CS 375 Homework 1 Anchu A. Lee

I have done this assignment completely on my own. I have not copied it, nor have I given my solution to anyone else. I understand that if I am involved in plagiarism or cheating I will have to sign an official form that I have cheated and that this form will be stored in my official university record. I also understand that I will receive a grade of 0 for the involved assignment for my first offense and that I will receive a grade of F for the course for any additional offense.

1. Given the pseudo code below for bubble sort:

   (a) Let `length[A] = n`. What is the count for `BubbleSort(A)`? Show the steps necessary to derive your final answer.

Line	Cost	Times
1	c_1	n
2	c_2	$\sum_{i+1}^{n} t_i$
3	c_3	$\sum_{i+1}^{n} (t_i - 1)$
4	c_4	0 (best), $\sum_{i+1}^{n} (t_i - 1)$ (worst)

   $\sum_{line}^{4}$(times *line* executed) = instruction count

   (b) Show the worst case and best case time complexity in term of instruction counts.
   Worst case scenario:
   $n + \sum_{i+1}^{n} t_i + \sum_{i+1}^{n} (t_i - 1) + \sum_{i+1}^{n} (t_i - 1)$
   Best case scenario:
   $n + \sum_{i+1}^{n} t_i + \sum_{i+1}^{n} (t_i - 1)$

2. Fill in all the missing values.

$f(n)$	$g(n)$	$f(n) = O(g(n))$	$f(n) = \Omega(g(n))$	$f(n) = \Theta(g(n))$
$n^{2.125}$	$n^2 \text{lg} n$	No	Yes	No
$\sqrt{n}$	n	Yes	No	No
$n!$	$(n+1)!$	Yes	No	No
$2^{n/2}$	2^n	Yes	No	No
$\sum_{i=1}^{n} i = \frac{n^2+n}{2}$	n^2	Yes	Yes	Yes
$\sum_{i=0}^{n-1} 4^i = \frac{1}{3}(4^n - 1)$	$n4^{n-1}$	No	Yes	No

3. Order the functions below by increasing growth rate.

$$n^n,\ n\text{ln}n,\ n^\epsilon (0 < \epsilon < 1),\ 2^{\text{ln}n},\ \text{ln}n,\ 10,\ n!,\ 2^n$$
$$10,\ \text{ln}n,\ 2^{\text{ln}n},\ n^\epsilon (0 < \epsilon < 1),\ n\text{ln}n,\ 2^n,\ n^n,\ n!$$

4. Let $f(n)$ and $g(n)$ be asymptotically positive functions. Prove or show a counter example for each of the following conjectures.

   (a) $f(n) \in O(g(n))$ implies $2^{f(n)} \in O(2^{g(n)})$
   False.

   Proof. Consider $f(n) = 2n$ and $g(n) = n$. $2n \in O(n)$
   Let's then assume $2^{2n} \notin O(2^n)$. By definition of big-O, there must exist some constant c, n_0 such that $0 \leq 2^{2n} \leq c \cdot 2^n$, $\forall n \geq n_0$. Let's try to prove $2^{2n} \leq c \cdot 2^n$ for some constant c.

$$2^{2n} \leq c \cdot 2^n$$
$$(2^n)^2 \leq c \cdot 2^n$$
$$2^n \nleq \sqrt{c} \cdot 2^{n/2}$$

As c is constant, as n approaches ∞ the LHS will become greater than the RHS. Therefore the implication is not true. $\square$

(b) $f(n) \in O(g(n))$ implies $g(n) \in \Omega(f(n))$
True.

Proof. The definition of $f(n) \in O(g(n))$ says there must exist a constant c such that $0 \leq f(n) \leq c \cdot g(n)$.
The definition of $g(n) \in \Omega(f(n))$ says there must exist a constant c such that $0 \leq c \cdot f(n) \leq g(n)$.
Therefore $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$

$\square$

5. Prove $n^2 - 3n - 20 \in \Theta(n^2)$ using the original definition of Θ

Proof. By definition of Θ,
$\Theta(n^2) = \{n^2 - 3n - 20$: there exist positive constants c_1, c_2, n_0 such that:
$0 \leq c_1(n^2) \leq n^2 - 3n - 20 \leq c_2(n^2), \forall n \geq n_0\}$

Let $c_1 = \frac{1}{2}$, $c_2 = 1$ for all $n \geq 10$. First, need to show that $\frac{1}{2}n^2 \leq n^2 - 3n - 20$
Base case: $n = 10$

$$\begin{aligned}
\frac{1}{2}(10)^2 &\leq (10)^2 - 3(10) - 20 \\
\frac{100}{2} &\leq 100 - 30 - 20 \\
50 &= 50
\end{aligned}$$

IH: Assume $\frac{1}{2}k^2 \leq k^2 - 3k - 20$, $k \geq 10$
Prove true for $k + 1$.

$$\begin{aligned}
\frac{1}{2}(k+1)^2 &\leq (k+1)^2 - 3(k+1) - 20 \\
\frac{1}{2}(k+1)^2 &\leq k^2 + 2k + 1 - 3k - 3 - 20 \\
\frac{1}{2}(k+1)^2 &\leq k^2 - k - 22 \\
\frac{1}{2}(k+1)^2 &\leq k^2 - k - 22 - 2k + 2k + 2 - 2 \\
\frac{1}{2}(k+1)^2 &\leq k^2 - 3k - 20 + 2k - 2 \\
\frac{1}{2}(k+1)^2 &\leq \frac{1}{2}k^2 + 2k - 2 && \text{Induction step} \\
\frac{1}{2}k^2 + k + \frac{1}{2} &\leq \frac{1}{2}k^2 + 2k - 2 \\
\frac{1}{2} &< k - 2
\end{aligned}$$

Now, show that $n^2 - 3n - 20 \leq n^2$
Base case: $n = 10$

$$\begin{aligned}
(10)^2 - 3(10) - 20 &\leq (10)^2 \\
100 - 30 - 20 &\leq 100 \\
50 &\leq 100
\end{aligned}$$

IH: Assume $k^2 - 3k - 20 \le k^2$ for $k \ge 10$
Prove true for $k+1$.

$$
\begin{aligned}
(k+1)^2 - 3(k+1) - 20 &\le (k+1)^2 \\
k^2 + 2k + 1 - 3k - 3 - 20 &\le (k+1)^2 \\
k^2 - k - 22 &\le (k+1)^2 \\
k^2 - k - 22 - 2k + 2k + 2 - 2 &\le (k+1)^2 \\
k^2 - 3k - 20 + 2k - 2 &\le (k+1)^2 \\
k^2 + 2k - 2 &\le (k+1)^2 && \text{Induction step} \\
k^2 + 2k - 2 &\le k^2 + 2k + 1 \\
-2 &\le 1
\end{aligned}
$$

This shows that $n^2 - 3n - 20 \in \Theta(n^2)$ is true with $c_1 = \frac{1}{2}$, $c_2 = 1$ for all $n \ge 10$ □

6. Disprove $n^3 \in O(n^2)$ using the original definition of O.

   Proof. By definition of O,
   $O(n^2) = \{n^3$: there exist positive constants c, n_0 such that:
   $0 \le n^3 \le c(n^2), \forall n \ge n_0\}$
   Need to show that $n^3 \le c(n^2)$ is never true. Let c be any constant number.

$$
\begin{aligned}
n^3 &\le c(n^2) \\
1 &\le \frac{c}{n} \\
n &\le c
\end{aligned}
$$

   As n approaches infinity it cannot be bounded by a constant. Thus $n^3 \notin O(n^2)$. □

7. Prove $n = \omega(\lg n^2)$ using limit.

   Proof. If $n = \omega(\lg n^2)$, then the limit $\lim_{n\to\infty} \frac{n}{lgn^2}$ approaches ∞.

$$
\lim_{n\to\infty} \frac{n}{\lg n^2}
$$
$$
\lim_{n\to\infty} \frac{1}{\frac{2}{n}}
$$
$$
\lim_{n\to\infty} \frac{n}{2} \Rightarrow \infty
$$

□

8. Prove $n^a = \omega(\lg^k n)$, where $k > 0$, $a > 0$, using limit.

   Proof. If $n^a = \omega(\lg^k n)$ then the limit $\lim_{n\to\infty} \frac{n^a}{lg^k n}$ approaches ∞.

$$
\lim_{n\to\infty} \frac{n^a}{\lg^k n}
$$
$$
\lim_{n\to\infty} \frac{an^a}{k \lg^{k-1}(n)}
$$

□