1 Introduction

1.1 Outline

This paper presents a comprehensive analysis of Limit Continuous Poker (LCP), a simplified poker variant. We begin by reviewing essential poker concepts and previous work on continuous poker variants, establishing the theoretical foundation for our analysis of LCP. After introducing the mathematical framework, we formally define Limit Continuous Poker, explaining its rules, strategic elements, and how it differs from existing variants through the addition of limited bet sizes.

Poker is a notoriously complex game, with many different variants and strategic elements. Simplified variants are used to study specific aspects of the game, allowing us to analytically solve for Nash equilibria and attempt to generalize these results to more complex variants. One such simplified variant is Continuous Poker, which is a two-player zero-sum game designed to isolate the strategic elements of bluffing and value betting. As the name implies, Continuous Poker abstracts poker hands to continuous numerical hand strengths and restricts the game to a single street of betting. We will discuss two continuous poker variants in depth: Fixed-Bet Continuous Poker (FBCP) and No-Limit Continuous Poker (NLCP), which differ in how the betting player is allowed to choose bet sizes. We then introduce Limit Continuous Poker (LCP), which generalizes these two variants by imposing upper and lower limits of the bet size.

The core of our analysis focuses on deriving and characterizing the Nash equilibrium strategies for LCP. We examine the structure of optimal play, determine a criteria for distinguishing multiple Nash equilibria, and solve for a unique Nash equilibrium strategy profile which fits this criteria. We then explore the payoffs to each player in this strategy profile and prove several lemmas about how payoffs vary with hand strength combinations.

The paper then focuses on the bet size limits as parameters. We discuss the effects of varying these parameters on the Nash equilibrium strategies and payoffs. We carefully walk through the ways in which varying parameters indirectly affect each player's optimal strategy and payoffs, and attempt to intuitively explain these effects.

Next, we return to the two motivating poker variants, FBCP and NLCP. We show that taking limiting values of the parameters LCP gives us the Nash equilibria of these two variants in an elegant way.

We end by solving for the value of LCP, which turns out to have a surprisingly elegant formula and several nice properties. We prove monotonicity of the game value with respect to the limit parameters, and show that the value converges to that of the two other poker variants, just like the optimal strategies. This leads to a beautiful symmetry property of the game value.

Throughout the paper, we employ both analytical techniques and numerical methods to provide a complete understanding of LCP's strategic landscape. Our analysis reveals that LCP exhibits unique strategic properties that make it an interesting intermediate case between the fixed-bet and no-limit variants, offering new insights into the relationship between betting flexibility and strategic complexity in continuous poker games.

1.2 Previous Work

1.2.1 Fixed-Bet Continuous Poker (FBCP)

Continuous Poker (also called Von Neumann Poker, and referred to in this paper as Fixed-Bet Continuous Poker or FBCP) is a simplified model of poker. It is a two-player zero-sum game designed to study strategic decision-making in competitive environments. The game abstracts away many complexities of real poker, focusing instead on the mathematical and strategic aspects of bluffing, betting, and optimal play.

Definition 1.1 (FBCP)**.** Two players, referred to as the bettor and the caller, each put a 0.5 unit ante into a pot[1]. They are each dealt a hand strength between 0 an 1 (referred to as x for bettor and y for caller). After seeing x, the bettor can either check - in which case, the higher hand between x and y wins the pot of 1 and the game ends - or they can bet, by putting a pre-determined amount B into the pot. The caller must now either call by matching the bet of B units, after which the higher hand wins the pot of $1 + 2B$, or fold, conceding the pot of $1 + B$ to the bettor and ending the game.

FBCP has many Nash equilibria, but it has a unique one in which the caller plays an admissible strategy[2], as shown by Ferguson and Ferguson [**?**, p. 2]. This strategy profile, parametrized by the bet size B, is as follows:

The bettor bets with hands x such that either

$$x > \frac{1 + 4B + 2B^2}{(1 + 2B)(2 + B)} \text{ or } x < \frac{B}{(1 + 2B)(2 + B)}.$$

We call the higher interval the value betting range and the lower interval the bluffing range. The caller calls with hands y above a calling threshold:

[1] An ante of 1 is often used, but since the pot size is the more relevant value, we use an ante of 0.5. All bet sizes scale proportionally.

[2] An admissible strategy is one which is not weakly dominated by any other strategy.

$$y > \frac{B(3+2B)}{(1+2B)(2+B)}.$$

Note that uniqueness in this context ignores the strictness of inequalities, since the endpoints of intervals occur with probability 0. The non-uniqueness of this Nash Equilibrium is due to the fact that given the bettor's strategy, the caller has many optimal responses. The caller must always fold with hands below the bluffing threshold, and must always call with hands above the value betting threshold, but with hands inbetween, they are indifferent between calling and folding. This is because with a hand strength in this range, the caller wins if and only if the bettor is bluffing, so their actual hand strength is irrelevant as long as it beats the bluffing threshold. To prevent the betting player from exploiting them, the caller need only call with exactly the right proportion of hands in this range. For example, the caller could take the strategy described above, but swap some calling and folding hands in the range between the bluffing and value betting thresholds.

Why is this Nash equilibrium special? We mentioned above that it is admissible, meaning that both players' strategies are not weakly dominated by any other strategy. Importantly, the caller's strategy is not weakly dominated. The same cannot be said for other Nash equilibria like the one described in the previous paragraph, for reasons that are beyond the scope of this introduction but relate to themes in section **??**.

FBCP also has a unique value as a function of the bet size B. The value of the game for the bettor is

$$V_{FB}(B) = \frac{B}{2(1+2B)(2+B)},$$

which is positive (advantageous to the bettor) and maximized at $B = 1$, when the bet size is exactly the pot size. It should not be surprising that the value is positive - at worst, the bettor can always check and turn the game into a coin flip, so the bettor will only deviate from this strategy if they have a positive expected value. The fact that $B = 1$ exactly maximizes the value is more subtle, but we should expect that some such maximal value of B exists. Forcing the bettor to bet too large relative to the pot would make betting too risky with most hands, and making the bet too small would simply give less profit to the bettor when they win a bet. As we will see later, part of the motivation for studying LCP is to understand this concept more generally.

1.2.2 No-Limit Continuous Poker (NLCP)

Another continuous poker variant allows the bettor to choose a bet size $s > 0$ after seeing their hand strength, as opposed to a fixed bet size B. This variant is called No-Limit Continuous Poker (or Newman Poker after Donald J. Newman, or NLCP in this paper). The Nash equilibrium strategy profile for this variant is discussed and solved by Bill Chen and Jerrod Ankenman [**?**, p. 154].

In Nash Equilibrium, the bettor should make large bets with their strongest and weakest hands and smaller bets or checks with their intermediate hands. It turns out that the optimal strategy is most elegantly described by a mapping from bet sizes s to hand strengths x for bluffing and value betting, respectively[3]. The caller simply has a calling threshold $c(s)$ for each possible bet size s. The full strategy profile is as follows:

The bettor bets s with hands x such that either

$$x = \frac{3s+1}{7(s+1)^3} \text{ or } x = 1 - \frac{3}{7(s+1)^2},$$

where the first condition represents bluffing hands and the second value betting hands. After seeing a bet of size s, the caller should call with hands y such that

$$y > 1 - \frac{6}{7(s+1)}.$$

See Figure 1 for a graphical representation of the strategy profile.

Note that the bettor uses all possible bet sizes and has exactly two hand strengths for each bet size[4]. On first inspection, this feels like the bettor is giving away too much information, but it turns out to still an optimal strategy. This concept appears again and is explained more thoroughly in section **??**.

The value of NLCP is

$$V_{NL} = \frac{1}{14},$$

for the bettor[5]. Thus, NLCP is again advantageous to the bettor. In fact, one can easily verify that NLCP is more advantageous to the bettor

[3]This feels backwards - mapping hand strengths to bet sizes would be more natural, but the math is more elegant this way.

[4]Seen visually in Figure 1 by the fact that a horizontal line intersects the bet function at exactly two points.

[5]Would be 1/7 for an ante of 1, but the value is halved with an ante of 0.5.

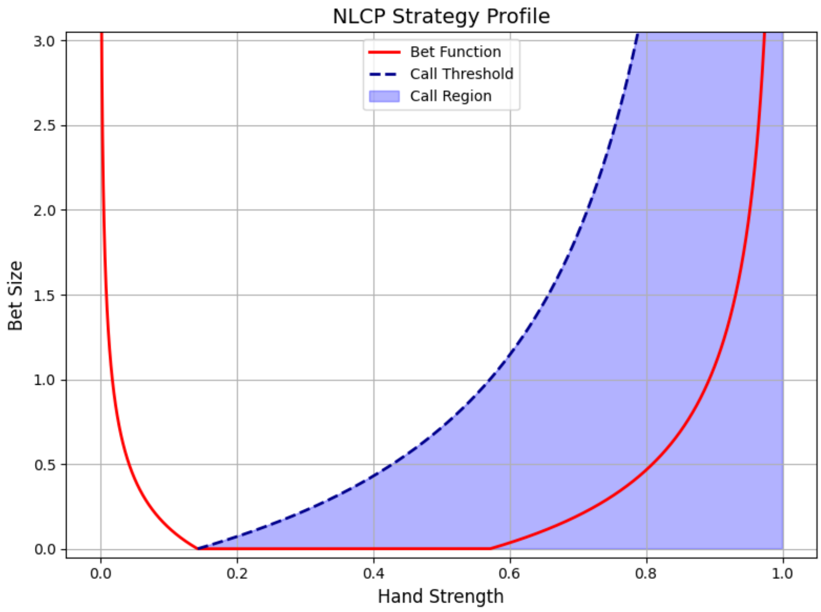

Figure 1: NLCP Strategy Profile

than FBCP for any bet size B by arguing that the bettor could artificially restrict themselves to a single bet size and achieve the same value as the bettor in FBCP.