Homework # 5

ME526/NSE526

Due: December 1

1. You are encouraged to work in a group of up to 2 students and submit one solution per group.
2. Your solution must be clearly legible. Illegible work may not be graded and returned without any points. Although not necessary, you may type your work.
3. All problems must be solved. However, all problems may not be graded. A random sample of problems will be selected for grading.
4. If you are required to write a computer program, attach your code with several comment statements on the code wherever possible.

1. The steady state temperature distribution $u(x, y)$ in the rectangular plate ($0 \le x \le 2$, $0 \le y \le 1$) satisfies the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1}$$

The upper and lower boundaries ($y = 0$ and $y = 1$) are perfectly insulated; the left side ($x = 0$) is kept at 0° C and the right side ($x = 2$) at $f(y) = y^\circ$ C. The exact solution can be obtained analytically:

$$u(x, y) = \frac{x}{4} - 4 \sum_{n=1,\ n \text{ odd}}^{\infty} \frac{1}{(n\pi)^2 \sinh(2n\pi)} \sinh(n\pi x) \cos(n\pi y). \tag{2}$$

(a) Write a program to compute the steady state solution to the second-order finit

e difference approximation of the heat equation using the point Jacobi iteration method. You should use N_x and N_y uniformly spaced points in the horizontal and vertical directions, respectively (this includes the points on the boundaries).

(b) Now with $N_x = 21$ and $N_y = 21$ apply the Jacobi iteration to the discrete equations until the solution reaches steady state. To start the iterations, initialize the array with zeroes except for the boundary elements corresponding to $u = y$.

You can monitor the progress of the solution by watching the value of the solution at the center of the plate: $(x, y) = (1, 0.5)$. How many iterations are required until the solution at $(1, 0.5)$ *steadily* varies by no more than 0.00005 between the iterations? At this point, how does the numerical approximation compare to the analytical solution? What is the absolute error? What is the error in the numerical approximation relative to the analytical solution (percentage error)?

Plot isotherms of the numerical and exact temperature distributions (say, 16 isotherms). Use different line styles for the numerical and analytical isotherms and put them on the same axes, but be sure to use the same temperature values for each set of isotherms (that is, the same contour levels).

(c) Repeat the above using Gauss Seidel and SOR. Compare the performance of the methods (number of iterations needed to reach the same accuracy).

2. Consider the convection-diffusion equation in 1D:

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2}; \qquad 0 \le x \le 1 \tag{3}$$

with the conditions $T(x, 0) = 0$ and $T(1, t) = 0$. Also consider the initial condition:

$$T(x, 0) = \begin{cases} 1 - (10x - 1)^2 & \text{for } 0 \le x \le 0.2, \\ 0 & \text{for } 0.2 \le x \le 1. \end{cases} \tag{4}$$

(a) **Pure Convection ($\alpha = 0$)**

i. Let $u = 0.08$. The exact solution is

$$T(x,t) = \begin{cases} 1 - (10(x-ut)-1)^2 & \text{for } 0 \le (x-ut) \le 0.2, \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

Solve the problem for $0 \le t \le 8$ using

A. Explicit Euler for time advancement and second-order central for spatial derivative.

B. Leapfrog for time advancement and second-order central for spatial derivative. Use RK2 as starter scheme.

Plot the solution for t = 0,4,8. Use at least 51 points in x direction. Discuss your solution in terms of the stability and accuracy of these schemes. Try different appropriate values for $u\Delta t/\Delta x = 0.02$, 0.2, 0.8, 1.

ii. With the results in (i)A as motivation, the following scheme is suggested (Lax-Wendroff) for the solution:

$$T_j^{(n+1)} = T_j^{(n)} - \frac{\gamma}{2}(T_{j+1}^{(n)} - T_{j-1}^{(n)}) + \frac{\gamma^2}{2}(T_{j+1}^{(n)} - 2T_j^{(n)} + T_{j-1}^{(n)}), \tag{6}$$

where $\gamma = u\Delta t/\Delta x$. What are the accuracy and stability of this scheme? Solve the problem using this scheme for $\gamma = 0.8$, 1, and 1.1 and plot and compare with i.A.

(b) **Convection-Diffusion** ($\alpha = 0.001$) Repeat parts i.A and i.B with the addition of second-order differencing for the diffusion term. Discuss you results and choices of time-steps. How the presence of physical diffusion affects the behavior of solution and the stability of the numerical scheme?

3. Consider the two-dimensional Burgers equation, which is a non-linear convection-diffusion equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{7}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{8}$$

We want to obtain the steady state solution to this problem in a unit square $0 \le x \le 1$, $0 \le y \le 1$ with the following boundary conditions,

$$u(0,y) = u(1,y) = v(x,1) = 0; \quad v(x,0) = 1 \tag{9}$$

$$u(x,0) = u(x,1) = \sin(2\pi x); \quad v(0,y) = v(1,y) = 1 - y \tag{10}$$

Solve the equations using an explicit method of your choice. Use central differencing for spacial derivatives. Using 40 points in each direction, solve the problem to steady state (to plotting accuracy). Use surface plotter in MATLAB to plot the steady state velocities u and v. Since we are interested in steady state, initial condition will not be important.