CS 373 Homework 9 Anchu A. Lee
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1. Let Some_{DFA} =$\{ \langle A\rangle \mid A$ is a DFA and $L(A)$ is not empty and $L(A)$ is not equal to Σ^* $\}$. Show that Some_{DFA} is decidable.

   Construct a Turing machine M to decide the Some_{DFA} problem.

   On input d that is a DFA:

   1. Run $\langle d\rangle$ on a Turing machine T that decides E_{DFA}, if T rejects then continue, if T accepts, then reject.
   2. Construct a DFA d^C that is the compliment of d.
   3. Run $\langle \text{d}^C\rangle$ on a turing machine T, if T rejects then accept d, if T accepts then reject d.

   If d is not a DFA, then M rejects. If $L(d)$ is empty, then M rejects. If the compliment of d is empty, meaning $L(d) = \Sigma^*$, then M rejects. Otherwise M accepts. All conditions are handled in M so $M = \text{Some}_{DFA}$.

2. Let Alot_{RE} =$\{ \langle A\rangle \mid A$ is a regular expression and $L(A)$ is infinite $\}$. Show that Alot_{RE} is decidable.

   Construct a Turing machine M to decide the Alot_{RE} problem.

   On input r that is a RE:

   1. Construct a DFA A that is equivalent to r.
   2. For s that is the number of states in A, construct DFA B that accepts all strings over the alphabet in A that are at least length s.
   3. Construct a DFA C so that $L(C) = L(A) \cap L(B)$.
   4. Run $\langle C\rangle$ on a turing machine T that decides E_{DFA}.
   5. If T accepts then reject r, if T rejects then accept r.

   In order for a DFA to accept an infinite language, it must contain a loop. If a DFA contains a string with a length greater than the number of states, then it contains a loop and therefore accepts an infinite language. The Turing machine M checks if a RE accepts an infinite language by intersecting the language accepted by r to the language of strings with length greater than the number of states of the DFA for r. M accepts if the intersection is non-empty and rejects otherwise. Therefore $M = \text{Alot}_{RE}$.

3. Let $\text{Complimentary}_{RE,DFA}$ =$\{ \langle A, B\rangle \mid A$ is a regular expression and B is a DFA such that $L(A) \cup L(B) = \Sigma^*$ and $L(A) \cap L(B) = \emptyset\}$. Show that $\text{Complimentary}_{RE,DFA}$ is decidable.

   Construct a Turing machine M to decide the $\text{Complimentary}_{RE,DFA}$ problem.

   On input r that is a RE and d that is a DFA:

   1. Create a DFA A that is equivalent to r.
   2. Construct a DFA B so that $L(B) = L(A) \cup L(d)$.
   3. Cosntruct a DFA B^C that is the compliment of B.
   4. Run $\langle B^C\rangle$ on a Turing machine T that decides E_{DFA}.
   5. If T accepts, then continue, if T rejects then reject r, d.
   6. Construct a DFA C so that $L(C) = L(A) \cap L(d)$.
   7. Run $\langle C\rangle$ on turing machine T.
   8. If T accepts, then accept r, d. Otherwise reject.

   M rejects if $L(A) \cup L(B) \neq \Sigma^*$. M rejects if $L(A) \cap L(B) \neq \emptyset$. M only accepts if both conditions are met, therefore $M = \text{Complimentary}_{RE,DFA}$.

4. Let ALL_{DFA} =$\{ \langle A\rangle \mid A$ is a DFA and $L(A) = \Sigma^*\}$. Show that ALL_{DFA} is decidable.

   Construct a Turing machine M to decide the ALL_{DFA} problem.

   On input d that is a DFA:

   1. asdf

5. Let $N_{\varepsilon CFG}$ =$\{ \langle G\rangle \mid G$ is a CFG and G does not generate the empty string $\}$. Show that $N_{\varepsilon CFG}$ is decidable.

6. Let X be the set $\{1, 2, 3, 4, 5\}$ and Y be the set $\{6, 7, 8, 9, 10\}$. We describe the functions $f : X \to Y$ and $g : X \to Y$ in the following tables. Answer each part and give a reason for each negative answer.

n	$f(n)$
1	6
2	7
3	6
4	7
5	6

n	$g(n)$
1	10
2	9
3	8
4	7
5	6

(a) Is f onto?

(b) Is f a correspondence?

(c) Is g onto?

(d) Is g a correspondence?

7. Let $U = \{ \langle A, B, C \rangle \mid A, B, C$ are DFA's and $|L(A)| = |L(B)| + |L(C)| \}$. Show that U is decidable.

8. Let $A = \{ \langle R \rangle \mid R$ is a regular expression describing a language containing at least one string w that has 111 as a substring (i.e., $w = x111y$ for some x and y)$\}$. Show that A is decidable.

9. Let $E_{PDA} = \{ \langle P \rangle \mid P$ is a pushdown automata and $L(P)$ is empty $\}$. Show E_{PDA} is decidable.

10. A ***useless state*** in a pushdown automaton is never entered on any input string. Consider the problem of determining whether a pushdown automaton has any useless states. Formulate this problem as a language and show that it is decidable.