Chapter 4 Notes: Cont Rand Var

Definitions and Terminologies

- Continuous Random Variable: takes values on intervals of $\mathbb{R}$
- probability density function: a nonnegative function $f : \mathbb{R} \to \mathbb{R}_+$ such that for every subset $B \subseteq \mathbb{R}$: $P(X \in B) = \int_B f(x)dx$; necessary that $\int_{\mathbb{R}} f(x)dx = 1$
- cumulative distribution function: for every $x \in \mathbb{R}$, $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(z)dz$

Properties of Cumulative Distribution Function

- F is increasing
- $F(-\infty) = 0$, $F(\infty) = 1$
- F is continuous
- $F'(x) = f(x)$

Expected Value

- $E[X] = \int_{-\infty}^{\infty} xf(x)dx$
- let a, b be real numbers, then $E[aX + b] = aE[X] + b$

Functions of Random Variables

- let h: $\mathbb{R} \to \mathbb{R}$ be a function, then $E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)dx$
- for any functions $h_1, ..., h_k : \mathbb{R} \to \mathbb{R}$, $E[h_1(X) + ... + h_k(X)] = E[h_1(X)] + ... + E[h_k(X)]$

Variance and Standard Deviation

- Variance: average of square deviations from the mean: $\mathrm{Var}[X] = E[(X - E[X])^2]$
- standard deviation: $\mathrm{Std}[X] = \sqrt{\mathrm{Var}[X]}$
- $\mathrm{Var}[X] = 0$ if and only if P(X=c)=1 for some constant c
- $\mathrm{Var}[X] = E[X^2] - E^2[X]$
- let a, b be real numbers, then $\mathrm{Var}[aX + b] = a^2\mathrm{Var}[X]$

Uniform distribution

- Uniform distribution on [a, b]: X takes values on [a, b] with density:

$$f(x) = \begin{cases} (b-a)^{-1} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

- $E[X] = \frac{1}{2}(a + b)$
- $\mathrm{Var}[X] = \frac{1}{12}(b - a)^2$

Exponential Distribution

- Exponential distribution with rate λ, exponential distribution with parameter $1/\lambda$; X is nonnegative with density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

- $E[X] = \frac{1}{\lambda}$
- $\mathrm{Var}[X] = \frac{1}{\lambda^2}$

Normal (Gaussian) Distribution

- normal distribution $N(\mu, \sigma^2)$; X takes values on $\mathbb{R}$ with density:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} exp - \frac{(x-\mu)^2}{2\sigma^2}$$

  for $x \in \mathbb{R}$

- the function f defines a probability density function on $\mathbb{R}$, and $E[X] = \mu$, $\mathrm{Var}[X] = \sigma^2$
- Properties of normal distribution
  - for every $x \in \mathbb{R}$: $\phi(x) + \phi(-x) = 1$
  - X has distribution $N(\mu, \sigma^2)$, then its standardization $Z = \frac{X-\mu}{\sigma}$ is a standard normal
  - let X be a random variable with distribution $N(\mu, \sigma^2)$, then for any constants a, b, $\in \mathbb{R}$, Y=aX+b is $N(a\mu + b, a^2\sigma^2)$

Standard Normal Distribution $N(0, 1)$

- density is $f(x) = \frac{1}{\sqrt{2\pi}}\exp - \frac{x^2}{2}$ for $x \in \mathbb{R}$
- CDF for N(0, 1): $\phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\exp - \frac{z^2}{2} dz$
- symmetry of standard normal: let Z be N(0, 1) and $x \in \mathbb{R}$, then: $P(Z > x) = P(Z \leq -x)$