Matrix Inversion

1. Iterative methods in matrix form

$$Ax = b \tag{1}$$
$$A_1 x = A_2 x + b \tag{2}$$
$$A_1 x^{(k+1)} = A_2 x^{(k)} + b \tag{3}$$

2. Properties of the matrix decomposition for convergence of the system of equations.

   - A_1 should be *easily* invertible.
   - Iterations should converge, hopefully quickly.
   - Let $\epsilon^{(k)} = x - x^{(k)}$ be the error at the kth iteration. Then, subtracting the above equations gives,

$$\begin{aligned}
A_1 x - A_1 x^{(k+1)} &= A_2(x - x^{(k)}) && (4)\\
A_1 \epsilon^{(k+1)} &= A_2 \epsilon^{(k)} && (5)\\
\epsilon^{(k+1)} &= A_1^{-1} A_2 \epsilon^{(k)} && (6)\\
\epsilon^{(k)} &= (A_1^{-1} A_2)^k \epsilon^{(0)} && (7)\\
\epsilon^{(k)} \to 0 \quad &\text{iff} \quad ? && (8)
\end{aligned}$$

3. Point-Jacobi: $A_1 = D$, $A = A_1 - A_2$, $A_2 = A_1 - A = D - A$.

4. Gauss-Seidel: $A_1 = D - L$, $A_2 = U$.

5. SOR: similar to Gauss-Seidel with an extra step for relaxation

$$
\begin{aligned}
\phi^{(k+1)} &= \phi^{(k)} + \omega(\tilde{\phi}^{(k+1)} - \phi^{(k)}) && (9)\\
D\phi^{(k+1)} &= L\phi^{(k+1)} + U\phi^{(k)} + b && (10)
\end{aligned}
$$

$$
\phi^{(k+1)} = \underbrace{[I - \omega D^{-1}L]^{-1}[(1-\omega)I + \omega D^{-1}U]}_{G_{\mathrm{SOR}}}\phi^{(k)} + [I - \omega D^{-1}L]^{-1}\omega D^{-1}b \qquad (11)
$$

6. Consider the parabolic PDE

$$\frac{\partial \phi}{\partial t} = \alpha\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}\right) \tag{12}$$

Develop finite difference approximation for trapezoidal method in time and show that it can be solved using direct inversions of two tri-diagonal matrices using a approximate factorization technique.