Numerical Solution of Ordinary Differential Equations (MTH 452/552)

Homework due Wednesday, February 22, 2017

Problem 1. (40 points). **a)** Download the file with the code for ode45v4.m from Canvas. This program uses a Runge-Kutta method and was part of an early version of Matlab. Step size control is implemented by means of an embedded pair of Runge-Kutta methods of order 4 and 5. Use the program to solve the astronomical orbit problem and report how many function evaluations are needed in order to obtain a qualitatively correct graph of the orbit.

b) Identify the Butcher array of the Runge-Kutta method that is used to carry forward the solution.

Problem 2. (40 points) Consider the following four numerical methods for solving an IVP $y'(x) = f(x, y(x), y(x_0) = y_0$. Note that the step size Δx is denoted by h.

$$
\begin{aligned}
y_{n+1} + y_n - 2y_{n-1} &= (h/4)\left(f(x_{n+1}, y_{n+1}) + 8f(x_n, y_n) + 3f(x_{n-1}, y_{n-1})\right) && (1)\\
y_{n+1} - y_n &= (h/3)\left(3f(x_n, y_n) - 2f(x_{n-1}, y_{n-1})\right) && (2)\\
y_{n+1} + \frac{1}{4}y_n - \frac{1}{2}y_{n-1} - \frac{3}{4}y_{n-2} &= (h/8)\left(19f(x_n, y_n) + 5f(x_{n-2}, y_{n-2})\right) && (3)\\
y_{n+1} - y_{n-1} &= h\Big(f(x_{n+1}, y^*_{n+1}) + f(x_{n-1}, y_{n-1})\Big) && (4)\\
\text{where } y^*_{n+1} - 3y_n + 2y_{n-1} &= \frac{h}{2}\left(f(x_n, y_n) - 3f(x_{n-1}, y_{n-1})\right)
\end{aligned}
$$

a) For each method determine whether it is convergent. *Hint. If needed, use the following theorem to check consistency.*

Theorem 1 *The numerical method*

$$\sum_{j=0}^{r} \alpha_j\, U^{n+j} = \Delta t\, \Phi_f(U^{n+r}, \ldots, U^n, t_n, \Delta t)$$

is consistent if and only if

$$\sum_{j=0}^{r} \alpha_j = 0, \quad \textit{and} \quad \Phi_f(u(t), \ldots, u(t), t, 0) = f(t, u(t)) \sum_{j=0}^{r} j\alpha_j.$$

b) For the linear multistep methods (1)-(3) determine the order of consistency. Explain why method (4) is not a linear multistep method.

c) Consider the IVP $y' = f(x,y)$, $y(0) = \eta$, $x \in [0,1]$, where

$$y = \begin{pmatrix} u \\ v \end{pmatrix}, \quad f(x,y) = \begin{pmatrix} v \\ v(v-1)/u \end{pmatrix}, \quad \eta = \begin{pmatrix} 1/2 \\ -3 \end{pmatrix}.$$

The exact solution is

$$u(x) = \frac{1}{8}\left(1 + 3e^{-8x}\right), \quad v(x) = -3e^{-8x}.$$

Use methods (1) through (3) to solve this IVP numerically. If additional starting values are required, use the exact solution. Use for each method the following values of h: 0.2, 0.1, 0.01. Present your results in form of a table for the L_2-error E_n defined by

$$E_n = \| y(x_n) - y_n \|_2$$

for $x_n = 0.2, 0.4, 0.6, 0.8, 1.0$. Are your results consistent with your findings in part a)? If you are unsure, conduct an additional experiment with $h = 10^{-4}$.

A text file mfiles.txt with M-files to help with the numerical experiments is available on Canvas.

c) For each method state how the error behaves for fixed x and h getting smaller, as well as for fixed h and x getting larger. Are your observations consistent with your findings in parts a) and b)?