IMPROVED BOUNDS FOR AVERAGE BENDING ON THE CONVEX CORE OF A KLEINIAN GROUP

ANDREW YARMOLA

Abstract. Let $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^3)$ be a finitely-generated Kleinian group and $N = \mathbb{H}^3/\Gamma$. In this note, we consider the case where the boundary $\partial C(N)$ of the convex core is non-empty and incompressible in N. Our main result improves bounds on the total bending of a geodesic arc on $\partial C(N)$ as a function of its length. Following the work of Bridgeman [Bri03], we use this to provide better bounds on the length and average bending of the measured lamination on $\partial C(N)$ and universal bounds on the optimal Lipschitz constant for a map from $\partial C(N)$ to the conformal structure at infinity $\partial_\infty N$. Additionally, we show that the shortest rectifiable path connecting four disjoint sequentially tangent hyperplanes in $\mathbb{H}^3$ is attained when the planes support four of the faces of a standard ideal octahedron. This work formed part of the author's thesis.

1. Introduction

Let $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^3)$ be a finitely-generated Kleinian group, $N = \mathbb{H}^3/\Gamma$ and assume that the boundary $\partial C(N)$ of the convex core is non-empty and incompressible in N. Thurston [Thu91] showed that the path metric on $\partial C(N)$ is a complete hyperbolic metric. In addition to this structure, one may consider the conformal structure $\partial_\infty N = \Omega_\Gamma/\Gamma$, where $\Omega_\Gamma \subset \hat{\mathbb{C}}$ is the domain of discontinuity of Γ as a subgroup of $\mathrm{PSL}(2,\mathbb{C})$. Since $\partial C(N)$ is non-empty and incompressible in N, Ω_Γ is non-empty and its components are simply connected hyperbolic domains. Let $CH(\cdot)$ denote taking the convex hull in $\mathbb{H}^3$. There is a well-defined Γ-equivariant map $\widetilde{r} : \Omega_\Gamma \to \partial CH(\hat{\mathbb{C}} \setminus \Omega_\Gamma)$ given by letting $\widetilde{r}(x)$ be the first point of intersection of a growing family of horoballs at $x \in \Omega_\Gamma$ with $\partial CH(\hat{\mathbb{C}} \setminus \Omega_\Gamma)$. This map projects to the *nearest point retraction* $r : \partial_\infty N \to \partial C(N) = \partial CH(\hat{\mathbb{C}} \setminus \Omega_\Gamma)/\Gamma$.

The hyperbolic structure on $\partial C(N)$ comes with a measured lamination μ_Γ on $\partial C(N)$, called the *bending lamination*. For any measured lamination μ on hyperbolic surface and $L > 0$, one defines the *L-roundness* $\|\mu\|_L$ as

$$\|\mu\|_L = \sup i(\alpha, \mu)$$

where the supremum is taken over all open geodesic arcs of length L. As seen in the work of Epstein, Marden, and Markovic [EMM04] and our previous work [BCY16] *L-roundness* for a measured lamination μ on $\mathbb{H}^2$ provides some control on the embeddedness of the associated pleating map $P_\mu : \mathbb{H}^2 \to \mathbb{H}^3$. Extending [BCY16, Theorem 3.1], we prove

In this note, we extend our previous results in [BCY16] to prove

Theorem 1.1. *If $L \in (0, 2\sinh^{-1}(2)]$, μ is a measured lamination on $\mathbb{H}^2$, and P_μ is an embedding, then $\|\mu\|_L \leq F(L)$ where*

$$F(L) = \begin{cases} 2\cos^{-1}(-\sinh(L/2)) & \text{for } L \in [0, 2\sinh^{-1}(1)] \\ 3\pi - 2\cos^{-1}\left(\left(\sqrt{\cosh(L)} - 1\right)/2\right) & \text{for } L \in (2\sinh^{-1}(1), 2\sinh^{-1}(2)] \end{cases}$$

As a corollary of the proof of Theorem 1.1, we obtain

Corollary 1.1. *The shortest rectifiable path $\alpha(t) \subset \mathbb{H}^3$ connecting four disjoint sequentially tangent hyperplanes in $\mathbb{H}^3$ has length $2\sinh^{-1}(2)$ and is attained when the planes support four of the faces of a standard ideal octahedron.*

Theorem 1.1 allows us to improve bounds by Bridgeman [Bri03, Theorem 1.2] on average bending and the Lipschitz constant for the homotopy inverse of the retraction map.

Theorem 1.2. *There exist universal constants K_0, K_1 with $K_0 \leq 2.494$ and $K_1 \leq 3.101$ such that if $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^3)$ is a finitely-generated Kleinian group, $N = \mathbb{H}^3/\Gamma$, and the boundary $\partial C(N)$ of the convex core is non-empty and incompressible in N, then*

(i) if μ_Γ is the bending lamination of $\partial C(N)$, then

$$\ell_{\partial C(N)}(\mu_\Gamma) \leq K_0\,\pi^2\,|\chi(\partial C(N))|$$

(ii) for any closed geodesic α on $\partial C(N)$,

$$B_\Gamma(\alpha) = \frac{i(\alpha, \mu_\Gamma)}{\ell(\alpha)} \leq K_1$$

where $B_\Gamma(\alpha)$ is called the average bending of α.

(iii) there exists a $(1+K_1)$-Lipschitz map $s : \partial C(N) \to \partial_\infty N$ that is a homotopy inverse to the nearest point retraction $r : \partial_\infty N \to \partial C(N)$.

The previous best constants in [Bri03, Theorem 1.2] were $K_0 \leq 2.8396$ and $K_1 \leq 3.4502$.

2. Background

2.1. **Kleinian Groups and Convex Hulls.** Let $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^3)$ be a discrete torsion free subgroup. Define the *limit set* of Γ to be $\Lambda_\Gamma = \overline{\Gamma\, x} \cap \partial_\infty \mathbb{H}^3$ for any $x \in \mathbb{H}^3$. This definition is independent of the choice of x. We say that Γ is a *Kleinian group* if Λ_Γ contains at least 3 points. The set $\Omega(\Gamma) = \partial_\infty \mathbb{H}^3 \setminus \Lambda_\Gamma$ is called the *domain of discontinuity* of Γ. It can be equivalently defined as the largest open subset in $\partial_\infty \mathbb{H}^3$ where Γ acts properly discontinuously.

The *convex hull* $CH(X)$ of a closed set $X \subset \partial_\infty \mathbb{H}^3$ is smallest convex subset of $\mathbb{H}^3$ such that $\overline{CH(X)} \cap \partial_\infty \mathbb{H}^3 = X$. We require that X contain more than two points. For a Kleinian group Γ, the convex hull of Γ is $CH(\Lambda_\Gamma)$ and the convex hull of $\mathbb{H}^3/\Gamma$ is $CH(\Lambda_\Gamma)/\Gamma$, which is the smallest π_1-injective convex submanifold.

One defines a projection $\widetilde{r} : \mathbb{H}^3 \cup \Omega(\Gamma) \to CH(X)$ as follows. For $x \in \overline{\mathbb{H}}^3$ (resp. $x \in \Omega(\Gamma)$) let $B_t(x)$ denote the 1-parameter family of hyperbolic balls (resp. horoballs) centered at x with $B_{t_0}(x) \subset B_{t_1}(x)$ for all $t_0 < t_1$. Then, for $x \in CH(X)$, we define $r(x) = x$ and for all other x, $r(x)$ is the first (unique) intersection point of $CH(X)$ with $B_t(x)$. See [EM87] for a proof that this is a well defined continuous distance decreasing map. For a Kleinian group Γ, this map projects to the *nearest point retraction* $r : \left(\mathbb{H}^3 \cup \Omega(\Gamma)\right)/\Gamma \to CH(\Lambda_\Gamma)/\Gamma$.

A *hyperbolic domain* Ω in $\hat{\mathbb{C}}$ is a connected open set such that $\hat{\mathbb{C}} \setminus \Omega$ is at least 3 points. In particular, when one identifies $\partial_\infty \mathbb{H}^3 \cong \hat{\mathbb{C}}$, a connected component of $\Omega(\Gamma)$ for a Kleinian group Γ is a hyperbolic domain. Let $X = \hat{\mathbb{C}} \setminus \Omega$. Epstein and Marden [EM87] show that if X is not contained in a circle, then $CH(X)$ has non empty interior and a well defined boundary, denoted $\mathrm{Dome}(\Omega) = \partial CH(X)$. If X lies in a circle, then $CH(X)$ lies in a hyperbolic plane and is bounded by a countable collection of complete geodesics. In this setting, $\mathrm{Dome}(\Omega)$ is defined as the double $CH(X)$ along those geodesics.

Points on $\mathrm{Dome}(\Omega)$ can be connected by rectifiable paths along $\mathrm{Dome}(\Omega)$ and so it inherits a path metric from $\mathbb{H}^3$. Thurston [Thu91] showed that this path metric is, in fact, a complete hyperbolic metric. Further, he demonstrates that the covering map $\mathbb{H}^2 \to \mathrm{Dome}(\Omega)$ as a very specific structure that we now describe.

2.2. **Laminations.** A *geodesic lamination* on $\mathbb{H}^2$ is a closed subset $\lambda \subset \mathscr{G}(\mathbb{H}^2)$ which does not contain any intersecting geodesics. It can be realized on $\mathbb{H}^2$ as a closed set foliated by complete geodesics and therefore the elements of λ are called *leaves.* A *measured lamination* μ on $\mathbb{H}^2$ is a non-negative countably additive measure μ on $\mathscr{G}(\mathbb{H}^2)$ supported on a geodesic lamination. A geodesic arc α in $\mathbb{H}^2$ is said to be *transverse to* μ, if it is transverse to every geodesic in $\mathrm{supp}(\mu)$. Whenever α is transverse to μ, we define

$$i(\mu, \alpha) = \mu\left(\{\gamma \in \mathscr{G}(\mathbb{H}^2) \mid \gamma \cap \alpha \neq \emptyset\}\right).$$

If α is not transverse to μ, then it is contained in a geodesic of $\mathrm{supp}(\mu)$ and we let $i(\mu, \alpha) = 0$.

Given a measured lamination μ on $\mathbb{H}^2$, we may construct a *pleated plane* $P_\mu : \mathbb{H}^2 \to \mathbb{H}^3$, well-defined up to post-composition with elements of $\mathrm{Isom}^+(\mathbb{H}^3)$. P_μ is an isometry on the components of $\mathbb{H}^2 \setminus \mathrm{supp}(\mu)$, which are called *flats.* If μ is a finite-leaved lamination, then P_μ is simply obtained by bending, consistently rightward, by the angle $\mu(\{l\})$ along each leaf l of μ. Since any measured lamination is a limit of finite-leaved laminations, one may define P_μ in general by taking limits (see [EM87, Theorem 3.11.9]).

Lemma 2.1. [EM87] *If Ω is a hyperbolic domain, there is a lamination μ on $\mathbb{H}^2$ such that P_μ is a locally isometric covering map with image* $\mathrm{Dome}(\Omega)$*.*

2.3. **Pleated Planes.** For any point $x \in \mathrm{Dome}(\Omega)$, a *support plane* P at x is a totally geodesic plane through x which is disjoint from the interior of the convex hull of $\hat{\mathbb{C}} \setminus \Omega$. At least one support plane exists at every point $x \in \mathrm{Dome}(\Omega)$ and $\mathrm{Dome}(\Omega) \cap P$ is either a geodesic line with endpoints in $\partial\Omega$, called a *bending line*, or a *flat*, which is the convex hull of a subset of ∂P containing at least 3 points. The boundary geodesics of a flat will also be called bending lines. Support planes come with a preferred normal direction pointing *away* from $CH(\hat{\mathbb{C}} \setminus \Omega)$. The closure of the complement of $\mathbb{H}^3 \setminus P$ that lies in this direction is called the associated *half space*, denoted H_P. A detailed discussion and proofs on these facts can be found in [EM87].

For a curve $\alpha : (a, b) \to \text{Dome}(\Omega)$, it is natural to consider the space of support planes at each point $\alpha(t)$. A theorem of Kulkarni and Pinkall [KP94] asserts that the space of support planes to $\text{Dome}(\Omega)$ is an $\mathbb{R}$-*tree* in the induced path metric from $\mathscr{P}(\mathbb{H}^3)$ whenever Ω is a simply connected hyperbolic domain. Recall that an $\mathbb{R}$-*tree* is a simply connected, geodesic metric space such that for any two points there is a unique embedded arc connecting them. Therefore, dual to any rectifiable path $\alpha : (a, b) \to \text{Dome}(\Omega)$, there is a continuous path $P_t : (c, d) \to \mathcal{P}(\mathbb{H}^3)$ and a map $p : (c, d) \to (a, b)$ such that P_t is a support plane at $\alpha(p(t))$. It also follows that we can define terminal support planes on the ends of α by $P_a = \lim_{t \to c^+} P_t$ and $P_b = \lim_{t \to d^-} P_t$.

Epstein and Marden further show that for every point $x \in \text{Dome}(\Omega)$, there is a neighborhood $W \subset \mathbb{H}^3$ of x such that if l_1, l_2 are bending lines that meet W, then any support plane that meets l_1 intersects all support planes that meet l_2 [EM87, Lemma 1.8.3]. The transverse intersection of two support planes P, Q is called a *ridge line*. Notice that if two support planes P, Q intersect, they either do so at a ridge line or $P = Q$. If $P = Q$ and the interiors of H_P, H_Q are not equal, then $\hat{\mathbb{C}} \setminus \Omega$ is contained in a the circle ∂P.

The exterior angle, denoted $\angle_{ext}(P, Q)$, between two intersecting or tangent support planes is the angle between their normal vectors at any point of intersection or tangency. We define the interior angle by $\angle_{int}(P, Q) = \pi - \angle_{ext}(P, Q)$.

Let μ be the measured lamination on $\text{Dome}(\Omega)$ such that $P_\mu : \mathbb{H}^2 \to \text{Dome}(\Omega)$ is the pleated plane. By a *transverse geodesic* arc $\alpha : (a, b) \to \text{Dome}(\Omega)$, we will mean arc such that $P_\mu^{-1}(\alpha)$ is a geodesic arc in $\mathbb{H}^2$ and transverse to $\text{supp}(\mu)$. We say the terminal support planes P_a, P_b form a *roof* over α if the interiors of the associated half spaces H_t intersects H_a for all t. Roofs play an important role in approximating the bending along α.

Lemma 2.2. (Lemmas 4.1 and 4.2 [BC03]) *Let μ be the measured lamination on* $\text{Dome}(\Omega)$. *If $\alpha : (a, b) \to \text{Dome}(\Omega)$ is a transverse geodesic arc such that the terminal support planes P_a, P_b form a roof over α then $i(\alpha, \mu) \le \angle_{ext}(P, Q) = \pi - \angle_{int}(P, Q)$.*

Lemma 2.3. *Let Ω be a simply connected hyperbolic domain and $\alpha : (a, b) \to \text{Dome}(\Omega)$ a transverse geodesic arc. If the interiors of the terminal half spaces H_a, H_b intersect, then P_a and P_b form a roof over α.*

Proof. Intuitively, this is a consequence of the fact that support planes can't form "loops" when Ω is simply connected. Recall that the space of support planes to $\text{Dome}(\Omega)$ is an $\mathbb{R}$-tree. Since α is geodesic, the of support planes P_t to α must be embedded, and therefore the unique path between P_a and P_b. As the interiors of H_a, H_b intersect, either $P_a = P_b$

or P_a, P_b intersect at a ridge line ℓ_r. In the former case, it follows that $P_t = P_a = P_b$ is constant and therefore $H_t = H_a$ for all t.

In the later case, consider the path β which goes from $\alpha(a)$ to ℓ_r along P_a and from ℓ_r to $\alpha(b)$ along P_b. We can project β to $r(\beta) \subset \mathrm{Dome}(\Omega)$. Since P_t is the unique path connecting P_a to P_b, it follows that the path of support planes along $r(\beta)$ must fun over all of P_t. By construction, every support plane to $r(\beta)$ must contain the ridge line ℓ_r. Thus, the interiors of H_t and H_a intersect for all t and P_a, P_b is a roof over α. $\square$

2.4. **L-roundness.** For a measured lamination μ on $\mathbb{H}^2$, Epstein, Marden and Markovic [EMM04] defined the *roundness* of μ to be $||\mu|| = \sup i(\mu, \alpha)$ where the supremum is taken over all open unit length geodesic arcs in $\mathbb{H}^2$. The roundness bounds the total bending of P_μ on any segment of length 1 and is closely related to average bending, which was introduced earlier by Bridgeman [Bri98].

In our work, we consider the *L-roundness* of a measured lamination for any $L > 0$

$$||\mu||_L = \sup i(\alpha, \mu)$$

where now the supremum is taken over all open geodesic arcs of length L in $\mathbb{H}^2$. We note that the supremum over open geodesic arcs of length L, is the same as that over *half* open geodesic arcs of length L.

3. Improved Upper Bound on L-roundness for Embedded Pleated Planes

In this section, we adapt the techniques of [Bri03] to obtain an improved bound on the L-roundness of an embedded pleated plane. As it appears here, Theorem 1.1 is an extended version of our work in [BCY16, Theorem 3.1].

Theorem 1.1. *If $L \in (0, 2\sinh^{-1}(2)]$, μ is a measured lamination on $\mathbb{H}^2$, and P_μ is an embedding, then $\|\mu\|_L \le F(L)$ where*

$$F(L) = \begin{cases} 2\cos^{-1}(-\sinh(L/2)) & \text{for } L \in [0, 2\sinh^{-1}(1)] \\ 3\pi - 2\cos^{-1}\left(\left(\sqrt{\cosh(L)} - 1\right)/2\right) & \text{for } L \in (2\sinh^{-1}(1), 2\sinh^{-1}(2)] \end{cases}$$

The proof relies on a careful analysis of minimal lengths of arcs joining a sequence of 3 or 4 pleated planes. We present these arguments as Lemmas 3.1, 3.2, 3.4.

Lemma 3.1. *Let P_0, P_1, P_2 be planes in $\mathbb{H}^3$ with boundary circles $C_i \in \partial_\infty \mathbb{H}^3$. Assume that $C_0 \cap C_2 = \{a\}$, $a \notin C_1$, and the minor angles $\angle_m(C_0, C_1) = \angle_m(C_1, C_2) = \theta < \pi/2$. If $\alpha : [0, 1] \to \mathbb{H}^3$ is a rectifiable path with $\alpha(0) \in P_0, \alpha(1) \in P_2$ and $\alpha(t_1) \in P_1$ for some $t_1 \in (0, 1)$. Then,*

$$\ell(\alpha) \ge 2\sinh^{-1}(\cos\theta).$$

Proof. Since $a \notin C_1$, there is a plane $T \subset \mathbb{H}^3$ perpendicular to all P_i. Let $\lambda_i = T \cap P_i$. Take $\overline{\alpha}$ to be the nearest point projection of α onto T. Since nearest point projections shrink distances, $\ell(\alpha) \geq \ell(\overline{\alpha})$. In addition, as T is perpendicular to P_i, we have $\overline{\alpha}(0) \in \lambda_0, \overline{\alpha}(1) \in \lambda_2$ and $\overline{\alpha}(t_1) \in \lambda_1$. We can identify T with the Poincare disk and conjugate λ_i as in Figure 1.

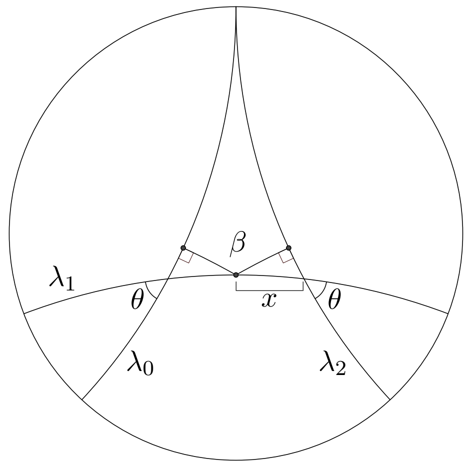

FIGURE 1. Configuration of $\lambda_i \subset T$ in the Poincare disk model for Lemma 3.1

By symmetry, the shortest curve connecting λ_0 to λ_2 via λ_1 is the symmetric piecewise geodesic β depicted in Figure 1. Let x be the sub-arc of λ_1 between $\lambda_1 \cap \lambda_2$ and $\lambda_1 \cap \beta$. Then, one may apply hyperbolic trigonometry formulae [Bea95, Theorem 7.9.1] and [Bea95, Theorem 7.11.2] to obtain

$$\sinh(x) \tan\theta = 1 \quad \text{and} \quad \sinh(\ell(\beta)/2) = \sinh(x)\sin\theta.$$

Therefore,

$$\ell(\alpha) \geq \ell(\beta) \geq 2\sinh^{-1}(\cos\theta).$$

$\square$

Lemma 3.2. *Let P_0, P_1, P_2, P_3 be planes in $\mathbb{H}^3$ with boundary circles $C_i \in \partial_\infty \mathbb{H}^3$. Assume*

(i) $P_0 \cap P_2 = P_1 \cap P_3 = P_0 \cap P_3 = \emptyset$

(ii) $C_0 \cap C_3 = \{a\}$ *and* $C_1 \cap C_2 = \{b\}$

(iii) $a \notin C_1 \cup C_2$ *and* $b \notin C_0 \cup C_3$.

(iv) let η_i be normal directions to C_i such that η_0, η_3 point away from each other and η_1, η_2 point toward each other, then $\angle(\eta_0, \eta_1) = \angle(\eta_2, \eta_3) = \theta < \pi/2$.

If $\alpha : [0,1] \to \mathbb{H}^3$ is a rectifiable path with $\alpha(0) \in P_0, \alpha(1) \in P_3$, $\alpha(t_1) \in P_1$, and $\alpha(t_2) \in P_2$ for some $t_1, t_2 \in (0,1)$ with $t_1 < t_2$. Then,

$$\ell(\alpha) \geq \cosh^{-1}\left((2\cos\theta + 1)^2\right).$$

Proof. Let ρ_i denote the reflection across P_i and $\rho_{i,j} = \rho_i \circ \rho_j$. Since α is supported by the planes P_i, we may look at pieces of α under a series of reflections. In particular, consider the curve

$$\beta = \alpha[0, t_1] \cup \rho_1(\alpha[t_1, t_2]) \cup \rho_{1,2}(\alpha[t_2, 1]).$$

Notice that β is a curve from P_0 to $\rho_{1,2}(P_3)$ and $\ell(\alpha) = \ell(\beta)$. Our goal is now to find a lower bound for $\ell(\beta)$ in terms of θ.

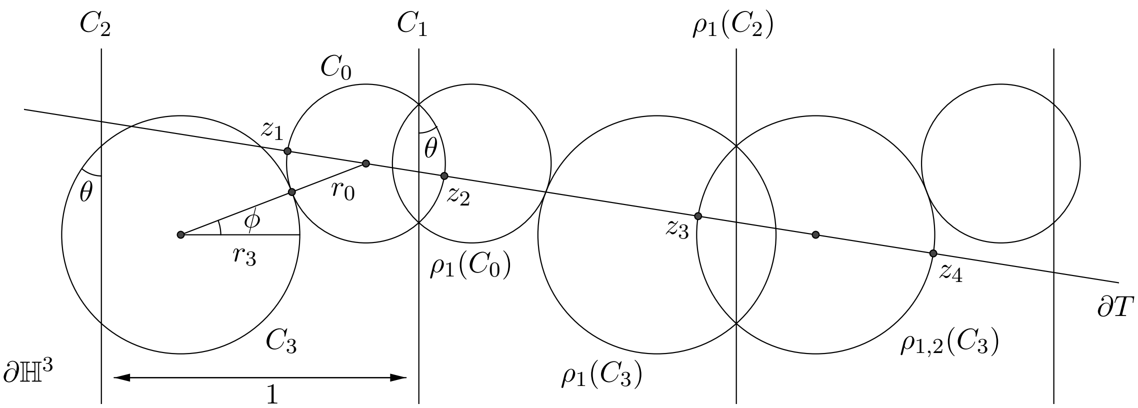

FIGURE 2. Boundaries of the planes P_i in Lemma 3.2 and their reflections in the upper half space model.

By construction, β is longer than the geodesic from P_0 to $\rho_{1,2}(P_3)$. Notice that this geodesic intersects P_1 and $\rho_1(P_2)$, so after reflecting some pieces, it satisfies the assumptions of the Lemma. Let T be the hyperplane going through the Euclidean centers of C_0 and $\rho_{1,2}(C_3)$. Since the geodesic between P_0 and $\rho_{1,2}(P_3)$ is unique, it must lie in T. Refer to Figure 2 for the generic configuration.

We need to say a few words about the validity of Figure 2 for our computations. Conjugating, we can map the points $a \to 0$ and $b \to \infty$. It follows from (ii) and (iii) that C_1, C_2 are parallel lines and C_0, C_3 are circles in the plane. Assumptions (i) and (iv) also guarantee that, maybe after flipping, $0 \le \phi \le \pi/2$. It is straightforward to check that assumption (iv) on a choice of normal directions guarantees that θ is correctly labeled in Figure 2.

Identify T with $\mathbb{U}^2$ so that the center of C_0 corresponds to 0. We compute the distance between the two disjoint geodesics $\lambda = T \cap P_0$ and $\gamma = T \cap \rho_{1,2}(P_3)$. Let $z_1 < z_2 < z_3 < z_4$, $z_i \in \mathbb{R} \subset \partial T$ be the points $\partial\lambda \cup \partial\gamma$. We can use the standard cross ration to compute

$$(z_1, z_3; z_2, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} = \coth^2\left(\frac{1}{2} d_{\mathbb{H}}(\lambda, \gamma)\right) > 0$$

$$d_{\mathbb{H}}(\lambda, \gamma) = \log\left(\frac{\sqrt{(z_1, z_3; z_2, z_4)} + 1}{\sqrt{(z_1, z_3; z_2, z_4)} - 1}\right).$$

Let r_0, r_1, ϕ, θ be as in Figure 2 and normalize the diagram as shown. By directly constructing a diagram from our parameters, one checks that a configuration satisfies out assumptions if and only if

$$0 \le \theta < \pi/2 \quad \text{and} \quad 0 \le \phi \le \pi/2$$

$$0 \le r_i + r_i \cos\theta \le 1 \text{ for } i = 0, 1$$

$$1 = (r_0 + r_1)(\cos\theta + \cos\phi)$$

To evaluate the cross ratio, let $z_1 = -r_0, z_2 = r_0, z_3 = c - r_0$, and $z_4 = c + r_0$, where c is the distance between the Euclidean centers of C_0 and $\rho_{1,2}(C_3)$. Computing, we have

$$c^2 = (r_0 + r_1)^2 \sin^2 \phi + (2 - (r_0 + r_1)\cos\phi)^2 = 4 - 4(r_0 + r_1)\cos\phi + (r_0 + r_1)^2.$$

The cross ratio of these point is then

$$x = (-r_0, c - r_1; r_0, c + r_1) = \frac{(r_0 - r_1)^2 - c^2}{(r_0 + r_1)^2 - c^2} = 1 + \frac{r_0 r_1}{1 - (r_0 + r_1)\cos\phi} = 1 + \frac{r_0 r_1}{(r_0 + r_1)\cos\theta}.$$

Therefore,

$$\ell(\alpha) \geq d_{\mathbb{H}}(P_0, \rho_{1,2}(P_3)) \geq \inf_{r_0, r_1, \phi} \log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) = \inf_{r_0, r_1, \phi} \log\left(1 + \frac{2}{\sqrt{x}-1}\right).$$

Since $\log(1 + 2/(\sqrt{x} - 1))$ is a decreasing function of x, our goal is to maximize x over all allowable configurations with fixed $0 \leq \theta < \pi/2$. Our parameter conditions imply

$$0 \leq r_i + r_i \cos\theta \leq 1, \quad \frac{1}{1 + \cos\theta} \leq (r_0 + r_1), \quad \text{and} \quad (r_0 + r_1) \leq \frac{1}{\cos\theta}.$$

Since $0 \leq \theta \leq \pi/2$, it is easy to see that this region is a triangle in the (r_0, r_1)-plane bounded by $r_i = 1/(1 + \cos\theta)$ and $(r_0 + r_1) = 1/(1 + \cos(\theta))$, see Figure 3.

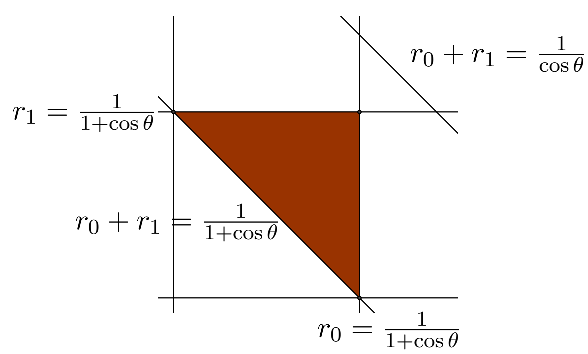

FIGURE 3. Constraints for maximizing $x = 1 + \frac{r_0 r_1}{(r_0+r_1)\cos\theta}$ in Lemma 3.2.

We also have

$$\frac{\partial x}{\partial r_i} = \frac{r_j^2}{(r_i + r_j)^2} > 0 \text{ for } r_i, r_j > 0 \text{ where } \{i, j\} = \{0, 1\},$$

so the maximum value of x is attained on the boundary of our triangle. On the edges corresponding to $r_i = 1/(1 + \cos\theta)$, we get a maximum when $r_0 = r_1 = 1/(1 + \cos\theta)$. For the edge corresponding to $(r_0 + r_1) = 1/(1 + \cos(\theta))$, we have a maximum at $r_0 = r_1 = 1/(2 + 2\cos\theta)$. Of these two points, x has the largest value at the former, so

$$\sup_{r_0, r_1, \phi} x = x\,|_{r_i = 1/(1+\cos\theta)} = 1 + \frac{1}{2(1 + \cos\theta)\cos\theta}$$

Lastly, note that using $\cosh(z) = (e^z + e^{-z})/2$, we have

$$\cosh\left(\log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right)\right) = \frac{1}{2}\left(\frac{\sqrt{x}+1}{\sqrt{x}-1} + \frac{\sqrt{x}-1}{\sqrt{x}+1}\right) = \frac{x+1}{x-1}$$

Our desired results follows,

$$\ell(\alpha) \geq \inf_{r_0,r_1,\phi} \log\left(\frac{\sqrt{x}+1}{\sqrt{x}-1}\right) = \inf_{r_0,r_1,\phi} \cosh^{-1}\left(\frac{x+1}{x-1}\right) = \cosh^{-1}\left((2\cos\theta+1)^2\right).$$

□

Corollary 3.3. *The shortest rectifiable path $\alpha(t) \subset \mathbb{H}^3$ connecting four sequentially tangent hyperplanes in $\mathbb{H}^3$ has length $2\sinh^{-1}(2)$ and is attained when the planes support four of the faces of a standard ideal octahedron.*

Proof. If $\theta = 0$, then the geodesic we have find in Lemma 3.2 has length

$$\cosh^{-1}\left((2\cos(0)+1)^2\right) = \cosh^{-1}(9) = 2\sinh^{-1}(2).$$

The critical values of r_0, r_1 were $r_i = 1/(1+\cos\theta) = 1/2$, so $1 = (r_0+r_1)(\cos\theta + \cos\phi) = 1 + \cos\phi$ and $\phi = \pi/2$. This configuration and the other four planes supporting a standard ideal octahedron are shown in Figure 4. □

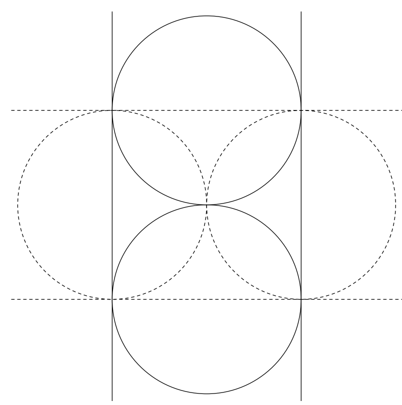

FIGURE 4. The supporting planes of a standard ideal octahedron in Cor 3.3.

Next, we prove a slight generalization of Lemma 3.2 where we replace the tangency of P_0 and P_3 for another condition.

Lemma 3.4. *Let P_0, P_1, P_2, P_3 be planes in $\mathbb{H}^3$ with boundary circles $C_i \in \partial_\infty \mathbb{H}^3$. Assume*

(i) $P_0 \cap P_2 = P_1 \cap P_3 = \emptyset$

(ii) $C_1 \cap C_2 = \{b\}$ *and* $b \notin C_0 \cup C_3$.

(iii) let $P_\star$ be the unique plane between P_1 and P_2 tangent to P_3, then $\partial P_\star \cap C_0 \neq \emptyset$

(iv) let η_i be normal directions to C_i such that η_0, η_3 point away from each other and η_1, η_2 point toward each other, then $\angle(\eta_0, \eta_1) = \angle(\eta_2, \eta_3) = \theta < \pi/2$.

If $\alpha : [0,1] \to \mathbb{H}^3$ is a rectifiable path with $\alpha(0) \in P_0, \alpha(1) \in P_3$, $\alpha(t_1) \in P_1$, and $\alpha(t_2) \in P_2$ for some $t_1, t_2 \in (0,1)$ with $t_1 < t_2$. Then,

$$\ell(\alpha) \geq \cosh^{-1}\left((2\cos\theta+1)^2\right).$$

Proof. We will reduce to the case of Lemma 3.2 as follows. We can conjugate $b \to \infty$ and build as similar diagram with C_3 "below" C_0 as before, except they may no longer be tangent. Condition (iii) implies that there is some "slide" of P_0 along $P_\star$ to a plane P_0' that is tangent to P_3, see Figure 5.

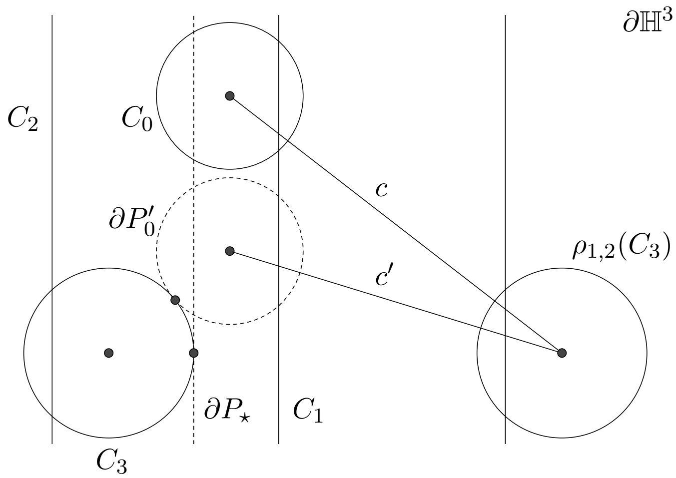

FIGURE 5. The "slide" move of P_0 to P_0' in Lemma 3.4. Notice that the Euclidean length $c \geq c'$.

Notice that the "slide" operation does not change radii of the circles in our configuration. In the proof of Lemma 3.2, the cross ratio was given as

$$x = \frac{(r_0 - r_1)^2 - c^2}{(r_0 + r_1)^2 - c^2}$$

This function is decreasing in c, so if we replace c with the shorter c' as in Figure 5. This gives a larger value of x and, therefore, a shorter geodesic. Thus, we replace P_0 with P_0' and apply Lemma 3.2. $\square$

Proof of Theorem 1.1. Fix $L \in (0, 2\sinh^{-1}(2)]$. If we fix $\|\mu\|_L$, then for every $\epsilon > 0$, we can find a geodesic arc $\alpha : (0,1) \to P_\mu$ with $\ell(\alpha) = L$ such that $\|\mu\|_L - \epsilon < i(\alpha, \mu) \leq \|\mu\|_L$. Let $\{P_t\}$ for $t \in [0,1]$ denote the path of support planes to α and $p : [0,1] \to [0,1]$ be such that P_t is a support plane at $\alpha(p(t))$. Here, we take $P_0 = \lim_{t \to 0^+} P_t$ and $P_1 = \lim_{t \to 1^-} P_t$. We will divide our argument into cases via bounds on $\|\mu\|_L$.

Case $\|\mu\|_L \leq \pi$**.** This is the trivial case as $0 \leq L$ implies $\|\mu\|_L \leq \pi = F(0) \leq F(L)$.

Case $\pi < \|\mu\|_L \leq 2\pi$**.** Fix $\epsilon > 0$ small enough and α of length L such that

$$\pi < \|\mu\|_L - \epsilon < i(\alpha, \mu) \leq \|\mu\|_L \leq 2\pi.$$

Let $2\theta = 2\pi - \|\mu\|_L + \epsilon < \pi$, then by assumption $2\pi - 2\theta < i(\alpha, \mu)$. As the interior angle between P_0 and P_t decreases continuously, it follows from the roof property (Lemma 2.2) that there must be a t_1 such that $\angle_{int}(P_0, P_{t_1}) = \theta$ as $i(\alpha, \mu) > \pi - \theta$. Similarly, there must be at t_2 such that $\angle_{int}(P_{t_1}, P_{t_2}) = \theta$ as $i(\alpha, \mu) > 2\pi - 2\theta$. Notice that $P_0 \cap P_{t_2} = \emptyset$, as otherwise either they form a roof over α by Lemma 2.3 and $i(\alpha, \mu) \leq \pi$, a contraction.

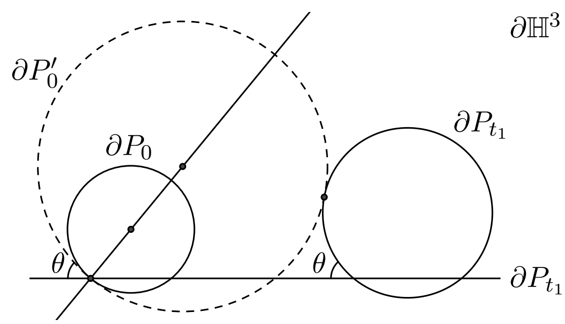

FIGURE 6. The "grow" move of P_0 to P_0' in **Case** $\pi < \|\mu\|_L \leq 2\pi$ of Theorem 1.1.

Since $2\theta < \pi$, our planes P_0, P_{t_1}, P_{t_2} *almost* satisfy the conditions of Lemma 3.1. By mapping P_{t_1} to a vertical plane in the upper half space model for $\mathbb{H}^3$, we easy see that we can "grow" P_0 to a plane P_0' that is tangent to P_{t_2} while keeping the interior angle with P_{t_1} equal to θ, see Figure 6. The plane P_0' is not a support plane, but a sub-arc of $\alpha[p(0), p(t_2)]$ joins it to P_{t_2}. Therefore, the shortest curve between P_0' and P_{t_2} with a point on P_{t_1} is shorter than α. We apply Lemma 3.1 to P_0', P_{t_1}, P_{t_3} and see

$$L \geq 2\sinh^{-1}(\cos\theta) \quad \Longrightarrow \quad \cos^{-1}\left(\sinh(L/2)\right) \leq \theta.$$

$$\|\mu\|_L = 2\pi - 2\theta + \epsilon \leq 2\pi - 2\cos^{-1}\left(\sinh(L/2)\right) + \epsilon = 2\cos^{-1}\left(-\sinh(L/2)\right) + \epsilon.$$

Since $\epsilon > 0$ can be taken arbitrarily small and $F(L)$ is an increasing function, $\|\mu\|_L \leq F(L)$.

Case $2\pi < \|\mu\|_L \leq 3\pi$**.** Fix $\epsilon > 0$ small enough and α of length L such that

$$2\pi < \|\mu\|_L - \epsilon < i(\alpha, \mu) \leq \|\mu\|_L \leq 3\pi.$$

Let $2\theta = 3\pi - \|\mu\|_L + \epsilon < \pi$, then by assumption $3\pi - 2\theta < i(\alpha, \mu)$. As before, since the interior angle decreases and we cannot violate the roof property, there exists t_1 such that $\angle_{int}(P_0, P_{t_1}) = \theta$, the smallest t_2 such that $\angle_{int}(P_{t_1}, P_{t_2}) = 0$, and t_3 such that $\angle_{int}(P_{t_2}, P_{t_3}) = \theta$.

We want to modify our set of planes slightly to satisfy the assumptions of Lemma 3.4. We see that $P_0 \cap P_{t_2} = P_{t_1} \cap P_{t_3} = P_0 \cap P_{t_3} = \emptyset$ by Lemma 2.3 as any roofs over subarcs of α would decrease its bending. Let $P_\star$ be the unique plane between P_{t_1} and P_{t_2} that is tangent to P_{t_3}. If $P_\star \cap P_0 = \emptyset$, we can then "grow" P_0 to P_0' so that $P_0' \cap P_\star \neq \emptyset$, see Figure 7. As before, P_0' is joined to P_{t_3} by a sub-arc of $\alpha[p(0), p(t_3)]$. As $\theta < \pi/2$, all the assumptions of Lemma 3.4 are satisfied, so we have

$$L \geq \cosh^{-1}\left((2\cos\theta + 1)^2\right) \quad \Longrightarrow \quad \cos^{-1}\left(\left(\sqrt{\cosh(L)} - 1\right)/2\right) \leq \theta.$$

$$\|\mu\|_L = 3\pi - 2\theta + \epsilon \leq 3\pi - 2\cos^{-1}\left(\left(\sqrt{\cosh(L)} - 1\right)/2\right) + \epsilon.$$

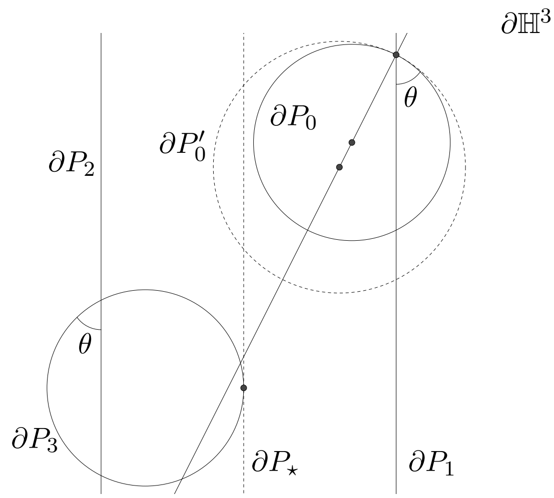

FIGURE 7. The "grow" move of P_0 to P_0' in **Case** $2\pi < \|\mu\|_L \leq 3\pi$ of Theorem 1.1.

Since $\epsilon > 0$ can be taken arbitrarily small, $\|\mu\|_L \leq F(L)$.

Case $\|\mu\|_L > 3\pi$**.** We can choose α of length L such that $i(\alpha, \mu) > 3\pi$. As before, we find the smallest t_1, t_2, t_3 (in that order) such that $\angle_{int}(P_0, P_{t_1}) = \angle_{int}(P_{t_1}, P_{t_2}) = \angle_{int}(P_{t_2}, P_{t_3}) = 0$. Notice that P_{t_3} is *not* the terminal support plane for α, as $i(\alpha, \mu) > 3\pi$. After a possible "grow" move, this configuration corresponds to the case of Lemma 3.4 with $\theta = 0$. This, however, implies

$$L > \cosh^{-1}\left((2\cos(0) + 1)^2\right) = \cosh^{-1}(9) = 2\sinh^{-1}(2)$$

which contradicts the fact that we fixed $L \in (0, 2\sinh^{-1}(2)]$. □

4. Improved Bounds on Average Bending and Lipschitz Constants

In this section, we improve the Lipschitz and average bending bounds of [Bri03, Theorem 1.2].

Theorem 1.2. *There exist universal constants K_0, K_1 with $K_0 \leq 2.494$ and $K_1 \leq 3.101$ such that if $\Gamma \leq \mathrm{Isom}^+(\mathbb{H}^3)$ is a finitely-generated Kleinian group, $N = \mathbb{H}^3/\Gamma$, and the boundary $\partial C(N)$ of the convex core is non-empty and incompressible in N, then*

(i) if μ_Γ is the bending lamination of $\partial C(N)$, then

$$\ell_{\partial C(N)}(\mu_\Gamma) \leq K_0\, \pi^2 \, |\chi\,(\partial C(N))|$$

(ii) for any closed geodesic α on $\partial C(N)$,

$$B_\Gamma(\alpha) = \frac{i(\alpha, \mu_\Gamma)}{\ell(\alpha)} \leq K_1$$

where $B_\Gamma(\alpha)$ is called the average bending of α.

(iii) there exists a $(1+K_1)$-Lipschitz map $s : \partial C(N) \to \partial_\infty N$ that is a homotopy inverse to the nearest point retraction $r : \partial_\infty N \to \partial C(N)$.

Proof. Our result is a direct generalization of [Bri03] by using our function $F(L)$ from Theorem 1.1. We provide an outline of the proof.

Let δ be a geodesic arc on P_{μ_Γ} and fix $L \in (0, 2\sinh^{-1}]$. Set $\lceil x \rceil$ to be the least integer $\geq x$. By subdividing δ into arcs or length $\leq L$, we see

$$B_\Gamma(\delta) \leq \frac{\|\mu_\Gamma\|_L}{\ell(\delta)} \left\lceil \frac{\ell(\delta)}{L} \right\rceil \leq \frac{\|\mu_\Gamma\|_L}{\ell(\delta)} \left(\frac{\ell(\delta)}{L} + 1 \right) = \frac{\|\mu_\Gamma\|_L}{L} \left(1 + \frac{L}{\ell(\delta)} \right) \leq \frac{F(L)}{L} \left(1 + \frac{L}{\ell(\delta)} \right)$$

For an infinite length geodesic β on P_{μ_Γ} and a point $x \in \beta$, let β_x^t denote the sub-arc centered at x of length $2t$. One can define average bending for β as

$$B_\Gamma(\beta_x) = \limsup_{t \to \infty} B_\Gamma(\beta_x^t).$$

In [Bri98], Bridgeman shows that this notion is well defined and independent of x. In particular, by taking $\ell(\delta) \to \infty$ in the bound on $B_\Gamma(\delta)$, we see that for any infinite length geodesic β on P_{μ_Γ},

$$B_\Gamma(\beta) \leq \frac{F(L)}{L} \text{ for all } L \in (0, 2\sinh^{-1}(2)].$$

Let

$$K_1 = \min \left[\left(3\pi - 2\cos^{-1}\left(\frac{\sqrt{\cosh(L)}}{2} - \frac{1}{2} \right) \right) / L \right] \text{ over } L \in (2\,\mathrm{arcsinh}(1), 2\sinh^{-1}(2)]$$

Then, $B_\Gamma(\beta) \leq K_1 \leq 3.101$, where the minimum is attained at $L \approx 2.74104$.

For a closed geodesic α on $\partial C(N)$, let $\widetilde{\alpha} \subset P_{\mu_\Gamma}$ be a lift. Then (ii) follows, as

$$B_\Gamma(\alpha) = B_\Gamma(\widetilde{\alpha}) \leq K_1.$$

The statement of (iii) can be derived from (ii). Let K_s be the minimal Lipschitz constant of $s : \partial C(N) \to \Omega(\Gamma)/\Gamma$. Then, Thurston characterized

$$K_s = \sup \left\{ \frac{\ell(s_*\alpha)}{\ell(\alpha)} \,\middle|\, \alpha \text{ is a simple closed curve on } \partial C(N) \right\}$$

and McMullen's showed that $\ell(s_*\alpha) \leq \ell(\alpha) + i(\alpha, \mu_\Gamma)$ (see [Thu98, Theorem 8.5] and [McM98, Theorem 3.1]). Combining these two facts gives $K_s \leq 1 + B_\Gamma(\widetilde{\alpha}) \leq 1 + K_1$, so (iii) holds.

For (i), we use a computation from [Bri03, Section 5] to bound $\ell(\mu_\Gamma)$ by integrating along the unit tangent bundle of $\partial C(N)$. Fix $L \in (0, 2\sinh^{-1}(2)]$ and for $v \in T_1(\partial C(N))$, let $\alpha_v : (0, L) \to \partial C(N)$ be the unit speed geodesic in the direction v. Then, Bridgeman and Canary [BC05] show

$$\ell(\mu_\Gamma) = \frac{1}{4L} \int_{T_1(\partial C(N))} i(\alpha_v, \mu_\Gamma)\, d\Omega$$

By taking a maximal lamination $\widetilde{\mu} \supset \mu_\Gamma$, one can integrate our bound $F(L) \geq i(\alpha_v, \mu_\Gamma)$ over the set of ideal triangles $\partial C(N) \setminus \widetilde{\mu}$. In [Bri03, Section 5], Bridgeman works out this integral and shows that

$$\frac{\ell(\mu_\Gamma)}{\pi^2 \left|\chi\left(\partial C(N)\right)\right|} \leq \frac{3}{\pi^2 L} \int_{(x,y)\in U} \frac{dx\,dy}{y^2} \int_0^{\cos^{-1}\left(\frac{D(x,y)}{\tanh(L)}\right)} F\left(L - \tanh^{-1}\left(\frac{D(x,y)}{\cos\theta}\right)\right) d\theta = K_0$$

where U is the ideal triangle

$$U = \{(x,y) \mid -1 \leq x \leq 1,\, y \geq \sqrt{1-x^2}\} \text{ and}$$

$$D(x,y) = \frac{x^2+y^2-1}{\sqrt{(x^2+y^2-1)^2+4y^2}}$$

computes the length of the unique perpendicular from (x,y) to the "bottom" edge of U. We compute this integral with using numerical approximation in Mathematica. We choose $L = \sinh^{-1}(89/10) < 2\sinh^{-1}(2)$ and find the upper bound

$$\frac{\ell(\mu_\Gamma)}{\pi^2 \left|\chi\left(\partial C(N)\right)\right|} \leq K_0 \leq 2.494.$$

□

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Department of Mathematics, Boston College, 140 Commonwealth Ave., Chestnut Hill, MA 02467

E-mail address: andrew.yarmola@bc.edu

URL: https://www2.bc.edu/andrew-v-yarmola