# Bayesian causal inference: day 2 practical solutions

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Yesterday, we studied the Lalonde experimental dataset from Dehejia and Wahba (1991) and Lalonde (1986). Today we will analyze a dataset with the same treated individuals combined with 2490 controls from the nonexperimental Panel Study of Income Dynamics (PSID), consisting of male household heads from 1975 to 1978 under the age of 55 who did not classify themselves as retired. The objective is to apply ideas from the lectures today to investigate whether we can replicate the experimental results by analyzing this dataset as if it came from an observational study. Again, it is recommended to read the first two sections of Imbens and Xu (2024) if you haven't already.

The skeleton code is found in `day2_practical.R`. We will need the `BART` R package for today's practical. Please install this package if you haven't already. Like yesterday, we will start by computing the difference-in-means estimator with the experimental data:

```
diff_means <- mean(ldw[ldw$treat == 1,Y]) - mean(ldw[ldw$treat == 0,Y])
```

The value of this estimator will act as our experimental benchmark to be compared with the results from our observational analyses.

We will run BART probit-link regression to fit the propensity score model on the observational data `ldw_psid`. As a default, we discard 2000 burn-in samples before obtaining 5000 posterior samples.

```
## run probit-link BART regression to fit propensity score model
Mskip <- 2000 # number of burn-in samples
M <- 5000 # number of posterior samples
pi_post <- pbart(x.train = ldw_psid[,covar], y.train = ldw_psid[,treat], ndpost = M,
nskip = Mskip, printevery = 500L)
```

Next, we extract the posterior draws of the propensity score:

```
pi_draws <- pi_post$prob.train
```

This gives us the posterior draws of $\pi(X_i)$ for each of the data covariate values $X_i$.

Now we proceed to fit the outcome regression model $\mu(t,x) = \mathbb{E}[Y \mid T = t, X = x]$ using BART. To compute the average treatment effects later, we need posterior draws of $\mu(1, X_i)$ and $\mu(0, X_i)$ for each $X_i$. So we start by preparing the test data:

```
## prepare test data for fitting BART outcome regression model
test_dat_tr <- cbind(rep(1, nrow(ldw_psid)), ldw_psid[,covar])
test_dat_co <- cbind(rep(0, nrow(ldw_psid)), ldw_psid[,covar])
names(test_dat_tr)[1] <- "treat"
names(test_dat_co)[1] <- "treat"
test_dat <- rbind(test_dat_tr, test_dat_co)
```

Then we fit the BART outcome regression model and extract the posterior samples for $\mu(1, X_i)$ and $\mu(0, X_i)$:

```
## fit BART outcome regression model
mu_post <- wbart(x.train = ldw_psid[,c(treat,covar)], y.train = ldw_psid[,Y],
x.test = test_dat, ndpost = M, nskip = Mskip, printevery = 500L)

## extract posterior samples for mu
mu_draws_tr <- mu_post$yhat.test[,1:n]
mu_draws_co <- mu_post$yhat.test[,(n+1):(2*n)]
```

The above routine should take 1-2 minutes.

(i) Use `mu_draws_tr` and `mu_draws_co` to obtain the marginal posterior draws of the CATE estimand

$$\chi_{CATE} = \frac{1}{n}\sum_{i=1}^{n} \mu(1, X_i) - \mu(0, X_i)$$

from Hill (2011). Compute the posterior mean and the 95% central credible interval (i.e., the 2.5% and 97.5% quantiles of the draws).

*Answer: Posterior mean = -$2598. Central 95% credible interval =* [−$6646, $1192].

Now we obtain samples from the one-step posterior for the average treatment effect. Recall that the one-step corrected parameter for the ATE takes the form

$$\tilde{\chi}_{ATE} = \tilde{P}\left[\frac{T(Y-\mu(1,\cdot))}{\pi} - \frac{(1-T)(Y-\mu(0,\cdot))}{1-\pi} + \mu(1,\cdot) - \mu(0,\cdot)\right],$$

where $\tilde{P}$ is drawn from the Bayesian bootstrap posterior. In practice, we sample $\tilde{\chi}_{ATE}$ via Algorithm 1, where $\pi^{(b)}$ and $\mu^{(b)}$ denote the posterior draws of $\pi$ and $\mu$ respectively. Following conventional practice, we "trim" the posterior draws of $\pi$ away from 0 and 1 to stabilize estimation in the presence of inverse weighting. Algorithm 1 is implemented in the script.

(ii) Compute the posterior mean and the 95% central credible interval for $\tilde{\chi}_{ATE}$. Compare the results to those of $\chi_{CATE}$ and the difference-in-means estimator.

*Answer: Posterior mean = -$2627. Central 95% credible interval =* [−$6325, $853]. *Comparisons of the densities can be found in Figure 1.*

The next section of the script uses the posterior mean of the propensity score (`pi_postmean`) as a point estimate to evaluate the overlap in the dataset. The following commands generate two histogram plots:

**Algorithm 1:** One-step posterior sampling for the ATE

**for** $b \leftarrow 1$ *to* $M$ **do**

  Draw $(W_1^{(b)}, \ldots, W_n^{(b)})$ from $\mathrm{Dir}(n; 1, \ldots, 1)$;

  $\pi^{(b)} \leftarrow \min(\pi^{(b)}, 0.9)$;

  $\pi^{(b)} \leftarrow \max(\pi^{(b)}, 0.1)$;

  Evaluate $\tilde{\chi}_{ATE}^{(b)} = \sum_{i=1}^{n} W_i^{(b)} \left[ \frac{T_i(Y_i - \mu^{(b)}(1, X_i))}{\pi^{(b)}(X_i)} - \frac{(1-T_i)(Y_i - \mu^{(b)}(0, X_i))}{1-\pi^{(b)}(X_i)} + \mu^{(b)}(1, X_i) - \mu^{(b)}(0, X_i) \right]$;

**end**

Return $\{\tilde{\chi}_{ATE}^{(1)}, \ldots, \tilde{\chi}_{ATE}^{(M)}\}$.

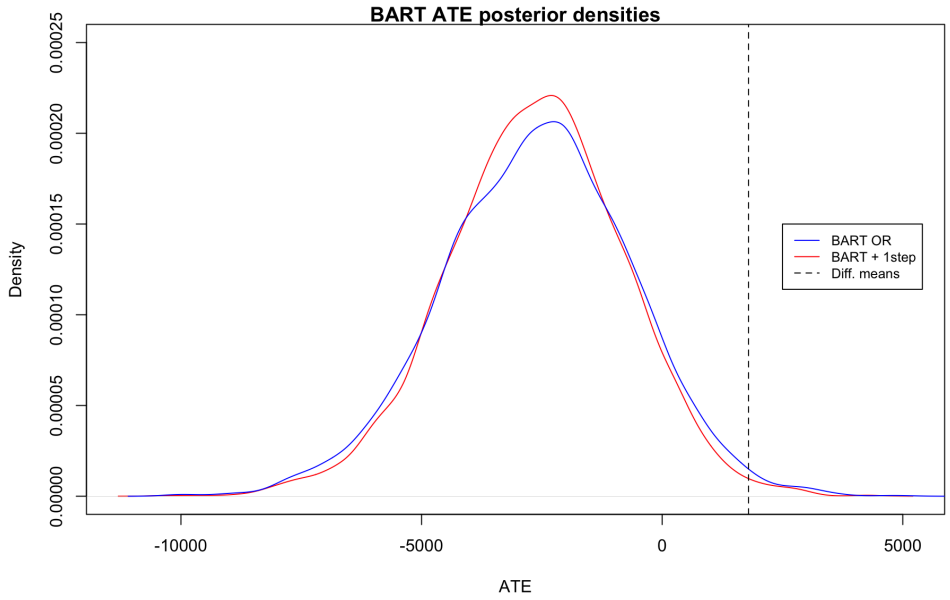


Figure 1: Comparisons of the ATE posterior densities.

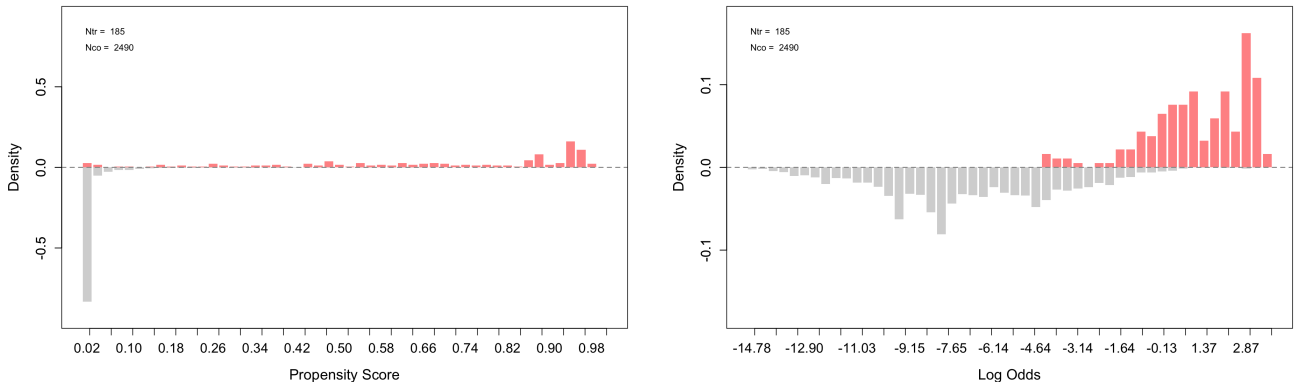


Figure 2: Propensity score densities for treated and controls on the standard scale (left) and the log-odds scale (right).

```
# overlap plot for propensity score
plot_hist(ldw_psid_prop, "pi_est", treat, odds = FALSE, breaks = 50,
          density = TRUE, main = "", xlim = c(0,1), ylim = NULL)

# overlap plot on log odds scale
plot_hist(ldw_psid_prop, "pi_est", treat, odds = TRUE, breaks = 50,
          density = TRUE, main = "", xlim = NULL, ylim = NULL)
```

The first plot compares the densities of the estimated propensity score values for treated and controls. The second plot does the same thing but on the log-odds scale (i.e. $\hat{\pi} \mapsto \log(\hat{\pi}/\{1-\hat{\pi}\})$).

(iii) Use these plots to argue that the average treatment effect on treated (ATT) $\mathbb{E}[Y^1 - Y^0 \mid T = 1]$ is in fact a more appropriate estimand than the ATE.

*Answer: The two plots can be found in Figure 2. We can make two key observations. First, many of the estimated propensity score values for the controls are very close to 0. This suggests that the ATE will be hard to estimate. To be more specific, recall from the lectures that the efficient influence function for the ATE involves inverse weighting by the propensity score, so its variance will be large when the propensity score has high density around 0, indicating that the ATE cannot be estimated precisely. Second, there is an alarming lack of overlap in the support of the densities even as we move away from 0, indicating that the population for PSID differs significantly from the experimental population. Given that the experimental population is of more relevant interest, it seems sensible to focus on the treated population.*

(iv) Write a routine to obtain marginal posterior draws of the CATT estimand

$$\chi_{CATT} = \frac{1}{n_t} \sum_{i:T_i=1} \mu(1, X_i) - \mu(0, X_i)$$

from Hill (2011), where $n_t = \sum_{i=1}^{n} T_i$. Compute the posterior mean and the 95% central credible interval.

*Answer: Posterior mean = \$1388. Central 95% credible interval =* [−\$1257, \$4080].

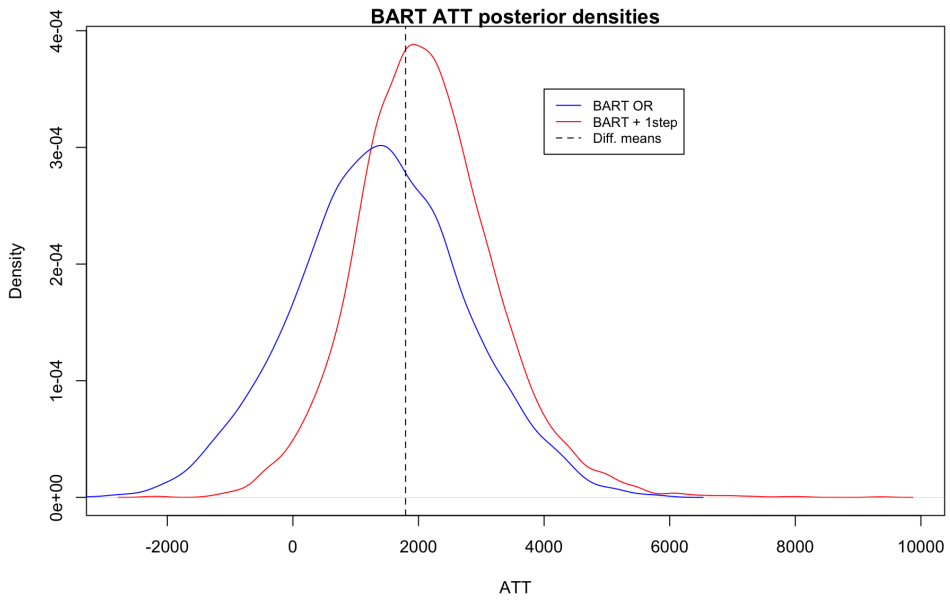


Figure 3: Comparisons of the ATT posterior densities.

(v) The one-step posterior parameter for the ATT has the following form:

$$\tilde{\chi}_{ATT} = \tilde{P}\left[\frac{(T-\pi)(Y-\mu(0,\cdot))}{1-\pi}\right].$$

Write a routine to draw posterior samples for $\tilde{\chi}_{ATT}$. Compute the posterior mean and the 95% central credible interval and compare the results to those of $\chi_{CATT}$ and the difference-in-means estimator.

*Answer: Posterior mean = \$2121. Central 95% credible interval =* [\$44, \$4470]. *Comparisons of the densities can be found in Figure 3.*

(vi) As described in the lectures, Hahn et al. (2020) proposed to include an estimate of the propensity score as an additional splitting covariate for the BART outcome regression model. Using `pi_postmean` (or a different propensity score estimate if you prefer, such as logistic regression), implement this idea and obtain the results for the new posteriors for $\chi_{CATT}$ and $\tilde{\chi}_{ATT}$.

*Answer:*

- *CATT: Posterior mean = \$917. Central 95% credible interval =* [−\$2406, \$4212]
- *One-step ATT: Posterior mean = \$1541. Central 95% credible interval =* [−\$765, \$4470]

*The densities are compared in Figure 4.*

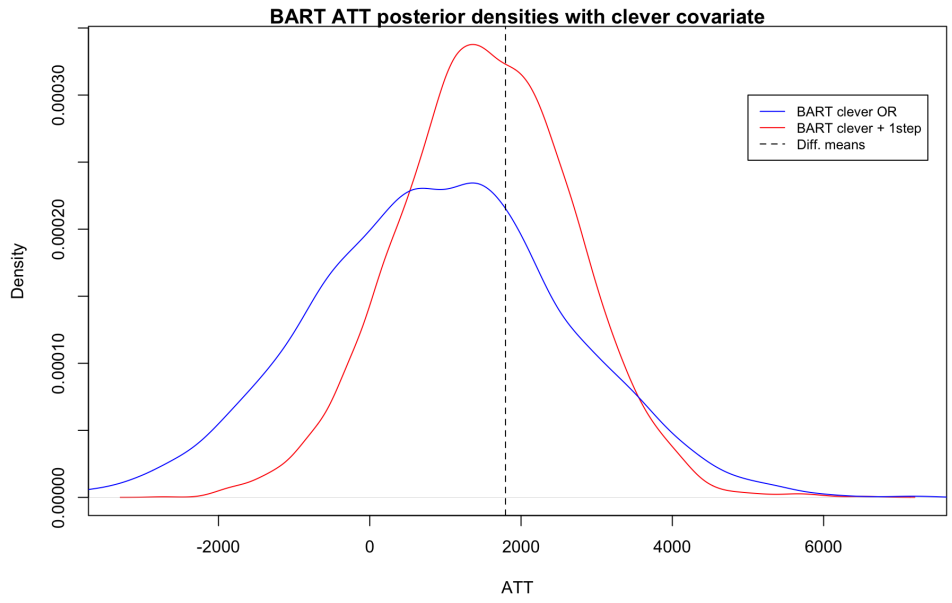


Figure 4: Comparisons of the ATT posterior densities with the estimated propensity score as a covariate.

# References

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R. Lalonde. Evaluating the Econometric Evaluations of Training Programs with Experimental Data. *The American Economic Review*, 76:604–620, 1986.