r-stage RK-method

$$U^{n+1} = U^n + \Delta t \sum_{j=1}^{r} b_j k_j \qquad (1)$$

$$k_j = f\Big(t_n + c_j \Delta t,\ U^n + \Delta t \sum_{\ell=1}^{r} a_{j\ell} k_\ell\Big), \quad j=1,\dots,r \qquad (2)$$

Butcher tableau

$$\begin{array}{c|ccc} c_1 & a_{11} & \cdots & a_{1r} \\ \vdots & & & \\ c_r & a_{r1} & \cdots & a_{rr} \\ \hline & b_1 & \cdots & b_r \end{array} \quad = \quad \begin{array}{c|c} c & A \\ \hline & b^T \end{array}$$

Method is

- explicit if A is strictly lower triangular, i.e., only $k_1, \dots, k_{j-1}$ occur on r.h.s. of (2).
- Semi-implicit if A is lower triangular but has some non-vanishing diagonal elements. So $k_1, \dots, k_j$ may occur on r.h.s. of (2), but not $k_{j+1}, \dots, k_r$. Then (2) can be solved separately for each j.
- implicit if A is not lower triangular.

<u>Order of consistency</u>:

<u>Def</u>: A numerical method is consistent if its LTE $\tau_n \to 0$ as $\Delta t \to 0$.
Consistent of order p if $\tau_n = O(\Delta t^p)$ as $\Delta t \to 0$.

We want to derive conditions for the RK-method to be consistent of order p.
We assume a scalar ODE.

Let u solve $u' = f(t, u)$, and $u(t_n) = U^n$.

$u = u(t_n)$, $u' = \frac{du}{dt}(t_n)$, $f = f(t_n, U^n)$ $f_t = \frac{\partial f}{\partial t}(t_n, U^n)$, ...

Taylor in 1 variable.

$$u(t+\Delta t) = u + \Delta t\, u' + \tfrac{1}{2}\Delta t^2 u'' + \tfrac{1}{6}\Delta t^3 u''' + O(\Delta t^4)$$

$$u'(t) = f, \quad u''(t) = f_t + f f_u$$

$$u'''(t) = \underbrace{(f_{tt} + 2 f f_{tu} + f^2 f_{uu})}_{G} + f_u \underbrace{(f_t + f f_u)}_{F = u''}$$

$$= G + f_u F$$

$$\Rightarrow u(t+\Delta t) = u + \Delta t\, f + \tfrac{1}{2}\Delta t^2 F + \tfrac{1}{6}\Delta t^3 (G + f_u F) + O(\Delta t^4)$$

(4).

Taylor in 2 variables:

$$f(x+\Delta x, y+\Delta y) = f + \Delta x f_x + \Delta y f_y + \tfrac{1}{2}\Delta x^2 f_{xx} + \Delta x \Delta y f_{xy}$$
$$+ \tfrac{1}{2}\Delta y^2 f_{yy} + O(h^3) \text{ if both } \Delta x, \Delta y \text{ are } O(h).$$

Here: $x = t_n = t$, $\Delta x = \Delta t$, $y = U^n$

$$K_j = f(t_n + c_j \Delta t, U^n + \underbrace{\Delta t \sum a_{jl} k_l}_{\Delta y})$$
$$= f + c_j \Delta t f_t + \Delta t (\textstyle\sum a_{jl} k_l) f_u$$
$$+ \tfrac{1}{2} c_j^2 \Delta t^2 f_{tt} + \Delta t^2 c_j \sum_l a_{jl} k_l f_{tu}$$
$$+ \tfrac{1}{2}\Delta t^2 (\textstyle\sum a_{jl} k_l)^2 f_{uu} + O(\Delta t^3)$$

In particular, $k_j = f + O(\Delta t)$.

$$U^{n+1} = U^n + \Delta t \sum_j b_j k_j$$

$$= U^n + \Delta t \sum_j b_j (\underbrace{f}_{u'} + O(\Delta t))$$

$$= U^n + \Delta t \left(\sum_j b_j\right) u' + O(\Delta t^2)$$

$$= u + \Delta t \left(\sum_j b_j\right) u' + O(\Delta t^2)$$

One step error:

$$u(t+\Delta t) - U^{n+1} = u + \Delta t\, u' + O(\Delta t^2)$$

$$- u - \Delta t \left(\sum b_j\right) u' - O(\Delta t^2)$$

$$= \Delta t \left(1 - \sum b_j\right) u' + O(\Delta t^2)$$

<u>LTE</u>: $\frac{1}{\Delta t}\left(u(t+\Delta t) - U^{n+1}\right) = \left(1 - \sum b_j\right) u' + O(\Delta t)$

<u>Thm</u>: A RK method is consistent if and only if $\sum_{j=1}^{r} b_j = 1$. In this case it is consistent of order at least one.

Now assume $\sum_j b_j = 1$.

$$k_j = f + c_j \Delta t f_t + \Delta t \big(\sum a_{j\ell} \underbrace{k_\ell}_{f+O(\Delta t)}\big) f_u + O(\Delta t^2)$$

$$= f + c_j \Delta t f_t + \Delta t (\sum a_{j\ell} f) f_u + O(\Delta t^2)$$

$$= f + \Delta t (c_j f_t + \sum a_{j\ell} f f_u) + O(\Delta t^2)$$

$$U^{n+1} = U^n + \sum_j b_j k_j$$

$$= U^n + \Delta t \underbrace{\sum_j b_j}_{=1} f + \Delta t^2 \big(\sum b_j c_j f_t + \sum_j b_j \sum_\ell a_{j\ell} f f_u\big) + O(\Delta t^3)$$

Compare

$$u(t+\Delta t) = u + \Delta t u' + \tfrac{1}{2} \Delta t^2 u'' + O(\Delta t^3)$$

$$= u + \Delta t f + \tfrac{1}{2} \Delta t^2 (f_t + f f_u) + O(\Delta t^3)$$

$\Rightarrow$ Need: $\sum b_j c_j f_t + \sum_j b_j \sum_\ell a_{j\ell} f f_u = \frac{1}{2}(f_t + f f_u)$

$\Rightarrow$ Need: $\sum_j b_j c_j = \frac{1}{2}$, $\sum_j b_j \sum_\ell a_{j\ell} = \frac{1}{2}$ for consistency of order 2, since then

$$u(t+\Delta t) - U^{n+1} = O(\Delta t^3)$$

From now on we will always assume that $\sum_\ell a_{j\ell} = c_j$, $j = 1, \dots, r$.
Then $\sum_j b_j \sum_\ell a_{j\ell} = \sum b_j c_j$, so the two conditions coincide

Theorem: For a RK method to have order of consistency up to 4 the following conditions are to be satisfied.
We assume $\sum_{\ell} a_{j\ell} = c_j$.

Order	
1	$\sum_j b_j = 1$
2	$\sum b_j c_j = \frac{1}{2}$
3	$\sum_j b_j c_j^2 = \frac{1}{3}$
	$\sum_{j,\ell} b_j a_{j\ell} c_\ell = \frac{1}{6}$
4	$\sum_j b_j c_j^3 = \frac{1}{4}$
	$\sum_{j,\ell} b_j c_j a_{j\ell} c_\ell = \frac{1}{8}$
	$\sum_{j,\ell} b_j a_{j\ell} c_\ell^2 = \frac{1}{12}$
	$\sum_{j,\ell,k} b_j a_{j\ell} a_{\ell k} c_k = \frac{1}{24}$