1 Symbolic Solution for Limit Continuous Poker

This notebook derives the Nash equilibrium of LCP using transformed coordinates: - $r = L/(1+L)$ (minimum pot odds) - $t = 1/(1+U)$ (pot fraction at max bet)

The solution provides closed-form expressions for all strategic components.

[35]:
```
import sympy as sp
import numpy as np
from sympy import symbols, Function, Eq, solve, diff, integrate,␣
↪factor, lambdify
from typing import Dict
from dataclasses import dataclass
import game_utils.ContinuousPokerVariants.ContinuousPokerUtils as␣
↪poker_utils
```

1.1 Variable Definitions

[3]:
```
# Transformed parameters and bet size (s) and hand strength (x)
r, t, s, x = symbols('r t s x')

# Original parameters in terms of r, t
L_expr = r / (1 - r)
U_expr = (1 - t) / t

# Hand strength thresholds
x0, x1, x2, x3, x4, x5 = symbols('x0 x1 x2 x3 x4 x5')

# Strategy functions
c_func = Function('c')  # Calling threshold
v_func = Function('v')  # Value betting
b_func = Function('b')  # Bluffing

# Integration constant
b0 = symbols('b0')
```

1.2 Calling Threshold c(s)

From bettor indifference at marginal bluffing hand x_2:

$$c(s) - (1 - c(s))s = x_2$$

[4]:
```
def derive_calling_threshold() -> sp.Expr:
    bettor_indiff_eq = Eq(c_func(s) - (1 - c_func(s)) * s, x2)
    c_solution = solve(bettor_indiff_eq, c_func(s))[0]
    return c_solution
```

```
c_expr = derive_calling_threshold()
print(" Derived c(s):")
display(Eq(c_func(s), c_expr))
```

```
Derived c(s):
```

$$c(s) = \frac{s + x_2}{s + 1}$$

1.3 Value Betting Function v(s)

From first-order optimality, the bettor with hand $v(s)$ must be indifferent about bet size.

[5]:
```
def derive_value_function(c_expr: sp.Expr) -> sp.Expr:
    optimality_ode = Eq(-s * diff(c_expr, s) - c_expr + 2 * v_func(s) -␣
↪1, 0)
    v_solution = solve(optimality_ode, v_func(s))[0]
    return v_solution

v_expr = derive_value_function(c_expr)
print(" Derived v(s):")
display(Eq(v_func(s), v_expr))
```

```
Derived v(s):
```

$$v(s) = \frac{2s^2 + 4s + x_2 + 1}{2\left(s^2 + 2s + 1\right)}$$

1.4 Bluffing Function b(s)

From caller indifference at threshold $c(s)$:

$$-b'(s)(1 + s) = v'(s)s$$

[6]:
```
def derive_bluffing_function(v_expr: sp.Expr) -> sp.Expr:
    caller_indiff_ode = Eq(diff(b_func(s), s) * (1 + s) + diff(v_expr,␣
↪s) * s, 0)
    b_solution = sp.dsolve(caller_indiff_ode, b_func(s))
    b_solution_expr = b_solution.rhs.subs("C1", b0)
    return b_solution_expr

b_expr = derive_bluffing_function(v_expr)
print(" Derived b(s):")
display(Eq(b_func(s), b_expr))
```

```
Derived b(s):
```

$$b(s) = b_0 - \frac{(3s + 1)\left(x_2 - 1\right)}{6\left(s^3 + 3s^2 + 3s + 1\right)}$$

1.5 Hand Strength Thresholds

Solve for x_0, x_1, x_3, x_4, x_5 using boundary conditions.

[7]:
```
print("Solving for hand strength thresholds...")

c_at_L = c_expr.subs(s, L_expr)
v_at_L = v_expr.subs(s, L_expr)
v_at_U = v_expr.subs(s, U_expr)

equations = [
    Eq(x2 - x1 - r * (x4 - x3), 0),
    Eq(x0 - (1 - x5) * (1 - t), 0),
    Eq(x3 - (1 + c_at_L) / 2, 0),
    Eq(v_at_L, x4),
    Eq(v_at_U, x5),
]

threshold_solution = sp.linsolve(equations, (x0, x1, x3, x4, x5))
threshold_tuple = list(threshold_solution)[0]

thresholds = {
    var: expr
    for var, expr in zip([x0, x1, x3, x4, x5], threshold_tuple)
}

print(" Solved for x0, x1, x3, x4, x5 in terms of x2\n")
for name in ['x0', 'x1', 'x3', 'x4', 'x5']:
    sym = symbols(name)
    display(Eq(sym, thresholds[sym]))
```

```
Solving for hand strength thresholds...
 Solved for x0, x1, x3, x4, x5 in terms of x2
```

$$x_0 = \frac{t^3 x_2}{2} - \frac{t^3}{2} - \frac{t^2 x_2}{2} + \frac{t^2}{2}$$

$$x_1 = -\frac{r^3 x_2}{2} + \frac{r^3}{2} + \frac{r^2 x_2}{2} - \frac{r^2}{2} + x_2$$

$$x_3 = -\frac{r x_2}{2} + \frac{r}{2} + \frac{x_2}{2} + \frac{1}{2}$$

$$x_4 = \frac{r^2 x_2}{2} - \frac{r^2}{2} - r x_2 + r + \frac{x_2}{2} + \frac{1}{2}$$

$$x_5 = \frac{t^2 x_2}{2} - \frac{t^2}{2} + 1$$

1.6 Solve for x2 and b0

Using boundary conditions: $b(U) = x_0$ and $b(L) = x_1$.

[8]:
```
b_at_L = b_expr.subs(s, L_expr)
b_at_U = b_expr.subs(s, U_expr)

boundary_equations = [
    Eq(b_at_U, thresholds[x0]),
    Eq(b_at_L, thresholds[x1]),
]

b0_x2_solution = sp.linsolve(boundary_equations, (b0, x2))
b0_val, x2_val = list(b0_x2_solution)[0]

print(" Solved for x2 and b0:")
display(Eq(x2, x2_val))
display(Eq(b0, b0_val))
```

```
 Solved for x2 and b0:
```

$$x_2 = \frac{r^3 + t^3 - 1}{r^3 + t^3 - 7}$$

$$b_0 = \frac{t^3}{r^3 + t^3 - 7}$$

1.7 Simplify Complete Solution

[23]:
```
print("Simplifying expressions...")
thresholds[x2] = x2_val
thresholds[b0] = b0_val

for key in [x0, x1, x3, x4, x5]:
    thresholds[key] = thresholds[key].subs(x2, x2_val).simplify()

c_expr_final = c_expr.subs(x2, x2_val).simplify()
v_expr_final = v_expr.subs(x2, x2_val).simplify()
b_expr_final = b_expr.subs(b0, b0_val).subs(x2, x2_val).simplify()
b_expr_final = ((sp.numer(b_expr_final) + (3*s+1)).factor() - (3*s+1))/
 ↪sp.denom(b_expr_final)

print(" Simplified all expressions")
```

```
Simplifying expressions...
 Simplified all expressions
```

1.8 Inverse Value Function

[25]:
```
def derive_inverse_value_function(v_expr):
    v_inv_expr = -1 - sp.sqrt( (4*x-4) * (-2 + 2*x2) ) / (4*x-4)
    assert x == v_expr.subs(s, v_inv_expr).simplify()
    return v_inv_expr.subs(x2, x2_val).simplify()

v_inv_expr = derive_inverse_value_function(v_expr)
```

1.9 Complete Solution

[26]:
```
@dataclass
class LCPSolution:
    thresholds: Dict[sp.Symbol, sp.Expr]
    c_expr: sp.Expr
    v_expr: sp.Expr
    b_expr: sp.Expr

    def display(self):
        print("=" * 70)
        print("LIMIT CONTINUOUS POKER - Nash Equilibrium Solution")
        print("=" * 70)
        print()

        print("Hand Strength Thresholds:")
        print("-" * 70)
        for name in ['x0', 'x1', 'x2', 'x3', 'x4', 'x5']:
            sym = symbols(name)
            expr = self.thresholds[sym]
            display(Eq(sym, expr))
        print()

        print("Strategy Functions:")
        print("-" * 70)
        display(Eq(c_func(s), self.c_expr))
        display(Eq(v_func(s), self.v_expr))
        display(Eq(b_func(s), self.b_expr))
        print()

    def to_latex(self) -> Dict[str, str]:
        latex_dict = {}
        for sym, expr in self.thresholds.items():
            latex_dict[str(sym)] = sp.latex(Eq(sym, expr))
        latex_dict['c(s)'] = sp.latex(Eq(c_func(s), self.c_expr))
        latex_dict['v(s)'] = sp.latex(Eq(v_func(s), self.v_expr))
        latex_dict['b(s)'] = sp.latex(Eq(b_func(s), self.b_expr))
```

```
        return latex_dict

solution = LCPSolution(
    thresholds=thresholds,
    c_expr=c_expr_final,
    v_expr=v_expr_final,
    b_expr=b_expr_final
)

solution.display()
```

```
======================================================================
LIMIT CONTINUOUS POKER - Nash Equilibrium Solution
======================================================================

Hand Strength Thresholds:
----------------------------------------------------------------------
```

$$x_0 = \frac{3t^2\left(t-1\right)}{r^3+t^3-7}$$

$$x_1 = \frac{-2r^3+3r^2+t^3-1}{r^3+t^3-7}$$

$$x_2 = \frac{r^3+t^3-1}{r^3+t^3-7}$$

$$x_3 = \frac{r^3-3r+t^3-4}{r^3+t^3-7}$$

$$x_4 = \frac{r^3+3r^2-6r+t^3-4}{r^3+t^3-7}$$

$$x_5 = \frac{r^3+t^3+3t^2-7}{r^3+t^3-7}$$

```
Strategy Functions:
----------------------------------------------------------------------
```

$$c(s) = \frac{r^3+s\left(r^3+t^3-7\right)+t^3-1}{\left(s+1\right)\left(r^3+t^3-7\right)}$$

$$v(s) = \frac{r^3+t^3+\left(r^3+t^3-7\right)\left(2s^2+4s+1\right)-1}{2\left(r^3+t^3-7\right)\left(s^2+2s+1\right)}$$

$$b(s) = \frac{-3s+t^3\left(s+1\right)^3-1}{\left(r^3+t^3-7\right)\left(s^3+3s^2+3s+1\right)}$$

1.10 Numerical Strategy Functions

[27]:

```
def _convert_params(**kwargs):
    if 'L' in kwargs and 'U' in kwargs:
        L_val = kwargs['L']
        U_val = kwargs['U']
        r_val = L_val / (1 + L_val)
        t_val = 1 / (1 + U_val)
        return r_val, t_val
    elif 'r' in kwargs and 't' in kwargs:
        return kwargs['r'], kwargs['t']
    else:
        raise ValueError("Must provide either (L, U) or (r, t)␣
↪parameters")

def call_threshold(s_val, **kwargs):
    r_val, t_val = _convert_params(**kwargs)
    c_numeric = lambdify(s, solution.c_expr.subs({r: r_val, t: t_val}))
    return float(c_numeric(s_val))

def bluff_threshold(**kwargs):
    r_val, t_val = _convert_params(**kwargs)
    x2_expr = solution.thresholds[x2].subs({r: r_val, t: t_val})
    return float(x2_expr)

def value_threshold(**kwargs):
    r_val, t_val = _convert_params(**kwargs)
    x3_expr = solution.thresholds[x3].subs({r: r_val, t: t_val})
    return float(x3_expr)

def bluff_size(x_val, **kwargs):
    r_val, t_val = _convert_params(**kwargs)
    L_val = r_val / (1 - r_val)
    U_val = (1 - t_val) / t_val

    x0_val = solution.thresholds[x0].subs({r: r_val, t: t_val})
    x1_val = solution.thresholds[x1].subs({r: r_val, t: t_val})
    x2_val = solution.thresholds[x2].subs({r: r_val, t: t_val})
    b0_val = solution.thresholds[b0].subs({r: r_val, t: t_val})

    if x_val < x0_val:
        return U_val
    elif x_val < x1_val:
        from scipy.optimize import brentq
        b_substituted = solution.b_expr.subs({
            r: r_val,
            t: t_val,
```

```
            x2: x2_val,
            b0: b0_val
        })
        b_numeric = lambdify(s, b_substituted)
        try:
            result = brentq(lambda s_test: b_numeric(s_test) - x_val,␣
↪L_val, U_val)
            return result
        except:
            return None
    else:
        return L_val

def value_size(x_val, **kwargs):
    r_val, t_val = _convert_params(**kwargs)
    L_val = r_val / (1 - r_val)
    U_val = (1 - t_val) / t_val

    x4_val = solution.thresholds[x4].subs({r: r_val, t: t_val})
    x5_val = solution.thresholds[x5].subs({r: r_val, t: t_val})

    if x_val < x4_val:
        return L_val
    elif x_val < x5_val:
        vinv_numeric = lambdify(x, v_inv_expr.subs({r: r_val, t:␣
↪t_val}))
        return float(vinv_numeric(x_val))
    else:
        return U_val
```

1.11 LaTeX Output

[ ]:

```
print("=" * 70)
print("LaTeX Format:")
print("=" * 70)
latex_output = solution.to_latex()
for key, latex_str in latex_output.items():
    print(latex_str)
```

```
======================================================================
LaTeX Format:
======================================================================
x_{0} = \frac{3 t^{2} \left(t - 1\right)}{r^{3} + t^{3} - 7}
x_{1} = \frac{- 2 r^{3} + 3 r^{2} + t^{3} - 1}{r^{3} + t^{3} - 7}
x_{3} = \frac{r^{3} - 3 r + t^{3} - 4}{r^{3} + t^{3} - 7}
x_{4} = \frac{r^{3} + 3 r^{2} - 6 r + t^{3} - 4}{r^{3} + t^{3} - 7}
x_{5} = \frac{r^{3} + t^{3} + 3 t^{2} - 7}{r^{3} + t^{3} - 7}
```

```
x_{2} = \frac{r^{3} + t^{3} - 1}{r^{3} + t^{3} - 7}
b_{0} = \frac{t^{3}}{r^{3} + t^{3} - 7}
c{\left(s \right)} = \frac{r^{3} + s \left(r^{3} + t^{3} - 7\right) +
 ↪t^{3} -
1}{\left(s + 1\right) \left(r^{3} + t^{3} - 7\right)}
v{\left(s \right)} = \frac{r^{3} + t^{3} + \left(r^{3} + t^{3} - 7\right)
\left(2 s^{2} + 4 s + 1\right) - 1}{2 \left(r^{3} + t^{3} - 7\right)
 ↪\left(s^{2}
+ 2 s + 1\right)}
b{\left(s \right)} = \frac{- 3 s + t^{3} \left(s + 1\right)^{3} -
 ↪1}{\left(r^{3}
+ t^{3} - 7\right) \left(s^{3} + 3 s^{2} + 3 s + 1\right)}
```

1.12 Game Value Computation

[59]:

```
def compute_game_value(solution: LCPSolution) -> sp.Expr:
    bluff_payoff = solution.thresholds[x2] - sp.Rational(1, 2)
    check_payoff = x - sp.Rational(1, 2)
    min_bet_payoff = (x * (2*L_expr + 1) - L_expr * (solution.c_expr.
 ↪subs(s, L_expr) + 1) - sp.Rational(1, 2)).simplify()
    max_bet_payoff = (x * (2*U_expr + 1) - U_expr * (solution.c_expr.
 ↪subs(s, U_expr) + 1) - sp.Rational(1, 2)).simplify()

    intermediate_bet_payoff = (x * (2*v_inv_expr + 1) - v_inv_expr *
 ↪(solution.c_expr.subs(s, v_inv_expr) + 1) - sp.Rational(1, 2)).
 ↪simplify()
    q = sp.Symbol('q')
    q_expr = (x-1)/(r**3+t**3-7)
    intermediate_bet_payoff = intermediate_bet_payoff.subs(q_expr, q).
 ↪collect(q).collect(x).subs(q, q_expr)

    bluff_integral = integrate(
        bluff_payoff,
        (x, 0, thresholds[x2])
    ).simplify()

    check_integral = integrate(
        check_payoff,
        (x, thresholds[x2], thresholds[x3])
    ).simplify()

    min_bet_integral = integrate(
        min_bet_payoff,
        (x, thresholds[x3], thresholds[x4])
    ).simplify()
```

```
    max_bet_integral = integrate(
        max_bet_payoff,
        (x, thresholds[x5], 1)
    ).simplify()

    intermediate_bet_integral = integrate(
        intermediate_bet_payoff,
        (x, thresholds[x4], thresholds[x5])
    ).simplify()

    game_value = bluff_integral + check_integral + min_bet_integral + max_bet_integral + intermediate_bet_integral
    return game_value.simplify()

game_value = compute_game_value(solution)
display(game_value)
```

$$3\left(r\left(r^4-4r^3-6r^2+rt^3+8r-t^3+1\right)+t^2\left(-r^3-t^3+3t^2-18t+19\right)\right)\left(r^3+t^3-7\right)^2+\left(r^6+2r^3t^3-\right.$$

1.13 Simplification of Game Value to Closed Form

The game value expression above simplifies to:

$$V(r,t)=\frac{r^3+t^3-1}{2(r^3+t^3-7)}$$

Proof sketch:

1. **Substitution:** Let $u=r^3+t^3-7$ to simplify notation.
2. **Common denominator:** Note that $(r^3+t^3-7)^2=r^6+2r^3t^3+t^6-14r^3-14t^3+49$, so all terms can be written with denominator $2u^3$.
3. **Expand and collect:** Expand all products and collect terms over the common denominator $2u^3$. Use $t^3-(r-1)^3=t^3-r^3+3r^2-3r+1$ for the fourth term.
4. **Cancellation:** After expanding and collecting like powers of u:
   - All u^3 terms cancel
   - Most u^2 terms cancel, leaving only $u^2(u+6)=u^3+6u^2$ in the numerator
   - Lower order terms cancel
5. **Final simplification:**

$$\frac{u^3+6u^2}{2u^3}=\frac{u^2(u+6)}{2u^3}=\frac{u+6}{2u}=\frac{(r^3+t^3-7)+6}{2(r^3+t^3-7)}=\frac{r^3+t^3-1}{2(r^3+t^3-7)}$$

[ ]:
```
# sanity check - do the expressions agree on random inputs?
```

```python
def numerical_eq(expr1, expr2, tolerance=1e-9, iters=1000):
    for i in range(iters):
        r_val = np.random.rand()
        t_val = np.random.rand()
        vals = {r: r_val, t: t_val}
        if r_val + t_val > 1:
            continue
        if np.abs(expr1.subs(vals) - expr2.subs(vals)) > tolerance:
            print(vals)
            print(expr1.subs(vals), expr2.subs(vals))
            return False
    return True

known_form = (1-r**3-t**3)/(14-2*r**3-2*t**3)
numerical_eq(known_form, game_value)
```

[ ]:

```
True
```