Ch.7 Notes: Symmetric Matrices and Quadratic Form

7.1: Diagonalization of Symmetric Matrices

Symmetric Matrices

- Definition: a symmetric matrix is a matrix A such that $A^T = A$; matrix must be square; entries on main diagonal must be symmetric across the main diagonal
- Theorem 1: If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal
- Definition: an nxn matrix A is said to be orthogonally diagonalizable if there are an orthogonal matrix P (with $P^{-1} = P^T$) and a diagonal matrix D such that

$$A = PDP^T = PDP^{-1}$$

- Theorem 2: An nxn matrix A is orthogonally diagonalizable if and only if A is a symmetric matrix

The Spectral Theorem

- Spectrum of a matrix: the set of eigenvalues of a matrix A is sometimes called the spectrum of A
- Theorem 3: The Spectral Theorem of Symmetric Matrices
  - An nxn symmetric matrix A has the following properties:
  - a: A has n real eigenvalues, counting multiplicities
  - b: The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation
  - c: The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal
  - d: A is orthogonally diagonalizable

Spectral Decomposition

- Suppose $A = PDP^{-1}$ where the columns of P are orthonormal eigenvectors $u_1, ..., u_n$ of A and the corresponding eigenvalues $\lambda_1, ..., \lambda_n$ are in the diagonal matrix D; also $P^{-1} = P^T$
- $A = PDP^{-1} = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + ... + \lambda_n u_n u_n^T$

7.2: Quadratic Forms

Quadratic forms

- Definition: a quadratic form on $\mathbb{R}^n$ is a function Q defined on $\mathbb{R}^n$ whose value at a vector x in $\mathbb{R}^n$ can be computed by an expression of the form $Q(x) = x^T Ax$ where A is an nxn symmetric matrix
- matrix A is called the matrix of the quadratic form
- simple example: $Q(x) = x^T Ix = ||x||^2$

Change of Variable in a Quadratic Form

- x represents a variable vector in $\mathbb{R}^n$
- P is an invertible matrix
- y is a new variable vector in $\mathbb{R}^n$, also the coordinate vector of x relative to the basis $\mathbb{R}^n$ determined by the columns of P
- change of variable is: $x = Py$ or $y = P^{-1}x$

Principle Axes

- Theorem 4: The Principal Axes Theorem
  - Let A be an nxn matrix
  - Then there is an orthogonal change of variable $x = Py$ that transforms the quadratic from $x^T Ax$ into a quadratic form $y^T Dy$ with no cross product term
- Definition: The columns of P are called the principal axes of the quadratic form $x^T Ax$; the vector y is the coordinate vector of x relative to the orthonormal basis of $\mathbb{R}^n$ given by these principal axes

Classifying Quadratic Forms

- When A is an nxn matrix, the quadratic form $Q(x) = x^T Ax$ is a real-valued function with domain $\mathbb{R}^n$
- Definition: A quadratic form Q is
  - positive definite if Q(x)¿0 for all $x \neq 0$
  - negative definite if Q(x)¡0 for all $x \neq 0$
  - indefinite if Q(x) assumes both positive and negative values
- Q is said to be positive semi-definite if $Q(x) \geq 0$ for all x; negative semi-definite if $Q(x) \leq 0$ for all x
- Theorem 5: Quadratic forms and Eigenvalues
  - Let A be an nxn symmetric matrix. Then a quadratic from $x^T Ax$ is:
  - a: positive definite if and only if the eigenvalues of A are all positive
  - b: negative definite if and only if the eigenvalues of A are all negative
  - c: indefinite if and only if A has both positive and negative eigenvalues