Measures of Accuracy

Order of Accuracy

The leading order truncation term decides the order of accuracy of a finite difference approximation. If the error E is second order, then we can write

$$E = Ch^2, \tag{1}$$

where C is some constant and h is the grid spacing. Taking the log to the base 10, we get

$$\begin{aligned} \log(E) &= \log(Ch^2) && (2) \\ &= \log C + 2\log(h). && (3) \end{aligned}$$

Plotting the error versus grid spacing on a log − log scale can then be used to compare different discretization schemes by evaluating the slopes that are indicative of the order of accuracy of the scheme.

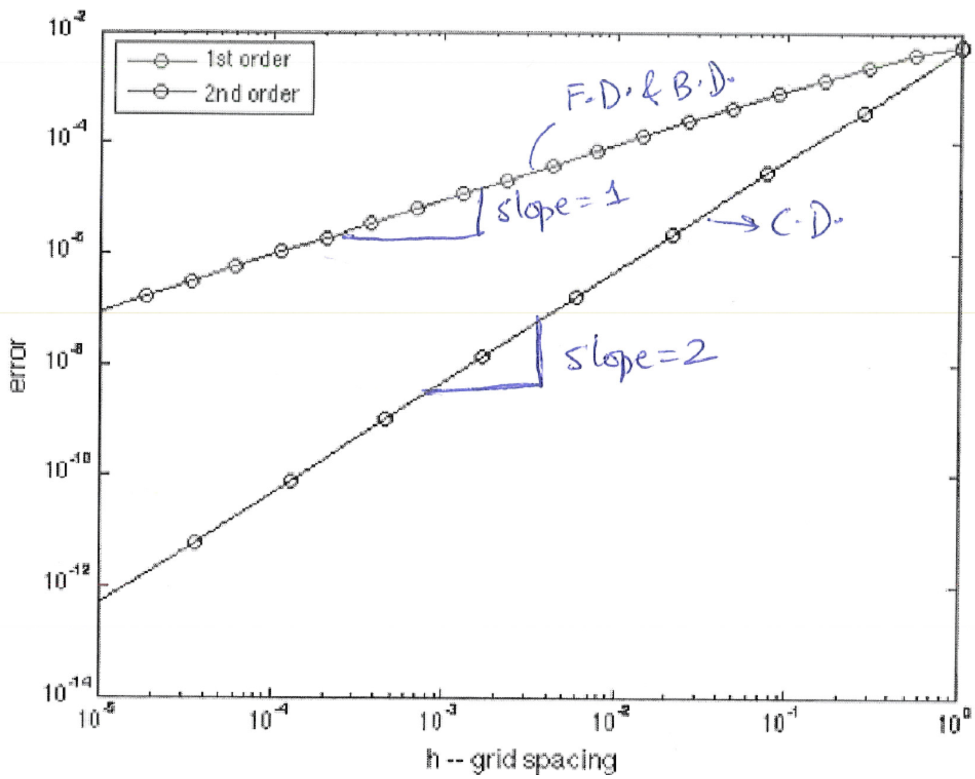

Figure 1: Error as a function of grid spacing for three different discretization schemes.

Measures to Quantify Error

Errors in numerical predictions can be defined using different measures or norms. The commonly used measures are the L_∞ norm, L_1 norm, or L_2 norm, defined below. These norms are defined with respect to the exact solution (if it is known), or an assumed true solution which could be the numerically predicted solution on a very fine grid.

Let f_j be a properly non-dimensionalized solution predicted by your numerical approach (could be velocity, temperature etc.) at different points x_j in your domain. Let f_{true} be the true solution at these same exact points. Then the errors in different norms are given as follows:

$$\begin{aligned}
L_\infty &= \text{Max}_{j=1}^{N} |f_j - f_{\text{exact}}| && (4)\\
L_1 &= \frac{1}{N}\sum_{j=1}^{N} |f_j - f_{\text{exact}}| && (5)\\
L_2 &= \frac{1}{N}\sqrt{\sum_{j=1}^{N} (f_j - f_{\text{exact}})^2} && (6)\\
& && (7)
\end{aligned}$$

Modified Wavenumber Analysis

This analysis presents another way of assessing the accuracy of a numerical scheme (or difference formula). The analysis proceeds by asking the question *How well does a finite-difference scheme approximate oscillatory or periodic function?*, e.g. sinusoidal functions are used and differentiated by using the finite-difference scheme. The main reason in using sinusoidal function is that, Fourier series consists of sinusoidal basis functions and is often used to represent arbitrary and complex functions.

Given a set of points (e.g. grid resolution in your computation) and a finite-difference operator, we try to find how well it approximates high wave number sinusoidal functions. The approximations are compared with exact formulae to evaluate accuracy at different wave-numbers.

Consider a harmonic function $f(x)$ over the domain [0,L]. For an *equally* spaced grid with $N+1$ points, $\Delta = L/N$. The locations of the grid points, x_j are readily obtained as $x_j = (L/N)j$ for $j = 0, 1, 2.....N-1$. The value at point N will be same as point $j = 0$ because of the assumed periodicity. The Fourier representation of $f(x)$ is

$$f(x) = \sum_{n=-N/2}^{n=N/2-1} \widehat{f}_k \exp{(ikx)}\,; \qquad i = \sqrt{-1}; \qquad k = \frac{2\pi}{L}n \tag{8}$$

Here, f_k are the Fourier coefficients, and $k = (2\pi/L)n$ is the wave number. On the uniform grid, this function is available only on *discrete* data points, x_j:

$$f(x_j) = \sum_{n=-N/2}^{n=N/2-1} \widehat{f}_k \exp{(ikx_j)}\,; \qquad x_j = \Delta j \tag{9}$$

For simplicity, we just consider one Fourier mode at a time. Dropping the summation and the Fourier coefficient, the continuous oscillatory function $f(x) = \exp{(ikx)}$ is represented on the discrete grid points to obtain

$$f(x_j) = f_j = \exp{(ikx_j)} \tag{10}$$

The exact expression for the first derivative of the function is $f' = ik\exp(ikx) = ikf$. The finite difference approximation (f.d.a) of the derivative following centered differencing formula is:

$$\begin{aligned}
f'_j = \frac{\delta f}{\delta x}\Big|_j &= \frac{f_{j+1} - f_{j-1}}{2\Delta} \\
&= \frac{e^{ikx_{j+1}} - e^{ikx_{j-1}}}{2\Delta} \\
&= e^{ikj\Delta}\frac{\left(e^{ik\Delta} - e^{-ik\Delta}\right)}{2\Delta} \\
&= ie^{ikx_j}\frac{2i\sin(k\Delta)}{2\Delta} \\
&= i\left(\frac{\sin(k\Delta)}{\Delta}\right) f_j \\
&= ik' f_j
\end{aligned} \tag{11}$$

where $k' = \frac{\sin(k\Delta)}{\Delta}$ is termed as the 'modified wave number'. Figure 2 shows the plot of the modified $(k'\Delta)$ versus the true wave-number times the grid resolution $(k\Delta)$ for $L = 2\pi$. A numerical scheme

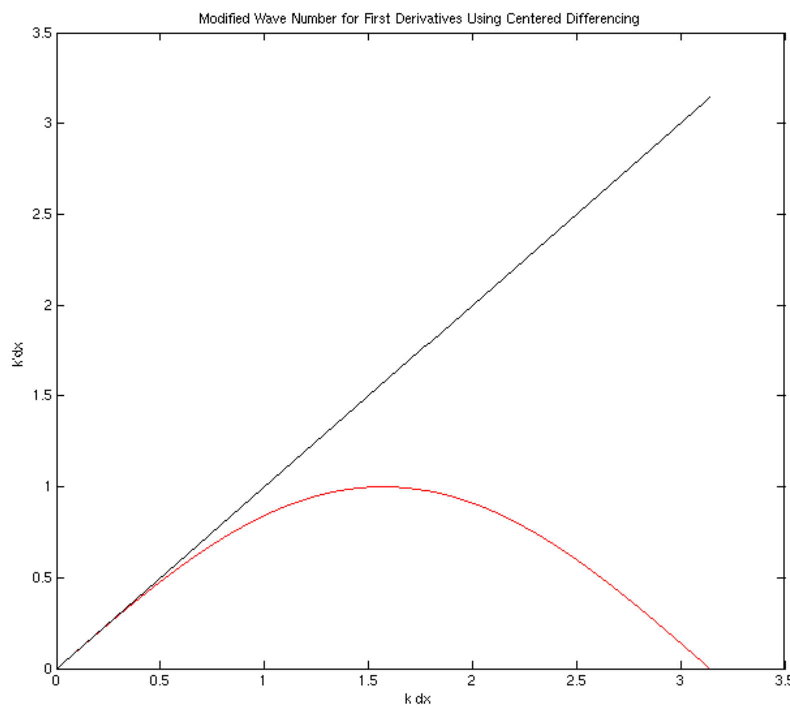

Figure 2: The Modified Wave Number for the first derivative using centered difference formula. The small wave numbers are well captured, whereas the large ones are not: straight line is exact, curve is the centered difference scheme

'truly' representing the sinusoidal function would result in exactly the same expression even in the discrete form. A class of numerical methods called 'spectral methods' allow such accuracy, however, are suitable only for simplified geometries and periodic boundary conditions. A finite-difference scheme representing the derivative of the sinusoidal function close to (as much as possible) the exact expression is desired. This is also considered as a measure of the accuracy of the finite-difference scheme. You will notice in most cases, however, that the finite-difference scheme approximates the first derivatives of the low wave number functions (slowly varying sinusoidal functions) very accurately on a given grid. However, higher wave number (rapidly varying) functions are not accurately represented.

Aliasing

Figure 3 shows a common problem involved in distinguishing periodic signals on a fixed grid. The 'circles' denote the actual grid points on which a particular function $f(x)$ is represented. The dotted line shows a slowly varying (smaller wave number), and solid line shows the rapidly varying function, respectively. Notice that at the grid points, the signal values are *exactly* same. This means both wave number modes would appear to be the same on the computational mesh. This is termed as aliasing. All centered differencing schemes involve aliasing errors inherently.

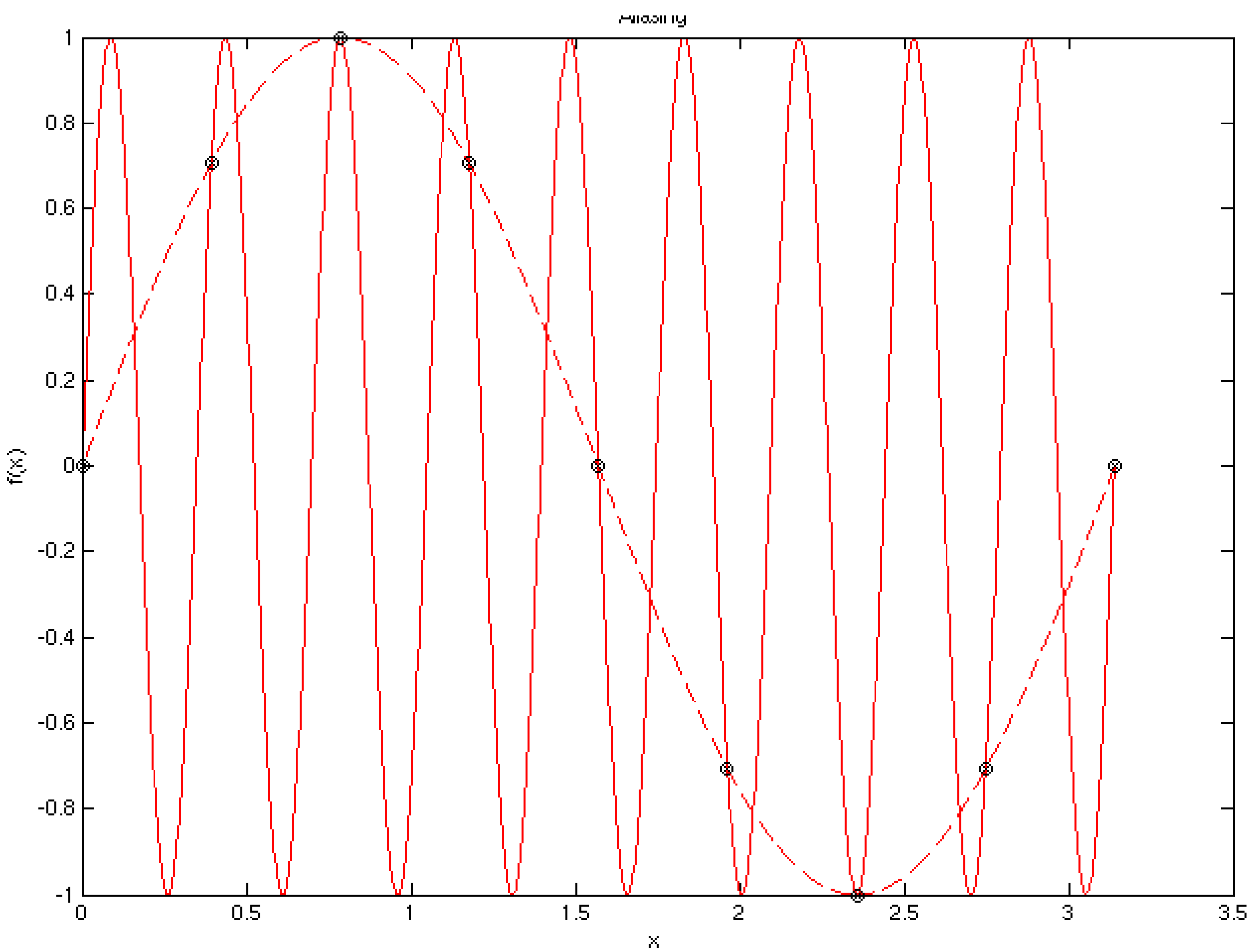

Figure 3: Problem of *aliasing* on a finite-difference stencil.