Homework # 1

ME526/NSE526

Due: October 11

1. You are encouraged to work in a group of up to 2 students and submit one solution per student.
2. Your solution must be clearly legible. Illegible work may not be graded and returned without any points. Although not necessary, you may type your work.
3. All problems must be solved. However, all problems may not be graded. A random sample of problems will be selected for grading.
4. If you are required to write a computer program, attach your code with several comment statements on the code wherever possible.

1. Find the most accurate formula for the first derivative at x_i utilizing known values of f at x_{i-1}, x_i, x_{i+1}, and x_{i+2}. The points are uniformly spaced with a spacing of h between them. Give the complete expression for the leading-order truncation error term, i.e. the coefficient sign and form. State the order of the finite difference approximation. You may use Taylor Table to do this. Show all details for the table.

2. Consider the central difference operator $\delta/\delta x$ defined by

$$\frac{\delta u_n}{\delta x} = \frac{u_{n+1} - u_{n-1}}{2h}. \tag{1}$$

   (a) In calculus, we have

$$\frac{duv}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \tag{2}$$

   It is important to see if our discrete representations of the derivatives give the same results as the continuous one does. Does the following analogous finite difference expression hold? Show details steps.

$$\frac{\delta u_n v_n}{\delta x} = u_n\frac{\delta v_n}{\delta x} + v_n\frac{\delta u_n}{\delta x} \tag{3}$$

   (b) Show that

$$\frac{\delta u_n v_n}{\delta x} = \bar{u}_n\frac{\delta v_n}{\delta x} + \bar{v}_n\frac{\delta u_n}{\delta x}, \tag{4}$$

   where

$$\bar{u}_n = \frac{1}{2}(u_{n+1} + u_{n-1}) \tag{5}$$

   (c) Show that

$$\phi_n\frac{\delta \psi_n}{\delta x} = \frac{\delta}{\delta x}(\bar{\phi}_n\psi_n) - \overline{\psi_n\frac{\delta \phi_n}{\delta x}}, \tag{6}$$

   where the $\bar{(\ )}$ indicates the averaging operator defined in the previous part.

3. Consider a finite difference approximation for the second derivative on a uniformly spaced grid of size h,

$$\frac{\delta^2 u_n}{\delta x^2} \approx \frac{u_{n+2} - 2u_n + u_{n-2}}{4h^2}. \tag{7}$$

   (a) Using Taylor series expansion, calculate the leading order truncation term and then state the order of accuracy of the formula.

   (b) Evaluate the second derivate of $\cos(5x)$ at $x = 1$ using the above finite difference approximation. Do the evaluation using different grid spacings, h values of 10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}, and 10^{-1}. Create a table containing the numerical evaluated value of the derivate, the exact value of the derivative, and the absolute error between the numerical and the exact solution for the different grid spacings. Make a plot of the absolute error versus grid spacing on a log-log plot. Discuss your plot in relation to the answer obtained in the first part.

4. A general *Pade′* type scheme (compact differences) at a boundary point $i = N$ for the first derivative on a uniform mesh of spacing h can be written as

$$f'_N + \alpha f'_{N-1} = \frac{1}{h}\left(a f_N + b f_{N-1} + c f_{N-2} + d f_{N-3}\right). \tag{8}$$

Using Taylor table find all coefficients such that the scheme would be fourth-order accurate. Write the expression for the fourth-order scheme and also evaluate the leading order truncation term.

5. Using Taylor series, find the leading order truncation term and the order of accuracy to obtain f''_j given by the following expression. Clearly show all steps.

$$\frac{1}{6} f''_{j-1} + \frac{2}{3} f''_j + \frac{1}{6} f''_{j+1} \approx \frac{f_{j+1} - 2f_j + f_{j-1}}{h^2} \tag{9}$$

6. Consider the function

$$f(x) = \sin((4-x)(4+x)), \quad 0 \le x \le 8. \tag{10}$$

Use a uniform grid of $N + 1$ points, where $N = 32$ to numerically calculate the second derivative of f as explained below.

   (a) Use the formula given in equation 9 to approximate the second derivative f''_j at any interior point x_j. Use the first-order one-sided schemes for the second derivatives at the left and right boundaries. For example, for the left boundary, construct a first order scheme for f''_0 using f_0, f_1 and f_2. Write down a system of equations for the unknown derivatives and show that it results in a tri-diagonal matrix. Note, to do this by hand, use a total of only $N + 1 = 5$, say, points to write the matrix. Once your have the proper form, then you can generalize this to $N + 1 = 32$ total points.

   (b) Solve the resulting tri-diagonal system and plot the exact and the numerical solutions for f''_j for $0 \le x_j \le 8$ on the same plot (i.e. plot of f''_j versus x_j) . Use different line styles. Calculate the absolute error between the two (exact and numerical solution) and plot the error as a function of x_j. Explain the behavior of the error.

   (c) Investigate the accuracy of your scheme at $x = 4$ by varying the grid spacing h. That is, plot %error versus h on a log-log plot. Use grid spacings of $h = 10^{-3}$, 5×10^{-3}, 10^{-2}, 5×10^{-2}, 10^{-1} and 0.25 for the plot. Verify the order of accuracy of the method by calculating the slope of the curve and by comparing it to the leading order truncation term derived in the previous problem.