520 Phase 2 Notes

1.3: Span

- Definition: $v_1, ..., v_p$ are in $\mathbb{R}^n$; set of all linear combinations of $v_1, ..., v_p$ is denoted by Span $\{v_1, ..., v_p\}$; this is called the subset of $\mathbb{R}^n$ spanned by $v_1, ..., v_p$
- Span $\{v_1, ..., v_p\}$ is the collection of all vectors that can be written in the form $c_1 v_1 + ... + c_p v_p$
- vector b is in Span $\{v_1, ..., v_p\}$ if augmented matrix [v_1 ... v_p b] has a solution
- zero vector must be in Span $\{v_1, ..., v_p\}$

1.5: Homogeneous Linear Systems

- system of linear equations is homogeneous if it can be written in form $Ax = 0$
- $Ax = 0$ has a nontrivial solution if and only if the equation as at least one free variable

1.7: Linear Independence and Dependence

- Dependent Definition: vector$\{v_1, ..., v_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if $Ax = 0$ has only the trivial solution
- if a set contains more vectors than entries in each vector (more columns than rows) the set is linearly dependent
- if a set contains the zero vector, the set is linearly dependent
- Independent Definition: vector$\{v_1, ..., v_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if $Ax = 0$ there exist non trivial solutions to the equation

1.8: Introduction to Linear Transformations

- Definition: Linear Transformation
  - A transformation T is linear if
  - (i) $T(u+v) = T(u)+T(v)$ for all u, v in the domain of T
  - (ii) $T(cu) = cT(u)$ for all scalars c and all u in the domain of T

1.9: The Matrix of Linear Transformation

- Definition: onto (surjective)
  - A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each b in $\mathbb{R}^m$ is the image of at least one x in $\mathbb{R}^n$
  - each y has at least one x
- Definition: one-to-one (injective)
  - each y has at most one x
- Theorem 11: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation; T is one-to-one if and only if the equation $T(x) = 0$ has only the trivial solution
- Theorem 12: Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the standard matrix for T, then:
  - a: T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ if and only if the columns of A span $\mathbb{R}^m$
  - b: T is one to one if and only if the columns of A are linearly independent

2.1: Matrix Operations

- Theorem 3: Transpose Properties
  - Let A and B denote matrices whose sizes are appropriate for the following sums and products
  - a: $(A^T)^T = A$
  - b: $(A + B)^T = A^T + B^T$
  - c: For any scalar r, $(rA^T) = rA^T$
  - d: $(AB)^T = B^T A^T$
- the tranpose of a product of matrices equals the product of their transposes in reverse order

2.2: The Inverse of a Matrix

- Theorem 5: If A is an invertible nxn matrix, then for each b in $\mathbb{R}^n$, the equation $Ax = b$ has the unique solution $x = A^{-1}b$
- Theorem 6:
  - $(A^{-1})^{-1} = A$
  - $(AB)^{-1} = B^{-1}A^{-1}$
  - $(A^T)^{-1} = (A^{-1})^T$
- Finding the inverse of a matrix:
  - Gauss-Jordan elimination [A I] to [I A^{-1}]
  - 2d: check visual notes

2.3: The Invertible Matrix Theorem

- A is an invertible matrix
- A is row equivalent to the nxn identity matrix
- A has n pivot positions
- The equation $Ax = 0$ has only the trivial solution
- The columns of A form a linearly independent set
- The linear transformation $x \to Ax$ is one to one
- The equation $Ax = b$ has at least one solution for each b in $\mathbb{R}^n$
- The columns of A span $\mathbb{R}^n$
- the linear transformation $x \to Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$
- There is an nxn matrix C such that $CA = I$
- there is an nxn matrix D such that $AD = I$
- A^T is an invertible matrix

2.8: Subspaces of $\mathbb{R}^n$

- Definition of subspace: A subspace of $\mathbb{R}^n$ is any set H in $\mathbb{R}^n$ that has three properties:

- a: The zero vector is in H
- b: For each u and v in H, the sum u+v is in H
- c: For each u in H and each scalar c, the vector cu is in H
- basically, a subspace is closed under addition and scalar multiplication

- Column space Definition: The Column space of a matrix A is the set Col A of all linear combinations of the columns of A
- column space of an mxn matrix is a subspace of $\mathbb{R}^m$; COl A equals $\mathbb{R}^m$ only when the columns of A span $\mathbb{R}^m$
- Null space definition: The null space of a matrix A is the set Nul A of all solutions of the homogeneous equation $Ax = 0$
- null space of A is the subset of $\mathbb{R}^n$
- Basis Definition: A basis for subspace H of $\mathbb{R}^n$ is linearly independent set in H that spans H

2.9: Dimension and Rank

- Coordinate vector definition:
  - given set B $\{b_1, ..., b_p\}$, a basis for a subspace H; for each vector x in H, the coordinates of x relative to the basis B are the weights $c_1, ..., c_p$ such that $x = c_1 b_1 + ... + c_p b_p$; also known as the B coordinate vector of x
- Dimension definition: the dimension of a nonzero subspace H, denoted by dim H, is the number of vectors in any basis for H. The dimension of the zero subspace 0 is defined to be 0.
- Rank Definition: The rank of a matrix A, denoted by rank A, is the dimension of the column space of A
- Theorem 14: The Rank Theorem
  - If a matrix A has n columns then rank A + dim Nul A = n
- Theorem 15: The Basis Theorem
  - Let H be a p-dimensional subspace of $\mathbb{R}^n$
  - Any linearly independent set of exactly p elements in H is automatically a basis for H
  - Any set of p elements of H that spans H is automatically a basis for H

The Invertible Matrix Theorem Continued

- Let A be an nxn matrix
- The columns of A form a basis for $\mathbb{R}^n$
- Col A = $\mathbb{R}^n$
- dim Col A = n
- rank A = n
- Nul A = $\{0\}$
- dim Nul A = 0

3.1: Intro to Determinants

- $detA = \Sigma_{j=1}^{n} (-1)^{i+j} a_{1j}$
- determinant can be computed by cofactor expansion across any row or down any column
- Theorem 2: If A is a triangular matrix, then det A is the product of the entries on the main diagonal of A

Properties of Determinants

- Theorem 3: Row Operations
  - Let A be a square matrix
  - a: multiple of one row added to another row $\rightarrow$ det B=det A
  - b: rows are interchanged $\rightarrow$ det B = - det A
  - c: if one row of A is multiplied by k to produce B, then det B = k $\cdot$ Det A
- Theorem 4: A square matrix A is invertible if and only if det A $\neq 0$
- Theorem 5: A is a nxn matrix, det A^T = det A

3.3: Cramer's Rule

- Let A be an invertible nxn matrix
- For any b in $\mathbb{R}^n$, the unique solution x of $Ax = b$ has entries given by $x_i = \frac{\det\ A_i(b)}{\det A}$ for $i = 1, ..., n$

3.3: Theorem 8: An Inverse Formula

- Let A be an invertible nxn matrix, then:

$$A - 1 = \frac{1}{\text{det A}} \text{adj A}$$

4.1: Vector Spaces and Subspaces

- Subspace definition:
  - A subspace of a vector space V is a subset H of v has 3 properties:
  - a: zero vector of V is in H
  - H is closed under vector addition: for each u and v in H, the sum (u+v) is in H
  - H is closed under multiplication by scalars: for each u in H and each scalar c, the vector cu is in H
- Theorem 1: If $v_1, ..., v_p$ are in a vector space V, then Span $\{v_1, ..., v_p\}$ is a subspace of V

4.2: Null spaces, column spaces, and linear transformations

- Null space definition: The null space of an mxn matrix A, written as Nul A, is the set of all solutions of the homogeneous equation Ax=0: Nul A $= \{x : x \text{ is in } \mathbb{R}^n \text{ and } Ax = 0\}$

- Column space definition: The column space of an mxn matrix A, written as Col A, is the set of all linear combinations of A; if A = $[a_1 ... a_n]$ then Col A = Span $\{a_1, ..., a_n\}$
- Theorem 3: the column space of an mxn matrix A is a subsapce of $\mathbb{R}^m$
- the column space of an mxn matrix A is all of $\mathbb{R}^m$ if and only if the equation Ax=b has a solution for each b in $\mathbb{R}^m$
- Linear Transformation Definition: A linear transformation T form a vector space V into a vector space W is a rule that assigns to each vector x in V a unique vector T(x) in W, such that
  - T(u+v) = T(u)+T(v) for all u, v in V
  - T(cu) = cT(u) for all u in V and all scalars c

4.3: Linearly Independent Sets; Bases

- Theorem 4: given a set of vectors, if one vector is a linear combination of any of the other vectors, the set is linearly dependent (and vice versa)
- Basis Definition: Let H be a subspace of a vector space V; An indexed set of vectors $B = \{b_1, ..., b_p\}$ in V is a basis for H if
  - B is a linearly independent set
  - the subspace spanned by B coincides with H -¿ $H = Span\{b_1, ..., b_p\}$
- Theorem 5: The Spanning Set Theorem
  - Let S = $\{v_1, ..., v_p\}$ be a set in V, let H=Span$\{v_1, ..., v_p\}$
  - if one of the vectors that is a linear combination of any of the other vectors is removed, the set without that vector still spans H
  - if $H \neq \{0\}$ some subset of S is a basis for H
- Theorem 6: the pivot columns of a matrix A form a basis for Col A

4.4: Coordinate Systems

- B-coordinates of X definition (see 2.9 section)
- Theorem 8: Let B = $\{b_1, ..., b_n\}$ b e a basis for a vector space V; then the coordinate mapping $x \mapsto [x]_B$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$
- Isomorphism definition: a one-to-one linear transformation from a vector space V onto a vector space W is called an isomorphism from V onto W

4.5: The Dimension of a Vector Space

- Dimension Definition:
  - If V is spanned by a finite set, V is finite-dimensional and dim V is the number of vectors in a basis for V
  - dimension of the zero vector space is defined to be 0
  - if V is not spanned by a finite set, V is infinite dimensional
- The dimension of Nul A is the number of free variables in the equation Ax=0
- The dimension of Col A is the number of pivot columns in A

4.6: Rank

- Row Space Definition: set of all linear combinations of the row vectors is called the row space of A (row A); Col A^T = row A
- Theorem 13: If 2 matrices A and B are row equivalent, their row spaces are the same; If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B
- Rank Definition: the rank of A is the dimension of the column space of A
- Theorem 13: The Rank Theorem
  - the dimensions of the column space and the row space of an mxn matrix A are equal
  - rank A = dim Nul A = n
  - number of pivot columns + number of non pivot columns = number of columns
- The Invertible Matrix Theorem (continued)
  - Let A be an nxn matrix
  - columns of A form a basis of $\mathbb{R}^n$
  - Col A = $\mathbb{R}^n$
  - dim Col A = n
  - rank A = n
  - Nul A = $\{0\}$
  - dim nul A = 0

4.7: Change of Basis

- Theorem 15:
  - Let B = $\{b_1, ..., b_n\}$ and C = $\{c_1, ..., c_n\}$ be bases of a vector space V
  - then there is a unique nxn matrix $P_{C \leftarrow B}[x]_B$
  - the columns of $P_{C \leftarrow B}$ are the C-coodrinate vectors of the vectors in the basis B
- Change of Basis in $\mathbb{R}^n$: $[c_1\ c_2 \mid b_1\ b_2] \rightarrow$ [I | $P_{C \leftarrow B}$]

5.1: Eigenvectors and Eigenvalues

- Definition: An eigenvector of an nxn matrix A is a nonzero vector x such that $Ax = \lambda x$ for some scalar λ
- Definition: A scalar λ is called an eigenvalue of A if there is a nontrivial solution x of $Ax = \lambda x$; such an x is called an eigenvector corresponding to λ

- Theorem 2: If $\{v_1, ..., v_r\}$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, ..., \lambda_r$ of an nxn matrix A, then the set $\{v_1, ..., v_r\}$ is linearly independent

5.2: Characteristic Equation

- The Invertible Matrix Theorem Continued
  - Let A be an nxn matrix, A is invertible if and only if
  - The number 0 is not an eigenvalue of A
  - The determinant of A is not zero
- Definition: A scalar λ is an eigenvalue of an nxn matrix A if and only if λ satisfies the characteristic equation $\det(A - \lambda I) = 0$
- Similarity Definition: A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$ or equivalently $A = PBP^{-1}$
- Theorem 4: If nxn matrices A and B are similar, they have the same characteristic polynomial and hence the same eigenvalues with the same multiplicities

5.3: Diagonalization

- Diagonalizable definition: A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix; that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D
- Theorem 5: Diagonalization Theorem
  - An nxn matrix A is diagonalizable if and only if A has n linearly independent eigenvectors
  - $A = PDP^{-1}$ with D a diagonal matrix if and only if the columns of P are n linearly independent eigenvectors of A; then the diagonal entries of D are eigenvalues of A that correspond respectively to the eigenvectors in P
- A is diagonalizable if and only if there are enough eigenvectors to form a basis of $\mathbb{R}^n$
- Diagonalizing Matrices:
  - 1: Find the eigenvalues of A
  - 2: Find 3 linearly independent eigenvectors of A
  - 3: Construct P form vectors in step 2
  - 4: Construct D from corresponding eigenvalues

5.4: Eigenvectors and Linear Transformations

- Theorem 8: Diagonal Matrix Representation
  - Suppose $A = PDP^{-1}$, D is the diagonal nxn matrix
  - B is the basis for $\mathbb{R}^n$ formed from the columns of P
  - D is teh B matrix for the transformation $x \mapsto Ax$

5.5: Complex Eigenvalues

- complex scalar λ satisfies $\det(A - \lambda I) = 0$ if and only if there is a nonzero vector x in $\mathbb{C}^n$ such that $Ax = \lambda x$; λ is the complex eigenvalue and x is the complex eigenvector corresponding to λ