Limit Continuous Poker: A Variant of Continuous Poker with Limited Bet Sizes

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Abstract

We introduce and analyze Limit Continuous Poker, a variant of Von Neumann's Continuous Poker with variable but limited bet sizes. This simplified variant of poker captures aspects of information asymmetry, bluffing, balancing, and the impact of bet size limits while still being simple enough to solve analytically. We derive the Nash equilibrium strategy profile for this game, showing how the bettor's and caller's strategies depend on the bet size limits. We demonstrate that as the bet size limits approach extreme values, the strategy profile converges to those of other continuous poker variants. Finally, we connect these results to strategic implications of limited bet sizing in real-world poker.

Contents

1 Introduction

Poker is a notoriously complex game, with many different variants and strategic elements that have challenged both players and theorists for decades. Simplified poker models play a crucial role in game theory research by isolating specific strategic aspects—such as bluffing, value betting, and bet sizing—while remaining analytically tractable. One such class of models is Continuous Poker, which abstracts poker hands to continuous numerical hand strengths and restricts play to a single betting round. This simplification allows for exact Nash equilibrium solutions and provides insights that can inform understanding of more complex poker variants.

In this paper, we introduce and analyze Limit Continuous Poker (LCP), a new variant that bridges two well-studied extremes: Fixed-Bet Continuous Poker (FBCP), where the bettor must choose a predetermined bet size, and No-Limit Continuous Poker (NLCP), where the bettor can choose any positive bet size. LCP generalizes both by imposing lower and upper bounds L and U on allowable bet sizes, creating a family of games parametrized by these limits.

1.1 Related Work and Background

The study of simplified poker models dates back to von Neumann's seminal work on game theory, which introduced the concept of optimal mixed strategies in competitive games. Continuous poker variants have since become standard examples in game theory textbooks and research, serving as tractable models for studying information asymmetry and strategic bluffing. Our work builds directly on two classical variants: Fixed-Bet Continuous Poker (FBCP) and No-Limit Continuous Poker (NLCP). We briefly review these games to establish context for LCP.

1.1.1 Fixed-Bet Continuous Poker (FBCP)

Continuous Poker (also called Von Neumann Poker or simply Continuous Poker, and referred to in this paper as Fixed-Bet Continuous Poker or FBCP) is a simplified model of poker introduced by von Neumann.

Definition 1.1 (FBCP)**.** Two players, referred to as the bettor and the caller, each put a 0.5 unit ante into a pot[1]. They are each dealt a hand strength between 0 an 1 (referred to as x for bettor and y for caller). After seeing x, the bettor can either check—in which case, the higher hand between x and y wins the pot of 1 and the game ends—or they can bet by putting a pre-determined amount B into

[1] An ante of 1 is often used, but since the total pot size is the more relevant value, we use an ante of 0.5. All bet sizes scale proportionally.

the pot. The caller must now either call by matching the bet of B units, after which the higher hand wins the pot of $1+2B$, or fold, conceding the pot of $1+B$ to the bettor and ending the game.

FBCP has many Nash equilibria, but it has a unique one in which the caller plays an admissible strategy[2], as shown by Ferguson and Ferguson [2, p. 2]. This strategy profile, parametrized by the bet size B, is as follows:

The bettor bets with hands x such that either

$$x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ or } x < \frac{B}{(1+2B)(2+B)}.$$

We call the higher interval the value betting range and the lower interval the bluffing range. The caller calls with hands y above a calling threshold:

$$y > \frac{B(3+2B)}{(1+2B)(2+B)}.$$

Note that uniqueness in this context ignores the strictness of inequalities, since the endpoints of intervals occur with probability 0. The non-uniqueness of this Nash Equilibrium is due to the fact that given the bettor's strategy, the caller has many optimal responses. The caller must always fold with hands below the bluffing threshold, and must always call with hands above the value betting threshold, but with hands inbetween, they are indifferent between calling and folding. This is because with a hand strength in this range, the caller wins if and only if the bettor is bluffing, so their actual hand strength is irrelevant as long as it beats the bluffing threshold. To prevent the betting player from exploiting them, the caller need only call with exactly the right proportion of hands in this range. For example, the caller could take the strategy described above, but swap some calling and folding hands in the range between the bluffing and value betting thresholds.

Why is this Nash equilibrium special? We mentioned above that it is admissible, meaning that both players' strategies are not weakly dominated by any other strategy. Importantly, the caller's strategy is not weakly dominated. The same cannot be said for other Nash equilibria like the one described in the previous paragraph, for reasons that are beyond the scope of this introduction but relate to themes in Section 4.2.

FBCP also has a unique value as a function of the bet size B. The value of the game for the bettor is

$$V_{FB}(B) = \frac{B}{2(1+2B)(2+B)},$$

[2] An admissible strategy is one which is not weakly dominated by any other strategy.

which is positive (advantageous to the bettor) and maximized at $B = 1$, when the bet size is exactly the pot size. It should not be surprising that the value is positive—at worst, the bettor can always check and turn the game into a coin flip, so the bettor will only deviate from this strategy if they have a positive expected value. It should also not be surprising that some finite value of B maximizes the game value (although $B = 1$ exactly is not obvious). Intuitively, this is because forcing the bettor to bet too large relative to the pot would make betting too risky with most hands, and making the bet too small would simply give less profit to the bettor when they win a bet. Part of the motivation for studying LCP is to understand this concept more generally.

1.1.2 No-Limit Continuous Poker (NLCP)

Another continuous poker variant allows the bettor to choose a bet size $s > 0$ after seeing their hand strength, as opposed to a fixed bet size B. This variant is called Newman Poker after Donald J. Newman, but referred to as No-Limit Continuous Poker (NLCP) in this paper. This variant is discussed and solved by Bill Chen and Jerrod Ankenman [1, p. 154].

Once again, there is no unique Nash Equilibrium, but there is a strategy profile which is uniquely optimal in a different sense, which we will explore later on. Specifically the bettor should make large bets with their strongest and weakest hands and smaller bets or checks with their intermediate hands. It turns out that the optimal strategy is most elegantly described by a mapping from bet sizes s to hand strengths x for bluffing and value betting, respectively[3]. The caller simply has a calling threshold $c(s)$ for each possible bet size s. The full strategy profile is as follows:

The bettor bets s with hands x such that either

$$x = \frac{3s+1}{7(s+1)^3} \text{ or } x = 1 - \frac{3}{7(s+1)^2},$$

where the first condition represents bluffing hands and the second value betting hands. After seeing a bet of size s, the caller should call with hands y such that

$$y > 1 - \frac{6}{7(s+1)}.$$

See Figure 1 for a graphical representation of the strategy profile.

Note that the bettor uses all possible bet sizes and has exactly two hand

[3]This feels backwards—mapping hand strengths to bet sizes would be more natural, but the math is more elegant this way.

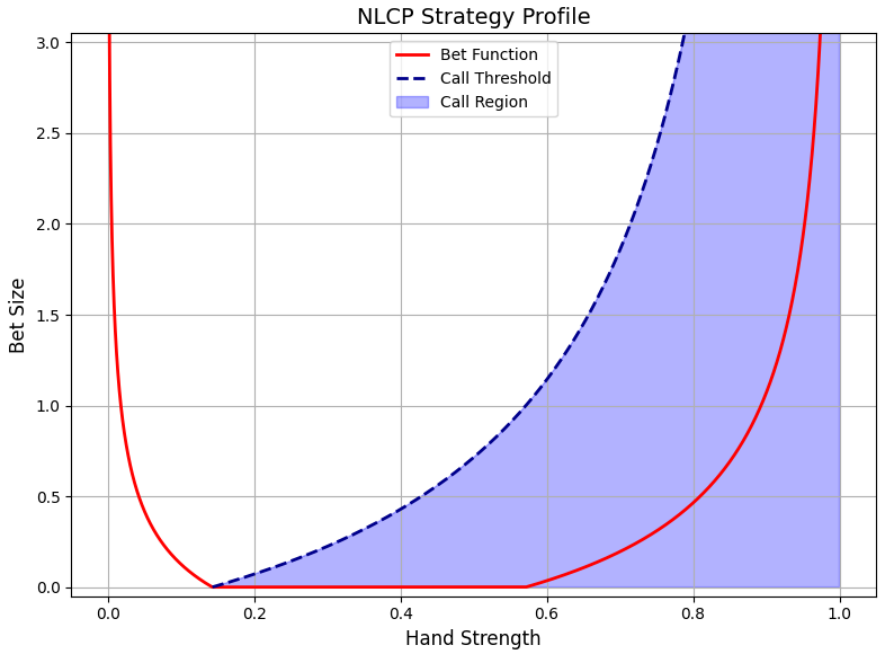

Figure 1: NLCP Nash Equilibrium Strategy Profile

strengths for each bet size[4]. On first inspection, this feels like the bettor is giving away too much information, but it turns out to still be an optimal strategy. This concept appears again and is explained more thoroughly in Section 4.3.

The value of NLCP is

$$V_{NL} = \frac{1}{14},$$

for the bettor[5]. Notice that NLCP is more advantageous to the bettor than FBCP for any bet size B because the bettor could artificially restrict themselves to a single bet size.

To recap: Ferguson and Ferguson [2] provided the comprehensive analysis of FBCP described above, establishing the unique admissible Nash equilibrium and deriving closed-form solutions for optimal strategies. Chen and Ankenman [1] extended this analysis to NLCP, demonstrating how unlimited bet sizing fundamentally changes the strategic landscape while maintaining analytical tractability. Our work builds on these foundations by introducing a parametric family of games

[4]Seen visually in Figure 1 by the fact that a horizontal line intersects the bet function at exactly two points.

[5]Chen and Ankenman find 1/7 using an ante of 1, but our value is halved using an ante of 0.5.

that interpolates between these extremes, allowing us to study how betting constraints affect optimal strategies and game value in a continuous fashion.

1.2 Our Contributions

This paper makes the following contributions:

- **Nash Equilibrium Solution:** We derive the unique admissible Nash equilibrium for LCP, characterized by six threshold parameters and two continuous bet-sizing functions. We establish the structure of optimal play and provide closed-form expressions for all strategic components (Sections 3, 4, and 5).
- **Game Value Analysis:** We compute the value of LCP as a function of the betting limits L and U, obtaining a surprisingly elegant rational formula. We prove monotonicity properties and establish a remarkable symmetry: $V_{LCP}(L, U) = V_{LCP}(1/U, 1/L)$ (Section 6).
- **Convergence Results:** We prove that LCP strategies and values converge to those of FBCP and NLCP in the appropriate limit cases, establishing LCP as a genuine generalization of both variants (Section 7).
- **Payoff and Parameter Analysis:** We analyze how changes in betting limits affect optimal strategies and payoffs, revealing counterintuitive effects where expanding betting options can reduce expected value for certain hand strengths due to strategic adjustments by the opponent (Section 8).

2 Preliminaries and Conventions

This section establishes the notation and conventions used throughout the paper. We assume familiarity with basic game theory concepts such as Nash equilibrium, zero-sum games, and mixed strategies, but we clarify our specific notation and modeling choices.

2.1 Game-Theoretic Concepts

All continuous poker variants studied in this paper are *two-player zero-sum games*. The *value* of such a game is the expected payoff to the first player (the bettor) when both players adopt Nash equilibrium strategies. Since the game is zero-sum, the caller's expected payoff is the negative of this value. A positive value indicates an advantage for the bettor.

We seek Nash equilibria where both players use *pure strategies* (deterministic mappings from information to actions). As is standard in continuous games, we treat sets of measure zero as negligible—for instance, the action taken at an exact threshold hand strength is irrelevant if that hand strength occurs with probability zero.

2.2 Conventions

- **Ante and Pot Size:** Each player contributes an ante of 0.5 units, creating an initial pot of 1 unit. All bet sizes are measured in units relative to this pot. This convention (ante = 0.5 rather than ante = 1) simplifies payoff calculations.

- **Payoffs:** Payoffs represent the net gain or loss including the initial ante. A check results in the winner receiving the pot of 1 minus their ante of 0.5, for a net payoff of 0.5. When a bet of size s is called, the winner receives $1 + 2s$ (pot plus both contributions) minus their ante of 0.5 and their contributed bet of s, for a net payoff of $0.5 + s$.

- **Inequalities and Measure Zero:** When we write, for example, "the caller calls with hands $y \geq c(s)$," the choice of $\geq$ versus $>$ is immaterial because the set $\{y : y = c(s)\}$ has measure zero. We use closed intervals $[a, b]$ and open intervals (a, b) interchangeably when the boundary points have probability zero. Note that this is more subtle when considering bet sizes, because the bettor can choose to make specific bet sizes with nonnegligible probability.

- **Monotone Strategies:** A calling strategy is *monotone* if (1) stronger hands are more likely to call than weaker hands for any fixed bet size, and (2) smaller bets are more likely to be called than larger bets for any fixed hand strength. This concept is formalized in Section 4.2.

- **Admissibility:** A strategy is *admissible* if it is not weakly dominated by any other strategy.

3 Limit Continuous Poker (LCP)

Limit Continuous Poker (LCP) is an extension of NLCP with the additional constraint of limiting bet sizes to a range. Just like NLCP, both players are dealt random hand strengths between 0 and 1. The bettor acts first, choosing either to check or to make a bet from an allowed range of sizes. If a bet is made, the caller must decide whether to call the bet or fold.

Definition 3.1 (LCP)**.** A two-player zero-sum game where:

- The bettor and caller are each dealt independent hand strengths $X, Y \sim \text{Uniform}[0, 1]$
- The bettor observes their hand strength x and chooses an action from $\mathcal{A}_1 = \{0\} \cup [L, U]$ (a bet of 0 is a check)
- If the bettor chooses an action from $[L, U]$ (a bet), then the caller observes the bettor's action along with their own hand strength y and chooses from $\mathcal{A}_2 = \{\text{call}, \text{fold}\}$
- Payoffs are determined as follows:
  - If the bettor checks: payoff is 0.5 to the player with higher hand strength (the pot of 1, minus the initial ante of 0.5)
  - If the bettor bets $s \in [L, U]$ and the caller calls: payoff is $0.5 + s$ (the pot of $1 + 2s$ minus the initial ante of 0.5 and the bettor's contribution of s) to the player with higher hand strength
  - If the bettor bets $s \in [L, U]$ and the caller folds: payoff is 0.5 to the bettor

We can describe a strategy for the bettor as a measurable function $\sigma_1 : [0, 1] \to \mathcal{A}_1$ mapping hand strengths to actions. A strategy for the caller is a measurable function $\sigma_2 : [L, U] \times [0, 1] \to \mathcal{A}_2$ mapping bet sizes and hand strengths to caller responses.

The motivation for studying this variant is twofold. First, it is a more realistic variant than NLCP or FBCP, where bets are not fixed but are also not unbounded. In most real variants of poker, bet sizes are constrained by the stack sizes of the players and by a minimum bet size. Strong poker players have intuition about how bet size constraints affect strategy, but rigorously proving this intuition is often impossible given the combinatorial complexity of the game.

Second, LCP can be seen as a generalization of FBCP and NLCP; specifically, as $L \to 0$ and $U \to \infty$, LCP approaches NLCP, and as $L \to B$ and $U \to B$ for some fixed value B, LCP approaches FBCP. Studying LCP can help us understand the relationship between these two and answer questions about why they produce the strategies they do (see Section 7.2 for formal convergence results).

In the next section, we develop the methodology for solving LCP and describing optimal strategies. Section 4.3 will describe the structure of this equilibrium, and the complete closed-form solution is presented in Section 5.

4 Solving for Nash Equilibrium

In this section, we develop the methodology for computing the Nash equilibrium of LCP. Our approach proceeds in three stages: (1) establishing the concept of *monotone calling strategies* and using admissibility to select among multiple equilibria, (2) characterizing the structure that any Nash equilibrium must satisfy, and (3) deriving a system of equations whose solution yields the equilibrium strategy profile. The complete derivation and verification that our solution constitutes a Nash equilibrium is provided in Appendix B.

4.1 Uniqueness and Equilibrium Selection

Like NLCP, LCP has an infinite class of Nash equilibria, differentiated by how the caller distributes hands AND by how the bettor sizes their bluffs. We resolve this non-uniqueness by imposing two natural refinements: first, we restrict the caller to *monotone* strategies; second, we require the bettor's strategy to be admissible against all such monotone calling strategies. These refinements uniquely determine the equilibrium we analyze.

4.2 Monotone Strategies

Definition 4.1 (Monotone Calling Strategy). A *monotone* calling strategy is a pure strategy which satisfies two conditions:

1. For any bet size s and any two hand strengths $y_1 < y_2$, if the caller calls a bet of size s with y_1, they must also call with y_2.
2. For any hand strength y and any two bet sizes $s_1 < s_2$, if the caller calls a bet of size s_2 with y, they must also call a bet of size s_1 with y.

This should sound intuitive. Clearly, calling with a stronger hand is weakly better than calling with a weaker hand. Restricting to pure strategies can be explained similarly—it is better to always call with a stronger hand and always fold a weaker one than to mix between the two.

Violating the first condition (in a non-negligible way) is actually weakly dominated: not only is a monotone strategy weakly better against all opponents, but there exists an opponent against which the monotone strategy is strictly better (see Appendix A for proof).

The second condition for a monotone calling strategy—that the caller must be more willing to call smaller bets—is more subtle. From a poker player's perspective, it aligns with intuition about pot odds: a larger bet is riskier and should require a stronger hand to call. While violating this condition is not dominated, it leads to exploitable calling strategies. If the caller calls less aggressively against smaller bets, the bettor can take smaller risks for higher returns. A monotone calling strategy is therefore less exploitable, and imposing this condition yields a unique Nash equilibrium.

Definition 4.2 (Monotone-Admissible Strategy)**.** A betting strategy σ_B is *monotone-admissible* if it is admissible in LCP against the set of monotone calling strategies. That is, there does not exist a betting strategy σ'_B that performs at least as well against all monotone calling strategies and strictly better against at least one.

This definition is necessary because variable bet sizes complicate bluffing strategy. In NLCP and LCP, the hand strength of a bluff is actually irrelevant when the caller plays optimally, since the caller never calls with a hand that will lose to a bluff. However, if the caller deviates to a suboptimal but still monotone strategy, the bettor's bluffing hand strength becomes important. Monotone-admissibility selects the equilibrium where the bettor bluffs larger with weaker hands and smaller with stronger hands, which is optimal against all monotone deviations. For proof that this refinement actually selects a unique equilibrium, see Appendix A.

4.3 Nash Equilibrium Structure

We will now describe the structure of the Nash equilibrium in terms of constants x_i and functions $c(s)$, $b(s)$, and $v(s)$. These turn out to be fully determined by the parameters L and U, but for now they are unknown. Notice that both players use pure strategies, like in NLCP: the bettor maps hand strengths directly to bet sizes, and the caller maps hand strengths and bet sizes to actions with no mixing. The structure of the equilibrium is as follows:

1. The caller has a calling threshold $c(s)$ that is continuous in s, including at endpoints L and U. They call with hands $y \geq c(s)$ and fold with hands $y < c(s)$[6].

2. The bettor partitions $[0, 1]$ into three regions: bluffing $x \in [0, x_2]$, checking $x \in [x_2, x_3]$, and value betting $x \in [x_3, 1]$.

3. Within the bluffing region, the bettor partitions hands into a max-betting region $x \in [0, x_0]$, an intermediate region $x \in [x_0, x_1]$, and a min-betting region $x \in [x_1, x_2]$.

4. Within the intermediate bluffing region $[x_0, x_1]$, the bettor bets according to a continuous, decreasing function $s = b^{-1}(x)$ with endpoints $b^{-1}(x_0) = U$ and $b^{-1}(x_1) = L$.

[6]The action taken at the threshold is irrelevant, since it occurs with probability zero.

5. Within the value betting region, the bettor partitions into a min-betting region $x \in [x_3, x_4]$, an intermediate region $x \in [x_4, x_5]$, and a max-betting region $x \in [x_5, 1]$.

6. Within the intermediate value betting region, the bettor bets according to a continuous, increasing function $s = v^{-1}(x)$ with endpoints $v^{-1}(x_4) = L$ and $v^{-1}(x_5) = U$.

See figure 2 for visual representations of the strategy profile.

4.4 Constraints and Indifference Equations

Having established the qualitative structure of the Nash equilibrium, we now derive the quantitative relationships that the strategy profile must satisfy. These constraints arise from two fundamental equilibrium conditions: players must be indifferent among actions they mix over (or, in our case, use with positive probability), and players must optimize when choosing among available actions. The resulting system of differential and algebraic equations will uniquely determine all threshold values $x_0, \ldots, x_5$ and the functions $b(s)$, $v(s)$, and $c(s)$.

The key conditions are:

- The caller must be indifferent between calling and folding at their calling threshold
- The bettor must be indifferent between checking and betting at their value betting and bluffing thresholds
- The bettor's bet size for a value bet must maximize their expected value
- The bettor's strategy must be continuous in bet size (in the regions where they bet)

These conditions give us the following system of equations:

Caller Indifference:

$$(x_2 - x_1) \cdot (1 + L) - (x_4 - x_3) \cdot L = 0 \tag{1}$$

$$x_0 \cdot (1 + U) - (1 - x_5) \cdot U = 0 \tag{2}$$

$$|b'(s)| \cdot (1 + s) - |v'(s)| \cdot s = 0 \tag{3}$$

Bettor Indifference and Optimality:

$$-sc'(s) - c(s) + 2v(s) - 1 = 0 \tag{4}$$

$$(x_3 - c(L)) \cdot (1 + L) - (1 - x_3) \cdot (L) + c(L) = x_3 \tag{5}$$

$$c(s) - (1 - c(s)) \cdot s = x_2 \tag{6}$$

Continuity Constraints:

$$b(U) = x_0 \tag{7}$$
$$b(L) = x_1 \tag{8}$$
$$v(U) = x_5 \tag{9}$$
$$v(L) = x_4. \tag{10}$$

We will now derive each of these equations in turn. Note that for this analysis, it is simpler to pretend that payoffs exclude the initial ante of 0.5, since this is a sunk cost to both players and we only care about relative payoffs between actions.

4.4.1 Caller Indifference

By definition, $c(s)$ is the threshold above which the caller calls and below which they fold. This means that in Nash Equilibrium, the caller must be indifferent between calling and folding with a hand strength of $c(s)$:

$$\mathbb{E}[\text{call } c(s)] = \mathbb{E}[\text{fold } c(s)]$$
$$\mathbb{P}[\text{bluff}|s] \cdot (1+s) - \mathbb{P}[\text{value bet}|s] \cdot s = 0.$$

We now split into cases based on the value of s.

Case 1: $s = L$. The hands the bettor value bets L with are $x \in (x_3, x_4)$, and the hands they bluff with are $x \in (x_1, x_2)$.

$$(x_2 - x_1) \cdot (1 + L) - (x_4 - x_3) \cdot L = 0. \tag{11}$$

Here, we are implicitly multiplying both sides by the common denominator of $(x_4 - x_3) + (x_2 - x_1)$.

Case 2: $s = U$. The hands the bettor value bets U with are $x \in (x_5, 1)$, and the hands they bluff with are $x \in (0, x_0)$.

$$(1 - x_5) \cdot (1 + U) - x_0 \cdot U = 0, \tag{12}$$

again, implicitly multiplying both sides by the common denominator of $(1 - x_5) + x_0$.

Case 3: $L \leq s \leq U$. In this case, the bettor has exactly one value hand and one bluffing hand, but somewhat paradoxically, they are not equally likely. The probability of a value bet given the size s is related to the inverse derivative of the value function $v(s)$ at s, and the same goes for a bluff. This gives us the following relation:

$$\frac{\mathbb{P}[\text{value bet}|s]}{\mathbb{P}[\text{bluff}|s]} = \frac{|b'(s)|}{|v'(s)|}$$

An intuitive interpretation of this is that for any small neighborhood around the bet size s, the bettor has more hands which use a bet size in the neighborhood if $v(s)$ does not change rapidly around s, that is, if $|v'(s)|$ is small. The same goes for bluffing hands, and as we limit the neighborhood to a single point, the ratio of the two probabilities approaches the ratio of the derivatives. We know that these are the only two possible bettor actions for such a bet size, so

$$\mathbb{P}[\text{value bet}|s] = \frac{|b'(s)|}{|b'(s)| + |v'(s)|}$$
$$\mathbb{P}[\text{bluff}|s] = \frac{|v'(s)|}{|b'(s)| + |v'(s)|}$$

Plugging this into the indifference equation and dividing out the common denominator, we get:

$$|b'(s)| \cdot (1+s) - |v'(s)| \cdot s = 0. \tag{13}$$

4.4.2 Bettor Indifference and Optimality

When the bettor makes a value bet, they are attempting to maximize the expected value of the bet. We can write the expected value of a value bet as:

$$\begin{aligned}\mathbb{E}[\text{value bet } s|x] &= \mathbb{P}[\text{call with worse}] \cdot (1+s) - \mathbb{P}[\text{call with better}] \cdot s + \mathbb{P}[\text{fold}] \cdot 1 \\ &= (x - c(s)) \cdot (1+s) - (1-x) \cdot (s) + c(s).\end{aligned}$$

To maximize this, we take the derivative with respect to s and set it equal to zero. Crucially, we are treating $c(s)$ as a function of s and using the chain rule, since changing the bet size s will also change the calling threshold $c(s)$. We want this optimality condition to hold for the bettor's Nash equilibrium strategy, so we set $x = v(s)$. This gives us:

$$\begin{aligned}&\frac{d}{ds}\mathbb{E}[\text{value bet } s|x = v(s)] = 0 \\ &-sc'(s) - c(s) + 2v(s) - 1 = 0.\end{aligned} \tag{14}$$

Additionally, when the bettor has the most marginal value betting hand at $x = x_3$, they should be indifferent between a minimum value bet and a check:

$$\begin{aligned}\mathbb{E}[\text{value bet } L|x = x_3] &= \mathbb{E}[\text{check}|x = x_3] \\ (x_3 - c(L)) \cdot (1+L) - (1-x_3) \cdot (L) + c(L) &= x_3.\end{aligned} \tag{15}$$

Finally, when the bettor has the most marginal bluffing hand at $x = x_2$, they should be indifferent between a minimum bluff and a check. However, as we discussed earlier, the bettor should be indifferent among all bluffing sizes, so the bettor should actually be indifferent between checking and making any bluffing size s at $x = x_2$. This gives us:

$$\begin{aligned} \mathbb{E}[\text{bluff } s | x = x_2] &= \mathbb{E}[\text{check} | x = x_2] \\ c(s) - (1 - c(s)) \cdot s &= x_2. \end{aligned} \tag{16}$$

4.4.3 Continuity Constraints

As discussed above, the bettor's strategy $s(x)$ should be continuous in x. Intuitively, this is because the payoff as a function of hand strength and bet size turns out to be smooth in both inputs in equilibrium. This means that for very similar hand strengths, the optimal bet sizes are also very similar, so a discontinuity cannot be optimal. This is proven more formally in Appendix **??**.

This gives us the following constraints at the endpoints:

$$b(U) = x_0, \quad b(L) = x_1, \quad v(U) = x_5, \quad v(L) = x_4. \tag{17}$$

5 Nash Equilibrium Strategy Profile

Having established the equilibrium structure in Section 4 and derived the indifference equations in Section 4.4, we now present the complete solution. The system of equations was solved symbolically using Sympy, yielding closed-form expressions for all threshold values and strategic functions. We prove formally in Appendix B that this strategy profile constitutes a Nash equilibrium.

Theorem 5.1 (LCP Nash Equilibrium)**.** *LCP has a unique Nash equilibrium strategy profile in which the caller's strategy is monotone and the bettor's strategy is monotone-admissible (up to measure zero sets of hands for each player). This strategy profile is given by:*

$$
\begin{aligned}
x_0 &= \frac{3t^2\,(t-1)}{r^3+t^3-7}\\
x_1 &= \frac{-2r^3+3r^2+t^3-1}{r^3+t^3-7}\\
x_2 &= \frac{r^3+t^3-1}{r^3+t^3-7}\\
x_3 &= \frac{r^3-3r+t^3-4}{r^3+t^3-7}\\
x_4 &= \frac{r^3+3r^2-6r+t^3-4}{r^3+t^3-7}\\
x_5 &= \frac{r^3+t^3+3t^2-7}{r^3+t^3-7}\\
b_0 &= \frac{t^3}{r^3+t^3-7}\\
b(s) &= \frac{t^3\,(s+1)^3-(3s+1)}{(r^3+t^3-7)\,(s+1)^3}\\
c(s) &= \frac{r^3+t^3-1+s\,(r^3+t^3-7)}{(s+1)\,(r^3+t^3-7)}\\
v(s) &= \frac{r^3+t^3-1+(r^3+t^3-7)\,(2s^2+4s+1)}{2\,(r^3+t^3-7)\,(s^2+2s+1)}
\end{aligned}
$$

where $r = L/(1+L)$ *and* $t = 1/(1+U)$[7].

Refer back to Section 4.3 for an explanation of how these values fit together to actually form the strategy profile. A proof of this theorem can be found in Appendix B.

This solution is more interpretable in graphical form. Figure 2 shows the strategy profile for various values of L and U ranging from very lenient ($L = 0, U = 10$) to very restricted ($L = 0.5, U = 1$). The more lenient bet size limits model something closer to NLCP, while the more restricted bet size limits model something closer to FBCP. Indeed, we see that the strategy profile for $L = 0, U = 10$ looks qualitatively similar to the strategy profile of NLCP—we will show in Section 7.2 that the strategy profile approaches the Nash equilibrium of NLCP as L and U approach 0 and ∞, respectively, and that the strategy profile approaches the Nash equilibrium of FBCP as L and U approach some fixed value s from either side.

[7]The change of variables to (r, t) significantly simplifies the expressions compared to the original (L, U) formulation. This transformation reveals underlying symmetries and makes many properties more transparent, as we will see in the analysis of game value and parameter effects.

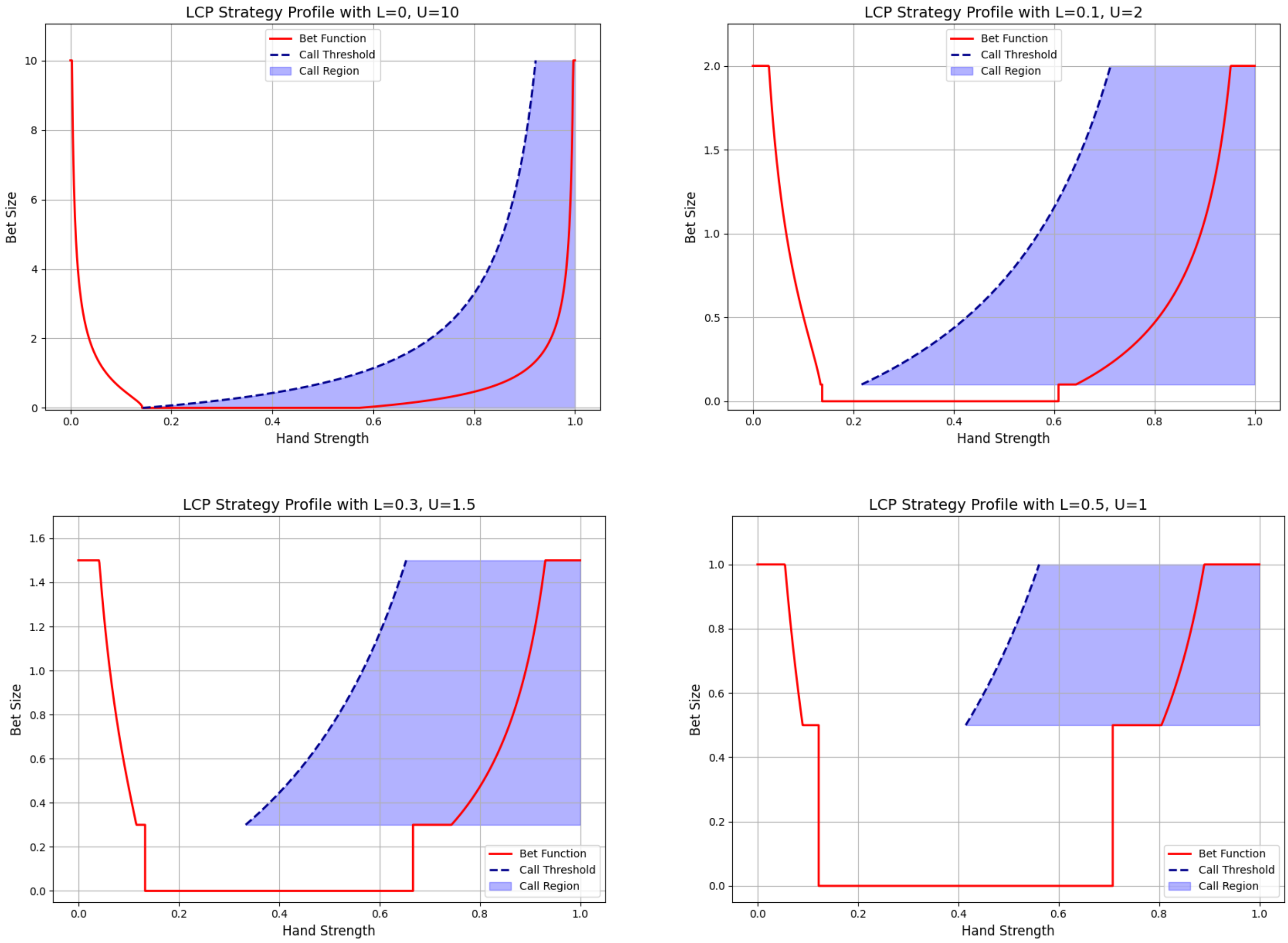

Figure 2: Nash equilibrium strategy profiles for different values of L and U, from very lenient to very restricted bet sizes. The bet function maps hand strengths to bet sizes, while the call function gives the minimum calling hand strength for a given bet size. The shaded regions represent the hand strengths for which the caller should call a given bet size.

6 Game Value

Having characterized the Nash equilibrium strategy profile in Section 5, we now turn to analyzing the expected payoff when both players employ these optimal strategies. In zero-sum games like Limit Continuous Poker, the concept of game value is fundamental. The game value represents the expected payoff that the first player (bettor) can guarantee when both players play optimally. Since this is a zero-sum game, the caller's expected payoff is simply the negative of this value. This value serves as a measure of how favorable the game is to the bettor under the given betting limits L and U.

Theorem 6.1. *The value of Limit Continuous Poker is given by:*

$$V_{LCP}(L, U) = \frac{(1 + L)^3(1 + U)^3 - ((1 + L)^3 + L^3(1 + U)^3)}{14(1 + L)^3(1 + U)^3 - 2((1 + L)^3 + L^3(1 + U)^3)}$$

Equivalently, using the change of variables $r = L/(1 + L)$ and $t = 1/(1 + U)$, this can be written more compactly as:

$$V(r, t) = \frac{1 - r^3 - t^3}{14 - 2r^3 - 2t^3}$$

Proof Sketch. Since LCP is a zero-sum game, all Nash equilibria yield the same payoff. We have explicitly constructed optimal strategies for both players, so the value of the game is the expected payoff of these strategies, averaged over all hand pairs (x, y). This reduces to an integral of the payoff over the unit square, like we saw in Figure 4.

The computation is extremely nontrivial because the bet size is only defined implicitly in terms of the hand strength x. The proof requires breaking the unit square into regions based on the strategies and bet sizes, then computing the expected value as a weighted sum of payoffs over these regions. The full details can be found in Sympy code on GitHub. □

This is a rational function of L and U with a surprisingly simple form. The change of variables to r and t reveals an even more elegant structure, along with the following symmetry property:

$$V_{LCP}(L, U) = V_{LCP}\left(\frac{1}{U}, \frac{1}{L}\right)$$

or equivalently,

$$V(r, t) = V(t, r)$$

The symmetry becomes immediately apparent in the (r, t) formulation: the numerator $1 - r^3 - t^3$ and denominator $14 - 2r^3 - 2t^3$ are both symmetric in r and t. This is not at all obvious from the game setup in terms of L and U.

Figure 3 shows the game value as a function of L and U on log-log scale as well as a function of r and t, making the symmetry $V(r, t) = V(t, r)$ visually apparent as a reflection across the diagonal. The plot also clearly illustrates the interpretations discussed above: the left edge ($r = 0$, corresponding to $L = 0$) represents the case where minimum bets are negligible, while the bottom edge ($t = 0$, corresponding to $U \to \infty$) represents the case where maximum bets become arbitrarily large. The diagonal ($r + t = 1$, or equivalently $L = U$) represents the boundary where the game reduces to fixed-bet continuous poker.

In terms of the original parameters, the symmetry $V_{LCP}(L, U) = V_{LCP}(1/U, 1/L)$ tells us that the benefit from increasing U is exactly equivalent to that of decreasing L in a reciprocal manner, centered around the pot size of 1. For example, suppose you are given the choice between playing LCP as the bettor with limits $L = 1/2$ and $U = 5$ or $L = 1/5$ and $U = 2$. You know you can play optimally, but it is unclear which game favors you. The symmetry property tells us that the value of the game is the same in both cases, so you should be indifferent between the two.

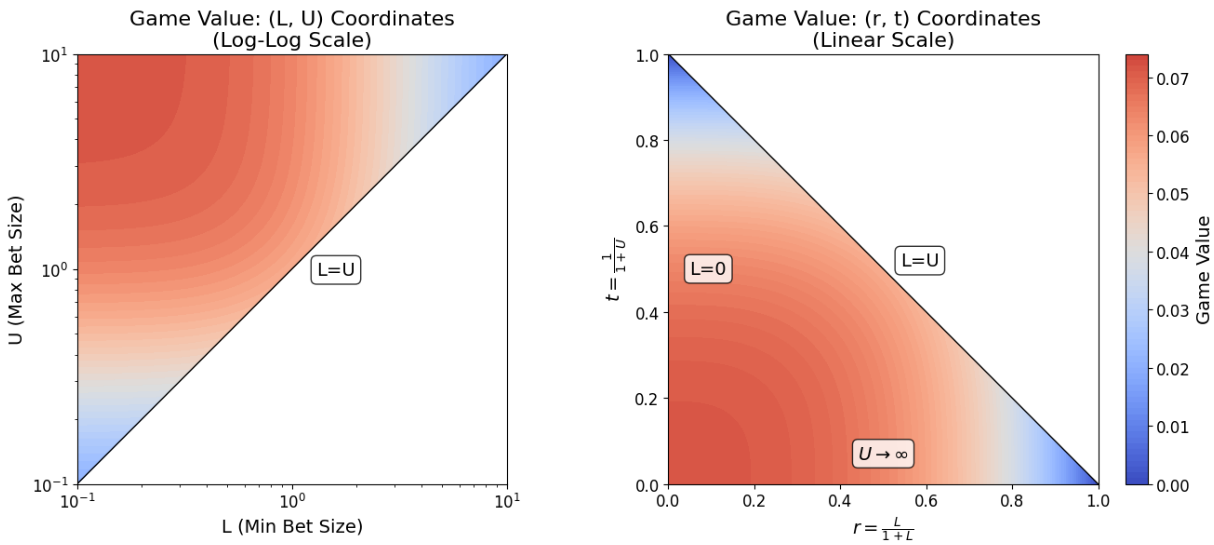

Figure 3: Game value as a function of both parametrizations. The symmetry about the diagonal is immediately visible, corresponding to $V(r, t) = V(t, r)$ or $V(L, U) = V(1/U, 1/L)$. Note that the left plot is cutting off extremely large and small values of L and U, while the right plot shows every possible parameter combination.

6.1 Interpreting the Parameters r and t

Before analyzing the properties of the game value, it is helpful to understand what the transformed parameters r and t represent in game-theoretic terms.

The parameter $r = L/(1 + L)$ as minimum pot odds: When the bettor makes a minimum bet of size L, the pot grows from 1 to $1 + L$. The caller must risk L to call and potentially win a pot of $1 + L$. Thus, $r = L/(1 + L)$ represents the *pot odds* the caller receives when facing a minimum bet—the ratio of what they risk to the total pot. This is the most favorable pot odds any caller ever faces in LCP, since larger bets offer worse pot odds. As $L \to 0$, we have $r \to 0$, meaning the minimum bet becomes negligible and calling becomes essentially free. As $L \to \infty$, we have $r \to 1$, meaning the minimum bet becomes prohibitively expensive relative to the pot.

The parameter $t = 1/(1 + U)$ as pot fraction at maximum bet: When the bettor makes a maximum bet of size U, the pot grows from 1 to $1 + U$. The parameter $t = 1/(1 + U)$ represents the original pot as a fraction of the total pot after a maximum bet. Equivalently, $1 - t = U/(1 + U)$ represents the pot odds the caller receives when facing a maximum bet. A small value of t (close to 0) indicates that U is very large relative to the pot, allowing the bettor to make very aggressive bets. As $U \to \infty$, we have $t \to 0$, meaning the maximum bet becomes arbitrarily large. As $U \to 0$, we have $t \to 1$, meaning the maximum bet becomes negligible.

The duality revealed by the symmetry: The parameter r fundamentally controls the *caller's incentive to call* with marginal hands: higher r means the minimum bet offers worse pot odds, discouraging calls. The parameter t fundamentally controls the *bettor's ability to apply pressure*: lower t means the maximum bet can be much larger relative to the pot, allowing more aggressive play. The symmetry $V(r, t) = V(t, r)$ reveals a deep duality in the game: swapping the "minimum calling incentive" (measured by r) with the "maximum betting freedom" (measured inversely by t) produces games with identical value. This is remarkable because r is primarily about the caller's decisions (pot odds when facing small bets) while t is primarily about the bettor's decisions (how large they can bet), yet these two forces are perfectly balanced in determining the game's value.

In the following sections, we investigate the properties and behavior of $V_{LCP}(L, U)$ in more detail. These include monotonicity, convergence to NLCP and FBCP, and the symmetry property.

6.2 Value Monotonicity

Intuitively, more options for the bettor should increase the game's value. Notice the higher value for more lenient limits (red regions of Figure 3) and lower value for more strict limits (blue corners). We can easily prove this formally:

Theorem 6.2. *The value of Limit Continuous Poker is weakly monotonically increasing in U and weakly monotonically decreasing in L:*

$$\frac{\partial V_{LCP}(L, U)}{\partial U} \geq 0, \quad \frac{\partial V_{LCP}(L, U)}{\partial L} \leq 0.$$

Proof. We can express the derivatives in terms of the cleaner (r, t) variables. Since $r = L/(1 + L)$ and $t = 1/(1 + U)$, we have:

$$\frac{dr}{dL} = \frac{1}{(1+L)^2}, \quad \frac{dt}{dU} = -\frac{1}{(1+U)^2}$$

Using the chain rule and the fact that $V(r,t) = \frac{1-r^3-t^3}{14-2r^3-2t^3}$:

$$\frac{\partial V}{\partial r} = \frac{-3r^2(14-2r^3-2t^3) + 2 \cdot 3r^2(1-r^3-t^3)}{(14-2r^3-2t^3)^2} = \frac{-18r^2(2-r^3-t^3)}{(14-2r^3-2t^3)^2} < 0$$
$$\frac{\partial V}{\partial t} = \frac{-18t^2(2-r^3-t^3)}{(14-2r^3-2t^3)^2} < 0$$

where the inequalities hold since $r, t \in (0, 1)$ implies $r^3 + t^3 < 2$ and $14 - 2r^3 - 2t^3 > 0$. Therefore:

$$\frac{\partial V_{LCP}}{\partial L} = \frac{\partial V}{\partial r}\frac{dr}{dL} = \frac{-18r^2(2-r^3-t^3)}{(14-2r^3-2t^3)^2} \cdot \frac{1}{(1+L)^2} < 0$$
$$\frac{\partial V_{LCP}}{\partial U} = \frac{\partial V}{\partial t}\frac{dt}{dU} = \frac{-18t^2(2-r^3-t^3)}{(14-2r^3-2t^3)^2} \cdot \left(-\frac{1}{(1+U)^2}\right) > 0$$

□

6.3 Value Convergence

The main diagonal of the plots in Figure 3 should represent $L = U$, which means the bet size is fixed. Thus, this diagonal should represent the value of FBCP for various values of B. We can prove this formally:

Theorem 6.3. *For any $B > 0$, the value of Limit Continuous Poker converges to the value of Fixed-Bet Continuous Poker as L and U approach B:*

$$\lim_{L\to B}\lim_{U\to B} V_{LCP}(L,U) = \lim_{U\to B}\lim_{L\to B} V_{LCP}(L,U) = V_{FB}(B)$$

Proof. $V_{LCP}(L, U)$ is a rational function of L and U, and no part of the expression is undefined for $L = U = B$. We can simply plug in and simplify to get

$$\frac{B}{2(1+2B)(2+B)},$$

which is exactly $V_{FB}(B)$. □

These plots also align with the known result that a fixed pot-size bet of $B = 1$ maximizes the expected value for the bettor in FBCP, as seen by the fact that $(1, 1)$ achieves the maximum value on the diagonal.

We can also show that the value of LCP converges to the value of NLCP as L and U approach their extremes, represented by the top left of any plot in Figure 3.

Theorem 6.4. *The value of Limit Continuous Poker converges to the value of NLCP as L and U approach 0 and ∞:*

$$\lim_{L\to 0}\lim_{U\to\infty} V_{LCP}(L,U) = \lim_{U\to\infty}\lim_{L\to 0} V_{LCP}(L,U) = V_{NL}$$

Proof. We can simply plug in $L = 0$ to get

$$\frac{(U+1)^3 - 1}{14(U+1)^3 - 2}.$$

Taking the limit as $U \to \infty$ gives $\frac{1}{14} = V_{NL}$. □

7 Strategic Comparison to Fixed-Bet and No-Limit Continuous Poker

As noted in the introduction, LCP is designed to interpolate between Fixed-Bet Continuous Poker (FBCP) and No-Limit Continuous Poker (NLCP). Having derived the Nash equilibrium (Section 5) and game value (Section 6) for LCP, we now make this interpolation precise by proving convergence results. Specifically, we show that as the betting limits L and U approach appropriate boundary values, both the strategies and the game value of LCP converge to those of FBCP and NLCP.

To facilitate comparison across variants, we model the bettor strategies for all three games as 'bet functions' from hand strengths to bets (with 0 representing a check), and caller strategies as 'call functions' from bet sizes to minimum calling thresholds. We also introduce notation to reference all three strategy profiles more efficiently.

7.1 Setup and Notation

To compare the strategy profiles across different variants of Continuous Poker, we introduce the following notation for the strategy functions of the three games:

Symbol	Meaning
$S_{FB}(x, B)$	Bettor's bet function in FBCP with fixed bet size B
$C_{FB}(s, B)$	Caller's call function in FBCP with fixed bet size B
$S_{NL}(x)$	Bettor's bet function in NLCP
$C_{NL}(s)$	Caller's call function in NLCP
$S_{LCP}(x, L, U)$	Bettor's bet function in LCP with limits L and U
$C_{LCP}(s, L, U)$	Caller's call function in LCP with limits L and U
$x_i\|_{L,U}$	Threshold x_i in LCP with limits L and U

Table 1: Notation for strategy functions across different variants of Continuous Poker. Recall that x_i are variables used to describe the LCP strategy profile in Section 4.3.

To recap, we will directly define these functions, although there will be overlap with the introduction. In FBCP, the bettor can only make a fixed bet size B or check. The bet function $S_{FB}(x, B)$ maps hand strengths to either 0 (check) or B (bet):

$$S_{FB}(x, B) = \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \text{ (bluffing range)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (checking range)} \\ B & x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (value betting range)} \end{cases} \tag{18}$$

The caller's strategy is defined by a single threshold $C_{FB}(s, B)$:

$$C_{FB}(s, B) = \frac{B(3 + 2B)}{(1 + 2B)(2 + B)} \tag{19}$$

In NLCP, the bettor can choose any positive bet size. The strategy is most naturally described by functions $v_{NL}(s)$ and $b_{NL}(s)$ that map bet sizes to hand strengths:

$$\begin{aligned} v_{NL}(s) &= 1 - \frac{3}{7(s+1)^2} \text{ (value betting function)} \\ b_{NL}(s) &= \frac{3s+1}{7(s+1)^3} \text{ (bluffing function).} \end{aligned}$$

The bet function $S_{NL}(x)$ is then defined in terms of the inverse functions:

$$S_{NL}(x) = \begin{cases} b_{NL}^{-1}(x) & x < \frac{1}{7} \text{ (bluffing range)} \\ 0 & \frac{1}{7} < x < \frac{4}{7} \text{ (checking range)} \\ v_{NL}^{-1}(x) & x > \frac{4}{7} \text{ (value betting range)} \end{cases}$$

The caller's strategy is defined by a continuous function $C_{NL}(s)$:

$$C_{NL}(s) = 1 - \frac{6}{7(s+1)}$$

In LCP, the bettor can choose any bet size between L and U. The strategy profile is defined by six thresholds x_0 through x_5 and functions $v(s)$ and $b(s)$ that map bet sizes to hand strengths. The bet function $S_{LCP}(x, L, U)$ and call function $C_{LCP}(s, L, U)$ are defined in terms of these values, which are given in Theorem 5.1.

7.2 Strategic Convergence

7.2.1 Bettor Strategy Convergence to FBCP

We expect that as L and U approach some fixed value s, the bet function $S_{LCP}(x, L, U)$ should converge to the bet function $S_{FB}(x, s)$ for Fixed-Bet Continuous Poker.

Theorem 7.1. *For any $B > 0$, $S_{LCP}(x, L, U)$ converges pointwise to $S_{FB}(x, B)$ as L and U approach B:*

$$\lim_{L\to B} \lim_{U\to B} S_{LCP}(x, L, U) = \lim_{U\to B} \lim_{L\to B} S_{LCP}(x, L, U) = S_{FB}(x, B)$$

for all hand strengths $x \in [0, 1]$.

Proof. We analyze the expressions for the x_i's, each of which is a rational function of L and U. Since these functions are defined and continuous for all positive $0 \le L \le U$, the limit as $L \to B$ and $U \to B$ can be found by simply substituting $L = U = B$:

$$
\begin{aligned}
x_0|_{B,B} = x_1|_{B,B} &= \frac{B}{2B^3 + 7B^2 + 7B + 2} \\
x_2|_{B,B} &= \frac{B}{(1+2B)(2+B)} \\
x_3|_{B,B} &= \frac{2B^2 + 4B + 1}{(1+2B)(2+B)} \\
x_4|_{B,B} = x_5|_{B,B} &= \frac{2B^2 + 5B + 1}{(1+2B)(2+B)}
\end{aligned}
$$

$x_0 = x_1$ and $x_4 = x_5$ are expected, since these intervals are where the bettor uses an intermediate bet size, and $L = U = B$ does not allow intermediate bet sizes. This reduces the bet function to

$$
\begin{aligned}
\lim_{L\to B}\lim_{U\to B} S_{LCP}(x, L, U) &= \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \\ B & x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \end{cases} \\
&= S_{FB}(x, B)
\end{aligned}
$$

□

7.2.2 Caller Strategy Convergence to FBCP

The calling function is easier to analyze. We want to show that the calling threshold $C_{LCP}(s, L, U)$ converges to the calling threshold $C_{FB}(s, B)$ for Fixed-Bet Continuous Poker as L and U approach B.

Theorem 7.2. *For any $B > 0$, $C_{LCP}(s, L, U)$ converges pointwise to $C_{FB}(s, B)$ as L and U approach B:*

$$
\lim_{L\to B}\lim_{U\to B} C_{LCP}(s, L, U) = \lim_{U\to B}\lim_{L\to B} C_{LCP}(s, L, U) = C_{FB}(s, B).
$$

for all bet sizes $s \in [L, U]$.

Proof. We already have the value of $x_2|_{B,B}$, so we can plug this into the expression for the calling threshold:

$$\begin{aligned}
\lim_{L\to B}\lim_{U\to B} C_{LCP}(s,L,U) &= \frac{x_2|_{B,B}+s}{1+s} \\
&= \frac{\frac{B}{(1+2B)(2+B)}+s}{1+s} \\
&= \frac{B(3+2B)}{(1+2B)(2+B)} \\
&= C_{FB}(s,B)
\end{aligned}$$

□

7.2.3 Bettor Strategy Convergence to NLCP

In a similar fashion, we expect that as L and U approach 0 and ∞ respectively, the bet function $S_{LCP}(x,L,U)$ should converge to the bet function $S_{NL}(x)$ for NLCP.

Theorem 7.3. *$S_{LCP}(x,L,U)$ converges pointwise to $S_{NL}(x)$ as L and U approach 0 and ∞:*

$$\lim_{L\to 0}\lim_{U\to\infty} S_{LCP}(x,L,U) = \lim_{U\to\infty}\lim_{L\to 0} S_{LCP}(x,L,U) = S_{NL}(x).$$

for all hand strengths $x \in [0,1]$.

Proof. We can analyze the expressions for the x_i's as L and U approach 0 and ∞. The limits are all well-defined:

$$\begin{aligned}
x_0|_{0,\infty} &= 0 \\
x_1|_{0,\infty} &= x_2|_{0,\infty} = \frac{1}{7} \\
x_3|_{0,\infty} &= x_4|_{0,\infty} = \frac{4}{7} \\
x_5|_{0,\infty} &= 1
\end{aligned}$$

$x_0|_{0,\infty} = 0$ and $x_5|_{0,\infty} = 1$ are expected, since these intervals are where the bettor uses a minimum bet size and a maximum bet size, respectively, both of which are impossible. The bettor now bets intermediate values for $x < \frac{1}{7}$ and $x > \frac{4}{7}$, and checks for $\frac{1}{7} < x < \frac{4}{7}$. But how much do they bet? We can take the limits of $v(s)$ and $b(s)$ as L and U approach 0 and ∞:

$$\lim_{L\to 0}\lim_{U\to\infty} b(s) = \frac{3s+1}{7(s+1)^3}$$
$$\lim_{L\to 0}\lim_{U\to\infty} v(s) = 1 - \frac{3}{7(s+1)^2}.$$

To summarize, the bettor bets s with hands $x < \frac{1}{7}$ such that $x = b(s)$ or hands $x > \frac{4}{7}$ such that $x = v(s)$. This is exactly the same as the bet function $S_{NL}(x)$ for NLCP.

□

7.2.4 Caller Strategy Convergence to NLCP

The calling function is again easier to analyze. We want to show that the calling threshold $C_{LCP}(s, L, U)$ converges to the calling threshold $C_{NL}(s)$ for NLCP as L and U approach 0 and ∞ respectively.

Theorem 7.4. *$C_{LCP}(s, L, U)$ converges pointwise to $C_{NL}(s)$ as L and U approach 0 and ∞:*

$$\lim_{L\to 0}\lim_{U\to\infty} C_{LCP}(s, L, U) = \lim_{U\to\infty}\lim_{L\to 0} C_{LCP}(s, L, U) = C_{NL}(s).$$

for all bet sizes $s \in [0, \infty)$.

Proof. Again, we already have the limiting value of $x_2|_{0,\infty}$, so we can plug this into the expression for the calling threshold:

$$\begin{aligned}\lim_{L\to 0}\lim_{U\to\infty} C_{LCP}(s, L, U) &= \lim_{L\to 0}\lim_{U\to\infty} \frac{x_2 + s}{1+s} \\ &= \frac{\frac{1}{7} + s}{1+s} \\ &= 1 - \frac{6}{7(1+s)} \\ &= C_{NL}(s).\end{aligned}$$

□

We have now shown that the bettor and caller strategies for LCP converge to those of FBCP and NLCP as the limits L and U approach their extreme values. In the next section, we explore the value of LCP in more detail, and in particular how it relates to that of FBCP and NLCP.

8 Parameter and Payoff Analysis

Having established the Nash equilibrium (Section B), analyzed the game value (Section 6), and proved convergence to FBCP and NLCP (Section 7), we now explore in greater detail how the parameters L and U affect player strategies and payoffs. This section summarizes key insights and presents visualizations that illuminate the strategic dynamics of LCP. Complete technical proofs are provided in Appendices **??** and **??**.

8.1 Visualizing Payoffs in Equilibrium

In Nash equilibrium, each hand combination (x, y) uniquely determines the bettor's payoff. Figure 4 shows how these payoffs vary across the unit square for different values of L and U, from strict limits ($L = U = 1$) to lenient limits ($L = 0, U \to \infty$).

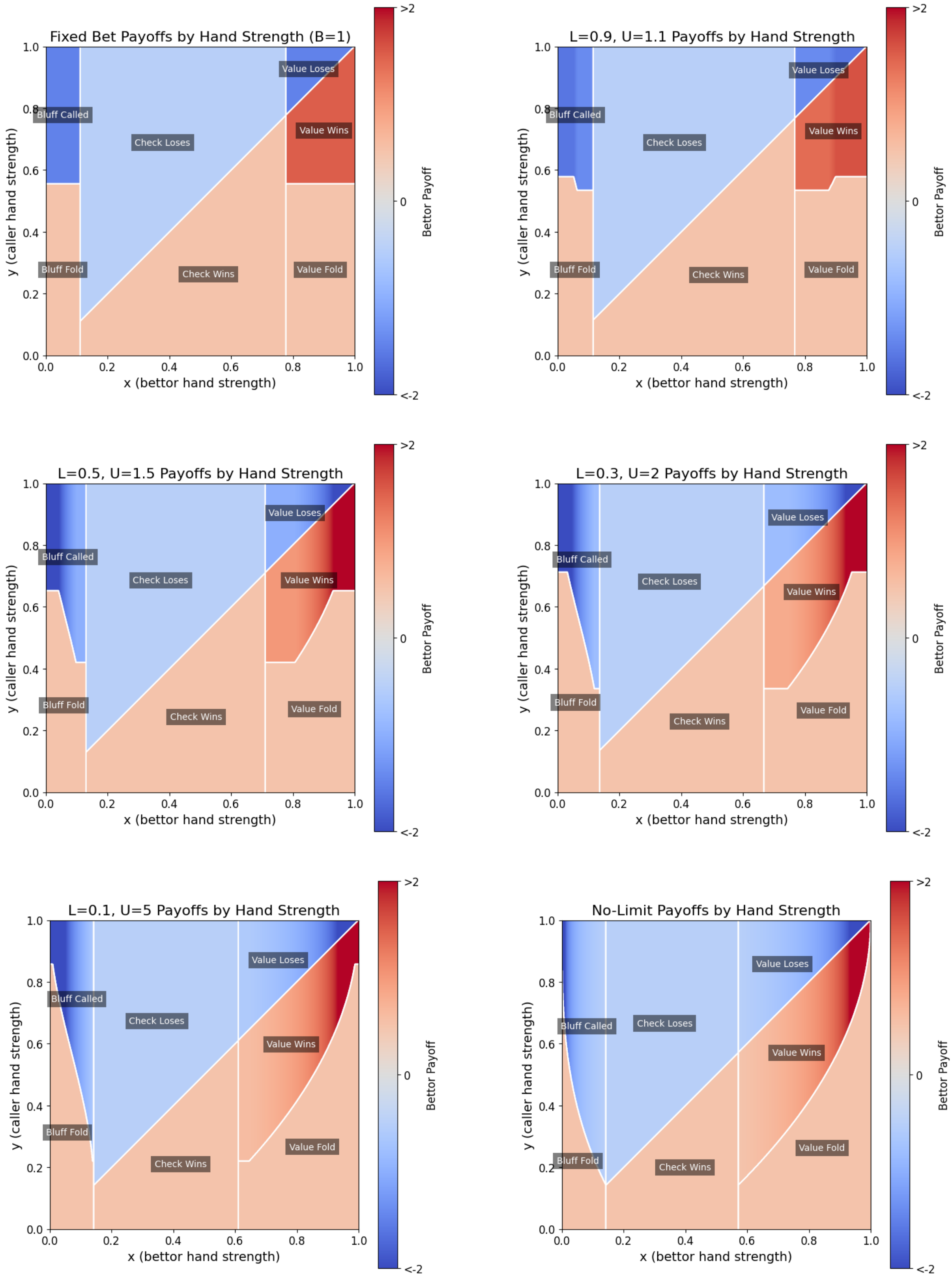

Figure 4: Bettor payoffs in Nash equilibrium as a function of hand strengths x, y for fixed bet size $B = 1$ (top left), and No-Limit Continuous Poker (bottom right). Intermediate plots show the payoffs for different values of L and U between the two extremes. Regions are labeled according to the outcome of the game in Nash equilibrium.

The visualization reveals that the biggest wins and losses occur when both hands are strong (top right), consistent with real poker intuition. Large payoffs also occur when a very weak bettor bluffs big and gets called by a strong caller (top left). As limits become more lenient, these extreme outcomes become more pronounced but also less likely, since making and calling maximum bets become riskier for both players.

It is worth noting that the bettor's strongest hands (right edge) actually become less likely to make any profit (more than the ante) as limits increase. These strongest hands make very large bets, which force all but the strongest hands to fold, but win huge pots when they do get called.

In more complicated poker variants, it is common to "slowplay" strong hands by checking or making small bets to induce bluffs from the opponent. In LCP, there is only one betting round and the caller is not allowed to raise, both of which make slowplaying obsolete. With extremely strong hands, the benefit of winning a large pot when betting big outweighs the lower likelihood of getting called.

This strategic pattern demonstrates a fundamental tension in poker: extracting maximum value from strong hands requires finding the optimal balance between bet size (which determines pot size when called) and calling frequency (which decreases as bet size increases). In LCP, the strongest hands resolve this tension by accepting a lower calling frequency in exchange for winning much larger pots.

8.2 Expected Value by Hand Strength

Beyond specific hand matchups, we can analyze the expected value $EV(x)$ of a bettor hand x averaged over all possible caller hands. This function characterizes how profitable each hand is in equilibrium (intuitively, how happy the bettor should be to see each specific hand).

We can calculate $EV(x)$ analytically by combining previous results:

$$EV(x) = \begin{cases} x_2 - \frac{1}{2} & \text{if } x \le x_2 \\ x - \frac{1}{2} & \text{if } x_2 < x \le x_3 \\ x(2L+1) - L(c(L)+1) - \frac{1}{2} & \text{if } x_3 < x < v(L) \\ x(2v^{-1}(x)+1) - v^{-1}(x)(c(v^{-1}(x))+1) - \frac{1}{2} & \text{if } v(L) \le x \le v(U) \\ x(2U+1) - U(c(U)+1) - \frac{1}{2} & \text{if } x > v(U). \end{cases} \tag{20}$$

This function is much easier to view graphically (Figure 5). Observations:

- The function $EV(x)$ is increasing in x (Appendix 8, Theorem **??**).
- All bluffing hands ($x \le x_2$) achieve the same expected value $x_2 - 1/2$, regardless of hand strength. This is because the caller will never call with a

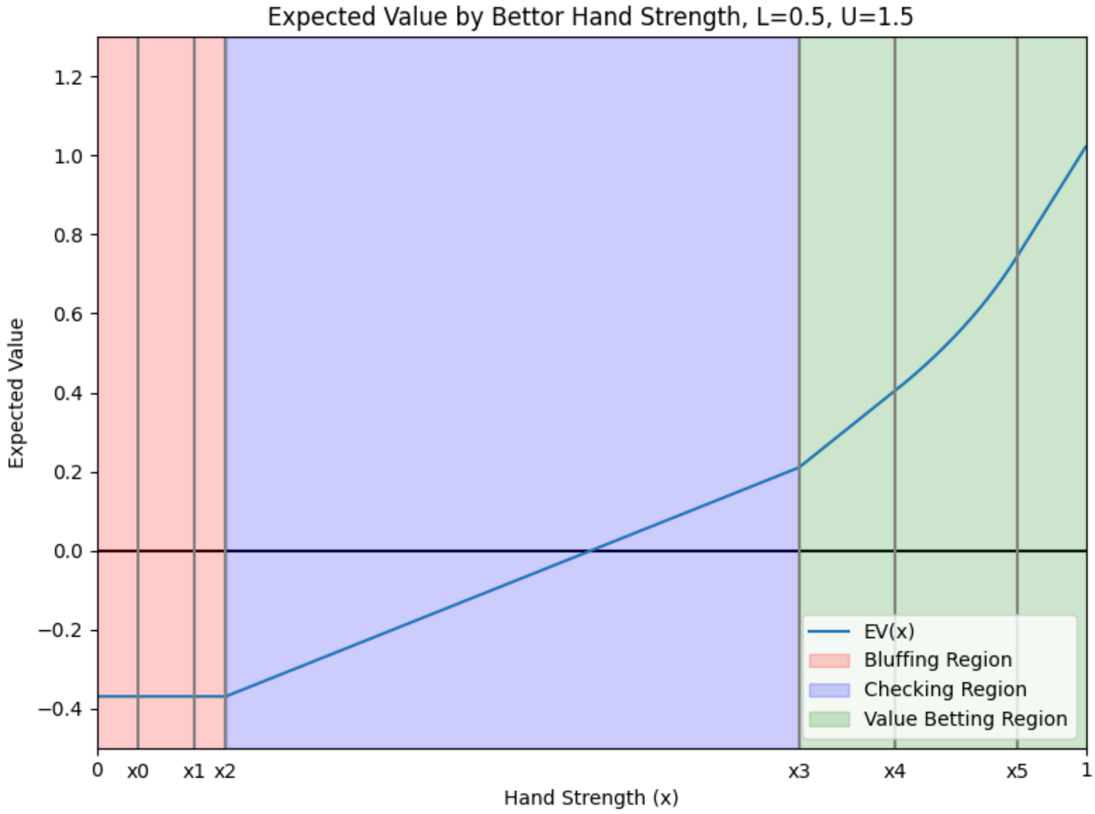

Figure 5: Expected value of bettor hand strength x under optimal play in LCP.

losing hand, so a bluff either induces a fold or loses the hand. If a bluffing action had lower EV, the bettor would simply not take this action, so all bluffing hands must have equal EV in equilibrium.

- Checking hands ($x_2 < x \leq x_3$) have EV equal to $x - 1/2$. Notice the slope of 1 from the pot being exactly 1 unit.
- Value betting hands ($x > x_3$) earn more steeply increasing returns as hand strength increases. This is because the pot is increasing in size as the bettor makes larger bets with stronger hands, so a marginal increase in hand strength becomes increasingly more profitable.

8.3 Effect of Increasing the Upper Limit U

A counterintuitive result emerges when examining how individual hand values change as we increase U: for strong hands, the expected value *decreases* beyond a certain threshold of U (see Figure 6). This occurs despite the bettor having strictly more strategic options.

Notice that this result is not contradictory to the previous section. There are three distinct statements being made:

- For any fixed parameters L, U, stronger hands have higher expected value (Figure 5 and Theorem **??**). This must be true because the bettor could always play as though they had the weaker hand.
- Averaged across all hand strengths $x \in [0, 1]$, increasing U or decreasing L increases the bettors' overall expected value. This must be true because the bettor could artificially restrict their bet sizes to a narrower range.
- For a fixed hand strength x, increasing U can *decrease* the expected value of that specific hand (Figure 6). This is the counterintuitive result being discussed here.

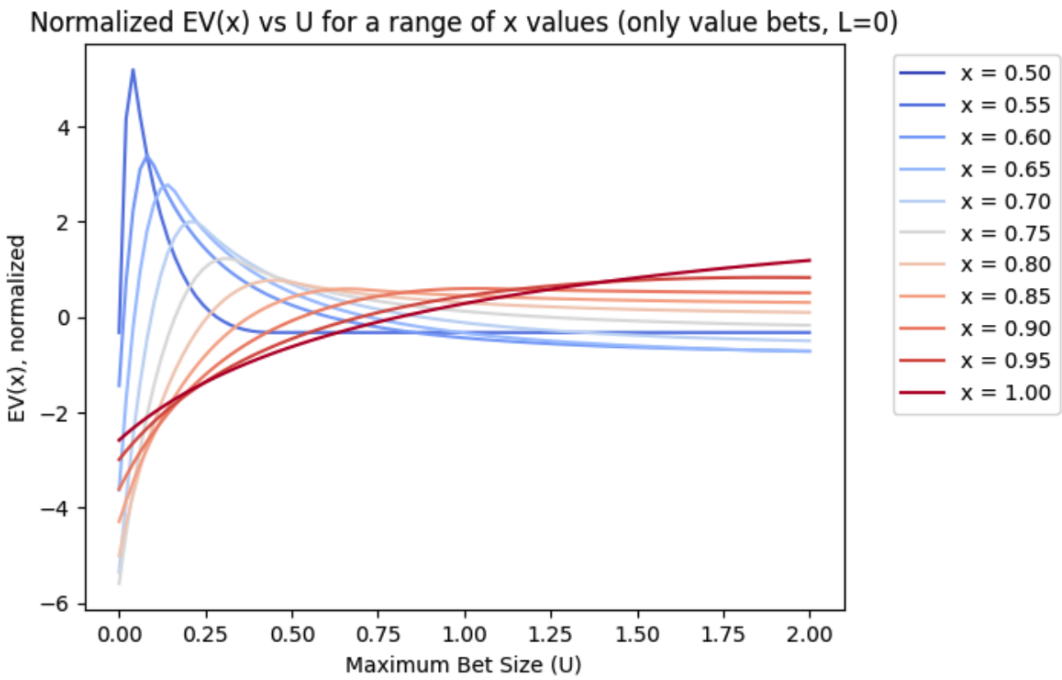

Figure 6: Expected value of a value-betting hand x versus the upper limit U under optimal play. Each curve increases in U up to some peak, after which it decreases, indicating that more flexibility decreases the expected value of that specific hand.

The explanation lies in strategic interdependence: as U increases, the bettor can make larger bets with their strongest hands, which forces the caller to become more conservative across *all* bet sizes. This defensive adjustment by the caller harms the expected value of intermediate-strength hands, even though they're betting less. Only the very strongest hands (above a threshold greater than $v(U)$) benefit from the increased flexibility, and the threshold for 'strongest' only increases further as we increase U.

We can state this formally:

Theorem 8.1. *For any any $0 < L \leq U$ and any value-betting hand strength $x > x_3$, the derivative $\frac{d}{dU}EV(x) < 0$ if*

$$x < \max\left(v(U), \frac{1}{2(1+U)}\left(U\frac{\partial x_2}{\partial U} + \frac{U^2 + x_2(1+2U)}{1+U}\right)\right),$$

and $\frac{d}{dU}EV(x) > 0$ otherwise.

This is equivalent to the statement above. The expression on the right is the threshold on x below which the expected value is decreasing in U. This results relies on several others, which are stated below:

Lemma 8.1 (Monotonicity in U)**.** *For fixed L and $x \in [x_3, v(U)]$, as U increases:*

1. *The bluffing threshold x_2 increases (more hands bluff).*
2. *The bet size $v^{-1}(x)$ decreases.*
3. *The calling cutoff for the chosen bet size $c(v^{-1}(x))$ increases despite smaller bets.*

A complete analysis and proof can be found in Appendix **??**. The crucial takeaway is that varying bet size limits has complex effects on optimal strategies and payoffs, and these effects are not always intuitive.

9 Conclusion

We have introduced and analyzed Limit Continuous Poker (LCP), a parametric family of simplified poker games that bridges the gap between Fixed-Bet Continuous Poker (FBCP) and No-Limit Continuous Poker (NLCP). By imposing lower and upper bounds L and U on bet sizes, LCP creates a rich spectrum of strategic environments that interpolate continuously between the fixed-bet and no-limit extremes.

9.1 Summary of Key Results

Our analysis has yielded several main contributions:

Nash Equilibrium Characterization: We first described concepts of monotonicity and monotone-admissibility to characterize a notion of optimality, then derived the unique Nash equilibrium of LCP satisfying this condition. In this Nash equilibrium, the bettor partitions hands into bluffing, checking, and value betting regions, with bet sizes varying continuously within the bluffing and value betting ranges. The caller responds with a calling threshold as a function of bet size.

Game Value Formula: We computed the value of LCP as a rational function of the betting limits, obtaining the surprisingly compact expression

$$V(r,t) = \frac{1 - r^3 - t^3}{14 - 2r^3 - 2t^3}$$

in the transformed coordinates $r = L/(1+L)$ and $t = 1/(1+U)$. This formula reveals a remarkable symmetry: $V(r,t) = V(t,r)$, meaning that swapping the roles of minimum and maximum bet constraints (in a specific reciprocal sense) leaves the game value unchanged. We proved that the value is monotonically increasing in U and decreasing in L, confirming the intuition that more betting flexibility favors the bettor.

Convergence to Limiting Cases: We established that LCP smoothly converges to both FBCP and NLCP in the appropriate limit regimes. As $L \to B$ and $U \to B$, the strategies and value converge to those of FBCP with fixed bet size B. As $L \to 0$ and $U \to \infty$, they converge to those of NLCP. These results validate LCP as a genuine generalization of both classical variants.

Parameter Sensitivity and Strategic Dynamics: Our analysis revealed counterintuitive strategic effects: increasing the upper limit U does not uniformly benefit all bettor hands. While the strongest hands gain from the ability to make larger bets, intermediate-strength hands can suffer because the caller adjusts by becoming more conservative across all bet sizes. This illustrates the complex strategic interdependencies in equilibrium play.

9.2 Strategic Insights and Connections to Real Poker

The theoretical results for LCP offer several insights relevant to practical poker strategy:

Bet Sizing and Stack Depth: In real poker, effective stack sizes create implicit upper bounds on bet sizes, analogous to our parameter U. Our analysis suggests that deeper stacks (higher U) create more strategic complexity and favor skilled players who can exploit the additional betting options. The symmetry property $V(L, U) = V(1/U, 1/L)$ suggests a duality between raising minimum bet requirements and constraining maximum bets.

Bluffing Frequency and Bet Size: The equilibrium strategy confirms poker wisdom that bluffing frequency should be calibrated to value betting. It shows that wider bet size limits should incentivize a wider range of bluff to balance possible value bets, as opposed to always bluffing big to scare off the opponent or always bluffing small to minimize risk. This formal description might inform bluffing frequencies and sizes for a variety of stack sizes in heads-up games.

Calling Thresholds and Pot Odds: The calling function $c(s)$ embodies the pot odds principle: the caller must be offered better odds (a larger pot relative to the cost of calling) to justify calling with weaker hands. The equilibrium precisely balances the bettor's bluffing and value betting frequencies against these pot odds.

9.3 Limitations

Several simplifications distinguish LCP from real poker:

- **Single Betting Round:** Real poker involves multiple streets of betting with community cards revealed between rounds, creating dynamic information revelation. LCP models only a single decision point.

- **Uniform Hand Distribution:** We assume hand strengths are uniformly distributed on $[0, 1]$. Real poker hand distributions are discrete and non-uniform, with specific card combinations determining hand strength.

- **Perfect Correlation:** In our model, hands are perfectly ordered (if $x > y$, the bettor always wins). Real poker has card removal effects and some hands have non-zero equity even when behind.

- **Symmetric Information:** Both players receive one hand each. Real poker often features asymmetric information structures (e.g., one player knows the other folded a certain range on an earlier street).

Despite these limitations, the tractability of LCP enables rigorous analysis that would be impossible in more complex settings, providing a foundation for understanding strategic principles.

9.4 Future Directions

Several natural extensions of this work could deepen our understanding of bet sizing in poker:

Multiple Betting Rounds: Extending LCP to multiple streets with information revelation between rounds would capture dynamic aspects of poker strategy. One could study how bet size limits in early rounds affect optimal play in later rounds.

Asymmetric Limits: Our model assumes both players face the same ante and pot size. Investigating games where players have different effective stack sizes (asymmetric U values) could model scenarios common in tournament poker.

Non-Uniform Hand Distributions: Relaxing the uniform distribution assumption to model more realistic hand strength distributions could test the robustness of our results.

Discrete Approximations: Real poker involves discrete bet sizing increments (e.g., betting in whole chips or minimum raise increments). Studying discrete approximations to LCP could bridge the gap between our continuous model and practical applications.

Computational Tools: The closed-form solutions for LCP could be used to validate numerical solvers for more complex poker variants. The smooth parameter dependence makes LCP an ideal test bed for computational game theory algorithms.

Multi-Player Extensions: While our analysis focuses on two-player games, extending to three or more players would introduce new strategic considerations such as collusion, side pots, and positional dynamics.

A Monotone and Continuous Strategy Proofs

This appendix provides the detailed proofs regarding monotone calling strategies, monotone-admissible and continuous betting strategies, and their relationship to the uniqueness of the Nash equilibrium in LCP.

A.1 Monotone Strategies and Weak Dominance

Lemma A.1. *If a calling strategy violates the first monotonicity condition (see 4.2) for a nonzero-measure set of hands for any bet size s, it is weakly dominated. Specifically, if there exists s and measurable sets $A, B \subseteq [0,1]$ such that:*

1. *The caller calls s with hands in A*
2. *The caller folds s with hands in B*
3. $\sup A \leq \inf B$
4. *A and B have positive measure*

then the strategy is weakly dominated.

Proof. Let σ_C be the non-monotone strategy described above. Since A and B are nonzero-measure, there exist subsets $A' \subseteq A$ and $B' \subseteq B$ such that:

1. A' and B' have positive measure
2. $|A'| = |B'|$

Where $|A|$ and $|B|$ denote the Lebesgue measure of A and B (see Figure 7).

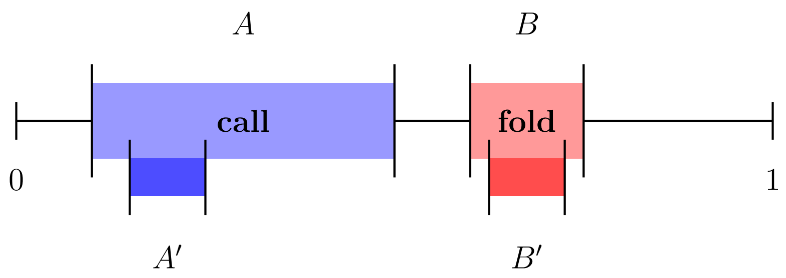

Figure 7: A simple case of sets A and B which violate monotonicity ($\sup A \leq \inf B$). We can find equal-measure subsets $A' \subseteq A$ and $B' \subseteq B$ to swap actions, improving the strategy.

The existence of such subsets follows from a fundamental property of nonatomic measures: since the uniform distribution on $[0, 1]$ is nonatomic (no single point has positive probability), for any two measurable sets with positive measure, we can always find measurable subsets of equal measure[3]. This property allows us to construct the strategy improvement described below.

Let σ'_C be the strategy which switches the actions for A' and B', i.e. calls with B' and folds with A' (and behaves identically for all other bet sizes). We now analyze how this change affects the caller's performance against any betting strategy.

Against a bet of size s, the key improvement occurs in two scenarios:

1. When $y \in B'$ and $x \in A'$: σ_C folds while σ'_C calls and wins (since $x \in A$ and $y \in B$ with $\sup A \leq \inf B$)
2. When $y \in A'$ and $x \in B'$: σ_C calls and loses while σ'_C folds (avoiding the loss)

For all other cases, σ_C and σ'_C behave identically, so σ'_C is weakly better than σ_C against every betting strategy.

To show that σ_C is strictly dominated, consider a betting strategy which always bets s. Against this strategy, both scenarios above occur with positive probability (since A' and B' have positive measure), so σ'_C is strictly better than σ_C. Thus, σ_C is weakly dominated. $\square$

We now show that the restriction to monotone-admissible betting strategies implies a natural ordering of bluff sizes.

Lemma A.2. *If a betting strategy σ_B is monotone-admissible, then for any two bluffing hands $x < x'$ with corresponding bet sizes s and s' respectively, we must have $s' \geq s$.*

Proof. Suppose for contradiction that there exist bluffing hands $x < x'$ with bet sizes $s > s'$. We will show that σ_B is not monotone-admissible by constructing a strategy σ'_B that weakly dominates σ_B against all monotone calling strategies.

Define σ'_B to be identical to σ_B except that it bets s' with hand x instead of s.

Let σ_C be an arbitrary monotone calling strategy, and let $c(s)$ denote the calling threshold for bet size s under σ_C. By Definition 4.1 (condition 2), monotonicity implies

$$c(s') \leq c(s).$$

We compare the expected payoffs of betting s versus s' with hand x against σ_C.

Case 1: $x' < c(s')$.

Since $c(s') \leq c(s)$, we have $x < x' < c(s') \leq c(s)$. Thus the caller never calls with a hand weaker than x or x' under either bet size. Both bets succeed as pure bluffs with probability $P(Y < c(\cdot))$, winning the pot of size 1. Since $s' < s$, betting s' risks less to win the same amount, so

$$\mathbb{E}[\text{payoff of } s' \mid x] > \mathbb{E}[\text{payoff of } s \mid x].$$

Case 2: $x' \geq c(s')$.

When betting s' with hand x, the bettor wins at showdown against any caller hand $y \in [c(s'), x']$, an event with positive probability. When betting s with hand x, since $c(s) \geq c(s') > x$, the bettor never wins at showdown (as $x < c(s)$ implies all calling hands beat x). Therefore,

$$\mathbb{E}[\text{payoff of } s' \mid x] > \mathbb{E}[\text{payoff of } s \mid x].$$

In both cases, σ'_B achieves strictly higher expected payoff than σ_B against σ_C. Since this holds for all monotone calling strategies σ_C, strategy σ_B is not monotone-admissible, contradicting our assumption. $\square$

A.2 Continuity

We stated in section 4 that the betting strategy should be continuous except at the bluffing and value-betting thresholds. Here we provide a formal justification for this requirement.

Consider the bettor's expected value function for a value hand x:

$$EV(s,x) = (x - c(s))(1 + s) - (1 - x)s + c(s) \tag{21}$$

which is continuous in both s and x. For a fixed value-betting hand strength x, the bettor chooses an optimal bet size $s^*(x)$ such that:

$$s^*(x) = \arg\max_{s\in[L,U]} EV(s,x) \tag{22}$$

Assuming $EV(s,x)$ is strictly concave in s (which follows if $c(s)$ is sufficiently smooth and monotone increasing), the maximizer $s^*(x)$ is unique. By the Theorem of the Maximum [4], since $EV(s,x)$ is continuous and the set of possible bet sizes S is compact, the optimal strategy $s^*(x)$ is a continuous function of x.

If $s(x)$ were discontinuous at some x_0, it would imply that $EV(s,x)$ possesses multiple global maxima at x_0, or that the EV function itself is not concave, allowing the optimal bet size to "jump" between distant peaks.

For the bluffing range, continuity is a consequence of the equilibrium balancing requirement. To maintain the caller's indifference at every $s \in [L,U]$, there must exist a corresponding bluffing hand strength such that the ratio of bluffs to value hands satisfies the Nash condition.

As previously established, the bluffing strategy $b^{-1}(x)$ is strictly decreasing in x as a consequence of being monotone-admissible. Since every $s \in [L,U]$ must be represented in the bluffing range to balance the value bets, the image of $b^{-1}(x)$ is the interval $[L,U]$.

By the Intermediate Value Theorem for Monotone Functions (the property that a monotonic function is continuous if and only if its image is an interval), $b^{-1}(x)$ must be continuous.

B Appendix: Proof of Nash Equilibrium

We now show formally that the strategy profile described in Section 5 is truly a Nash Equilibrium.

Proof. To show that this is a Nash equilibrium, we need to show that no player can improve their payoff by unilaterally deviating from the strategy profile.

In the proof, we assume that all of the constraints outlined in the previous section are satisfied. The solution was obtained by solving for $c(s)$ in terms of

x_2, then using this to solve for $v(s)$, and finally solving for $b(s)$ up to a constant of integration. The resulting system of 7 equations in 7 unknowns was solved symbolically using Sympy. The full python script is available on GitHub.

1. **Caller's Deviation:** Fix the bet size s and consider the caller's payoff from either calling or folding for each hand strength y.

$$\begin{aligned}
&\mathbb{E}[\text{call}|y, s] = \mathbb{P}[x < y|s](1 + s) + \mathbb{P}[x \geq y|s](-s) \\
&\mathbb{E}[\text{fold}|y, s] = 0
\end{aligned}$$

   From section 4.4.1, we know that the expected value of a call is exactly 0 for $y = c(s)$ (by design). The expected value of calling is weakly increasing in y, so it must be weakly greater than 0 for $y > c(s)$ and weakly less than 0 for $y < c(s)$.

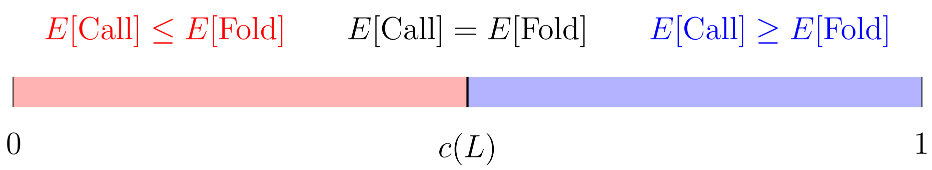

Figure 8: Caller's decision threshold $c(L)$. At hand strength of exactly $c(L)$, the caller is indifferent between calling and folding. Since the value of calling weakly increases with y, it must be weakly greater than 0 for $y > c(L)$ and weakly less than 0 for $y < c(L)$. Folding always has value 0.

   This proves that calling is weakly better than folding for all $y > c(s)$, and that folding is weakly better than calling for all $y < c(s)$, so the caller cannot improve their payoff by deviating from the strategy profile.

2. **Bettor's Deviation:** We need to consider a few cases.

   (a) $x < c(s)$: These are hands and bet sizes for which the caller will call with only stronger hands (potential bluffs). The expected value of betting here is

$$\begin{aligned}
\mathbb{E}[\text{bet } s|x] &= \mathbb{P}[\text{call with worse}] \cdot (1 + s) - \mathbb{P}[\text{call with better}] \cdot s + \mathbb{P}[\text{fold}] \cdot 1 \\
&= 0 - (1 - c(s)) \cdot (s) + c(s) \\
&= c(s) - (1 - c(s)) \cdot s \\
&= x_2,
\end{aligned}$$

with the last line coming from equation 6. The value of checking is always

$$\mathbb{E}[\text{check}|x] = x.$$

This means that (by design), the bettor is indifferent between checking and betting any amount at $x = x_2$. Importantly, the value of betting is independent of the hand strength x while the value of checking is strictly increasing in x, so checking must be preferable for $x_2 < x < c(L)$ and betting must be preferable for $x < x_2$, which is exactly what our strategy profile does. Because the value of bluffing is simply x_2 no matter the bet size or hand strength, we also know that the bettor cannot improve their payoff by bluffing with different bet sizes.

(b) $c(s) \leq x < x_3$: These are hands and bet sizes for which the caller will at least sometimes call with weaker hands (potential value bets), but where the optimal strategy still checks. The expected value of betting here is

$$\begin{aligned}\mathbb{E}[\text{bet } s|x] &= \mathbb{P}[\text{call with worse}] \cdot (1+s) - \mathbb{P}[\text{call with better}] \cdot s + \mathbb{P}[\text{fold}] \cdot 1 \\ &= (x - c(s))(1+s) - (1-x) \cdot (s) + c(s) \\ &= s(2x - c(s) - 1) + x,\end{aligned}$$

while that of checking is

$$\mathbb{E}[\text{check}|x] = x.$$

We know from 5 that

$$\begin{aligned}(x_3 - c(L)) \cdot (1+L) - (1-x_3) \cdot (L) + c(L) &= x_3 \\ 2x_3 - c(L) - 1 &= 0\end{aligned}$$

Using our inequality $c(s) \leq x < x_3$ and the fact that $c(L)$ is the minimum of $c(s)$, we get

$$\begin{aligned}2x_3 - c(L) - 1 &= 0 \\ 2x - c(s) - 1 &\leq 0.\end{aligned}$$

Substituting this into the expected value of betting, we get

$$\begin{aligned}\mathbb{E}[\text{bet } s|x] &= s(2x - c(s) - 1) + x \\ &\leq s \cdot 0 + x \\ &= x \\ &= \mathbb{E}[\text{check}|x].\end{aligned}$$

So no value can be gained by deviating from checking here.

(c) $x_3 \leq x < x_4$: These are value bets where the bettor should bet the minimum. We need to show that the bettor cannot improve their payoff by either checking or by betting more.

The expected value of betting the minimum is

$$\mathbb{E}[\text{bet } L|x] = L(2x - c(L) - 1) + x.$$

Again, we can use 5 to get

$$\begin{aligned} 2x_3 - c(L) - 1 &= 0 \\ 2x - c(L) - 1 &\geq 0, \end{aligned}$$

since $x \geq x_3$. Substituting like before,

$$\begin{aligned} \mathbb{E}[\text{bet } L|x] &= L(2x - c(L) - 1) + x \\ &\geq L \cdot 0 + x \\ &= x \\ &= \mathbb{E}[\text{check}|x]. \end{aligned}$$

So no value can be gained by deviating from betting to checking.

What about betting more? To show that this cannot improve the bettor's payoff, we show that the expected value of betting is weakly decreasing in s for $x < x_4$, and must therefore be maximized at the lowest possible bet of L.

$$\frac{d}{ds}\mathbb{E}[\text{bet } s|x] = -sc'(s) - c(s) + 2x - 1$$

We know from 4 that this equals 0 when $x = v(s)$. We also know that $v(s)$ is at least x_4 for all $s \in [L, U]$, so $x < x_4 \leq v(s)$ for any such s. This means that

$$\begin{aligned} \frac{d}{ds}\mathbb{E}[\text{bet } s|x] &= -sc'(s) - c(s) + 2x - 1 \\ &\leq -sc'(s) - c(s) + 2x_4 - 1 \\ &= 0 \end{aligned}$$

for any $s \in [L, U]$. Therefore, the expected value of betting is decreasing in s for $x < x_4$, and must therefore be maximized at the lowest possible bet of L, so the bettor cannot improve their payoff by betting more.

(d) $x_4 \leq x < x_5$: These are value bets where the bettor should bet an intermediate amount between L and U. We need to show that the bettor cannot improve their payoff by either checking, betting less, or by betting more.

Rather than showing that checking is inferior to the optimal bet size, we show that checking is inferior to betting the minimum, which we will later show is inferior to the optimal bet size. Like the previous cases, the expected value of betting the minimum is

$$\mathbb{E}[\text{bet } L|x] = L(2x - c(L) - 1) + x.$$

Like before, we know that $2x - c(L) - 1 \geq 0$ for $x \geq x_3$ (and in this case, $x \geq x_4 \geq x_3$). This means that

$$\begin{aligned}\mathbb{E}[\text{bet } L|x] &= L(2x - c(L) - 1) + x \\ &\geq L \cdot 0 + x \\ &= x \\ &= \mathbb{E}[\text{check}|x].\end{aligned}$$

So betting the minimum is at least as good as checking.

Now we show that betting any amount other than $v^{-1}(x)$ cannot gain value. Let's again consider the derivative of the expected value with respect to s.

$$\frac{d}{ds}\mathbb{E}[\text{bet } s|x] = -sc'(s) - c(s) + 2x - 1$$

We know from 4 that this is equal to 0 when $x = v(s)$:

$$-sc'(s) - c(s) + 2v(s) - 1 = 0.$$

This derivative is clearly an increasing function of x, so for $x < v(s)$, the expected value is decreasing in s. But since v is an increasing function, $x < v(s)$ is equivalent to $v^{-1}(x) < s$ (essentially, our bet size is too large for the hand strength).

This should make sense - when our bet size is too large for the hand strength, the expected value of that bet is decreasing in the bet size, so smaller bets are more profitable. We can show the same for bets too small: when $x > v(s)$, the expected value is increasing in s. This is equivalent to saying that $v^{-1}(x) > s$, or our bet size is too small for the hand strength, so larger bets are more profitable.

We have shown that the expected value of betting is increasing for $s < v^{-1}(x)$, equal to 0 at $s = v^{-1}(x)$, and decreasing for $s > v^{-1}(x)$, so the expected value of betting is maximized at $s = v^{-1}(x)$. This means that the bettor cannot improve their payoff by deviating from this bet size. In particular, they cannot benefit by betting the minimum, which in turn proves that they cannot benefit by checking, as we showed above.

(e) $x_5 \leq x \leq 1$: These are value bets where the bettor should bet the maximum. We need to show that the bettor cannot improve their payoff by either checking or betting less.

The expected value of betting the maximum is

$$\mathbb{E}[\text{bet } U|x] = U(2x - c(U) - 1) + x.$$

Plugging $s = U$, $x = v(U) = x_5$ into 4, we get

$$-Uc'(U) - c(U) + 2x_5 - 1 = 0.$$

What happens when $x > x_5$? This expression must be greater than 0, meaning the expected value of betting is increasing in s for $x > x_5$. If it is always increasing in s for such x, then it must be maximized at the largest possible bet of U, so the bettor cannot improve their payoff by betting less.

We can use the exact same logic as the previous case to show that checking cannot improve the payoff either.

□

C Payoff Analysis: Complete Proofs

Below are proofs of results from Section 8, specifically Lemma 8.1 and Theorem 8.1.

C.1 Proof of Lemma 8.1

Recall the lemma statement:

For fixed L and $x \in [x_3, v(U)]$, as U increases:

1. *The bluffing threshold x_2 increases (more hands bluff).*
2. *The bet size $v^{-1}(x)$ decreases.*

3. *The calling cutoff for the chosen bet size* $c(v^{-1}(x))$ *increases despite smaller bets.*

Proof of Lemma 8.1.

We show part 1 first. Recall that x_2 is given by:

$$x_2 = \frac{r^3 + t^3 - 1}{r^3 + t^3 - 7}$$

where $r = L/(1 + L)$ and $t = 1/(1 + U)$. We can use the chain rule to differentiate x_2 with respect to U:

$$\frac{dx_2}{dU} = \frac{\partial x_2}{\partial t}\frac{dt}{dU}.$$

Note that r has no dependence on U. We compute:

$$\frac{\partial x_2}{\partial t} = \frac{-18t^2}{(r^3 + t^3 - 7)^2}, \quad \frac{dt}{dU} = -\frac{1}{(1 + U)^2}.$$

Therefore,

$$\frac{dx_2}{dU} = \frac{-18t^2}{(r^3 + t^3 - 7)^2} \cdot \left(-\frac{1}{(1 + U)^2}\right) = \frac{18t^2}{(1 + U)^2\,(r^3 + t^3 - 7)^2} > 0,$$

which is positive since $r, t \in (0, 1)$ implies $r^3 + t^3 - 7 < 0$.

Now part 2. If we fix x at any intermediate value-betting hand strength (betting neither the minimum nor maximum bet size) and then increase U, the bet size s made by x decreases. The intermediate value-betting hands are exactly $x \in [x_3, v(U)]$ and their bet sizes are given by $s = v^{-1}(x)$, so we need to show that $\frac{d}{dU}v^{-1}(x) < 0$.

Recall that

$$v^{-1}(x) = -\frac{\sqrt{(4x - 4)(2x_2 - 2)}}{4x - 4} - 1,$$

where $x_2 = \frac{r^3+t^3-1}{r^3+t^3-7}$ with $t = 1/(1 + U)$. Importantly, $v^{-1}(x)$ is only dependent on U through x_2, which in turn depends on U only through t. Using the chain rule:

$$\frac{d}{dU}v^{-1}(x) = \frac{\partial v^{-1}(x)}{\partial x_2}\frac{\partial x_2}{\partial t}\frac{dt}{dU}$$

We compute each factor:

$$\frac{\partial v^{-1}(x)}{\partial x_2} = -\frac{1}{\sqrt{(4x - 4)(2x_2 - 2)}} = -\frac{1}{(v^{-1}(x) + 1)(4 - 4x)} < 0$$

which is negative since $x \in [0, 1]$ and $v^{-1}(x) > 0$. From Lemma **??**, we know that $\frac{\partial x_2}{\partial t}\frac{dt}{dU} > 0$.

Therefore, the product of the three terms is always negative, so the bet size of intermediate bets is decreasing in U.

Finally part 3. Recall that $c(s)$ is defined as the minimum hand strength y which should call a bet of size s and is given in Nash equilibrium by:

$$c(s) = \frac{x_2 + s}{s + 1}$$

We are specifically interested in how $c(v^{-1}(x))$ varies with U for $x \in [x_3, v(U)]$, since this represents how the calling cutoff changes both directly from a strategic change, as well as indirectly due to the lower bet size s. It turns out that the calling cutoff is increasing in U for all $x \in [x_3, v(U)]$. This is surprising because we just showed that the bet size s is decreasing in U, and we expect smaller bets to be called more often. For reasons we will see later, this effect is overpowered by a strategic shift for the caller, who calls less often for all bet sizes as U increases.

Explictly, we are showing that for any fixed $x \in [x_3, v(U)]$,

$$\frac{d}{dU}c(v^{-1}(x)) > 0$$

As mentioned above, $c(s) = \frac{x_2+s}{s+1}$ is dependent on U in two distinct ways: directly through $x_2 = x_2(t)$ where $t = 1/(1 + U)$, and indirectly through the bet size $s = v^{-1}(x)$. We use the multivariate chain rule:

$$\frac{d}{dU}c(v^{-1}(x)) = \frac{\partial c(s)}{\partial s}\frac{dv^{-1}(x)}{dU} + \frac{\partial c(s)}{\partial x_2}\frac{\partial x_2}{\partial t}\frac{dt}{dU}$$

The partial derivatives of $c(s)$ are:

$$\frac{\partial c(s)}{\partial s} = \frac{1 - x_2}{(s + 1)^2} \quad \text{and} \quad \frac{\partial c(s)}{\partial x_2} = \frac{1}{s + 1}.$$

Substituting $s = v^{-1}(x)$, we can simplify the first term using the fact that $(v^{-1}(x) + 1)^2 = (1 - x_2)/(2 - 2x)$:

$$\left.\frac{\partial c(s)}{\partial s}\right|_{s=v^{-1}(x)} = \frac{1 - x_2}{(v^{-1}(x) + 1)^2} = 2 - 2x$$

From part 2, we have:

$$\frac{dv^{-1}(x)}{dU} = \frac{\partial v^{-1}(x)}{\partial x_2}\frac{\partial x_2}{\partial t}\frac{dt}{dU} = \frac{-1}{(v^{-1}(x) + 1)(4 - 4x)}\frac{\partial x_2}{\partial t}\frac{dt}{dU}.$$

Substituting everything:

$$\begin{aligned}
\frac{d}{dU}c(v^{-1}(x)) &= (2-2x)\cdot\frac{-1}{(v^{-1}(x)+1)(4-4x)}\cdot\frac{\partial x_2}{\partial t}\frac{dt}{dU}+\frac{1}{v^{-1}(x)+1}\cdot\frac{\partial x_2}{\partial t}\frac{dt}{dU}\\
&= \frac{1}{v^{-1}(x)+1}\cdot\frac{\partial x_2}{\partial t}\frac{dt}{dU}\cdot\left(-\frac{2-2x}{4-4x}+1\right)\\
&= \frac{1}{v^{-1}(x)+1}\cdot\frac{\partial x_2}{\partial t}\frac{dt}{dU}\cdot\frac{1}{2}
\end{aligned}$$

All three factors are positive: $v^{-1}(x) > 0$, and by part 1, $\frac{\partial x_2}{\partial t}\frac{dt}{dU} > 0$. Therefore, the calling cutoff is increasing in U for all $x \in [x_3, v(U)]$.

□

C.1.1 Proof of Theorem 8.1

Having these tools, we can now finally return to the proof of Theorem 8.1.
Proof. Recall the expected payoff of a value-betting hand x from equation 20:

$$EV(x) = \frac{1}{2}c(s) + (x - c(s))\left(s + \frac{1}{2}\right) + (1-x)\left(-s - \frac{1}{2}\right)$$

We break the proof into two cases:

Case 1 ($x > v(U)$): In this case, hand x bets the maximum amount U. Recall that $c(s)$ is implicitly a function of s and x_2, which is itself a function of U. Using the multivariate chain rule, the derivative at $s = U$ is:

$$\begin{aligned}
\frac{d}{dU}EV(x) &= \left.\frac{\partial EV(x)}{\partial s}\right|_{s=U} + \left.\frac{\partial EV(x)}{\partial c(s)}\left(\frac{\partial c(s)}{\partial s} + \frac{\partial c(s)}{\partial x_2}\frac{\partial x_2}{\partial U}\right)\right|_{s=U}\\
&= \left.\left(\frac{\partial EV(x)}{\partial s} + \frac{\partial EV(x)}{\partial c(s)}\frac{\partial c(s)}{\partial s}\right)\right|_{s=U} + \left.\left(\frac{\partial EV(x)}{\partial c(s)}\frac{\partial c(s)}{\partial x_2}\frac{\partial x_2}{\partial U}\right)\right|_{s=U}
\end{aligned}$$

We want to know exactly when the above expression is positive. The partial derivatives to plug in are:

$$\begin{aligned}
\frac{\partial EV(x)}{\partial s} &= 2x - 1 - c(s) \\
\frac{\partial EV(x)}{\partial c(s)} &= -s \\
\frac{\partial c(s)}{\partial s} &= \frac{1 - x_2}{(s+1)^2} \\
\frac{\partial c(s)}{\partial x_2} &= \frac{1}{s+1}
\end{aligned}$$

We can leave $\frac{\partial x_2}{\partial U}$ as a free variable for now, since it is always positive by Lemma 8.1 (1), and is independent of x. Plugging these in and rearranging terms, we can say that $EV(x)$ is increasing in U if

$$x > \frac{1}{2(1+U)} \left(U \frac{\partial x_2}{\partial U} + \frac{U^2 + x_2(1+2U)}{1+U} \right)$$

Where nothing on the right hand side is dependent on x. This means that for any fixed L, U, this gives a threshold value for x below which $EV(x)$ is decreasing in U, and above which it is increasing in U.

Case 2 ($x_3 < x < v(U)$): In this case, hand x makes an intermediate-sized bet $s = v^{-1}(x)$. There are two distinct factors influencing the derivative $\frac{d}{dU}EV(x)$, namely the change in bet size $s = v^{-1}(x)$ and the change in calling cutoff $c(v^{-1}(x))$. By the multivariate chain rule, we can express the derivative as:

$$\frac{d}{dU}EV(x) = \frac{\partial EV(x)}{\partial s} \frac{dv^{-1}(x)}{dU} + \frac{\partial EV(x)}{\partial c(s)} \frac{dc(v^{-1}(x))}{\partial U}$$

The partial derivatives of $EV(x)$ are:

$$\begin{aligned}
\frac{\partial EV(x)}{\partial s} &= 2x - 1 - c(s) \\
\frac{\partial EV(x)}{\partial c(s)} &= -s
\end{aligned}$$

The second is clearly negative. We can verify that the first must be positive if we go back to the constraints which gave us the Nash equilibrium. For the bet size $v^{-1}(x)$ to be optimal, we required that

$$-s\frac{\partial c(s)}{\partial s} - c(s) + 2v(s) - 1 = 0,$$

or equivalently, if we substitute $s = v^{-1}(x)$ and $v(s) = x$ and rearrange:

$$2x - 1 - c(v^{-1}(x)) = v^{-1}(x)\frac{\partial c(s)}{\partial s} > 0,$$

since $\frac{\partial c(s)}{\partial s} = \frac{1-x_2}{(s+1)^2} > 0$, and s is positive by definition.

We know from Lemma 8.1 that $\frac{dv^{-1}(x)}{dU} < 0$ and $\frac{dc(v^{-1}(x))}{\partial U} > 0$.

Combining everything, we see that both terms in $\frac{d}{dU}EV(x)$ are products of negative and positive, making both terms negative. Therefore, the expected payoff is decreasing in U for all $x \in [x_3, v(U)]$.

□

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