Linear Programs Zero Sum Game Formulation

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Consider a normal-form zero-sum game with payoff matrix A, where player 1 (P1) chooses a mixed strategy represented by the vector x, and player 2 (P2) chooses a mixed strategy y.

The expected utility for player 1 is given by:

$$v = x^T A y$$

where v represents the value of the game. It turns out that it makes more sense to frame v as an independent decision variable.

Trivial Constraints

The strategies x and y must satisfy the standard probability constraints:

$$\sum_i x_i = 1, \quad 0 \le x_i \quad \forall i$$

$$\sum_j y_j = 1, \quad 0 \le y_j \quad \forall j$$

Equilibrium Constraints

A necessary condition for Nash equilibrium is that player 2 should be indifferent between all actions they play with positive probability. This is equivalent to requiring that the mixed strategy y should do at least as well for player 2 as any pure strategy, <u>against the specific strategy x</u> (or rather, that the value v for player 1 cannot be decreased by deviating y to any pure strategy):

$$v \le x^T A e_j, \quad \forall j$$

which can be rewritten as:

$$v \cdot \mathbf{1} \leq x^T A I = x^T A$$
$$x^T A \geq v \cdot \mathbf{1}$$

Where **1** is a vector of all 1s. Notice y and the indices j do not appear in this constraint.

All in all, we get the linear program:

$$\begin{aligned}
&\text{Maximize } v\\
&\text{Subject to}\\
&\qquad x^T A \geq v \cdot \mathbf{1}\\
&\qquad \sum_i x_i = 1\\
&\qquad 0 \leq x_i \quad \forall i
\end{aligned}$$

The solution to this linear program will give the value v of the game and the Nash equilibrium strategy x for player 1[1].

[1]Supposedly, Player 2's strategy y can be found similarly by solving the dual problem