2-17-17

Stepsize control using embedded RK methods.

Idea: Find 2 RK methods that share A, c

$\begin{array}{c|c} c & A \\ \hline & b \end{array}$ has order q, $\begin{array}{c|c} c & A \\ \hline & \hat{b} \end{array}$ has order $q+1$

Use the (supposedly) more accurate result from the order $q+1$ method to estimate the one-step error of the order q method.

Let U^{n+1}, $\hat{U}^{n+1}$ be results from these two methods.

$$u(t_n + \Delta t) - U^{n+1} \approx \hat{U}^{n+1} - U^{n+1}$$

Then proceed as when $\hat{U}^{n+1}$ was found via Richardson extrapolation:

$$E_{n+1} = | \hat{U}^{n+1} - U^{n+1} | \le \tau = \text{tolerance}$$

Δt_1 = new Δt.

$$\Delta t_1 = s \cdot \left(\frac{\tau}{E_{n+1}} \right)^{\frac{1}{q+1}} \Delta t.$$

s = safety factor, $0 < s < 1$.

Embedded methods require fewer evaluations of f for the error estimate.

Note: We can only estimate the error for the lower order method.

If the higher method is used to carry the solution, this is called local extrapolation (RK-Fehlberg methods).

Ex for counting evaluations of $f(t,u)$;

let $f(t,u) = t - u^2$.

```
function uprime = myfct(t,u)
  global fcount
  uprime = t.^2 + u^2;
  fcount = fcount + 1;
end
```

In the main program,

Start with:

```
global fcount   % define global variable
fcount = 0;     % initialize only once at
                % start of main program.
```

Systems of linear ODEs and linear difference equations with constant coefficients.

Def: A linear system of ODEs has the form

$$u' = Au + \varphi(t) \qquad (1)$$

where $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times m}$, A may depend on t but does not depend on u. We only consider A with constant coefficients.

Note: $f(t,u) = Au + \varphi(t)$ is linear with respect to u but not necessarily w.r. to t.

A set of M solutions to (1) $\{u_k(t), k=1,\dots,M\}$ is called linearly independent if

$$\sum_{k=1}^{M} c_k u_k(t) = 0 \text{ for all } t \iff c_1 = \dots = c_n = 0.$$

In words: The linear combination $\sum c_k u_k(t)$ is equal to the zero-function if and only if all c_k are zero.

A set of m lin. ind. solutions $\{\tilde{u}_k, 1,\dots,m\}$ of the homogeneous system

$$u' = Au \qquad (2)$$

is called a fundamental system of solution. Every solution of (2) can be written as

$$u(t) = \sum_{k=1}^{m} c_k \tilde{u}_k(t)$$

If $\psi(t)$ solves (1) (i.e. ψ is a "particular solution") then every solution of (1) can be written as

$$u(t) = \psi(t) + \sum_k c_k \tilde{u}_k(t)$$

Thm: Let λ be an eigenvalue of A with algebraic multiplicity σ. Then there are σ linearly ind. solutions of (2) that have the form

$$u_1(t) = P_0(t)\, e^{\lambda t}, \ldots, u_\sigma(t) = P_{\sigma-1}(t)\, e^{\lambda t}$$

where each component ${}^{e}P_j(t)$ of

$$P_j(t) = \begin{bmatrix} {}^{1}P_j(t) \\ \vdots \\ {}^{n}P_j(t) \end{bmatrix}$$

is a polynomial of degree $\leq j$.

If this construction is carried out for every eigenvalue of A, the set of all these solutions forms a fundamental system.

Corollary: If A has n distinct eigenvalues λ_j, $j=1,\ldots,n$ with corresponding eigenvectors v_j, $j=1,\ldots,n$, then the functions $u_j(t) = v_j\, e^{\lambda_j t}$, $j = 1,\ldots,n$ form a fundamental system.