Taylor Tables of Differencing Schemes

- Notation: Consider $u(x,t)$ for fixed t and $x = j\Delta x$ so that, $u(x + k\Delta x) = u(j\Delta x + k\Delta x) = u_{j+k}$.
- The generalized form of the Taylor Series Expansions is given by

$$u_{j+k} = u_j + (k\Delta x)\left(\frac{\partial u}{\partial x}\right)_j + \frac{1}{2}(k\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \ldots + \frac{1}{n!}(k\Delta x)^n\left(\frac{\partial^n u}{\partial x^n}\right)_j + \ldots$$

- For example, consider the Taylor series expansion for u_{j+1}:

$$u_{j+1} = u_j + (\Delta x)\left(\frac{\partial u}{\partial x}\right)_j + \frac{1}{2}(\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \ldots + \frac{1}{n!}(\Delta x)^n\left(\frac{\partial^n u}{\partial x^n}\right)_j + \ldots$$

- Or for u_{j-2}:

$$u_{j-2} = u_j + (-2\Delta x)\left(\frac{\partial u}{\partial x}\right)_j + \frac{1}{2}(-2\Delta x)^2\left(\frac{\partial^2 u}{\partial x^2}\right)_j + \ldots + \frac{1}{n!}(-2\Delta x)^n\left(\frac{\partial^n u}{\partial x^n}\right)_j + \ldots$$

Finite Difference Formulas

- Take the expansion for u_{j-1}

$$u_{j-1} = u_j - \Delta x \left(\frac{\partial u}{\partial x}\right)_j + \frac{1}{2}(\Delta x)^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_j + \ldots$$

- Rearrange terms to

$$\left(\frac{\partial u}{\partial x}\right)_j - \frac{(u_j - u_{j-1})}{\Delta x} = \; er_t$$

- Truncation error term is $er_t = +\frac{1}{2}\Delta x \left(\frac{\partial^2 u}{\partial x^2}\right)_j$

- The truncation error er_t is made up of 4 important pieces

$$er_t = \text{Sign} \quad \text{Coefficient} \quad \Delta x^p \quad (p+q)^{th}\text{Derivative}$$

Taylor Table For the 1^{st} Order Backward Difference

- Given

$$\left(\frac{\partial u}{\partial x}\right)_j - \frac{(u_j - u_{j-1})}{\Delta x} = er_t$$

- Each term expanded in Taylor Series and placed in a table simplifing algebra.
- Note the multiplication by Δx to again simplify the table.

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$
$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$		1			
u_{j-1}	1	(-1) $\frac{1}{1!}$	$(-1)^2 \frac{1}{2!}$	$(-1)^3 \frac{1}{3!}$	$(-1)^4 \frac{1}{4!}$
$-u_j$	-1	0	0	0	0
$=$					
$\Delta x \cdot er_t$	0	0	$\frac{1}{2}$	?	?

- Truncation error term $er_t = \frac{1}{2}\Delta x \left(\frac{\partial^2 u}{\partial x^2}\right)_j$ defined from the first non-zero column.
- Don't forget the division by the Δx to undo the previous multiplication.
- Order of accuracy is defined as the exponent on the Δx term in er_t.

Taylor Table For the 2^{nd} Order Central Difference

- Given

$$\left(\frac{\partial u}{\partial x}\right)_j - \frac{(u_{j+1} - u_{j-1})}{2\Delta x} = er_t$$

- The Taylor Table

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$
$2\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$		2			
u_{j-1}	1	$(-1)\,\frac{1}{1!}$	$(-1)^2\,\frac{1}{2!}$	$(-1)^3\,\frac{1}{3!}$	$(-1)^4\,\frac{1}{4!}$
$-\ u_{j+1}$	-1	$-(1)\,\frac{1}{1!}$	$-(1)^2\,\frac{1}{2!}$	$-(1)^3\,\frac{1}{3!}$	$-(1)^4\,\frac{1}{4!}$
$=$					
$2\Delta x \cdot er_t$	0	0	0	$-\frac{1}{3}$	?

- The truncation error term $er_t = -\frac{1}{6}\Delta x^2 \left(\frac{\partial^3 u}{\partial x^3}\right)_j$ is defined from the first non-zero column.

- Accuracy is 2^{nd} Order.

Taylor Table For A General 3 Point Difference Scheme

- Starting with

$$\left(\frac{\partial u}{\partial x}\right)_j - \frac{1}{\Delta x}(c\,u_{j-2} + b\,u_{j-1} + a\,u_j) = \; er_t$$

- The Taylor Table

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$
$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$		1			
$-cu_{j-2}$	$-c$	$-c \cdot (\text{-}2) \cdot \frac{1}{1}$	$-c \cdot (\text{-}2)^2 \cdot \frac{1}{2}$	$-c \cdot (\text{-}2)^3 \cdot \frac{1}{6}$	$-c \cdot (\text{-}2)^4 \cdot \frac{1}{24}$
$-bu_{j-1}$	$-b$	$-b \cdot (\text{-}1) \cdot \frac{1}{1}$	$-b \cdot (\text{-}1)^2 \cdot \frac{1}{2}$	$-b \cdot (\text{-}1)^3 \cdot \frac{1}{6}$	$-b \cdot (\text{-}1)^4 \cdot \frac{1}{24}$
$-au_j$	$-a$				
$=$					
$\Delta x \cdot er_t$	?	?	?	?	?

- Now instead of having colums sum to zero, we set enough colums to zero to satisfy the number of unknowns.

Taylor Table For A General 3 Point Difference Scheme

- This time the first three columns sum to zero if

$$\begin{bmatrix} -1 & -1 & -1 \\ 2 & 1 & 0 \\ -4 & -1 & 0 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

- Note we put the linear equations into a matrix form, let Matlab do the work for you.
- Which gives $[c, b, a] = \frac{1}{2}[1, -4, 3]$.
- In this case the fourth column provides the leading truncation

$$er_t = \frac{1}{\Delta x}\left[\frac{8c}{6} + \frac{b}{6}\right]\Delta x^3 \left(\frac{\partial^3 u}{\partial x^3}\right)_j = \frac{\Delta x^2}{3}\left(\frac{\partial^3 u}{\partial x^3}\right)_j$$

- Thus we have derived a second-order backward-difference approximation of a first derivative:

$$\left(\frac{\partial u}{\partial x}\right)_j = \frac{1}{2\Delta x}(u_{j-2} - 4u_{j-1} + 3u_j) + O(\Delta x^2)$$

Taylor Table For Other Derivatives, e.g. 2^{nd}

- Consider a gerneral 3 point formula for the 2^{nd} derivative

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j - \frac{1}{\Delta x^2}(a\,u_{j-1} + b\,u_j + c\,u_{j+1}) = \; er_t$$

- The Taylor Table is

	u_j	$\Delta x \cdot \left(\frac{\partial u}{\partial x}\right)_j$	$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$	$\Delta x^3 \cdot \left(\frac{\partial^3 u}{\partial x^3}\right)_j$	$\Delta x^4 \cdot \left(\frac{\partial^4 u}{\partial x^4}\right)_j$
$\Delta x^2 \cdot \left(\frac{\partial^2 u}{\partial x^2}\right)_j$			1		
$-a u_{j-1}$	$-a$	$-a \cdot (-1) \cdot \frac{1}{1}$	$-a \cdot (-1)^2 \cdot \frac{1}{2}$	$-a \cdot (-1)^3 \cdot \frac{1}{6}$	$-a \cdot (-1)^4 \cdot \frac{1}{24}$
$-b u_j$	$-b$				
$-c u_{j+1}$	$-c$	$-c \cdot (1) \cdot \frac{1}{1}$	$-c \cdot (1)^2 \cdot \frac{1}{2}$	$-c \cdot (1)^3 \cdot \frac{1}{6}$	$-c \cdot (1)^4 \cdot \frac{1}{24}$
$=$					
$\Delta x^2 \cdot er_t$	?	?	?	?	?

- Setting the first 3 colums to 0 leads to

$$\begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}$$

- The solution is given by $[a, b, c] = [1, -2, 1]$.

Taylor Table For 2^{nd} Derivative

- In this case er_t occurs at the fifth column in the table (for this example all even columns will vanish by symmetry) and one finds

$$er_t = \frac{1}{\Delta x^2}\left[\frac{-a}{24} + \frac{-c}{24}\right]\Delta x^4 \left(\frac{\partial^4 u}{\partial x^4}\right)_j = \frac{-\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_j$$

- Note that Δx^2 has been divided through to make the error term consistent.
- We have just derived the familiar 3-point central-differencing point operator for a second derivative

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_j = \frac{1}{\Delta x^2}(u_{j-1} - 2u_j + u_{j+1}) + O(\Delta x^2)$$