Taylor's theorem

In 1 variable.

Thm Assume f has $(n+1)$ continuous derivatives in $[\alpha, \beta]$ and assume t and $t+h$ lie in $[\alpha, \beta]$.

Let $P_n(h) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!} h^k$. (Note $P_n(0) = f(t)$). Then

$$f(t+h) = P_n(h) + \frac{f^{(n+1)}(t+\vartheta h)}{(n+1)!} h^{n+1} \text{ for some } \vartheta \in (0,1).$$

Note: $0! = 1$, $f^{(0)}(t) = f(t)$, ϑ depends on t and h. ϑ is unknown.

$$f(t+h) = P_n(h) + O(h^{n+1})$$

In several variables:

$$t = \begin{bmatrix} t_1 \\ \vdots \\ t_N \end{bmatrix}, \quad h = \begin{bmatrix} h_1 \\ \vdots \\ h_N \end{bmatrix}, \quad f: \mathbb{R}^N \to \mathbb{R}.$$

$$\nabla = \begin{bmatrix} \partial/\partial t_1 \\ \vdots \\ \partial/\partial t_N \end{bmatrix} \qquad \nabla f = \begin{bmatrix} \partial f/\partial t_1 \\ \vdots \\ \partial f/\partial t_N \end{bmatrix}$$

$$h \cdot \nabla = h_1 \frac{\partial}{\partial t_1} + h_2 \frac{\partial}{\partial t_2} + \dots + h_N \frac{\partial}{\partial t_N}$$

$$(h \cdot \nabla) f = h_1 \frac{\partial f}{\partial t_1} + \dots + h_N \frac{\partial f}{\partial t_N} = \sum_{k=1}^{N} h_k \frac{\partial f}{\partial t_k} = (\nabla f) \cdot h$$

$(h \cdot \nabla)^{(k)} = h \cdot \nabla$ applied successively k times. Then

$$f(t+h) = \sum_{k=0}^{n} \frac{1}{k!} (h \cdot \nabla)^{(k)} f(t) + \frac{1}{(n+1)!} (h \cdot \nabla)^{(n+1)} f(t+\vartheta h)$$

for some $\vartheta \in (0,1)$.

Example: $N=2$.

$$t = \begin{bmatrix} x \\ y \end{bmatrix}, \quad h = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}, \quad f_x = \frac{\partial f}{\partial x}, \quad f_y = \frac{\partial f}{\partial y}, \quad f_{xy} = \frac{\partial^2 f}{\partial x \partial y}$$

etc.

Abbreviate $f(t)$, $\frac{\partial f}{\partial x}(t)$, ... by f, f_x, etc.

$$f(t+h) = f(x+\Delta x, y+\Delta y) =$$

$$f(x,y) + \underbrace{\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)}_{\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} \cdot \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \end{bmatrix} = h \cdot D} f(x,y)$$

$$+ \frac{1}{2} (h \cdot D)^{(2)} f(x,y) + \frac{1}{6} (h \cdot D)^{(3)} f(x,y) + O(h^4)$$

$$= f + \Delta x f_x + \Delta y f_y$$

$$+ \frac{1}{2} \left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right)\left(\Delta x \frac{\partial}{\partial x} + \Delta y \frac{\partial}{\partial y}\right) f + \frac{1}{6} (h \cdot D)^{(3)} f + O(h^4)$$

$$= f + \Delta x f_x + \Delta y f_y$$

$$+ \frac{1}{2} \left(\Delta x^2 \frac{\partial^2}{\partial x^2} + 2 \Delta x \Delta y \frac{\partial^2}{\partial x \partial y} + \Delta y^2 \frac{\partial^2}{\partial y^2}\right) f + \frac{1}{6} (h \cdot D)^{(3)} f + O(h^4)$$

$$= f + \Delta x f_x + \Delta y f_y + \frac{1}{2} \Delta x^2 f_{xx} + \Delta x \Delta y f_{xy} + \Delta y^2 f_{yy}$$

$$+ \frac{1}{6} \left(\Delta x^3 \frac{\partial^3}{\partial x^3} + 3 \Delta x^2 \Delta y \frac{\partial^3}{\partial x^2 \partial y} + 3 \Delta x \Delta y^2 \frac{\partial^3}{\partial x \partial y^2} + \Delta y^3 \frac{\partial^3}{\partial y^3}\right) f + O(h^4)$$

$$= f + \Delta x f_x + \Delta y f_y + \frac{1}{2} \Delta x^2 f_{xx} + \Delta x \Delta y f_{xy} + \frac{1}{2} \Delta y^2 f_{yy}$$

$$+ \frac{1}{6} \Delta x^3 f_{xxx} + \frac{1}{2} \Delta x^2 \Delta y f_{xxy} + \frac{1}{2} \Delta x \Delta y^2 f_{xyy} + \frac{1}{6} \Delta y^3 f_{yyy} + O(h^4)$$

Here $O(h^k)$ means $O(\|h\|^k)$, $\|h\| = \sqrt{h_1^2 + \dots + h_N^2}$

Chain rule:

$$u' = f(t, u(t)), \quad u \in \mathbb{R}^1$$

$$u'' = \frac{d}{dt} f(t, u(t)) = \frac{\partial f}{\partial t}(t, u(t)) + \frac{\partial f}{\partial u}(t, u(t)) \frac{du}{dt}$$

$$= f_t + f_u \underbrace{\frac{du}{dt}}_{f} = f_t + f_u f$$