is the probability that the system will be found in its initial state?

Problem 5

Now let's combine what you learned in problems 2 and 4 above. Consider two systems of spins in a uniform magnetic field. The energy of a state with spin excess s is $U = -2smB$, where m is the magnetic moment of a spin and B is the magnetic field. System 1 has $N_1 = 10^4$ spins, while system 2 has $N_2 = 2 \times 10^4$ spins. Initially, system 1 has half its spins up and half down, so its spin excess is $s_1 = 0$. Initially, system 2 has a spin excess $s_2 = 1500$. Use the Gaussian approximation to the multiplicity that we derived in class for all parts of this problem, and ignore the first term $\sqrt{\frac{2}{\pi N}}$.

(a): 1 point

What is the initial energy and entropy of each system?

(b): 2 points

What is the initial temperature of each system? (Your answer for system 1 may be counter-intuitive.) Notice that the temperature τ in our textbook has units of energy.

(c): 1 point

Bring the two systems into thermal contact with each other. In thermal equilibrium, what is the average spin excess for each system, $\hat{s}_1$ and $\hat{s}_2$? What is the final (common) temperature of the two systems?

(d): 2 points

By how much did the total entropy increase during the process of thermal equilibration? (Hint: Ignore the constant terms $N \ln 2$ and just look at the part of the entropy that changed. You may approximate the entropy by $\ln(g_1 g_2)_{\max}$, so don't do any sums or integrals!) By what factor did the total multiplicity of the combined system increase?

After you do part (d), think about what it means. Does energy (heat) always have to flow from the hotter system to the colder system? It wouldn't violate energy conservation to have energy flow the other way, but the probability for that to happen in a macroscopic system is incredibly small!

You should also have noticed in doing this problem that spin temperatures can be infinite or even negative! What does that mean? When there are more spins in the high-energy state than the low-energy state, we call that an "inverted" spin population, and it has a negative temperature. But you can never use such a system to cool another system, because a system with an inverted spin population has higher energy than a system with a normal (noninverted) spin population. The moral of the story is, be careful about using equilibrium concepts such as temperature to describe nonequilibrium situations! By the way, inverted populations are the basis for lasers.