

# 1 Strategic Comparison to Fixed-Bet and No-Limit Continuous Poker

The most natural way to think of Limit Continuous Poker is as a generalization between Fixed-Bet and No-Limit Continuous Poker, on a spectrum from strict to lenient bet sizing. In this light, we begin by asking whether the Nash equilibrium strategies approach those of NLCP and FCP as the limits  $L$  and  $U$  approach their extreme cases. To make this more explicit, we model the bettor strategies for all three games as ‘bet functions’ from hand strengths to bets (with 0 representing a check), and caller strategies as ‘call functions’ from bet sizes to minimum calling thresholds. We also introduce notation to reference all three strategy profiles more efficiently.

## 1.1 Setup and Notation

To compare the strategy profiles across different variants of Continuous Poker, we introduce the following notation for the strategy functions of the three games:

Symbol	Meaning
$S_{FB}(x, B)$	Bettor’s bet function in FBCP with fixed bet size $B$
$C_{FB}(s, B)$	Caller’s call function in FBCP with fixed bet size $B$
$S_{NL}(x)$	Bettor’s bet function in NLCP
$C_{NL}(s)$	Caller’s call function in NLCP
$S_{LCP}(x, L, U)$	Bettor’s bet function in LCP with limits $L$ and $U$
$C_{LCP}(s, L, U)$	Caller’s call function in LCP with limits $L$ and $U$
$x_i _{L,U}$	Threshold $x_i$ in LCP with limits $L$ and $U$

Table 1: Notation for strategy functions across different variants of Continuous Poker

In FBCP, the bettor can only make a fixed bet size  $B$  or check. The bet function  $S_{FB}(x, B)$  maps hand strengths to either 0 (check) or  $B$  (bet):

$$S_{FB}(x, B) = \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \text{ (bluffing range)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (checking range)} \\ B & x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (value betting range)} \end{cases} \quad (1)$$

The caller’s strategy is defined by a single threshold  $C_{FB}(s, B)$ :

$$C_{FB}(s, B) = \frac{B(3 + 2B)}{(1 + 2B)(2 + B)} \quad (2)$$

In NLCP, the bettor can choose any positive bet size. The strategy is most naturally described by functions  $v_{NL}(s)$  and  $b_{NL}(s)$  that map bet sizes to hand strengths:

$$\begin{aligned} v_{NL}(s) &= 1 - \frac{3}{7(s+1)^2} \text{ (value betting function)} \\ b_{NL}(s) &= \frac{3s+1}{7(s+1)^3} \text{ (bluffing function).} \end{aligned}$$

The bet function  $S_{NL}(x)$  is then defined in terms of the inverse functions:

$$S_{NL}(x) = \begin{cases} b_{NL}^{-1}(x) & x < \frac{1}{7} \text{ (bluffing range)} \\ 0 & \frac{1}{7} < x < \frac{4}{7} \text{ (checking range)} \\ v_{NL}^{-1}(x) & x > \frac{4}{7} \text{ (value betting range)} \end{cases}$$

The caller's strategy is defined by a continuous function  $C_{NL}(s)$ :

$$C_{NL}(s) = 1 - \frac{6}{7(s+1)}$$

In LCP, the bettor can choose any bet size between  $L$  and  $U$ . The strategy profile is defined by six thresholds  $x_0$  through  $x_5$  and functions  $v(s)$  and  $b(s)$  that map bet sizes to hand strengths. The bet function  $S_{LCP}(x, L, U)$  and call function  $C_{LCP}(s, L, U)$  are defined in terms of these values, which are given in Theorem ??.

## 1.2 Strategic Convergence

### 1.2.1 Bettor Strategy Convergence to Continuous Poker

We expect that as  $L$  and  $U$  approach some fixed value  $s$ , the bet function  $S_{LCP}(x, L, U)$  should converge to the bet function  $S_{FB}(x, s)$  for Fixed-Bet Continuous Poker with a fixed bet size  $s$ .

**Theorem 1.1.** *For any  $B > 0$ , the bet function  $S_{LCP}(x, L, U)$  for Limit Continuous Poker converges to the bet function  $S_{FB}(x, B)$  for Fixed-Bet Continuous Poker with a fixed bet size  $B$  as  $L$  and  $U$  approach  $B$ :*

$$\lim_{L \rightarrow B} \lim_{U \rightarrow B} S_{LCP}(x, L, U) = \lim_{U \rightarrow B} \lim_{L \rightarrow B} S_{LCP}(x, L, U) = S_{FB}(x, B).$$

**Proof.** We analyze the expressions for the  $x_i$ 's, each of which is a rational function<sup>1</sup> of  $L$  and  $U$ . Since these functions are defined and continuous for all positive values of  $L$  and  $U$ , the limit as  $L \rightarrow B$  and  $U \rightarrow B$  can be found by simply substituting  $L = U = B$ :

$$\begin{aligned} x_0|_{B,B} &= x_1|_{B,B} = \frac{B}{2B^3 + 7B^2 + 7B + 2} \\ x_2|_{B,B} &= \frac{B}{(1+2B)(2+B)} \\ x_3|_{B,B} &= \frac{2B^2 + 4B + 1}{(1+2B)(2+B)} \\ x_4|_{B,B} &= x_5|_{B,B} = \frac{2B^2 + 5B + 1}{(1+2B)(2+B)} \end{aligned}$$

$x_0 = x_1$  and  $x_4 = x_5$  are expected, since these intervals are where the bettor uses an intermediate bet size, and  $L = U = B$  does not allow intermediate bet sizes. This reduces the bet function to

$$\begin{aligned} \lim_{L \rightarrow B} \lim_{U \rightarrow B} S_{LCP}(x, L, U) &= \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \\ B & x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \end{cases} \\ &= S_{FB}(x, B) \end{aligned}$$

□

### 1.2.2 Caller Strategy Convergence to Continuous Poker

The calling function is easier to analyze. We want to show that the calling threshold  $C_{LCP}(s, L, U)$  converges to the calling threshold  $C_{FB}(s, B)$  for Fixed-Bet Continuous Poker with a fixed bet size  $B$  as  $L$  and  $U$  approach  $B$ .

**Theorem 1.2.** *For any  $B > 0$ , the call function  $C_{LCP}(s, L, U)$  for Limit Continuous Poker converges to the call function  $C_{FB}(s, B)$  for Fixed-Bet Continuous Poker with a fixed bet size  $B$  as  $L$  and  $U$  approach  $B$ :*

$$\lim_{L \rightarrow B} \lim_{U \rightarrow B} C_{LCP}(s, L, U) = \lim_{U \rightarrow B} \lim_{L \rightarrow B} C_{LCP}(s, L, U) = C_{FB}(s, B).$$

<sup>1</sup>A ratio of polynomials in  $L$  and  $U$ .

**Proof.** We already have the value of  $x_2|_{B,B}$ , so we can plug this into the expression for the calling threshold:

$$\begin{aligned}
\lim_{L \rightarrow B} \lim_{U \rightarrow B} C_{LCP}(s, L, U) &= \frac{x_2|_{B,B} + s}{1 + s} \\
&= \frac{\frac{B}{(1+2B)(2+B)} + s}{1 + s} \\
&= \frac{B(3 + 2B)}{(1 + 2B)(2 + B)} \\
&= C_{FB}(s, B)
\end{aligned}$$

□

### 1.2.3 Bettor Strategy Convergence to NLCP

In a similar fashion, we expect that as  $L$  and  $U$  approach 0 and  $\infty$ , the bet function  $S_{LCP}(x, L, U)$  should converge to the bet function  $S_{NL}(x)$  for NLCP.

**Theorem 1.3.** *The bet function  $S_{LCP}(x, L, U)$  for Limit Continuous Poker converges to the bet function  $S_{NL}(x)$  for NLCP as  $L$  and  $U$  approach 0 and  $\infty$ :*

$$\lim_{L \rightarrow 0} \lim_{U \rightarrow \infty} S_{LCP}(x, L, U) = \lim_{U \rightarrow \infty} \lim_{L \rightarrow 0} S_{LCP}(x, L, U) = S_{NL}(x).$$

**Proof.** We can analyze the expressions for the  $x_i$ 's as  $L$  and  $U$  approach 0 and  $\infty$ . The limit is well-defined, and we can substitute  $L = 0$  and  $U = \infty$  into the expressions for the  $x_i$ s.

$$\begin{aligned}
x_0|_{0,\infty} &= 0 \\
x_1|_{0,\infty} &= x_2|_{0,\infty} = \frac{1}{7} \\
x_3|_{0,\infty} &= x_4|_{0,\infty} = \frac{4}{7} \\
x_5|_{0,\infty} &= 1
\end{aligned}$$

$x_0|_{0,\infty} = 0$  and  $x_5|_{0,\infty} = 1$  are expected, since these intervals are where the bettor uses a minimum bet size and a maximum bet size, respectively, both of which are impossible. The bettor now bets intermediate values for  $x < \frac{1}{7}$  and  $x > \frac{4}{7}$ , and checks for  $\frac{1}{7} < x < \frac{4}{7}$ . But how much do they bet? We can take the limits of  $v(s)$  and  $b(s)$  as  $L$  and  $U$  approach 0 and  $\infty$ :

$$\lim_{L \rightarrow 0} \lim_{U \rightarrow \infty} b(s) = \frac{3s + 1}{7(s + 1)^3}$$

$$\lim_{L \rightarrow 0} \lim_{U \rightarrow \infty} v(s) = 1 - \frac{3}{7(s + 1)^2}.$$

To summarize, the better bets  $s$  with hands  $x < \frac{1}{7}$  such that  $x = b(s)$  or hands  $x > \frac{4}{7}$  such that  $x = v(s)$ . This is exactly the same as the bet function  $S_{NL}(x)$  for NLCP.

□

#### 1.2.4 Caller Strategy Convergence to NLCP

The calling function is again easier to analyze. We want to show that the calling threshold  $C_{LCP}(s, L, U)$  converges to the calling threshold  $C_{NL}(s)$  for NLCP as  $L$  and  $U$  approach 0 and  $\infty$ .

**Theorem 1.4.** *The call function  $C_{LCP}(s, L, U)$  for Limit Continuous Poker converges to the call function  $C_{NL}(s)$  for NLCP as  $L$  and  $U$  approach 0 and  $\infty$ :*

$$\lim_{L \rightarrow 0} \lim_{U \rightarrow \infty} C_{LCP}(s, L, U) = \lim_{U \rightarrow \infty} \lim_{L \rightarrow 0} C_{LCP}(s, L, U) = C_{NL}(s).$$

**Proof.** Again, we already have the limiting value of  $x_2|_{0,\infty}$ , so we can plug this into the expression for the calling threshold:

$$\begin{aligned} \lim_{L \rightarrow 0} \lim_{U \rightarrow \infty} C_{LCP}(s, L, U) &= \lim_{L \rightarrow 0} \lim_{U \rightarrow \infty} \frac{x_2 + s}{1 + s} \\ &= \frac{\frac{1}{7} + s}{1 + s} \\ &= 1 - \frac{6}{7(1 + s)} \\ &= C_{NL}(s). \end{aligned}$$

□

We have now shown that the better and caller strategies for LCP converge to those of FBCP and NLCP as the limits  $L$  and  $U$  approach their extreme values. In the next section, we explore the value of LCP in more detail, and in particular how it relates to that of FBCP and NLCP.