1 Solving LCP

Before diving into a solution for the Nash equilibrium of LCP, we first address the uniqueness of such a solution. LCP has an infinite class of Nash equilibria, differentiated only by how the bettor sizes their bluffing hands. In this section, we define a way to distinguish between these equilibria. This involves two steps: first, defining a class of *monotone* calling strategies which are, in some sense, more reasonable than non-monotone strategies; and second, restricting the bettor's strategy to be admissible (not weakly dominated) against these calling strategies. This turns out to be enough to uniquely determine the Nash equilibrium.

1.1 Monotone Strategies

Definition 1.1 (Monotone Calling Strategy)**.** A *monotone* calling strategy is a pure strategy which satisfies two conditions:

1. For any bet size s and any two hand strengths $y_1 < y_2$, if the caller calls a bet of size s with y_1, they must also call with y_2.
2. For any hand strength y and any two bet sizes $s_1 < s_2$, if the caller calls a bet of size s_2 with y, they must also call a bet of size s_1 with y.

This should sound intuitive. Clearly, calling with a stronger hand is weakly better than calling with a weaker hand. Restricting to pure strategies can be explained similarly - it is better to always call with a stronger hand and always fold a weaker one than to mix between the two.

Violating the first condition (in a non-negligible way) is actually weakly dominated - not only is a monotone strategy weakly better against all opponents, but there exists an opponent against which the non-monotone strategy is strictly better. We prove this in lemma 1.1.

Lemma 1.1. *If a calling strategy violates the first monotonicity condition for a nonzero-measure set of hands for any bet size s, it is weakly dominated. Specifically, if there exists s and measurable sets $A, B \subseteq [0, 1]$ such that:*

1. *The caller calls s with hands in A*
2. *The caller folds s with hands in B*
3. $\sup A \leq \inf B$
4. *A and B have positive measure*

then the strategy is weakly dominated.

Proof. Let σ_C be the non-monotone strategy described above. Since A and B are nonzero-measure, there exist subsets $A' \subseteq A$ and $B' \subseteq B$ such that:

1. A' and B' have positive measure
2. $|A'| = |B'|$

Where $|A|$ and $|B|$ denote the Lebesgue measure of A and B (see figure 1).

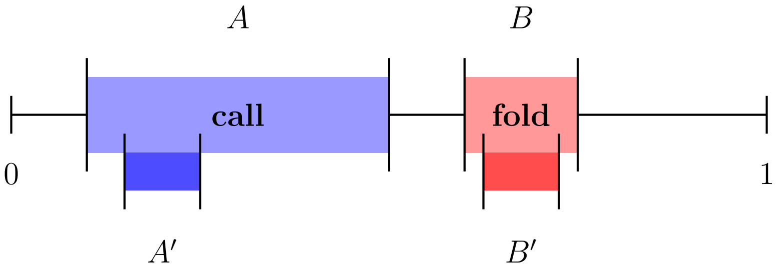

Figure 1: A simple case of sets A and B which violate monotonicity ($\sup A \leq \inf B$). We can find equal-measure subsets $A' \subseteq A$ and $B' \subseteq B$ to swap actions, improving the strategy.

The existence of such subsets follows from a fundamental property of nonatomic measures: since the uniform distribution on $[0, 1]$ is nonatomic (no single point has positive probability), for any two measurable sets with positive measure, we can always find measurable subsets of equal measure[**?**]. This property allows us to construct the strategy improvement described below.

Let σ'_C be the strategy which switches the actions for A' and B', i.e. calls with B' and folds with A' (and behaves identically for all other bet sizes). We now analyze how this change affects the caller's performance against any betting strategy.

Against a bet of size s, the key improvement occurs in two scenarios:

1. When $y \in B'$ and $x \in A'$: σ_C folds while σ'_C calls and wins (since $x \in A$ and $y \in B$ with $\sup A \leq \inf B$)
2. When $y \in A'$ and $x \in B'$: σ_C calls and loses while σ'_C folds (avoiding the loss)

For all other cases, σ_C and σ'_C behave identically, so σ'_C is weakly better than σ_C against every betting strategy.

To show that σ_C is strictly dominated, consider a betting strategy which always bets s. Against this strategy, both scenarios above occur with positive probability (since A' and B' have positive measure), so σ'_C is strictly better than σ_C. Thus, σ_C is weakly dominated. $\square$

The second condition for a monotone calling strategy - that the caller must be more willing to call smaller bets - is more subtle. From a poker player's perspective, it aligns with intuition about pot odds - a larger bet is more risky, so it should require a stronger hand to call. However, violating this condition is not dominated - in fact, it is common in real poker. Often, a larger bet is perceived as a bluff while a smaller bet is perceived as a value bet. Whether or not it is optimal, a caller might call more aggressively against large bets based on this perception. To motivate this condition from more fundamental game-theoretic principles, we consider how it affects the bettor's response. If the caller violates this condition by calling less aggressively against smaller bets, then the bettor gets away with taking smaller risks for higher returns. Because of this, a monotone calling strategy is less exploitable. Imposing this condition on the caller restricts the betting strategies in a way that gives us a unique Nash equilibrium.

Definition 1.2 (Monotone-Admissible Strategy)**.** A betting strategy σ_B is *monotone-admissible* if it is admissible in LCP against the set of monotone calling strategies. More explicitly, σ_B is monotone-admissible if there does not exist a betting strategy σ'_B such that both of the following hold:

1. $\pi_B(\sigma'_B, \sigma_C) \geq \pi_B(\sigma_B, \sigma_C)$ for all monotone calling strategies σ_C
2. $\pi_B(\sigma'_B, \sigma_C) > \pi_B(\sigma_B, \sigma_C)$ for at least one monotone calling strategy σ_C

This is useful in distinguishing bettor strategies which differ only in how they bluff. We will see that the hand strength of a bluff is irrelevant if the caller plays optimally because the caller will never call with a hand weaker than any bluff. However, if the caller deviates to a suboptimal but still monotone strategy, then the bettor's bluffing hand strength becomes important. If the caller becomes too loose, i.e. calling too often with weak hands, then the bettor ends up winning some pots "accidentally" if they make their smallest bluffs with their strongest bluffing hands. This is where monotone-admissibility differentiates between equilibria.

1.2 Nash Equilibrium Structure

We will now describe the structure of the Nash equilibrium in terms of constants x_i and functions $c(s)$, $b(s)$, and $v(s)$. These turn out to be fully determined by the parameters L and U, but for now they are unknown. Notice that both players use pure strategies, like in NLCP: the bettor maps hand strengths directly to bet sizes, and the caller maps hand strengths and bet sizes to actions with no mixing.

1. The caller has a calling threshold $c(s)$ that is continuous in s, including at endpoints L and U. They call with hands $y \geq c(s)$ and fold with hands

$y < c(s)$[1].

2. The bettor partitions $[0, 1]$ into three regions: bluffing $x \in [0, x_2]$, checking $x \in [x_2, x_3]$, and value betting $x \in [x_3, 1]$.

3. Within the bluffing region, the bettor's partitions into a max-betting region $x \in [x_0, x_1]$, an intermediate region $x \in [x_1, x_2]$, and a min-betting region $x \in [x_2, x_3]$.

4. Within the intermediate bluffing region, the bettor bets according to a continuous, decreasing function $s = b^{-1}(x)$ with endpoints $b^{-1}(x_0) = U$ and $b^{-1}(x_3) = L$.

5. Within the value betting region, the bettor partitions into a min-betting region $x \in [x_3, x_4]$, an intermediate region $x \in [x_4, x_5]$, and a max-betting region $x \in [x_5, 1]$.

6. Within the intermediate value betting region, the bettor bets according to a continuous, increasing function $s = v^{-1}(x)$ with endpoints $v^{-1}(x_3) = L$ and $v^{-1}(x_5) = U$.

See figure **??** for visual representations of the strategy profile.

1.3 Constraints and Indifference Equations

The Nash equilibrium strategy profile must satisfy several constraints and indifference conditions, which we will derive and use to solve for the strategy profile. The key conditions are:

- The caller must be indifferent between calling and folding at their calling threshold
- The bettor must be indifferent between checking and betting at their value betting and bluffing thresholds
- The bettor's bet size for a value bet must maximize their expected value
- The bettor's strategy must be continuous in bet size (in the regions where they bet)

[1]The action taken at the threshold is irrelevant, since it occurs with probability zero.

These conditions give us the following system of equations:
Caller Indifference:

$$(x_2 - x_1) \cdot (1 + L) - (x_4 - x_3) \cdot L = 0 \tag{1}$$
$$x_0 \cdot (1 + U) - (1 - x_5) \cdot U = 0 \tag{2}$$
$$|b'(s)| \cdot (1 + s) - |v'(s)| \cdot s = 0 \tag{3}$$

Bettor Indifference and Optimality:

$$- sc'(s) - c(s) + 2v(s) - 1 = 0 \tag{4}$$
$$(x_3 - c(L)) \cdot (1 + L) - (1 - x_3) \cdot (L) + c(L) = x_3 \tag{5}$$
$$c(s) - (1 - c(s)) \cdot s = x_2 \tag{6}$$

Continuity Constraints:

$$b(U) = x_0 \tag{7}$$
$$b(L) = x_1 \tag{8}$$
$$v(U) = x_5 \tag{9}$$
$$v(L) = x_4. \tag{10}$$

We will now derive each of these equations in turn. Note that for this analysis, it is simpler to pretend that payoffs exclude the initial ante of 0.5, since this is a sunk cost to both players and we only care about relative payoffs between actions.

1.3.1 Caller Indifference

By definition, $c(s)$ is the threshold above which the caller calls and below which they fold. This means that in Nash Equilibrium, the caller must be indifferent between calling and folding with a hand strength of $c(s)$:

$$\mathbb{E}[\text{call } c(s)] = \mathbb{E}[\text{fold } c(s)]$$
$$\mathbb{P}[\text{bluff}|s] \cdot (1 + s) - \mathbb{P}[\text{value bet}|s] \cdot s = 0.$$

We now split into cases based on the value of s.

Case 1: $s = L$. The hands the bettor value bets L with are $x \in (x_3, x_4)$, and the hands they bluff with are $x \in (x_1, x_2)$.

$$(x_2 - x_1) \cdot (1 + L) - (x_4 - x_3) \cdot L = 0. \tag{11}$$

Here, we are implicitly multiplying both sides by the common denominator of $(x_4 - x_3) + (x_2 - x_1)$.

Case 2: $s = U$. The hands the bettor value bets U with are $x \in (x_5, 1)$, and the hands they bluff with are $x \in (0, x_0)$.

$$(1 - x_5) \cdot (1 + U) - x_0 \cdot U = 0, \tag{12}$$

again, implicitly multiplying both sides by the common denominator of $(1 - x_5) + x_0$.

Case 3: $L \leq s \leq U$. In this case, the bettor has exactly one value hand and one bluffing hand, but somewhat paradoxically, they are not equally likely. The probability of a value bet given the size s is related to the inverse derivative of the value function $v(s)$ at s, and the same goes for a bluff. This gives us the following relation:

$$\frac{\mathbb{P}[\text{value bet}|s]}{\mathbb{P}[\text{bluff}|s]} = \frac{|b'(s)|}{|v'(s)|}$$

An intuitive interpretation of this is that for any small neighborhood around the bet size s, the bettor has more hands which use a bet size in the neighborhood if $v(s)$ does not change rapidly around s, that is, if $|v'(s)|$ is small. The same goes for bluffing hands, and as we limit the neighborhood to a single point, the ratio of the two probabilities approaches the ratio of the derivatives. We know that these are the only two possible bettor actions for such a bet size, so

$$\mathbb{P}[\text{value bet}|s] = \frac{|b'(s)|}{|b'(s)| + |v'(s)|}$$
$$\mathbb{P}[\text{bluff}|s] = \frac{|v'(s)|}{|b'(s)| + |v'(s)|}$$

Plugging this into the indifference equation and dividing out the common denominator, we get:

$$|b'(s)| \cdot (1 + s) - |v'(s)| \cdot s = 0. \tag{13}$$

1.3.2 Bettor Indifference and Optimality

When the bettor makes a value bet, they are attempting to maximize the expected value of the bet. We can write the expected value of a value bet as:

$$\begin{aligned}\mathbb{E}[\text{value bet } s|x] &= \mathbb{P}[\text{call w ith worse}] \cdot (1 + s) - \mathbb{P}[\text{call with better}] \cdot s + \mathbb{P}[\text{fold}] \cdot 1 \\ &= (x - c(s)) \cdot (1 + s) - (1 - x) \cdot (s) + c(s).\end{aligned}$$

To maximize this, we take the derivative with respect to s and set it equal to zero. Crucially, we are treating $c(s)$ as a function of s and using the chain rule, since changing the bet size s will also change the calling threshold $c(s)$. We want this optimality condition to hold for the bettor's Nash equilibrium strategy, so we set $x = v(s)$. This gives us:

$$\begin{aligned} &\frac{d}{ds}\mathbb{E}[\text{value bet } s|x = v(s)] = 0 \\ &-sc'(s) - c(s) + 2v(s) - 1 = 0. \end{aligned} \tag{14}$$

Additionally, when the bettor has the most marginal value betting hand at $x = x_3$, they should be indifferent between a minimum value bet and a check:

$$\begin{aligned} \mathbb{E}[\text{value bet } L|x = x_3] &= \mathbb{E}[\text{check}|x = x_3] \\ (x_3 - c(L)) \cdot (1 + L) - (1 - x_3) \cdot (L) + c(L) &= x_3. \end{aligned} \tag{15}$$

Finally, when the bettor has the most marginal bluffing hand at $x = x_2$, they should be indifferent between a minimum bluff and a check. However, as we discussed earlier, the bettor should be indifferent among all bluffing sizes, so the bettor should actually be indifferent between checking and making any bluffing size s at $x = x_2$. This gives us:

$$\begin{aligned} &\mathbb{E}[\text{bluff } s|x = x_2] = \mathbb{E}[\text{check}|x = x_2] \\ &c(s) - (1 - c(s)) \cdot s = x_2. \end{aligned} \tag{16}$$

1.3.3 Continuity Constraints

As discussed above, the bettor's strategy is continuous in s and x (except when checking). This means that the endpoints of the functions $v(s)$ and $b(s)$ are constrained as follows:

$$b(U) = x_0, \quad b(L) = x_1, \quad v(U) = x_5, \quad v(L) = x_4. \tag{17}$$