



# Optimization Design Homework 1

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1 FIND THE STATIONARY POINTS FOR THE FOLLOWING FUNCTIONS. ALSO IDENTIFY (FOR EACH STATIONARY POINT), THE LOCAL MAXIMUM, MINIMUM, OR NEITHER (BY USING SECOND ORDER DERIVATIVE OR HESSIAN MATRIX.)

1.1  $f(x) = x^3 \exp(-x^2)$ , for  $-2 < x < 2$ .

The stationary points are those where  $f'(x) = 0$ , so we first find the first derivative. By using the product rule,

$$f'(x) = 3x^2 \exp(-x^2) - 2x^4 \exp(-x^2) = -x^2 \exp(-x^2)(2x^4 - 3x^2) \quad (1.1)$$

which is zero at  $x = 0$ , and by solving  $2x^4 - 3x^2$  we get the other roots  $x = \pm \frac{\sqrt{3}}{\sqrt{2}}$ . Theorem 2.2 gives the sufficient condition for a minimum or maximum point for single variables. We must find a point where  $f''(x^*) \neq 0$ , so we find  $f''(x)$ . We again use the product rule on Equation 1.1 and factor the result to obtain

$$f''(x) = -x \exp(-x^2)(4x^4 + 2x^2 - 6) \quad (1.2)$$

where  $f''(\frac{\sqrt{3}}{\sqrt{2}}) < 0$  and  $f''(-\frac{\sqrt{3}}{\sqrt{2}}) > 0$  However,  $f''(0) = 0$ , so we find the third derivative by using the same procedure

$$f'''(x) = -x \exp(-x^2)(20x^3 + 6x + 8x^4 + 4x^2 - 12) - 6 \exp(-x^2) \quad (1.3)$$

where  $f'''(0) = 6$ , however since  $n = 3$ , 0 does not correspond to either a maximum nor minimum point. In the end,  $-\frac{\sqrt{3}}{\sqrt{2}}$  is a minimum, 0 is neither, and  $\frac{\sqrt{3}}{\sqrt{2}}$  is a maximum.

1.2  $f(x, y) = -x^2 - 3y^2 + 12xy$

For a multivariable equation, solving for  $x$  and  $y$  in the partial derivatives will give us the stationary points.

$$\frac{\partial f}{\partial x} = -2x + 12y \quad (1.4)$$

$$\frac{\partial f}{\partial y} = -6y + 12x \quad (1.5)$$

We first solve for  $y$  in Equation 1.5 and we get  $y = 2x$ . Substituting for  $y$  in Equation 1.4, we get

$$-2x + 24x = 22x$$

$$22x = 0$$

which means  $x = 0$ . Substituting again  $x = 0$  in Equation 1.5 we get  $-6y = 0$  which means  $y = 0$ . The stationary point for  $f$  is  $(0, 0)$ .

To determine the nature of the stationary points, we have to find the Hessian matrix, which involves finding the second order partial derivatives.

$$\frac{\partial^2 f}{\partial^2 x} = -2 \quad \frac{\partial^2 f}{\partial^2 y} = -6 \quad (1.6)$$

$$\frac{\partial^2 f}{\partial y \partial x} = 12 \quad \frac{\partial^2 f}{\partial x \partial y} = 12 \quad (1.7)$$

and build the Hessian matrix

$$\mathbf{H} = \begin{bmatrix} -2 & 12 \\ 12 & -6 \end{bmatrix}$$

We can check for positive or negative definiteness by looking at the sign of  $|\mathbf{H}_1|$  and  $|\mathbf{H}_2|$ . Here  $|\mathbf{H}_1| = -2$  and  $|\mathbf{H}_2| = (-6)(-2) - (12)(12) = -136$ . Since both determinants are negative, we conclude  $(0, 0)$  is neither a maximum nor minimum of  $f$ .

## 2 A QUADRATIC FUNCTION OF $n$ VARIABLES HAS THE FOLLOWING STANDARD FORM

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{A} \mathbf{x} / 2 + \mathbf{b}^T \mathbf{x} + c$$

, WHERE  $\mathbf{x}$  IS THE VECTOR CONTAINING THE  $n$  VARIABLES. VECTOR  $\mathbf{b}(n \times 1)$  AND SYMMETRIC MATRIX  $\mathbf{A}(n \times n)$  CONTAIN CONSTANT COEFFICIENTS. FOR THE FOLLOWING TWO QUADRATIC FUNCTIONS  $f_a(\mathbf{x}) = x_1^2 + 2x_1x_2 + 3x_2^2$  AND  $f_b(\mathbf{x}) = -x_1^2 - x_2^2 - x_3^2 + 2x_1x_2 + 6x_1x_3 + 4x_1 - 5x_3 + 7$ , PLEASE

### 2.1 Rewrite these functions in the standard quadratic function form.

In the standard quadratic form, the first product  $\mathbf{x}^T \mathbf{A} \mathbf{x}$  produces the quadratic terms and the combinations  $\mathbf{x}_i \mathbf{x}_j, i \neq j$ . For  $f_a$ , on the diagonals of  $\mathbf{A}$  we place the coefficients of the quadratic terms, in this case 2 and 6. Outside of the diagonals we place the half of the combinations  $\mathbf{x}_1 \mathbf{x}_2$ , or 2 in this case. The product  $\mathbf{b}^T \mathbf{x}$  would take care of any single first order variables  $x_i, i = 1, 2$ ; however  $f_a$  does not have such factors. The final quadratic form for  $f_a(\mathbf{x})$  is

$$f_a(\mathbf{x}) = \frac{1}{2} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \quad (2.1)$$

The process is similar for  $f_b$ . On the diagonals we place the coefficients of the second order components, and off the diagonals the coefficients of  $x_i x_j, i \neq j$ . Next,  $\mathbf{b}$  will contain

the first order coefficients, here  $4, 0, -5$  for  $x_1, x_2, x_3$ , respectively. The standard quadratic formula for  $f_b(\mathbf{x})$  is

$$f_b(\mathbf{x}) = \frac{1}{2} \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} -2 & 2 & 6 \\ 2 & -2 & 0 \\ 6 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 7 \quad (2.2)$$

## 2.2 For these functions, show that $\nabla f = A\mathbf{x} + \mathbf{b}$ .

The gradient vector  $\nabla f(\mathbf{x})$  is the vector containing the partial derivatives  $\frac{\partial f}{\partial x_i}$ .

$$\nabla f_a(\mathbf{x}) = \begin{bmatrix} 2x_1 + 2x_2 \\ 2x_1 + 6x_2 \end{bmatrix}$$

and the product  $A\mathbf{x} + \mathbf{b}$  is

$$\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_1 + 2x_2 \\ 2x_1 + 6x_2 \end{bmatrix}$$

which shows that  $\nabla f_a(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ .

For the second equation

$$\nabla f_b(\mathbf{x}) = \begin{bmatrix} -2x_1 + 2x_2 + 6x_3 + 4 \\ -2x_2 + 2x_1 \\ -2x_3 + 6x_1 - 5 \end{bmatrix}$$

and the product  $A\mathbf{x} + \mathbf{b}$  is

$$\begin{bmatrix} -2 & 2 & 6 \\ 2 & -2 & 0 \\ 6 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ -5 \end{bmatrix} = \begin{bmatrix} -2x_1 + 2x_2 + 6x_3 + 4 \\ -2x_2 + 2x_1 \\ -2x_3 + 6x_1 - 5 \end{bmatrix}$$

which again shows  $\nabla f_b(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$ .

## 2.3 For these functions, determine whether the matrix $A$ is positive definite, negative definite, or indefinite.

The textbook mentions two methods to check whether the matrix is positive definite, negative definite, or semi-definite. The first method involves checking the signs of the eigenvalues of  $A$ . The second involves checking the sign of the determinants of the submatrices  $A_1, A_2, \dots, A_n$ . For  $f_a(\mathbf{x})$  this would be

$$|A_1| = |2| \quad \text{and} \quad |A_2| = \begin{vmatrix} 2 & 2 \\ 2 & 6 \end{vmatrix} = (2)(6) - (2)(2) = 8$$

Since  $|A_1|$  and  $|A_2|$  are positive,  $A$  is positive definite.

For  $f_b(\mathbf{x})$  we repeat the same procedure, we check the signs of  $|A_1|, |A_2|$  and  $|A_3|$ .

$$|A_1| = -2$$

$$|A_2| = (-2)(-2) - (2)(2) = 0$$

$$|A_3| = -2(4) - 2(-4) + 6(-12) = 72$$

Since there are both negative and positive determinants for the submatrices of  $A$ , the matrix of  $f_b(\mathbf{x})$  is indefinite.

**3 MINIMIZE THE FOLLOWING FUNCTIONS SUBJECT TO EQUALITY AND/OR INEQUALITY CONSTRAINTS. FOR PROBLEMS WITH INEQUALITY CONSTRAINTS, IDENTIFY THE ACTIVE ONES. TO KNOW WHETHER A POINT IS LOCAL MINIMUM, PERFORM FUNCTION EVALUTATION AT 2 NEARBY POINTS (WHICH SATISFY CONSTRAINTS) TO SHOW THAT WHAT YOU FOUND IS LOCAL MINIMUM.**

**3.1  $f(x, y) = 1/(xy)^2$ , subject to  $x^2 + y^2 = 1$**

Using the Lagrange multiplier method, we first define our Lagrange function  $L$

$$L(x, y, \lambda) = \frac{1}{(xy)^2} + \lambda(1 - x^2 - y^2)$$

and find the partial derivatives of  $x, y$ , and  $\lambda$ .

$$\frac{\partial L}{\partial x} = -2x^{-3}y^{-2} - 2\lambda x \quad (3.1)$$

$$\frac{\partial L}{\partial y} = -2x^{-2}y^{-3} - 2\lambda y \quad (3.2)$$

$$\frac{\partial L}{\partial \lambda} = 1 - x^2 - y^2 \quad (3.3)$$

When we solve for  $\lambda$  in Equation 3.1, we get  $\lambda = -x^{-4}y^{-2}$ . Substituting for  $\lambda$  in Equation 3.2, we get

$$-2x^{-2}y^{-3} - 2y(-x^{-4}y^{-2})$$

which in the end gives us  $y = x$ . Plugging this result in Equation 3.3, we get the result

$$x = y = \frac{1}{\sqrt{2}}$$

which gives us a value of 4 in  $f$ . To show  $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$  is a local minimum, we plot some nearby points. We chose the points (0.69, 0.7238) and (0.72, 0.694), which lie on either side of our proposed minimum, and evaluate

$$f(0.69, 0.7238) = 4.009$$

$$f(0.72, 0.694) = 4.005$$

which supports our claim that  $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$  is a minimum.

**3.2  $f(x, y) = x^2 - 2x + y^2 - 10y$ , subject to  $-x^2 + y \leq 4$  and  $-(x - 2)^2 + y \leq 3$**