Numerical Solution of Ordinary Differential Equations (MTH 452/552)

Homework due Wednesday, February 1, 2017

Problem 1. Consider the Runge-Kutta method that is defined as follows.

$$\begin{aligned} K_1 &= f(t_n, U^n) \\ K_2 &= f(t_n + \frac{\Delta t}{2}, U^n + \frac{\Delta t}{2} K_1) \\ K_3 &= f(t_n + \frac{\Delta t}{2}, U^n + \frac{\Delta t}{2} K_2) \\ K_4 &= f(t_n + \Delta t, U^n + \Delta t\, K_3) \\ U^{n+1} &= U^n + \frac{\Delta t}{6}(K_1 + 2K_2 + 2K_3 + K_4) \end{aligned} \tag{1}$$

a) (10 points) Identify the Butcher array of the method.

b) (20 points) Modify the code for the Euler method to implement a general explicit Runge-Kutta method with Butcher array A, b, c. You may use the M-file RKexplicitstep.m that has been posted with the lecture notes on Canvas. Test your code by writing the Euler method as a Runge-Kutta method and then compare the results with those for the original code for the Euler method for the initial value problem

$$u'' = -u, \quad u(0) = 1,\ u'(0) = 2 \tag{2}$$

Compute the solutions on the interval $0 \leq t \leq 1$.

c) (10 points) Now conduct a numerical experiment to experimentally determine the order of consistency of the Runge-Kutta method (1). Do this by solving the IVP (2) on the interval $0 \leq t \leq 1$ with the Runge-Kutta method (1). Remember that the Euler code provided on Canvas can compute the exact solution for this IVP. Tabulate the maximum error for NSTEP = $1, 2, 4, 8, 16, 32, 64$. What order of convergence can you infer from this experiment? Based on this, what order of consistency would you expect the method to have? You may use the fact that the order of convergence and the order of consistency will be integers.

d) (20 points) Write a Matlab function M-file that returns the order of consistency of a Runge-Kutta method up to and including order 4. When coding, practice the use of Matlab's vector operations (both elementwise operations and matrix multiplications) by using these to compute the sums occurring in the order conditions instead of using for-loops. Test your function on the Euler method as well as the method (1). Is the result in agreement with the result from your experiment in part **c)**?

Problem 2. (10 points) Consider the so-called predictor-corrector method

$$\begin{aligned} V^{n+1} &= U^n + \Delta t\, f(t_n, U^n) \\ U^{n+1} &= U^n + \frac{\Delta t}{2}\Big(f(t_n, U_n) + f(t_n + \Delta t, V^{n+1})\Big) \end{aligned} \tag{3}$$

(cf. page 135 in our textbook). Show that this method is also a Runge-Kutta method, find its Butcher array, and determine its order of consistency.