1 Solving LCP

1.1 Nash Equilibrium Structure

Theorem 1.1. *LCP has a unique admissible-strategy Nash equilibrium, which has the following structure:*

1. *The caller has a single calling threshold $c(s)$ that is non-decreasing and continuous in s, including at endpoints L and U. They call with hands $y \geq c(s)$ and fold with hands $y < c(s)$.*

2. *The bettor partitions $[0, 1]$ into seven regions with thresholds $0 \leq x_0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq 1$:*

    - *Value bets: hands $x \in (x_5, 1)$ bet U, hands $x \in (x_4, x_5)$ bet $s \in (L, U)$ according to $x = v(s)$, hands $x \in (x_3, x_4)$ bet L*
    - *Checks: hands $x \in (x_2, x_3)$*
    - *Bluffs: hands $x \in (x_1, x_2)$ bet L, hands $x \in (x_0, x_1)$ bet $s \in (L, U)$ according to $x = b(s)$, hands $x \in (0, x_0)$ bet U*

Intuitively, the strategy profile is similar to NLCP, with the bettor partitioning $[0, 1]$ into value betting, checking, and bluffing regions, and the caller partitioning into calling and folding regions.
Proof. We will prove the structure of the Nash equilibrium by establishing several key claims:

1. The caller must use a threshold strategy with cutoff $c(s)$

2. The cutoff $c(s)$ must be non-decreasing in s

3. The cutoff $c(s)$ must be continuous, even at endpoints L and U

4. The bettor must bet for value with hands stronger than $c(s)$, bluff with hands weaker than $c(s)$, and check some range of intermediate hands

5. Value betting sizes must increase with hand strength

6. When bluffing, the bettor must be indifferent among exactly the sizes which are used for value betting

7. Bluffing sizes should decrease with hand strength

Claim 1: The caller must use a threshold strategy with cutoff $c(s)$. For any bet size s, if the caller calls with hand strength y and folds with hand strength $y' > y$, they are strictly better off calling with y' and folding with y. Therefore, they must call with all hands above some threshold $c(s)$ and fold with all hands below it.

Claim 2: The cutoff $c(s)$ must be non-decreasing in s. If $c(s)$ were decreasing at any point, the bettor could exploit this by bluffing with a slightly smaller size than they would otherwise use. This would cause the caller to fold more often while risking less money, contradicting equilibrium.

Claim 3: The cutoff $c(s)$ must be continuous, even at endpoints L and U. If $c(s)$ had a discontinuity, the bettor's expected value from bluffing would also be discontinuous. They could then exploit by bluffing with sizes just below the discontinuity, contradicting equilibrium.

Claim 4: Value bets must be made with hands stronger than $c(s)$, bluffs with hands weaker than $c(s)$. When betting size s, the bettor wants to get called by weaker hands when value betting (requiring $x > c(s)$) and get stronger hands to fold when bluffing (requiring $x < c(s)$).

Claim 5: Value betting sizes must increase with hand strength. Since $c(s)$ is non-decreasing, stronger hands can profitably bet larger sizes and get called by a more restricted range of strong hands. Weaker value betting hands must bet smaller to get called by a wider range.

Claim 6: The bettor must be indifferent among bluffing sizes that are also used for value bets. The expected value of bluffing size s is:

$$\mathbb{E}[\text{bluff } s] = c(s) - (1 - c(s)) \cdot s \tag{1}$$

This is independent of the bettor's hand strength. If the bettor strictly preferred certain sizes for bluffing, they would never bluff with other sizes. If any other size were used for value betting, then the caller could exploit by only calling those sizes with hands stronger than the value betting range. Additionally, the bettor cannot bluff with sizes which are not used for value betting. In this case, the caller can similarly exploit by always calling this size with hands stronger than the bluffing range.

Claim 7: Bluffing sizes should decrease with hand strength. This claim is not necessary to have a Nash equilibrium, but it is what makes the strategy profile uniquely admissible. If the caller deviates by calling too loosely but maintains consistency (never calling with weaker hands and folding with stronger hands to the same bet size), the bettor uniquely benefits by bluffing larger with their weakest hands and bluffing smaller with their strongest hands. This gives them a possiblity of winning showdowns with their strongest bluffing hands, which would not happen if they bluffed large with the strongest hands. □ admissiblity?

diagram of strategy profile

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1.2 Constraints and Indifference Equations

The Nash equilibrium strategy profile must satisfy several constraints and indifference conditions, which we will derive and use to solve for the strategy profile. The key conditions are:

- The caller must be indifferent between calling and folding at their calling threshold
- The bettor must be indifferent between checking and betting at their value betting and bluffing thresholds
- The bettor's bet size for a value bet must maximize their expected value
- The bettor's strategy must be continuous in bet size (in the regions where they bet)

These conditions give us the following system of equations:

$$\begin{aligned}
\text{Caller Indifference:}& \\
&(x_4 - x_3) \cdot (1 + L) - (x_2 - x_1) \cdot L = 0 \\
&(1 - x_5) \cdot (1 + U) - x_0 \cdot U = 0 \\
&|b'(s)| \cdot (1 + s) + |v'(s)| \cdot s = 0 \\
\text{Bettor Indifference and Optimality:}& \\
&- sc'(s) - c(s) + 2v(s) - 1 = 0 \\
&c(L) - (1 - c(L)) \cdot L = x_3 \\
&c(s) - (1 - c(s)) \cdot s = x_2 \\
\text{Continuity Constraints:}& \\
&b(U) = x_0 \\
&b(L) = x_1 \\
&v(U) = x_5 \\
&v(L) = x_4
\end{aligned}$$

We will now derive each of these equations in turn.

1.2.1 Caller Indifference

By definition, $c(s)$ is the threshold above which the caller calls and below which they fold. This means that in Nash Equilibrium, the caller must be indifferent between calling and folding with a hand strength of $c(s)$:

$$\mathbb{E}[\text{call } c(s)] = \mathbb{E}[\text{fold } c(s)]$$

$$\mathbb{P}[\text{bluff}|s] \cdot (1+s) - \mathbb{P}[\text{value bet}|s] \cdot s = 0$$

We now split into cases based on the value of s.

Case 1: $s = L$. The hands the bettor value bets L with are $x \in (x_3, x_4)$, and the hands they bluff with are $x \in (x_1, x_2)$.

$$(x_4 - x_3) \cdot (1 + L) - (x_2 - x_1) \cdot L = 0 \tag{2}$$

Case 2: $s = U$. The hands the bettor value bets U with are $x \in (x_5, 1)$, and the hands they bluff with are $x \in (0, x_0)$.

$$(1 - x_5) \cdot (1 + U) - x_0 \cdot U = 0 \tag{3}$$

Case 3: $L \leq s \leq U$. In this case, the bettor has exactly one value hand and one bluffing hand, but somewhat paradoxically, they are not equally likely. The probability of a value bet given the size s is related to the inverse derivative of the value function $v(s)$ at s, and the same goes for a bluff. This gives us the following relation:

$$\frac{\mathbb{P}[\text{value bet}|s]}{\mathbb{P}[\text{bluff}|s]} = \frac{|b'(s)|}{|v'(s)|}$$

An intuitive interpretation of this is that for any small neighborhood around the bet size s, the bettor has more hands which use a bet size in the neighborhood if $v(s)$ does not change rapidly around s, that is, if $|v'(s)|$ is small. The same goes for bluffing hands, and as we limit the neighborhood to a single point, the ratio of the two probabilities approaches the ratio of the derivatives. Plugging this into the indifference equation, we get:

$$|b'(s)| \cdot (1 + s) + |v'(s)| \cdot s = 0 \tag{4}$$

1.2.2 Bettor Indifference and Optimality

When the bettor makes a value bet, they are attempting to maximize the expected value of the bet. We can write the expected value of a value bet as:

$$\begin{aligned}\mathbb{E}[\text{value bet } s|x] &= \mathbb{P}[\text{call with worse}] \cdot (1+s) - \mathbb{P}[\text{call with better}] \cdot s + \mathbb{P}[\text{fold}] \cdot 1 \\ &= (x - c(s)) \cdot (1+s) - (1-x) \cdot (s) + c(s)\end{aligned}$$

To maximize this, we take the derivative with respect to s and set it equal to zero. Crucially, we are treating $c(s)$ as a function of s and using the chain rule, since changing the bet size s will also change the calling threshold $c(s)$. We want this optimality condition to hold for the bettor's Nash equilibrium strategy, so we set $x = v(s)$. This gives us:

$$\begin{aligned}&\frac{d}{ds}\mathbb{E}[\text{value bet } s|x = v(s)] = 0 \\ &-sc'(s) - c(s) + 2v(s) - 1 = 0\end{aligned} \tag{5}$$

Additionally, when the bettor has the most marginal value betting hand at $x = x_3$, they should be indifferent between a minimum value bet and a check:

$$\begin{aligned}\mathbb{E}[\text{value bet } L|x = x_3] &= \mathbb{E}[\text{check}|x = x_3] \\ (x_3 - c(L)) \cdot (1+L) - (1-x_3) \cdot (L) + c(L) &= x_3\end{aligned} \tag{6}$$

Finally, when the bettor has the most marginal bluffing hand at $x = x_2$, they should be indifferent between a minimum bluff and a check. However, as we discussed earlier, the bettor should be indifferent among all bluffing sizes, so the bettor should actually be indifferent between checking and making any bluffing size s at $x = x_2$. This gives us:

$$\begin{aligned}\mathbb{E}[\text{bluff } s|x = x_2] &= \mathbb{E}[\text{check}|x = x_2] \\ c(s) - (1 - c(s)) \cdot s &= x_2\end{aligned} \tag{7}$$

1.2.3 Continuity Constraints

As discussed above, the bettor's strategy is continuous in s and x (except when checking). This means that the endpoints of the functions $v(s)$ and $b(s)$ are constrained as follows:

$$b(U) = x_0, \;\; b(L) = x_1, \;\; v(U) = x_5, \;\; v(L) = x_4 \tag{8}$$