Math 327 Homework 6 Anchu A. Lee
October 24, 2017

Question 6.2: Suppose X follows a continuous uniform distribution from 1 to 5. Determine the conditional probability $P(X > 2.5 \mid X \leq 4)$

$$\frac{P(2.5<X\geq 4)}{P(X\leq 4)} = \frac{4-2.5}{4-1} = \frac{1.5}{3}$$

Question 6.4: A bus arrives every 10 minutes at a bus stop. It is assumed that the waiting time for a particular individual is a random variable with a continuous uniform distribution.

(a) What is the probability that the individual waits more than 7 minutes?

$$P(X > 7) = \frac{10-7}{10} = \frac{3}{10}$$

(b) What is the probability that the individual waits between 2 and 7 minutes?

$$P(2 < X < 7) = \frac{7-2}{10} = \frac{5}{10}$$

Question 6.6: Find the value of z if the area under a standard normal curve

(a) to the right of z is 0.3622;

The left of z is then $1 - 0.3622 = 0.6378$. 0.6378 can be approximated as 0.6368 so $z = 0.35$ by Table A.3.

(b) to the left of z is 0.1131;

The right of z is then $0.1131 - 1 = -0.8869$. So by table A.3, $z = -1.21$

(c) between 0 and z, with $z > 0$, is 0.4838;

Area left of z is $0.5 + 0.4838 = 0.9838$. So $z = 2.14$.

(d) between z and z, with $z > 0$, is 0.9500.

Area left of z is $0.25 + 0.95 = 0.975$, so $z = 1.96$.

Question 6.10: According to Chebyshevs theorem, the probability that any random variable assumes a value within 3 standard deviations of the mean is at least 8/9. If it is known that the probability distribution of a random variable X is normal with mean μ and variance σ^2, what is the exact value of $P(\mu - 3\sigma < X < \mu + 3\sigma)$?

$$\begin{aligned}
z_1 &= \frac{((\mu - 3\sigma) - \mu)}{\sigma} = -3 \\
z_2 &= \frac{((\mu + 3\sigma) - \mu)}{\sigma} = 3 && \text{So then:} \\
P(\mu - 3\sigma < X < \mu + 3\sigma) &= P(-3 < Z < 3) \\
&= 0.9987 - 0.0013 && \text{by Table A.3} \\
&= 0.9974
\end{aligned}$$

Question 6.12: The loaves of rye bread distributed to local stores by a certain bakery have an average length of 30 centimeters and a standard deviation of 2 centimeters. Assuming that the lengths are normally distributed, what percentage of the loaves are

(a) longer than 31.7 centimeters?

$$\begin{aligned}
z &= \frac{31.7 - 30}{2} = 0.85 \\
P(X > 31.7) &= P(Z > 0.85) = 0.1977 && \text{by Table A.3}
\end{aligned}$$

19.77% of loaves are longer than 31.7 cm.

(b) between 29.3 and 33.5 centimeters in length?

$$\begin{aligned}
z_1 &= \frac{29.3-30}{2} = -0.35 \\
z_2 &= \frac{33.5-30}{2} = 1.75 \\
P(29.3 < X < 33.5) &= P(-0.35 < Z < 1.75) \\
&= 0.9599 - 0.3632 && \text{by Table A.3} \\
&= 0.5967
\end{aligned}$$

59.67% of loaves are within 29.3 and 33.5 cm.

(c) shorter than 25.5 centimeters?

$$\begin{aligned}
z &= \frac{25.5-30}{2} = -2.25 \\
P(X < 25.5) &= P(Z < 2.25) \\
&= 0.0122 && \text{by Table A.3}
\end{aligned}$$

1.22% of loaves are shorter than 25.5 cm.

Question 6.22: If a set of observations is normally distributed, what percent of these differ from the mean by

(a) more than 1.3σ?

$$\begin{aligned}
x_1 &= \mu + 1.3\sigma \\
x_2 &= \mu - 1.3\sigma && \text{Therefore:} \\
z_1 &= 1.3 \\
z_2 &= -1.3 \\
P(X > \mu + 1.3\sigma) + P(X < 1.3\sigma) &= P(Z > 1.3) + P(Z < -1.3) \\
&= 0.0968 + 0.0968 && \text{by Table A.3} \\
&= 0.1936
\end{aligned}$$

19.36% differ from the mean by more than 1.3σ.

(b) less than 0.52σ?

$$\begin{aligned}
x_1 &= \mu + 0.52\sigma && z_1 = 0.52 \\
x_2 &= \mu - 0.52\sigma && z_2 = -0.52 \\
P(\mu - 0.52\sigma < X < \mu + 0.52\sigma) &= P(-0.52 < Z < 0.52) \\
&= 0.6985 - 0.3015 && \text{by Table A.3} \\
&= 0.3970
\end{aligned}$$

39.70% differ from the mean by less than 0.52σ

Question 6.26: A process yields 10% defective items. If 100 items are randomly selected from the process, what is the probability that the number of defectives

(a) exceeds 13?

$$\begin{aligned}
\mu &= np = (100)(0.1) = 10 \\
\sigma &= \sqrt{(100)(0.1)(0.9)} = 3 \\
z &= \frac{13.5-10}{3} = 1.17 \\
P(X > 13.5) &= P(Z > 1.17) && \text{by Table A.3} \\
&= 0.1210
\end{aligned}$$

12.10% number of defects exceeds 13.

(b) is less than 8?

$$\begin{aligned} z &= \frac{7.5-10}{3} = -0.83 \\ P(X < 7.5) &= P(Z < -0.83) && \text{by Table A.3} \\ &= 0.2033 \end{aligned}$$

20.33% number of defects is below 8.

Question 6.30: A drug manufacturer claims that a certain drug cures a blood disease, on the average, 80% of the time. To check the claim, government testers use the drug on a sample of 100 individuals and decide to accept the claim if 75 or more are cured.

(a) What is the probability that the claim will be rejected when the cure probability is, in fact, 0.8?

$$\begin{aligned} \mu &= (100)(0.8) = 80 \\ \sigma &= \sqrt{(100)(0.8)(0.2)} = 4 \\ z &= \frac{74.5-80}{4} = -1.38 && \text{by Table A.3} \\ &= 0.0838 && (8.38\%) \end{aligned}$$

(b) What is the probability that the claim will be accepted by the government when the cure probability is as low as 0.7?

$$\begin{aligned} \mu &= (100)(0.7) = 70 \\ \sigma &= \sqrt{(100)(0.7)(0.3)} = 4.583 \\ z &= \frac{74.5-70}{4.583} = 0.98 && \text{by Table A.3} \\ &= 1 - 0.8365 = 0.1635 && (16.35\%) \end{aligned}$$

Question 6.34: A pair of dice is rolled 180 times. What is the probability that a total of 7 occurs

(a) at least 25 times?

$$\begin{aligned} \mu &= (180)(1/6) = 30 \\ \sigma &= \sqrt{(180)(1/2)(5/6)} = 5 \\ z &= \frac{24.5-30}{5} = -1.1 \\ P(X > 24.5) &= P(Z > -1.1) && \text{by Table A.3} \\ &= 1 - 0.1357 = 0.8643 && (86.43\%) \end{aligned}$$

(b) between 33 and 41 times inclusive?

$$\begin{aligned} z_1 &= \frac{32.5-30}{5} = 0.5 \\ z_2 &= \frac{41.5-30}{5} = 2.3 \\ P(32.5 < X < 41.5) &= P(0.5 < Z < 2.3) && \text{by Table A.3} \\ &= 0.9893 - 0.6915 = 0.2978 && (29.78\%) \end{aligned}$$

(c) exactly 30 times?

$$\begin{aligned} z_1 &= \frac{29.5-30}{5} = -0.1 \\ z_2 &= \frac{30.5-30}{5} = 0.1 \\ P(29.5 < X < 30.5) &= P(-0.1 < Z < 0.1) && \text{by Table A.3} \\ &= 0.5398 - 0.4602 = 0.0796 && (7.96\%) \end{aligned}$$

Question 6.40: In a certain city, the daily consumption of water (in millions of liters) follows approximately a gamma distribution with $\alpha = 2$ and $\beta = 3$. If the daily capacity of that city is 9 million liters of water, what is the probability that on any given day the water supply is inadequate?

$$\begin{aligned} P(X > 9) &= \frac{1}{9}\int_9^\infty xe^{-x/3}dx \\ &\approx 0.19915 \end{aligned}$$

Question 6.42: Suppose that the time, in hours, required to repair a heat pump is a random variable X having a gamma distribution with parameters $\alpha = 2$ and $\beta = 1/2$. What is the probability that on the next service call

(a) at most 1 hour will be required to repair the heat pump?

$$\begin{aligned} P(X < 1) &= 4\int_0^1 xe^{-2x}dx \\ &\approx 0.5940 \end{aligned}$$

(b) at least 2 hours will be required to repair the heat pump?

$$\begin{aligned} P(X > 2) &= 4\int_2^\infty xe^{-2x}dx \\ &\approx 0.0916 \end{aligned}$$

Question 6.46: The life, in years, of a certain type of electrical switch has an exponential distribution with an average life $\beta = 2$. If 100 of these switches are installed in different systems, what is the probability that at most 30 fail during the first year?

$$\begin{aligned} P(X < 1) &= \frac{1}{2}\int_0^1 e^{-x/2}dx \approx 0.39347 \\ \mu &= (100)(0.39347) = 39.347 \\ \sigma &= \sqrt{(100)(0.39347)(0.6065)} = 4.885 \\ z &= \frac{30.5 - 39.347}{4.885} = -1.81 \\ P(X \leq 30) &= P(Z < -1.81) = 0.0352 && \text{by Table A.3} \end{aligned}$$

Question 6.50: If the proportion of a brand of television set requiring service during the first year of operation is a random variable having a beta distribution with $\alpha = 3$ and $\beta = 2$, what is the probability that at least 80% of the new models of this brand sold this year will require service during their first year of operation?

$$\begin{aligned} f(x) &= 12x^2(1-x) && \text{for } 0 < x < 1 \\ P(X > 0.8) &= 12\int_{0.8}^1 x^2(1-x)dx \\ &\approx 0.1808 \end{aligned}$$

Question 6.54: The lifetime, in weeks, of a certain type of transistor is known to follow a gamma distribution with mean 10 weeks and standard deviation $\sqrt{50}$ weeks.

(a) What is the probability that a transistor of this type will last at most 50 weeks?

$$\begin{aligned} \alpha\beta &= 10 \\ \sigma &= \sqrt{\alpha\beta^2} = \sqrt{50} = 7.07 \end{aligned}$$

$$\begin{aligned} P(X \leq 50) &= \frac{1}{\beta^\alpha \Gamma(\alpha)}\int_0^{50} x^{\alpha-1}e^{-x/\beta}dx \\ &= \frac{1}{25}\int_0^{50} xe^{-x/5}dx \\ &\approx 0.9995 \end{aligned}$$

(b) What is the probability that a transistor of this type will not survive the first 10 weeks?

$$P(X < 10) = \frac{1}{\beta^\alpha \Gamma(\alpha)} \int_0^{10} x^{\alpha-1} e^{-x/\beta} dx$$
$$y = \frac{x}{\beta}$$
$$P(Y < 2) = \int_0^2 y e^{-y} dy$$
$$\approx 0.59399$$

Question 6.56: Rate data often follow a lognormal distribution. Average power usage (dB per hour) for a particular company is studied and is known to have a lognormal distribution with parameters $\mu = 4$ and $\sigma = 2$. What is the probability that the company uses more than 270 dB during any particular hour?

$$P(X > 270) = 1 - \Phi(\frac{ln(270) - 4}{2})$$
$$= 1 - \Phi(0.7992)$$
$$= 0.2119$$

Question 6.58: The number of automobiles that arrive at a certain intersection per minute has a Poisson distribution with a mean of 5. Interest centers around the time that elapses before 10 automobiles appear at the intersection.

(a) What is the probability that more than 10 automobiles appear at the intersection during any given minute of time?

$$P(X > 10) = 1 - P(X \leq 10)$$
$$= 1 - 0.9863 \qquad \text{by Table A.2}$$
$$= 0.0137$$

(b) What is the probability that more than 2 minutes elapse before 10 cars arrive?

$$P(X \leq 2) = \int_0^2 \frac{1}{\beta^\alpha} \cdot \frac{x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)} dx$$
$$y = \frac{x}{\beta}$$
$$P(X \leq 2) = P(Y \leq 10)$$
$$= \int_0^{10} \frac{y^{\alpha-1} e^{-y}}{\Gamma(\alpha)} dy$$
$$= \int_0^{10} \frac{y^{10-1} e^{-y}}{\Gamma(10)} dy$$
$$= 0.542 \qquad \text{by Table A.23}$$
$$1 - P(X \leq 2) = 1 - 0.542$$
$$= 0.458$$