# Limit Continuous Poker: A Variant of Continuous Poker with Limited Bet Sizes

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**Abstract**

We introduce and analyze Limit Continuous Poker, a variant of Von Neumann's Continuous Poker with variable but limited bet sizes. This simplified variant of poker captures aspects of information asymmetry, bluffing, balancing, and the impact of bet size limits while still being simple enough to solve analytically. We derive the Nash equilibrium strategy profile for this game, showing how the bettor's and caller's strategies depend on the bet size limits. We demonstrate that as the bet size limits approach extreme values, the strategy profile converges to those of other continuous poker variants. Finally, we connect these results to strategic implications of limited bet sizing in real-world poker.

## Contents

# 1 Introduction

Poker is a notoriously complex game, with many different variants and strategic elements that have challenged both players and theorists for decades. Simplified poker models play a crucial role in game theory research by isolating specific strategic aspects—such as bluffing, value betting, and bet sizing—while remaining analytically tractable. One such class of models is Continuous Poker, which abstracts poker hands to continuous numerical hand strengths and restricts play to a single betting round. This simplification allows for exact Nash equilibrium solutions and provides insights that can inform understanding of more complex poker variants.

In this paper, we introduce and analyze Limit Continuous Poker (LCP), a new variant that bridges two well-studied extremes: Fixed-Bet Continuous Poker (FBCP), where the bettor must choose a predetermined bet size, and No-Limit Continuous Poker (NLCP), where the bettor can choose any positive bet size. LCP generalizes both by imposing lower and upper bounds $L$ and $U$ on allowable bet sizes, creating a spectrum of games parametrized by these limits.

## 1.1 Related Work and Background

The study of simplified poker models dates back to von Neumann's seminal work on game theory, which introduced the concept of optimal mixed strategies in competitive games. Continuous poker variants have since become standard examples in game theory textbooks and research, serving as tractable models for studying information asymmetry and strategic bluffing. Our work builds directly on two classical variants: Fixed-Bet Continuous Poker (FBCP) and No-Limit Continuous Poker (NLCP). We briefly review these games to establish context for LCP.

### 1.1.1 Fixed-Bet Continuous Poker (FBCP)

Continuous Poker (also called Von Neumann Poker, and referred to in this paper as Fixed-Bet Continuous Poker or FBCP) is a simplified model of poker introduced by von Neumann. It is a two-player zero-sum game designed to study strategic decision-making in competitive environments. The game abstracts away many complexities of real poker, focusing instead on the mathematical and strategic aspects of bluffing, betting, and optimal play.

**Definition 1.1** (FBCP)**.** Two players, referred to as the bettor and the caller, each put a 0.5 unit ante into a pot[1]. They are each dealt a hand strength between

[1] An ante of 1 is often used, but since the pot size is the more relevant value, we use an ante of 0.5. All bet sizes scale proportionally.

0 an 1 (referred to as $x$ for bettor and $y$ for caller). After seeing $x$, the bettor can either check - in which case, the higher hand between $x$ and $y$ wins the pot of 1 and the game ends - or they can bet, by putting a pre-determined amount $B$ into the pot. The caller must now either call by matching the bet of $B$ units, after which the higher hand wins the pot of $1 + 2B$ minus their ante of 0.5, or fold, conceding the pot of $1 + B$ to the bettor and ending the game.

FBCP has many Nash equilibria, but it has a unique one in which the caller plays an admissible strategy[2], as shown by Ferguson and Ferguson [2, p. 2]. This strategy profile, parametrized by the bet size $B$, is as follows:

The bettor bets with hands $x$ such that either

$$x > \frac{1 + 4B + 2B^2}{(1 + 2B)(2 + B)} \text{ or } x < \frac{B}{(1 + 2B)(2 + B)}.$$

We call the higher interval the value betting range and the lower interval the bluffing range. The caller calls with hands $y$ above a calling threshold:

$$y > \frac{B(3 + 2B)}{(1 + 2B)(2 + B)}.$$

Note that uniqueness in this context ignores the strictness of inequalities, since the endpoints of intervals occur with probability 0. The non-uniqueness of this Nash Equilibrium is due to the fact that given the bettor's strategy, the caller has many optimal responses. The caller must always fold with hands below the bluffing threshold, and must always call with hands above the value betting threshold, but with hands inbetween, they are indifferent between calling and folding. This is because with a hand strength in this range, the caller wins if and only if the bettor is bluffing, so their actual hand strength is irrelevant as long as it beats the bluffing threshold. To prevent the betting player from exploiting them, the caller need only call with exactly the right proportion of hands in this range. For example, the caller could take the strategy described above, but swap some calling and folding hands in the range between the bluffing and value betting thresholds.

Why is this Nash equilibrium special? We mentioned above that it is admissible, meaning that both players' strategies are not weakly dominated by any other strategy. Importantly, the caller's strategy is not weakly dominated. The same cannot be said for other Nash equilibria like the one described in the previous paragraph, for reasons that are beyond the scope of this introduction but relate to themes in Section 4.2.

FBCP also has a unique value as a function of the bet size $B$. The value of the game for the bettor is

[2] An admissible strategy is one which is not weakly dominated by any other strategy.

$$V_{FB}(B) = \frac{B}{2(1+2B)(2+B)},$$

which is positive (advantageous to the bettor) and maximized at $B = 1$, when the bet size is exactly the pot size. It should not be surprising that the value is positive - at worst, the bettor can always check and turn the game into a coin flip, so the bettor will only deviate from this strategy if they have a positive expected value. The fact that $B = 1$ exactly maximizes the value is more subtle, but we should expect that some such maximal value of $B$ exists. Forcing the bettor to bet too large relative to the pot would make betting too risky with most hands, and making the bet too small would simply give less profit to the bettor when they win a bet. As we will see later, part of the motivation for studying LCP is to understand this concept more generally.

### 1.1.2 No-Limit Continuous Poker (NLCP)

Another continuous poker variant allows the bettor to choose a bet size $s > 0$ after seeing their hand strength, as opposed to a fixed bet size $B$. This variant is called No-Limit Continuous Poker (or Newman Poker after Donald J. Newman, or NLCP in this paper). The Nash equilibrium strategy profile for this variant is discussed and solved by Bill Chen and Jerrod Ankenman [1, p. 154].

In Nash Equilibrium, the bettor should make large bets with their strongest and weakest hands and smaller bets or checks with their intermediate hands. It turns out that the optimal strategy is most elegantly described by a mapping from bet sizes $s$ to hand strengths $x$ for bluffing and value betting, respectively[3]. The caller simply has a calling threshold $c(s)$ for each possible bet size $s$. The full strategy profile is as follows:

The bettor bets $s$ with hands $x$ such that either

$$x = \frac{3s+1}{7(s+1)^3} \text{ or } x = 1 - \frac{3}{7(s+1)^2},$$

where the first condition represents bluffing hands and the second value betting hands. After seeing a bet of size $s$, the caller should call with hands $y$ such that

$$y > 1 - \frac{6}{7(s+1)}.$$

See Figure 1 for a graphical representation of the strategy profile.

[3]This feels backwards - mapping hand strengths to bet sizes would be more natural, but the math is more elegant this way.

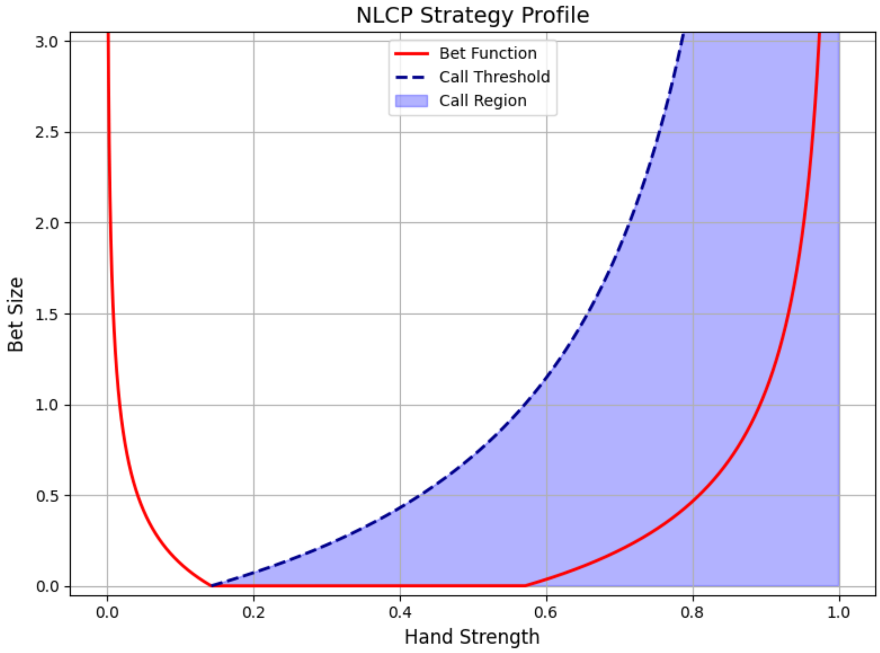


Figure 1: NLCP Strategy Profile

Note that the bettor uses all possible bet sizes and has exactly two hand strengths for each bet size[4]. On first inspection, this feels like the bettor is giving away too much information, but it turns out to still be an optimal strategy. This concept appears again and is explained more thoroughly in Section 4.3.

The value of NLCP is

$$V_{NL} = \frac{1}{14},$$

for the bettor[5]. Thus, NLCP is again advantageous to the bettor. In fact, one can easily verify that NLCP is more advantageous to the bettor than FBCP for any bet size $B$ by arguing that the bettor could artificially restrict themselves to a single bet size and achieve the same value as the bettor in FBCP.

Ferguson and Ferguson [2] provided the comprehensive analysis of FBCP described above, establishing the unique admissible Nash equilibrium and deriving closed-form solutions for optimal strategies. Chen and Ankenman [1] extended this analysis to NLCP, demonstrating how unlimited bet sizing fundamentally changes the strategic landscape while maintaining analytical tractability. Our work builds

[4]Seen visually in Figure 1 by the fact that a horizontal line intersects the bet function at exactly two points.

[5]Would be 1/7 for an ante of 1, but the value is halved with an ante of 0.5.

on these foundations by introducing a parametric family of games that interpolates between these extremes, allowing us to study how betting constraints affect optimal strategies and game value in a continuous fashion.

## 1.2 Our Contributions

This paper makes the following contributions:

- **Nash Equilibrium Solution:** We derive the unique admissible Nash equilibrium for LCP, characterized by six threshold parameters and two continuous bet-sizing functions. We establish the structure of optimal play and provide closed-form expressions for all strategic components (Section 4.3 and Section 3).
- **Game Value Analysis:** We compute the value of LCP as a function of the betting limits $L$ and $U$, obtaining a surprisingly elegant rational formula. We prove monotonicity properties and establish a remarkable symmetry: $V_{LCP}(L, U) = V_{LCP}(1/U, 1/L)$ (Section 6).
- **Convergence Results:** We prove that LCP strategies and values converge to those of FBCP and NLCP in the appropriate limit cases, establishing LCP as a genuine generalization of both variants (Section 7.2).
- **Parameter Sensitivity Analysis:** We analyze how changes in betting limits affect optimal strategies and payoffs, revealing counterintuitive effects where expanding betting options can reduce expected value for certain hand strengths due to strategic adjustments by the opponent (Section 5).

## 1.3 Outline

The remainder of this paper is organized as follows. Section 2 establishes notation and conventions. Section 3 formally defines LCP and develops the methodology for solving for Nash equilibrium, including the concepts of monotone and admissible strategies. Section 4 presents the Nash equilibrium strategy profile with closed-form solutions. Section 5 analyzes the game value, proving monotonicity and symmetry properties. Section 6 establishes convergence to FBCP and NLCP. Section 7 examines how parameters affect strategies and payoffs. Section 8 concludes with a discussion of implications and future work. Detailed proofs are provided in the appendices.

# 2 Preliminaries and Conventions

This section establishes the notation and conventions used throughout the paper. We assume familiarity with basic game theory concepts such as Nash equilibrium, zero-sum games, and mixed strategies, but we clarify our specific notation and modeling choices.

## 2.1 Game-Theoretic Concepts

All continuous poker variants studied in this paper are *two-player zero-sum games*. The *value* of such a game is the expected payoff to the first player (the bettor) when both players adopt Nash equilibrium strategies. Since the game is zero-sum, the caller's expected payoff is the negative of this value. A positive value indicates an advantage for the bettor.

We seek Nash equilibria where both players use *pure strategies* (deterministic mappings from information to actions). As is standard in continuous games, we treat sets of measure zero as negligible—for instance, the action taken at an exact threshold value is irrelevant since it occurs with probability zero.

## 2.2 Conventions

**Ante and Pot Size:** Each player contributes an ante of 0.5 units, creating an initial pot of 1 unit. All bet sizes are measured in units relative to this pot. This convention (ante = 0.5 rather than ante = 1) simplifies payoff calculations.

**Payoffs:** Payoffs represent the net gain or loss relative to the initial ante. A check results in the winner receiving the pot of 1 minus their ante of 0.5, for a net payoff of $\pm 0.5$. When a bet of size $s$ is called, the winner receives $1 + 2s$ (pot plus both contributions) minus their ante of 0.5, for a net payoff of $\pm(0.5 + s)$.

**Inequalities and Measure Zero:** When we write, for example, "the caller calls with hands $y \geq c(s)$," the choice of $\geq$ versus $>$ is immaterial because the set $\{y : y = c(s)\}$ has measure zero. We use closed intervals $[a, b]$ and open intervals $(a, b)$ interchangeably when the boundary points have probability zero.

**Monotone Strategies:** A calling strategy is *monotone* if (1) stronger hands are more likely to call for any fixed bet size, and (2) smaller bets are more likely to be called for any fixed hand strength. This concept is formalized in Section 3.

**Admissibility:** A strategy is *admissible* if it is not weakly dominated by any other strategy. Among multiple Nash equilibria, we focus on those where the bettor's strategy is admissible against all monotone calling strategies (Section 3).

# 3 Limit Continuous Poker (LCP)

We now introduce a variant where the bettor may choose a bet size $s$ after seeing their hand strength, but where $s$ is bounded by an upper limit $U$ and a lower limit $L$, referred to as the maximum and minimum bet sizes. We call this variant Limit Continuous Poker (or LCP).

In Limit Continuous Poker, two players compete in a simplified poker game where each receives a hand strength represented by a real number between 0 and 1. The bettor acts first, choosing either to check (pass) or to make a bet of some size within the allowed range. If a bet is made, the caller must decide whether to call the bet or fold. The game rewards the player with the stronger hand, with the size of the bet affecting the magnitude of the payoff.

**Definition 3.1** (LCP)**.** A two-player zero-sum game where:

- The bettor and caller are each dealt independent hand strengths $X, Y \sim \text{Uniform}[0, 1]$
- The bettor observes their hand strength $x$ and chooses an action from $\mathcal{A}_1 = \{0\} \cup [L, U]$ (a bet of 0 is a check)
- If the bettor chooses an action from $[L, U]$ (a bet), then the caller observes the bettor's action along with their own hand strength $y$ and chooses from $\mathcal{A}_2 = \{\text{call}, \text{fold}\}$
- Payoffs are determined as follows:
  - If the bettor checks: payoff is 0.5 to the player with higher hand strength (the pot of 1, minus the initial ante of 0.5)
  - If the bettor bets $s \in [L, U]$ and the caller calls: payoff is $0.5 + 2s$ to the player with higher hand strength
  - If the bettor bets $s \in [L, U]$ and the caller folds: payoff is 0.5 to the bettor

A strategy for the bettor is a measurable function $\sigma_1 : [0, 1] \to \mathcal{A}_1$ mapping hand strengths to actions. A strategy for the caller is a measurable function $\sigma_2 : [L, U] \times [0, 1] \to \mathcal{A}_2$ mapping bettor actions and caller hand strengths to caller responses.

The motivation for studying this variant is twofold. First, it is a more realistic variant of poker, where bets are not fixed but are also not unbounded. In most real variants of poker, bet sizes are constrained by the stack sizes of the players and by a minimum bet size. Analytically solving LCP can give insight into the effect of bet size constraints on more complex variants of poker. Strong poker players have intuition about how bet size constraints affect strategy, but rigorously proving this intuition is often impossible given the combinatorial complexity of the game. Second, LCP can be seen as a generalization of FBCP and NLCP; specifically, as $L \to 0$ and $U \to \infty$, LCP approaches NLCP, and as $L \to B$ and $U \to B$ for some fixed value $B$, LCP approaches FBCP. Studying LCP can help us understand the relationship between these two and answer questions about why they produce the strategies they do (see Section 7.2 for formal convergence results).

In the next section, we develop the methodology for solving LCP and identifying its unique admissible Nash equilibrium. Section 4.3 will describe the structure of this equilibrium, and the complete closed-form solution is presented in Section 4.

# 4 Solving for Nash Equilibrium

In this section, we develop the methodology for computing the Nash equilibrium of LCP. Our approach proceeds in three stages: (1) establishing the concept of *monotone calling strategies* and using admissibility to select among multiple equilibria, (2) characterizing the structure that any Nash equilibrium must satisfy, and (3) deriving a system of equations whose solution yields the equilibrium strategy profile. The complete derivation and verification that our solution constitutes a Nash equilibrium is provided in Appendix B.

## 4.1 Uniqueness and Equilibrium Selection

Like FBCP, LCP has an infinite class of Nash equilibria, differentiated primarily by how the bettor sizes their bluffs. We resolve this non-uniqueness by imposing two natural refinements: first, we restrict the caller to *monotone* strategies which are more robust and less exploitable; second, we require the bettor's strategy to be admissible (not weakly dominated) against all such monotone calling strategies. These refinements uniquely determine the equilibrium we analyze.

## 4.2 Monotone Strategies

**Definition 4.1** (Monotone Calling Strategy)**.** A *monotone* calling strategy is a pure strategy which satisfies two conditions:

1. For any bet size $s$ and any two hand strengths $y_1 < y_2$, if the caller calls a bet of size $s$ with $y_1$, they must also call with $y_2$.

2. For any hand strength $y$ and any two bet sizes $s_1 < s_2$, if the caller calls a bet of size $s_2$ with $y$, they must also call a bet of size $s_1$ with $y$.

This should sound intuitive. Clearly, calling with a stronger hand is weakly better than calling with a weaker hand. Restricting to pure strategies can be explained similarly - it is better to always call with a stronger hand and always fold a weaker one than to mix between the two.

Violating the first condition (in a non-negligible way) is actually weakly dominated - not only is a monotone strategy weakly better against all opponents, but there exists an opponent against which the non-monotone strategy is strictly better (see Appendix A for proof).

The second condition for a monotone calling strategy - that the caller must be more willing to call smaller bets - is more subtle. From a poker player's perspective, it aligns with intuition about pot odds: a larger bet is riskier and should require a stronger hand to call. While violating this condition is not dominated, it leads to exploitable calling strategies. If the caller calls less aggressively against smaller bets, the bettor can take smaller risks for higher returns. A monotone calling strategy is therefore less exploitable, and imposing this condition yields a unique Nash equilibrium.

**Definition 4.2** (Monotone-Admissible Strategy)**.** A betting strategy $\sigma_B$ is *monotone-admissible* if it is admissible in LCP against the set of monotone calling strategies. That is, there does not exist a betting strategy $\sigma'_B$ that performs at least as well against all monotone calling strategies and strictly better against at least one.

This definition distinguishes bettor strategies that differ only in how they bluff. The hand strength of a bluff is irrelevant when the caller plays optimally, since the caller never calls with a hand weaker than any bluff. However, if the caller deviates to a suboptimal but still monotone strategy, the bettor's bluffing hand strength matters. Monotone-admissibility selects the equilibrium where the bettor bluffs larger with weaker hands and smaller with stronger hands, which is optimal against all monotone deviations. See Appendix A for detailed analysis.

## 4.3 Nash Equilibrium Structure

We will now describe the structure of the Nash equilibrium in terms of constants $x_i$ and functions $c(s)$, $b(s)$, and $v(s)$. These turn out to be fully determined by the parameters $L$ and $U$, but for now they are unknown. Notice that both players use pure strategies, like in NLCP: the bettor maps hand strengths directly to bet sizes, and the caller maps hand strengths and bet sizes to actions with no mixing.

1. The caller has a calling threshold $c(s)$ that is continuous in $s$, including at endpoints $L$ and $U$. They call with hands $y \geq c(s)$ and fold with hands $y < c(s)$[6].

2. The bettor partitions $[0, 1]$ into three regions: bluffing $x \in [0, x_2]$, checking $x \in [x_2, x_3]$, and value betting $x \in [x_3, 1]$.

3. Within the bluffing region, the bettor's partitions into a max-betting region $x \in [x_0, x_1]$, an intermediate region $x \in [x_1, x_2]$, and a min-betting region $x \in [x_2, x_3]$.

4. Within the intermediate bluffing region, the bettor bets according to a continuous, decreasing function $s = b^{-1}(x)$ with endpoints $b^{-1}(x_0) = U$ and $b^{-1}(x_3) = L$.

5. Within the value betting region, the bettor partitions into a min-betting region $x \in [x_3, x_4]$, an intermediate region $x \in [x_4, x_5]$, and a max-betting region $x \in [x_5, 1]$.

6. Within the intermediate value betting region, the bettor bets according to a continuous, increasing function $s = v^{-1}(x)$ with endpoints $v^{-1}(x_3) = L$ and $v^{-1}(x_5) = U$.

See figure 2 for visual representations of the strategy profile.

## 4.4 Constraints and Indifference Equations

Having established the qualitative structure of the Nash equilibrium, we now derive the quantitative relationships that the strategy profile must satisfy. These constraints arise from two fundamental equilibrium conditions: players must be indifferent among actions they mix over (or, in our case, use with positive probability), and players must optimize when choosing among available actions. The resulting system of differential and algebraic equations will uniquely determine all threshold values $x_0, \ldots, x_5$ and the functions $b(s)$, $v(s)$, and $c(s)$.

The Nash equilibrium strategy profile must satisfy several constraints and indifference conditions, which we will derive and use to solve for the strategy profile. The key conditions are:

- The caller must be indifferent between calling and folding at their calling threshold

[6] The action taken at the threshold is irrelevant, since it occurs with probability zero.

- The bettor must be indifferent between checking and betting at their value betting and bluffing thresholds
- The bettor's bet size for a value bet must maximize their expected value
- The bettor's strategy must be continuous in bet size (in the regions where they bet)

These conditions give us the following system of equations:
**Caller Indifference:**

$$(x_2 - x_1) \cdot (1 + L) - (x_4 - x_3) \cdot L = 0 \tag{1}$$
$$x_0 \cdot (1 + U) - (1 - x_5) \cdot U = 0 \tag{2}$$
$$|b'(s)| \cdot (1 + s) - |v'(s)| \cdot s = 0 \tag{3}$$

**Bettor Indifference and Optimality:**

$$-sc'(s) - c(s) + 2v(s) - 1 = 0 \tag{4}$$
$$(x_3 - c(L)) \cdot (1 + L) - (1 - x_3) \cdot (L) + c(L) = x_3 \tag{5}$$
$$c(s) - (1 - c(s)) \cdot s = x_2 \tag{6}$$

**Continuity Constraints:**

$$b(U) = x_0 \tag{7}$$
$$b(L) = x_1 \tag{8}$$
$$v(U) = x_5 \tag{9}$$
$$v(L) = x_4. \tag{10}$$

We will now derive each of these equations in turn. Note that for this analysis, it is simpler to pretend that payoffs exclude the initial ante of 0.5, since this is a sunk cost to both players and we only care about relative payoffs between actions.

### 4.4.1 Caller Indifference

By definition, $c(s)$ is the threshold above which the caller calls and below which they fold. This means that in Nash Equilibrium, the caller must be indifferent between calling and folding with a hand strength of $c(s)$:

$$\mathbb{E}[\text{call } c(s)] = \mathbb{E}[\text{fold } c(s)]$$
$$\mathbb{P}[\text{bluff}|s] \cdot (1 + s) - \mathbb{P}[\text{value bet}|s] \cdot s = 0.$$

We now split into cases based on the value of $s$.

**Case 1:** $s = L$. The hands the bettor value bets $L$ with are $x \in (x_3, x_4)$, and the hands they bluff with are $x \in (x_1, x_2)$.

$$(x_2 - x_1) \cdot (1 + L) - (x_4 - x_3) \cdot L = 0. \tag{11}$$

Here, we are implicitly multiplying both sides by the common denominator of $(x_4 - x_3) + (x_2 - x_1)$.

**Case 2:** $s = U$. The hands the bettor value bets $U$ with are $x \in (x_5, 1)$, and the hands they bluff with are $x \in (0, x_0)$.

$$(1 - x_5) \cdot (1 + U) - x_0 \cdot U = 0, \tag{12}$$

again, implicitly multiplying both sides by the common denominator of $(1 - x_5) + x_0$.

**Case 3:** $L \leq s \leq U$. In this case, the bettor has exactly one value hand and one bluffing hand, but somewhat paradoxically, they are not equally likely. The probability of a value bet given the size $s$ is related to the inverse derivative of the value function $v(s)$ at $s$, and the same goes for a bluff. This gives us the following relation:

$$\frac{\mathbb{P}[\text{value bet}|s]}{\mathbb{P}[\text{bluff}|s]} = \frac{|b'(s)|}{|v'(s)|}$$

An intuitive interpretation of this is that for any small neighborhood around the bet size $s$, the bettor has more hands which use a bet size in the neighborhood if $v(s)$ does not change rapidly around $s$, that is, if $|v'(s)|$ is small. The same goes for bluffing hands, and as we limit the neighborhood to a single point, the ratio of the two probabilities approaches the ratio of the derivatives. We know that these are the only two possible bettor actions for such a bet size, so

$$\mathbb{P}[\text{value bet}|s] = \frac{|b'(s)|}{|b'(s)| + |v'(s)|}$$

$$\mathbb{P}[\text{bluff}|s] = \frac{|v'(s)|}{|b'(s)| + |v'(s)|}$$

Plugging this into the indifference equation and dividing out the common denominator, we get:

$$|b'(s)| \cdot (1 + s) - |v'(s)| \cdot s = 0. \tag{13}$$

### 4.4.2 Bettor Indifference and Optimality

When the bettor makes a value bet, they are attempting to maximize the expected value of the bet. We can write the expected value of a value bet as:

$$\begin{aligned}\mathbb{E}[\text{value bet } s|x] &= \mathbb{P}[\text{call w ith worse}] \cdot (1+s) - \mathbb{P}[\text{call with better}] \cdot s + \mathbb{P}[\text{fold}] \cdot 1 \\ &= (x - c(s)) \cdot (1+s) - (1-x) \cdot (s) + c(s).\end{aligned}$$

To maximize this, we take the derivative with respect to $s$ and set it equal to zero. Crucially, we are treating $c(s)$ as a function of $s$ and using the chain rule, since changing the bet size $s$ will also change the calling threshold $c(s)$. We want this optimality condition to hold for the bettor's Nash equilibrium strategy, so we set $x = v(s)$. This gives us:

$$\begin{aligned}&\frac{d}{ds}\mathbb{E}[\text{value bet } s|x = v(s)] = 0 \\ &-sc'(s) - c(s) + 2v(s) - 1 = 0.\end{aligned} \tag{14}$$

Additionally, when the bettor has the most marginal value betting hand at $x = x_3$, they should be indifferent between a minimum value bet and a check:

$$\begin{aligned}\mathbb{E}[\text{value bet } L|x = x_3] &= \mathbb{E}[\text{check}|x = x_3] \\ (x_3 - c(L)) \cdot (1+L) - (1-x_3) \cdot (L) + c(L) &= x_3.\end{aligned} \tag{15}$$

Finally, when the bettor has the most marginal bluffing hand at $x = x_2$, they should be indifferent between a minimum bluff and a check. However, as we discussed earlier, the bettor should be indifferent among all bluffing sizes, so the bettor should actually be indifferent between checking and making any bluffing size $s$ at $x = x_2$. This gives us:

$$\begin{aligned}\mathbb{E}[\text{bluff } s|x = x_2] &= \mathbb{E}[\text{check}|x = x_2] \\ c(s) - (1 - c(s)) \cdot s &= x_2.\end{aligned} \tag{16}$$

### 4.4.3 Continuity Constraints

As discussed above, the bettor's strategy is continuous in $s$ and $x$ (except when checking). This means that the endpoints of the functions $v(s)$ and $b(s)$ are constrained as follows:

$$b(U) = x_0, \quad b(L) = x_1, \quad v(U) = x_5, \quad v(L) = x_4. \tag{17}$$

# 5 Nash Equilibrium Strategy Profile

Having established the equilibrium structure in Section 4 and derived the indifference equations in Section 4.4, we now present the complete solution. The system of equations from Section 4.4 was solved symbolically using Mathematica (see Appendix **??** for the complete code), yielding closed-form expressions for all threshold values and strategic functions.

**Theorem 5.1** (LCP Nash Equilibrium)**.** *LCP has a unique Nash equilibrium strategy profile in which the bettor's strategy is monotone-admissible (up to measure zero sets of hands for each player). This strategy profile is given by:*

$$
\begin{aligned}
x_0 &= \frac{3t^2(t-1)}{r^3+t^3-7}\\
x_1 &= \frac{-2r^3+3r^2+t^3-1}{r^3+t^3-7}\\
x_2 &= \frac{r^3+t^3-1}{r^3+t^3-7}\\
x_3 &= \frac{r^3-3r+t^3-4}{r^3+t^3-7}\\
x_4 &= \frac{r^3+3r^2-6r+t^3-4}{r^3+t^3-7}\\
x_5 &= \frac{r^3+t^3+3t^2-7}{r^3+t^3-7}\\
b_0 &= \frac{t^3}{r^3+t^3-7}\\
b(s) &= \frac{t^3(s+1)^3-(3s+1)}{(r^3+t^3-7)(s+1)^3}\\
c(s) &= \frac{r^3+t^3-1+s(r^3+t^3-7)}{(s+1)(r^3+t^3-7)}\\
v(s) &= \frac{r^3+t^3-1+(r^3+t^3-7)(2s^2+4s+1)}{2(r^3+t^3-7)(s^2+2s+1)}
\end{aligned}
$$

*where $r = L/(1+L)$ and $t = 1/(1+U)$.*

***Remark:*** *The change of variables to $(r,t)$ significantly simplifies the expressions compared to the original $(L,U)$ formulation. This transformation reveals underlying symmetries and makes many properties more transparent, as we will see in the analysis of game value and parameter effects.*

Refer back to Section 4.3 for an explanation of how these values fit together to actually form the strategy profile.

A proof of this theorem can be found in Appendix B.

This solution is more interpretable in graphical form. Figure 2 shows the strategy profile for various values of $L$ and $U$ ranging from very lenient ($L = 0, U = 10$) to very restricted ($L = 0.5, U = 1$). The more lenient bet size limits model something closer to NLCP, while the more restricted bet size limits model something closer to FBCP with a fixed bet size. Indeed, we see that the strategy profile for $L = 0, U = 10$ looks qualitatively similar to the strategy profile of NLCP—we will show in Section 7.2 that the strategy profile approaches the Nash equilibrium of NLCP as $L$ and $U$ approach 0 and $\infty$, respectively, and that the strategy profile approaches the Nash equilibrium of FBCP as $L$ and $U$ approach some fixed value $s$ from either side.

# 6 Game Value

Having characterized the Nash equilibrium strategy profile in Section 4, we now turn to analyzing the expected payoff when both players employ these optimal strategies. In zero-sum games like Limit Continuous Poker, the concept of game value is fundamental. The game value represents the expected payoff that the first player (bettor) can guarantee when both players play optimally. Since this is a zero-sum game, the caller's expected payoff is simply the negative of this value. This value serves as a measure of how favorable the game is to the bettor under the given betting limits $L$ and $U$.

**Theorem 6.1.** *The value of Limit Continuous Poker is given by:*

$$V_{LCP}(L,U) = \frac{(1+L)^3(1+U)^3 - ((1+L)^3 + L^3(1+U)^3)}{14(1+L)^3(1+U)^3 - 2((1+L)^3 + L^3(1+U)^3)}$$

*Equivalently, using the change of variables $r = L/(1+L)$ and $t = 1/(1+U)$, this can be written more compactly as:*

$$V(r,t) = \frac{1 - r^3 - t^3}{14 - 2r^3 - 2t^3}$$

**Proof.** Since LCP is a zero-sum game, all Nash equilibria yield the same payoff. We have explicitly constructed optimal strategies for both players, so the value of the game is the expected payoff of these strategies, averaged over all hand pairs $(x, y)$. This reduces to an integral of the payoff over the unit square, like we saw in Figure 4.

The computation is extremely nontrivial because the bet size is only defined implicitly in terms of the hand strength $x$. The proof requires breaking the unit square into regions based on the strategies and bet sizes, then computing the

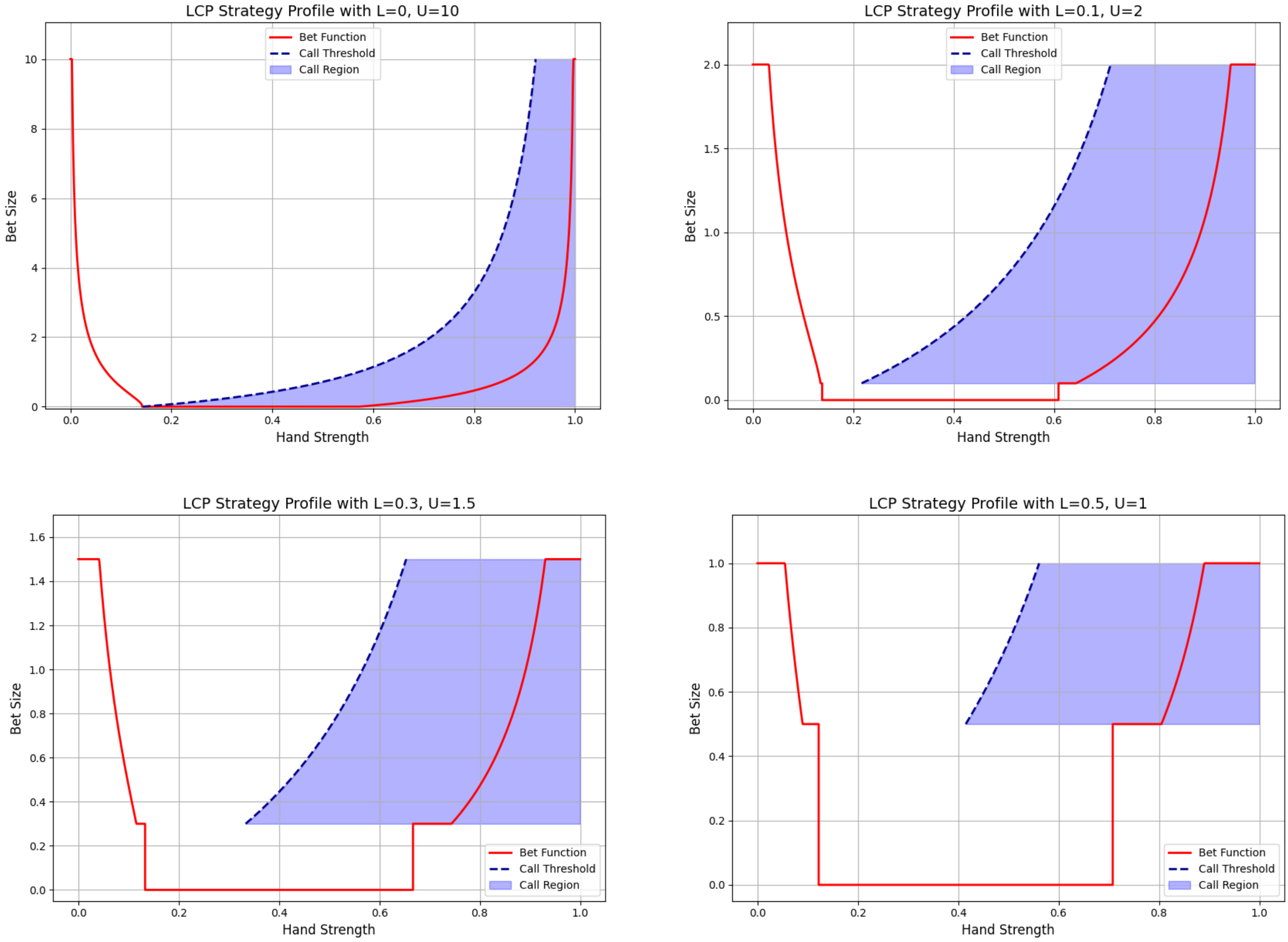


Figure 2: Nash equilibrium strategy profiles for different values of $L$ and $U$, from very lenient to very restricted bet sizes. The bet function maps hand strengths to bet sizes, while the call function gives the minimum calling hand strength for a given bet size. The shaded regions represent the hand strengths for which the caller should call a given bet size.

expected value as a weighted sum of payoffs over these regions. The full technical details are provided in Appendix **??**. $\square$

This is a rational function of $L$ and $U$ with a surprisingly simple form. The change of variables to $r$ and $t$ reveals an even more elegant structure, along with the following symmetry property:

$$V_{LCP}(L, U) = V_{LCP}\left(\frac{1}{U}, \frac{1}{L}\right)$$
$$\text{or equivalently, } V(r, t) = V(t, r)$$

The symmetry becomes immediately apparent in the $(r, t)$ formulation: the numerator $1 - r^3 - t^3$ and denominator $14 - 2r^3 - 2t^3$ are both symmetric in $r$ and $t$. This is not at all obvious from the game setup in terms of $L$ and $U$.

Figure **??** shows the game value as a function of $r$ and $t$, making the symmetry $V(r, t) = V(t, r)$ visually apparent as a reflection across the diagonal. The plot also clearly illustrates the interpretations discussed above: the left edge ($r = 0$, corresponding to $L = 0$) represents the case where minimum bets are negligible, while the bottom edge ($t = 0$, corresponding to $U \to \infty$) represents the case where maximum bets become arbitrarily large. The diagonal ($r + t = 1$, or equivalently $L = U$) represents the boundary where the game reduces to fixed-bet continuous poker.

In terms of the original parameters, the symmetry $V_{LCP}(L, U) = V_{LCP}(1/U, 1/L)$ tells us that the benefit from increasing $U$ is exactly equivalent to that of decreasing $L$ in a reciprocal manner, centered around the pot size of 1. For example, suppose you are given the choice between playing LCP as the bettor with limits $L = 1/2$ and $U = 5$ or $L = 1/5$ and $U = 2$. You know you can play optimally, but it is unclear which game favors you. The symmetry property tells us that the value of the game is the same in both cases, so you should be indifferent between the two.

## 6.1 Interpreting the Parameters r and t

Before analyzing the properties of the game value, it is helpful to understand what the transformed parameters $r$ and $t$ represent in game-theoretic terms.

**The parameter $r = L/(1 + L)$ as minimum pot odds:** When the bettor makes a minimum bet of size $L$, the pot grows from 1 to $1 + L$. The caller must risk $L$ to call and potentially win a pot of $1 + L$. Thus, $r = L/(1 + L)$ represents the *pot odds* the caller receives when facing a minimum bet—the ratio of what they risk to the total pot. This is the most favorable pot odds any caller ever faces in LCP, since larger bets offer worse pot odds. As $L \to 0$, we have $r \to 0$, meaning the minimum bet becomes negligible and calling becomes essentially free. As $L \to \infty$, we have $r \to 1$, meaning the minimum bet becomes prohibitively expensive relative to the pot.

**The parameter $t = 1/(1 + U)$ as pot fraction at maximum bet:** When the bettor makes a maximum bet of size $U$, the pot grows from 1 to $1 + U$. The

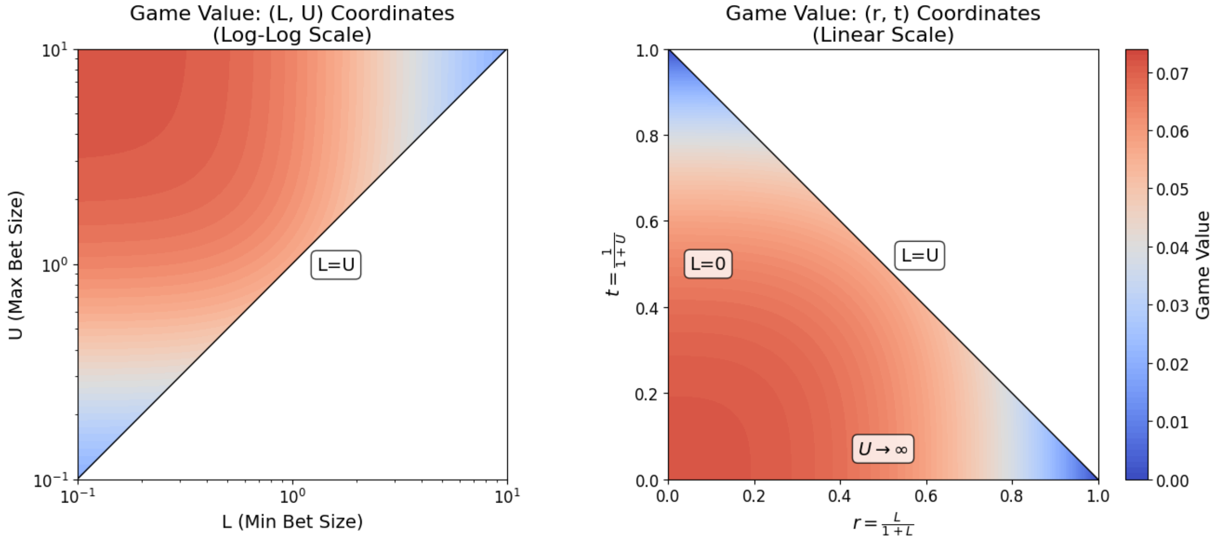


Figure 3: Game value as a function of both parametrizations. The symmetry about the diagonal is immediately visible, corresponding to $V(r,t) = V(t,r)$ or $V(L,U) = V(1/U, 1/L)$. Note that the left plot is cutting off extremely large and small values of $L$ and $U$, while the right plot shows every possible parameter combination.

parameter $t = 1/(1+U)$ represents the original pot as a fraction of the total pot after a maximum bet. Equivalently, $1-t = U/(1+U)$ represents the pot odds the caller receives when facing a maximum bet. A small value of $t$ (close to 0) indicates that $U$ is very large relative to the pot, allowing the bettor to make very aggressive bets. As $U \to \infty$, we have $t \to 0$, meaning the maximum bet becomes arbitrarily large. As $U \to 0$, we have $t \to 1$, meaning the maximum bet becomes negligible.

**The duality revealed by the symmetry:** The parameter $r$ fundamentally controls the *caller's incentive to call* with marginal hands: higher $r$ means the minimum bet offers worse pot odds, discouraging calls. The parameter $t$ fundamentally controls the *bettor's ability to apply pressure*: lower $t$ means the maximum bet can be much larger relative to the pot, allowing more aggressive play. The symmetry $V(r,t) = V(t,r)$ reveals a deep duality in the game: swapping the "minimum calling incentive" (measured by $r$) with the "maximum betting freedom" (measured inversely by $t$) produces games with identical value. This is remarkable because $r$ is primarily about the caller's decisions (pot odds when facing small bets) while $t$ is primarily about the bettor's decisions (how large they can bet), yet these two forces are perfectly balanced in determining the game's value.

In the following sections, we investigate the properties and behavior of $V_{LCP}(L,U)$

in more detail. These include monotonicity, convergence to NLCP and FBCP, and the symmetry property.

## 6.2 Value Monotonicity

Intuitively, more options for the bettor should increase the game's value. Notice the higher value for more lenient limits (red regions of Figure 3) and lower value for more strict limits (blue corners). We can easily prove this formally:

**Theorem 6.2.** *The value of Limit Continuous Poker is weakly monotonically increasing in $U$ and weakly monotonically decreasing in $L$:*

$$\frac{\partial V_{LCP}(L,U)}{\partial U} \geq 0, \quad \frac{\partial V_{LCP}(L,U)}{\partial L} \leq 0.$$

**Proof.** We can express the derivatives in terms of the cleaner $(r,t)$ variables. Since $r = L/(1+L)$ and $t = 1/(1+U)$, we have:

$$\frac{dr}{dL} = \frac{1}{(1+L)^2}, \quad \frac{dt}{dU} = -\frac{1}{(1+U)^2}$$

Using the chain rule and the fact that $V(r,t) = \frac{1-r^3-t^3}{14-2r^3-2t^3}$:

$$\frac{\partial V}{\partial r} = \frac{-3r^2(14-2r^3-2t^3) + 2\cdot 3r^2(1-r^3-t^3)}{(14-2r^3-2t^3)^2} = \frac{-18r^2(2-r^3-t^3)}{(14-2r^3-2t^3)^2} < 0$$
$$\frac{\partial V}{\partial t} = \frac{-18t^2(2-r^3-t^3)}{(14-2r^3-2t^3)^2} < 0$$

where the inequalities hold since $r, t \in (0,1)$ implies $r^3 + t^3 < 2$ and $14 - 2r^3 - 2t^3 > 0$. Therefore:

$$\frac{\partial V_{LCP}}{\partial L} = \frac{\partial V}{\partial r}\frac{dr}{dL} = \frac{-18r^2(2-r^3-t^3)}{(14-2r^3-2t^3)^2} \cdot \frac{1}{(1+L)^2} < 0$$
$$\frac{\partial V_{LCP}}{\partial U} = \frac{\partial V}{\partial t}\frac{dt}{dU} = \frac{-18t^2(2-r^3-t^3)}{(14-2r^3-2t^3)^2} \cdot \left(-\frac{1}{(1+U)^2}\right) > 0$$

□

## 6.3 Value Convergence

The main diagonal of the plots in Figure **??** should represent $L = U$, which means the bet size is fixed. Thus, this diagonal should represent the value of FBCP for various values of $B$. We can prove this formally:

**Theorem 6.3.** *For any $B > 0$, the value of Limit Continuous Poker converges to the value of Fixed-Bet Continuous Poker as $L$ and $U$ approach $B$:*

$$\lim_{L\to B}\lim_{U\to B} V_{LCP}(L,U) = \lim_{U\to B}\lim_{L\to B} V_{LCP}(L,U) = V_{FB}(B)$$

**Proof.** $V_{LCP}(L,U)$ is a rational function of $L$ and $U$, and no part of the expression is undefined for $L = U = B$. We can simply plug in and simplify to get

$$\frac{B}{2(1+2B)(2+B)},$$

which is exactly $V_{FB}(B)$. □

These plots also align with the known result that a fixed pot-size bet of $B = 1$ maximizes the expected value for the bettor in FBCP, as seen by the fact that $(1,1)$ achieves the maximum value on the diagonal.

We can also show that the value of LCP converges to the value of NLCP as $L$ and $U$ approach their extremes, represented by the top left of any plot in Figure **??**.

**Theorem 6.4.** *The value of Limit Continuous Poker converges to the value of NLCP as $L$ and $U$ approach $0$ and $\infty$:*

$$\lim_{L\to 0}\lim_{U\to\infty} V_{LCP}(L,U) = \lim_{U\to\infty}\lim_{L\to 0} V_{LCP}(L,U) = V_{NL}$$

**Proof.** We can simply plug in $L = 0$ to get

$$\frac{(U+1)^3-1}{14(U+1)^3-2}.$$

Taking the limit as $U \to \infty$ gives $\frac{1}{14} = V_{NL}$. □

# 7 Strategic Comparison to Fixed-Bet and No-Limit Continuous Poker

As noted in the introduction (Section 1), LCP is designed to interpolate between Fixed-Bet Continuous Poker (FBCP) and No-Limit Continuous Poker (NLCP). Having derived the Nash equilibrium (Section 4) and game value (Section 6) for LCP, we now make this interpolation precise by proving convergence results. Specifically, we show that as the betting limits $L$ and $U$ approach appropriate boundary values, both the strategies and the game value of LCP converge to those of FBCP and NLCP.

To facilitate comparison across variants, we model the bettor strategies for all three games as 'bet functions' from hand strengths to bets (with 0 representing a check), and caller strategies as 'call functions' from bet sizes to minimum calling thresholds. We also introduce notation to reference all three strategy profiles more efficiently.

## 7.1 Setup and Notation

To compare the strategy profiles across different variants of Continuous Poker, we introduce the following notation for the strategy functions of the three games:

| **Symbol** | **Meaning** |
|---|---|
| $S_{FB}(x, B)$ | Bettor's bet function in FBCP with fixed bet size $B$ |
| $C_{FB}(s, B)$ | Caller's call function in FBCP with fixed bet size $B$ |
| $S_{NL}(x)$ | Bettor's bet function in NLCP |
| $C_{NL}(s)$ | Caller's call function in NLCP |
| $S_{LCP}(x, L, U)$ | Bettor's bet function in LCP with limits $L$ and $U$ |
| $C_{LCP}(s, L, U)$ | Caller's call function in LCP with limits $L$ and $U$ |
| $x_i\|_{L,U}$ | Threshold $x_i$ in LCP with limits $L$ and $U$ |

Table 1: Notation for strategy functions across different variants of Continuous Poker

In FBCP, the bettor can only make a fixed bet size $B$ or check. The bet function $S_{FB}(x, B)$ maps hand strengths to either 0 (check) or $B$ (bet):

$$S_{FB}(x, B) = \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \text{ (bluffing range)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (checking range)} \\ B & x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (value betting range)} \end{cases} \tag{18}$$

The caller's strategy is defined by a single threshold $C_{FB}(s, B)$:

$$C_{FB}(s, B) = \frac{B(3 + 2B)}{(1 + 2B)(2 + B)} \tag{19}$$

In NLCP, the bettor can choose any positive bet size. The strategy is most naturally described by functions $v_{NL}(s)$ and $b_{NL}(s)$ that map bet sizes to hand strengths:

$$v_{NL}(s) = 1 - \frac{3}{7(s+1)^2} \text{ (value betting function)}$$
$$b_{NL}(s) = \frac{3s+1}{7(s+1)^3} \text{ (bluffing function).}$$

The bet function $S_{NL}(x)$ is then defined in terms of the inverse functions:

$$S_{NL}(x) = \begin{cases} b_{NL}^{-1}(x) & x < \frac{1}{7} \text{ (bluffing range)} \\ 0 & \frac{1}{7} < x < \frac{4}{7} \text{ (checking range)} \\ v_{NL}^{-1}(x) & x > \frac{4}{7} \text{ (value betting range)} \end{cases}$$

The caller's strategy is defined by a continuous function $C_{NL}(s)$:

$$C_{NL}(s) = 1 - \frac{6}{7(s+1)}$$

In LCP, the bettor can choose any bet size between $L$ and $U$. The strategy profile is defined by six thresholds $x_0$ through $x_5$ and functions $v(s)$ and $b(s)$ that map bet sizes to hand strengths. The bet function $S_{LCP}(x, L, U)$ and call function $C_{LCP}(s, L, U)$ are defined in terms of these values, which are given in Theorem 5.1.

## 7.2 Strategic Convergence

### 7.2.1 Bettor Strategy Convergence to Continuous Poker

We expect that as $L$ and $U$ approach some fixed value $s$, the bet function $S_{LCP}(x, L, U)$ should converge to the bet function $S_{FB}(x, s)$ for Fixed-Bet Continuous Poker with a fixed bet size $s$.

**Theorem 7.1.** *For any $B > 0$, the bet function $S_{LCP}(x, L, U)$ for Limit Continuous Poker converges to the bet function $S_{FB}(x, B)$ for Fixed-Bet Continuous Poker with a fixed bet size $B$ as $L$ and $U$ approach $B$:*

$$\lim_{L\to B} \lim_{U\to B} S_{LCP}(x, L, U) = \lim_{U\to B} \lim_{L\to B} S_{LCP}(x, L, U) = S_{FB}(x, B).$$

**Proof.** We analyze the expressions for the $x_i$'s, each of which is a rational function[7] of $L$ and $U$. Since these functions are defined and continuous for all positive values

[7] A ratio of polynomials in $L$ and $U$.

of $L$ and $U$, the limit as $L \to B$ and $U \to B$ can be found by simply substituting $L = U = B$:

$$\begin{aligned}
x_0|_{B,B} = x_1|_{B,B} &= \frac{B}{2B^3 + 7B^2 + 7B + 2} \\
x_2|_{B,B} &= \frac{B}{(1+2B)(2+B)} \\
x_3|_{B,B} &= \frac{2B^2 + 4B + 1}{(1+2B)(2+B)} \\
x_4|_{B,B} = x_5|_{B,B} &= \frac{2B^2 + 5B + 1}{(1+2B)(2+B)}
\end{aligned}$$

$x_0 = x_1$ and $x_4 = x_5$ are expected, since these intervals are where the bettor uses an intermediate bet size, and $L = U = B$ does not allow intermediate bet sizes. This reduces the bet function to

$$\begin{aligned}
\lim_{L\to B}\lim_{U\to B} S_{LCP}(x, L, U) &= \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \\ B & x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \end{cases} \\
&= S_{FB}(x, B)
\end{aligned}$$

□

### 7.2.2 Caller Strategy Convergence to Continuous Poker

The calling function is easier to analyze. We want to show that the calling threshold $C_{LCP}(s, L, U)$ converges to the calling threshold $C_{FB}(s, B)$ for Fixed-Bet Continuous Poker with a fixed bet size $B$ as $L$ and $U$ approach $B$.

**Theorem 7.2.** *For any $B > 0$, the call function $C_{LCP}(s, L, U)$ for Limit Continuous Poker converges to the call function $C_{FB}(s, B)$ for Fixed-Bet Continuous Poker with a fixed bet size $B$ as $L$ and $U$ approach $B$:*

$$\lim_{L\to B}\lim_{U\to B} C_{LCP}(s, L, U) = \lim_{U\to B}\lim_{L\to B} C_{LCP}(s, L, U) = C_{FB}(s, B).$$

**Proof.** We already have the value of $x_2|_{B,B}$, so we can plug this into the expression for the calling threshold:

$$\begin{aligned}
\lim_{L\to B}\lim_{U\to B} C_{LCP}(s,L,U) &= \frac{x_2|_{B,B}+s}{1+s} \\
&= \frac{\frac{B}{(1+2B)(2+B)}+s}{1+s} \\
&= \frac{B(3+2B)}{(1+2B)(2+B)} \\
&= C_{FB}(s,B)
\end{aligned}$$

□

### 7.2.3 Bettor Strategy Convergence to NLCP

In a similar fashion, we expect that as $L$ and $U$ approach 0 and $\infty$, the bet function $S_{LCP}(x,L,U)$ should converge to the bet function $S_{NL}(x)$ for NLCP.

**Theorem 7.3.** *The bet function $S_{LCP}(x,L,U)$ for Limit Continuous Poker converges to the bet function $S_{NL}(x)$ for NLCP as $L$ and $U$ approach 0 and $\infty$:*

$$\lim_{L\to 0}\lim_{U\to\infty} S_{LCP}(x,L,U) = \lim_{U\to\infty}\lim_{L\to 0} S_{LCP}(x,L,U) = S_{NL}(x).$$

**Proof.** We can analyze the expressions for the $x_i$'s as $L$ and $U$ approach 0 and $\infty$. The limit is well-defined, and we can substitute $L=0$ and $U=\infty$ into the expressions for the $x_i$s.

$$\begin{aligned}
x_0|_{0,\infty} &= 0 \\
x_1|_{0,\infty} = x_2|_{0,\infty} &= \frac{1}{7} \\
x_3|_{0,\infty} = x_4|_{0,\infty} &= \frac{4}{7} \\
x_5|_{0,\infty} &= 1
\end{aligned}$$

$x_0|_{0,\infty} = 0$ and $x_5|_{0,\infty} = 1$ are expected, since these intervals are where the bettor uses a minimum bet size and a maximum bet size, respectively, both of which are impossible. The bettor now bets intermediate values for $x < \frac{1}{7}$ and $x > \frac{4}{7}$, and checks for $\frac{1}{7} < x < \frac{4}{7}$. But how much do they bet? We can take the limits of $v(s)$ and $b(s)$ as $L$ and $U$ approach 0 and $\infty$:

$$\begin{aligned}
\lim_{L\to 0}\lim_{U\to\infty} b(s) &= \frac{3s+1}{7(s+1)^3} \\
\lim_{L\to 0}\lim_{U\to\infty} v(s) &= 1 - \frac{3}{7(s+1)^2}.
\end{aligned}$$

To summarize, the bettor bets $s$ with hands $x < \frac{1}{7}$ such that $x = b(s)$ or hands $x > \frac{4}{7}$ such that $x = v(s)$. This is exactly the same as the bet function $S_{NL}(x)$ for NLCP.

$\square$

### 7.2.4 Caller Strategy Convergence to NLCP

The calling function is again easier to analyze. We want to show that the calling threshold $C_{LCP}(s, L, U)$ converges to the calling threshold $C_{NL}(s)$ for NLCP as $L$ and $U$ approach 0 and $\infty$.

**Theorem 7.4.** *The call function $C_{LCP}(s, L, U)$ for Limit Continuous Poker converges to the call function $C_{NL}(s)$ for NLCP as $L$ and $U$ approach 0 and $\infty$:*

$$\lim_{L\to 0}\lim_{U\to\infty} C_{LCP}(s, L, U) = \lim_{U\to\infty}\lim_{L\to 0} C_{LCP}(s, L, U) = C_{NL}(s).$$

**Proof.** Again, we already have the limiting value of $x_2|_{0,\infty}$, so we can plug this into the expression for the calling threshold:

$$\begin{aligned}
\lim_{L\to 0}\lim_{U\to\infty} C_{LCP}(s, L, U) &= \lim_{L\to 0}\lim_{U\to\infty} \frac{x_2 + s}{1 + s} \\
&= \frac{\frac{1}{7} + s}{1 + s} \\
&= 1 - \frac{6}{7(1 + s)} \\
&= C_{NL}(s).
\end{aligned}$$

$\square$

We have now shown that the bettor and caller strategies for LCP converge to those of FBCP and NLCP as the limits $L$ and $U$ approach their extreme values. In the next section, we explore the value of LCP in more detail, and in particular how it relates to that of FBCP and NLCP.

# 8 Parameter and Payoff Analysis

Having established the Nash equilibrium (Section 4), analyzed the game value (Section 6), and proved convergence to FBCP and NLCP (Section 7), we now explore in greater detail how the parameters $L$ and $U$ affect player strategies and payoffs. This section summarizes key insights and presents visualizations that illuminate the strategic dynamics of LCP. Complete technical proofs are provided in Appendices D and E.

## 8.1 Visualizing Payoffs in Equilibrium

In Nash equilibrium, each hand combination $(x, y)$ uniquely determines the bettor's payoff. Figure 4 shows how these payoffs vary across the unit square for different values of $L$ and $U$, from strict limits (Fixed-Bet, $L = U = 1$) to lenient limits (No-Limit, $U \to \infty$).

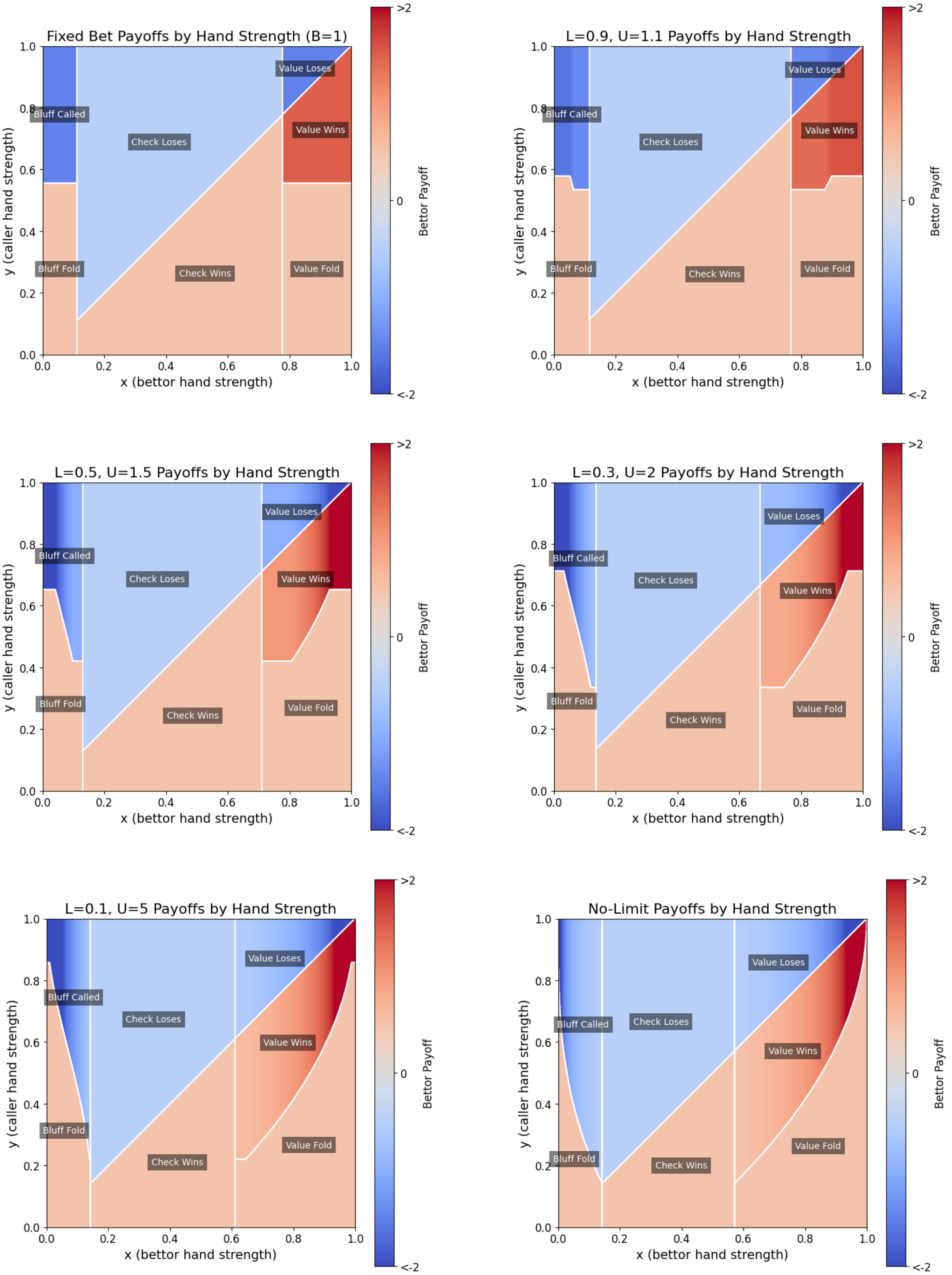


Figure 4: Bettor payoffs in Nash equilibrium as a function of hand strengths $x, y$ for fixed bet size $B = 1$ (top left), and No-Limit Continuous Poker (bottom right). Intermediate plots show the payoffs for different values of $L$ and $U$ ranging from strict (fixed bet size $B = 1$) to lenient (No limits). Regions are labeled according to the outcome of the game in Nash equilibrium.

The visualization reveals that the biggest wins and losses occur when both hands are strong (top right), consistent with real poker intuition. Large payoffs also occur when a very weak bettor bluffs big and gets called by a strong caller (top left). As limits become more lenient, these extreme outcomes become more pronounced but also less likely, since making and calling maximum bets become riskier for both players.

## 8.2 Expected Value by Hand Strength

Beyond specific hand matchups, we can analyze the expected value $EV(x)$ of a bettor hand $x$ averaged over all possible caller hands. This function characterizes how profitable each hand is in equilibrium.

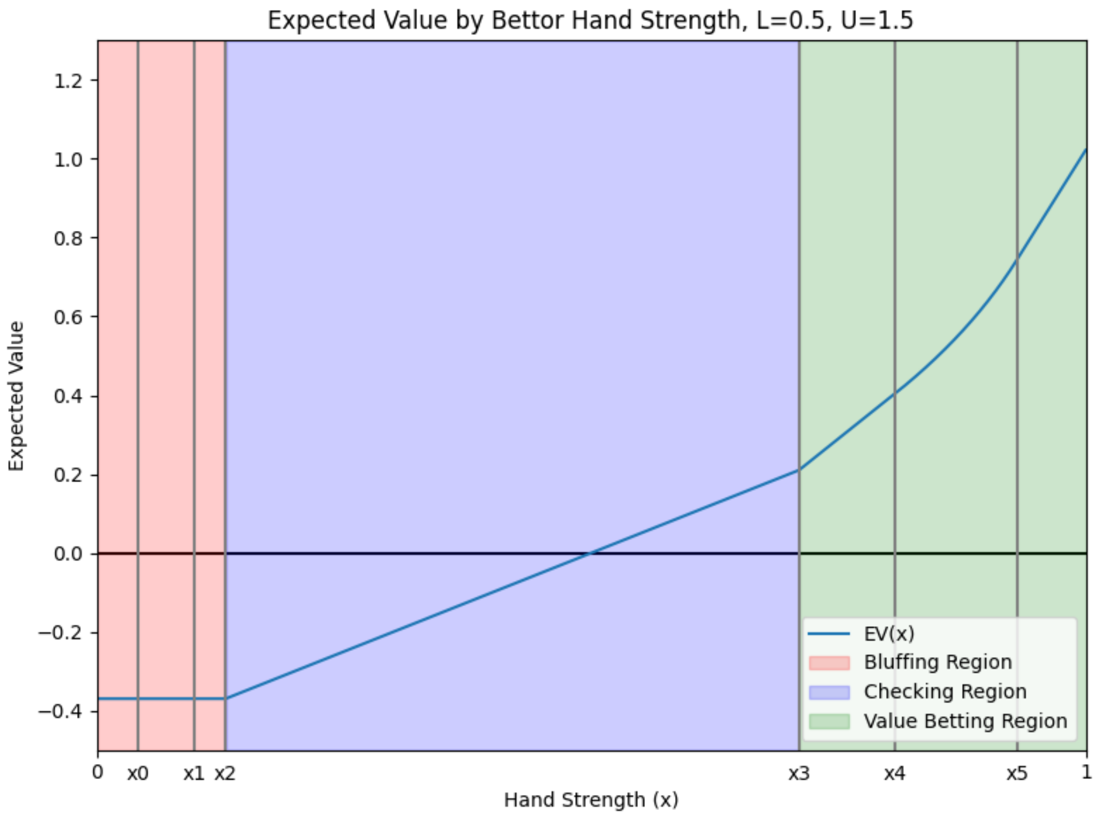


Figure 5: Expected value of bettor hand strength $x$ in the unique admissible Nash equilibrium for various parameter settings. The bettor's expected value is increasing in $x$ (stronger hands are more profitable), with discontinuities at the bluffing threshold $x_2$ and value betting threshold $x_3$.

Key observations:

- All bluffing hands ($x \leq x_2$) achieve the same expected value $x_2 - 1/2$, regardless of hand strength
- Checking hands ($x_2 < x \leq x_3$) earn $x - 1/2$ (the ante, won when the bettor has the best hand)
- Value betting hands ($x > x_3$) earn increasing returns, with the strongest hands making large bets that win huge pots when called
- The function $EV(x)$ is increasing in $x$ (Appendix E, Theorem E.2)

## 8.3 Effect of Increasing the Upper Limit $U$

A counterintuitive result emerges when examining how individual hand values change as we increase $U$: for most hand strengths, the expected value *decreases* beyond a certain threshold of $U$ (see Figure 6). This occurs despite the bettor having strictly more strategic options.

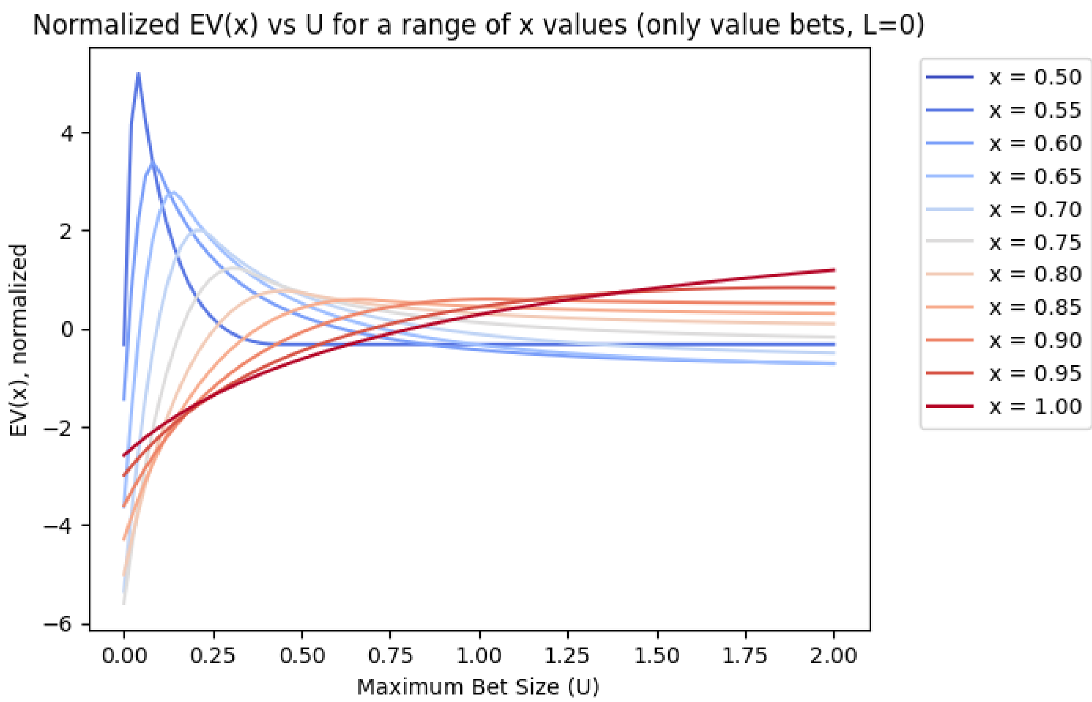


Figure 6: Expected value of a value-betting hand $x$ versus the upper limit $U$ in Nash equilibrium. Each curve increases in $U$ up to some maximum, after which it decreases. This counterintuitive phenomenon is explained by strategic adjustments: as $U$ increases, the caller becomes more conservative (Appendix D).

The explanation lies in strategic interdependence: as $U$ increases, the bettor can make larger bets with their strongest hands, which forces the caller to become more conservative across *all* bet sizes. This defensive adjustment by the caller harms the expected value of intermediate-strength hands, even though they're betting less. Only the very strongest hands (above a threshold greater than $v(U)$) benefit from the increased flexibility.

The complete analysis in Appendix D proves:

- The bluffing threshold $x_2$ increases with $U$ (more hands bluff)
- For fixed hand $x \in [x_3, v(U)]$, the bet size $v^{-1}(x)$ decreases with $U$
- The calling cutoff $c(v^{-1}(x))$ increases with $U$ despite smaller bets
- There exists a threshold hand strength above which $EV(x)$ increases with $U$, and below which it decreases

These results demonstrate the rich strategic dynamics of LCP, where expanding betting options creates complex ripple effects throughout the equilibrium strategy profile.

# 9 Conclusion

We have introduced and analyzed Limit Continuous Poker (LCP), a parametric family of simplified poker games that bridges the gap between Fixed-Bet Continuous Poker (FBCP) and No-Limit Continuous Poker (NLCP). By imposing lower and upper bounds $L$ and $U$ on bet sizes, LCP creates a rich spectrum of strategic environments that interpolate continuously between the fixed-bet and no-limit extremes.

## 9.1 Summary of Key Results

Our analysis has yielded several main contributions:

**Nash Equilibrium Characterization:** We derived the unique admissible Nash equilibrium for LCP, providing closed-form expressions for all strategic components. The equilibrium exhibits a elegant structure where the bettor partitions hands into bluffing, checking, and value betting regions, with bet sizes varying continuously within the bluffing and value betting ranges. The caller responds with a calling threshold that depends on bet size, creating a delicate balance where both players are indifferent among their equilibrium actions.

**Game Value Formula:** We computed the value of LCP as a rational function of the betting limits, obtaining the surprisingly compact expression

$$V(r,t) = \frac{1 - r^3 - t^3}{14 - 2r^3 - 2t^3}$$

in the transformed coordinates $r = L/(1+L)$ and $t = 1/(1+U)$. This formula reveals a remarkable symmetry: $V(r,t) = V(t,r)$, meaning that swapping the roles of minimum and maximum bet constraints (in a specific reciprocal sense) leaves the game value unchanged. We proved that the value is monotonically increasing in $U$ and decreasing in $L$, confirming the intuition that more betting flexibility favors the bettor.

**Convergence to Limiting Cases:** We established that LCP smoothly converges to both FBCP and NLCP in the appropriate limit regimes. As $L \to B$ and $U \to B$, the strategies and value converge to those of FBCP with fixed bet size $B$. As $L \to 0$ and $U \to \infty$, they converge to those of NLCP. These results validate LCP as a genuine generalization of both classical variants.

**Parameter Sensitivity and Strategic Dynamics:** Our analysis revealed counterintuitive strategic effects: increasing the upper limit $U$ does not uniformly benefit all bettor hands. While the strongest hands gain from the ability to make larger bets, intermediate-strength hands can suffer because the caller adjusts by becoming more conservative across all bet sizes. This illustrates the complex strategic interdependencies in equilibrium play.

## 9.2 Strategic Insights and Connections to Real Poker

The theoretical results for LCP offer several insights relevant to practical poker strategy:

**Bet Sizing and Stack Depth:** In real poker, effective stack sizes create implicit upper bounds on bet sizes, analogous to our parameter $U$. Our analysis suggests that deeper stacks (higher $U$) create more strategic complexity and favor skilled players who can exploit the additional betting options. The symmetry property $V(L,U) = V(1/U, 1/L)$ suggests a duality between raising minimum bet requirements and constraining maximum bets.

**Bluffing Frequency and Bet Size:** The equilibrium strategy confirms poker wisdom that bluffing frequency should be calibrated to bet size—larger bluffs require fewer bluffing combinations to remain balanced. Moreover, the monotone-admissible equilibrium prescribes bluffing larger with weaker hands and smaller with stronger bluffs, which aligns with the practical consideration that stronger bluffs have showdown value.

**Calling Thresholds and Pot Odds:** The calling function $c(s)$ embodies the pot odds principle: the caller must be offered better odds (a larger pot relative to the cost of calling) to justify calling with weaker hands. The equilibrium precisely balances the bettor's bluffing and value betting frequencies against these pot odds.

## 9.3 Limitations

Several simplifications distinguish LCP from real poker:

- **Single Betting Round:** Real poker involves multiple streets of betting with community cards revealed between rounds, creating dynamic information revelation. LCP models only a single decision point.
- **Uniform Hand Distribution:** We assume hand strengths are uniformly distributed on $[0, 1]$. Real poker hand distributions are discrete and non-uniform, with specific card combinations determining hand strength.
- **Perfect Correlation:** In our model, hands are perfectly ordered (if $x > y$, the bettor always wins). Real poker has card removal effects and some hands have non-zero equity even when behind.
- **Symmetric Information:** Both players receive one hand each. Real poker often features asymmetric information structures (e.g., one player knows the other folded a certain range on an earlier street).

Despite these limitations, the tractability of LCP enables rigorous analysis that would be impossible in more complex settings, providing a foundation for understanding strategic principles.

## 9.4 Future Directions

Several natural extensions of this work could deepen our understanding of bet sizing in poker:

**Multiple Betting Rounds:** Extending LCP to multiple streets with information revelation between rounds would capture dynamic aspects of poker strategy. How do bet size limits in early rounds affect optimal play in later rounds?

**Asymmetric Limits:** Our model assumes both players face the same ante and pot size. Investigating games where players have different effective stack sizes (asymmetric $U$ values) could model scenarios common in tournament poker.

**Non-Uniform Hand Distributions:** Relaxing the uniform distribution assumption to model more realistic hand strength distributions could test the robustness of our results. Do the qualitative features of the equilibrium persist?

**Discrete Approximations:** Real poker involves discrete bet sizing increments (e.g., betting in whole chips or minimum raise increments). Studying discrete approximations to LCP could bridge the gap between our continuous model and practical applications.

**Computational Tools:** The closed-form solutions for LCP could be used to validate numerical solvers for more complex poker variants. The smooth parameter dependence makes LCP an ideal test bed for computational game theory algorithms.

**Multi-Player Extensions:** While our analysis focuses on two-player games, extending to three or more players would introduce new strategic considerations such as collusion, side pots, and positional dynamics.

The analytical tractability of Limit Continuous Poker makes it a valuable model system for exploring fundamental questions about betting, bluffing, and strategic bet sizing. We hope this work inspires further research into the rich interplay between game structure and optimal strategy in poker and related competitive decision-making environments.

# A Monotone Strategy Proofs

This appendix provides the detailed proofs regarding monotone calling strategies and their relationship to the uniqueness of the Nash equilibrium in LCP.

## A.1 Monotone Strategies and Weak Dominance

**Lemma A.1.** *If a calling strategy violates the first monotonicity condition for a nonzero-measure set of hands for any bet size $s$, it is weakly dominated. Specifically, if there exists $s$ and measurable sets $A, B \subseteq [0,1]$ such that:*

1. *The caller calls $s$ with hands in $A$*
2. *The caller folds $s$ with hands in $B$*
3. $\sup A \leq \inf B$
4. *$A$ and $B$ have positive measure*

*then the strategy is weakly dominated.*

*Proof.* Let $\sigma_C$ be the non-monotone strategy described above. Since $A$ and $B$ are nonzero-measure, there exist subsets $A' \subseteq A$ and $B' \subseteq B$ such that:

1. $A'$ and $B'$ have positive measure

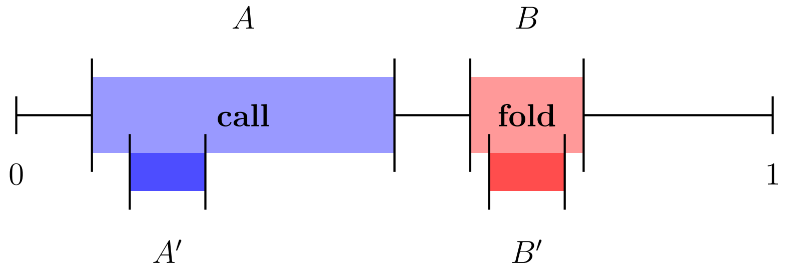


Figure 7: A simple case of sets $A$ and $B$ which violate monotonicity ($\sup A \leq \inf B$). We can find equal-measure subsets $A' \subseteq A$ and $B' \subseteq B$ to swap actions, improving the strategy.

2. $|A'| = |B'|$

Where $|A|$ and $|B|$ denote the Lebesgue measure of $A$ and $B$ (see Figure 7).

The existence of such subsets follows from a fundamental property of nonatomic measures: since the uniform distribution on $[0, 1]$ is nonatomic (no single point has positive probability), for any two measurable sets with positive measure, we can always find measurable subsets of equal measure[3]. This property allows us to construct the strategy improvement described below.

Let $\sigma'_C$ be the strategy which switches the actions for $A'$ and $B'$, i.e. calls with $B'$ and folds with $A'$ (and behaves identically for all other bet sizes). We now analyze how this change affects the caller's performance against any betting strategy.

Against a bet of size $s$, the key improvement occurs in two scenarios:

1. When $y \in B'$ and $x \in A'$: $\sigma_C$ folds while $\sigma'_C$ calls and wins (since $x \in A$ and $y \in B$ with $\sup A \leq \inf B$)

2. When $y \in A'$ and $x \in B'$: $\sigma_C$ calls and loses while $\sigma'_C$ folds (avoiding the loss)

For all other cases, $\sigma_C$ and $\sigma'_C$ behave identically, so $\sigma'_C$ is weakly better than $\sigma_C$ against every betting strategy.

To show that $\sigma_C$ is strictly dominated, consider a betting strategy which always bets $s$. Against this strategy, both scenarios above occur with positive probability (since $A'$ and $B'$ have positive measure), so $\sigma'_C$ is strictly better than $\sigma_C$. Thus, $\sigma_C$ is weakly dominated. $\square$

## A.2 Monotone-Admissibility and Uniqueness

The second monotonicity condition - that the caller must be more willing to call smaller bets - cannot be derived from dominance arguments alone. However, it plays a crucial role in selecting among the infinite class of Nash equilibria that differ only in how the bettor sizes their bluffs.

**Motivation:** From a poker player's perspective, this condition aligns with intuition about pot odds: a larger bet is riskier and should require a stronger hand to call. While violating this condition is not strictly dominated (and indeed occurs in real poker when players perceive larger bets as bluffs), it leads to exploitable calling strategies.

If the caller violates this condition by calling less aggressively against smaller bets, the bettor can exploit this by taking smaller risks for higher returns. A monotone calling strategy is therefore less exploitable. Imposing this condition restricts the strategy space in a way that yields a unique Nash equilibrium.

**Definition A.1** (Monotone-Admissible Strategy)**.** A betting strategy $\sigma_B$ is *monotone-admissible* if it is admissible in LCP against the set of monotone calling strategies. More explicitly, $\sigma_B$ is monotone-admissible if there does not exist a betting strategy $\sigma'_B$ such that both of the following hold:

1. $\pi_B(\sigma'_B, \sigma_C) \geq \pi_B(\sigma_B, \sigma_C)$ for all monotone calling strategies $\sigma_C$
2. $\pi_B(\sigma'_B, \sigma_C) > \pi_B(\sigma_B, \sigma_C)$ for at least one monotone calling strategy $\sigma_C$

**Application to LCP:** This definition is particularly useful in distinguishing bettor strategies which differ only in how they bluff. The hand strength of a bluff is irrelevant when the caller plays optimally, since the caller will never call with a hand weaker than any bluff. However, if the caller deviates to a suboptimal but still monotone strategy, then the bettor's bluffing hand strength becomes important.

Specifically, if the caller becomes too loose (calling too often with weak hands), then the bettor wins some pots "accidentally" when they make their smallest bluffs with their strongest bluffing hands. This is where monotone-admissibility differentiates between equilibria: the unique monotone-admissible equilibrium has the bettor bluffing larger with weaker hands and bluffing smaller with stronger hands, which is optimal against all monotone deviations by the caller.

# B Appendix: Proof of Nash Equilibrium

**Proof.** To show that this is a Nash equilibrium, we need to show that no player can improve their payoff by unilaterally deviating from the strategy profile.

In the proof, we assume that all of the constraints outlined in the previous section are satisfied. The solution was obtained by solving for $c(s)$ in terms of $x_2$, then using this to solve for $v(s)$, and finally solving for $b(s)$ up to a constant of integration. The resulting system of 7 equations in 7 unknowns was solved symbolically using Mathematica and simplified by finding common subexpressions $A_0, A_1, A_2, A_3, A_4, A_5$. The full Mathematica script is available in Appendix **??**.

1. **Caller's Deviation:** Fix the bet size $s$ and consider the caller's payoff from either calling or folding for each hand strength $y$.

$$\begin{aligned}
&\mathbb{E}[\text{call}|y, s] = \mathbb{P}[x < y|s](1 + s) + \mathbb{P}[x \geq y|s](-s) \\
&\mathbb{E}[\text{fold}|y, s] = 0
\end{aligned}$$

   From section 4.4.1, we know that the expected value of a call is exactly 0 for $y = c(s)$ (by design). The expected value of calling is weakly increasing in $y$, so it must be weakly greater than 0 for $y > c(s)$ and weakly less than 0 for $y < c(s)$.

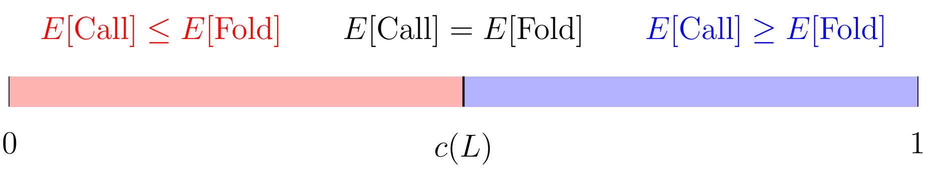


Figure 8: Caller's decision threshold $c(L)$. At hand strength of exactly $c(L)$, the caller is indifferent between calling and folding. Since the value of calling weakly increases with $y$, it must be weakly greater than 0 for $y > c(L)$ and weakly less than 0 for $y < c(L)$. Folding always has value 0.

   This proves that calling is weakly better than folding for all $y > c(s)$, and that folding is weakly better than calling for all $y < c(s)$, so the caller cannot improve their payoff by deviating from the strategy profile.

2. **Bettor's Deviation:** We need to consider a few cases.

   (a) $x < c(s)$: These are hands and bet sizes for which the caller will call with only stronger hands (potential bluffs). The expected value of betting here is

$$\begin{aligned}
\mathbb{E}[\text{bet } s|x] &= \mathbb{P}[\text{call with worse}] \cdot (1 + s) - \mathbb{P}[\text{call with better}] \cdot s + \mathbb{P}[\text{fold}] \cdot 1 \\
&= 0 - (1 - c(s)) \cdot (s) + c(s) \\
&= c(s) - (1 - c(s)) \cdot s \\
&= x_2,
\end{aligned}$$

with the last line coming from equation 6. The value of checking is always

$$\mathbb{E}[\text{check}|x] = x.$$

This means that (by design), the bettor is indifferent between checking and betting any amount at $x = x_2$. Importantly, the value of betting is independent of the hand strength $x$ while the value of checking is strictly increasing in $x$, so checking must be preferable for $x_2 < x < c(L)$ and betting must be preferable for $x < x_2$, which is exactly what our strategy profile does. Because the value of bluffing is simply $x_2$ no matter the bet size or hand strength, we also know that the bettor cannot improve their payoff by bluffing with different bet sizes.

(b) $c(s) \leq x < x_3$: These are hands and bet sizes for which the caller will at least sometimes call with weaker hands (potential value bets), but where the optimal strategy still checks. The expected value of betting here is

$$\begin{aligned}
\mathbb{E}[\text{bet } s|x] &= \mathbb{P}[\text{call with worse}] \cdot (1+s) - \mathbb{P}[\text{call with better}] \cdot s + \mathbb{P}[\text{fold}] \cdot 1 \\
&= (x - c(s))(1+s) - (1-x) \cdot (s) + c(s) \\
&= s(2x - c(s) - 1) + x,
\end{aligned}$$

while that of checking is

$$\mathbb{E}[\text{check}|x] = x.$$

We know from 5 that

$$\begin{aligned}
(x_3 - c(L)) \cdot (1+L) - (1-x_3) \cdot (L) + c(L) &= x_3 \\
2x_3 - c(L) - 1 &= 0
\end{aligned}$$

Using our inequality $c(s) \leq x < x_3$ and the fact that $c(L)$ is the minimum of $c(s)$, we get

$$\begin{aligned}
2x_3 - c(L) - 1 &= 0 \\
2x - c(s) - 1 &\leq 0.
\end{aligned}$$

Substituting this into the expected value of betting, we get

$$\begin{aligned}
\mathbb{E}[\text{bet } s|x] &= s(2x - c(s) - 1) + x \\
&\leq s \cdot 0 + x \\
&= x \\
&= \mathbb{E}[\text{check}|x].
\end{aligned}$$

So no value can be gained by deviating from checking here.

(c) $x_3 \leq x < x_4$: These are value bets where the bettor should bet the minimum. We need to show that the bettor cannot improve their payoff by either checking or by betting more.

The expected value of betting the minimum is

$$\mathbb{E}[\text{bet } L|x] = L(2x - c(L) - 1) + x.$$

Again, we can use 5 to get

$$\begin{aligned} 2x_3 - c(L) - 1 &= 0 \\ 2x - c(L) - 1 &\geq 0, \end{aligned}$$

since $x \geq x_3$. Substituting like before,

$$\begin{aligned} \mathbb{E}[\text{bet } L|x] &= L(2x - c(L) - 1) + x \\ &\geq L \cdot 0 + x \\ &= x \\ &= \mathbb{E}[\text{check}|x]. \end{aligned}$$

So no value can be gained by deviating from betting to checking.

What about betting more? To show that this cannot improve the bettor's payoff, we show that the expected value of betting is weakly decreasing in $s$ for $x < x_4$, and must therefore be maximized at the lowest possible bet of $L$.

$$\frac{d}{ds}\mathbb{E}[\text{bet } s|x] = -sc'(s) - c(s) + 2x - 1$$

We know from 4 that this equals 0 when $x = v(s)$. We also know that $v(s)$ is at least $x_4$ for all $s \in [L, U]$, so $x < x_4 \leq v(s)$ for any such $s$. This means that

$$\begin{aligned} \frac{d}{ds}\mathbb{E}[\text{bet } s|x] &= -sc'(s) - c(s) + 2x - 1 \\ &\leq -sc'(s) - c(s) + 2x_4 - 1 \\ &= 0 \end{aligned}$$

for any $s \in [L, U]$. Therefore, the expected value of betting is decreasing in $s$ for $x < x_4$, and must therefore be maximized at the lowest possible bet of $L$, so the bettor cannot improve their payoff by betting more.

(d) $x_4 \le x < x_5$: These are value bets where the bettor should bet an intermediate amount between $L$ and $U$. We need to show that the bettor cannot improve their payoff by either checking, betting less, or by betting more.

Rather than showing that checking is inferior to the optimal bet size, we show that checking is inferior to betting the minimum, which we will later show is inferior to the optimal bet size. Like the previous cases, the expected value of betting the minimum is

$$\mathbb{E}[\text{bet } L|x] = L(2x - c(L) - 1) + x.$$

Like before, we know that $2x - c(L) - 1 \ge 0$ for $x \ge x_3$ (and in this case, $x \ge x_4 \ge x_3$). This means that

$$\begin{aligned}\mathbb{E}[\text{bet } L|x] &= L(2x - c(L) - 1) + x \\ &\ge L \cdot 0 + x \\ &= x \\ &= \mathbb{E}[\text{check}|x].\end{aligned}$$

So betting the minimum is at least as good as checking.

Now we show that betting any amount other than $v^{-1}(x)$ cannot gain value. Let's again consider the derivative of the expected value with respect to $s$.

$$\frac{d}{ds}\mathbb{E}[\text{bet } s|x] = -sc'(s) - c(s) + 2x - 1$$

We know from 4 that this is equal to 0 when $x = v(s)$:

$$-sc'(s) - c(s) + 2v(s) - 1 = 0.$$

This derivative is clearly an increasing function of $x$, so for $x < v(s)$, the expected value is decreasing in $s$. But since $v$ is an increasing function, $x < v(s)$ is equivalent to $v^{-1}(x) < s$ (essentially, our bet size is too large for the hand strength).

This should make sense - when our bet size is too large for the hand strength, the expected value of that bet is decreasing in the bet size, so smaller bets are more profitable. We can show the same for bets too small: when $x > v(s)$, the expected value is increasing in $s$. This is equivalent to saying that $v^{-1}(x) > s$, or our bet size is too small for the hand strength, so larger bets are more profitable.

We have shown that the expected value of betting is increasing for $s < v^{-1}(x)$, equal to 0 at $s = v^{-1}(x)$, and decreasing for $s > v^{-1}(x)$, so the expected value of betting is maximized at $s = v^{-1}(x)$. This means that the bettor cannot improve their payoff by deviating from this bet size. In particular, they cannot benefit by betting the minimum, which in turn proves that they cannot benefit by checking, as we showed above.

(e) $x_5 \leq x \leq 1$: These are value bets where the bettor should bet the maximum. We need to show that the bettor cannot improve their payoff by either checking or betting less.

The expected value of betting the maximum is

$$\mathbb{E}[\text{bet } U|x] = U(2x - c(U) - 1) + x.$$

Plugging $s = U$, $x = v(U) = x_5$ into 4, we get

$$-Uc'(U) - c(U) + 2x_5 - 1 = 0.$$

What happens when $x > x_5$? This expression must be greater than 0, meaning the expected value of betting is increasing in $s$ for $x > x_5$. If it is always increasing in $s$ for such $x$, then it must be maximized at the largest possible bet of $U$, so the bettor cannot improve their payoff by betting less.

We can use the exact same logic as the previous case to show that checking cannot improve the payoff either.

□

# C Symbolic Solution for Limit Continuous Poker

This notebook derives the Nash equilibrium of LCP using transformed coordinates: - $r = L/(1 + L)$ (minimum pot odds) - $t = 1/(1 + U)$ (pot fraction at max bet)

The solution provides closed-form expressions for all strategic components.

[35]:

```
import sympy as sp
import numpy as np
from sympy import symbols, Function, Eq, solve, diff, integrate, factor, lambdify
from typing import Dict
from dataclasses import dataclass
import game_utils.ContinuousPokerVariants.ContinuousPokerUtils as poker_utils
```

## C.1 Variable Definitions

[3]:

```
# Transformed parameters and bet size (s) and hand strength (x)
r, t, s, x = symbols('r t s x')

# Original parameters in terms of r, t
L_expr = r / (1 - r)
U_expr = (1 - t) / t

# Hand strength thresholds
x0, x1, x2, x3, x4, x5 = symbols('x0 x1 x2 x3 x4 x5')

# Strategy functions
c_func = Function('c')  # Calling threshold
v_func = Function('v')  # Value betting
b_func = Function('b')  # Bluffing

# Integration constant
b0 = symbols('b0')
```

## C.2 Calling Threshold c(s)

From bettor indifference at marginal bluffing hand $x_2$:

$$c(s) - (1 - c(s))s = x_2$$

[4]:

```
def derive_calling_threshold() -> sp.Expr:
    bettor_indiff_eq = Eq(c_func(s) - (1 - c_func(s)) * s, x2)
    c_solution = solve(bettor_indiff_eq, c_func(s))[0]
    return c_solution

c_expr = derive_calling_threshold()
print(" Derived c(s):")
display(Eq(c_func(s), c_expr))
```

```
Derived c(s):
```

$$c(s) = \frac{s + x_2}{s + 1}$$

## C.3 Value Betting Function v(s)

From first-order optimality, the bettor with hand $v(s)$ must be indifferent about bet size.

[5]:

```
def derive_value_function(c_expr: sp.Expr) -> sp.Expr:
    optimality_ode = Eq(-s * diff(c_expr, s) - c_expr + 2 * v_func(s) - 1, 0)
    v_solution = solve(optimality_ode, v_func(s))[0]
    return v_solution

v_expr = derive_value_function(c_expr)
print(" Derived v(s):")
display(Eq(v_func(s), v_expr))
```

```
Derived v(s):
```

$$v(s) = \frac{2s^2 + 4s + x_2 + 1}{2\left(s^2 + 2s + 1\right)}$$

## C.4 Bluffing Function b(s)

From caller indifference at threshold $c(s)$:

$$-b'(s)(1+s) = v'(s)s$$

[6]:

```
def derive_bluffing_function(v_expr: sp.Expr) -> sp.Expr:
    caller_indiff_ode = Eq(diff(b_func(s), s) * (1 + s) + diff(v_expr, s) * s, 0)
    b_solution = sp.dsolve(caller_indiff_ode, b_func(s))
    b_solution_expr = b_solution.rhs.subs("C1", b0)
    return b_solution_expr

b_expr = derive_bluffing_function(v_expr)
print(" Derived b(s):")
display(Eq(b_func(s), b_expr))
```

```
Derived b(s):
```

$$b(s) = b_0 - \frac{(3s+1)\left(x_2 - 1\right)}{6\left(s^3 + 3s^2 + 3s + 1\right)}$$

## C.5 Hand Strength Thresholds

Solve for $x_0, x_1, x_3, x_4, x_5$ using boundary conditions.

[7]:

```
print("Solving for hand strength thresholds...")

c_at_L = c_expr.subs(s, L_expr)
v_at_L = v_expr.subs(s, L_expr)
v_at_U = v_expr.subs(s, U_expr)

equations = [
    Eq(x2 - x1 - r * (x4 - x3), 0),
    Eq(x0 - (1 - x5) * (1 - t), 0),
    Eq(x3 - (1 + c_at_L) / 2, 0),
    Eq(v_at_L, x4),
    Eq(v_at_U, x5),
]

threshold_solution = sp.linsolve(equations, (x0, x1, x3, x4, x5))
threshold_tuple = list(threshold_solution)[0]

thresholds = {
    var: expr
    for var, expr in zip([x0, x1, x3, x4, x5], threshold_tuple)
}

print(" Solved for x0, x1, x3, x4, x5 in terms of x2\n")
for name in ['x0', 'x1', 'x3', 'x4', 'x5']:
    sym = symbols(name)
    display(Eq(sym, thresholds[sym]))
```

```
Solving for hand strength thresholds...
 Solved for x0, x1, x3, x4, x5 in terms of x2
```

$$x_0 = \frac{t^3 x_2}{2} - \frac{t^3}{2} - \frac{t^2 x_2}{2} + \frac{t^2}{2}$$

$$x_1 = -\frac{r^3 x_2}{2} + \frac{r^3}{2} + \frac{r^2 x_2}{2} - \frac{r^2}{2} + x_2$$

$$x_3 = -\frac{r x_2}{2} + \frac{r}{2} + \frac{x_2}{2} + \frac{1}{2}$$

$$x_4 = \frac{r^2 x_2}{2} - \frac{r^2}{2} - r x_2 + r + \frac{x_2}{2} + \frac{1}{2}$$
$$x_5 = \frac{t^2 x_2}{2} - \frac{t^2}{2} + 1$$

## C.6 Solve for x2 and b0

Using boundary conditions: $b(U) = x_0$ and $b(L) = x_1$.

[8]:

```
b_at_L = b_expr.subs(s, L_expr)
b_at_U = b_expr.subs(s, U_expr)

boundary_equations = [
    Eq(b_at_U, thresholds[x0]),
    Eq(b_at_L, thresholds[x1]),
]

b0_x2_solution = sp.linsolve(boundary_equations, (b0, x2))
b0_val, x2_val = list(b0_x2_solution)[0]

print(" Solved for x2 and b0:")
display(Eq(x2, x2_val))
display(Eq(b0, b0_val))
```

```
 Solved for x2 and b0:
```

$$x_2 = \frac{r^3 + t^3 - 1}{r^3 + t^3 - 7}$$
$$b_0 = \frac{t^3}{r^3 + t^3 - 7}$$

## C.7 Simplify Complete Solution

[23]:

```
print("Simplifying expressions...")
thresholds[x2] = x2_val
thresholds[b0] = b0_val

for key in [x0, x1, x3, x4, x5]:
    thresholds[key] = thresholds[key].subs(x2, x2_val).simplify()
```

```
c_expr_final = c_expr.subs(x2, x2_val).simplify()
v_expr_final = v_expr.subs(x2, x2_val).simplify()
b_expr_final = b_expr.subs(b0, b0_val).subs(x2, x2_val).simplify()
b_expr_final = ((sp.numer(b_expr_final) + (3*s+1)).factor() - (3*s+1))/ sp.denom(b

print(" Simplified all expressions")
```

```
Simplifying expressions...
 Simplified all expressions
```

## C.8 Inverse Value Function

[25]:

```
def derive_inverse_value_function(v_expr):
    v_inv_expr = -1 - sp.sqrt( (4*x-4) * (-2 + 2*x2) ) / (4*x-4)
    assert x == v_expr.subs(s, v_inv_expr).simplify()
    return v_inv_expr.subs(x2, x2_val).simplify()

v_inv_expr = derive_inverse_value_function(v_expr)
```

## C.9 Complete Solution

[26]:

```
@dataclass
class LCPSolution:
    thresholds: Dict[sp.Symbol, sp.Expr]
    c_expr: sp.Expr
    v_expr: sp.Expr
    b_expr: sp.Expr

    def display(self):
        print("=" * 70)
        print("LIMIT CONTINUOUS POKER - Nash Equilibrium Solution")
        print("=" * 70)
        print()

        print("Hand Strength Thresholds:")
        print("-" * 70)
        for name in ['x0', 'x1', 'x2', 'x3', 'x4', 'x5']:
            sym = symbols(name)
```

```
            expr = self.thresholds[sym]
            display(Eq(sym, expr))
        print()

        print("Strategy Functions:")
        print("-" * 70)
        display(Eq(c_func(s), self.c_expr))
        display(Eq(v_func(s), self.v_expr))
        display(Eq(b_func(s), self.b_expr))
        print()

    def to_latex(self) -> Dict[str, str]:
        latex_dict = {}
        for sym, expr in self.thresholds.items():
            latex_dict[str(sym)] = sp.latex(Eq(sym, expr))
        latex_dict['c(s)'] = sp.latex(Eq(c_func(s), self.c_expr))
        latex_dict['v(s)'] = sp.latex(Eq(v_func(s), self.v_expr))
        latex_dict['b(s)'] = sp.latex(Eq(b_func(s), self.b_expr))
        return latex_dict

solution = LCPSolution(
    thresholds=thresholds,
    c_expr=c_expr_final,
    v_expr=v_expr_final,
    b_expr=b_expr_final
)

solution.display()
```

```
======================================================================
LIMIT CONTINUOUS POKER - Nash Equilibrium Solution
======================================================================

Hand Strength Thresholds:
----------------------------------------------------------------------
```

$$x_0 = \frac{3t^2\left(t-1\right)}{r^3+t^3-7}$$

$$x_1 = \frac{-2r^3+3r^2+t^3-1}{r^3+t^3-7}$$

$$x_2 = \frac{r^3+t^3-1}{r^3+t^3-7}$$

$$x_3 = \frac{r^3 - 3r + t^3 - 4}{r^3 + t^3 - 7}$$
$$x_4 = \frac{r^3 + 3r^2 - 6r + t^3 - 4}{r^3 + t^3 - 7}$$
$$x_5 = \frac{r^3 + t^3 + 3t^2 - 7}{r^3 + t^3 - 7}$$

```
Strategy Functions:
----------------------------------------------------------------------
```

$$c(s) = \frac{r^3 + s\left(r^3 + t^3 - 7\right) + t^3 - 1}{(s+1)\left(r^3 + t^3 - 7\right)}$$
$$v(s) = \frac{r^3 + t^3 + \left(r^3 + t^3 - 7\right)\left(2s^2 + 4s + 1\right) - 1}{2\left(r^3 + t^3 - 7\right)\left(s^2 + 2s + 1\right)}$$
$$b(s) = \frac{-3s + t^3 (s+1)^3 - 1}{\left(r^3 + t^3 - 7\right)\left(s^3 + 3s^2 + 3s + 1\right)}$$

## C.10 Numerical Strategy Functions

[27]:

```
def _convert_params(**kwargs):
    if 'L' in kwargs and 'U' in kwargs:
        L_val = kwargs['L']
        U_val = kwargs['U']
        r_val = L_val / (1 + L_val)
        t_val = 1 / (1 + U_val)
        return r_val, t_val
    elif 'r' in kwargs and 't' in kwargs:
        return kwargs['r'], kwargs['t']
    else:
        raise ValueError("Must provide either (L, U) or (r, t) parameters")

def call_threshold(s_val, **kwargs):
    r_val, t_val = _convert_params(**kwargs)
    c_numeric = lambdify(s, solution.c_expr.subs({r: r_val, t: t_val}))
    return float(c_numeric(s_val))

def bluff_threshold(**kwargs):
    r_val, t_val = _convert_params(**kwargs)
```

```
    x2_expr = solution.thresholds[x2].subs({r: r_val, t: t_val})
    return float(x2_expr)

def value_threshold(**kwargs):
    r_val, t_val = _convert_params(**kwargs)
    x3_expr = solution.thresholds[x3].subs({r: r_val, t: t_val})
    return float(x3_expr)

def bluff_size(x_val, **kwargs):
    r_val, t_val = _convert_params(**kwargs)
    L_val = r_val / (1 - r_val)
    U_val = (1 - t_val) / t_val

    x0_val = solution.thresholds[x0].subs({r: r_val, t: t_val})
    x1_val = solution.thresholds[x1].subs({r: r_val, t: t_val})
    x2_val = solution.thresholds[x2].subs({r: r_val, t: t_val})
    b0_val = solution.thresholds[b0].subs({r: r_val, t: t_val})

    if x_val < x0_val:
        return U_val
    elif x_val < x1_val:
        from scipy.optimize import brentq
        b_substituted = solution.b_expr.subs({
            r: r_val,
            t: t_val,
            x2: x2_val,
            b0: b0_val
        })
        b_numeric = lambdify(s, b_substituted)
        try:
            result = brentq(lambda s_test: b_numeric(s_test) - x_val, L_val, U_val
            return result
        except:
            return None
    else:
        return L_val

def value_size(x_val, **kwargs):
    r_val, t_val = _convert_params(**kwargs)
    L_val = r_val / (1 - r_val)
```

```
    U_val = (1 - t_val) / t_val

    x4_val = solution.thresholds[x4].subs({r: r_val, t: t_val})
    x5_val = solution.thresholds[x5].subs({r: r_val, t: t_val})

    if x_val < x4_val:
        return L_val
    elif x_val < x5_val:
        vinv_numeric = lambdify(x, v_inv_expr.subs({r: r_val, t: t_val}))
        return float(vinv_numeric(x_val))
    else:
        return U_val
```

## C.11 LaTeX Output

[ ]:

```
print("=" * 70)
print("LaTeX Format:")
print("=" * 70)
latex_output = solution.to_latex()
for key, latex_str in latex_output.items():
    print(latex_str)
```

```
======================================================================
LaTeX Format:
======================================================================
x_{0} = \frac{3 t^{2} \left(t - 1\right)}{r^{3} + t^{3} - 7}
x_{1} = \frac{- 2 r^{3} + 3 r^{2} + t^{3} - 1}{r^{3} + t^{3} - 7}
x_{3} = \frac{r^{3} - 3 r + t^{3} - 4}{r^{3} + t^{3} - 7}
x_{4} = \frac{r^{3} + 3 r^{2} - 6 r + t^{3} - 4}{r^{3} + t^{3} - 7}
x_{5} = \frac{r^{3} + t^{3} + 3 t^{2} - 7}{r^{3} + t^{3} - 7}
x_{2} = \frac{r^{3} + t^{3} - 1}{r^{3} + t^{3} - 7}
b_{0} = \frac{t^{3}}{r^{3} + t^{3} - 7}
c{\left(s \right)} = \frac{r^{3} + s \left(r^{3} + t^{3} - 7\right) + t^{3} -
1}{\left(s + 1\right) \left(r^{3} + t^{3} - 7\right)}
v{\left(s \right)} = \frac{r^{3} + t^{3} + \left(r^{3} + t^{3} - 7\right)
\left(2 s^{2} + 4 s + 1\right) - 1}{2 \left(r^{3} + t^{3} - 7\right) \left(s^{2}
+ 2 s + 1\right)}
b{\left(s \right)} = \frac{- 3 s + t^{3} \left(s + 1\right)^{3} - 1}{\left(r^{3}
+ t^{3} - 7\right) \left(s^{3} + 3 s^{2} + 3 s + 1\right)}
```

## C.12 Game Value Computation

[59]:

```
def compute_game_value(solution: LCPSolution) -> sp.Expr:
    bluff_payoff = solution.thresholds[x2] - sp.Rational(1, 2)
    check_payoff = x - sp.Rational(1, 2)
    min_bet_payoff = (x * (2*L_expr + 1) - L_expr * (solution.c_expr.subs(s, L_exp
    max_bet_payoff = (x * (2*U_expr + 1) - U_expr * (solution.c_expr.subs(s, U_exp

    intermediate_bet_payoff = (x * (2*v_inv_expr + 1) - v_inv_expr * (solution.c_e
    q = sp.Symbol('q')
    q_expr = (x-1)/(r**3+t**3-7)
    intermediate_bet_payoff = intermediate_bet_payoff.subs(q_expr, q).collect(q).c

    bluff_integral = integrate(
        bluff_payoff,
        (x, 0, thresholds[x2])
    ).simplify()

    check_integral = integrate(
        check_payoff,
        (x, thresholds[x2], thresholds[x3])
    ).simplify()

    min_bet_integral = integrate(
        min_bet_payoff,
        (x, thresholds[x3], thresholds[x4])
    ).simplify()

    max_bet_integral = integrate(
        max_bet_payoff,
        (x, thresholds[x5], 1)
    ).simplify()

    intermediate_bet_integral = integrate(
        intermediate_bet_payoff,
        (x, thresholds[x4], thresholds[x5])
    ).simplify()

    game_value = bluff_integral + check_integral + min_bet_integral + max_bet_inte
```

```
    return game_value.simplify()

game_value = compute_game_value(solution)
display(game_value)
```

$$3\left(r\left(r^4-4r^3-6r^2+rt^3+8r-t^3+1\right)+t^2\left(-r^3-t^3+3t^2-18t+19\right)\right)\left(r^3+t^3-7\right)^2+\left(r^6+2\right.$$

## C.13 Simplification of Game Value to Closed Form

The game value expression above simplifies to:

$$V(r,t)=\frac{r^3+t^3-1}{2(r^3+t^3-7)}$$

**Proof sketch:**

1. **Substitution:** Let $u=r^3+t^3-7$ to simplify notation.
2. **Common denominator:** Note that $(r^3+t^3-7)^2=r^6+2r^3t^3+t^6-14r^3-14t^3+49$, so all terms can be written with denominator $2u^3$.
3. **Expand and collect:** Expand all products and collect terms over the common denominator $2u^3$. Use $t^3-(r-1)^3=t^3-r^3+3r^2-3r+1$ for the fourth term.
4. **Cancellation:** After expanding and collecting like powers of $u$:
   - All $u^3$ terms cancel
   - Most $u^2$ terms cancel, leaving only $u^2(u+6)=u^3+6u^2$ in the numerator
   - Lower order terms cancel
5. **Final simplification:**

$$\frac{u^3+6u^2}{2u^3}=\frac{u^2(u+6)}{2u^3}=\frac{u+6}{2u}=\frac{(r^3+t^3-7)+6}{2(r^3+t^3-7)}=\frac{r^3+t^3-1}{2(r^3+t^3-7)}$$

[ ]:

```
# sanity check - do the expressions agree on random inputs?
def numerical_eq(expr1, expr2, tolerance=1e-9, iters=1000):
    for i in range(iters):
        r_val = np.random.rand()
```

```
        t_val = np.random.rand()
        vals = {r: r_val, t: t_val}
        if r_val + t_val > 1:
            continue
        if np.abs(expr1.subs(vals) - expr2.subs(vals)) > tolerance:
            print(vals)
            print(expr1.subs(vals), expr2.subs(vals))
            return False
    return True

known_form = (1-r**3-t**3)/(14-2*r**3-2*t**3)
numerical_eq(known_form, game_value)
```

[ ]:

```
True
```

# D Parameter Analysis: Complete Proofs

This appendix provides detailed proofs of how the upper limit $U$ and lower limit $L$ affect the Nash equilibrium strategies and expected payoffs in Limit Continuous Poker.

## D.1 Effect of Increasing $U$

### D.1.1 Expected Payoff of Value-Betting Hands

It may seem unsurprising that strong hands become less likely to get called as limits increase, but what about the actual expected payoff of these hands? Does the expected value of a specific hand strength in the unique admissible Nash equilibrium continue increasing as we increase $U$? The answer is no. In fact, for any fixed hand strength $x$, the expected payoff of that hand increases in $U$ only up to a certain threshold, after which it decreases (see Figure 9). This feels counterintuitive; increasing $U$ only gives the bettor more options, so how is it possible that the expected payoff of individual hands decreases? And which hands are gaining expected payoff to offset this? This is a surprising result, and it is worth exploring in more detail.

**Theorem D.1.** *For any value-betting hand strength $x$ and any $L, U$, $\frac{d}{dU} EV(x) < 0$ if*

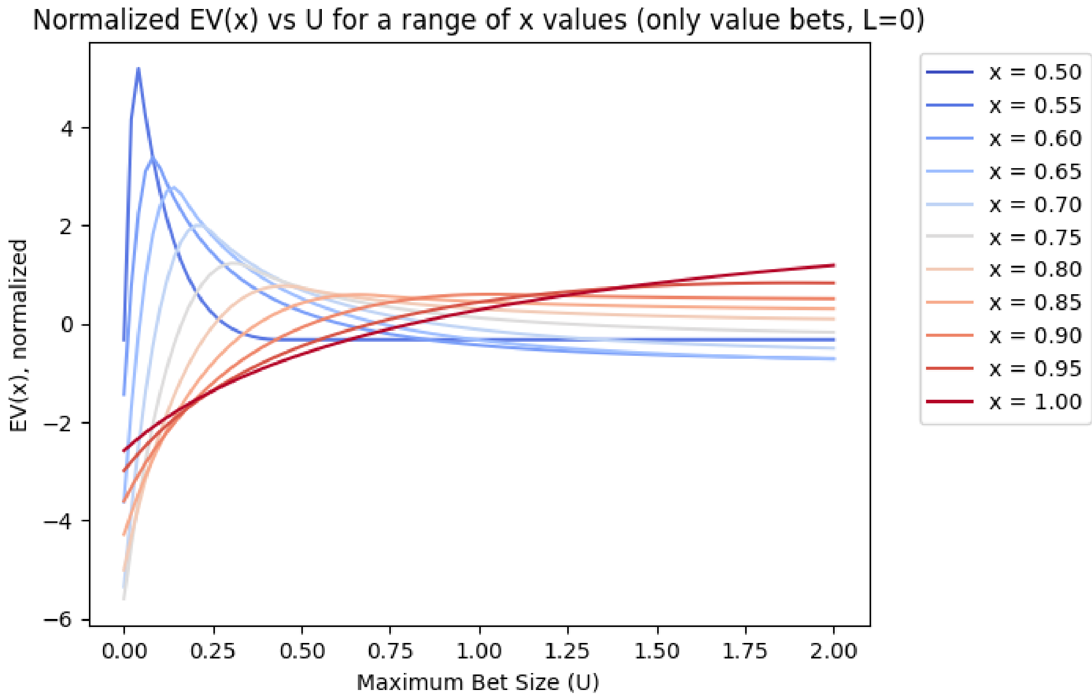


Figure 9: Expected value of a value-betting hand $x$ versus the upper limit $U$ in Nash equilibrium. Notice that each curve is increasing in $U$ up to some maximum, after which it decreases. Recall that $EV(x)$ denotes the expected payoff of a value-betting hand $x$ in the unique admissible Nash equilibrium.

$$x < \max\left(v(U), \frac{1}{2(1+U)}\left(U\frac{\partial x_2}{\partial U} + \frac{U^2 + x_2(1+2U)}{1+U}\right)\right),$$

*and* $\frac{d}{dU}EV(x) > 0$ *otherwise.*

To parse this in English: if we fix the bettor's hand strength $x$, the expected payoff to the bettor in the unique admissible Nash equilibrium is decreasing in $U$ if $x$ falls below a certain threshold, but increasing in $U$ if $x$ is above this threshold. Specifically, this threshold is greater than the hand strength $v(U)$ which bets the maximum.

Before proving the theorem, we will walk through some lemmas which explore how all the relevant variables change as we increase $U$, including the bluffing threshold $x_2$, the bet size $v^{-1}(x)$, and the calling cutoff $c(s)$.

### D.1.2 Bluffing Threshold

We begin by showing that $x_2$, the boundary hand strength between bluffing and checking, is increasing in $U$. This means that for fixed $L$, increasing the upper limit $U$ makes the bettor bluff with more hands.

**Lemma D.1.** *For any $L, U$,*

$$\frac{dx_2}{dU} > 0.$$

**Proof.** Recall that $x_2$ is given by:

$$x_2 = \frac{r^3 + t^3 - 1}{r^3 + t^3 - 7}$$

where $r = L/(1+L)$ and $t = 1/(1+U)$. We can use the chain rule to differentiate $x_2$ with respect to $U$:

$$\frac{dx_2}{dU} = \frac{\partial x_2}{\partial t}\frac{dt}{dU}.$$

Note that $r$ has no dependence on $U$. We compute:

$$\frac{\partial x_2}{\partial t} = \frac{-18t^2}{(r^3 + t^3 - 7)^2}, \quad \frac{dt}{dU} = -\frac{1}{(1+U)^2}.$$

Therefore,

$$\frac{dx_2}{dU} = \frac{-18t^2}{(r^3 + t^3 - 7)^2} \cdot \left(-\frac{1}{(1+U)^2}\right) = \frac{18t^2}{(1+U)^2\,(r^3 + t^3 - 7)^2} > 0,$$

which is positive since $r, t \in (0, 1)$ implies $r^3 + t^3 - 7 < 0$.

□

### D.1.3 Bet Size

We now show that if we fix $x$ at any intermediate value-betting hand strength (betting neither the minimum nor maximum bet size) and then increase $U$, the bet size $s$ made by $x$ decreases. The intermediate value-betting hands are exactly $x \in [x_3, v(U)]$ and their bet sizes are given by $s = v^{-1}(x)$, so we get the following lemma:

**Lemma D.2.** *For any fixed $x \in [x_3, v(U)]$,*

$$\frac{d}{dU}v^{-1}(x) < 0$$

**Proof.** Recall that

$$v^{-1}(x) = -\frac{\sqrt{(4x-4)(2x_2-2)}}{4x-4} - 1,$$

where $x_2 = \frac{r^3+t^3-1}{r^3+t^3-7}$ with $t = 1/(1+U)$. Importantly, $v^{-1}(x)$ is only dependent on $U$ through $x_2$, which in turn depends on $U$ only through $t$. Using the chain rule:

$$\frac{d}{dU}v^{-1}(x) = \frac{\partial v^{-1}(x)}{\partial x_2}\frac{\partial x_2}{\partial t}\frac{dt}{dU}$$

We compute each factor:

$$\frac{\partial v^{-1}(x)}{\partial x_2} = -\frac{1}{\sqrt{(4x-4)(2x_2-2)}} = -\frac{1}{(v^{-1}(x)+1)(4-4x)} < 0$$

which is negative since $x \in [0,1]$ and $v^{-1}(x) > 0$. From Lemma D.1, we know that $\frac{\partial x_2}{\partial t}\frac{dt}{dU} > 0$.

Therefore, the product of the three terms is always negative, so the bet size of intermediate bets is decreasing in $U$. □

### D.1.4 Calling Cutoff

Recall that $c(s)$ is defined as the minimum hand strength $y$ which should call a bet of size $s$ and is given in Nash equilibrium by:

$$c(s) = \frac{x_2+s}{s+1}$$

We are specifically interested in how $c(v^{-1}(x))$ varies with $U$ for $x \in [x_3, v(U)]$, since this represents how the calling cutoff changes both directly from a strategic change, as well as indirectly due to the lower bet size $s$. It turns out that the calling cutoff is increasing in $U$ for all $x \in [x_3, v(U)]$. This is surprising because we just showed that the bet size $s$ is decreasing in $U$, and we expect smaller bets to be called more often. For reasons we will see later, this effect is overpowered by a strategic shift for the caller, who calls less often for all bet sizes as $U$ increases.

**Lemma D.3.** *For any fixed $x \in [x_3, v(U)]$,*

$$\frac{d}{dU}c(v^{-1}(x)) > 0$$

**Proof.** As mentioned above, $c(s) = \frac{x_2+s}{s+1}$ is dependent on $U$ in two distinct ways: directly through $x_2 = x_2(t)$ where $t = 1/(1+U)$, and indirectly through the bet size $s = v^{-1}(x)$. We use the multivariate chain rule:

$$\frac{d}{dU}c(v^{-1}(x)) = \frac{\partial c(s)}{\partial s}\frac{dv^{-1}(x)}{dU} + \frac{\partial c(s)}{\partial x_2}\frac{\partial x_2}{\partial t}\frac{dt}{dU}$$

The partial derivatives of $c(s)$ are:

$$\frac{\partial c(s)}{\partial s} = \frac{1-x_2}{(s+1)^2} \quad \text{and} \quad \frac{\partial c(s)}{\partial x_2} = \frac{1}{s+1}.$$

Substituting $s = v^{-1}(x)$, we can simplify the first term using the fact that $(v^{-1}(x)+1)^2 = (1-x_2)/(2-2x)$:

$$\left.\frac{\partial c(s)}{\partial s}\right|_{s=v^{-1}(x)} = \frac{1-x_2}{(v^{-1}(x)+1)^2} = 2-2x$$

From Lemma D.2, we have:

$$\frac{dv^{-1}(x)}{dU} = \frac{\partial v^{-1}(x)}{\partial x_2}\frac{\partial x_2}{\partial t}\frac{dt}{dU} = \frac{-1}{(v^{-1}(x)+1)(4-4x)}\frac{\partial x_2}{\partial t}\frac{dt}{dU}.$$

Substituting everything:

$$\begin{aligned}
\frac{d}{dU}c(v^{-1}(x)) &= (2-2x)\cdot\frac{-1}{(v^{-1}(x)+1)(4-4x)}\cdot\frac{\partial x_2}{\partial t}\frac{dt}{dU} + \frac{1}{v^{-1}(x)+1}\cdot\frac{\partial x_2}{\partial t}\frac{dt}{dU} \\
&= \frac{1}{v^{-1}(x)+1}\cdot\frac{\partial x_2}{\partial t}\frac{dt}{dU}\cdot\left(-\frac{2-2x}{4-4x}+1\right) \\
&= \frac{1}{v^{-1}(x)+1}\cdot\frac{\partial x_2}{\partial t}\frac{dt}{dU}\cdot\frac{1}{2}
\end{aligned}$$

All three factors are positive: $v^{-1}(x) > 0$, and by Lemma D.1, $\frac{\partial x_2}{\partial t}\frac{dt}{dU} > 0$. Therefore, the calling cutoff is increasing in $U$ for all $x \in [x_3, v(U)]$.

□

### D.1.5 Proof of Theorem D.1

Having these tools, we can now finally return to the proof of Theorem D.1.
**Proof.** Recall the expected payoff of a value-betting hand $x$:

$$EV(x) = \frac{1}{2}c(s) + (x - c(s))\left(s + \frac{1}{2}\right) + (1 - x)\left(-s - \frac{1}{2}\right)$$

We break the proof into two cases:

**Case 1 ($x > v(U)$):** In this case, hand $x$ bets the maximum amount $U$. Recall that $c(s)$ is implicitly a function of $s$ and $x_2$, which is itself a function of $U$. Using the multivariate chain rule, the derivative at $s = U$ is:

$$\begin{aligned}
\frac{d}{dU}EV(x) &= \frac{\partial EV(x)}{\partial s}\bigg|_{s=U} + \frac{\partial EV(x)}{\partial c(s)}\left(\frac{\partial c(s)}{\partial s} + \frac{\partial c(s)}{\partial x_2}\frac{\partial x_2}{\partial U}\right)\bigg|_{s=U} \\
&= \left(\frac{\partial EV(x)}{\partial s} + \frac{\partial EV(x)}{\partial c(s)}\frac{\partial c(s)}{\partial s}\right)\bigg|_{s=U} + \left(\frac{\partial EV(x)}{\partial c(s)}\frac{\partial c(s)}{\partial x_2}\frac{\partial x_2}{\partial U}\right)\bigg|_{s=U}
\end{aligned}$$

We want to know exactly when the above expression is positive. The partial derivatives to plug in are:

$$\begin{aligned}
\frac{\partial EV(x)}{\partial s} &= 2x - 1 - c(s) \\
\frac{\partial EV(x)}{\partial c(s)} &= -s \\
\frac{\partial c(s)}{\partial s} &= \frac{1 - x_2}{(s+1)^2} \\
\frac{\partial c(s)}{\partial x_2} &= \frac{1}{s+1}
\end{aligned}$$

We can leave $\frac{\partial x_2}{\partial U}$ as a free variable for now, since it is always positive by Lemma D.1, and is independent of $x$. Plugging these in and rearranging terms, we can say that $EV(x)$ is increasing in $U$ if

$$x > \frac{1}{2(1+U)}\left(U\frac{\partial x_2}{\partial U} + \frac{U^2 + x_2(1+2U)}{1+U}\right)$$

Where nothing on the right hand side is dependent on $x$. This means that for any fixed $L, U$, this gives a threshold value for $x$ below which $EV(x)$ is decreasing in $U$, and above which it is increasing in $U$.

**Case 2 ($x_3 < x < v(U)$):** In this case, hand $x$ makes an intermediate-sized bet $s = v^{-1}(x)$. There are two distinct factors influencing the derivative $\frac{d}{dU}EV(x)$, namely the change in bet size $s = v^{-1}(x)$ and the change in calling cutoff $c(v^{-1}(x))$. By the multivariate chain rule, we can express the derivative as:

$$\frac{d}{dU}EV(x) = \frac{\partial EV(x)}{\partial s}\frac{dv^{-1}(x)}{dU} + \frac{\partial EV(x)}{\partial c(s)}\frac{dc(v^{-1}(x))}{\partial U}$$

The partial derivatives of $EV(x)$ are:

$$\frac{\partial EV(x)}{\partial s} = 2x - 1 - c(s)$$
$$\frac{\partial EV(x)}{\partial c(s)} = -s$$

The second is clearly negative. We can verify that the first must be positive if we go back to the constraints which gave us the Nash equilibrium. For the bet size $v^{-1}(x)$ to be optimal, we required that

$$-s\frac{\partial c(s)}{\partial s} - c(s) + 2v(s) - 1 = 0,$$

or equivalently, if we substitute $s = v^{-1}(x)$ and $v(s) = x$ and rearrange:

$$2x - 1 - c(v^{-1}(x)) = v^{-1}(x)\frac{\partial c(s)}{\partial s} > 0,$$

since $\frac{\partial c(s)}{\partial s} = \frac{1-x_2}{(s+1)^2} > 0$, and $s$ is positive by definition.

We know from Lemma D.2 that $\frac{dv^{-1}(x)}{dU} < 0$ and from Lemma D.3 that $\frac{dc(v^{-1}(x))}{\partial U} > 0$.

Combining everything, we see that both terms in $\frac{d}{dU}EV(x)$ are products of negative and positive, making both terms negative. Therefore, the expected payoff is decreasing in $U$ for all $x \in [x_3, v(U)]$.

□

# E Payoff Analysis: Complete Proofs

This appendix provides detailed proofs regarding the expected payoffs and values of different hand strengths in the Nash equilibrium of Limit Continuous Poker.

## E.1 Expected Value of Bettor's Hand Strengths

In addition to considering payoffs for a specific bettor-caller hand combination, we can also consider the expected value of a given hand for the bettor, not knowing the hand of the caller.

**Theorem E.1.** *Let $EV(x)$ denote the expected value of a value-betting hand $x$ in the unique admissible Nash equilibrium:*

$$EV(x) = \begin{cases} x_2 - \frac{1}{2} & \text{if } x \leq x_2 \\ x - \frac{1}{2} & \text{if } x_2 < x \leq x_3 \\ x(2L+1) - L(c(L)+1) - \frac{1}{2} & \text{if } x_3 < x < v(L) \\ x(2v^{-1}(x)+1) - v^{-1}(x)(c(v^{-1}(x))+1) - \frac{1}{2} & \text{if } v(L) \leq x \leq v(U) \\ x(2U+1) - U(c(U)+1) - \frac{1}{2} & \text{if } x > v(U). \end{cases} \tag{20}$$

**Proof.** For bluffing hands $x \leq x_2$, we can use a simple argument to show that the hand strength $x$ is actually irrelevant. The bettor never gets called by worse hands, so either the caller folds or calls with the best hand. In either case, the bettor's payoff has no dependence on $x$, so the payoff must be the same for all $x \leq x_2$ (otherwise, the lower-payoff hands would imitate the strategy of the higher-payoff hands). We know that at $x = x_2$, the bettor is indifferent between bluffing and checking, so the payoff must be $x_2 - \frac{1}{2}$ for all $x \leq x_2$.

For checking hands $x_2 \leq x \leq x_3$, the bettor wins only the ante exactly when they have the best hand, which happens with probability $x$. The value of the ante is $\frac{1}{2}$, so the bettor's expected value is $x - \frac{1}{2}$.

For any value betting hand, the bettor has three cases to consider: the caller folds, the caller calls with a worse hand, or the caller calls with the best hand. We simply sum the expected value of each of these cases.

$$EV(x) = \frac{1}{2}c(s) + (x - c(s))\left(s + \frac{1}{2}\right) + (1-x)\left(-s - \frac{1}{2}\right)$$

The last three cases come from substituting $L, v^{-1}(x), U$ for $s$ in the expression above and simplifying. $\square$

## E.2 Monotonicity of Expected Value

We can quickly verify that the bettor's expected value is increasing in $x$. This must be the case, since with any given hand strength, the bettor can always choose to imitate the Nash equilibrium strategy of a weaker hand, so the stronger hand must be at least as good in expectation.

**Theorem E.2.** *For any fixed $L, U$, the bettor's expected value $EV(x)$ is increasing in $x$.*

**Proof.** It is clear from inspection that any checking EV is higher than that of a bluff and that the checking EV is increasing in $x$. We also know that at $x = x_3$, the bettor is indifferent between checking and betting, so the EV of checking and betting must be equal at this point. Therefore, we only need to show that within the checking and value betting regions, the EV is increasing in $x$. This is obvious for checking hands. For value betting hands, we can take the derivative of the expression for $EV(x)$ with respect to $x$ and show that it is always positive. We consider the max and min betting hands first:

$$s = U \implies \frac{d}{dx}EV(x) = 2U + 1 > 0$$
$$s = L \implies \frac{d}{dx}EV(x) = 2L + 1 > 0$$

For intermediate value betting hands, we can use the chain rule to show that the EV is increasing in $x$:

$$\frac{d}{dx}EV(x) = \frac{\partial EV(x)}{\partial x} + \frac{dEV(x)}{ds}\frac{ds}{dx}$$

From the value optimality condition (equation 4 in the main text), we know that $\frac{dEV(x)}{ds} = 0$ (otherwise, the bettor could gain value by varying the bet size). Clearly, $\frac{\partial EV(x)}{\partial x} > 0$. Therefore, the derivative is positive, and the EV is increasing in $x$ for all value betting hands. $\square$

## E.3 Discussion: Strong Hands and Risk-Reward Tradeoffs

It is worth noting that the bettor's strongest hands (right edge) actually seem to become less likely to make any profit more than the ante as limits increase. These strongest hands make very large bets, which force all but the strongest hands to fold, but win huge pots when they do get called.

In more complicated poker variants, it is common to "slowplay" strong hands by checking or making small bets to induce bluffs from the opponent. In LCP, there is only one betting round and the caller is not allowed to raise, both of which make slowplaying obsolete. With extremely strong hands, the benefit of winning a large pot when betting big outweighs the lower likelihood of getting called.

This strategic pattern demonstrates a fundamental tension in poker: extracting maximum value from strong hands requires finding the optimal balance between bet size (which determines pot size when called) and calling frequency (which decreases as bet size increases). In LCP's Nash equilibrium, the strongest hands resolve this tension by accepting a lower calling frequency in exchange for winning much larger pots.

# References


[1] Bill Chen and Jerrod Ankenman. *The Mathematics of Poker*. Conjelco, Pittsburgh, PA, 2006.

[2] Chris Ferguson and Thomas Ferguson. On the borel and von neumann poker models. *Game Theory and Applications*, 9:2, 01 2003.

[3] W. Sierpiński. Sur une mesure non atomique. *Fundamenta Mathematicae*, 1922. Classical result: for every measurable set $X$ and $a \in [0, \mu(X)]$, there exists a subset $A \subseteq X$ with $\mu(A) = a$.