1 Approximation Methods

1.1 Linear Approximation

Can form a tangent line of f(x) at a with the derivative f'(x)

$$L(x) = f(a) + f'(a)(x - a) \tag{1}$$

1.1.1 Solution Methods

Bisection Method

1. Determine interval (a,b) on which function has a root.
2. Divide interval into sub intervals: $(a, \frac{a_b}{2}), (\frac{a+b}{2}, b)$
3. Determine which interval solution lives in.
4. Apply 2 again with smaller interval

Repeat until reach desired accuracy.

Newtons method Rearrange linearization to $x_(n+1) = x_n - \frac{f(x_n)}{f'(x_n)}$

Fixed Point Iteration Let f(x) be differentiable and suppose f(x) has a solution. If —f'(x)—¡1 for all values of x within an interval of the fixed point then FPI converges for any x.

Basically rearrange equation for x and use result of one solution as argument for next.

If guess is not close though function WILL diverge.

1.2 Polynomial Interpolation

2 points can determine a line in R^2 3 points determine a quadratic in R^2

1.2.1 Newton Forward Difference Formula

$$y = y_0 + (x - x_0)\frac{\triangle y_0}{\triangle x} + \ldots \frac{x(x - x_0)(x - x_1)...(x - (n-1))\triangle^n y_0}{h^n n!} \tag{2}$$

Where h is the difference in space between elements and $x_n = x_0 + nh$

1.2.2 Taylor Polynomial

$$P_(k, x_0)(x) = \sum_{n=0}^{k} \frac{f^n x_0 (x - x_0)^n}{n!} \tag{3}$$

Far away from x_0 the approximations can get bad.

$$a_k = \frac{1}{k!} f^{(k)}(x_0) \tag{4}$$

Example $f(x) = e^{x^2}$ Find for $P_{(4,0)}(x)$ for e^t

$$P_{(4,0)}(t) = 1 + t + 1/2t^2 + 1/6t^3 + a/24t^4 \quad (5)$$
$$P_{(8,0)}(t) = 1 + x^2 + 1/2x^4 + 1/6x^6 + a/24x^8 \quad (6)$$

Mclaurin TP with $x_0 = 0$

Fundamental theorem of Calculus: if F'(x)=f(x) then

$$\int_a^b f(t)\mathrm{d}t = F(b) - F(a) \quad (7)$$

Taylor Theorem

$f(x) = P(n, x_0)(x) + \int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{n+1}(t)\mathrm{d}t$

We know that the remainder must be equal to the actual function minus the taylor polynomial by definition. Accept that $|f^{n+1}(t)| <= k$ for all $t\epsilon[x_0, x]$

Error is equal to :

$|f(x) - P_{(n,x_0)}(x)| \leq |\int_{x_0}^{x} \frac{(x-t)^n}{n!} f^{n+1}(t)\mathrm{d}t| \leq \frac{k}{(n+1)!}|x - x_0|^{n+1}$