Homework # 2

ME526/NSE526

Due: October 25

1. You are encouraged to work in a group of up to 2 students and submit one solution per student.
2. Your solution must be clearly legible. Illegible work may not be graded and returned without any points. Although not necessary, you may type your work.
3. All problems must be solved. However, all problems may not be graded. A random sample of problems will be selected for grading.
4. If you are required to write a computer program, attach your code with several comment statements on the code wherever possible.

1. The objective is to use the modified wave-number analysis to identify the accuracy of a finite-difference formula.

   (a) Consider the function $f(x) = \exp(ikx)$ where $i = \sqrt{-1}$. Let $f_j = \exp(ikx_j)$ be the discrete representation of the continuous function. Consider the following expressions for the approximations of the first derivative[1]

$$f'_j = \frac{f_j - f_{j-1}}{\Delta x} \tag{1}$$

$$f'_j = \frac{f_{j+1} - f_{j-1}}{2\Delta x} \tag{2}$$

$$f'_j = \frac{3f_j - 4f_{j-1} + f_{j-2}}{2\Delta x} \tag{3}$$

$$f'_j = \frac{f_{j-3} - 6f_{j-2} + 3f_{j-1} + 2f_j}{6\Delta x} \tag{4}$$

$$f'_j = \frac{-f_{j+2} + 8f_{j+1} - 8f_{j-1} + f_{j-2}}{12\Delta x} \tag{5}$$

   Using the above discretization schemes, compute the modified wavenumbers (k^*). Note that you will have to express the first derivatives in the form $f'_j = ik^* f_j$ to evaluate k^*.

   (b) For some schemes these wave-numbers may be complex numbers (having real and imaginary parts). Plot the imaginary part of $(ik^*\Delta x)$ versus $k\Delta x$ for each scheme on the same plot (along with the straight line obtained for the exact scheme). Clearly identify your curves for each scheme (if curves overlap and you cannot distinguish them clearly, you may plot them on separate figures). In order to plot this, choose L (the domain length) to be 2π. This gives $k = 2\pi n/L = n$, where n is the wave number. Define (N+1) grid points to give $\Delta x = L/N = 2\pi/N$. Then to plot $k^*\Delta x$ versus $k\Delta x$, vary k from 0, $N/2$ (which represent the modes in a Fourier expansion). The x-axis will thus span from $[0, \pi]$. *Use large enough N to get smooth curves.* Compare different schemes.

   (c) Plot exp(Real Part$(-ik^*\Delta x)$) versus $k\Delta x$ for each scheme together with that for the exact scheme. Briefly explain what this plot indicates and compare different schemes.

   (d) The modified wave number can be used to determine the total error as a function of the resolution. In order to 'quantify' the error in the discrete approximation of the first derivative, let us define the error as

$$\epsilon = \frac{\|k^*\Delta x - k\Delta x\|}{k\Delta x} \times 100 \tag{6}$$

   where $\| \ \|$ is the absolute value. Again, let $(L = 2\pi)$ and now choose one particular mode, say $k = 1$. For this mode, vary the mesh spacing $\Delta x = 2\pi/N$ by changing N (start from $N = 2$ to $N = 100$). For each scheme evaluate the error and plot the 'error versus N' on a log–log scale. Again all should be on the same figure and use different line styles or symbols to indicate each scheme clearly. Comment on order of accuracy of each scheme.

[1] Hint: Start with $f_j = \exp(ikx_j)$ and replace derivatives as $f'_j = ik^* f'_j$ as well as any other derivatives by similar equations, where k^* is the modified wavenumber that you are trying to find.

2. Consider the first-order ordinary differential equation:

$$\frac{dv}{dt} = -0.2v - 2\cos(2t)v^2; \qquad v(0) = 1. \tag{7}$$

   (a) Solve the equation analytically.

   (b) For a Forward Euler scheme, find the finite difference approximation and predict any time-step restrictions for obtaining stable solution around the initial condition of $v(0) = 1$. Clearly show all steps.

   (c) Write a computer program based on Forward Euler to solve the above equation over $0 \le t \le 7$. Use the following time steps $\Delta t = 0.2,\ 0.05,\ 0.025,\ 0.006$. Plot the four numerical solutions together with the analytical solution on the 'same' graph (with clear labeling of all curves). Set the x axis from 0 to 7 and the y axis from 0 o 1.4. Discuss your results.

   (d) In practice, it is not easy to find the exact solution to the problem. In such situations, to obtain good confidence in our numerical prediction, we reduce the truncation errors by reducing the time step (by factor of 2), until the solutions based on two consecutive time-steps are nearly identical. This is called grid (or time-step) convergence study. Do you think that the solution corresponding to $\Delta t = 0.006$ is accurate (to plotting accuracy?). Justify your answer. In case you find it is not accurate, find a better one and indicate the time-step needed.

3. Consider the first-order ordinary equation:

$$\frac{dy}{dx} = -2y - 0.01x^2 y; \qquad y(0) = 4; \qquad 0 \le x \le 10. \tag{8}$$

   (a) Solve the differential equation using the following numerical schemes:

      i. Forward Euler

      ii. Backward Euler

      iii. Trapezoidal

   Clearly write down the finite difference approximations for each scheme. Use $\Delta x = 0.1$, 0.5, and 1.0 and compare to the exact solution.

   (b) For each of the above schemes, estimate the maximum Δx for stable solution (over the given domain). Discuss your estimate in terms of the results of the above part.

4. Consider the ODE

$$\frac{dy}{dt} = e^{y-t}, \quad y(0) = y_0, \tag{9}$$

which has an analytical solution given as

$$y(t) = -\ln(e^{-y_0} + e^{-t} - 1). \tag{10}$$

The goal is to compare fully implicit Euler scheme and a linearized implicit Euler scheme applied to the above problem.

(a) Derive a finite difference approximation for the Euler Implicit method and the linearized Euler Implicit method.

(b) For $y_0 = -1 \times 10^{-5}$ and with $h = 0.2$, solve the above equation using both methods. Plot their solutions against the analytical solution. Estimate the absolute error at each time step as $\epsilon = |y_{\text{num}} - y_{\text{exact}}|$ for each scheme and plot the error versus time for each method.

(c) Repeat the above part for $y_0 = -1$. Comment on the sensitivity of the linearized solution to the initial condition.