1 Monotone and Continuous Strategy Proofs

This appendix provides the detailed proofs regarding monotone calling strategies, monotone-admissible and continuous betting strategies, and their relationship to the uniqueness of the Nash equilibrium in LCP.

1.1 Monotone Strategies and Weak Dominance

Lemma 1.1. *If a calling strategy violates the first monotonicity condition (see* **??***) for a nonzero-measure set of hands for any bet size s, it is weakly dominated. Specifically, if there exists s and measurable sets $A, B \subseteq [0,1]$ such that:*

1. *The caller calls s with hands in A*
2. *The caller folds s with hands in B*
3. $\sup A \leq \inf B$
4. *A and B have positive measure*

then the strategy is weakly dominated.

Proof. Let σ_C be the non-monotone strategy described above. Since A and B are nonzero-measure, there exist subsets $A' \subseteq A$ and $B' \subseteq B$ such that:

1. A' and B' have positive measure
2. $|A'| = |B'|$

Where $|A|$ and $|B|$ denote the Lebesgue measure of A and B (see Figure 1).

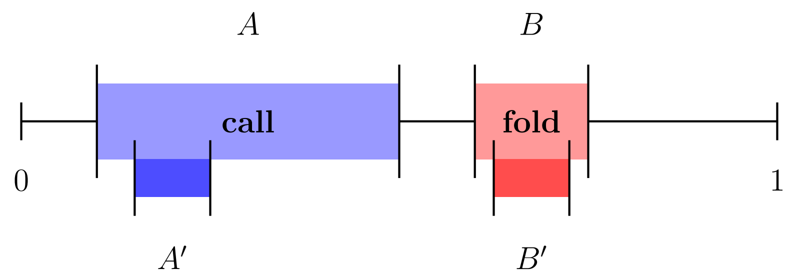

Figure 1: A simple case of sets A and B which violate monotonicity ($\sup A \leq \inf B$). We can find equal-measure subsets $A' \subseteq A$ and $B' \subseteq B$ to swap actions, improving the strategy.

The existence of such subsets follows from a fundamental property of nonatomic measures: since the uniform distribution on $[0,1]$ is nonatomic (no single point has positive probability), for any two measurable sets with positive measure, we can always find measurable subsets of equal measure[**?**]. This property allows us to construct the strategy improvement described below.

Let σ'_C be the strategy which switches the actions for A' and B', i.e. calls with B' and folds with A' (and behaves identically for all other bet sizes). We now analyze how this change affects the caller's performance against any betting strategy.

Against a bet of size s, the key improvement occurs in two scenarios:

1. When $y \in B'$ and $x \in A'$: σ_C folds while σ'_C calls and wins (since $x \in A$ and $y \in B$ with $\sup A \leq \inf B$)

2. When $y \in A'$ and $x \in B'$: σ_C calls and loses while σ'_C folds (avoiding the loss)

For all other cases, σ_C and σ'_C behave identically, so σ'_C is weakly better than σ_C against every betting strategy.

To show that σ_C is strictly dominated, consider a betting strategy which always bets s. Against this strategy, both scenarios above occur with positive probability (since A' and B' have positive measure), so σ'_C is strictly better than σ_C. Thus, σ_C is weakly dominated. □

We now show that the restriction to monotone-admissible betting strategies implies a natural ordering of bluff sizes.

Lemma 1.2. *If a betting strategy σ_B is monotone-admissible, then for any two bluffing hands $x < x'$ with corresponding bet sizes s and s' respectively, we must have $s' \geq s$.*

Proof. Suppose for contradiction that there exist bluffing hands $x < x'$ with bet sizes $s > s'$. We will show that σ_B is not monotone-admissible by constructing a strategy σ'_B that weakly dominates σ_B against all monotone calling strategies.

Define σ'_B to be identical to σ_B except that it bets s' with hand x instead of s.

Let σ_C be an arbitrary monotone calling strategy, and let $c(s)$ denote the calling threshold for bet size s under σ_C. By Definition **??** (condition 2), monotonicity implies

$$c(s') \leq c(s).$$

We compare the expected payoffs of betting s versus s' with hand x against σ_C.

Case 1: $x' < c(s')$.

Since $c(s') \leq c(s)$, we have $x < x' < c(s') \leq c(s)$. Thus the caller never calls with a hand weaker than x or x' under either bet size. Both bets succeed as pure bluffs with probability $P(Y < c(\cdot))$, winning the pot of size 1. Since $s' < s$, betting s' risks less to win the same amount, so

$$\mathbb{E}[\text{payoff of } s' \mid x] > \mathbb{E}[\text{payoff of } s \mid x].$$

Case 2: $x' \geq c(s')$.

When betting s' with hand x, the bettor wins at showdown against any caller hand $y \in [c(s'), x']$, an event with positive probability. When betting s with hand x, since $c(s) \geq c(s') > x$, the bettor never wins at showdown (as $x < c(s)$ implies all calling hands beat x). Therefore,

$$\mathbb{E}[\text{payoff of } s' \mid x] > \mathbb{E}[\text{payoff of } s \mid x].$$

In both cases, σ'_B achieves strictly higher expected payoff than σ_B against σ_C. Since this holds for all monotone calling strategies σ_C, strategy σ_B is not monotone-admissible, contradicting our assumption. □

1.2 Continuity

We stated in section **??** that the betting strategy should be continuous except at the bluffing and value-betting thresholds. Here we provide a formal justification for this requirement.

Consider the bettor's expected value function for a value hand x:

$$EV(s, x) = (x - c(s))(1 + s) - (1 - x)s + c(s) \tag{1}$$

which is continuous in both s and x. For a fixed value-betting hand strength x, the bettor chooses an optimal bet size $s^*(x)$ such that:

$$s^*(x) = \arg \max_{s \in [L,U]} EV(s, x) \tag{2}$$

Assuming $EV(s, x)$ is strictly concave in s (which follows if $c(s)$ is sufficiently smooth and monotone increasing), the maximizer $s^*(x)$ is unique. By the Theorem of the Maximum [**?**], since $EV(s, x)$ is continuous and the set of possible bet sizes S is compact, the optimal strategy $s^*(x)$ is a continuous function of x.

If $s(x)$ were discontinuous at some x_0, it would imply that $EV(s, x)$ possesses multiple global maxima at x_0, or that the EV function itself is not concave, allowing the optimal bet size to "jump" between distant peaks.

For the bluffing range, continuity is a consequence of the equilibrium balancing requirement. To maintain the caller's indifference at every $s \in [L, U]$, there must exist a corresponding bluffing hand strength such that the ratio of bluffs to value hands satisfies the Nash condition.

As previously established, the bluffing strategy $b^{-1}(x)$ is strictly decreasing in x as a consequence of being monotone-admissible. Since every $s \in [L, U]$ must be represented in the bluffing range to balance the value bets, the image of $b^{-1}(x)$ is the interval $[L, U]$.

By the Intermediate Value Theorem for Monotone Functions (the property that a monotonic function is continuous if and only if its image is an interval), $b^{-1}(x)$ must be continuous.