1 Strategic Comparison to Fixed-Bet and No-Limit Continuous Poker

As noted in the introduction, LCP is designed to interpolate between Fixed-Bet Continuous Poker (FBCP) and No-Limit Continuous Poker (NLCP). Having derived the Nash equilibrium (Section **??**) and game value (Section **??**) for LCP, we now make this interpolation precise by proving convergence results. Specifically, we show that as the betting limits L and U approach appropriate boundary values, both the strategies and the game value of LCP converge to those of FBCP and NLCP.

To facilitate comparison across variants, we model the bettor strategies for all three games as 'bet functions' from hand strengths to bets (with 0 representing a check), and caller strategies as 'call functions' from bet sizes to minimum calling thresholds. We also introduce notation to reference all three strategy profiles more efficiently.

1.1 Setup and Notation

To compare the strategy profiles across different variants of Continuous Poker, we introduce the following notation for the strategy functions of the three games:

Symbol	**Meaning**
$S_{FB}(x, B)$	Bettor's bet function in FBCP with fixed bet size B
$C_{FB}(s, B)$	Caller's call function in FBCP with fixed bet size B
$S_{NL}(x)$	Bettor's bet function in NLCP
$C_{NL}(s)$	Caller's call function in NLCP
$S_{LCP}(x, L, U)$	Bettor's bet function in LCP with limits L and U
$C_{LCP}(s, L, U)$	Caller's call function in LCP with limits L and U
$x_i\|_{L,U}$	Threshold x_i in LCP with limits L and U

Table 1: Notation for strategy functions across different variants of Continuous Poker. Recall that x_i are variables used to describe the LCP strategy profile in Section **??**.

To recap, we will directly define these functions, although there will be overlap with the introduction. In FBCP, the bettor can only make a fixed bet size B or check. The bet function $S_{FB}(x, B)$ maps hand strengths to either 0 (check) or B (bet):

$$S_{FB}(x,B) = \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \text{ (bluffing range)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (checking range)} \\ B & x > \frac{1+4B+2B^2}{(1+2B)(2+B)} \text{ (value betting range)} \end{cases} \quad (1)$$

The caller's strategy is defined by a single threshold $C_{FB}(s,B)$:

$$C_{FB}(s,B) = \frac{B(3+2B)}{(1+2B)(2+B)} \quad (2)$$

In NLCP, the bettor can choose any positive bet size. The strategy is most naturally described by functions $v_{NL}(s)$ and $b_{NL}(s)$ that map bet sizes to hand strengths:

$$v_{NL}(s) = 1 - \frac{3}{7(s+1)^2} \text{ (value betting function)}$$
$$b_{NL}(s) = \frac{3s+1}{7(s+1)^3} \text{ (bluffing function).}$$

The bet function $S_{NL}(x)$ is then defined in terms of the inverse functions:

$$S_{NL}(x) = \begin{cases} b_{NL}^{-1}(x) & x < \frac{1}{7} \text{ (bluffing range)} \\ 0 & \frac{1}{7} < x < \frac{4}{7} \text{ (checking range)} \\ v_{NL}^{-1}(x) & x > \frac{4}{7} \text{ (value betting range)} \end{cases}$$

The caller's strategy is defined by a continuous function $C_{NL}(s)$:

$$C_{NL}(s) = 1 - \frac{6}{7(s+1)}$$

In LCP, the bettor can choose any bet size between L and U. The strategy profile is defined by six thresholds x_0 through x_5 and functions $v(s)$ and $b(s)$ that map bet sizes to hand strengths. The bet function $S_{LCP}(x,L,U)$ and call function $C_{LCP}(s,L,U)$ are defined in terms of these values, which are given in Theorem **??**.

1.2 Strategic Convergence

1.2.1 Bettor Strategy Convergence to FBCP

We expect that as L and U approach some fixed value s, the bet function $S_{LCP}(x,L,U)$ should converge to the bet function $S_{FB}(x,s)$ for Fixed-Bet Continuous Poker.

Theorem 1.1. *For any $B > 0$, $S_{LCP}(x, L, U)$ converges pointwise to $S_{FB}(x, B)$ as L and U approach B:*

$$\lim_{L\to B}\lim_{U\to B} S_{LCP}(x, L, U) = \lim_{U\to B}\lim_{L\to B} S_{LCP}(x, L, U) = S_{FB}(x, B)$$

for all hand strengths $x \in [0, 1]$.

Proof. We analyze the expressions for the x_i's, each of which is a rational function of L and U. Since these functions are defined and continuous for all positive $0 \le L \le U$, the limit as $L \to B$ and $U \to B$ can be found by simply substituting $L = U = B$:

$$\begin{aligned}
x_0|_{B,B} = x_1|_{B,B} &= \frac{B}{2B^3 + 7B^2 + 7B + 2} \\
x_2|_{B,B} &= \frac{B}{(1 + 2B)(2 + B)} \\
x_3|_{B,B} &= \frac{2B^2 + 4B + 1}{(1 + 2B)(2 + B)} \\
x_4|_{B,B} = x_5|_{B,B} &= \frac{2B^2 + 5B + 1}{(1 + 2B)(2 + B)}
\end{aligned}$$

$x_0 = x_1$ and $x_4 = x_5$ are expected, since these intervals are where the bettor uses an intermediate bet size, and $L = U = B$ does not allow intermediate bet sizes. This reduces the bet function to

$$\begin{aligned}
\lim_{L\to B}\lim_{U\to B} S_{LCP}(x, L, U) &= \begin{cases} B & x < \frac{B}{(1+2B)(2+B)} \\ 0 & \frac{B}{(1+2B)(2+B)} > x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \\ B & x > \frac{2B^2+4B+1}{(1+2B)(2+B)} \end{cases} \\
&= S_{FB}(x, B)
\end{aligned}$$

□

1.2.2 Caller Strategy Convergence to FBCP

The calling function is easier to analyze. We want to show that the calling threshold $C_{LCP}(s, L, U)$ converges to the calling threshold $C_{FB}(s, B)$ for Fixed-Bet Continuous Poker as L and U approach B.

Theorem 1.2. *For any $B > 0$, $C_{LCP}(s, L, U)$ converges pointwise to $C_{FB}(s, B)$ as L and U approach B:*

$$\lim_{L\to B}\lim_{U\to B} C_{LCP}(s, L, U) = \lim_{U\to B}\lim_{L\to B} C_{LCP}(s, L, U) = C_{FB}(s, B).$$

for all bet sizes $s \in [L, U]$.

Proof. We already have the value of $x_2|_{B,B}$, so we can plug this into the expression for the calling threshold:

$$\begin{aligned}
\lim_{L\to B}\lim_{U\to B} C_{LCP}(s, L, U) &= \frac{x_2|_{B,B} + s}{1+s} \\
&= \frac{\frac{B}{(1+2B)(2+B)} + s}{1+s} \\
&= \frac{B(3+2B)}{(1+2B)(2+B)} \\
&= C_{FB}(s, B)
\end{aligned}$$

□

1.2.3 Bettor Strategy Convergence to NLCP

In a similar fashion, we expect that as L and U approach 0 and ∞ respectively, the bet function $S_{LCP}(x, L, U)$ should converge to the bet function $S_{NL}(x)$ for NLCP.

Theorem 1.3. *$S_{LCP}(x, L, U)$ converges pointwise to $S_{NL}(x)$ as L and U approach 0 and ∞:*

$$\lim_{L\to 0}\lim_{U\to\infty} S_{LCP}(x, L, U) = \lim_{U\to\infty}\lim_{L\to 0} S_{LCP}(x, L, U) = S_{NL}(x).$$

for all hand strengths $x \in [0, 1]$.

Proof. We can analyze the expressions for the x_i's as L and U approach 0 and ∞. The limits are all well-defined:

$$\begin{aligned}
x_0|_{0,\infty} &= 0 \\
x_1|_{0,\infty} &= x_2|_{0,\infty} = \frac{1}{7} \\
x_3|_{0,\infty} &= x_4|_{0,\infty} = \frac{4}{7} \\
x_5|_{0,\infty} &= 1
\end{aligned}$$

$x_0|_{0,\infty} = 0$ and $x_5|_{0,\infty} = 1$ are expected, since these intervals are where the bettor uses a minimum bet size and a maximum bet size, respectively, both of which are impossible. The bettor now bets intermediate values for $x < \frac{1}{7}$ and $x > \frac{4}{7}$, and checks for $\frac{1}{7} < x < \frac{4}{7}$. But how much do they bet? We can take the limits of $v(s)$ and $b(s)$ as L and U approach 0 and ∞:

$$\lim_{L\to 0}\lim_{U\to\infty} b(s) = \frac{3s+1}{7(s+1)^3}$$
$$\lim_{L\to 0}\lim_{U\to\infty} v(s) = 1 - \frac{3}{7(s+1)^2}.$$

To summarize, the bettor bets s with hands $x < \frac{1}{7}$ such that $x = b(s)$ or hands $x > \frac{4}{7}$ such that $x = v(s)$. This is exactly the same as the bet function $S_{NL}(x)$ for NLCP.

□

1.2.4 Caller Strategy Convergence to NLCP

The calling function is again easier to analyze. We want to show that the calling threshold $C_{LCP}(s, L, U)$ converges to the calling threshold $C_{NL}(s)$ for NLCP as L and U approach 0 and ∞ respectively.

Theorem 1.4. *$C_{LCP}(s, L, U)$ converges pointwise to $C_{NL}(s)$ as L and U approach 0 and ∞:*

$$\lim_{L\to 0}\lim_{U\to\infty} C_{LCP}(s, L, U) = \lim_{U\to\infty}\lim_{L\to 0} C_{LCP}(s, L, U) = C_{NL}(s).$$

for all bet sizes $s \in [0, \infty)$.

Proof. Again, we already have the limiting value of $x_2|_{0,\infty}$, so we can plug this into the expression for the calling threshold:

$$\begin{aligned}
\lim_{L\to 0}\lim_{U\to\infty} C_{LCP}(s, L, U) &= \lim_{L\to 0}\lim_{U\to\infty} \frac{x_2 + s}{1+s} \\
&= \frac{\frac{1}{7} + s}{1+s} \\
&= 1 - \frac{6}{7(1+s)} \\
&= C_{NL}(s).
\end{aligned}$$

□

We have now shown that the bettor and caller strategies for LCP converge to those of FBCP and NLCP as the limits L and U approach their extreme values. In the next section, we explore the value of LCP in more detail, and in particular how it relates to that of FBCP and NLCP.