1-20-17

One-step error

IVP $\quad u' = f(t,u), \qquad u(t_0) = u_0 \qquad (1)$

Want to find $u(t)$ for $t \in [t_0, t_1]$.

Our methods compute approximations U^n for $u(t_n)$, with $t_n \in [t_0, t_1]$.

Consider the Euler method. What is the error of a single step?

Let $u(t)$ be the exact sol. of the IVP

$$u' = f(t,u), \quad u(t_n) = U^n.$$

Note: u depends on t_n and U^n, it is not necessarily the sol. of the original IVP (1).

$$\begin{aligned} U^{n+1} &= U^n + \Delta t\, f(t_n, U^n) \\ &= u(t_n) + \Delta t\, f(t_n, u(t_n)) \qquad (2) \\ &= u(t_n) + \Delta t\, u'(t_n) \qquad (U^{n+1} \text{ approximates } u(t_n+\Delta t)) \quad (3) \end{aligned}$$

On the other hand

$$u(t_n+\Delta t) = u(t_n) + \Delta t\, u'(t_n) + \frac{1}{2}\Delta t^2 u''(t_n) + \frac{1}{6}\Delta t^3 u'''(\xi), \quad t_n \le \xi \le t_n + \Delta t$$

It follows that

$$u(t_n+\Delta t) - U^{n+1} = \frac{1}{2}\Delta t^2 u''(t_n) + O(\Delta t^3) = O(\Delta t^2)$$

Accumulation of errors: Expected to be Proportional to # of steps $\sim \frac{t_1 - t_0}{\Delta t}$

Def: We call $u(t_n+\Delta t) - U^{n+1}$ the "one-step error"

and
$$\frac{u(t_{n+1}) - U^{n+1}}{\Delta t} = \frac{u(t_{n+1}) - (u(t_n) + \Delta t\, f(t_n, u(t_n)))}{\Delta t}$$

$$= \frac{u(t_{n+1}) - u(t_n)}{\Delta t} - f(t_n, u(t_n)) = \tau_n \quad \Big|\ (t_{n+1} = t_n + \Delta t)$$

the "local truncation error". Note: If $u(t_n)$, $u(t_{n+1})$ are replaced by U^n, U^{n+1} the expression for τ_n would be zero.

Richardson extrapolation

Idea: Combine results from using Δt and $\frac{1}{2}\Delta t$ to cancel out the leading term in the one-step error.
For Euler's method this is the $O(\Delta t^2)$ term.

Let $t_n = t$, $t_{n+1} = t + \Delta t$, $U^n = u(t)$

Euler step with Δt:

$$V^{n+1} = u(t) + \Delta t\, f(t, u(t)) = u(t) + \Delta t\, u'(t)$$

Now take 2 steps with stepsize $\frac{1}{2}\Delta t$.

$$\left.\begin{aligned} U^{n+1/2} &= U^n + \tfrac{1}{2}\Delta t\, f(t_n, U^n) \\ &= u(t) + \tfrac{1}{2}\Delta t\, u'(t) \end{aligned}\right\} \text{first step}$$

$$U^{n+1} = U^{n+1/2} + \tfrac{1}{2}\Delta t\, f\left(t + \tfrac{1}{2}\Delta t, U^{n+1/2}\right)$$

$$= U^n + \tfrac{1}{2}\Delta t\, f(t, U^n) + \tfrac{1}{2}\Delta t\, \underbrace{f\Big(t + \tfrac{1}{2}\Delta t, U^n + \tfrac{\Delta t}{2}\overbrace{f(t, U^n)}^{u'(t)}\Big)}_{\text{Taylor: } f + \frac{1}{2}\Delta t\, f_t + \frac{1}{2}\Delta t\, f_u f + O(\Delta t^2)}$$

Here $f = f(t, U^n)$, $f_t = f_t(t, U^n)$, etc.
Note: $U^n = u(t) \Rightarrow f(t, U^n) = f(t, u(t)) = u'(t)$.

$$U^{n+1} = u(t) + \tfrac{1}{2}\Delta t\, u'(t) + \tfrac{1}{2}\Delta t\Big[f + \underbrace{\tfrac{1}{2}\Delta t\, f_t + \tfrac{1}{2}\Delta t\, f_u f}_{\frac{1}{2}\Delta t (f_t + f_u f) = \frac{1}{2}\Delta t\, u''(t)\ \text{(HW)}} + O(\Delta t^2)\Big]$$

$$= u(t) + \tfrac{1}{2}\Delta t\, u'(t) + \tfrac{1}{4}\Delta t^2 u''(t) + O(\Delta t^3)$$