Modified Equation and Consistency

1. Consider the pure advection equation:

$$u_t + cu_x = 0; \qquad c > 0 \tag{1}$$

Now consider the following scheme:

$$u_i^{n+1} = u_i^n - \frac{c\Delta t}{\Delta x}\left(u_i^n - u_{i-1}^n\right) \tag{2}$$

(a) Derive the modified equation and write it in the form:

$$u_t + cu_x = \alpha u_{xx} + \text{higher order terms} \tag{3}$$

(b) By comparing the truncation terms (i.e α coefficients) for the two schemes, determine which scheme is more accurate?

(c) Determine whether the scheme is consistent.

(d) What are the constraints on Δt and/or Δx for stable solution?

Consider the following parabolic PDE:

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} \tag{4}$$

(a) Using Leap Frog for time discretization and CD for space, obtain the finite difference approximation.

(b) Find stability constraints

(c) Modify the scheme by replacing $\phi_j^{(n)} = 1/2(\phi_j^{(n+1)} + \phi_j^{(n)})$ only on RIGHT hand side. Find stability limits. The new scheme is called Du Fort-Frankel.

(d) Find out the modified equation for the Du Fort-Frankel scheme and identify if the method is consistent or not.