Rock the Boat
On the Dynamics of Floating Rigid Bodies

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1 Introduction

Floating rigid bodies and their stability are of great concern to Marine Engineers. Metacritic stability analysis, a practice commonly used today and taught in many undergraduate fluid mechanics classes, is well suited to this engineering application, being heavily supported by empirical data. Although these simple equations are widely used, there are some applications in which the model begin to break down and non-linear Lagrangian analysis becomes necessary (Holden et al., 2011). In this project, We will use tools developed in Physics 350 to go beyond the standard practices and characterize these floating systems in a similar manner to Massa and Vignolo (2016).

2 The Lagrangian of a Floating Rigid Body

In order to characterize our system using the Lagrangian formalism, we must consider the energy of our system.

$$E = T + U_G$$

Note that this quantity alone is not a constant, as our system is not closed. The hydrostatic forces exerted on the rigid body by the fluid change its height and velocity of the rigid body, so a constant energy cannot be defined.

We can, however, consider a quantity, here called $\tilde{E}$, which is defined to be the sum of E and the energy imparted by our system onto the fluid from some reference time to t. This imparted energy can be expressed as the integral over time of the power imparted, π. This is equal to the energy of the system at the starting reference point, and by conservation of energy in the closed system of the boat *and* the fluid, this quantity is invariant with time.

$$\tilde{E} = T + U - \int \pi dt$$

The kinetic energy of a rigid planar cross-section, expressed in cartesian coordinates, is known to be $\frac{1}{2}m\left(\dot{x_G} + \dot{y_G}\right)^2 + \frac{1}{2}I_{zz}\omega^2$, and the gravitational potential is mgy_G. Manipulation of the power (see section 3.1) term gives

$$\tilde{E} = \frac{1}{2}m\left(\dot{x} + \dot{y}\right)^2 + \frac{1}{2}I_{zz}\omega^2 + mgy - \rho g V_{sub}\left(\theta, y\right) y_B\left(\theta, y\right)$$

In this expression the last two terms are entirely dependent on the coordinates of our system, and thus it makes sense to define to quantity

$$U_{eff} = mgy - \rho g V_{sub}\left(\theta, y\right) y_B\left(\theta, y\right)$$

For any specific geometry, this U_{eff} can be calculated numerically with only θ and y_B. This is the computationaly strategy used to completely determine the potential energy for the circular and elliptical cross-sections in part 5.

3 Derivation of Linearized Boat Stability Analysis

3.1 Proof of Conservativeness of the Hydrostatic Force

This report aims to model the dynamics of extruded floating bodies with lagrangian mechanics. In order to do this, a langrangian must be defined for the body, which amounts to showing that the forces acting on it are conservative.

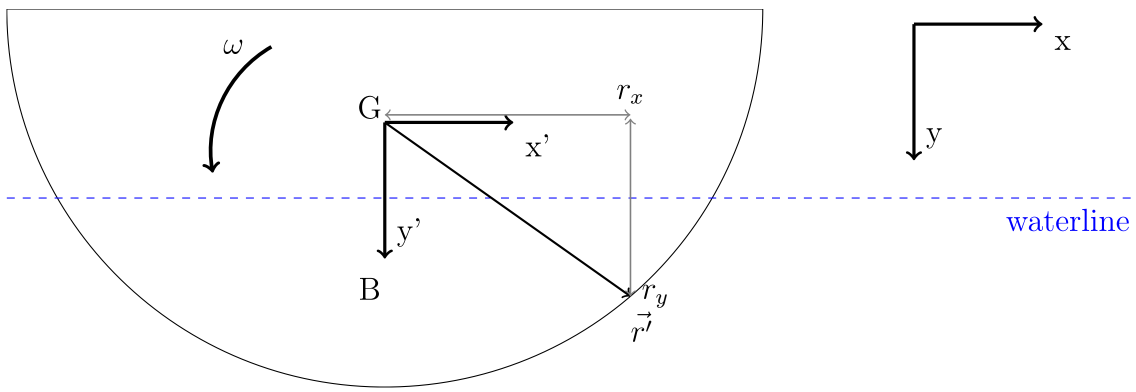
The gravtiational force is clearly conservative, but for the hydrostatic force on the hull, this must be shown.

Before we proceed, let's state the required assumptions. We will assume that the fluid is perfectly inviscid, that is, it will only exert normal forces (pressure) and no shear forces or surface tension on bodies immersed in it. In addition, we will assume that, because the body is an extruded cross section, it will remain completely parallel to the fluid's surface along the axis of extrusion (i.e. it may roll, but not pitch or yaw). We also will assume the fluid pressure is hydrostatic everywhere (i.e. proportional only to depth and independent of the motion of the body). Finally, we will model the fluid as a perfectly flat plane, the surface of which does not rise, fall or deform with time (i.e. no waves are present).

These assumptions are widely employed, as in Massa and Vignolo (2016) to simplify analysis of floating bodies in marine engineering, where, on calm water, deviations of the ocean from a flat, fully hydrostatic medium are extremely small and surface tension is negligible.

We require an inertial reference frame to perform the analysis, so we define all distances with respect to a set of arbitrary stationary coordinates x, y, z. In addition, we define a set non-inertial coordinates x', y', z' fixed to the body itself. G represents the center of gravity, and B the centroid of the submerged region. The system is diagrammed in **Figure 1.**

Figure 1. Consider a 3-dimensional extruded floating body with an arbitrary cross-section



We will take an interesting approach to defining a potential energy for the buoyant force, by first calculating the power exerted by this force on the body. Let V_{sub} be the submerged volume, π power, p pressure, and $\vec{v}$ hull element velocity

$$\pi = -\int\limits_{\partial V_{sub}} p\hat{\mathbf{n}} \cdot \vec{v} d\vec{S}$$

Applying the divergence theorem and exploiting the incompressibility of fluids, we rewrite this as

$$\pi = -\int\limits_{V_{sub}} \nabla \cdot (p\vec{v}) dV = -\int\limits_{V_{sub}} \big(\vec{v} \cdot \nabla p + p(\nabla \cdot \vec{v})\big) dV = -\int\limits_{V_{sub}} \vec{v} \cdot \nabla p dV$$

Now, assuming hydrostatic pressure, substitute $\nabla p = \rho g \hat{\mathbf{y}}$ and $\vec{v} = \vec{v_G} + \hat{\mathbf{z}}\omega \times \vec{r'}$

$$\begin{aligned}\pi = -\rho g \int\limits_{V_{sub}} \hat{\mathbf{y}} \cdot \vec{v} dV &= -\rho g \int\limits_{V_{sub}} (\hat{\mathbf{y}} \cdot \vec{v_G} + \hat{\mathbf{y}} \cdot \hat{\mathbf{z}}\omega \times \vec{r'}) \\ &= -\rho g \int\limits_{V_{sub}} (\dot{y}_G + \hat{\mathbf{y}} \cdot \hat{\mathbf{z}}\omega \times \vec{r'})\end{aligned}$$

Recall that $\{x', y', z'\}$ represent the noninertial coordinates defined on the floating body, and the body rotates only about $\hat{\mathbf{z}}$. Therefore, $\frac{\partial}{\partial t}\hat{\mathbf{x}}\text{'} = \omega \hat{\mathbf{y}}\text{'}$ and $\frac{\partial}{\partial t}\hat{\mathbf{y}}\text{'} = -\omega \hat{\mathbf{x}}\text{'}$

$$\begin{aligned}\hat{\mathbf{y}} \cdot \hat{\mathbf{z}}\omega \times (r_x\hat{\mathbf{x}}\text{'} + r_y\hat{\mathbf{y}}\text{'} + r_z\hat{\mathbf{z}}\text{'}) &= \hat{\mathbf{y}} \cdot (-\omega r_y \hat{\mathbf{x}}\text{'} + \omega r_x \hat{\mathbf{y}}\text{'}) \\ &= \hat{\mathbf{y}} \cdot (\frac{\partial}{\partial t}(r_y\hat{\mathbf{y}}\text{'}) + \frac{\partial}{\partial t}(r_x\hat{\mathbf{x}}\text{'}))\end{aligned}$$

So, the expression for power can be reduced to

$$\pi = -\rho g \int\limits_{V_{sub}} \left(\dot{y}_G + \frac{\partial}{\partial t}(r_x(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}\text{'})) + \frac{\partial}{\partial t}(r_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}\text{'}))\right) dV = -\rho g \int\limits_{V_{sub}} \frac{\partial}{\partial t}\Psi(t) dV,$$

where $\Psi(t) = y_G + r_x(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}\text{'}) + r_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}\text{'})$.

Lemma 1. For an arbitrary function $\Psi(t)$ integrated over a non-constant volume

$$
\begin{aligned}
D_t\left[\int\limits_{\Omega(t)} \Psi(t)dV(t)\right] &= \int\limits_{\Omega(t)} D_t[\Psi(t)dx_1(t)dx_2(t)dx_3(t)] \\
&= \int\limits_{\Omega(t)} \left(\dot{\Psi}dx_1dx_2dx_3 + \Psi d\dot{x}_1dx_2dx_3 + \Psi dx_1d\dot{x}_2dx_3 + \Psi dx_1dx_2d\dot{x}_3\right) \\
&= \int\limits_{\Omega(t)} \left(\dot{\Psi} + \Psi(\frac{d\dot{x}_1}{dx_1} + \frac{d\dot{x}_2}{dx_2} + \frac{d\dot{x}_3}{dx_3})\right)dV \\
&= \int\limits_{\Omega(t)} \left(\dot{\Psi} + \Psi\nabla\cdot\vec{v}\right)dV = \int\limits_{\Omega(t)} \left(\dot{\Psi} + \nabla(\Psi\vec{v}) - \vec{v}\cdot\nabla\Psi\right)dV
\end{aligned}
$$

Note that since $\dot{\Psi} = D_t\Psi = \frac{\partial\Psi}{\partial t} + \vec{v}\cdot\nabla\Psi$,

$$
\begin{aligned}
&= \int\limits_{\Omega(t)} \left(\frac{\partial}{\partial t}\Psi + \nabla(\Psi\vec{v})\right)dV(t) \\
&= \int\limits_{\Omega(t)} \frac{\partial}{\partial t}\Psi dV + \int\limits_{\partial\Omega(t)} \Psi\vec{v}\cdot\hat{\mathbf{n}}d\vec{S}
\end{aligned}
$$

Let us interpret the meaning of these integrals. The first term concerns only the explicit time-dependence of Ψ, and therefore accounts for the change of Ψ within the volume Ω.

The second term integrates $\vec{v}\cdot\hat{\mathbf{n}}d\vec{S}$ over $\partial\Omega$, or the *amount by which each surface element changes the volume* Ω. Therefore, this term represents all change in the integral resulting from the change of the volume itself.

Critically, because the body is rigid, although the submerged volume Ω changes with time, this change *only* occurs along the waterline. The body's outer surfaces are fixed relative to eachother, so the $\vec{v}\cdot\hat{\mathbf{n}}d\vec{S}$ elements will cancel on all the interfaces between the water and the hull. Along the waterline, regions of the body's volume pass in and out of Ω, so the integral over $\partial\Omega$ can be replaced by one over $Waterline$.

Applying this lemma, we can rewrite the expression for π

$$
\begin{aligned}
\pi &= -\rho g D_t\left[\int\limits_{V_{sub}} \Psi dV + \rho g \int\limits_{Waterline} \Psi\vec{v}\cdot\hat{\mathbf{n}}dV\right] \\
&= \rho g\left(- D_t\left[\int\limits_{V_{sub}} \left(y_G + r_x(\hat{\mathbf{y}}\cdot\hat{\mathbf{x}}\text{'}) + r_y(\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}\text{'})\right)dV\right]\right. \\
&+ \left.\int\limits_{Waterline} \left(y_G + r_x(\hat{\mathbf{y}}\cdot\hat{\mathbf{x}}\text{'}) + r_y(\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}\text{'})\right)\vec{v}\cdot\hat{\mathbf{n}}dS\right)
\end{aligned}
$$

Recall from earlier in this derivation that $\hat{\mathbf{y}} \cdot \vec{v}$ can be expanded:

$$\hat{\mathbf{y}} \cdot \vec{v_G} + \hat{\mathbf{y}} \cdot (\hat{\mathbf{z}}\omega \times \vec{r'}) = \dot{y}_G + \hat{\mathbf{y}} \cdot (-\omega r_y \hat{\mathbf{x}}\text{'} + \omega r_x \hat{\mathbf{y}}\text{'}) = \dot{y}_G + \frac{\partial}{\partial t} r_x(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}\text{'}) + \frac{\partial}{\partial t} r_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}\text{'})$$

The waterline boundary is simply a flat surface in the xz-plane, so $\hat{\mathbf{n}} = \hat{\mathbf{y}}$. Therefore, the second integrand in the expression for π above can be rewritten

$$\Big(y_G + r_x(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}\text{'}) + r_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}\text{'})\Big)\vec{v} \cdot \hat{\mathbf{y}} = \frac{\partial}{\partial t}(y_G + r_x(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}\text{'}) + r_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}\text{'}))^2$$

Note that if the boat's fixed axes are rotated an angle θ counterclockwise from $\{x, y, z\}$, then

$$r_x(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}\text{'}) + r_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}\text{'}) = r_x sin(\theta) + r_y cos(\theta)$$

This is the negative projection of $\vec{r}$ onto the y-axis (normal to the water's surface). A quick glance at the diagram will reveal that, along the waterline, this is simply $-y_G$. Thus, the entire waterline integral reduces to 0.

Furthermore, the expression for the power exerted by the fluid on the hull is reduced to

$$\pi = \frac{d}{dt} U_B(t),$$

where

$$U_B(t) = -\rho g \int\limits_{V_{sub}} \Big(y_G + r_x(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}\text{'}) + r_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}\text{'})\Big) dV.$$

Note that $U_b(t)$ is not explicitly time-dependent

By definition, power π is equal to the derivative of the potential energy of a system for systems in which a potential energy can be defined. Therefore, our $U_B(t)$ can be interpreted as the potential energy of the floating body.

The time-independace of U_B implies that the hydrostatic force is indeed conservative for an extruded body floating in an inviscid fluid, with potential energy

$$U_B = -\rho g \int\limits_{V_{sub}} \Big(y_G + r_x(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}\text{'}) + r_y(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}\text{'})\Big) dV.$$

3.2 Derivation of Lagrangian

Consider the buoyancy potential derived in the previous section, where $(\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}\text{'}) = -sin\theta$ and $(\hat{\mathbf{y}} \cdot \hat{\mathbf{y}}\text{'}) = cos\theta$,

$$\begin{aligned} U_B &= -\rho g \int\limits_{V_{sub}} (y_G - r_x sin\theta + r_y cos\theta) dV \\ &= -\rho g y_G \int\limits_{V_{sub}} dV - \rho g sin\theta \int\limits_{V_{sub}} r_x dV - \rho g cos\theta \int\limits_{V_{sub}} r_y dV \end{aligned}$$

For convenience we define the arms $B_x = \frac{\int_{V_{sub}} r_x dV}{V_{sub}}$, and $B_y = \frac{\int_{V_{sub}} r_y dV}{V_{sub}}$ which denote the center of buoyancy in boat coordinates {x', y'}. Projecting these to general coordinates, we get

$$U_B = -\rho g V_{sub}(y_G - B_x cos\theta + B_y sin\theta) = -\rho g V_{sub} y_B,$$

where y_B is the center of buoyancy in the global coordinates. Note that both V_{sub} and y_B are functions of θ and y, rotation angle and distance between waterline and center of gravity respectively.

Combining this potential with the gravitational potential energy, we can define

$$U_{eff} = U_G + U_B = mgy - \rho g V_{sub}(\theta, y) y_B(\theta, y)$$

This gives us the lagrangian

$$\mathcal{L} = \frac{1}{2} m (\dot{x} + \dot{y})^2 + \frac{1}{2} I_{zz} \omega^2 - mgy + \rho g V_{sub}(\theta, y) y_B(\theta, y)$$

4 Simulation Methods

Using the Lagrangian derived in the previous section, we can find generalized forces for our system, as defined by the second order partial differential Euler-Lagrange equations, by way of finite difference calculations. We employ such techniques in our simulation wherein we use finite time steps to find the forces on the boat and render the result using WebGL. You can find the simulation at `https://rahatchd.github.io/rock-the-boat`.

4.1 Motivations and Usage

Most people have a general intuition about how a boat floats in water. However, as our derivation is only valid for an inviscid fluid, there are certain modes and behaviours that are different from what people would expect with a traditional boat. The simulation is designed so that users are able to draw and test any shape with longitudinal symmetry for its stability and behaviour in a fluid. There is a damping parameter in our simulation that creates a drag force proportional to instantaneous velocity. Users are meant to begin the simulation with the damping set to the default value. They should then draw a few boats to convince themselves that the simulation is correct and that the behaviours of the boats are consistent with their intuition. Next, users should adjust the damping to zero: which is the scenario we have studied and derived in our investigation of the buoyancy force. This set of steps will allow the users to have more confidence in the simulation and help to contrast the motion of a boat in inviscid fluid with their own expectations.

5 Potential Energy of the Bouyant Force

To apply lagrangian methods to the problem of a floating rigid body, an effective potential energy taking into account the action of hydrostatic pressure was defined

$$U_{eff} = mgy - \rho g V_{sub}(\theta, y) y_B(\theta, y)$$

For general bodies, this must be computed iteratively as the submerged volume changes. However, for specific geometries, U_{eff} may be explicitly computed as a function of the water height and angle of rotation.

$$U_{eff} = U_{eff}(\theta, y_G)$$

In this section, we analyze the effective potential of several specific geometries.

5.1 The Effective Potential of a Circle

Since the shape of a circle is rotationally invariant, it represents a very simplified version of our problem, where our potential is not a function of θ. Nonetheless, it still serves as an effective starting point for analysis of rigid body motion at inviscid fluid boundaries. Let us begin with the case of a uniform density circle, on the surface of an inviscid fluid with $\rho_{fluid} = 2\rho_{circle}$

This potential was calculated numerically in MATLAB, and was found to have the following shape.

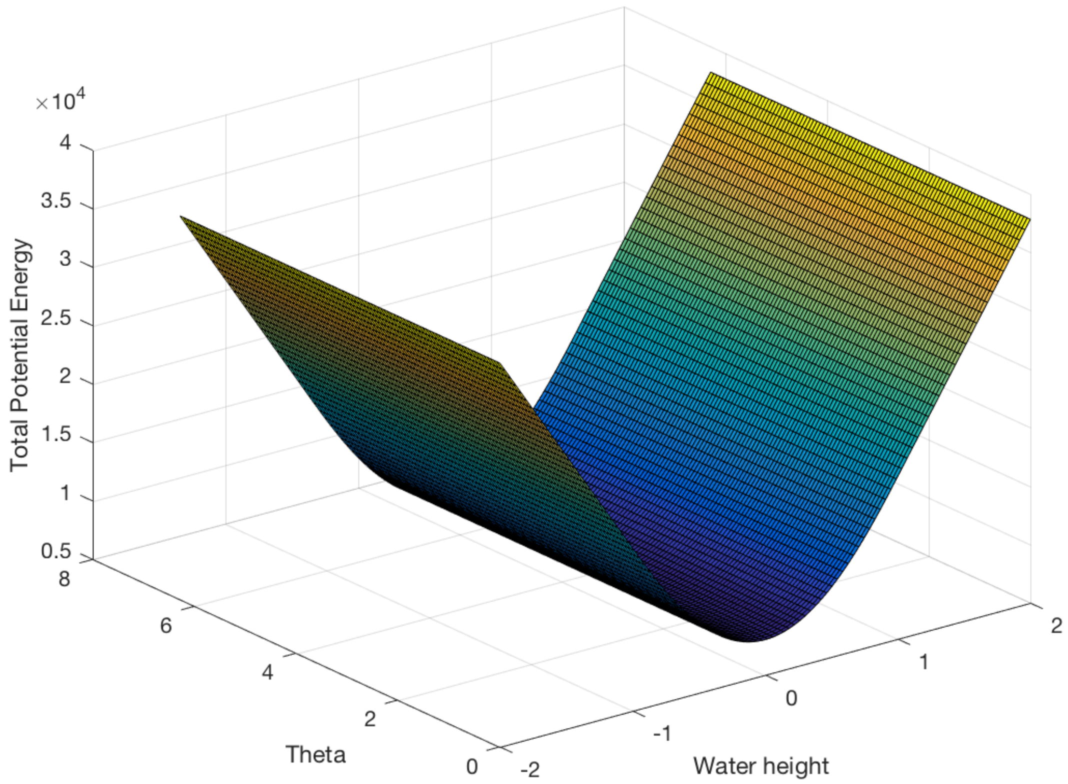


Figure 1: Potential Energy of a Circular Hull.

It is also useful to consider how varying the density of the cylinder will affect the form of U_{eff}. the following graph plots potential against water height and the density of the fluid relative to that of the floating body.

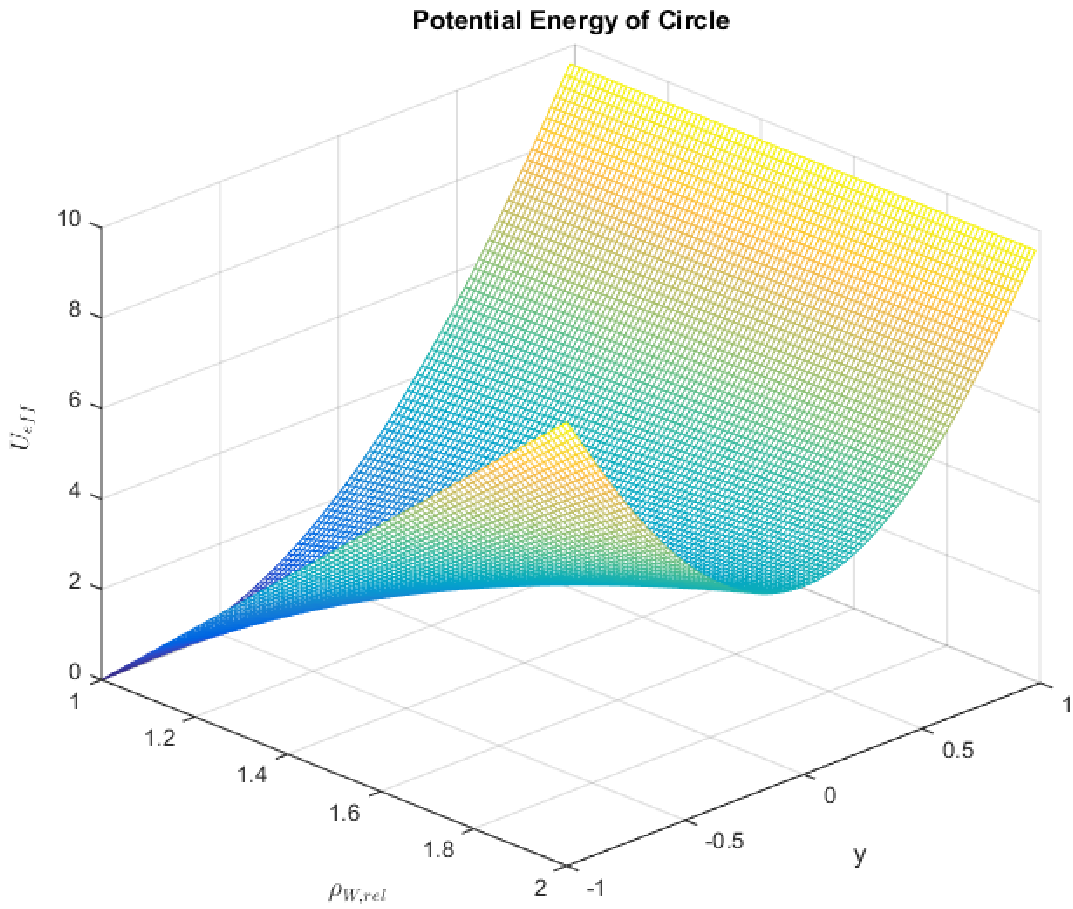


Figure 2: A boat.

As we expect to see, the figure shows that the effective potential has a minimum at $y_G = 0$ when $\rho_{rel} = 2$,that is, the circle rests with exactly half of it's volume submerged in equilibrium when the density of the inviscid fluid is exactly twice of it's own. The figure also shows how the potential function shift as the relative density of the fluid is altered. As expected, the minimum of the potential function shifts down as the relative density of the fluid is decreased, until the point where the fluids and the body have the same density, at this point, the potential is constant for all configurations where the fluid is completely submerged. This tells us that if a circular rigid body of equal density to the inviscid fluid is placed in the fluid, there will be no motion, as Archimedes principle would predict.

The relationship between relative density and equilibrium height is graphed below.

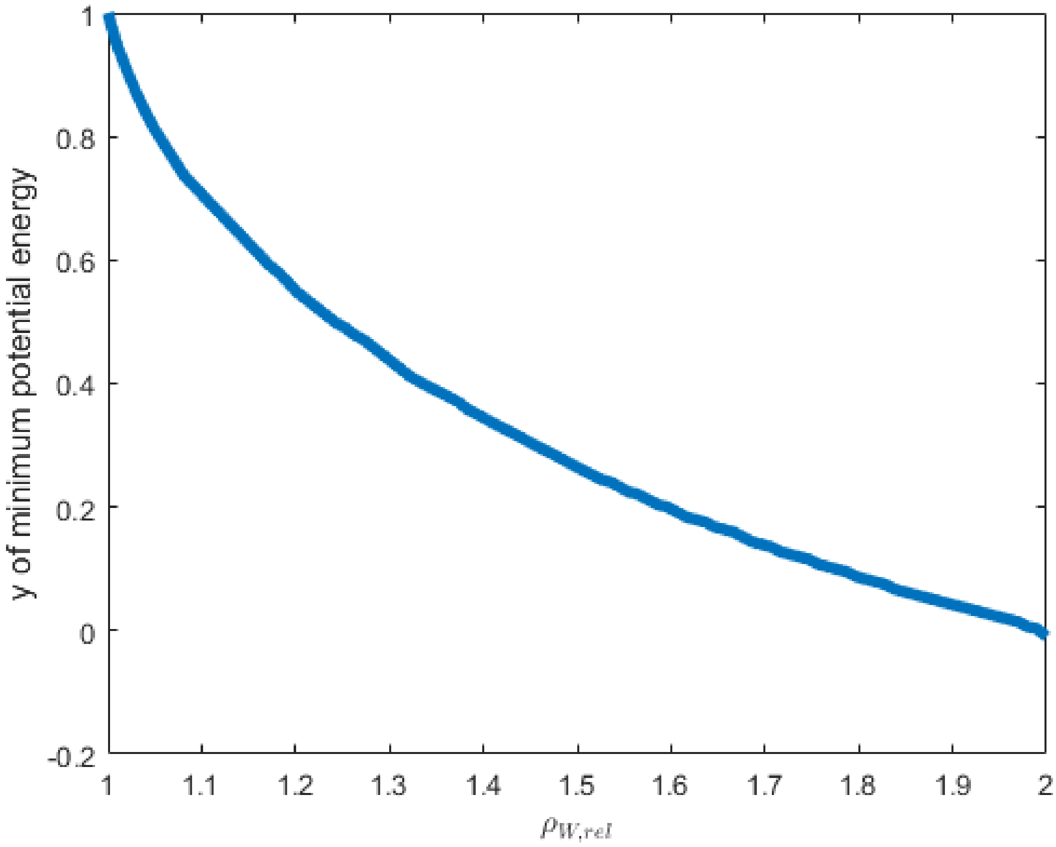


Figure 3: A boat.

5.2 The Effective Potential of an Ellipse

In contrast to the case of the circle, the effective potential of a uniform density ellipse has a dependence on the θ coordinate of the system. This can be easily seen by generating a parametric ellipse surface in MATLAB and numerically obtaining the effective potential energy.

The following figure is the potential energy of the parametric ellipse $\frac{X^2}{1^2} + \frac{Y^2}{4^2} = 1$ as a function of both y_B and θ. In this figure, we measure y_B with respect to the center of the ellipse and begin with the major axis of the ellipse perpendicular to the waterline. As described in the previous section, the waterline will rest at the origin of the ellipse due to the relative density of the ellipse relative to the water.

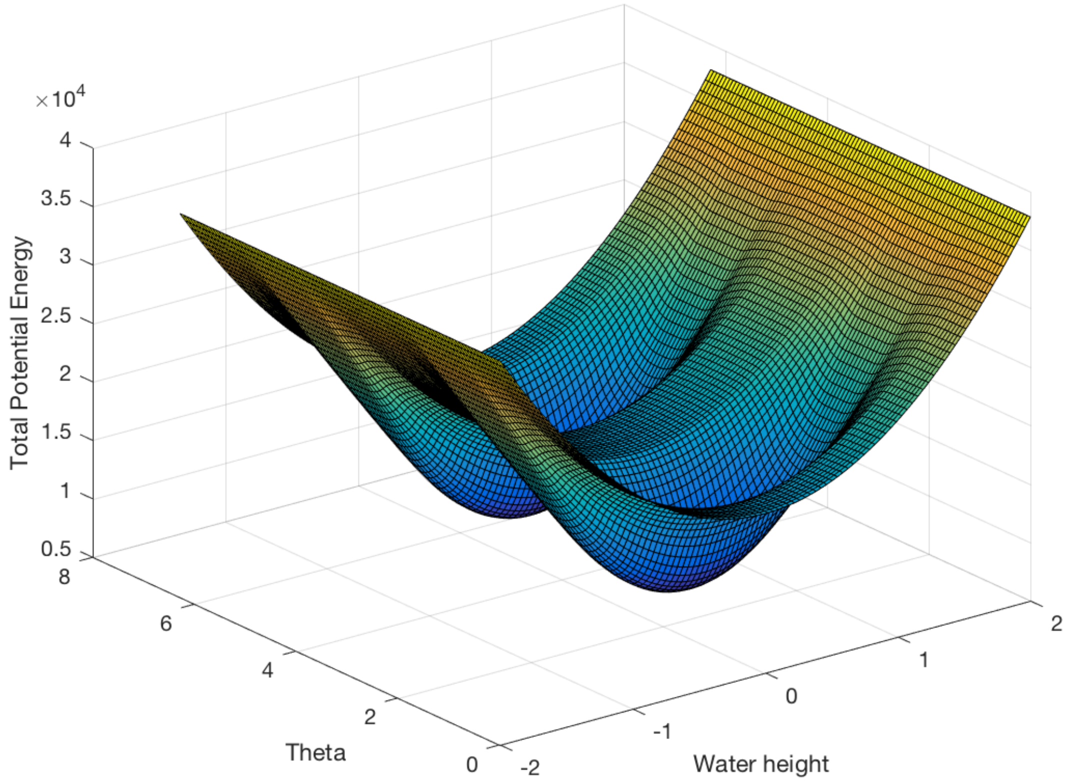


Figure 4: Potential Energy of a Elliptical Hull

In the water height axis we see parabolas of varying slopes; Information about the relative change in volume submerged with respect to the waterline height determines their curvature. The parabolic regions with the shallowest slope correspond to the ellipse's major axis being normal to the waterline. The steepest regions correspond to the minor axis instead. There are no regions with non-positive second derivate, indicating that ellipses will have complete y stability granted that $0 < \rho_{ellipse} < \rho_{fluid}$.

In the θ axis, we can see a π periodic potential which corresponds to the ellipses rotation. The potential is minimized when the minor axis is perpendicular to the waterline. This is because a small deviation in θ in this position will cause the center of buoyancy to move downwards, increasing potential energy. Similarly, the potential is maximized when the major axis is normal to the waterline due to perturbations raising the center of buoyancy.

We can extend these ideas naturally to other floating rigid bodies. If we model the submerged volumes and relative heights of an arbitrary rigid body, we can create plots such as the one above to determine the maximum θ a boat will undergo prior to tipping. Our implementation of these concepts can be found at `https://rahatchd.github.io/rock-the-boat`.

References

C. Holden, T. Perez, and T. I. Fossen. A lagrangian approach to nonlinear modeling of anti-roll tanks. *Ocean Engineering*, 38(2–3):341 – 359, 2011. ISSN 0029-8018. doi: http://dx.doi.org/10.1016/j.oceaneng.2010.11.012. URL `http://www.sciencedirect.com/science/article/pii/S002980181000257X`.

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