Diophantine Equations to the Power of n

MATC15 - Project - Draft 2

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Conjecture 1:

$$x^n = \sum_{i=1}^{n} y_i^n \text{ has an integer solution such that } y_i \neq x, \forall i.$$

Andrew D'Amario (A.D.), February 18, 2021

1 Introduction

The objective of this project is to investigate the conjecture above: whether or not we can always find at least one integer solution to equations of the form $x^n = y_0^n + \cdots + y_n^n$ given any x, excluding trivial solutions involving y_i's$= 0$ or x. This project will be a Type II project.
Some of this investigation and research will involve:

- Finding parameters and conditions for possible valid solutions
- Computational analysis on random integers raised to the power of n and finding an integer solution to the sum.
- Noting differences between even and odd n.
- Identifying different families of solutions that take on a similar form.

Though this conjecture may be false, we hope to investigate as much as we can on the matter and provide some deeper research to the subject.

2 Patterns of Powers

In order to find solutions for different n we investigated checking the reduced residues if x^n mod N so that we could eliminate solutions that would not lead to a possible solution.

While investigating different positive integers n and N we found certain patterns for reduced residues.

Let A be the sequence $A = \{0^n, 1^n, 2^n, 3^n, 4^n, 5^n, ...\}$, and $A_r = \{a \bmod N\}_{a \in A}$ be the reduced residue of A mod N. Here may be some potential conjetures on the set A given the data we have collected:

Conjecture 2: If n is even, every other element starting with the first in A_r is 0, $A_r = \{0, \_, 0, \_, 0, \_, ...\}$, for $N = 2^d$ for all integers d.
i.e. $a_{2i} = 0$, where $a_{2i} \in A_r$ is the $2i^{th}$ element in A_r.

A.D.

Collected data for Conjecture 2, A_r for even n:

- n = 24, N = 2: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 24, N = 4: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 24, N = 8: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 24, N = 16: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 24, N = 32: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 24, N = 64: 0, 1, 0, 33, 0, 33, 0, 1, 0, 1, 0, 33, 0, 33, 0, 1, ...
  n = 24, N = 128: 0, 1, 0, 97, 0, 33, 0, 65, 0, 65, 0, 33, 0, 97, 0, 1, ...
  n = 24, N = 256: 0, 1, 0, 225, 0, 161, 0, 65, 0, 193, 0, 33, 0, 97, 0, 129, ...
  n = 24, N = 512: 0, 1, 0, 225, 0, 417, 0, 65, 0, 449, 0, 33, 0, 97, 0, 129, ...
  n = 24, N = 1024: 0, 1, 0, 225, 0, 417, 0, 65, 0, 449, 0, 545, 0, 97, 0, ...
  n = 24, N = 8192: 0, 1, 0, 6369, 0, 3489, 0, 5185, 0, 5569, 0, 7713, 0, ...
- n = 10, N = 32: 0, 1, 0, 9, 0, 25, 0, 17, 0, 17, 0, 25, 0, 9, 0, 1, ...
- n = 12, N = 64: 0, 1, 0, 49, 0, 17, 0, 33, 0, 33, 0, 17, 0, 49, 0, 1, ...
- n = 14, N = 512: 0, 1, 0, 377, 0, 489, 0, 81, 0, 305, 0, 137, 0, 473, 0, 33, ...
- n = 18, N = 2: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
- n = 34, N = 128: 0, 1, 0, 9, 0, 25, 0, 49, 0, 81, 0, 121, 0, 41, 0, 97, ...
- n = 235676, N = 128: 0, 1, 0, 49, 0, 17, 0, 33, 0, 97, 0, 81, 0, 113, 0, 65, ...

Conjecture 2.1: If $n = 2^k$ for some integer k, there exists a natural number N such that A_r is of the form $\{0, 1, 0, 1, 0, 1, ...\}$.
Moreover, for $k > 1$, A_r has this form for all $N = 2^d$, $d \in [1, k + 2]$.

A.D.

Collected data for Conjecture 2.1, A_r for $n = 2^k$:

- n = 1, N = 2: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
- n = 2, N = 2: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 2, N = 4: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
- n = 4, N = 2: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 4, N = 4: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 4, N = 8: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 4, N = 16: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...

- n = 8, N = 2: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 8, N = 4: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 8, N = 8: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 8, N = 16: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 8, N = 32: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...

- n = 16, N = 2: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 16, N = 4: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 16, N = 8: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 16, N = 16: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 16, N = 32: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 16, N = 64: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...

- n = 32, N = 2: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 32, N = 4: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 32, N = 8: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 32, N = 16: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 32, N = 32: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 32, N = 64: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 32, N = 128: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...

- n = 64, N = 2: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 64, N = 4: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 64, N = 8: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 64, N = 16: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 64, N = 32: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 64, N = 64: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 64, N = 128: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...
  n = 64, N = 256: 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, ...

Conjecture 3: If n is prime, there exists a natural number N such that A_r is of the form: $A_r = \{0, 1, 2, 3, ..., n-1, 0, 1, 2, 3, ..., n-1, ...\}$.

A.D.

Collected data for Conjecture 3, A_r for prime n:

- n = 3, N = 3: 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, 1, 2, 0, ...

- n = 5, N = 5: 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 0, 1, 2, 3, 4, 0, ...

- n = 7, N = 7: 0, 1, 2, 3, 4, 5, 6, 0, 1, 2, 3, 4, 5, 6, 0, 1, 2, 3, 4, 5, 6, ...

- n = 11, N = 11: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 0, 1, 2, ...

3 References

- Drago Bajc, **Power solutions of some Diophantine equations**,
  The Mathematical Gazette, 97:538, 107-110 (2013).
  https://www.jstor.org/stable/24496765

  Mentions form of above conjecture and states that solutions have been found in some cases but not in other cases, such as $n = 6$. Considers above conjecture with x^k instead of x^n, where $(k, n) = 1$ and provides a general form for these solutions.

- **Computing Minimal Equal Sums Of Like Powers**,
  http://euler.free.fr/index.htm

  Website dedicated to finding and compiling examples and counterexamples of Euler's sums of powers conjecture, which states that if a sum of n positive kth powers equals one kth power, then $n >= k$. Includes many resources we can look into.

- **BEST KNOWN SOLUTIONS**,
  http://euler.free.fr/records.htm

  Extensive list of aforementioned examples and counterexamples to Euler's sums of powers conjecture.

- L. Jacobi, D. Madden, **On** $a^4 + b^4 + c^4 + d^4 = (a + b + c + d)^4$,
  The American Mathematical Monthly, 115:3, 230-236 (2008).
  https://doi.org/10.1080/00029890.2008.11920519

  Discusses specific case of the conjecture with $n = 4$. Also discusses relation of Euler's conjecture and related Diophantine equations to the topic of elliptic curves.

- T. Roy and F. J. Sonia, **A Direct Method To Generate Pythagorean Triples And Its Generalization To Pythagorean Quadruples And n-tuples**,
  https://arxiv.org/ftp/arxiv/papers/1201/1201.2145.pdf

  Gives methods for finding Pythagorean n-tuples, sums of n squares that result in a square. Might be able to reduce some cases into one of these cases.

- D. R. Heath-Brown, W. M. Lioen and H. J. J. Te Riele, **On Solving the Diophantine Equation** $x^3 + y^3 + z^3 = k$ **on a Vector Computer**,
  Mathematics of Computation, 61:203, 235-244 (1993)

  Presents detailed algorithm for the $n = 3$ case, might be able to apply similar principles with higher n values.

- L. J. Lander, T. R. Parkin and J. L. Selfridge, **A survey of equal sums of like powers**,
  Mathematics of Computation, 21, 446-459 (1967).

https://www.ams.org/journals/mcom/1967-21-099/S0025-5718-1967-0222008-0/S0025-5718-1967-0222008-0.pdf

Presents various solutions to powers of Diophantine equations, including the $n = 4$ and $n = 5$ cases of the conjecture.

- J. Leech, **On** $A^4 + B^4 + C^4 + D^4 = E^4$,
  Mathematical Proceedings of the Cambridge Philosophical Society, 54(4), 554-555, (1958).
  doi.org/10.1017/S0305004100003091

  Brief paper outlining found solutions for the $n = 4$ case and considerations that reduce the number of possible solutions that need to be checked.