Multi-step methods

Higher order of accuracy for RK was obtained by multiple evaluations of the function f (between t^n & t^{n+1}). However, one may obtain higher accuracy by using f at points earlier to t_n so t_{n-1}, t_{n-2} ... These are called multistep methods.

$f(t^{n+1})$ depends on $f(t^n)$, $f(t^{n-1})$ and so...on. These methods are not self-starting. e.g. @ $t=1$, we need a sol^n @ $t=0$ and $(t=-1)$. Explicit Euler is used for the first time step.

second-order Adams-Bashforth

$y' = f(y,t)$

$h = \Delta t$

Consider Taylor series

$$y_{n+1} = y_n + h y_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' + \cdots$$

let us use $y_n'' = \dfrac{f(y_n,t_n) - f(y_{n-1},t_{n-1})}{h} + O(h)$

Then

$$y_{n+1} = y_n + h f(y_n,t_n) + \frac{h}{2}\left[f(y_n,t_n) - f(y_{n-1},t_{n-1})\right] + O(h^3)$$

$$\boxed{\therefore \quad y_{n+1} = y_n + \frac{3h}{2} f(y_n,t_n) - \frac{h}{2} f(y_{n-1},t_{n-1}) + O(h^3)}$$

This is Adams-Bashforth method

we require t_n & t_{n-1} solutions $\rightarrow$ 2 step.

second-order globally

for $t=1$ $\quad y_{n+1} = y_n + h f(y_n,t_n)$

$\Rightarrow f(y_{-1},t_{-1}) = f(y_0,t_0)$

Let us apply this to our model problem

$$y' = \lambda y$$

$$y^{n+1} = y^n + \frac{3h}{2} f(y_n, t_n) - \frac{h}{2} f(y_{n-1}, t_{n-1})$$

$$= y^n + \frac{3h}{2} \lambda y^n - \frac{h}{2} \lambda y^{n-1}$$

$$\boxed{y^{n+1} = y^n \left(1 + \frac{3h}{2}\lambda\right) - \frac{h}{2} \lambda y^{n-1}}$$

$\sigma = \dfrac{y^{n+1}}{y^n}$ is Amplification factor

$$\Rightarrow \quad \sigma = \frac{y^n}{y^{n-1}} \quad \Rightarrow \quad \sigma = \frac{y^{n+1}}{y^n} = 1 + \frac{3}{2}\lambda h - \frac{\lambda h}{2}\frac{1}{\sigma}$$

$$\therefore \boxed{\sigma^2 - \left(1 + \frac{3}{2}\lambda h\right)\sigma + \frac{\lambda h}{2} = 0}$$

Notice that there will be more than one roots for the amplification factor! This is key of all multistep methods. The stability depends on both roots

solving for σ

$$\sigma_{1,2} = \frac{1}{2}\left[1 + \frac{3}{2}\lambda h \pm \sqrt{1 + \lambda h + \frac{9}{4}\lambda^2 h^2}\right]$$

For pure stability both roots should have

$$|\sigma_{1,2}| \le 1$$

$$\sqrt{1 + \lambda h + \frac{9}{4}\lambda^2 h^2} = 1 + \frac{1}{2}\left(\lambda h + \frac{9}{4}\lambda^2 h^2\right) - \frac{1}{8}\left(\lambda h + \frac{9}{4}\lambda^2 h^2\right)^2 + \frac{3}{48}\left(\lambda h + \frac{9}{4}\lambda^2 h^2\right)^3 + \cdots$$

Higher-order odes:

$$y^{(n)} = f(t, y, y', \cdots, y^{(n-1)})$$

$y_1 = y$

$y_2 = y' = y_1$, $y_n = y^{(n-1)} = y_{n-1}'$

$\Rightarrow y_1' = y_2$

$y_2' = y_3$

$\vdots$

$y_n' = f_n(t, y_1, y_2, \ldots y_n)$

$\Rightarrow \vec{y'} = \vec{f}(\vec{y}, t)$ $\rightarrow$ stands for the vector

solving systems of odes $\Rightarrow$ solving each ode sequentially with use of "n" level values for unknown variables in the vector

e.g. explicit Euler $\Rightarrow$ $\boxed{y_i^{(n+1)} = y_i^{(n)} + hf_i\underbrace{(y_1^{(n)}, y_2^{(n)}, \ldots y_m^{(n)}, t_n)}_{\text{all known}}}$

Problem? stiffness.

consider a model system of ode of m variables

$$\vec{y'} = Ay \quad , A \text{ is a constant coeff. } m \times m \text{ matrix}$$

This is equivalent to our model problem $(y' = \lambda y)$

Using Euler method

$$\vec{y}_{n+1} = \vec{y}_n + hA\vec{y}_n = (I + hA)\vec{y}_n$$

$$\Rightarrow \vec{y}_{n+1} = (I + hA)^n \vec{y}_0$$

$$\vec{y}_{n+1} = B^n \vec{y}_n$$

$$B^n = (I + hA)^n$$

To have a bounded solⁿ, the magnitudes of eigenvalues of B should be < 1

If λ_i are eigen values of A

Then eigenvalues of B are $\alpha_i = 1 + \lambda_i h$

$|1 + \lambda_i h| \leq 1 \Rightarrow$ real part of (λ_i) should be -ve.

If λ_i are all real and -ve, then

$$\boxed{h \leq \frac{2}{|\lambda|_{max}}}$$

$|\lambda|_{max}$ is the restrictive eigen value. The ode which gives highest eigen value $\Rightarrow$ information propagates fastest for that variable. In order to capture this, Δt has to be smaller.

If $\dfrac{|\lambda|_{max}}{|\lambda|_{min}} >> 1$, we have a stiff system

This poses numerical issues, even if we are interested in long-term behavior, one may be forced to use time-steps resolving the time-scales of the fastest moving eigen-value $\Rightarrow$ rate-limiting eqⁿ

$y' = -1000e^{-1000t} + 1 = -1000(y - t) + 1$

consider

$$y' = -1000(y - t - 2) + 1 \Rightarrow y = -e^{-1000t} + t + 2$$

Increasing soln

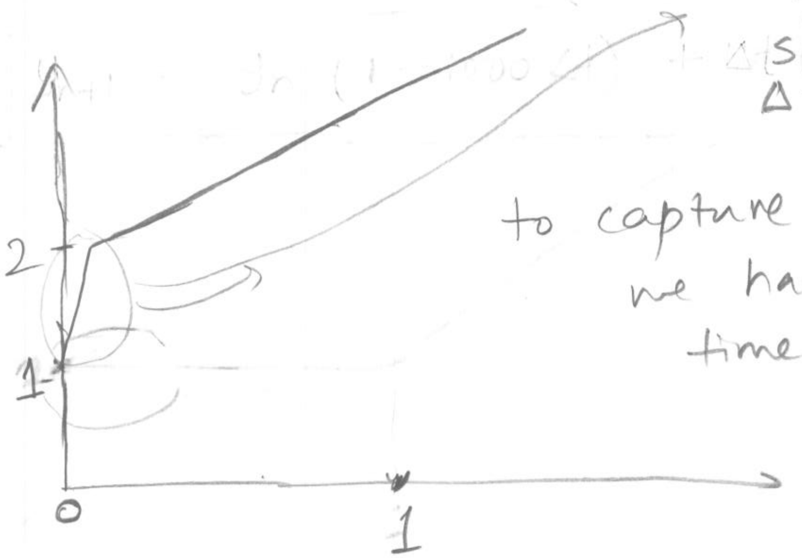
decaying soln

$y(0) = 1$

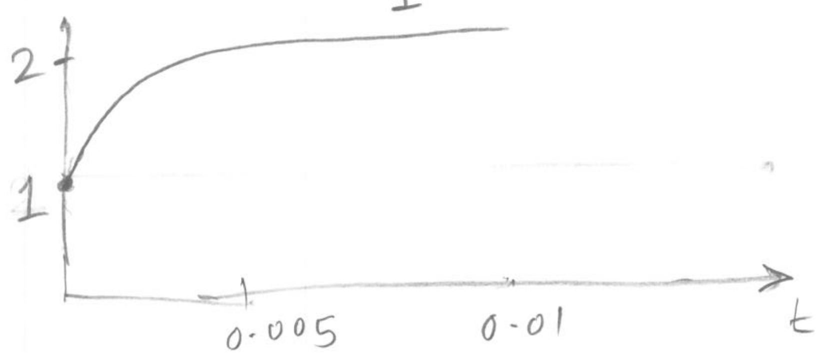
Apply Explicit Euler

Because of this sharp transient $\Delta t \le \frac{2}{1000}$ for Euler explicit

to capture this transient we have to reduce time step..



If not interested in transients use implicit scheme!



Usually Implicit schemes for f are used for stability. However, if we need to resolve the transients (time-dependent problem) we have to use small $\underline{\underline{\Delta t}}$ →

reading 7.14

Problem with implicit schemes

Requires sol^n of a non-linear algebraic eq. $\Rightarrow$ iterative sol^n procedure (Newton Raphson)

Linearization is a technique used in several schemes

$$\frac{d\vec{u}}{dt} = \vec{f}(u_1, u_2, \ldots u_m)$$

$$u_i^{n+1} = u_i^n + f_i(u_i^{n+1})\Delta t$$

$$f_i(u_i^{n+1}) = f_i(u_i^n) + \left.\frac{\partial f_i}{\partial u_1}\right|_n (u_1^{n+1} - u_1^n) + \left.\frac{\partial f_i}{\partial u_2}\right|_n (u_2^{n+1} - u_2^n) + \ldots \left.\frac{\partial f_i}{\partial u_m}\right|_n (u_m^{n+1} - u_m^n)$$

$$f_i(u_i^{n+1}) = f_i(u_i^n) + \sum_{j=1}^{m} \left.\frac{\partial f_i}{\partial u_j}\right|_n (u_j^{n+1} - u_j^n) + O(h^2)$$

$$\Rightarrow \vec{f}(\vec{u}^{n+1}) = \vec{f}(\vec{u}^n) + A_n(\vec{u}^{n+1} - \vec{u}^n)$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \frac{\partial f_m}{\partial u_1} & \cdots & & \frac{\partial f_m}{\partial u_m} \end{bmatrix}_n$$

$\Rightarrow$ time dependent Jacobian

$$\vec{u}^{n+1} = \vec{u}^{n} + A_n \left(\vec{u}^{n+1} - \vec{u}^{n}\right)\Delta t + \vec{f}^{n}\Delta t$$

$$\boxed{\left(\underline{I} - A_n\Delta t\right)\vec{u}^{n+1} = \left[\underline{I} - A_n\Delta t\right]\vec{u}^{n} + \vec{f}_n\Delta t}$$

Easier to solve than non-linear system

A_n. time-dependent.