11.4 CASE 4: - find shear stress at wall
- relate stress to body force

Flow down along a rod (gravity forced flow)

$$v_z = \frac{g a^2}{2\nu}\left[\frac{b^2}{a^2}\ln\left(\frac{r}{a}\right) + \frac{1}{2}\left(1-\frac{r^2}{a^2}\right)\right] \qquad (11.11)$$

Can take $\left.\frac{dv_z}{dr}\right|_{r=a}$ or let's start from

the differential N-S eqn: (simplified)

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = -\frac{g}{\nu}$$

integrate: $$\int_a^b d\left(r\frac{dv}{dr}\right) = \int_a^b -\frac{g}{\nu}\, r\,dr$$

$$-a\left.\frac{dv}{dr}\right|_a = -\frac{g}{\nu}\left(\frac{b^2}{2}-\frac{a^2}{2}\right)$$

$$\tau_a = \mu\left.\frac{dv}{dr}\right|_a = \gamma\left(\frac{b^2}{2a}-\frac{a}{2}\right)$$

$$\tau_a = \frac{\gamma}{2a}\left(b^2-a^2\right)$$

Over length l along the rod:

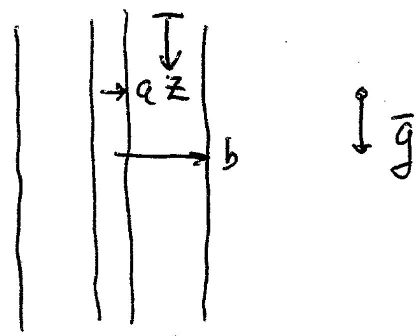
Friction force $= \tau_a 2\pi a l$.

fluid Weight $= \gamma V_{fluid} = \gamma\pi l\left(b^2-a^2\right)$

$$F_{shear} = \tau_a 2\pi a l = W = \gamma\pi l\left(b^2-a^2\right)$$

insert expression for τ_a — they match

11.9



Fully developed flow
$P =$ const.
(gravity driven flow).

Reduce N-S eqn. & give B.C.

- Use cylindrical coor. z direction (down)

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) =$$

$$-\frac{\partial P}{\partial z} + \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right]$$

From given conditions:

$$v_r = v_\theta = 0$$

$$\frac{\partial v_z}{\partial z} = 0 \qquad \frac{\partial v_z}{\partial t} = 0$$

$$\frac{\partial P}{\partial z} = 0$$

so: no accel. terms

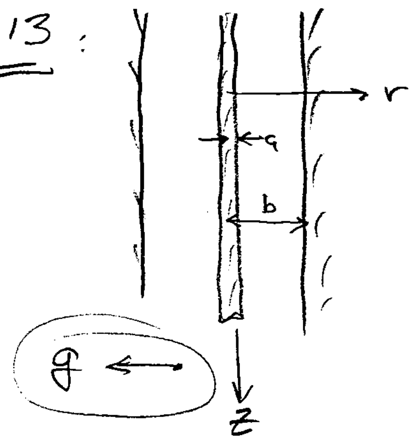
$$0 = \rho g_z + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)\right]$$

can also write as:

$$0 = \gamma + \mu\left[\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right]$$

by expanding visc. term derivative

11.13 :



- Thin wire drawn down the centerline of a pipe.
- Steady, incompressible, laminar fully developed flow.

- Use cylindrical coordinates for flow in a pipe.
- z mom. eqn: (10.22)

$$\underbrace{\cancel{\frac{\partial v_z}{\partial t}}}_{0\ (\text{steady})} + \underbrace{\cancel{v_r}}_{0}\frac{\partial v_z}{\partial r} + \underbrace{\cancel{\frac{v_\theta}{r}}}_{0}\frac{\partial v_z}{\partial \theta} + v_z \underbrace{\cancel{\frac{\partial v_z}{\partial r}}}_{0\ (\text{fully developed.})} = -\gamma \overbrace{\cancel{\frac{\partial h}{\partial z}}}^{0\ (\text{horizontal})} - \frac{\partial P}{\partial z} + \mu \nabla^2 v_z$$

$$0 = -\underbrace{\cancel{\frac{\partial P}{\partial z}}}_{0\ (\text{stated as zero})} + \mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \underbrace{\cancel{\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}}}_{0} + \underbrace{\cancel{\frac{\partial^2 v_z}{\partial z^2}}}_{0}\right)$$

Final Form

$$0 = \mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r}\right)$$

$$0 = \mu\left(\frac{1}{r}\frac{d}{dr}\left(r\frac{dv_z}{dr}\right)\right)$$

B.C. :
$r = b \quad v_z = 0$
$r = a \quad v_z = V_w$ (velocity of moving wire).

11.15

$$F = \int_0^{l} \tau_w \, 2\pi a \, dz = 2\pi a l \tau_w$$

$$\tau_w = -\mu \frac{\partial v_z}{\partial r} = -\mu \frac{c_1}{r} = -\mu \frac{V_w}{r \ln(r/b)}$$

$$F = 2\mu \pi l \frac{V_w}{\ln(a/b)}$$

c_1 is from integration const. (see 11.14)

$$\frac{\partial}{\partial r}\left(r \frac{\partial v_z}{\partial r}\right) = 0$$

$$r \frac{\partial v_z}{\partial r} = c_1 \quad \rightarrow \quad v_z = c_1 \ln r + C_2$$

at $r = a \quad v_z = V_w$

at $r = b \quad v_z = 0 \quad$ so $\quad C_2 = -C_1 \ln b.$