Limit Continuous Poker: A Variant of Continuous Poker with Limited Bet Sizes

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1 Abstract

We introduces and analyze Limit Continuous Poker, a variant of Von Neumann's Continuous Poker with variable but limited bet sizes. This simplified variant of poker captures aspects of information asymmetry, bluffing, balancing, and the impact of bet size limits while still being simple enough to solve analytically. We derive the Nash equilibrium strategy profile for this game, showing how the bettor's and caller's strategies depend on the bet size limits. We demonstrate that as the bet size limits approach extreme values, the strategy profile converges to those of other continuous poker variants. Finally, we connect these results to strategic implications of limited bet sizing in real-world poker.

2 Introduction

explain poker, general ideas of bluffing, value betting, etc.

2.1 Previous Work

2.1.1 Fixed-Bet Continuous Poker (FBCP)

Continuous Poker (also called Von Neumann Poker, and referred to in this paper as Fixed-Bet Continuous Poker or FBCP) is a simplified model of poker. It is a two-player zero-sum game designed to study strategic decision-making in competitive environments. The game abstracts away many complexities of real poker, focusing instead on the mathematical and strategic aspects of bluffing, betting, and optimal play.

Definition 2.1 (FBCP)**.** Two players, referred to as the bettor and the caller, each put a 0.5 unit ante into a pot[1]. They are each dealt a 'hand strength' uniformly and independently from the interval $[0, 1]$ (referred to as x for bettor and y for caller). After seeing x, the bettor can either check - in which case, the higher hand between x and y wins the pot of 1 and the game ends - or they can bet by putting a pre-determined amount $B > 0$ into the pot. The caller can now either call by matching the bet of B, after which the higher hand wins the pot of $1 + 2B$, or fold, conceding the pot of $1 + B$ to the bettor and ending the game.

FBCP has many Nash equilibria, but it has a unique one in which the caller plays an admissible strategy[2]. This strategy profile, parametrized by the bet size B, is as follows:

The bettor bets with hands x such that either

$$x > \frac{1 + 4s + 2B^2}{(1 + 2B)(2 + B)} \text{ or } x < \frac{B}{(1 + 2B)(2 + B)}$$

We call the higher interval the value betting range and the lower interval the bluffing range. The caller calls with hands y such that

$$y > \frac{B(3 + 2B)}{(1 + 2B)(2 + B)}$$

The non-uniqueness of this Nash Equilibrium is due to the fact that given the bettor's strategy, the caller can achieve the same expected payoff with any calling threshold between the value betting range and the bluffing range. However, any calling strategy other than this one incentivizes the bettor to deviate from the Nash Equilibrium strategy and leads to a lower payoff for the caller.

The value of FBCP for the bettor is

$$V_{FB}(B) = \frac{B}{2(1 + 2B)(2 + B)}$$

Which is positive and maximized at $B = 1$, when the bet size is exactly the pot size.

[1] An ante of 1 is often used, but since the pot size is the more relevant value, we use an ante of 0.5. All bet sizes simply scale proportionally.

[2] An admissible strategy is one which is not strictly dominated by any other strategy

2.1.2 No-limit Continuous Poker (NLCP)

Another continuous poker variant allows the bettor to choose a bet size $s > 0$ after seeing their hand strength, as opposed to a fixed bet size B. This variant is called No-Limit Continuous Poker (or Newman Poker after Donald J. Newman, or NLCP in this paper). The Nash equilibrium strategy profile for this variant is discussed and solved in *The Mathematics of Poker* by Bill Chen and Jerrod Ankenman (see page 154).

citation

generate a graph of the strategies

In Nash Equilibrium, the bettor should make large bets with their strongest and weakest hands and smaller bets or checks with their intermediate hands. It turns out that the optimal strategy is most elegantly described by a mapping from bet sizes s to hand strengths x for bluffing and value betting, respectively. The caller simply has a calling threshold $c(s)$ for each possible bet size s. The full strategy profile is as follows:

The bettor bets s with hands x such that either

$$x = \frac{3s+1}{7(s+1)^3} \text{ or } x = 1 - \frac{3}{7(s+1)^2}$$

Where the first condition represents bluffing hands and the second value betting hands. After seeing a bet of size s, the caller should call with hands y such that

$$y > 1 - \frac{6}{7(s+1)}$$

Note that the bettor uses all possible bet sizes and has exactly two hand strengths for each bet size. On first inspection, this feels like the bettor is giving away too much information, but it turns out to still an optimal strategy. This concept appears again and is explained more thoroughly in section **??**.

The value of NLCP is

$$V_{NL} = \frac{1}{14}$$

for the bettor[3].

3 Limit Continuous Poker (LCP)

We now introduce a variant where the bettor may choose a bet size s after seeing their hand strength, but where s is bounded by an upper limit U

[3] Would be 1/7 for an ante of 1, but the value is halved with an ante of 0.5

and a lower limit L, referred to as the maximum and minimum bet sizes. This variant, called Limit Continuous Poker (or LCP), retains similar rules to NLCP:

Definition 3.1 (LCP)**.** Two players, the bettor and the caller, are each dealt uniform, independent hand strengths $x, y \in [0, 1]$. After seeing x, the bettor chooses between checking (giving payoff 1 to the higher hand) or betting an amount $s \in [L, U]$. If the bettor bets, the caller must either call (giving payoff $1 + 2s$ to the higher hand) or fold (giving payoff 1 to the bettor).

4 Solving LCP

4.1 Nash Equilibrium Structure

Theorem 4.1. *LCP has a unique admissible-strategy Nash equilibrium, which has the following structure:*

1. *The caller has a single calling threshold $c(s)$ that is non-decreasing and continuous in s, including at endpoints L and U. They call with hands $y \geq c(s)$ and fold with hands $y < c(s)$.*
2. *The bettor partitions $[0, 1]$ into seven regions with thresholds $0 \leq x_0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq 1$:*
   - *Value bets: hands $x \in (x_5, 1)$ bet U, hands $x \in (x_4, x_5)$ bet $s \in (L, U)$ according to $x = v(s)$, hands $x \in (x_3, x_4)$ bet L*
   - *Checks: hands $x \in (x_2, x_3)$*
   - *Bluffs: hands $x \in (x_1, x_2)$ bet L, hands $x \in (x_0, x_1)$ bet $s \in (L, U)$ according to $x = b(s)$, hands $x \in (0, x_0)$ bet U*

Intuitively, the strategy profile is similar to NLCP, with the bettor partitioning $[0, 1]$ into value betting, checking, and bluffing regions, and the caller partitioning into calling and folding regions.

Proof. We will prove the structure of the Nash equilibrium by establishing several key claims:

1. The caller must use a threshold strategy with cutoff $c(s)$
2. The cutoff $c(s)$ must be non-decreasing in s
3. The cutoff $c(s)$ must be continuous, even at endpoints L and U

4. The bettor must bet for value with hands stronger than $c(s)$, bluff with hands weaker than $c(s)$, and check some range of intermediate hands

5. Value betting sizes must increase with hand strength

6. When bluffing, the bettor must be indifferent among exactly the sizes which are used for value betting

7. Bluffing sizes should decrease with hand strength

Claim 1: The caller must use a threshold strategy with cutoff $c(s)$. For any bet size s, if the caller calls with hand strength y and folds with hand strength $y' > y$, they are strictly better off calling with y' and folding with y. Therefore, they must call with all hands above some threshold $c(s)$ and fold with all hands below it.

Claim 2: The cutoff $c(s)$ must be non-decreasing in s. If $c(s)$ were decreasing at any point, the bettor could exploit this by bluffing with a slightly smaller size than they would otherwise use. This would cause the caller to fold more often while risking less money, contradicting equilibrium.

Claim 3: The cutoff $c(s)$ must be continuous, even at endpoints L and U. If $c(s)$ had a discontinuity, the bettor's expected value from bluffing would also be discontinuous. They could then exploit by bluffing with sizes just below the discontinuity, contradicting equilibrium.

Claim 4: Value bets must be made with hands stronger than $c(s)$, bluffs with hands weaker than $c(s)$. When betting size s, the bettor wants to get called by weaker hands when value betting (requiring $x > c(s)$) and get stronger hands to fold when bluffing (requiring $x < c(s)$).

Claim 5: Value betting sizes must increase with hand strength. Since $c(s)$ is non-decreasing, stronger hands can profitably bet larger sizes and get called by a more restricted range of strong hands. Weaker value betting hands must bet smaller to get called by a wider range.

Claim 6: The bettor must be indifferent among bluffing sizes that are also used for value bets. The expected value of bluffing size s is:

$$\mathbb{E}[\text{bluff } s] = c(s) - (1 - c(s)) \cdot s \tag{1}$$

This is independent of the bettor's hand strength. If the bettor strictly preferred certain sizes for bluffing, they would never bluff with other sizes. If any other size were used for value betting, then the caller could exploit by only calling those sizes with hands stronger than the value betting range. Additionally, the bettor cannot bluff with sizes which are not used for value betting. In this case, the caller can similarly exploit by always calling this size with hands stronger than the bluffing range.

Claim 7: Bluffing sizes should decrease with hand strength. This claim is not necessary to have a Nash equilibrium, but it is what makes the strategy profile uniquely admissible. If the caller deviates by calling too loosely but maintains consistency (never calling with weaker hands and folding with stronger hands to the same bet size), the bettor uniquely benefits by bluffing larger with their weakest hands and bluffing smaller with their strongest hands. This gives them a possiblity of winning showdowns with their strongest bluffing hands, which would not happen if they bluffed large with the strongest hands. □

admissiblity?

diagram of strategy profile

flow between paragraphs

4.2 Constraints and Indifference Equations

The Nash equilibrium strategy profile must satisfy several constraints and indifference conditions, which we will derive and use to solve for the strategy profile. The key conditions are:

- The caller must be indifferent between calling and folding at their calling threshold
- The bettor must be indifferent between checking and betting at their value betting and bluffing thresholds
- The bettor's bet size for a value bet must maximize their expected value
- The bettor's strategy must be continuous in bet size (in the regions where they bet)

These conditions give us the following system of equations:

Caller Indifference:

$$
\begin{aligned}
&(x_4 - x_3) \cdot (1 + L) - (x_2 - x_1) \cdot L = 0 \\
&(1 - x_5) \cdot (1 + U) - x_0 \cdot U = 0 \\
&|b'(s)| \cdot (1 + s) + |v'(s)| \cdot s = 0
\end{aligned}
$$

Bettor Indifference and Optimality:

$$
\begin{aligned}
&- sc'(s) - c(s) + 2v(s) - 1 = 0 \\
&c(L) - (1 - c(L)) \cdot L = x_3 \\
&c(s) - (1 - c(s)) \cdot s = x_2
\end{aligned}
$$

Continuity Constraints:

$$
\begin{aligned}
&b(U) = x_0 \\
&b(L) = x_1 \\
&v(U) = x_5 \\
&v(L) = x_4
\end{aligned}
$$

We will now derive each of these equations in turn.

4.2.1 Caller Indifference

By definition, $c(s)$ is the threshold above which the caller calls and below which they fold. This means that in Nash Equilibrium, the caller must be indifferent between calling and folding with a hand strength of $c(s)$:

$$\mathbb{E}[\text{call } c(s)] = \mathbb{E}[\text{fold } c(s)]$$

$$\mathbb{P}[\text{bluff}|s] \cdot (1 + s) - \mathbb{P}[\text{value bet}|s] \cdot s = 0$$

We now split into cases based on the value of s.

Case 1: $s = L$. The hands the bettor value bets L with are $x \in (x_3, x_4)$, and the hands they bluff with are $x \in (x_1, x_2)$.

$$(x_4 - x_3) \cdot (1 + L) - (x_2 - x_1) \cdot L = 0 \tag{2}$$

Case 2: $s = U$. The hands the bettor value bets U with are $x \in (x_5, 1)$, and the hands they bluff with are $x \in (0, x_0)$.

$$(1 - x_5) \cdot (1 + U) - x_0 \cdot U = 0 \tag{3}$$

Case 3: $L \leq s \leq U$. In this case, the bettor has exactly one value hand and one bluffing hand, but somewhat paradoxically, they are not equally likely. The probability of a value bet given the size s is related to the inverse derivative of the value function $v(s)$ at s, and the same goes for a bluff. This gives us the following relation:

$$\frac{\mathbb{P}[\text{value bet}|s]}{\mathbb{P}[\text{bluff}|s]} = \frac{|b'(s)|}{|v'(s)|}$$

An intuitive interpretation of this is that for any small neighborhood around the bet size s, the bettor has more hands which use a bet size in the neighborhood if $v(s)$ does not change rapidly around s, that is, if $|v'(s)|$ is small. The same goes for bluffing hands, and as we limit the neighborhood to a single point, the ratio of the two probabilities approaches the ratio of the derivatives. Plugging this into the indifference equation, we get:

$$|b'(s)| \cdot (1+s) + |v'(s)| \cdot s = 0 \tag{4}$$

4.2.2 Bettor Indifference and Optimality

When the bettor makes a value bet, they are attempting to maximize the expected value of the bet. We can write the expected value of a value bet as:

$$\begin{aligned}\mathbb{E}[\text{value bet } s|x] &= \mathbb{P}[\text{call with worse}] \cdot (1+s) - \mathbb{P}[\text{call with better}] \cdot s + \mathbb{P}[\text{fold}] \cdot 1 \\ &= (x - c(s)) \cdot (1+s) - (1-x) \cdot (s) + c(s)\end{aligned}$$

To maximize this, we take the derivative with respect to s and set it equal to zero. Crucially, we are treating $c(s)$ as a function of s and using the chain rule, since changing the bet size s will also change the calling threshold $c(s)$. We want this optimality condition to hold for the bettor's Nash equilibrium strategy, so we set $x = v(s)$. This gives us:

$$\begin{aligned}\frac{d}{ds}\mathbb{E}[\text{value bet } s|x = v(s)] &= 0 \\ -sc'(s) - c(s) + 2v(s) - 1 &= 0 \end{aligned} \tag{5}$$

Additionally, when the bettor has the most marginal value betting hand at $x = x_3$, they should be indifferent between a minimum value bet and a check:

$$\begin{aligned}\mathbb{E}[\text{value bet } L|x = x_3] &= \mathbb{E}[\text{check}|x = x_3] \\ (x_3 - c(L)) \cdot (1+L) - (1-x_3) \cdot (L) + c(L) &= x_3\end{aligned} \tag{6}$$

Finally, when the bettor has the most marginal bluffing hand at $x = x_2$, they should be indifferent between a minimum bluff and a check. However, as we discussed earlier, the bettor should be indifferent among all bluffing sizes, so the bettor should actually be indifferent between checking and making any bluffing size s at $x = x_2$. This gives us:

$$\begin{aligned} \mathbb{E}[\text{bluff } s|x = x_2] &= \mathbb{E}[\text{check}|x = x_2] \\ c(s) - (1 - c(s)) \cdot s &= x_2 \end{aligned} \tag{7}$$

4.2.3 Continuity Constraints

As discussed above, the bettor's strategy is continuous in s and x (except when checking). This means that the endpoints of the functions $v(s)$ and $b(s)$ are constrained as follows:

$$b(U) = x_0, \quad b(L) = x_1, \quad v(U) = x_5, \quad v(L) = x_4 \tag{8}$$

5 Nash Equilibrium Strategy Profile

The full solution process is detailed in Appendix A. Here we present the final solution, which was obtained by solving for $c(s)$ in terms of x_2, then using this to solve for $v(s)$, and finally solving for $b(s)$ up to a constant of integration. The resulting system of 7 equations in 7 unknowns was solved symbolically using Mathematica and simplified by finding common subexpressions $A_0, A_1, A_2, A_3, A_4, A_5$.

Theorem 5.1 (Nash Equilibrium Strategy Profile). *The unique admissible Nash equilibrium strategy profile for Limit Continuous Poker with minimum bet size L and maximum bet size U is given by:*

$$
\begin{aligned}
x_0 &= \frac{3(L+1)^3U}{A_4} \\
x_1 &= \frac{3A_0LU + A_0U - L^3 - 3L^2}{A_4} \\
x_2 &= \frac{A_5}{A_4} \\
x_3 &= \frac{A_2L^3 + 3A_2L^2 + 3L\left(5U^3 + 15U^2 + 15U + 4\right) + 4U^3 + 12U^2 + 12U + 3}{A_4} \\
x_4 &= \frac{3A_1L^2 + A_2L^3 + 3A_2L + 4U^3 + 12U^2 + 12U + 3}{A_4} \\
x_5 &= \frac{3A_3L^2 + 3A_3L + A_3 + L^3\left(6U^3 + 18U^2 + 15U + 2\right)}{A_4} \\
b_0 &= -\frac{(L+1)^3}{A4} \\
b(s) &= b_0 - \frac{(1+3s)(x_2-1)}{6(1+s)^3} \\
c(s) &= \frac{x_2+s}{s+1} \\
v(s) &= \frac{x_2 + 2s^2 + 4s + 1}{2(s+1)^2}
\end{aligned}
$$

where the common subexpressions are:

$$
\begin{aligned}
A_0 &= U^2 + 3U + 3 \\
A_1 &= 7U^3 + 21U^2 + 21U + 6 \\
A_2 &= 6U^3 + 18U^2 + 18U + 5 \\
A_3 &= 7U^3 + 21U^2 + 18U + 3 \\
A_4 &= 3A_1L^2 + 3A_1L + A_1 + A_2L^3 \\
A_5 &= 3A_0L^2U + 3A_0LU + A_0U - L^3
\end{aligned}
$$

Proof given in Appendix A. Refer to section **??** for an explanation of how these values fit together to actually form the strategy profile.

This solution is more interpretable in graphical form. Figure 1 shows the strategy profile for various values of L and U ranging from very lenient ($L = 0, U = 10$) to very restricted ($L = 0.5, U = 1$). The more lenient bet size limits model something closer to NLCP, while the more restricted bet size limits model something closer FBCP with a fixed bet size. Indeed, we see that the strategy profile of for $L = 0, U = 10$ looks qualitatively similar to the strategy profile of NLCP - we will show in section 8.2 that the strategy profile approaches the Nash equilibrium of NLCP as L and U approach 0 and ∞, respectively, and that the strategy profile approaches the Nash equilibrium of FBCP as L and U approach some fixed value s from either side.

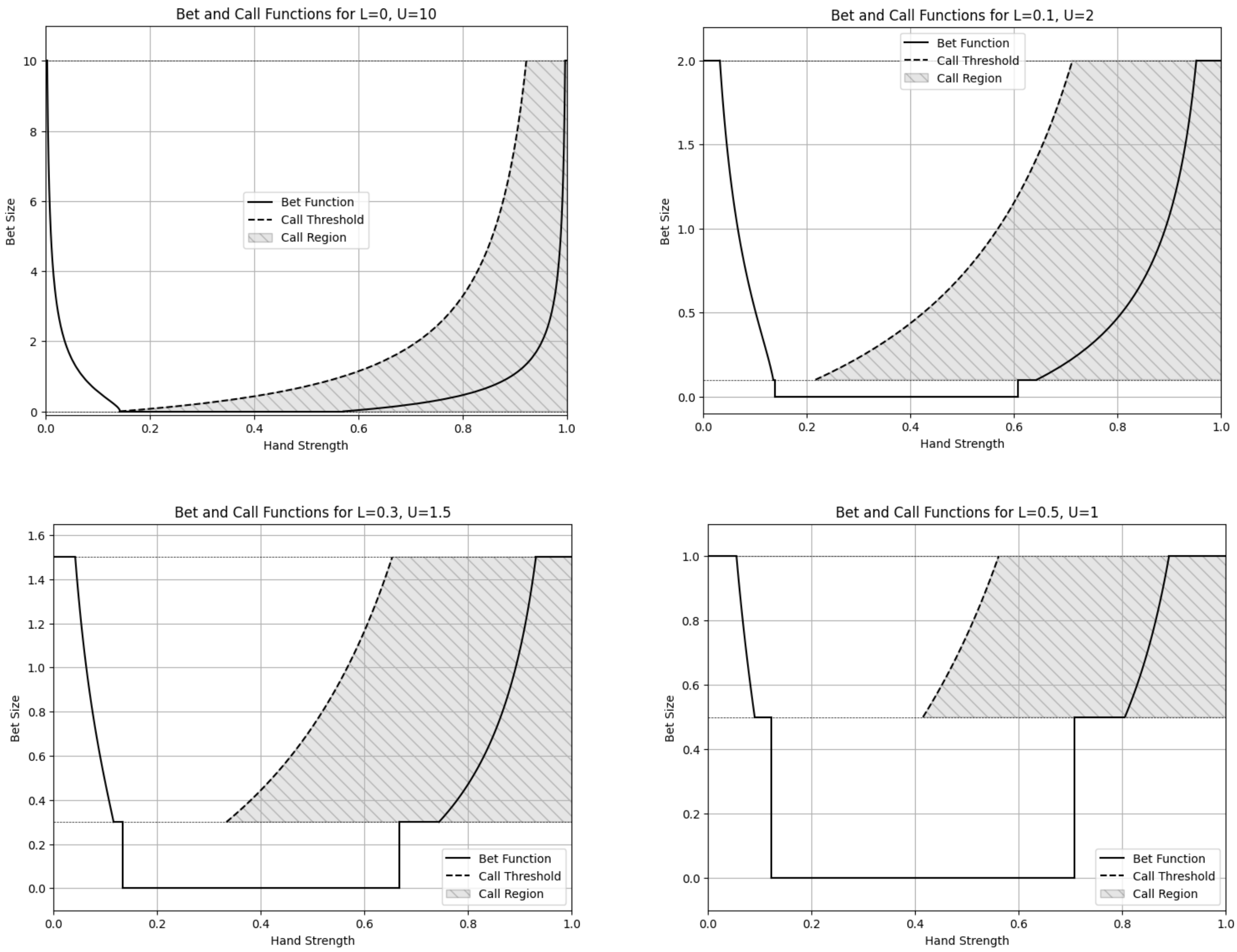


Figure 1: Nash equilibrium strategy profiles for different values of L and U, from very lenient to very restricted bet sizes. The bet function maps hand strengths to bet sizes, while the call function gives the minimum calling hand strength for a given bet size. The shaded regions represent the hand strengths for which the caller should call a given bet size.

6 Payoff Analysis

In Nash Equilibrium, a given hand combination (x, y) uniquely determines the bettor's payoff, since both players play pure strategies. We can better understand the strategy profile by examining payoffs for all hand combinations over the unit square $[0, 1]^2$ (see Figure 2). For comparison, the first and last plots show the payoffs in Fixed-Bet Continuous Poker with a fixed bet size $B = 1$ and No-Limit Continuous Poker, respectively. We will explore the relationship between the three games in more detail in section 8.2.

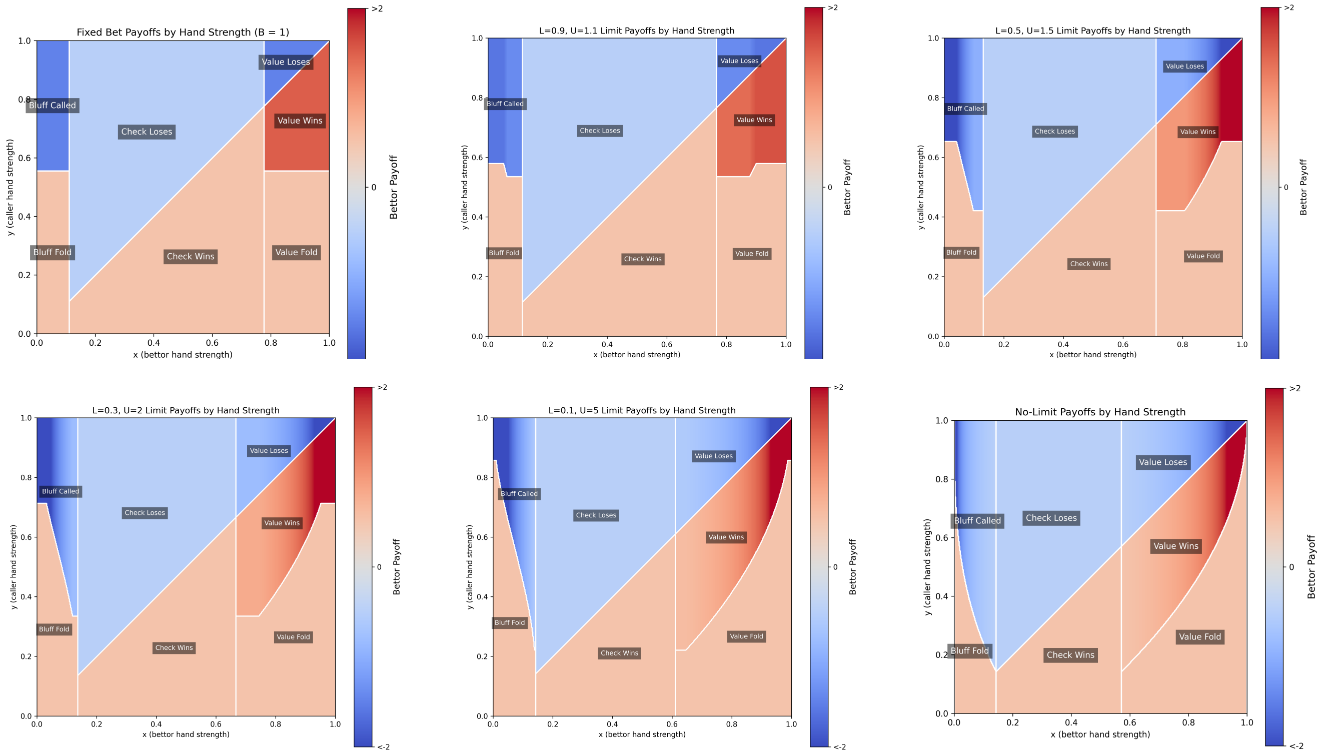


Figure 2: Bettor payoffs in Nash equilibrium as a function of hand strengths x, y for fixed bet size $B = 1$ (top left), and No-Limit Continuous Poker (bottom right). Intermediate plots show the payoffs for different values of L and U ranging from strict (fixed bet size $B = 1$) to lenient (No limits). Regions are labeled according to the outcome of the game in Nash equilibrium.

This visualization gives insight into exactly how the bettor is gaining value from the game. In line with experience of real poker, the biggest wins and losses occur when both hands are strong (top right of any plot), with the stronger of the two hands winning a large pot. However, we also see large payoffs when a very weak bettor bluffs big and gets called by a strong caller (top left). As the limits become more lenient, these cases become more extreme but also less likely, since making and calling maximum bets become more risky for both players.

6.1 Expected Value of Bettor's Hand Strengths

plots

In addition to considering payoffs for a specific bettor-caller hand combination, we can also consider the expected value of given hand for the bettor, not knowing the hand of the caller. Let $EV(x)$ denote the expected value of a value-betting hand x in the unique admissible Nash equilibrium.

Theorem 6.1.

$$EV(x) = \begin{cases} x_2 - \frac{1}{2} & \textit{if } x \le x_2 \\ x - \frac{1}{2} & \textit{if } x_2 < x \le x_3 \\ x(2L+1) - L(c(L)+1) - \frac{1}{2} & \textit{if } x_3 < x < v(L) \\ x(2v^{-1}(x)+1) - v^{-1}(x)(c(v^{-1}(x))+1) - \frac{1}{2} & \textit{if } v(L) \le x \le v(U) \\ x(2U+1) - U(c(U)+1) - \frac{1}{2} & \textit{if } x > v(U) \end{cases} \tag{9}$$

Proof. For bluffing hands $x \le x_2$, we can use a simple argument to show that the hand strength x is actually irrelevant. The bettor never gets called by worse hands, so either the caller folds or calls with the best hand. In either case, the bettor's payoff has no dependence on x, so the payoff must be the same for all $x \le x_2$ (otherwise, a the lower-payoff hands would imitate the strategy of the higher-payoff hands). We know that at $x = x_2$, the bettor is indifferent between bluffing and checking, so the payoff must be $x_2 - \frac{1}{2}$ for all $x \le x_2$.

For checking hands $x_2 \le x \le x_3$, the bettor wins only the ante exactly when they have the best hand, which happens with probability x. The value of the ante is $\frac{1}{2}$, so the bettor's expected value is $x - \frac{1}{2}$.

For any value betting hand, the bettor has three case to consider: the caller folds, the callers calls with a worse hand, or the caller calls with the best hand. We simply sum the expected value of each of these cases.

$$EV(x) = \frac{1}{2}c(s) + (x - c(s))\left(s + \frac{1}{2}\right) + (1-x)\left(-s - \frac{1}{2}\right)$$

The last three cases come from substituting $L, v^{-1}(x), U$ for s in the expression above and simplifying. $\square$

We can quickly verify that the bettor's expected value is increasing in x. This must be the case, since with any given hand strength, the bettor can always choose to imitate the Nash equilibrium strategy of a weaker hand, so the stronger hand must be at least as good in expectation.

Theorem 6.2. *For any fixed L, U, the bettor's expected value $EV(x)$ is increasing in x:*

$$\frac{d}{dx}EV(x) > 0$$

Proof. It is clear from inspection that any checking EV is higher than that of a bluff and that the checking EV is increasing in x. We also know that at $x = x_3$, the bettor is indifferent between checking and betting, so the EV of checking and betting must be equal at this point. Therefore, we only need to show that within the checking and value betting regions, the EV is increasing in x. This is obvious for checking hands. For value betting hands, we can take the derivative of the expression for $EV(x)$ with respect to x and show that it is always positive. We consider the max and min betting hands first:

$$s = U \implies \frac{d}{dx}EV(x) = 2U + 1 > 0$$
$$s = L \implies \frac{d}{dx}EV(x) = 2L + 1 > 0$$

For intermediate value betting hands, we can use the chain rule to show that the EV is increasing in x:

$$\frac{d}{dx}EV(x) = \frac{\partial EV(x)}{\partial x} + \frac{dEV(x)}{ds}\frac{ds}{dx}$$

From the value optimality condition 5, we know that $\frac{dEV(x)}{ds} = 0$ (otherwise, the bettor could gain value by varying the bet size). Clearly, $\frac{\partial EV(x)}{\partial x} > 0$. Therefore, the derivative is positive, and the EV is increasing in x for all value betting hands. □

It is worth noting that the bettors strongest hands (right edge) actually seem to become less likely to make any profit more than the ante as limits increase. These strongest hands make very large bets, which force all but the strongest hands to fold, but win huge pots when they do get called. In more complicated poker variants, it is common to 'slowplay' strong hands by checking or making small bets to induce bluffs from the opponent. In LCP, there is only one betting round and the caller is not allowed to raise, both of which make slowplaying obsolete. With extremely strong hands, the benefit of winning a large pot when betting big outweighs the lower likelihood of getting called.

7 Parameter Analysis

7.1 Effect of Increasing U

7.1.1 Expected Payoff of Value-Betting Hands

It may seem unsurprising that strong hangs become less likely to get called as limits increase, but what about the actual expected payoff of these hands? Does the expected value of each hand continue increasing as we increase U? The answer is no. In fact, for any fixed hand strength x, the expected payoff of that hand increases in U only for large enough x or small enough U, after which it decreases. This feels counterintuitive; increasing U only gives the bettor more options, so how is it possible that the expected payoff of a wide range of their hands decreases? And which hands are gaining expected payoff to offset this? This is a surprising result, and it is worth exploring in more detail.

Recall from section 6.1 that $EV(x)$ denotes the expected payoff of a value-betting hand x in the unique admissible Nash equilibrium.

Theorem 7.1. *For any value-betting hand strength x and any L, U, $EV(x)$ is decreasing in U for all x which make intermediate-sized bets and weak hands which bet the maximum, and increasing in U for strong hands which bet the maximum, above a certain threshold. Specifically, $\frac{d}{dU}EV(x) < 0$ if*

$$x < \max\left(v(U), \frac{1}{2(1+U)}\left(U\frac{\partial x_2}{\partial U} + \frac{U^2 + x_2(1+2U)}{1+U}\right)\right)$$

and $\frac{d}{dU}EV(x) > 0$ otherwise.

Before proving the theorem, we will walk through some lemmas which explore how all the relevant variables change as we increase U, including the bluffing threshold x_2, the bet size $v^{-1}(x)$, and the calling cutoff $c(s)$.

7.1.2 Bluffing Threshold

We begin by showing that x_2, the boundary hand strength between bluffing and checking, is increasing in U. This means that for fixed L, increasing the upper limit U makes the bettor bluff with more hands.

Lemma 7.1. *For any fixed L, U,*

$$\frac{\partial x_2}{\partial U} > 0$$

Proof. Taking the partial derivative of x_2 with respect to U and rearranging terms gives:

$$\frac{\partial x_2}{\partial U} = \frac{18(L+1)^6(U+1)^2}{(A_1 + L^3 A_2 + 3L^2 A_1 + 3LA_1)^2}$$

Which is always positive since $L \in [0, U]$, $U \in [0, \infty)$, and A_1, A_2 are both positive-coefficient polynomials in L and U. $\square$

7.1.3 Bet Size

We now show that if we fix x at any intermediate value-betting hand strength (betting neither the minimum nor maximum bet size) and then increase U, the bet size s made by x decreases. The intermediate value-betting hands are exactly $x \in [x_3, v(U)]$ and their bet sizes are given by $s = v^{-1}(x)$, so we get the following lemma:

Lemma 7.2. *For any fixed $x \in [x_3, v(U)]$,*

$$\frac{d}{dU} v^{-1}(x) < 0$$

Proof. Recall that

$$v^{-1}(x) = -\frac{\sqrt{(4x-4)(2x_2-2)}}{4x-4} - 1$$

Where x_2 is a function of L and U. Importantly, $v^{-1}(x)$ is only dependent on U through x_2, so we can use the chain rule to take the derivative with respect to U:

$$\frac{d}{dU} v^{-1}(x) = \frac{\partial v^{-1}(x)}{\partial x_2} \frac{\partial x_2}{\partial U}$$

The first term is

$$\frac{\partial v^{-1}(x)}{\partial x_2} = -\frac{1}{\sqrt{(4x-4)(2x_2-2)}}$$
$$= -\frac{1}{(v^{-1}(x)+1)(4-4x)}$$

Which is always negative since $x \in [0,1]$ and $v^{-1}(x) > 0$. We know that the second term is positive by Lemma 7.1.

Therefore, the product of the two terms is always negative, so the bet size of intermediate bets is decreasing in U. □

7.1.4 Calling Cutoff

Recall that $c(s)$ is defined as the minimum hand strength y which should call a bet of size s and is given in Nash equilibrium by:

$$c(s) = \frac{x_2 + s}{s+1}$$

We are specifically interested in how $c(v^{-1}(x))$ varies with U for $x \in [x_3, v(U)]$, since this represents how the calling cutoff changes both directly from a strategic change, as well as indirectly due to the lower bet size s. It turns out that the calling cutoff is increasing in U for all $x \in [x_3, v(U)]$. This is surprising because we just showed that the bet size s is decreasing in U, and we expect smaller bets to be called more often. For reasons we will see later, this effect is overpowered by a strategic shift for the caller, who calls less often for all bet sizes as U increases.

Lemma 7.3. *For any fixed $x \in [x_3, v(U)]$,*

$$\frac{d}{dU} c(v^{-1}(x)) > 0$$

Proof. As mentioned above, $c(s)$ is dependent on U in two distinct ways - directly through x_2, which can be interpreted as the caller changing strategy as the game changes - but also indirectly in response to how the bet size $s = v^{-1}(x)$ is dependent on U. We use the multivariate chain rule to express $\frac{d}{dU} c(v^{-1}(x))$ in terms of these two dependencies:

$$\frac{d}{dU} c(v^{-1}(x)) = \frac{\partial c(s)}{\partial s} \frac{dv^{-1}(x)}{dU} + \frac{\partial c(s)}{\partial x_2} \frac{\partial x_2}{\partial U}$$

The two partial derivatives of $c(s)$ are:

$$\frac{\partial c(s)}{\partial s} = \frac{1 - x_2}{(s+1)^2} \quad \text{and} \quad \frac{\partial c(s)}{\partial x_2} = \frac{1}{s+1}$$

Substituting $s = v^{-1}(x)$, we can further simplify the first term:

$$\begin{aligned} \left.\frac{\partial c(s)}{\partial s}\right|_{s=v^{-1}(x)} &= \frac{1 - x_2}{(v^{-1}(x)+1)^2} \\ &= 2 - 2x \end{aligned}$$

And as we showed in the proof of Lemma 7.1, $\frac{dv^{-1}(x)}{dU}$ is given by:

$$\frac{dv^{-1}(x)}{dU} = \frac{-1}{(v^{-1}(x)+1)(4-4x)} \frac{\partial x_2}{\partial U}$$

Substituting all of these with $s = v^{-1}(x)$:

$$\begin{aligned} \frac{d}{dU} c(v^{-1}(x)) &= (2-2x)\left(\frac{-1}{(v^{-1}(x)+1)(4-4x)}\right)\left(\frac{\partial x_2}{\partial U}\right) + \left(\frac{1}{v^{-1}(x)+1}\right)\left(\frac{\partial x_2}{\partial U}\right) \\ &= \left(\frac{1}{v^{-1}(x)+1}\right)\left(\frac{\partial x_2}{\partial U}\right)\left(-\frac{2-2x}{4-4x}+1\right) \\ &= \left(\frac{1}{v^{-1}(x)+1}\right)\left(\frac{\partial x_2}{\partial U}\right)\left(\frac{1}{2}\right) \end{aligned}$$

The first term is positive since $v^{-1}(x) > 0$ and the second term is positive by Lemma 7.1. Therefore, the product of the two terms is positive, so the calling cutoff is increasing in U for all $x \in [x_3, v(U)]$.

□

7.1.5 Proof of Theorem 7.1

Having these tools, we can now finally return to the proof of Theorem 7.1.
Proof. Recall the expected payoff of a value-betting hand x:

$$EV(x) = \frac{1}{2}c(s) + (x - c(s))\left(s + \frac{1}{2}\right) + (1-x)\left(-s - \frac{1}{2}\right)$$

We break the proof into two cases:

Case 1 ($x > v(U)$): In this case, hand x bets the maximum amount U. The expected value is:

$$EV(x) = \frac{1}{2}c(U) + (x - c(U))\left(U + \frac{1}{2}\right) + (1 - x)\left(-U - \frac{1}{2}\right)$$

and the derivative is:

$$\begin{aligned}
\frac{d}{dU}EV(x) &= \left.\frac{\partial EV(x)}{\partial s}\right|_{s=U} + \frac{\partial EV(x)}{\partial c(s)}\left(\left.\frac{\partial c(s)}{\partial s}\right|_{s=U} + \frac{\partial c(s)}{\partial x_2}\frac{\partial x_2}{\partial U}\right) \\
&= \left.\left(\frac{\partial EV(x)}{\partial s} + \frac{\partial EV(x)}{\partial c(s)}\frac{\partial c(s)}{\partial s}\right)\right|_{s=U} + \left(\frac{\partial EV(x)}{\partial c(s)}\frac{\partial c(s)}{\partial x_2}\frac{\partial x_2}{\partial U}\right) \\
&= \left.\frac{dEV(x)}{ds}\right|_{s=U} + \left(\frac{\partial EV(x)}{\partial c(s)}\frac{\partial c(s)}{\partial x_2}\frac{\partial x_2}{\partial U}\right) \\
&= \left.\frac{dEV(x)}{ds}\right|_{s=U} - \frac{U}{1+U}\cdot\frac{\partial x_2}{\partial U}
\end{aligned}$$

We want to know exactly when the above expression is positive. $\frac{\partial x_2}{\partial U}$ is always positive by Lemma 7.1, and is independent of x. At $x = v(U)$, we know that $\frac{dEV(x)}{ds} = 0$ from 5, so at $x = v(U)$, the entire expression is negative and $EV(x)$ is decreasing in U. What about for larger x? We can expand the expression further. The partial derivatives of $EV(x)$ are:

$$\begin{aligned}
\frac{\partial EV(x)}{\partial s} &= 2x - 1 - c(s) \\
\frac{\partial EV(x)}{\partial c(s)} &= -s
\end{aligned}$$

Plugging these in and rearranging terms, we can say that $EV(x)$ is increasing in U if

$$x > \frac{1}{2(1+U)}\left(U\frac{\partial x_2}{\partial U} + \frac{U^2 + x_2(1+2U)}{1+U}\right)$$

Where nothing on the right hand side is dependent on x. This means that for any fixed L, U, this gives a threshold value for x below which $EV(x)$ is decreasing in U, and above which it is increasing in U.

Case 2 ($x_3 < x < v(U)$): In this case, hand x makes an intermediate-sized bet $s = v^{-1}(x)$. There are two distinct factors influencing the derivative

$\frac{d}{dU}EV(x)$, namely the change in bet size $s = v^{-1}(x)$ and the change in calling cutoff $c(v^{-1}(x))$. By the multivariate chain rule, we can express the derivative as:

$$\frac{d}{dU}EV(x) = \frac{\partial EV(x)}{\partial s}\frac{dv^{-1}(x)}{dU} + \frac{\partial EV(x)}{\partial c(s)}\frac{dc(v^{-1}(x))}{\partial U}$$

The partial derivatives of $EV(x)$ are:

$$\frac{\partial EV(x)}{\partial s} = 2x - 1 - c(s)$$
$$\frac{\partial EV(x)}{\partial c(s)} = -s$$

The second is clearly negative. We can verify that the first must be positive if we go back to the constraints which gave us the Nash equilibrium. For the bet size $v^{-1}(x)$ to be optimal, we required that

$$-s\frac{\partial c(s)}{\partial s} - c(s) + 2v(s) - 1 = 0$$

Or equivalently, if we substitute $s = v^{-1}(x)$ and $v(s) = x$ and rearrange:

$$2x - 1 - c(v^{-1}(x)) = v^{-1}(x)\frac{\partial c(s)}{\partial s} > 0$$

Since $\frac{\partial c(s)}{\partial s} = \frac{1-x_2}{(s+1)^2} > 0$, and s is positive by definition.

We know from Lemma 7.2 that $\frac{dv^{-1}(x)}{dU} < 0$ and from Lemma 7.3 that $\frac{dc(v^{-1}(x))}{\partial U} > 0$.

Combining everything, we see that both terms in $\frac{d}{dU}EV(x)$ are products of negative and positive, making both terms negative. Therefore, the expected payoff is decreasing in U for all $x \in [x_3, v(U)]$.

□

8 Strategic Comparison to Fixed-Bet and No-Limit Continuous Poker

The most natural way to think of Limit Continuous Poker is as a generalization between Fixed-Bet and No-Limit Continuous Poker, on a spectrum from strict to lenient bet sizing. In this light, we begin by asking whether the Nash equilibrium strategies approach those of NLCP and FCP as the limits L and U approach their extreme cases. To make this more explicit, we model the bettor strategies for all three games as 'bet functions' from hand strengths to bets (with 0 representing a check), and caller strategies as 'call functions' from bet sizes to minimum calling thresholds. We also introduce notation to reference all three strategy profiles more efficiently.

8.1 Setup and Notation

To compare the strategy profiles across different variants of Continuous Poker, we introduce the following notation for the strategy functions of the three games:

Symbol	**Meaning**
$S_{FB}(x, B)$	Bettor's bet function in FBCP with fixed bet size B
$C_{FB}(s, B)$	Caller's call function in FBCP with fixed bet size B
$S_{NL}(x)$	Bettor's bet function in NLCP
$C_{NL}(s)$	Caller's call function in NLCP
$S_{LCP}(x, L, U)$	Bettor's bet function in LCP with limits L and U
$C_{LCP}(s, L, U)$	Caller's call function in LCP with limits L and U
$x_i\|_{L,U}$	Threshold x_i in LCP with limits L and U

Table 1: Notation for strategy functions across different variants of Continuous Poker

In FBCP, the bettor can only make a fixed bet size B or check. The bet function $S_{FB}(x, B)$ maps hand strengths to either 0 (check) or B (bet):

$$S_{FB}(x, B) = \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \text{ (bluffing range)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (checking range)} \\ B & x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (value betting range)} \end{cases} \tag{10}$$

The caller's strategy is defined by a single threshold $C_{FB}(s, B)$:

$$C_{FB}(s, B) = \frac{B(3 + 2B)}{(1 + 2B)(2 + B)} \tag{11}$$

In NLCP, the bettor can choose any positive bet size. The strategy is most naturally described by functions $v_{NL}(s)$ and $b_{NL}(s)$ that map bet sizes to hand strengths:

$$v_{NL}(s) = 1 - \frac{3}{7(s+1)^2} \text{ (value betting function)}$$
$$b_{NL}(s) = \frac{3s+1}{7(s+1)^3} \text{ (bluffing function)}$$

The bet function $S_{NL}(x)$ is then defined in terms of the inverse functions:

$$S_{NL}(x) = \begin{cases} b_{NL}^{-1}(x) & x < \frac{1}{7} \text{ (bluffing range)} \\ 0 & \frac{1}{7} < x < \frac{4}{7} \text{ (checking range)} \\ v_{NL}^{-1}(x) & x > \frac{4}{7} \text{ (value betting range)} \end{cases}$$

The caller's strategy is defined by a continuous function $C_{NL}(s)$:

$$C_{NL}(s) = 1 - \frac{6}{7(s+1)}$$

In LCP, the bettor can choose any bet size between L and U. The strategy profile is defined by six thresholds x_0 through x_5 and functions $v(s)$ and $b(s)$ that map bet sizes to hand strengths. The bet function $S_{LCP}(x, L, U)$ and call function $C_{LCP}(s, L, U)$ are defined in terms of these values, which are given in Theorem 5.1.

8.2 Strategic Convergence

8.2.1 Bettor Strategy Convergence to Continuous Poker

We expect that as L and U approach some fixed value s, the bet function $S_{LCP}(x, L, U)$ should converge to the bet function $S_{FB}(x, s)$ for Fixed-Bet Continuous Poker with a fixed bet size s.

Theorem 8.1. *For any $B > 0$, the bet function $S_{LCP}(x, L, U)$ for Limit Continuous Poker converges to the bet function $S_{FB}(x, B)$ for Fixed-Bet Continuous Poker with a fixed bet size B as L and U approach B:*

$$\lim_{L \to B} \lim_{U \to B} S_{LCP}(x, L, U) = \lim_{U \to B} \lim_{L \to B} S_{LCP}(x, L, U) = S_{FB}(x, B).$$

Proof. We analyze the expressions for the x_i's, each of which is a rational function[4] of L and U. Since these functions are defined and continuous for

[4] A ratio of polynomials in L and U.

all positive values of L and U, the limit as $L \to B$ and $U \to B$ can be found by simply substituting $L = U = B$:

$$\begin{aligned}
x_0|_{B,B} = x_1|_{B,B} &= \frac{B}{2B^3 + 7B^2 + 7B + 2} \\
x_2|_{B,B} &= \frac{B}{(1+2B)(2+B)} \\
x_3|_{B,B} &= \frac{2B^2 + 4B + 1}{(1+2B)(2+B)} \\
x_4|_{B,B} = x_5|_{B,B} &= \frac{2B^2 + 5B + 1}{(1+2B)(2+B)}
\end{aligned}$$

$x_0 = x_1$ and $x_4 = x_5$ are expected, since these intervals are where the bettor uses an intermediate bet size, and $L = U = B$ does not allow intermediate bet sizes. This reduces the bet function to

$$\begin{aligned}
\lim_{L \to B} \lim_{U \to B} S_{LCP}(x, L, U) &= \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \\ B & x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \end{cases} \\
&= S_{FB}(x, B)
\end{aligned}$$

□

8.2.2 Caller Strategy Convergence to Continuous Poker

The calling function is easier to analyze. We want to show that the calling threshold $C_{LCP}(s, L, U)$ converges to the calling threshold $C_{FB}(s, B)$ for Fixed-Bet Continuous Poker with a fixed bet size B as L and U approach B.

Theorem 8.2. *For any $B > 0$, the call function $C_{LCP}(s, L, U)$ for Limit Continuous Poker converges to the call function $C_{FB}(s, B)$ for Fixed-Bet Continuous Poker with a fixed bet size B as L and U approach B:*

$$\lim_{L \to B} \lim_{U \to B} C_{LCP}(s, L, U) = \lim_{U \to B} \lim_{L \to B} C_{LCP}(s, L, U) = C_{FB}(s, B).$$

Proof. We already have the value of $x_2|_{B,B}$, so we can plug this into the expression for the calling threshold:

$$\begin{aligned}
\lim_{L\to B}\lim_{U\to B} C_{LCP}(s,L,U) &= \frac{x_2|_{B,B}+s}{1+s} \\
&= \frac{\frac{B}{(1+2B)(2+B)}+s}{1+s} \\
&= \frac{B(3+2B)}{(1+2B)(2+B)} \\
&= C_{FB}(s,B)
\end{aligned}$$

□

8.2.3 Bettor Strategy Convergence to NLCP

In a similar fashion, we expect that as L and U approach 0 and ∞, the bet function $S_{LCP}(x,L,U)$ should converge to the bet function $S_{NL}(x)$ for NLCP.

Theorem 8.3. *The bet function $S_{LCP}(x,L,U)$ for Limit Continuous Poker converges to the bet function $S_{NL}(x)$ for NLCP as L and U approach* 0 *and* ∞*:*

$$\lim_{L\to 0}\lim_{U\to\infty} S_{LCP}(x,L,U) = \lim_{U\to\infty}\lim_{L\to 0} S_{LCP}(x,L,U) = S_{NL}(x).$$

Proof. We can analyze the expressions for the x_i's as L and U approach 0 and ∞. The limit is well-defined, and we can substitute $L=0$ and $U=\infty$ into the expressions for the x_is.

$$\begin{aligned}
x_0|_{0,\infty} &= 0 \\
x_1|_{0,\infty} &= x_2|_{0,\infty} = \frac{1}{7} \\
x_3|_{0,\infty} &= x_4|_{0,\infty} = \frac{4}{7} \\
x_5|_{0,\infty} &= 1
\end{aligned}$$

$x_0|_{0,\infty}=0$ and $x_5|_{0,\infty}=1$ are expected, since these intervals are where the bettor uses a minimum bet size and a maximum bet size, respectively, both of which are impossible. The bettor now bets intermediate values for $x<\frac{1}{7}$ and $x>\frac{4}{7}$, and checks for $\frac{1}{7}<x<\frac{4}{7}$. But how much do they bet? We can take the limits of $v(s)$ and $b(s)$ as L and U approach 0 and ∞:

$$\lim_{L\to 0}\lim_{U\to\infty} b(s) = \frac{3s+1}{7(s+1)^3}$$
$$\lim_{L\to 0}\lim_{U\to\infty} v(s) = 1 - \frac{3}{7(s+1)^2}$$

To summarize, the bettor bets s with hands $x < \frac{1}{7}$ such that $x = b(s)$ or hands $x > \frac{4}{7}$ such that $x = v(s)$. This is exactly the same as the bet function $S_{NL}(x)$ for NLCP.

□

8.2.4 Caller Strategy Convergence to NLCP

The calling function is again easier to analyze. We want to show that the calling threshold $C_{LCP}(s, L, U)$ converges to the calling threshold $C_{NL}(s)$ for NLCP as L and U approach 0 and ∞.

Theorem 8.4. *The call function $C_{LCP}(s, L, U)$ for Limit Continuous Poker converges to the call function $C_{NL}(s)$ for NLCP as L and U approach 0 and ∞:*

$$\lim_{L\to 0}\lim_{U\to\infty} C_{LCP}(s, L, U) = \lim_{U\to\infty}\lim_{L\to 0} C_{LCP}(s, L, U) = C_{NL}(s).$$

Proof. Again, we already have the limiting value of $x_2|_{0,\infty}$, so we can plug this into the expression for the calling threshold:

$$\begin{aligned}
\lim_{L\to 0}\lim_{U\to\infty} C_{LCP}(s, L, U) &= \lim_{L\to 0}\lim_{U\to\infty} \frac{x_2 + s}{1+s} \\
&= \frac{\frac{1}{7} + s}{1+s} \\
&= 1 - \frac{6}{7(1+s)} \\
&= C_{NL}(s)
\end{aligned}$$

□

We have now shown that the bettor and caller strategies for LCP converge to those of FBCP and NLCP as the limits L and U approach their extreme values. In the next section, we explore the value of LCP in more detail, and in particular how it relates to that of FBCP and NLCP.

9 Game Value

LCP is a zero-sum game, so the value of the game can be found by the minimax theorem. Specifically, it is the maximum expected payoff that the bettor can gain when the caller plays optimally, and the minimum expected payoff that the caller can lose when the bettor plays optimally.

Since we have already found optimal strategies for both players, we can compute the value of the game as the average of the payoff in Nash equilibrium over all possible hand combinations (x, y). This reduces to an integral of the payoff over the unit square, like we saw in Figure 2. Because the bet size is only defined implicitly in terms of the hand strength x, this is extremely nontrivial and requires breaking the square into regions based on the strategies and bet size. This method is fully specified in Appendix **??**, but the resulting expression is given below.

$$V_{LCP}(L, U) = \frac{(1+L)^3(1+U)^3 - ((1+L)^3 + L^3(1+U)^3)}{14(1+L)^3(1+U)^3 - 2((1+L)^3 + L^3(1+U)^3)}$$

This is a rational function of L and U with a surprisingly simple form. Importantly, this makes it easy to compute $V_{LCP}(L, U)$ as L, U approach their extreme values, which we use to relate the game value to that of NLCP and FBCP in the next section.

Most interestingly, the value of LCP has the following symmetry:

$$V_{LCP}(L, U) = V_{LCP}\left(\frac{1}{U}, \frac{1}{L}\right)$$

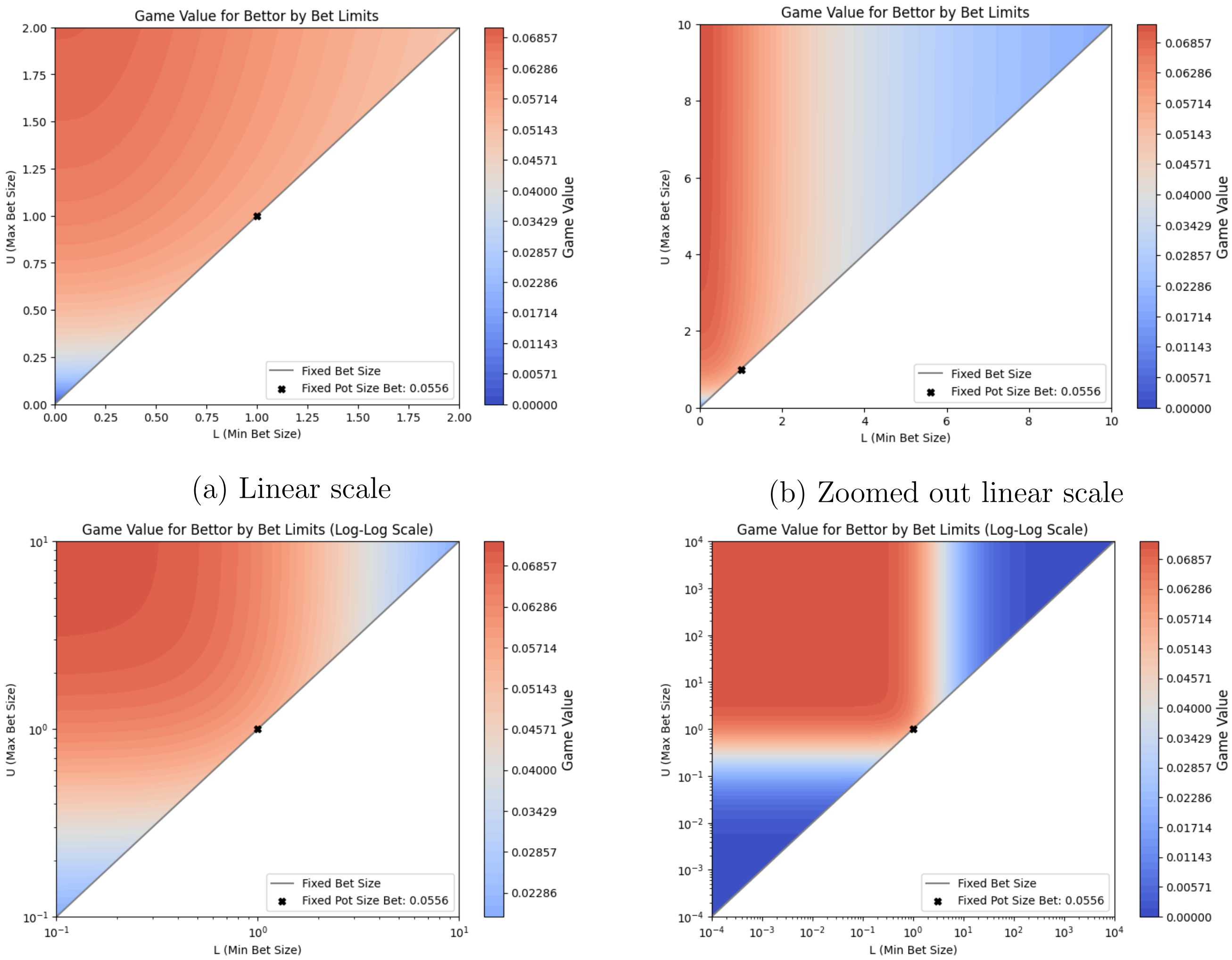
This symmetry is easily checked from the definition of $V_{LCP}(L, U)$, but is not at all obvious from the game setup.

In the following sections, we investigate the properties and behavior of $V_{LCP}(L, U)$ in more detail. These include monotonicity, convergence to NLCP and FBCP, and the symmetry property.

9.1 Value Monotonicity

Intuitively, more options for the bettor should increase the game's value. To see this visually, we can plot the value of LCP as a function of the limits L and U for a range of values, as in Figure 3.

Notice the higher value for more lenient limits (top left of any plot in Figure 3) and lower value for more strict limits (moving down/right). We can easily prove this formally:



(a) Linear scale

(b) Zoomed out linear scale

(c) Log-log scale

(d) Zoomed out log-log scale

Figure 3: Game value of Limit Continuous Poker as a function of the limits L and U. (a) Linear scale, (b) Zoomed out linear scale, (c) Log-log scale, (d) Zoomed out log-log scale. The restriction $L \leq U$ keeps the plots above the diagonal.

Theorem 9.1. *The value of Limit Continuous Poker is monotonically increasing in U and decreasing in L:*

$$\frac{\partial V_{LCP}(L,U)}{\partial U} \geq 0, \quad \frac{\partial V_{LCP}(L,U)}{\partial L} \leq 0$$

Proof. $\square$ prove this

9.2 Value Convergence

The main diagonal of the plots in Figure 3 should represent $L = U$, which means the bet size is fixed. Thus, this diagonal should represent the value of FBCP for various values of B. We can prove this formally:

Theorem 9.2. *For any $B > 0$, the value of Limit Continuous Poker converges to the value of Fixed-Bet Continuous Poker as L and U approach B:*

$$\lim_{L \to B} \lim_{U \to B} V_{LCP}(L, U) = \lim_{U \to B} \lim_{L \to B} V_{LCP}(L, U) = V_{FB}(B)$$

Proof. $\square$ prove this

These plots also align with the known result that a fixed pot-size bet of $B = 1$ maximizes the expected value for the bettor in FBCP, as seen by the fact that $(1, 1)$ achieves the maximum value on the diagonal.

We can also show that the value of LCP converges to the value of NLCP as L and U approach their extremes, represented by the top left of any plot in Figure 3.

Theorem 9.3. *The value of Limit Continuous Poker converges to the value of NLCP as L and U approach 0 and ∞:*

$$\lim_{L \to 0} \lim_{U \to \infty} V_{LCP}(L, U) = \lim_{U \to \infty} \lim_{L \to 0} V_{LCP}(L, U) = V_{NL}$$

Proof. $\square$ prove this

9.3 Value Symmetry

Looking again at the log-log plots in Figure 3, we see the symmetry property of $V_{LCP}(L, U)$ represented by symmetry about the reverse diagonal (the line $\log(L) = -\log(U)$).

This symmetry can be interpreted in terms of how changing the limits affects the game to the bettor. We discussed earlier how the bettor benefits from increasing U or from decreasing L, but it was unclear exactly how these benefits related quantitatively. The symmetry property tells us that the benefit from increasing U is exactly equivalent to that of decreasing L in a reciprocal manner, centered exactly around the pot size of 1. Similarly, the loss in value from decreasing U is equivalent to the loss from increasing L in a reciprocal manner.

For example, suppose you are given the choice between playing LCP as the bettor with limits $L = 1/2$ and $U = 5$ or $L = 1/5$ and $U = 2$. You know you can play optimally, but it is unclear which game favors you. The symmetry property tells us that the value of the game is the same in both cases, so you should be indifferent between the two.

A Derivation of Nash Equilibrium Strategy Profile

B Computation of Game Value