Numerical Solution of ODEs (MTH 452/552)
Notes on Linear Systems of ODEs

This is a brief review of some aspects regarding linear systems of ODEs. Let

$$u' = Au$$

where $u \in \mathbf{R}^m$ and A is a diagonalizable $m \times m$ matrix with constant entries. For example, let $m = 2$ and consider

$$\begin{aligned} u_1' &= 3u_1 + u_2 \\ u_2' &= u_1 + 3u_2 \end{aligned}$$

which can be written as

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}' = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Since A is assumed diagonalizable, there exists an invertible $m \times m$ matrix V such that the columns of V are a basis of $\mathbf{R}^m$ that consists of eigenvectors of A. Let v_j denote the j-th column vector of V. Then $Av_j = \lambda_j v_j$, where λ_j is the corresponding eigenvalue. Let D be the $m \times m$ diagonal matrix such that $D_{jj} = \lambda_j$, $j = 1, \ldots, m$. Then $Av_j = \lambda_j v_j$, $j = 1, \ldots, m$ is equivalent to the matrix equation $AV = VD$, which in turn is equivalent to $A = VDV^{-1}$.

In the above example, the eigenvalues and corresponding eigenvectors are

$$\lambda_1 = 4, \; v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \qquad \lambda_2 = 2, \; v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

so

$$V = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}.$$

Now define y by the equation $u = Vy$. Then

$$Vy' = u' = Au = VDV^{-1}u = VDV^{-1}Vy = VDy.$$

Multiplying both sides of the equation by V^{-1} from the left gives

$$y' = Dy$$

which is equivalent to $y_j' = \lambda_j y_j$, $j = 1, \ldots, m$. With initial values $y_j(t_0) = \eta_j$ one obtains the solutions $y_j(t) = e^{\lambda_j (t - t_0)} \eta_j$, $j = 1, \ldots, m$.

Now going back to u gives

$$u(t) = Vy(t) = \sum_{j=1}^{m} y_j(t)\, v_j = \sum_{j=1}^{m} \eta_j e^{\lambda_j (t-t_0)} v_j.$$

If one has the initial conditions $u(t_0) = u_0$, then η_j is the j-th entry of the vector $\eta = V^{-1}u_0$. Alternatively, η can be found by solving the linear system $V\eta = u_0$.

As an illustration we consider the ODE above with initial conditions

$$u_1(0) = 3, \quad u_2(0) = 4.$$

One has

$$V^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

so $\eta = V^{-1}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 7/2 \\ -1/2 \end{bmatrix}$ and

$$u(t) = \frac{7}{2}e^{4t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{2}e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

or

$$\begin{aligned} u_1(t) &= \frac{1}{2}\left(7e^{4t} - e^{2t}\right) \\ u_2(t) &= \frac{1}{2}\left(7e^{4t} + e^{2t}\right). \end{aligned}$$