

December 6, 2017

1. Let $\text{Some}_{DFA} = \{ \langle A \rangle \mid A \text{ is a DFA and } L(A) \text{ is not empty and } L(A) \text{ is not equal to } \Sigma^* \}$. Show that Some_{DFA} is decidable.

Construct a Turing machine M to decide the Some_{DFA} problem.

On input d that is a DFA:

1. Run $\langle d \rangle$ on a Turing machine T that decides E_{DFA} , if T rejects then continue, if T accepts, then reject.
2. Construct a DFA d^C that is the compliment of d .
3. Run $\langle d^C \rangle$ on a Turing machine T , if T rejects then accept d , if T accepts then reject d .

If d is not a DFA, then M rejects. If $L(d)$ is empty, then M rejects. If the compliment of d is empty, meaning $L(d) = \Sigma^*$, then M rejects. Otherwise M accepts. All conditions are handled in M so $M = \text{Some}_{DFA}$.

2. Let $\text{Alot}_{RE} = \{ \langle A \rangle \mid A \text{ is a regular expression and } L(A) \text{ is infinite} \}$. Show that Alot_{RE} is decidable.

Construct a Turing machine M to decide the Alot_{RE} problem.

On input r that is a RE:

1. Construct a DFA A that is equivalent to r .
2. For s that is the number of states in A , construct DFA B that accepts all strings over the alphabet in A that are at least length s .
3. Construct a DFA C so that $L(C) = L(A) \cap L(B)$.
4. Run $\langle C \rangle$ on a Turing machine T that decides E_{DFA} .
5. If T accepts then reject r , if T rejects then accept r .

In order for a DFA to accept an infinite language, it must contain a loop. If a DFA contains a string with a length greater than the number of states, then it contains a loop and therefore accepts an infinite language. The Turing machine M checks if a RE accepts an infinite language by intersecting the language accepted by r to the language of strings with length greater than the number of states of the DFA for r . M accepts if the intersection is non-empty and rejects otherwise. Therefore $M = \text{Alot}_{RE}$.

3. Let $\text{Complimentary}_{RE,DFA} = \{ \langle A, B \rangle \mid A \text{ is a regular expression and } B \text{ is a DFA such that } L(A) \cup L(B) = \Sigma^* \text{ and } L(A) \cap L(B) = \emptyset \}$. Show that $\text{Complimentary}_{RE,DFA}$ is decidable.

Construct a Turing machine M to decide the $\text{Complimentary}_{RE,DFA}$ problem.

On input r that is a RE and d that is a DFA:

1. Create a DFA A that is equivalent to r .
2. Construct a DFA B so that $L(B) = L(A) \cup L(d)$.
3. Construct a DFA B^C that is the compliment of B .
4. Run $\langle B^C \rangle$ on a Turing machine T that decides E_{DFA} .
5. If T accepts, then continue, if T rejects then reject r, d .
6. Construct a DFA C so that $L(C) = L(A) \cap L(d)$.
7. Run $\langle C \rangle$ on Turing machine T .
8. If T accepts, then accept r, d . Otherwise reject.

M rejects if $L(A) \cup L(B) \neq \Sigma^*$. M rejects if $L(A) \cap L(B) \neq \emptyset$. M only accepts if both conditions are met, therefore $M = \text{Complimentary}_{RE,DFA}$.

4. Let $\text{ALL}_{DFA} = \{ \langle A \rangle \mid A \text{ is a DFA and } L(A) = \Sigma^* \}$. Show that ALL_{DFA} is decidable.

Construct a Turing machine M to decide the ALL_{DFA} problem.

On input d that is a DFA:

1. Create a DFA d^C that is the compliment of d .
2. Run $\langle d^C \rangle$ on a Turing machine T that decides E_{DFA} .
3. If T rejects, then reject d . If T accepts, then accept d .

M only accepts if the language of the compliment of the input DFA is an empty language. For a DFA to equal Σ^* , it must accept all possible strings over the alphabet and therefore the compliment of such a DFA must be empty. This shows that $M = \text{ALL}_{DFA}$.

5. Let $N_{\varepsilon CFG} = \{ \langle G \rangle \mid G \text{ is a CFG and } G \text{ does not generate the empty string} \}$. Show that $N_{\varepsilon CFG}$ is decidable.

Construct a Turing machine M to decide the $N_{\varepsilon CFG}$ problem.

On input g that is a CFG:

1. Construct A that is a equivalent CFG to g except in Chomsky normal form.
2. If A contains the rule $S \rightarrow \epsilon$ reject, otherwise accept.

The CFG A can only generate the empty string if there exists a rule $S \rightarrow \epsilon$, so if A does not include the rule then $\epsilon \notin L(A)$. So $M = N_{\varepsilon CFG}$.

6. Let X be the set $\{1, 2, 3, 4, 5\}$ and Y be the set $\{6, 7, 8, 9, 10\}$. We describe the functions $f : X \rightarrow Y$ and $g : X \rightarrow Y$ in the following tables. Answer each part and give a reason for each negative answer.

n	$f(n)$	n	$g(n)$
1	6	1	10
2	7	2	9
3	6	3	8
4	7	4	7
5	6	5	6

- (a) Is f onto?
- (b) Is f a correspondence?
- (c) Is g onto?
- (d) Is g a correspondence?

7. Let $U = \{ \langle A, B, C \rangle \mid A, B, C \text{ are DFA's and } |L(A)| = |L(B)| + |L(C)| \}$. Show that U is decidable.
8. Let $A = \{ \langle R \rangle \mid R \text{ is a regular expression describing a language containing at least one string } w \text{ that has } 111 \text{ as a substring (i.e., } w = x111y \text{ for some } x \text{ and } y) \}$. Show that A is decidable.
9. Let $E_{PDA} = \{ \langle P \rangle \mid P \text{ is a pushdown automata and } L(P) \text{ is empty} \}$. Show E_{PDA} is decidable.
10. A **useless state** in a pushdown automaton is never entered on any input string. Consider the problem of determining whether a pushdown automaton has any useless states. Formulate this problem as a language and show that it is decidable.