# 1 Strategic Comparison to Fixed-Bet and No-Limit Continuous Poker

The most natural way to think of Limit Continuous Poker is as a generalization between Fixed-Bet and No-Limit Continuous Poker, on a spectrum from strict to lenient bet sizing. In this light, we begin by asking whether the Nash equilibrium strategies approach those of NLCP and FCP as the limits $L$ and $U$ approach their extreme cases. To make this more explicit, we model the bettor strategies for all three games as 'bet functions' from hand strengths to bets (with 0 representing a check), and caller strategies as 'call functions' from bet sizes to minimum calling thresholds. We also introduce notation to reference all three strategy profiles more efficiently.

## 1.1 Setup and Notation

To compare the strategy profiles across different variants of Continuous Poker, we introduce the following notation for the strategy functions of the three games:

| **Symbol** | **Meaning** |
|---|---|
| $S_{FB}(x, B)$ | Bettor's bet function in FBCP with fixed bet size $B$ |
| $C_{FB}(s, B)$ | Caller's call function in FBCP with fixed bet size $B$ |
| $S_{NL}(x)$ | Bettor's bet function in NLCP |
| $C_{NL}(s)$ | Caller's call function in NLCP |
| $S_{LCP}(x, L, U)$ | Bettor's bet function in LCP with limits $L$ and $U$ |
| $C_{LCP}(s, L, U)$ | Caller's call function in LCP with limits $L$ and $U$ |
| $x_i\|_{L,U}$ | Threshold $x_i$ in LCP with limits $L$ and $U$ |

Table 1: Notation for strategy functions across different variants of Continuous Poker

In FBCP, the bettor can only make a fixed bet size $B$ or check. The bet function $S_{FB}(x, B)$ maps hand strengths to either 0 (check) or $B$ (bet):

$$S_{FB}(x, B) = \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \text{ (bluffing range)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (checking range)} \\ B & x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (value betting range)} \end{cases} \quad (1)$$

The caller's strategy is defined by a single threshold $C_{FB}(s, B)$:

$$C_{FB}(s,B) = \frac{B(3+2B)}{(1+2B)(2+B)} \tag{2}$$

In NLCP, the bettor can choose any positive bet size. The strategy is most naturally described by functions $v_{NL}(s)$ and $b_{NL}(s)$ that map bet sizes to hand strengths:

$$v_{NL}(s) = 1 - \frac{3}{7(s+1)^2} \text{ (value betting function)}$$
$$b_{NL}(s) = \frac{3s+1}{7(s+1)^3} \text{ (bluffing function)}$$

The bet function $S_{NL}(x)$ is then defined in terms of the inverse functions:

$$S_{NL}(x) = \begin{cases} b_{NL}^{-1}(x) & x < \frac{1}{7} \text{ (bluffing range)} \\ 0 & \frac{1}{7} < x < \frac{4}{7} \text{ (checking range)} \\ v_{NL}^{-1}(x) & x > \frac{4}{7} \text{ (value betting range)} \end{cases}$$

The caller's strategy is defined by a continuous function $C_{NL}(s)$:

$$C_{NL}(s) = 1 - \frac{6}{7(s+1)}$$

In LCP, the bettor can choose any bet size between $L$ and $U$. The strategy profile is defined by six thresholds $x_0$ through $x_5$ and functions $v(s)$ and $b(s)$ that map bet sizes to hand strengths. The bet function $S_{LCP}(x,L,U)$ and call function $C_{LCP}(s,L,U)$ are defined in terms of these values, which are given in Theorem **??**.

## 1.2 Strategic Convergence

### 1.2.1 Bettor Strategy Convergence to Continuous Poker

We expect that as $L$ and $U$ approach some fixed value $s$, the bet function $S_{LCP}(x,L,U)$ should converge to the bet function $S_{FB}(x,s)$ for Fixed-Bet Continuous Poker with a fixed bet size $s$.

**Theorem 1.1.** *For any $B > 0$, the bet function $S_{LCP}(x,L,U)$ for Limit Continuous Poker converges to the bet function $S_{FB}(x,B)$ for Fixed-Bet Continuous Poker with a fixed bet size $B$ as $L$ and $U$ approach $B$:*

$$\lim_{L\to B}\lim_{U\to B} S_{LCP}(x,L,U) = \lim_{U\to B}\lim_{L\to B} S_{LCP}(x,L,U) = S_{FB}(x,B).$$

**Proof.** We analyze the expressions for the $x_i$'s, each of which is a rational function[1] of $L$ and $U$. Since these functions are defined and continuous for all positive values of $L$ and $U$, the limit as $L \to B$ and $U \to B$ can be found by simply substituting $L = U = B$:

$$\begin{aligned}
x_0|_{B,B} = x_1|_{B,B} &= \frac{B}{2B^3 + 7B^2 + 7B + 2} \\
x_2|_{B,B} &= \frac{B}{(1+2B)(2+B)} \\
x_3|_{B,B} &= \frac{2B^2 + 4B + 1}{(1+2B)(2+B)} \\
x_4|_{B,B} = x_5|_{B,B} &= \frac{2B^2 + 5B + 1}{(1+2B)(2+B)}
\end{aligned}$$

$x_0 = x_1$ and $x_4 = x_5$ are expected, since these intervals are where the bettor uses an intermediate bet size, and $L = U = B$ does not allow intermediate bet sizes. This reduces the bet function to

$$\begin{aligned}
\lim_{L\to B}\lim_{U\to B} S_{LCP}(x, L, U) &= \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \\ B & x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \end{cases} \\
&= S_{FB}(x, B)
\end{aligned}$$

□

### 1.2.2 Caller Strategy Convergence to Continuous Poker

The calling function is easier to analyze. We want to show that the calling threshold $C_{LCP}(s, L, U)$ converges to the calling threshold $C_{FB}(s, B)$ for Fixed-Bet Continuous Poker with a fixed bet size $B$ as $L$ and $U$ approach $B$.

**Theorem 1.2.** *For any $B > 0$, the call function $C_{LCP}(s, L, U)$ for Limit Continuous Poker converges to the call function $C_{FB}(s, B)$ for Fixed-Bet Continuous Poker with a fixed bet size $B$ as $L$ and $U$ approach $B$:*

$$\lim_{L\to B}\lim_{U\to B} C_{LCP}(s, L, U) = \lim_{U\to B}\lim_{L\to B} C_{LCP}(s, L, U) = C_{FB}(s, B).$$

[1] A ratio of polynomials in $L$ and $U$.

**Proof.** We already have the value of $x_2|_{B,B}$, so we can plug this into the expression for the calling threshold:

$$\begin{aligned}
\lim_{L\to B}\lim_{U\to B} C_{LCP}(s,L,U) &= \frac{x_2|_{B,B}+s}{1+s} \\
&= \frac{\frac{B}{(1+2B)(2+B)}+s}{1+s} \\
&= \frac{B(3+2B)}{(1+2B)(2+B)} \\
&= C_{FB}(s,B)
\end{aligned}$$

□

### 1.2.3 Bettor Strategy Convergence to NLCP

In a similar fashion, we expect that as $L$ and $U$ approach 0 and $\infty$, the bet function $S_{LCP}(x,L,U)$ should converge to the bet function $S_{NL}(x)$ for NLCP.

**Theorem 1.3.** *The bet function $S_{LCP}(x,L,U)$ for Limit Continuous Poker converges to the bet function $S_{NL}(x)$ for NLCP as $L$ and $U$ approach 0 and $\infty$:*

$$\lim_{L\to 0}\lim_{U\to\infty} S_{LCP}(x,L,U) = \lim_{U\to\infty}\lim_{L\to 0} S_{LCP}(x,L,U) = S_{NL}(x).$$

**Proof.** We can analyze the expressions for the $x_i$'s as $L$ and $U$ approach 0 and $\infty$. The limit is well-defined, and we can substitute $L=0$ and $U=\infty$ into the expressions for the $x_i$s.

$$\begin{aligned}
x_0|_{0,\infty} &= 0 \\
x_1|_{0,\infty} = x_2|_{0,\infty} &= \frac{1}{7} \\
x_3|_{0,\infty} = x_4|_{0,\infty} &= \frac{4}{7} \\
x_5|_{0,\infty} &= 1
\end{aligned}$$

$x_0|_{0,\infty}=0$ and $x_5|_{0,\infty}=1$ are expected, since these intervals are where the bettor uses a minimum bet size and a maximum bet size, respectively, both of which are impossible. The bettor now bets intermediate values for $x<\frac{1}{7}$ and $x>\frac{4}{7}$, and checks for $\frac{1}{7}<x<\frac{4}{7}$. But how much do they bet? We can take the limits of $v(s)$ and $b(s)$ as $L$ and $U$ approach 0 and $\infty$:

$$\lim_{L\to 0}\lim_{U\to\infty} b(s) = \frac{3s+1}{7(s+1)^3}$$
$$\lim_{L\to 0}\lim_{U\to\infty} v(s) = 1 - \frac{3}{7(s+1)^2}$$

To summarize, the bettor bets $s$ with hands $x < \frac{1}{7}$ such that $x = b(s)$ or hands $x > \frac{4}{7}$ such that $x = v(s)$. This is exactly the same as the bet function $S_{NL}(x)$ for NLCP.

□

### 1.2.4 Caller Strategy Convergence to NLCP

The calling function is again easier to analyze. We want to show that the calling threshold $C_{LCP}(s, L, U)$ converges to the calling threshold $C_{NL}(s)$ for NLCP as $L$ and $U$ approach 0 and $\infty$.

**Theorem 1.4.** *The call function $C_{LCP}(s, L, U)$ for Limit Continuous Poker converges to the call function $C_{NL}(s)$ for NLCP as L and $U$ approach 0 and $\infty$:*

$$\lim_{L\to 0}\lim_{U\to\infty} C_{LCP}(s, L, U) = \lim_{U\to\infty}\lim_{L\to 0} C_{LCP}(s, L, U) = C_{NL}(s).$$

**Proof.** Again, we already have the limiting value of $x_2|_{0,\infty}$, so we can plug this into the expression for the calling threshold:

$$\begin{aligned}
\lim_{L\to 0}\lim_{U\to\infty} C_{LCP}(s, L, U) &= \lim_{L\to 0}\lim_{U\to\infty} \frac{x_2 + s}{1+s} \\
&= \frac{\frac{1}{7} + s}{1+s} \\
&= 1 - \frac{6}{7(1+s)} \\
&= C_{NL}(s)
\end{aligned}$$

□

We have now shown that the bettor and caller strategies for LCP converge to those of FBCP and NLCP as the limits $L$ and $U$ approach their extreme values. In the next section, we explore the value of LCP in more detail, and in particular how it relates to that of FBCP and NLCP.