Solution to Assignment # 2

ME575/NSE526

1. Modified Wavenumber Analysis

   (a)

$$\begin{aligned} f(x) &= e^{ikx} \\ f'(x) &= ike^{ikx} = ikf(x) \end{aligned}$$

Let $f_j = e^{ikx_j}$ be the discrete representation of the continuous function. So $(f')_j = ik^* f_j$. Also for a uniform grid with spacing Δ between [0,1], $x_j = j\Delta$.

- Forward Difference[1]

$$\begin{aligned} f'_j &= \frac{f_{j+1} - f_j}{\Delta} \\ ik^* f_j &= \frac{e^{ik(j+1)\Delta} - e^{ikj\Delta}}{\Delta} \\ ik^* f_j &= f_j \frac{e^{ik\Delta} - 1}{\Delta} \\ ik^*\Delta &= [cos(k\Delta) - 1] + i[sin(k\Delta)] \end{aligned}$$

- Backward Difference

$$\begin{aligned} f'_j &= \frac{f_j - f_{j-1}}{\Delta} \\ ik^* f_j &= \frac{e^{ikj\Delta} - e^{ik(j-1)\Delta}}{\Delta} \\ ik^* f_j &= f_j \frac{1 - e^{-ik\Delta}}{\Delta} \\ ik^*\Delta &= [1 - cos(k\Delta)] + i[sin(k\Delta)] \end{aligned}$$

- Centered Difference

$$\begin{aligned} f'_j &= \frac{f_{j+1} - f_{j-1}}{2\Delta} \\ ik^* f_j &= \frac{e^{ik(j+1)\Delta} - e^{ik(j-1)\Delta}}{2\Delta} \end{aligned}$$

[1] Note that the actual modified wavenumber is k^* which can be obtained by dividing the expressions given here by $i\Delta$.

$$
\begin{aligned}
ik^* f_j &= f_j \frac{e^{ik\Delta} - e^{-ik\Delta}}{2\Delta} \\
ik^* \Delta &= i[sin(k\Delta)]
\end{aligned}
$$

- One Sided Difference

$$
\begin{aligned}
f'_j &= \frac{3f_j - 4f_{j-1} + f_{j-2}}{2\Delta} \\
ik^* f_j &= f_j \frac{3 - 4e^{-ik\Delta} + e^{-2ik\Delta}}{2\Delta} \\
ik^* \Delta &= \frac{3 - 4cos(k\Delta) + cos(2k\Delta)}{2} + i\frac{4sin(k\Delta) - sin(2k\Delta)}{2}
\end{aligned}
$$

- 5-Point Difference (The Higher-Order Centered difference scheme)

$$
\begin{aligned}
f'_j &= \frac{-f_{j+2} + 8f_{j+1} - 8f_{j-1} + f_{j-2}}{12\Delta} \\
ik^* f_j &= f_j \frac{-e^{2ik\Delta} + 8e^{ik\Delta} - 8e^{-ik\Delta} + e^{2ik\Delta}}{12\Delta} \\
ik^* \Delta &= i\left(\frac{4}{3}sin(k\Delta) - \frac{1}{6}sin(2k\Delta)\right)
\end{aligned}
$$

- Pade$'$ Formula (Compact Scheme)

$$
\begin{aligned}
\frac{1}{6}f'_{j+1} + \frac{2}{3}f'_j + \frac{1}{6}f'_{j-1} &= \frac{f_{j+1} - f_{j-1}}{2\Delta} \\
\frac{1}{6}ik^* f_j \left(e^{ik\Delta} + 4 + e^{-ik\Delta}\right) &= f_j \frac{\left(e^{ik\Delta} - e^{-ik\Delta}\right)}{2\Delta} \\
ik^* \Delta &= i\frac{3sin(k\Delta)}{2 + cos(k\Delta)}
\end{aligned}
$$

(b) See figure showing imaginary part of $ik^*\Delta$

(c) **Significance of the real part:** It is observed that the non-central schemes have a non-zero real part for $ik^*\Delta$, whereas for the ideal scheme, it is zero. One consequence for having the non-zero real part for $ik^*\Delta$ falls within the scope of PDEs, which we will learn later in this course. Consider the wave equation,

$$
\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0; \quad c > 0 \tag{1}
$$

Here u is the dependent variable and t and x are independent variables. If we seek periodic solutions, with the initial condition $u(x, 0) = e^{ikx}$, then solution to this equation is of the form $u(t, x) = e^{ikx} f(t)$, where $f(t)$ is just function of t. Substituting this

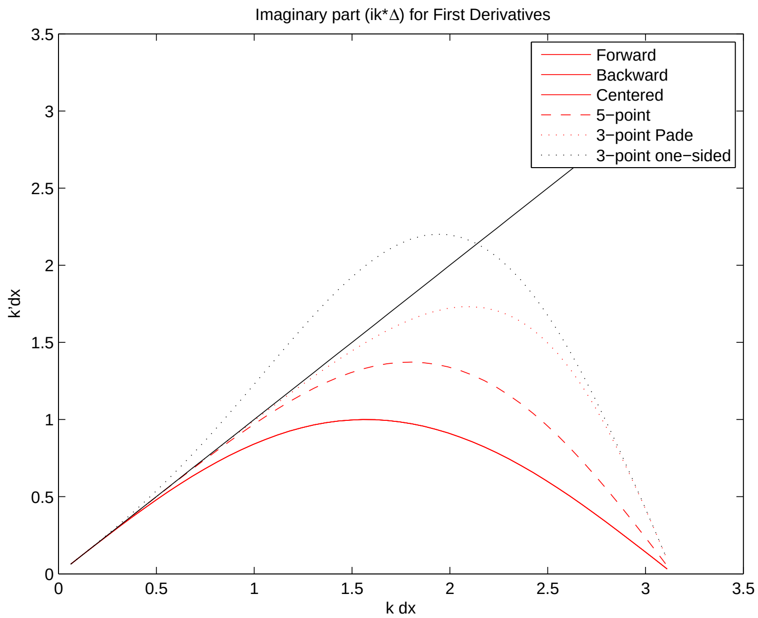

Figure 1: Imaginary part of $ik^*\Delta$ for different schemes compared with ideal scheme.

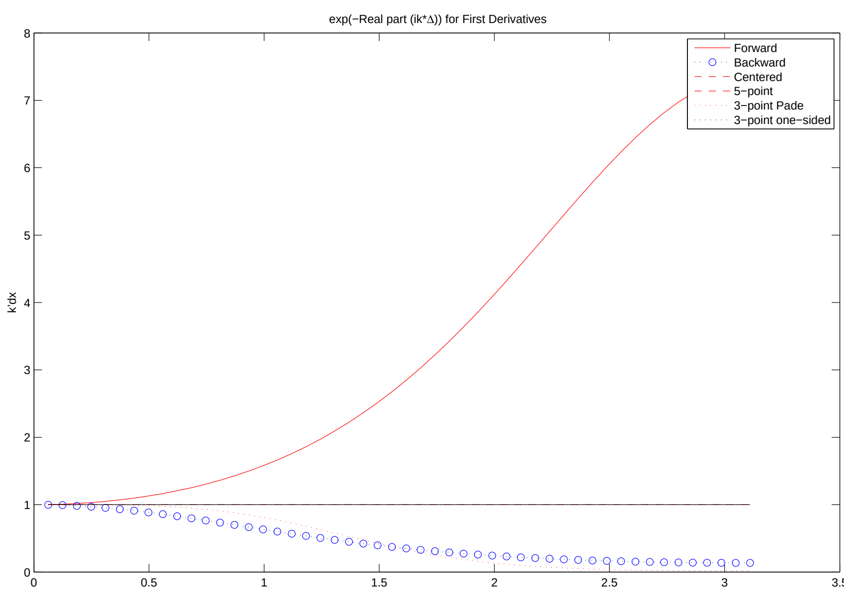

Figure 2: Exp(Real part of $ik^*\Delta$) for different schemes compared with ideal scheme.

solution into the PDE, we get,

$$\begin{aligned}
\frac{\partial e^{ikx} f(t)}{\partial t} + c\frac{\partial e^{ikx} f(t)}{\partial x} &= 0 \qquad (2)\\
e^{ikx}\frac{df}{dt} + ikce^{ikx} f &= 0 \qquad (3)\\
\frac{df}{dt} + ikcf &= 0 \qquad (4)\\
f(t) &= f(0)e^{-ikct}. \qquad (5)
\end{aligned}$$

Thus the exact solution will be,

$$u(t,x) = e^{ikx}e^{-ikct}; \qquad f(0) = 1. \qquad (6)$$

If we use any of the finite-difference approximations to solve for this ODE in f, our solution will be affected by errors in the finite-difference approximations. The solution can be written as,

$$u(t,x_j) = e^{ikx_j}e^{-ik^*ct} \qquad (7)$$

The imaginary part of ik^* creates error in phase or frequency and is called dispersion error. The real part of ik^* results in amplitude error or amplification error.

Note that $|e^{-ik^*ct} = 1|$. If we have exact solution with $k^* = k$ (no modification to wavenumber), then there is no change in the amplitude of the initial wave. From the discrete solution,

$$|e^{-ik^*ct}| = |e^{Real(-ik^*ct)}|. \qquad (8)$$

Since the centered (or symmetric) schemes do not have real part for ik^*, these schemes do not create any amplitude error, but only phase error. The one-sided schemes have both the dispersion error and amplitude error.

2. (a) Analytical solution is

$$v = \left[a(-0.1\mathrm{cos}2t + \mathrm{sin}2t) + be^{0.2t}\right]^{-1}; \qquad a = 1/1.01, \qquad b = 1.11/1.01 \qquad (9)$$

(b) Stability analysis gives $\lambda = \frac{\partial f}{\partial v} = -0.2 - 4\mathrm{cos}(2t)v$. Using the initial value for v gives $\lambda_{\max} = 4.2$ and $\Delta t_{max} = 2/4.2 = 0.476$, which is larger than all time-steps used below.

(c) Results are stable for all values of Δt, but it takes a very small value of Δt to get the write answer owing to the wandering nature of the solution.

(d) Solutions with $\Delta t = 0.006, 0.006/2$ and $0.006/4$ differ slightly near $t = 2.25$. Solutions wih $\Delta t = 0.006/4$ and $0.006/8$ are approximately same on the plot and are thus considered better solns.

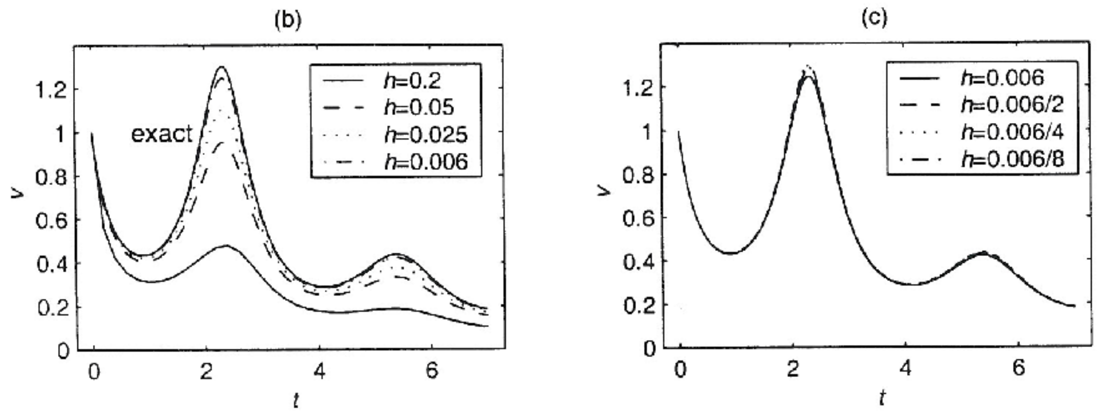

Figure 3: Solution for Problem 2.

3. Expanding $f(y_{n+1}, t_{n+1})$ using TS,

$$y_{n+1} = y_n + h\left(f(y_n, t_{n+1}) + (y_{n+1} - y_n)\left.\frac{\partial f}{\partial y}\right|_{y_n, t_{n+1}}\right) \tag{10}$$

which after rearrangement gives

$$y_{n+1} = \frac{y_n + h\left(f(y_n, t_{n+1}) - y_n \left.\frac{\partial f}{\partial y}\right|_{y_n, t_{n+1}}\right)}{1 - h\left.\frac{\partial f}{\partial y}\right|_{y_n, t_{n+1}}} \tag{11}$$

FIgures are shown. When the linearization error dominates the time-discretization error, the linearized solution differs markedly from the exact and fully implicit solution. When the two errors are nearly identical, all three solutions are quite similar. The difference in the ratio of errors is due solely to the different initial conditions, so the linearized solution can be quite senstitive to initial condition!

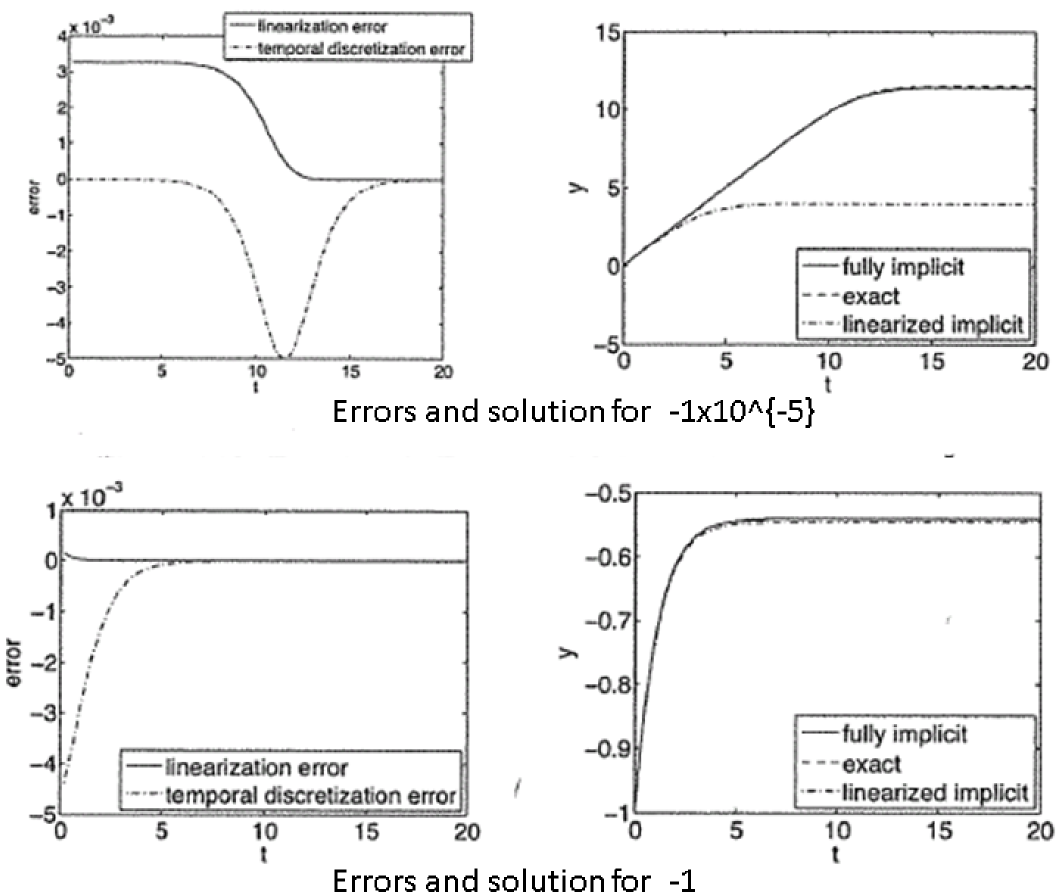

Figure 4: Problem 4.