Richardson extrapolation

(updated 1-23-17, renamed U^{n+1} into W^{n+1})

Idea: Combine results from using Δt and $\frac{1}{2}\Delta t$ to cancel out the leading term in the one-step error.
For Euler's method this is the $O(\Delta t^2)$ term.

Let $t_n = t$, $t_{n+1} = t + \Delta t$, $U^n = u(t)$

Euler step with Δt:

$$V^{n+1} = u(t) + \Delta t\, f(t, u(t)) = u(t) + \Delta t\, u'(t)$$

Now take 2 steps with stepsize $\frac{1}{2}\Delta t$.

$$\left.\begin{aligned} U^{n+1/2} &= U^n + \tfrac{1}{2}\Delta t\, f(t_n, U^n) \\ &= u(t) + \tfrac{1}{2}\Delta t\, u'(t) \end{aligned}\right\} \text{first step}$$

$$W^{n+1} = U^{n+1/2} + \frac{1}{2}\Delta t\, f\left(t + \tfrac{1}{2}\Delta t, U^{n+1/2}\right)$$

$$= U^n + \frac{1}{2}\Delta t\, f(t, U^n) + \frac{1}{2}\Delta t\, \underbrace{f\Big(t + \tfrac{1}{2}\Delta t, U^n + \tfrac{\Delta t}{2}\overbrace{f(t, U^n)}^{u'(t)}\Big)}_{\text{Taylor: } f + \frac{1}{2}\Delta t f_t + \frac{1}{2}\Delta t f_u f + O(\Delta t^2)}$$

Here $f = f(t, U^n)$, $f_t = f_t(t, U^n)$, etc.
Note: $U^n = u(t) \Rightarrow f(t, U^n) = f(t, u(t)) = u'(t)$.

$$W^{n+1} = u(t) + \frac{1}{2}\Delta t\, u'(t) + \frac{1}{2}\Delta t\Big[f + \underbrace{\tfrac{1}{2}\Delta t\, f_t + \tfrac{1}{2}\Delta t\, f_u f}_{\frac{1}{2}\Delta t (f_t + f_u f) = \frac{1}{2}\Delta t\, u''(t) \text{ (HW)}} + O(\Delta t^2)\Big]$$

$$= u(t) + \Delta t\, u'(t) + \frac{1}{4}\Delta t^2 u''(t) + O(\Delta t^3)$$

1-23-17

Compare: $u(t+\Delta t) = u + \Delta t\, u' + \frac{1}{2}\Delta t^2 u'' + O(\Delta t^3)$

We have $V^{n+1} = u + \Delta t\, u'$

$W^{n+1} = u + \Delta t\, u' + \frac{1}{4}\Delta t^2 u'' + O(\Delta t^3)$

$\Rightarrow U^{n+1} = 2W^{n+1} - V^{n+1} = u + \Delta t\, u' + \frac{1}{2}\Delta t^2 u'' + O(\Delta t^3)$

$= u(t+\Delta t) + O(\Delta t^3)$

The new extrapolated method

$$U^{n+1} = U^n + \Delta t\, f\left(t_n + \tfrac{1}{2}\Delta t,\ U^n + \tfrac{\Delta t}{2} f(t_n, U_n)\right)$$

has one-step error $O(\Delta t^3)$ and local truncation error (LTE) of order $O(\Delta t^2)$.

This is also an example of a Runge-Kutta method. Rewrite as

$$K_1 = f(t_n, U^n)$$

$$K_2 = f\left(t_n + \tfrac{1}{2}\Delta t,\ U^n + \tfrac{1}{2}\Delta t K_1\right)$$

$$U^{n+1} = U^n + \Delta t K_2$$

General r-stage Runge-Kutta (RK) method:

$$K_j = f\left(t_n + c_j \Delta t,\ U^n + \Delta t \sum_{\ell=1}^{r} a_{j\ell} K_\ell\right), \quad j = 1, \dots, r$$

$$U^{n+1} = U^n + \Delta t \sum_{j=1}^{r} b_j K_j$$

Recap

1-23/17

$$V^{n+1} = U^n + \Delta t\, f(t_n, U^n) \quad \text{(1 Euler step with stepsize } \Delta t\text{)}$$

$$U^{n+1/2} = U^n + \tfrac{1}{2}\Delta t\, f(t_n, U^n)$$

$$W^{n+1} = U^{n+1/2} + \tfrac{1}{2}\Delta t\, f(t_n + \tfrac{1}{2}\Delta t, U^{n+1/2})$$

[2 Consecutive Euler steps with stepsize $\Delta t/2$]

Extrapolated method:

$$\begin{aligned}
U^{n+1} &= 2W^{n+1} - V^{n+1} \\
&= 2U^{n+1/2} + \Delta t\, f(t_n + \tfrac{1}{2}\Delta t, U^{n+1/2}) \\
&\quad - U^n - \Delta t\, f(t_n, U^n) \\
&= 2U^n + \Delta t\, f(t_n, U^n) + \Delta t\, f(t_n + \tfrac{1}{2}\Delta t, U^{n+1/2}) \\
&\quad - U^n - \Delta t\, f(t_n, U^n) \\
&= U^n + \Delta t\, f(t_n + \tfrac{1}{2}\Delta t, U^n + \tfrac{1}{2}\Delta t\, f(t_n, U^n))
\end{aligned}$$

Let $K_1 = f(t_n, U^n)$, $K_2 = f(t_n + \tfrac{1}{2}\Delta t, U^n + \Delta t \tfrac{1}{2} K_1)$.

Then: $U^{n+1} = U^n + \Delta t\, K_2$

General form: $U^{n+1} = U^n + \Delta t \sum_{j=1}^{r} b_j K_j$ (r-stage RK-method)

$$K_j = f(t_n + c_j \Delta t, U^n + \Delta t \sum_{\ell=1}^{r} a_{j\ell} K_\ell)$$

Butcher array (tableau)

$$\begin{array}{c|ccc}
c_1 & a_{11} & \cdots & a_{1r} \\
\vdots & \vdots & & \\
c_r & a_{r1} & \cdots & a_{rr} \\
\hline
 & b_1 & \cdots & b_r
\end{array}$$

Choose $a_{j\ell}$, b_j, c_j appropriately. We will derive conditions for the method to be consistent of order 1, 2, 3.

We already know 3 RK methods:

1) Euler method: $K_1 = f(t_n, U^n)$, $U^{n+1} = U^n + \Delta t\, K_1$.

$r = 1$. $c_1 = 0$, $a_{11} = 0$

$$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$$ = Butcher tableau for Euler method

2) 2 Euler steps of $\frac{1}{2}\Delta t$:

$$K_1 = f(t_n, U^n), \quad K_2 = f(t_n + \tfrac{1}{2}\Delta t, U^n + \tfrac{1}{2}\Delta t K_1)$$

$$U^{n+1} = U^n + \Delta t(\tfrac{1}{2}K_1 + \tfrac{1}{2}K_2)$$

$$\begin{array}{c|cc} 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & \frac{1}{2} & \frac{1}{2} \end{array}$$

3) Extrapolated method.

$$\begin{array}{c|cc} 0 & 0 & \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array}$$

K_1, K_2 as in 2) above.

$$U^{n+1} = U^n + \Delta t K_2 .$$