Let $A$ be diagonalizable and have eigenvalues $\lambda_j$ with $\mathrm{Re}\,\lambda_j < 0$ for all $j$.
We use a num. method with region of abs. stability $S$ to solve $u' = Au$, $u(0) = \eta$ with constant step size $h$.
What conditions must $h$ satisfy for the num. sol. $U^n$ to converge to $0$ as $n \to \infty$?
Answer: $\hat{h} = h\lambda_j$ must be in the interior of $S$ for all $j$.

Solution has the form:

$$u(t) = \sum_j c_j v_j e^{\lambda_j t}, \quad v_j = \text{eigenvector for } \lambda_j$$

Let $V = [v_1 \dots v_m]$ = matrix whose columns are the $v_j$

Then

$$AV = [\lambda_1 v_1 \dots \lambda_m v_m] = [v_1 \dots v_m] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}$$

$$AV = [Av_1 \dots Av_m]$$

$$= VD, \quad D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_m \end{bmatrix}.$$

$$(\Rightarrow) \quad V^{-1}AV = D.$$

Let $y = V^{-1}u$, so $u = Vy$.

Then $y' = V^{-1}u' = V^{-1}Au = \underbrace{V^{-1}AV}_{D}y = Dy$

i.e.,

$$y_1' = \lambda_1 y_1$$
$$y_2' = \lambda_2 y_2$$
$$y_m' = \lambda_m y_m$$

So the system of ODEs has been decoupled.

Solving the system is equivalent to solving the $m$ decoupled equations. Each of these has a stability condition of the form $\lambda_j h \in S$. We need to satisfy every one of these conditions, so we need $\lambda_j h \in S$, $j = 1, \dots, m$.

Region of absolute stability for a lin. multistep method:

$$\sum_{j=0}^{r} \alpha_j U^{n+j} = h \sum_{j=0}^{r} \beta_j f_{n+j}, \text{ where } f_{n+j} = f(t_{n+j}, U^{n+j})$$
$$t_{n+j} = t_n + jh$$

Apply to $u' = \lambda u$, $u(0) = 1$. $f(t,u) = \lambda u$

$$\Rightarrow f_{n+j} = \lambda U^{n+j}$$

$$\Rightarrow \sum_j \alpha_j U^{n+j} = h \sum_j \beta_j \lambda . U^{n+j}$$

$$\Leftrightarrow \sum_{j=0}^{r} (\alpha_j - \underbrace{\lambda h}_{\hat{h}} \beta_j) U^{n+j} = 0 \qquad (1)$$

Let $\hat{h} = \lambda h$ and $\pi(\hat{h}, t) = \sum_{j=0}^{r} (\alpha_j - \hat{h}\beta_j) t^j$ = stability polynomial.

$$\rho(t) = \sum_{j=0}^{r} \alpha_j t^j, \quad \sigma(t) = \sum_{j=0}^{r} \beta_j t^j.$$

$$\Rightarrow \pi(\hat{h}, t) = \rho(t) - \hat{h}\sigma(t)$$

Solutions of the difference eqn (1) are linear combinations of terms of the form $n^\ell (R_k)^n$, where $R_k$ is a root of $\pi(\hat{h}, \cdot)$ with multiplicity $\mu$ and $0 \le \ell \le \mu - 1$.

For sol. to be guaranteed to go to zero we need $|R_k| < 1$ for all roots $R_k$. This is true in the interior of the region of abs. stability $S$.

At the boundary of $S$ there is a root $R_k$ with $|R_k| = 1$.

<u>Boundary locus method:</u> Find $\hat{h}$ for which $\pi(\hat{h}, t)$ has a root $R_k$ with modulus 1.

We write $R_k = e^{i\theta}$.

$$0 = \pi(\hat{h}, R_k) = \pi(\hat{h}, e^{i\theta})$$

$$= \rho(e^{i\theta}) - \hat{h}\,\sigma(e^{i\theta})$$

and solve for $\hat{h}$:

$\hat{h} = \dfrac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}$. For this $\hat{h}$, $\pi(\hat{h}, t)$ has the root $R = e^{i\theta}$. Now varying $\theta$ from 0 to $2\pi$ gives all possibilities. We get a graph of the boundary by plotting $Re(\hat{h}(\theta))$ vs. $Im(\hat{h}(\theta))$, $0 \le \theta \le 2\pi$.

<u>Example:</u> Euler method.

$$u^{n+1} = u^n + h \underbrace{f(t_n, u^n)}_{= f_n} \iff$$

$$-u^n + u^{n+1} = h f_n$$

$\alpha_0 = -1$, $\alpha_1 = 1$, $\beta_0 = 1$, $\beta_1 = 0$

$\rho(t) = -1 + t$, $\sigma(t) = 1$

$$\hat{h}(\theta) = \frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})} = -1 + e^{i\theta}$$

For $0 \le \theta \le 2\pi$, $\hat{h}(\theta) = -1 + e^{i\theta}$ traverses the circle with radius 1 and center $-1$ in the complex plane.

$Re(\hat{h}(\theta)) = \cos\theta - 1$, $Im(\hat{h}(\theta)) = \sin\theta$.