Fluid Mechanics

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Chapter 2

Cartesian Tensors

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1. *Scalars and Vectors*

In fluid mechanics we need to deal with quantities of various complexities. Some of these are defined by only one component and are called scalars, some others are defined by three components and are called vectors, and certain other variables called tensors need as many as nine components for a complete description. We shall assume that the reader is familiar with a certain amount of algebra and calculus of vectors. The concept and manipulation of tensors is the subject of this chapter.

A *scalar* is any quantity that is completely specified by a magnitude only, along with its unit. It is independent of the coordinate system. Examples of scalars are temperature and density of the fluid. A *vector* is any quantity that has a magnitude and a direction, and can be completely described by its components along three specified coordinate directions. A vector is usually denoted by a boldface symbol, for example, **x** for position and **u** for velocity. We can take a Cartesian coordinate system x_1, x_2, x_3, with unit vectors $\mathbf{a}^1$, $\mathbf{a}^2$, and $\mathbf{a}^3$ in the three mutually perpendicular directions (Figure 2.1). (In texts on vector analysis, the unit vectors are usually denoted

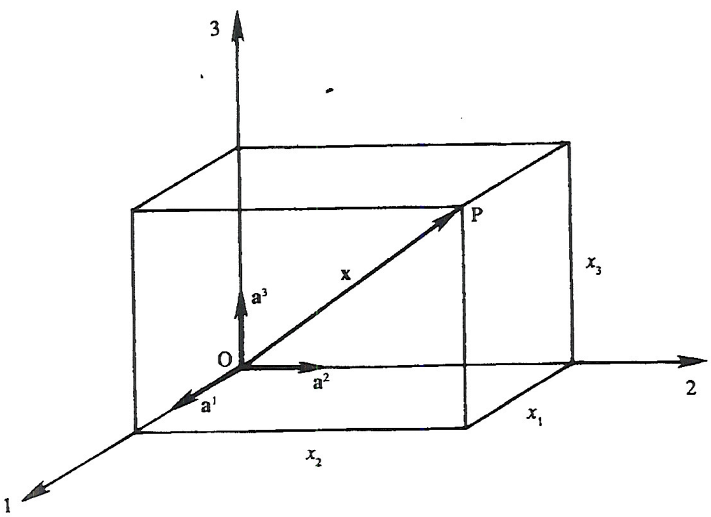


Figure 2.1 Position vector OP and its three Cartesian components (x_1, x_2, x_3). The three unit vectors are $\mathbf{a}^1$, $\mathbf{a}^2$, and $\mathbf{a}^3$.

by **i**, **j**, and **k**. We cannot use this simple notation here because we shall use ijk to denote *components* of a vector.) Then the position vector is written as

$$\mathbf{x} = \mathbf{a}^1 x_1 + \mathbf{a}^2 x_2 + \mathbf{a}^3 x_3,$$

where (x_1, x_2, x_3) are the components of $\mathbf{x}$ along the coordinate directions. (The superscripts on the unit vectors $\mathbf{a}$ do *not* denote the components of a vector; the $\mathbf{a}$'s are vectors themselves.) Instead of writing all three components explicitly, we can indicate the three Cartesian components of a vector by an index that takes all possible values of 1, 2, and 3. For example, the components of the position vector can be denoted by x_i, where i takes all of its possible values, namely, 1, 2, and 3. To obey the laws of algebra that we shall present, the components of a vector should be written as a column. For example,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

In matrix algebra, one defines the *transpose* as the matrix obtained by interchanging rows and columns. For example, the transpose of a column matrix $\mathbf{x}$ is the row matrix

$$\mathbf{x}^T = [x_1 \quad x_2 \quad x_3].$$

2. *Rotation of Axes: Formal Definition of a Vector*

A vector can be formally defined as any quantity whose components change similarly to the components of a position vector under the rotation of the coordinate system. Let $x_1\, x_2\, x_3$ be the original axes, and $x_1'\, x_2'\, x_3'$ be the rotated system (Figure 2.2). The

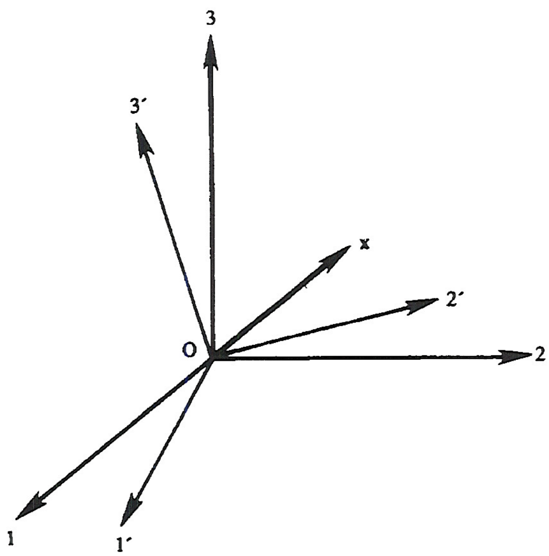


Figure 2.2 Rotation of coordinate system O 1 2 3 to O 1′ 2′ 3′.

components of the position vector **x** in the original and rotated systems are denoted by x_i and x'_i, respectively. The cosine of the angle between the old i and new j axes is represented by C_{ij}. Here, the *first* index of the **C** matrix refers to the *old* axes, and the second index of **C** refers to the new axes. It is apparent that $C_{ij} \neq C_{ji}$. A little geometry shows that the components in the rotated system are related to the components in the original system by

$$x'_j = x_1 C_{1j} + x_2 C_{2j} + x_3 C_{3j} = \sum_{i=1}^{3} x_i C_{ij}. \tag{2.1}$$

For simplicity, we shall verify the validity of Eq. (2.1) in two dimensions only. Referring to Figure 2.3, let α_{ij} be the angle between old i and new j axes, so that $C_{ij} = \cos\alpha_{ij}$. Then

$$x'_1 = \mathrm{OD} = \mathrm{OC} + \mathrm{AB} = x_1 \cos\alpha_{11} + x_2 \sin\alpha_{11}. \tag{2.2}$$

As $\alpha_{11} = 90° - \alpha_{21}$, we have $\sin\alpha_{11} = \cos\alpha_{21} = C_{21}$. Equation (2.2) then becomes

$$x'_1 = x_1 C_{11} + x_2 C_{21} = \sum_{i=1}^{2} x_i C_{i1}. \tag{2.3}$$

In a similar manner

$$x'_2 = \mathrm{PD} = \mathrm{PB} - \mathrm{DB} = x_2 \cos\alpha_{11} - x_1 \sin\alpha_{11}.$$

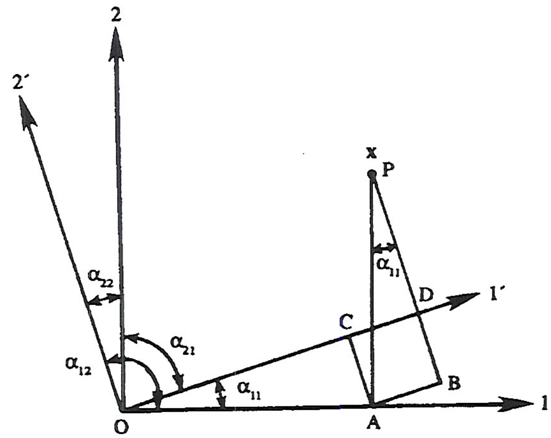


Figure 2.3 Rotation of a coordinate system in two dimensions.

As $\alpha_{11} = \alpha_{22} = \alpha_{12} - 90°$ (Figure 2.3), this becomes

$$x'_2 = x_2 \cos\alpha_{22} + x_1 \cos\alpha_{12} = \sum_{i=1}^{2} x_i C_{i2}. \tag{2.4}$$

In two dimensions, Eq. (2.1) reduces to Eq. (2.3) for $j = 1$, and to Eq. (2.4) for $j = 2$. This completes our verification of Eq. (2.1).

Note that the index i appears twice in the same term on the right-hand side of Eq. (2.1), and a summation is carried out over all values of this repeated index. This type of summation over repeated indices appears frequently in tensor notation. A convention is therefore adopted that, *whenever an index occurs twice in a term, a summation over the repeated index is implied, although no summation sign is explicitly written*. This is frequently called the *Einstein summation convention.* Equation (2.1) is then simply written as

summation rule.

$$x'_j = x_i C_{ij}, \tag{2.5}$$

where a summation over i is understood on the right-hand side.

The free index on both sides of Eq. (2.5) is j, and i is the repeated or dummy index. Obviously any letter (other than j) can be used as the dummy index without changing the meaning of this equation. For example, Eq. (2.5) can be written equivalently as

$$x_i C_{ij} = x_k C_{kj} = x_m C_{mj} = \cdots,$$

because they all mean $x'_j = C_{1j}x_1 + C_{2j}x_2 + C_{3j}x_3$. Likewise, any letter can also be used for the free index, as long as the same free index is used on *both* sides of the equation. For example, denoting the free index by i and the summed index by k, Eq. (2.5) can be written as

$$x'_i = x_k C_{ki}. \tag{2.6}$$

This is because the set of three equations represented by Eq. (2.5) corresponding to all values of j is the same set of equations represented by Eq. (2.6) for all values of i.

It is easy to show that the components of $\mathbf{x}$ in the old coordinate system are related to those in the rotated system by

$$x_j = C_{ji} x'_i. \tag{2.7}$$

Note that the indicial positions on the right-hand side of this relation are different from those in Eq. (2.5), because the first index of $\mathbf{C}$ is summed in Eq. (2.5), whereas the second index of $\mathbf{C}$ is summed in Eq. (2.7).

We can now formally define a Cartesian vector as any quantity that transforms like a position vector under the rotation of the coordinate system. Therefore, by analogy with Eq. (2.5), $\mathbf{u}$ is a vector if its components transform as

$$\boxed{u'_j = u_i C_{ij}.} \tag{2.8}$$

3. *Multiplication of Matrices*

In this chapter we shall generally follow the convention that 3×3 matrices are represented by uppercase letters, and column vectors are represented by lowercase letters. (An exception will be the use of lowercase $\boldsymbol{\tau}$ for the stress matrix.) Let $\mathbf{A}$ and $\mathbf{B}$ be two 3×3 matrices. The product of $\mathbf{A}$ and $\mathbf{B}$ is defined as the matrix $\mathbf{P}$ whose elements are related to those of $\mathbf{A}$ and $\mathbf{B}$ by

$$P_{ij} = \sum_{k=1}^{3} A_{ik} B_{kj},$$

or, using the summation convention

$$\boxed{P_{ij} = A_{ik} B_{kj}.} \tag{2.9}$$

Symbolically, this is written as

$$\mathbf{P} = \mathbf{A} \cdot \mathbf{B}. \tag{2.10}$$

A single dot between $\mathbf{A}$ and $\mathbf{B}$ is included in Eq. (2.10) to signify that a single index is summed on the right-hand side of Eq. (2.9). The important thing to note in Eq. (2.9) is that the elements are summed over the inner or *adjacent* index k. It is sometimes useful to write Eq. (2.9) as

$$P_{ij} = A_{ik} B_{kj} = (\mathbf{A} \cdot \mathbf{B})_{ij},$$

where the last term is to be read as the "ij-element of the product of matrices $\mathbf{A}$ and $\mathbf{B}$."

In explicit form, Eq. (2.9) is written as

$$\begin{bmatrix} P_{11} & \boxed{P_{12}} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \tag{2.11}$$

Note that Eq. (2.9) signifies that the ij-element of **P** is determined by multiplying the elements in the i-row of **A** and the j-column of **B**, and summing. For example,

$$P_{12} = A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32}.$$

This is indicated by the dotted lines in Eq. (2.11). It is clear that we can define the product $\mathbf{A} \cdot \mathbf{B}$ only if the number of columns of **A** equals the number of rows of **B**.

Equation (2.9) can be used to determine the product of a 3×3 matrix and a vector, if the vector is written as a column. For example, Eq. (2.6) can be written as $x_i' = C_{ik}^{\mathrm{T}} x_k$, which is now of the form of Eq. (2.9) because the summed index k is adjacent. In matrix form Eq. (2.6) can therefore be written as

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Symbolically, the preceding is

$$\mathbf{x}' = \mathbf{C}^{\mathrm{T}} \cdot \mathbf{x},$$

whereas Eq. (2.7) is

$$\mathbf{x} = \mathbf{C} \cdot \mathbf{x}'.$$

4. Second-Order Tensor

We have seen that scalars can be represented by a single number, and a Cartesian vector can be represented by three numbers. There are other quantities, however, that need more than three components for a complete description. For example, the stress (equal to force per unit area) at a point in a material needs nine components for a complete specification because *two* directions (and, therefore, *two* free indices) are involved in its description. One direction specifies the orientation of the *surface* on which the stress is being sought, and the other specifies the direction of the *force* on that surface. For example, the j-component of the force on a surface whose outward normal points in the i-direction is denoted by τ_{ij}. (Here, we are departing from the convention followed in the rest of the chapter, namely, that tensors are represented by uppercase letters. It is customary to denote the stress tensor by the lowercase τ.) The first index of τ_{ij} denotes the direction of the normal, and the second index denotes the direction in which the force is being projected.

This is shown in Figure 2.4, which gives the normal and shear stresses on an infinitesimal cube whose surfaces are parallel to the coordinate planes. The stresses are positive if they are directed as in this figure. The sign convention is that, on a surface whose outward normal points in the positive direction of a coordinate axis, the normal and shear stresses are positive if they point in the positive direction of the axes. For example, on the surface ABCD, whose outward normal points in the positive x_2 direction, the positive stresses τ_{21}, τ_{22}, and τ_{23} point toward the x_1, x_2 and x_3 directions, respectively. (Clearly, the normal stresses are positive if they are tensile and negative if they are compressive.) On the opposite face EFGH the stress components have the same value as on ABCD, but their directions are reversed. This

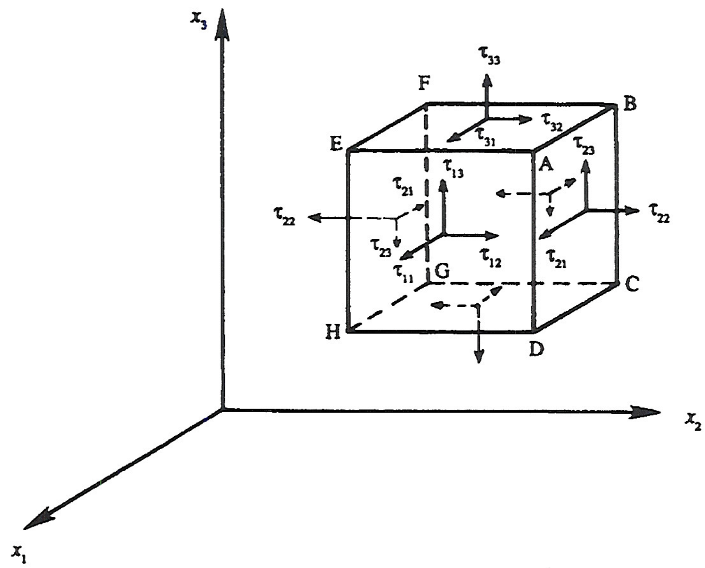


Figure 2.4 Stress field at a point. Positive normal and shear stresses are shown. For clarity, the stresses on faces FBCG and CDHG are not labeled.

is because Figure 2.4 shows the stresses *at a point*. The cube shown is supposed to be of "zero" size, so that the faces ABCD and EFGH are just opposite faces of a plane perpendicular to the x_2-axis. That is why the stresses on the opposite faces are equal and opposite.

Recall that a vector $\mathbf{u}$ can be completely specified by the three components u_i (where $i = 1, 2, 3$). We say "completely specified" because the components of $\mathbf{u}$ in any direction other than the original axes can be found from Eq. (2.8). Similarly, the state of stress at a point can be completely specified by the nine components τ_{ij} (where $i, j = 1, 2, 3$), which can be written as the matrix

$$\boldsymbol{\tau} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}.$$

The specification of the preceding nine components of the stress on surfaces parallel to the coordinate axes completely determines the state of stress at a point, because the stresses on any arbitrary plane can then be determined. To find the stresses on any arbitrary surface, we shall consider a rotated coordinate system $x_1' \, x_2' \, x_3'$ one of whose axes is perpendicular to the given surface. It can be shown by a force balance on a tetrahedron element (see, e.g., Sommerfeld (1964), page 59) that the components of $\boldsymbol{\tau}$ in the rotated coordinate system are

$$\boxed{\tau_{mn}' = C_{im} C_{jn} \tau_{ij}.} \tag{2.12}$$

Note the similarity between the transformation rule Eq. (2.8) for a vector, and the rule Eq. (2.12). In Eq. (2.8) the first index of $\mathbf{C}$ is summed, while its second index is free. The rule Eq. (2.12) is identical, except that this happens twice. A quantity that obeys the transformation rule Eq. (2.12) is called a *second-order tensor*.

The transformation rule Eq. (2.12) can be expressed as a matrix product. Rewrite Eq. (2.12) as

$$\tau'_{mn} = C^{\mathrm{T}}_{mi} \tau_{ij} C_{jn},$$

which, with adjacent dummy indices, represents the matrix product

$$\boldsymbol{\tau}' = \mathbf{C}^{\mathrm{T}} \cdot \boldsymbol{\tau} \cdot \mathbf{C}.$$

This says that the tensor $\boldsymbol{\tau}$ in the rotated frame is found by multiplying $\mathbf{C}$ by $\boldsymbol{\tau}$ and then multiplying the product by $\mathbf{C}^{\mathrm{T}}$.

The concepts of tensor and matrix are not quite the same. A matrix is any *arrangement* of elements, written as an array. The elements of a matrix represent the components of a tensor only if they obey the transformation rule Eq. (2.12).

Tensors can be of any order. In fact, a scalar can be considered a tensor of zero order, and a vector can be regarded as a tensor of first order. The number of free indices correspond to the order of the tensor. For example, $\mathbf{A}$ is a fourth-order tensor if it has four free indices, and the associated 81 components change under the rotation of the coordinate system according to

$$A'_{mnpq} = C_{im} C_{jn} C_{kp} C_{lq} A_{ijkl}. \tag{2.13}$$

Tensors of various orders arise in fluid mechanics. Some of the most frequently used are the stress tensor τ_{ij} and the velocity gradient tensor $\partial u_i/\partial x_j$. It can be shown that the nine products $u_i v_j$ formed from the components of the two vectors $\mathbf{u}$ and $\mathbf{v}$ also transform according to Eq. (2.12), and therefore form a second-order tensor. In addition, certain "isotropic" tensors are also frequently used; these will be discussed in Section 7.

5. Contraction and Multiplication

When the two indices of a tensor are equated, and a summation is performed over this repeated index, the process is called *contraction*. An example is

$$A_{jj} = A_{11} + A_{22} + A_{33},$$

which is the sum of the diagonal terms. Clearly, A_{jj} is a scalar and therefore independent of the coordinate system. In other words, A_{jj} is an *invariant*. (There are three independent invariants of a second-order tensor, and A_{jj} is one of them; see Exercise 5.)

Higher-order tensors can be formed by multiplying lower tensors. If $\mathbf{u}$ and $\mathbf{v}$ are vectors, then the nine components $u_i v_j$ form a second-order tensor. Similarly, if $\mathbf{A}$ and $\mathbf{B}$ are two second-order tensors, then the 81 numbers defined by $P_{ijkl} \equiv A_{ij} B_{kl}$ transform according to Eq. (2.13), and therefore form a fourth-order tensor.

Lower-order tensors can be obtained by performing contraction on these multiplied forms. The four contractions of $A_{ij}B_{kl}$ are

$$\begin{aligned} A_{ij}B_{ki} &= B_{ki}A_{ij} = (\mathbf{B}\cdot\mathbf{A})_{kj}, \\ A_{ij}B_{ik} &= A^{\mathrm{T}}_{ji}B_{ik} = (\mathbf{A}^{\mathrm{T}}\cdot\mathbf{B})_{jk}, \\ A_{ij}B_{kj} &= A_{ij}B^{\mathrm{T}}_{jk} = (\mathbf{A}\cdot\mathbf{B}^{\mathrm{T}})_{ik}, \\ A_{ij}B_{jk} &= (\mathbf{A}\cdot\mathbf{B})_{ik}. \end{aligned} \tag{2.14}$$

All four products in the preceding are second-order tensors. Note in Eq. (2.14) how the terms have been rearranged until the summed index is adjacent, at which point they can be written as a product of matrices.

The contracted product of a second-order tensor $\mathbf{A}$ and a vector $\mathbf{u}$ is a vector. The two possibilities are

$$\begin{aligned} A_{ij}u_j &= (\mathbf{A}\cdot\mathbf{u})_i, \\ A_{ij}u_i &= A^{\mathrm{T}}_{ji}u_i = (\mathbf{A}^{\mathrm{T}}\cdot\mathbf{u})_j. \end{aligned}$$

The doubly contracted product of two second-order tensors $\mathbf{A}$ and $\mathbf{B}$ is a scalar. The two possibilities are $A_{ij}B_{ji}$ (which can be written as $\mathbf{A}:\mathbf{B}$ in boldface notation) and $A_{ij}B_{ij}$ (which can be written as $\mathbf{A}:\mathbf{B}^{\mathrm{T}}$).

6. *Force on a Surface*

A surface area has a magnitude and an orientation, and therefore should be treated as a vector. The orientation of the surface is conveniently specified by the direction of a unit vector normal to the surface. If dA is the magnitude of an element of surface and $\mathbf{n}$ is the unit vector normal to the surface, then the surface area can be written as the vector

$$d\mathbf{A} = \mathbf{n}\,dA.$$

Suppose the nine components of the stress tensor with respect to a given set of Cartesian coordinates are given, and we want to find the force per unit area on a surface of given orientation $\mathbf{n}$ (Figure 2.5). One way of determining this is to take a

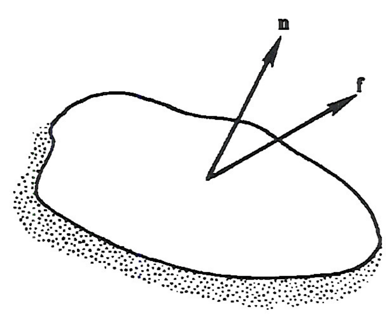


Figure 2.5 Force $\mathbf{f}$ per unit area on a surface element whose outward normal is $\mathbf{n}$.

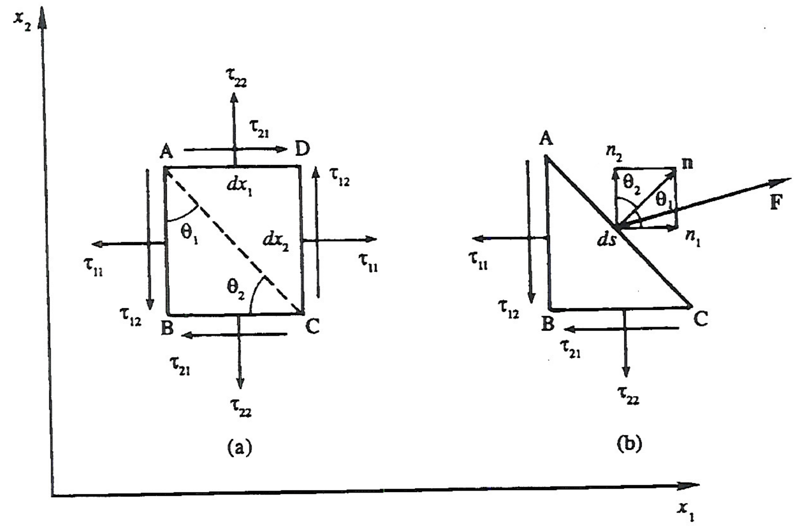


Figure 2.6 (a) Stresses on surfaces of a two-dimensional element; (b) balance of forces on element ABC.

rotated coordinate system, and use Eq. (2.12) to find the normal and shear stresses on the given surface. An alternative method is described in what follows.

For simplicity, consider a two-dimensional case, for which the known stress components with respect to a coordinate system $x_1\, x_2$ are shown in Figure 2.6a. We want to find the force on the face AC, whose outward normal $\mathbf{n}$ is known (Figure 2.6b). Consider the balance of forces on a triangular element ABC, with sides AB $= dx_2$, BC $= dx_1$, and AC $= ds$; the thickness of the element in the x_3 direction is unity. If $\mathbf{F}$ is the force on the face AC, then a balance of forces in the x_1 direction gives the component of $\mathbf{F}$ in that direction as

$$F_1 = \tau_{11}\, dx_2 + \tau_{21}\, dx_1.$$

(2.0) see Fig 2.6 (b)

Dividing by ds, and denoting the force per unit area as $\mathbf{f} = \mathbf{F}/ds$, we obtain

$$\begin{aligned} f_1 &= \frac{F_1}{ds} = \tau_{11}\frac{dx_2}{ds} + \tau_{21}\frac{dx_1}{ds} \\ &= \tau_{11}\cos\theta_1 + \tau_{21}\cos\theta_2 = \tau_{11}n_1 + \tau_{21}n_2, \end{aligned}$$

*

where $n_1 = \cos\theta_1$ and $n_2 = \cos\theta_2$ because the magnitude of $\mathbf{n}$ is unity (Figure 2.6b). Using the summation convention, the foregoing can be written as $f_1 = \tau_{j1}n_j$, where j is summed over 1 and 2. A similar balance of forces in the x_2 direction gives $f_2 = \tau_{j2}n_j$. Generalizing to three dimensions, it is clear that

$$f_i = \tau_{ji}n_j.$$

Because the stress tensor is symmetric (which will be proved in the next chapter), that is, $\tau_{ij} = \tau_{ji}$, the foregoing relation can be written in boldface notation as

$$\boxed{\mathbf{f} = \mathbf{n}\cdot\boldsymbol{\tau}.} \qquad (2.15)$$

Therefore, the contracted or "inner" product of the stress tensor τ and the unit outward vector $\mathbf{n}$ gives the force per unit area on a surface. Equation (2.15) is analogous to $u_n = \mathbf{u} \cdot \mathbf{n}$, where u_n is the component of the vector $\mathbf{u}$ along unit normal $\mathbf{n}$; however, whereas u_n is a scalar, $\mathbf{f}$ in Eq. (2.15) is a vector.

Example 2.1. Consider a two-dimensional parallel flow through a channel. Take x_1, x_2 as the coordinate system, with x_1 parallel to the flow. The viscous stress tensor at a point in the flow has the form

$$\tau = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix},$$

where the constant a is positive in one half of the channel, and negative in the other half. Find the magnitude and direction of force per unit area on an element whose outward normal points at 30° to the direction of flow.

Solution by Using Eq. (2.15): Because the magnitude of $\mathbf{n}$ is 1 and it points at 30° to the x_1 axis (Figure 2.7), we have

$$\mathbf{n} = \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix}.$$

The force per unit area is therefore

$$\mathbf{f} = \tau \cdot \mathbf{n} = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} a/2 \\ \sqrt{3}\,a/2 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}.$$

The magnitude of $\mathbf{f}$ is

$$f = (f_1^2 + f_2^2)^{1/2} = |a|.$$

If θ is the angle of $\mathbf{f}$ with the x_1 axis, then

$$\sin\theta = \frac{f_2}{f} = \frac{\sqrt{3}}{2}\frac{a}{|a|} \quad \text{and} \quad \cos\theta = \frac{f_1}{f} = \frac{1}{2}\frac{a}{|a|}.$$

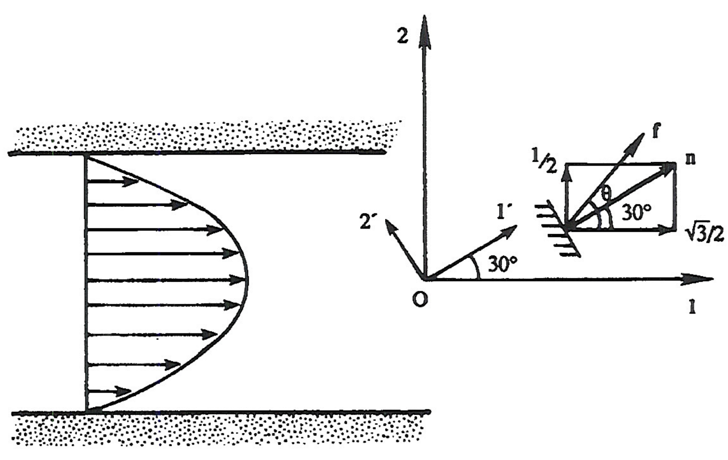


Figure 2.7 Determination of force on an area element (Example 2.1).

Thus $\theta = 60°$ if a is positive (in which case both $\sin\theta$ and $\cos\theta$ are positive), and $\theta = 240°$ if a is negative (in which case both $\sin\theta$ and $\cos\theta$ are negative).

Solution by Using Eq. (2.12) : Take a rotated coordinate system x_1', x_2', with x_1' axis coinciding with $\mathbf{n}$ (Figure 2.7). Using Eq. (2.12), the components of the stress tensor in the rotated frame are

$$\tau_{11}' = C_{11}C_{21}\tau_{12} + C_{21}C_{11}\tau_{21} = \tfrac{\sqrt{3}}{2}\tfrac{1}{2}a + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}a = \tfrac{\sqrt{3}}{2}a,$$
$$\tau_{12}' = C_{11}C_{22}\tau_{12} + C_{21}C_{12}\tau_{21} = \tfrac{\sqrt{3}}{2}\tfrac{\sqrt{3}}{2}a - \tfrac{1}{2}\tfrac{1}{2}a = \tfrac{1}{2}a.$$

The normal stress is therefore $\sqrt{3}\,a/2$, and the shear stress is $a/2$. This gives a magnitude a and a direction 60° or 240° depending on the sign of a. □

7. *Kronecker Delta and Alternating Tensor*

The *Kronecker delta* is defined as

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \tag{2.16}$$

which is written in the matrix form as

$$\boldsymbol{\delta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The most common use of the Kronecker delta is in the following operation: If we have a term in which one of the indices of δ_{ij} is repeated, then it simply replaces the dummy index by the other index of δ_{ij}. Consider

$$\delta_{ij}u_j = \delta_{i1}u_1 + \delta_{i2}u_2 + \delta_{i3}u_3.$$

The right-hand side is u_1 when $i = 1$, u_2 when $i = 2$, and u_3 when $i = 3$. Therefore

$$\boxed{\delta_{ij}u_j = u_i.} \tag{2.17}$$

From its definition it is clear that δ_{ij} is an *isotropic tensor* in the sense that its components are unchanged by a rotation of the frame of reference, that is, $\delta_{ij}' = \delta_{ij}$. Isotropic tensors can be of various orders. There is no isotropic tensor of first order, and δ_{ij} is the only isotropic tensor of second order. There is also only one isotropic tensor of third order. It is called the *alternating tensor* or *permutation symbol*, and is defined as

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, \text{ or } 312 \text{ (cyclic order)}, \\ 0 & \text{if any two indices are equal}, \\ -1 & \text{if } ijk = 321, 213, \text{ or } 132 \text{ (anticyclic order)}. \end{cases} \tag{2.18}$$

From the definition, it is clear that *an index on ε_{ijk} can be moved two places (either to the right or to the left) without changing its value*. For example, $\varepsilon_{ijk} = \varepsilon_{jki}$ where

i has been moved two places to the right, and $\varepsilon_{ijk} = \varepsilon_{kij}$ where k has been moved two places to the left. For a movement of one place, however, the sign is reversed. For example, $\varepsilon_{ijk} = -\varepsilon_{ikj}$ where j has been moved one place to the right.

A very frequently used relation is the *epsilon delta relation*

$$\boxed{\varepsilon_{ijk}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.} \tag{2.19}$$

The reader can verify the validity of this relationship by taking some values for $ijlm$. Equation (2.19) is easy to remember by noting the following two points: (1) The adjacent index k is summed; and (2) the first two indices on the right-hand side, namely, i and l, are the first index of ε_{ijk} and the first *free* index of ε_{klm}. The remaining indices on the right-hand side then follow immediately.

8. Dot Product

The dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as the scalar

$$\mathbf{u}\cdot\mathbf{v} = \mathbf{v}\cdot\mathbf{u} = u_1v_1 + u_2v_2 + u_3v_3 = u_iv_i.$$

It is easy to show that $\mathbf{u}\cdot\mathbf{v} = uv\cos\theta$, where u and v are the magnitudes and θ is the angle between the vectors. The dot product is therefore the magnitude of one vector times the component of the other in the direction of the first. Clearly, the dot product $\mathbf{u}\cdot\mathbf{v}$ is equal to the sum of the diagonal terms of the tensor u_iv_j.

9. Cross Product

The cross product between two vectors $\mathbf{u}$ and $\mathbf{v}$ is defined as the vector $\mathbf{w}$ whose magnitude is $uv\sin\theta$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, and whose direction is perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form a right-handed system. Clearly, $\mathbf{u}\times\mathbf{v} = -\mathbf{v}\times\mathbf{u}$, and the unit vectors obey the cyclic rule $\mathbf{a}^1 \times \mathbf{a}^2 = \mathbf{a}^3$. It is easy to show that

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{a}^1 + (u_3v_1 - u_1v_3)\mathbf{a}^2 + (u_1v_2 - u_2v_1)\mathbf{a}^3, \tag{2.20}$$

which can be written as the symbolic determinant

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{a}^1 & \mathbf{a}^2 & \mathbf{a}^3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

In indicial notation, the k-component of $\mathbf{u}\times\mathbf{v}$ can be written as

$$\boxed{(\mathbf{u}\times\mathbf{v})_k = \varepsilon_{ijk}u_iv_j = \varepsilon_{kij}u_iv_j.} \tag{2.21}$$

As a check, for $k = 1$ the nonzero terms in the double sum in Eq. (2.21) result from $i = 2$, $j = 3$, and from $i = 3$, $j = 2$. This follows from the definition equation (2.18) that the permutation symbol is zero if any two indices are equal. Then Eq. (2.21) gives

$$(\mathbf{u}\times\mathbf{v})_1 = \varepsilon_{ij1}u_iv_j = \varepsilon_{231}u_2v_3 + \varepsilon_{321}u_3v_2 = u_2v_3 - u_3v_2,$$

which agrees with Eq. (2.20). Note that the second form of Eq. (2.21) is obtained from the first by moving the index k two places to the left; see the remark below Eq. (2.18).

10. Operator ∇: Gradient, Divergence, and Curl

The vector operator "del"[1] is defined symbolically by

$$\nabla \equiv \mathbf{a}^1 \frac{\partial}{\partial x_1} + \mathbf{a}^2 \frac{\partial}{\partial x_2} + \mathbf{a}^3 \frac{\partial}{\partial x_3} = \mathbf{a}^i \frac{\partial}{\partial x_i}. \tag{2.22}$$

When operating on a scalar function of position ϕ, it generates the vector

$$\nabla\phi = \mathbf{a}^i \frac{\partial \phi}{\partial x_i},$$

whose i-component is

$$\boxed{(\nabla\phi)_i = \frac{\partial \phi}{\partial x_i}.}$$

The vector $\nabla\phi$ is called the *gradient* of ϕ. It is clear that $\nabla\phi$ is perpendicular to the ϕ = constant lines and gives the magnitude and direction of the *maximum* spatial rate of change of ϕ (Figure 2.8). The rate of change in any other direction $\mathbf{n}$ is given by

$$\frac{\partial \phi}{\partial n} = (\nabla\phi)\cdot\mathbf{n}.$$

The *divergence* of a vector field $\mathbf{u}$ is defined as the scalar

$$\nabla\cdot\mathbf{u} \equiv \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}. \tag{2.23}$$

So far, we have defined the operations of the gradient of a scalar and the divergence of a vector. We can, however, generalize these operations. For example, we can define the divergence of a second-order tensor $\boldsymbol{\tau}$ as the vector whose i-component is

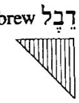
$$(\nabla\cdot\boldsymbol{\tau})_i = \frac{\partial \tau_{ij}}{\partial x_j}.$$

It is evident that the divergence operation *decreases* the order of the tensor by one. In contrast, the gradient operation *increases* the order of a tensor by one, changing a zero-order tensor to a first-order tensor, and a first-order tensor to a second-order tensor.

The *curl* of a vector field $\mathbf{u}$ is defined as the vector $\nabla\times\mathbf{u}$, whose i-component can be written as (using Eqs. (2.21) and (2.22))

$$\boxed{(\nabla\times\mathbf{u})_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j}.} \tag{2.24}$$

[1]The inverted Greek delta is called a "nabla" ($\nu\alpha\beta\lambda\alpha$). The origin of the word is from the Hebrew דֵבֶל (pronounced nabel), which means lyre, an ancient harp-like stringed instrument. It was on this instrument that the boy, David, entertained King Saul (Samuel II) and it is mentioned repeatedly in Psalms as a musical instrument to use in the praise of God.



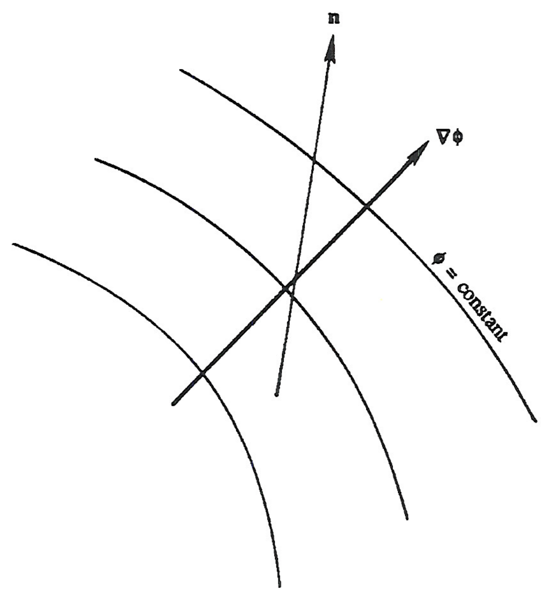


Figure 2.8 Lines of constant ϕ and the gradient vector $\nabla\phi$.

The three components of the vector $\nabla \times \mathbf{u}$ can easily be found from the right-hand side of Eq. (2.24). For the $i = 1$ component, the nonzero terms in the double sum in Eq. (2.24) result from $j = 2$, $k = 3$, and from $j = 3$, $k = 2$. The three components of $\nabla \times \mathbf{u}$ are finally found as

$$\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right), \quad \left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right), \quad \text{and} \quad \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right). \tag{2.25}$$

A vector field $\mathbf{u}$ is called *solenoidal* if $\nabla \cdot \mathbf{u} = 0$, and *irrotational* if $\nabla \times \mathbf{u} = 0$. The word "solenoidal" refers to the fact that the magnetic induction $\mathbf{B}$ always satisfies $\nabla \cdot \mathbf{B} = 0$. This is because of the absence of magnetic monopoles. The reason for the word "irrotational" will be clear in the next chapter.

11. *Symmetric and Antisymmetric Tensors*

A tensor $\mathbf{B}$ is called *symmetric* in the indices i and j if the components do not change when i and j are interchanged, that is, if $B_{ij} = B_{ji}$. The matrix of a second-order tensor is therefore symmetric about the diagonal and made up of only six distinct components. On the other hand, a tensor is called *antisymmetric* if $B_{ij} = -B_{ji}$. An antisymmetric tensor must have zero diagonal terms, and the off-diagonal terms must be mirror images; it is therefore made up of only three distinct components. Any tensor can be represented as the sum of a symmetric part and an antisymmetric part. For if we write

$$B_{ij} = \tfrac{1}{2}(B_{ij} + B_{ji}) + \tfrac{1}{2}(B_{ij} - B_{ji})$$

then the operation of interchanging i and j does not change the first term, but changes the sign of the second term. Therefore, $(B_{ij} + B_{ji})/2$ is called the symmetric part of B_{ij}, and $(B_{ij} - B_{ji})/2$ is called the antisymmetric part of B_{ij}.

Every vector can be associated with an antisymmetric tensor, and vice versa. For example, we can associate the vector

$$\boldsymbol{\omega} = \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix},$$

with an antisymmetric tensor defined by

$$\mathbf{R} \equiv \begin{bmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{bmatrix}, \tag{2.26}$$

where the two are related as

$$\begin{aligned} R_{ij} &= -\varepsilon_{ijk}\omega_k \\ \omega_k &= -\tfrac{1}{2}\varepsilon_{ijk}R_{ij}. \end{aligned} \tag{2.27}$$

As a check, Eq. (2.27) gives $R_{11} = 0$ and $R_{12} = -\varepsilon_{123}\omega_3 = -\omega_3$, which is in agreement with Eq. (2.26). (In Chapter 3 we shall call $\mathbf{R}$ the "rotation" tensor corresponding to the "vorticity" vector $\boldsymbol{\omega}$.)

A very frequently occurring operation is the doubly contracted product of a *symmetric* tensor $\boldsymbol{\tau}$ and any tensor $\mathbf{B}$. The doubly contracted product is defined as

$$P \equiv \tau_{ij}B_{ij} = \tau_{ij}(S_{ij} + A_{ij}),$$

where $\mathbf{S}$ and $\mathbf{A}$ are the symmetric and antisymmetric parts of $\mathbf{B}$, given by

$$S_{ij} \equiv \tfrac{1}{2}(B_{ij} + B_{ji}) \quad \text{and} \quad A_{ij} \equiv \tfrac{1}{2}(B_{ij} - B_{ji}).$$

Then

$$\begin{aligned} P &= \tau_{ij}S_{ij} + \tau_{ij}A_{ij} && && (2.28) \\ &= \tau_{ij}S_{ji} - \tau_{ij}A_{ji} && \text{because } S_{ij} = S_{ji} \text{ and } A_{ij} = -A_{ji}, \\ &= \tau_{ji}S_{ji} - \tau_{ji}A_{ji} && \text{because } \tau_{ij} = \tau_{ji}, \\ &= \tau_{ij}S_{ij} - \tau_{ij}A_{ij} && \text{interchanging dummy indices.} && (2.29) \end{aligned}$$

Comparing the two forms of Eqs. (2.28) and (2.29), we see that $\tau_{ij}A_{ij} = 0$, so that

$$\boxed{\tau_{ij}B_{ij} = \tfrac{1}{2}\tau_{ij}(B_{ij} + B_{ji}).}$$

The important rule we have proved is that the *doubly contracted product of a symmetric tensor* $\boldsymbol{\tau}$ *with any tensor* $\mathbf{B}$ *equals* $\boldsymbol{\tau}$ *times the symmetric part of* $\mathbf{B}$. In the process, we have also shown that the doubly contracted product of a symmetric tensor and an antisymmetric tensor is zero. This is analogous to the result that the definite integral over an even (symmetric) interval of the product of a symmetric and an antisymmetric function is zero.

12. *Eigenvalues and Eigenvectors of a Symmetric Tensor*

The reader is assumed to be familiar with the concepts of eigenvalues and eigenvectors of a matrix, and only a brief review of the main results is given here. Suppose $\boldsymbol{\tau}$ is a symmetric tensor with real elements, for example, the stress tensor. Then the following facts can be proved:

(1) There are three real eigenvalues λ^k ($k = 1, 2, 3$), which may or may not be all distinct. (The superscripted λ^k does not denote the k-component of a vector.) The eigenvalues satisfy the third-degree equation

$$\det |\tau_{ij} - \lambda \delta_{ij}| = 0,$$

which can be solved for λ^1, λ^2, and λ^3.

(2) The three eigenvectors $\mathbf{b}^k$ corresponding to distinct eigenvalues λ^k are mutually orthogonal. These are frequently called the *principal axes* of $\boldsymbol{\tau}$. Each $\mathbf{b}$ is found by solving a set of three equations

$$(\tau_{ij} - \lambda \delta_{ij})\, b_j = 0,$$

where the superscript k on λ and $\mathbf{b}$ has been omitted.

(3) If the coordinate system is rotated so as to coincide with the eigenvectors, then $\boldsymbol{\tau}$ has a diagonal form with elements λ^k. That is,

$$\boldsymbol{\tau}' = \begin{bmatrix} \lambda^1 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^3 \end{bmatrix}$$

in the coordinate system of the eigenvectors.

(4) The elements τ_{ij} change as the coordinate system is rotated, but they cannot be larger than the largest λ or smaller than the smallest λ. That is, the eigenvalues are the extremum values of τ_{ij}.

Example 2.2. The strain rate tensor $\mathbf{E}$ is related to the velocity vector $\mathbf{u}$ by

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

For a two-dimensional parallel flow

$$\mathbf{u} = \begin{bmatrix} u_1(x_2) \\ 0 \end{bmatrix},$$

show how $\mathbf{E}$ is diagonalized in the frame of reference coinciding with the principal axes.

Solution: For the given velocity profile $u_1(x_2)$, it is evident that $E_{11} = E_{22} = 0$, and $E_{12} = E_{21} = \frac{1}{2}(du_1/dx_2) = \Gamma$. The strain rate tensor in the unrotated coordinate system is therefore

$$\mathbf{E} = \begin{bmatrix} 0 & \Gamma \\ \Gamma & 0 \end{bmatrix}.$$

The eigenvalues are given by

$$\det |E_{ij} - \lambda\delta_{ij}| = \begin{vmatrix} -\lambda & \Gamma \\ \Gamma & -\lambda \end{vmatrix} = 0,$$

whose solutions are $\lambda^1 = \Gamma$ and $\lambda^2 = -\Gamma$. The first eigenvector $\mathbf{b}^1$ is given by

$$\begin{bmatrix} 0 & \Gamma \\ \Gamma & 0 \end{bmatrix} \begin{bmatrix} b_1^1 \\ b_2^1 \end{bmatrix} = \lambda^1 \begin{bmatrix} b_1^1 \\ b_2^1 \end{bmatrix},$$

whose solution is $b_1^1 = b_2^1 = 1/\sqrt{2}$, thus normalizing the magnitude to unity. The first eigenvector is therefore $\mathbf{b}^1 = [1/\sqrt{2}, 1/\sqrt{2}]$, writing it in a row. The second eigenvector is similarly found as $\mathbf{b}^2 = [-1/\sqrt{2}, 1/\sqrt{2}]$. The eigenvectors are shown in Figure 2.9. The direction cosine matrix of the original and the rotated coordinate system is therefore

$$\mathbf{C} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix},$$

which represents rotation of the coordinate system by 45°. Using the transformation rule (2.12), the components of $\mathbf{E}$ in the rotated system are found as follows:

$$\begin{aligned} E'_{12} &= C_{i1}C_{j2}E_{ij} = C_{11}C_{22}E_{12} + C_{21}C_{12}E_{21} \\ &= \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\Gamma - \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\Gamma = 0 \end{aligned}$$

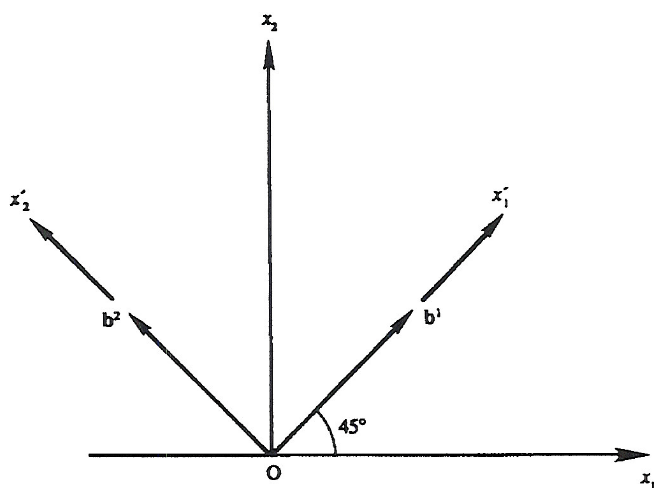


Figure 2.9 Original coordinate system Ox_1x_2 and rotated coordinate system $Ox'_1x'_2$ coinciding with the eigenvectors (Example 2.2).

The most common form of Gauss' theorem is when **Q** is a vector, in which case the theorem is

$$\int_V \frac{\partial Q_i}{\partial x_i}\, dV = \int_A dA_i\, Q,$$

which is called the *divergence theorem*. In vector notation, the divergence theorem is

$$\int_V \nabla \cdot \mathbf{Q}\, dV = \int_A d\mathbf{A} \cdot \mathbf{Q}.$$

Physically, it states that the volume integral of the divergence of **Q** is equal to the surface integral of the outflux of **Q**. Alternatively, Eq. (2.30), when considered in its limiting form for an infintesmal volume, can define a generalized field derivative of Q by the expression

$$\mathcal{D}Q = \lim_{V\to 0} \frac{1}{V} \int_A dA_i Q. \qquad (2.31)$$

This includes the gradient, divergence, and curl of any scalar, vector, or tensor Q. Moreover, by regarding Eq. (2.31) as a definition, the recipes for the computation of the vector field derivatives may be obtained in any coordinate system. For a tensor Q of any order, Eq. (2.31) as written defines the gradient. For a tensor of order one (vector) or higher, the divergence is defined by using a dot (scalar) product under the integral

$$\text{div}\, \mathbf{Q} = \lim_{V\to 0} \frac{1}{V} \int_A d\mathbf{A} \cdot \mathbf{Q}, \qquad (2.32)$$

and the curl is defined by using a cross (vector) product under the integral

$$\text{curl}\mathbf{Q} = \lim_{V\to 0} \frac{1}{V} \int_A d\mathbf{A} \times \mathbf{Q}. \qquad (2.33)$$

In Eqs. (2.31), (2.32), and (2.33), A is the closed surface bounding the volume V.

Example 2.3. Obtain the recipe for the divergence of a vector **Q**(**x**) in cylindrical polar coordinates from the integral definition equation (2.32). Compare with Appendix B.1.

Solution: Consider an elemental volume bounded by the surfaces $\mathrm{R} - \Delta\mathrm{R}/2$, $\mathrm{R} + \Delta\mathrm{R}/2$, $\theta - \Delta\theta/2$, $\theta + \Delta\theta/2$, $\mathrm{x} - \Delta\mathrm{x}/2$ and $\mathrm{x} + \Delta\mathrm{x}/2$. The volume enclosed $\Delta\mathrm{V}$ is $\mathrm{R}\Delta\theta\Delta\mathrm{R}\Delta\mathrm{x}$. We wish to calculate $\text{div}\,\mathbf{Q} = \lim_{\Delta\mathrm{V}\to 0} \frac{1}{\Delta\mathrm{V}} \int_A d\mathbf{A} \cdot \mathbf{Q}$ at the central point R, θ, x by integrating the net outward flux through the bounding surface A of $\Delta\mathrm{V}$:

$$\mathbf{Q} = \mathbf{i}_R Q_R(\mathrm{R}, \theta, \mathrm{x}) + \mathbf{i}_\theta Q_\theta(\mathrm{R}, \theta, \mathrm{x}) + \mathbf{i}_x Q_x(\mathrm{R}, \theta, \mathrm{x}).$$

In evaluating the surface integrals, we can show that in the limit taken, each of the six surface integrals may be approximated by the product of the value at the center of the surface and the surface area. This is shown by Taylor expanding each of the scalar products in the two variables of each surface, carrying out the integrations, and applying the limits. The result is

$$\begin{aligned}\operatorname{div}\mathbf{Q} = \lim_{\substack{\Delta R\to 0\\ \Delta\theta\to 0\\ \Delta x\to 0}} \Bigg\{\frac{1}{R\Delta\theta\Delta R\Delta x}\Bigg[& Q_R\left(R+\frac{\Delta R}{2},\theta,x\right)\left(R+\frac{\Delta R}{2}\right)\Delta\theta\Delta x\\ &- Q_R\left(R-\frac{\Delta R}{2},\theta,x\right)\left(R-\frac{\Delta R}{2}\right)\Delta\theta\Delta x\\ &+ Q_x\left(R,\theta,x+\frac{\Delta x}{2}\right)R\Delta\theta\Delta R - Q_x\left(R,\theta,x-\frac{\Delta x}{2}\right)R\Delta\theta\Delta R\\ &+\mathbf{Q}\left(R,\theta+\frac{\Delta\theta}{2},x\right)\cdot\left(\mathbf{i}_\theta-\mathbf{i}_R\frac{\Delta\theta}{2}\right)\Delta R\Delta x\\ &+\mathbf{Q}\left(R,\theta-\frac{\Delta\theta}{2},x\right)\cdot\left(-\mathbf{i}_\theta-\mathbf{i}_R\frac{\Delta\theta}{2}\right)\Delta R\Delta x\Bigg]\Bigg\},\end{aligned}$$

where an additional complication arises because the normals to the two planes $\theta \pm \Delta\theta/2$ are not antiparallel:

$$\begin{aligned}\mathbf{Q}\left(R,\theta\pm\frac{\Delta\theta}{2},x\right) = {}& Q_R\left(R,\theta\pm\frac{\Delta\theta}{2},x\right)\mathbf{i}_R\left(R,\theta\pm\frac{\Delta\theta}{2},x\right)\\ &+ Q_\theta\left(R,\theta\pm\frac{\Delta\theta}{2},x\right)\mathbf{i}_\theta\left(R,\theta\pm\frac{\Delta\theta}{2},x\right)\\ &+ Q_x\left(R,\theta\pm\frac{\Delta\theta}{2},x\right)\mathbf{i}_x.\end{aligned}$$

Now we can show that

$$\mathbf{i}_R\left(\theta\pm\frac{\Delta\theta}{2}\right) = \mathbf{i}_R(\theta)\pm\frac{\Delta\theta}{2}\mathbf{i}_\theta(\theta), \qquad \mathbf{i}_\theta\left(\theta\pm\frac{\Delta\theta}{2}\right) = \mathbf{i}_\theta(\theta)\mp\frac{\Delta\theta}{2}\mathbf{i}_R(\theta).$$

Evaluating the last pair of surface integrals explicitly,

$$\begin{aligned}\operatorname{div}\mathbf{Q} = \lim_{\substack{\Delta R\to 0\\ \Delta\theta\to 0\\ \Delta x\to 0}} \Bigg\{\frac{1}{R\Delta\theta\Delta R\Delta x}\Bigg[& Q_R\left(R+\frac{\Delta R}{2},\theta,x\right)\left(R+\frac{\Delta R}{2}\right)\Delta\theta\Delta x\\ &- Q_R\left(R-\frac{\Delta R}{2},\theta,x\right)\left(R-\frac{\Delta R}{2}\right)\Delta\theta\Delta x\\ &+\left(Q_x\left(R,\theta,x+\frac{\Delta x}{2}\right)-Q_x\left(R,\theta,x-\frac{\Delta x}{2}\right)\right)R\Delta\theta\Delta R\\ &+\left(Q_R\left(R,\theta+\frac{\Delta\theta}{2},x\right)\frac{\Delta\theta}{2}-Q_R\left(R,\theta+\frac{\Delta\theta}{2},x\right)\frac{\Delta\theta}{2}\right)\Delta R\Delta x\\ &+\left(Q_\theta\left(R,\theta+\frac{\Delta\theta}{2},x\right)-Q_\theta(R,\theta-\frac{\Delta\theta}{2},x\right)\Delta R\Delta x\\ &-\left(Q_R\left(R,\theta-\frac{\Delta\theta}{2},x\right)\frac{\Delta\theta}{2}-Q_R\left(R,\theta-\frac{\Delta\theta}{2},x\right)\frac{\Delta\theta}{2}\right)\Delta R\Delta x\Bigg]\Bigg\},\end{aligned}$$

where terms of second order in the increments have been neglected as they will vanish in the limits. Carrying out the limits, we obtain

$$\operatorname{div}\mathbf{Q} = \frac{1}{R}\frac{\partial}{\partial R}(RQ_R) + \frac{1}{R}\frac{\partial Q_\theta}{\partial \theta} + \frac{\partial Q_x}{\partial x}.$$

Here, the physical interpretation of the divergence as the net outward flux of a vector field per unit volume has been made apparent by its evaluation through the integral definition.

This level of detail is required to obtain the gradient correctly in these coordinates.

14. *Stokes' Theorem*

Stokes' theorem relates a surface integral over an open surface to a line integral around the boundary curve. Consider an open surface A whose bounding curve is C (Figure 2.11). Choose one side of the surface to be the outside. Let $d\mathbf{s}$ be an element of the bounding curve whose magnitude is the length of the element and whose direction is that of the tangent. The positive sense of the tangent is such that, when seen from the "outside" of the surface in the direction of the tangent, the interior is on the left. Then the theorem states that

$$\boxed{\int_A (\nabla \times \mathbf{u}) \cdot d\mathbf{A} = \int_C \mathbf{u} \cdot d\mathbf{s},} \qquad (2.34)$$

which signifies that the surface integral of the curl of a vector field $\mathbf{u}$ is equal to the line integral of $\mathbf{u}$ along the bounding curve.

The line integral of a vector $\mathbf{u}$ around a closed curve C (as in Figure 2.11) is called the "circulation of $\mathbf{u}$ about C." This can be used to define the curl of a vector through

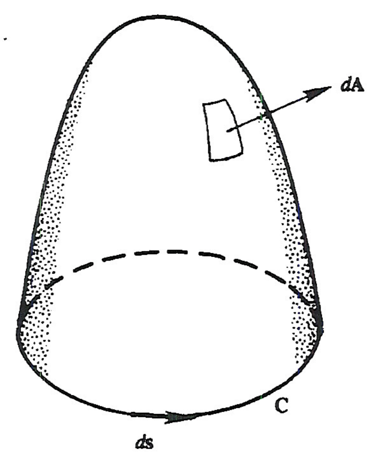


Figure 2.11 Illustration of Stokes' theorem.

the limit of the circulation integral bounding an infinitesmal surface as follows:

$$\mathbf{n} \cdot \operatorname{curl} \mathbf{u} = \lim_{A \to 0} \frac{1}{A} \int_C \mathbf{u} \cdot d\mathbf{s}, \tag{2.35}$$

where $\mathbf{n}$ is a unit vector normal to the local tangent plane of A. The advantage of the integral definitions of the field derivatives is that they may be applied regardless of the coordinate system.

Example 2.4. Obtain the recipe for the curl of a vector $\mathbf{u}(\mathbf{x})$ in Cartesian coordinates from the integral definition given by Eq. (2.35).

Solution: This is obtained by considering rectangular contours in three perpendicular planes intersecting at the point (x, y, z). First, consider the elemental rectangle in the x = const. plane. The central point in this plane has coordinates (x, y, z) and the area is $\Delta y\ \Delta z$. It may be shown by careful integration of a Taylor expansion of the integrand that the integral along each line segment may be represented by the product of the integrand at the center of the segment and the length of the segment with attention paid to the direction of integration $d\mathbf{s}$. Thus we obtain

$$(\operatorname{curl} \mathbf{u})_x = \lim_{\substack{\Delta y \to 0 \\ \Delta z \to 0}} \left\{ \frac{1}{\Delta y \Delta z}\left[u_z\left(x, y + \frac{\Delta y}{2}, z\right) - u_z\left(x, y - \frac{\Delta y}{2}, z\right)\right]\Delta z \right.$$
$$\left. + \frac{1}{\Delta y \Delta z}\left[u_y\left(x, y, z - \frac{\Delta z}{2}\right) - u_y\left(x, y, z + \frac{\Delta z}{2}\right)\right]\Delta y \right\}.$$

Taking the limits,

$$(\operatorname{curl} \mathbf{u})_x = \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}.$$

Similarly, integrating around the elemental rectangles in the other two planes

$$(\operatorname{curl} \mathbf{u})_y = \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x},$$
$$(\operatorname{curl} \mathbf{u})_z = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}.$$

15. Comma Notation

Sometimes it is convenient to introduce the notation

$$A_{,i} \equiv \frac{\partial A}{\partial x_i}, \tag{2.36}$$

where A is a tensor of any order. In this notation, therefore, the comma denotes a spatial derivative. For example, the divergence and curl of a vector $\mathbf{u}$ can be written, respectively, as

$$\nabla \cdot \mathbf{u} = \frac{\partial u_i}{\partial x_i} = u_{i,i},$$
$$(\nabla \times \mathbf{u})_i = \varepsilon_{ijk}\frac{\partial u_k}{\partial x_j} = \varepsilon_{ijk}u_{k,j}.$$