Charges and Fields

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Coulomb's law

The magnitude of the force F between two point charges q_1, q_2 is given by

$$F = k\frac{|q_1 q_2|}{r^2}$$

where k is a constant and r^2 is the distance between q_1 and q_2 squared.*

To obtain both the direction and magnitude of such force, in this case from q_1 to q_2, we could find

$$\vec{F}_{12} = k\frac{|q_1 q_2|\hat{r}_{12}}{r^2}$$

where $\hat{r}_{12}$ is a unit vector that points from q_1 to q_2.

Electric Fields

A point charge q exerts a electric field $\vec{E}$ described by

$$\boxed{\vec{E} = \frac{kq}{r^2}\hat{r}.}$$

where r^2 is the distance between q and the position at which the field is measured, and $\hat{r}$ is the unit vector that points from q to said position.

A test charge q_0 would experience a force $\vec{F}$ by $\vec{E}$ described by

$$\boxed{\vec{F} = q_0\vec{E}.}$$

*k is sometimes given as $k = \frac{1}{4\pi\epsilon_0}$ where ϵ_0 is the permittivity of vacuum.

Continuous Charge Distribution

Suppose we want to describe the total electric field $\vec{E}_{tot}$ exerted by a body of charge. The principle of superposition tells us that

$$\begin{aligned}\vec{E}_{tot} &= \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + ... + \vec{E}_n \\ &= \sum_{i=1}^{n} \vec{E}_i. \\ &= \int_V d\vec{E}.\end{aligned}$$

$\vec{E}_{tot}$ is the sum of all electric fields from each point charge in said body of charge. We then say each infinitesimal point charge dq exerts a field

$$d\vec{E} = \frac{k\ dq}{r^2}\hat{r}.$$

In most cases dq has different values as the location changes. Therefore, we map dV, the infinitesimal volume or intuitively understood as the coordinates of dq, to dq with

$$dq = \lambda dV,$$

where λ is the charge density. Now we replace dq with λdV and integrate $d\vec{E}$ over volume V to obtain

$$\vec{E}_{tot} = \int_V d\vec{E} = \int_V \frac{k\lambda}{r^2}\hat{r} dV.$$

A useful proposition is that for any charged wire of length $2l$ and charge Q measured x distance away from the midpoint, we have

$$\boxed{E_x = \frac{2k\lambda l}{x\sqrt{x^2 + l^2}}}$$

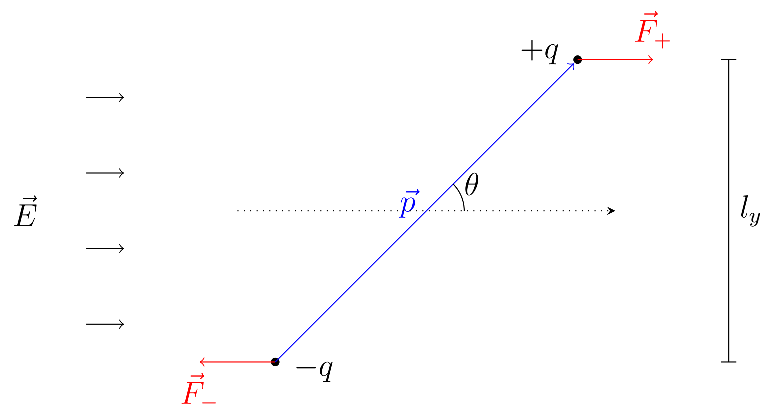
where $\lambda = \frac{Q}{2l}$.

Dipole Moment

The dipole moment $\vec{p}$ given by two charges $-q$ and $+q$ is described by

$$\boxed{\vec{p} = q\vec{l}.}$$

where $\vec{l}$ is the position vector that points from $-q$ to $+q$. The dipole moment also points from $-q$ to $+q$.



Suppose we are interested in finding out the torque $\vec{\tau}$ applied to an electric dipole $\vec{p}$ by an uniform electric field $\vec{E}$. The magnitude of the torque vector, τ, is described by a familiar formula from classical mechanics

$$\tau = Fl_y,$$

where F is the magnitude of either $\vec{F}_+$ or $\vec{F}_-$, and l_y is the component of distance between $q-$ and $q+$ perpendicular to the direction of $\vec{E}$. We can obtain F using

$$F = qE$$

where q is the magnitude of point charge and E is the magnitude of $\vec{E}$. We could also obtain l using

$$l_y = d \sin\theta$$

where d (not shown in figure) is the distance between $-q$, $+q$ and θ is the angle between $\vec{p}$ and $\vec{E}$. Reconstructing the equations, we have

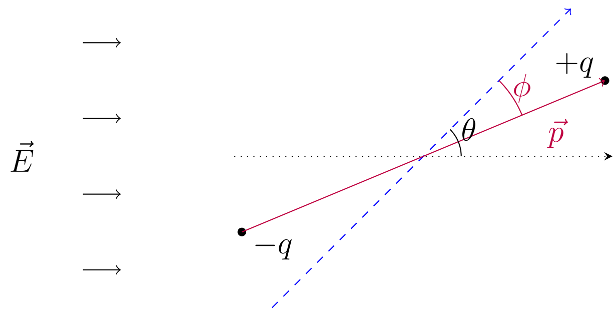
$$\begin{aligned}\tau &= Fl_y \\ &= qdE\sin\theta \\ &= pE\sin\theta\end{aligned}$$

where $p = qd$ is the magnitude of the dipole moment. To obtain the torque vector $\vec{\tau}$, we have

$$\boxed{\vec{\tau} = \vec{p} \times \vec{E}}$$

since the cross product of the dipole moment $\vec{p}$ and the electric field $\vec{E}$ describes the orientation of $\vec{\tau}$. The magnitude of the product also aligns with what we have arrived at before

$$|\vec{p} \times \vec{E}| = pE \sin\theta.$$



Consider an electric dipole $\vec{p}$ that started off with an angle θ with $\vec{E}$. Under the effect of torque $\vec{\tau}$, $\vec{p}$ rotated. We denote the angular displacement caused by the rotation ϕ. Suppose we are interested in finding out the work W that was done by $\vec{E}$ on $\vec{p}$

$$W = \tau \cdot \phi.$$

Since τ is a function of $\theta - \phi$, we need to integrate dW from the starting angle θ to the final angle $\theta - \phi$ with $d\phi$ in order to get W.

$$\begin{aligned} W = \int_{\theta-\phi}^{\theta} dW &= \int_{\theta-\phi}^{\theta} \tau d\phi \\ &= \int_{\theta-\phi}^{\theta} \vec{p} \times \vec{E} d\phi \\ &= \int_{\theta-\phi}^{\theta} pE \sin{(\theta - \phi)} d\phi \\ &= pE \int_{\theta-\phi}^{\theta} \sin{(\theta - \phi)} d\phi \\ &= pE \cos{(\theta - \phi)}|_{\theta-\phi}^{\theta} \\ &= pE(1 - \cos\phi). \end{aligned}$$

Suppose we are interested in finding out the potential energy U of $\vec{p}$ that forms an angle θ with $\vec{E}$. Conventionally, we consider $\vec{p}$ being perpendicular to $\vec{E}$ as $\vec{p}$ having 0 potential energy. That is, to calculate the potential energy related to $\pi/2$, we find the work needed to turn $\vec{p}$ from θ to $\pi/2$.

$$\boxed{\begin{aligned} U = \int_{\pi/2}^{\theta} dW &= \int_{\pi/2}^{\theta} pE\sin(\theta)d\theta \\ &= -pE\cos\theta|_{\pi/2}^{\theta} \\ &= -pE\cos\theta \\ &= -\vec{p}\cdot\vec{E}. \end{aligned}}$$