more than 1 independent variables

PDE (partial differential equations)

2 or more independent variables

Consider general pde

$$A f_{xx} + B f_{xy} + C f_{yy} + D f_x + E f_y + F f = G$$

$$A x^2 + B xy + C y^2 + D x + E y + F = 0$$ algebraic eq.

determinant $B^2 - 4AC$

(1) $B^2 - 4AC \Rightarrow -ve \rightarrow$ Ellipse / Elliptic pde

e.g. $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0 \Rightarrow A = 1,\ B = 0,\ C = 1$

heat diffusion (Laplace) $\quad B^2 - 4AC < 0$

(2) $B^2 - 4AC \Rightarrow 0 \rightarrow$ parabola / parabolic pde

$\dfrac{\partial f}{\partial t} - \alpha \dfrac{\partial^2 f}{\partial x^2} = 0 \Rightarrow A = -\alpha,\ B = 0,\ C = 0$

unsteady heat eqn $\quad B^2 - 4AC = 0$

(3) $B^2 - 4AC = -ve \rightarrow$ hyperbola / hyperbolic

$\dfrac{\partial^2 f}{\partial t^2} = c^2 \dfrac{\partial^2 f}{\partial x^2} \Rightarrow A = 1,\ B = 0,\ C = -c^2$

$B^2 - 4AC > 0$

wave equation

$$A f_{xx} + B f_{xy} + C f_{yy} = -D f_x - E f_y - F + G$$

we can write total derivates

$$d(f_x) = f_{xx}\,dx + f_{xy}\,dy$$

$$\& \quad d(f_y) = f_{yx}\,dx + f_{yy}\,dy$$

$$\Rightarrow \begin{bmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix} \begin{bmatrix} f_{xx} \\ f_{xy} \\ f_{yy} \end{bmatrix} = \begin{bmatrix} -D f_x - E f_y - F + G \\ d(f_x) \\ d(f_y) \end{bmatrix}$$

determinant $= 0$

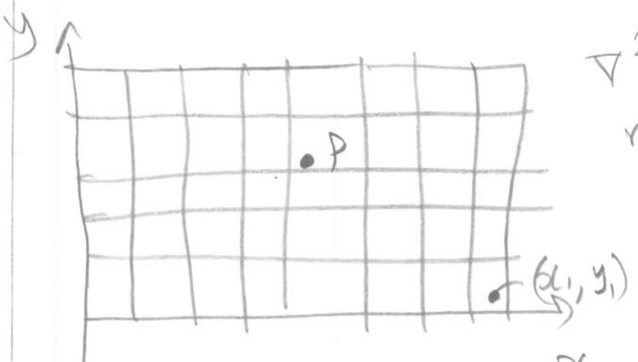
$$\Rightarrow A(dy)^2 - B(dx)(dy) + C(dx)^2 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

If $B^2 - 4AC < 0$, complex $\frac{dy}{dx}$

$B^2 - 4AC = 0$, $\frac{dy}{dx} = \frac{B}{2A}$ real & (identical)

$B^2 - 4AC > 0$ $\frac{dy}{dx} =$ real separate



$\nabla^2 f = 0$

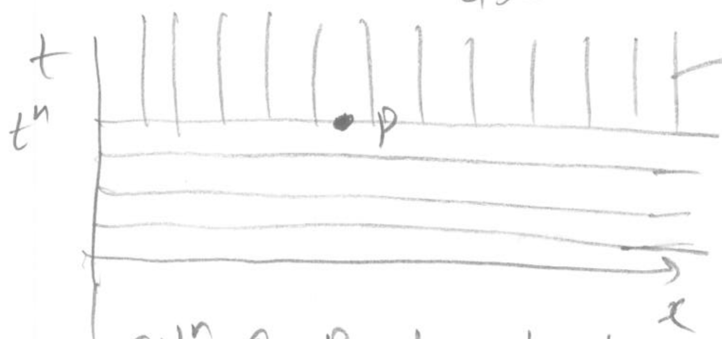
region of influence & region of dependence are same & whole domain for elliptic eq.

If you change some value at (x_1, y_1), this change will propagate over the entire domain "instantaneously" and change the whole solution.

consider $\frac{\partial f}{\partial t} - \alpha \frac{\partial^2 f}{\partial x^2} = 0$

$A = -\alpha$
$B = 0$
$C = 0$

$$\frac{dy}{dx} = \frac{dt}{dx} = 0$$



→ zone of influence

→ region of dependence

Marching methods

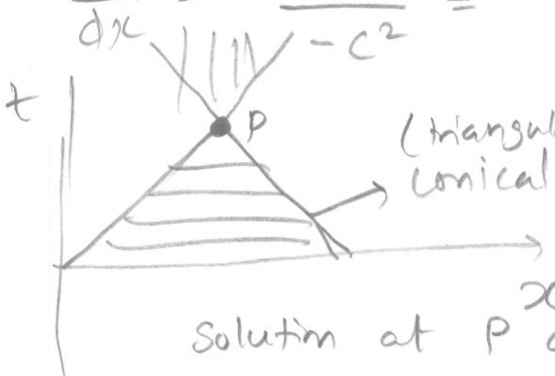
sol^n @ P dependends on all points in x and all sol^ns in t prior to t^n. Subsequently, sol^n at P affects the region in t above it!

conside $\frac{\partial^2 f}{\partial t^2} - c^2 \frac{\partial^2 f}{\partial x^2} = 0$

$A = -c^2$
$B = 0$
$C = 1$

$$\frac{dt}{dx} = \frac{\pm 2c}{-c^2} = \pm \frac{2}{c}$$

$B^2 - 4AC > 0$
$= 4c^2$



(triangular) conical region of influence

Solution at P dependent only on the triagular region characterized by 2 different sol^ns.

Elliptic Equations:

These arise in steady-state equilibrium problems

Laplace eq: $\nabla^2 \phi = 0$

Poisson eq: $\nabla^2 \phi = f$

Helmholtz $\nabla^2 \phi + \alpha^2 \phi = 0$

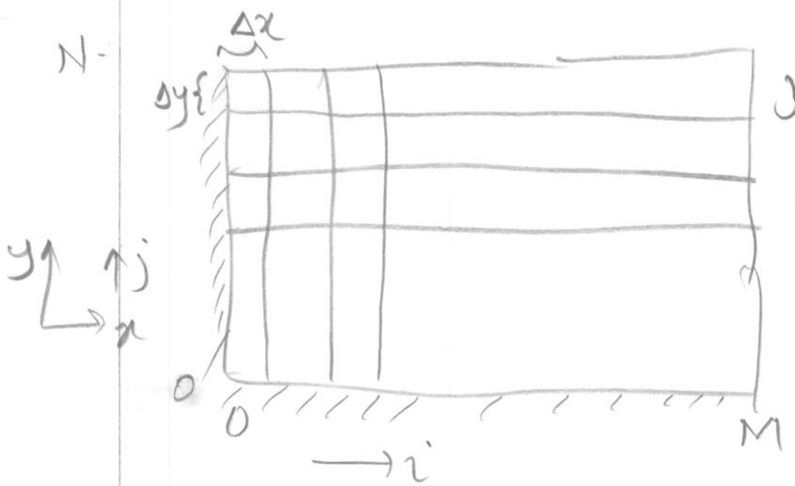
B.C.s. dirichlet ϕ describe on boundary

Neumann $\frac{\partial \phi}{\partial n}$ described on boundary

or mixed $c_1 \phi + c_2 \frac{\partial \phi}{\partial n} = g$

n is normal to the boundary

Rectangular domain



$j = 0, 1, 2, \ldots N$

$i = 0, 1, 2, \ldots M$

Let $\Delta x = \Delta y = \Delta$

Then, centered differences gives **Poisson eq:**

$$\frac{\phi_{i+1,j} - 2\phi_{i,j} + \phi_{i-1,j}}{\Delta x^2} + \frac{\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1}}{\Delta y^2} = f_{i,j}$$

$$\boxed{\phi_{i+1,j} - 4\phi_{i,j} + \phi_{i-1,j} + \phi_{i,j+1} + \phi_{i,j-1} = \Delta^2 f_{i,j}}$$

$i = 1, 2, \ldots M-1$ & $j = 1, 2, \ldots N-1$

$\phi_{i,j}$

consider $\nabla^2 \phi = 0$

Let us arrange unknown ϕ as

$$\left[\phi_{11} \ \phi_{21} \ \phi_{31} \ \cdots \ \phi_{M-1,1} \ \phi_{12} \ \phi_{22} \ \cdots \ \phi_{M-12} \ \cdots \right]^T$$

Then we can write the f.d. system in $A\vec{x} = \vec{b}$

$x =$

Consider $M = 6, \quad N = 4$

one block

$$\left[\begin{array}{ccccc|ccccc}
-4 & 1 & & & & 1 & & & & \\
1 & -4 & 1 & & & & 1 & & & \\
 & 1 & -4 & 1 & & & & 1 & & \\
 & & 1 & -4 & 1 & & & & 1 & \\
 & & & & -4 & 1 & & & & 1 \\
\hline
1 & & & & -4 & 1 & & & & 1 \\
 & \ddots & & & & & \ddots & & & \ddots
\end{array}\right]
\begin{bmatrix} \phi_{11} \\ \phi_{21} \\ \phi_{31} \\ \phi_{41} \\ \phi_{51} \\ \phi_{12} \\ \phi_{22} \\ \phi_{32} \\ \phi_{42} \\ \phi_{52} \\ \phi_{13} \\ \phi_{23} \\ \phi_{33} \\ \phi_{43} \\ \phi_{53} \end{bmatrix}$$

$(M-1) \times (M-1)$ block with $(N-1)$ of them.

b.c.s modify the first entry:

e.g. consider $\phi = \hat{\phi}(x,y)$ on the boundaries

Then at $i=1$,

$$\phi_{2j} - 4\phi_{1j} + \phi_{1j+1} + \phi_{1j-1} = -\phi_{0j}$$

The RHS of the $A\vec{x} = \vec{b}$ changes. It will be $(-\phi_{0j})$

For derivative bcs

$$\frac{\partial \phi}{\partial x} = g(y)$$

Then using one-sided differences

$$\frac{-3\phi_{0j} + 4\phi_{1j} - \phi_{2j}}{2\Delta} = g_j \qquad \text{second order}$$

Eliminating ϕ_{0j} we get

$$\frac{2}{3}\phi_{2j} - \frac{8}{3}\phi_{1j} + \phi_{1j+1} + \phi_{1j-1} = -\frac{2}{3}\Delta g_j.$$

coefficients change & rhs changes.

one may use first order differencing

$$\frac{-\phi_{0j} + \phi_{1j}}{\Delta x} = g_j$$

Inversion by iterative schemes:

$$A\vec{x} = \vec{b}$$

Let $A = A_1 - A_2$

$$A_1\vec{x} = A_2\vec{x} + \vec{b}$$

One can construct an iterative method by writing

$$A_1 \vec{x}^{(k+1)} = A_2 \vec{x}^{(k)} + \vec{b}$$

k is the iteration index. Starting with some guess $\vec{x}^{(0)}$ this scheme can be used to obtain solution to the system $A\vec{x} = \vec{b}$

The idea is as $k \to \infty$ $\quad \vec{x}^{(k+1)} \longrightarrow \vec{x}^{(k)}$

Then we get $\quad A_1 \vec{x}^{(k+1)} - A_2 \vec{x}^{(k)} \approx (A_1 - A_2)\vec{x}^{(k)}$

$$= A\vec{x}^{(k)} = \vec{b}$$

Requirements:

1. A_1 should be easily invertable
2. Iterations should converge

$$\lim_{k\to\infty} \vec{x}^{(k)} = \vec{x}$$

Let the error at iteration k be $\epsilon^{(k)} = x - x^{(k)}$

we can write

$$A_1(\vec{x} - \vec{x}^{(k+1)}) = A_2(\vec{x} - \vec{x}^{(k)}) + (\vec{b} - \vec{b})$$

$$\Rightarrow \quad A_1 \epsilon^{(k+1)} = A_2 \epsilon^{(k)}$$

$$\epsilon^{(k+1)} = A_1^{-1} A_2 \epsilon^{(k)}$$

$$\epsilon^{(k)} = (A_1^{-1} A_2)^k \epsilon^{(0)}$$

so we need

$$\lim_{k\to\infty} \epsilon^{(k)} = 0$$

$A_1^{-1} A_2 = B \qquad B^K$ will tend to zero

iff $|\lambda_i|_{max} \leq 1 \qquad \lambda_i$ are eigenvalues

$\rho = |\lambda_i|_{max}$ is the spectral radius

Point- Jacobi

To have easy inversion of a matrix a diagonal matrix is preferred.

So let A_1 be Diagonal matrix

$$A_1 = \begin{bmatrix} -4 & & & \\ & -4 & & \\ & & -4 & \\ & & & \ddots \end{bmatrix}$$

in the Laplace problem

$$\text{Then } A_2 = A_1 - A = \begin{bmatrix} 0 & & & -1 & \\ -1 & 0 & -1 & & -1 \\ & -1 & 0 & -1 & \\ & & & \ddots & \end{bmatrix}$$

The other elements

$$A_1^{-1} = \begin{bmatrix} -1/4 & & \\ & -1/4 & \\ & & \ddots \end{bmatrix}$$

$A_1 \phi^{(K+1)} = A_2 \phi^{(K)} + \vec{b}$ where ϕ is the unknown vector

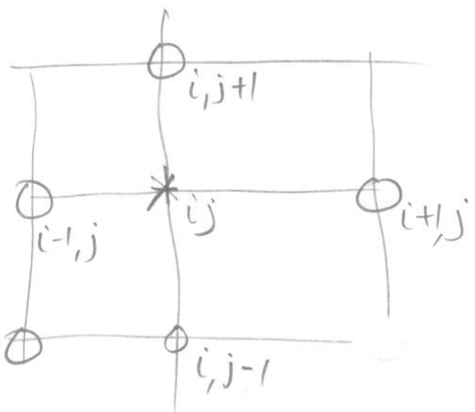
For each ϕ,

$$\boxed{\phi^{(k+1)} = -\frac{1}{4} A_2 \, \phi^{(k)} - \frac{1}{4} \vec{b}}$$

using A_2 and the index notation

$$\boxed{\phi_{ij}^{(k+1)} = \frac{1}{4}\left[\phi_{i-1,j}^{(k)} + \phi_{i+1,j}^{(k)} + \phi_{i,j-1}^{(k)} + \phi_{i,j-1}^{(k)}\right] - \frac{1}{4} b_{ij}}$$

i,j are used in the same order as the matrix eq^n in the Laplace eq^n (1)



ϕ_{ij} depends one values of ϕ at "immediate" neighbors. No matrix storage involve

So starting with $\phi_{ij}^{(0)}$ guess (could be $=0$) one just updates $\phi_{ij}^{(k+1)}$ using above eq^n. (1).

eigenvalues of $A_1^{-1} A_2 = -\frac{1}{4} A_2$ should be such that $|\lambda_i|_{max} < 1$

From Linear Algebra for (uniform grids) eigen values are

$$\lambda_{mn} = \frac{1}{2}\left[\cos\frac{\pi m}{M} + \cos\frac{\pi n}{N}\right]$$

$M = 1, 2, 3, \ldots M-1$ & $N = 1, 2, 3, \ldots N-1$ are grid points in x & y

for all m & n $|\lambda_{mn}| \leq 1$ Method converges!

For rate of convergence we look at $|\lambda_{mn}|_{max}$

using power series expansion

$$|\lambda_{mn}|_{max} = 1 - \frac{1}{4}\left[\frac{\pi^2}{M^2} + \frac{\pi^2}{N^2}\right] + \cdots$$

For large M & N $|\lambda_{mn}|_{max}$ is only slightly lower than 1. Large iterations

So let us say we want to reduce the error $\epsilon^{(0)}$ by a factor of 10^{-n}.

Then we need $|\lambda|_{max}^{K} \leq 10^{-n}$

Taking logs $K \geq \dfrac{-m}{\log|\lambda|_{max}}$

Now, let $M = 20$, $N = 20$

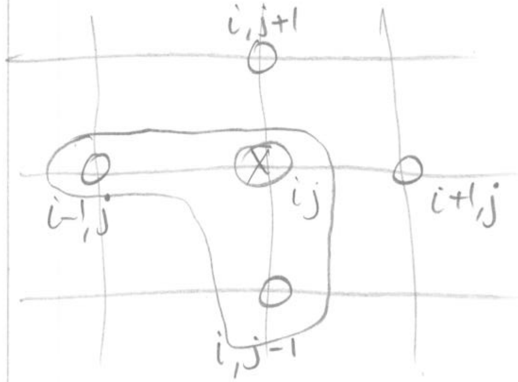
$$|\lambda|_{max} = \cos\frac{\pi}{20} = 0.988$$

To reduce error by a factor of 1000, $m = 3$

$K = 558$. For $M = 100$, $N = 100$, $K = 14000$ for $m = 3$!

<u>very slow.</u>

Gauss-Seidel



In point-Jacobi iterative scheme notice that we solve ϕ_{ij} in an orderly fashion. In other words $\phi_{i-1,j}$ & $\phi_{i,j-1}$ will be solved before ϕ_{ij}

So we will have $\phi_{i-1,j}^{(k+1)}$, $\phi_{i,j-1}^{(k+1)}$ available before we try to obtain $\phi_{ij}^{(k+1)}$. Gauss-Seidel makes use of "latest" or most updated values of ϕ_{ij}

Then

$$\phi_{ij}^{(k+1)} = \frac{1}{4}\left[\phi_{i-1,j}^{(k+1)} + \phi_{i+1,j}^{(k)} + \phi_{ij-1}^{(k+1)} + \phi_{ij+1}^{(k)}\right] - \frac{1}{4} b_{ij}$$

In the matrix form $A = A_1 - A_2$

Here $A_1 = D - L$, $A_2 = U$

$$A_1 = \begin{bmatrix} -4 & & & & \\ -1 & -4 & & & \\ & -1 & -4 & & \\ & & \ddots & \ddots & \\ -1 & & & \ddots & \end{bmatrix} \qquad A_2 = \begin{bmatrix} 0 & -1 & & & -1 & \\ & 0 & -1 & & & -1 \\ & & 0 & -1 & & \\ & & & \ddots & \ddots & \\ & & & & & 0 \end{bmatrix}$$

lower triangular
easy to invert

for Gauss-Seidel

$$\lambda_{mn} = \frac{1}{4}\left[\cos\frac{\pi m}{M} + \cos\frac{n\pi}{N}\right]^2$$

Square of the eigenvalues of point-Jacobi
Twice as fast

SOR (Successive over relaxation)

Note $\quad A_1 \vec{x}^{(k+1)} = A_2 \vec{x}^{(k)} + \vec{b}$

$\phi^{(k+1)} = \phi^{(k)} + \vec{d}$ or change in ϕ at each iteration is $\phi^{(k+1)} - \phi^{(k)} = \vec{d}$.

The idea of SOR is to "accelerate" this change at each iteration in order to reduce # of iterations.
This is done by saying

$$\phi^{(k+1)} - \phi^{(k)} = \omega \vec{d}$$

ω is the relaxation factor

$\omega > 1$ for accelerated convergence. Hence over-relaxation

$\omega = 1$ Gauss-Seidel

$\omega < 1$ slower convergence

We first use Gauss-Seidel, which can be written as

$$D\tilde{\phi}^{(k+1)} = L\phi^{(k+1)} + U\phi^{(k)} + \vec{b}$$

$\tilde{\phi}^{(k+1)}$ is not considered as the "accepted" value at $k+1$.

$\phi^{(k+1)} = \phi^{(k)} + \omega(\tilde{\phi}^{(k+1)} - \phi^{(k)})$ is used for weighted convergence

$$\phi^{(k+1)} = (1-\omega)\,\phi^{(k)} + \omega\,\tilde{\phi}^{(k+1)}$$

$\omega = 0 \Rightarrow$ Gauss-seidel (GS)

$\omega > 1$ over-relaxation.

So we weight the predicted value for GS more than the old iteration value

Combining the two eqns

$$\tilde{\phi}^{(k+1)} = D^{-1}L\phi^{(k+1)} + D^{-1}U\,\phi^{(k)} + D^{-1}b$$

$$\phi^{(k+1)} = (1-\omega)I\phi^{(k)} + \omega D^{-1}U\,\phi^{(k)} + \omega D^{-1}b + \omega D^{-1}L\,\phi^{(k+1)}$$

$$\Rightarrow \left(I - \omega D^{-1}L\right)\phi^{(k+1)} = \left[(1-\omega)I + \omega D^{-1}U\right]\phi^{(k)} + \omega D^{-1}b$$

$$\phi^{(k+1)} = \underbrace{\left[I - \omega D^{-1}L\right]^{-1}\left[(1-\omega)I + \omega D^{-1}U\right]}_{G_{SOR}}\phi^{(k)} + \left[I - \omega D^{-1}L\right]^{-1}\omega D^{-1}\vec{b}$$

To optimize convergence means selecting ω to minimize the largest eigenvalue

For rectangular regions with uniform mesh ω_{opt} can be obtained analytically

$$\omega_{opt} = 2\left\{\frac{1-\sqrt{1-\mu_{max}^2}}{\mu_{max}^2}\right\}$$, μ_{max} is max. eigen value of point-Jacobi

$\omega \sim 1.7$ to 1.9 is usually used.

For complex grids use numerical experiments.

Parabolic equations

diffusion equation

$$\frac{\partial f}{\partial t} = \alpha \frac{\partial^2 f}{\partial x^2}$$

convection-diffusion

$$\frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} = \alpha \frac{\partial^2 f}{\partial x^2}$$

reading → characteristics etc 10·2 – 10·3·3

consider $\dfrac{\partial \phi}{\partial t} = \alpha \dfrac{\partial^2 \phi}{\partial x^2}$, $\phi(0,t) = \phi(L,t) = 0$

$\phi(x,0) = g(x)$

$j = 1, 2, \dots N$

Von-Neumann stability analysis:

→ does not take into account bcs

→ periodic bcs are assumed
i.e. sol^n & its derivatives are the same on both ends of the domain

→ It works for linear, constant coefficient pdes

→ uniform grid spacing