Linearization of system of equations

Stability

Consider a function $f(t, y)$. The local behavior of the solution to a differential equation near any point (t_0, y_0) can be analyzed by expanding $f(t, y)$ using a *two*- dimensional Taylor series:

$$f(t, y) = f(t_0, y_0) + \frac{\partial f}{\partial t}\,|_{t_0, y_0}\,(t - t_0) + \frac{\partial f}{\partial y}\,|_{t_0, y_0}\,(y - y_0) + ... \tag{1}$$

The most important term in the series is usually the one involving $\frac{\partial f}{\partial y}$. It is termed as the Jacobian denoted by J.

Stability of the System of Non-Linear ODEs

Consider a system of differential equations with n components,

$$\frac{d}{dt}\begin{bmatrix} u_1(t) \\ u_2(t) \\ . \\ . \\ . \\ u_n(t) \end{bmatrix} = \begin{bmatrix} f_1(t, u_1, u_2, ...u_n) \\ f_2(t, u_1, u_2, ...u_n) \\ . \\ . \\ . \\ f_n(t, u_1, u_2, ...u_n) \end{bmatrix} \tag{2}$$

The Jacobian for the system of equations is given as:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_n} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_n} \\ . & . & & . \\ . & . & & . \\ . & . & & . \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_n} \end{bmatrix} \tag{3}$$

The influence of this Jacobian on the local behavior of the system of equations is determined by solution of the linearized ordinary differential equations:

$$\frac{d\overline{u}}{dt} = J\overline{u} \tag{4}$$

where $\overline{u}$ is a vector of variables $[u_1, u_2, u_3, ...u_n]^T$, and the Jacobian J is evaluated at any point where the solution is known, e.g. the initial point $[u_1, u_2, u_3, ...u_n]^T|_{t=0}$. Notice that in arriving at this linearized differential equation, we neglected some terms involving derivatives with respect to t and also products of some some constants. Of particular interest is the change in the behavior of the solution near the starting point and is characterized by the above system of equations. With known J (at a given point) the above system of equations can be decoupled using the eigen value analysis and its stability evaluated. Note, however, that the stability analysis is relevant only in the neighborhood of the initial point. If we want to evaluate the stability at some other point, the Jacobian matrix J will be different.

Linearized Approximation of Implicit Schemes for Non-linear Problems

The above linearization concept can be used for implicit schemes applied to non-linear problems. Consider a coupled system as:

$$\frac{du}{dt} = f(u,v,t); \qquad \frac{dv}{dt} = g(u,v,t), \tag{5}$$

where f and g are non-linear functions. Then each of these functions can be expanded around the solution at time t_n or around point (u_n, v_n, t_n).

$$\begin{aligned} f(u,v,t) &= f(u_n,v_n,t_n) + \left.\frac{\partial f}{\partial u}\right|_{u_n,v_n,t_n}(u-u_n) + \left.\frac{\partial f}{\partial v}\right|_{u_n,v_n,t_n}(v-v_n) + \text{higher order terms} \quad (6) \\ &\approx f(u_n,v_n,t_n) + \left.\frac{\partial f}{\partial u}\right|_{u_n,v_n,t_n}(u-u_n) + \left.\frac{\partial f}{\partial v}\right|_{u_n,v_n,t_n}(v-v_n) \quad (7) \end{aligned}$$

and similarly for $g(u,v,t)$ Now, if we want to use an implicit scheme, such as backward Euler or Trapezoidal, then we can apply the scheme to the original differential equation, with the non-linear right-hand side linearized as above. This way, the solution procedure for implicit scheme can be much easier, rather than solving coupled, non-linear algebraic equations. This strategy is employed in most cases. Linearization is an approximation, it does introduce an error; however, if our step size is small enough, it is possible to get good stability with linearized implicit methods.

Example

Consider the following coupled, non-linear system of ordinary differential equations.

$$\frac{du}{dt} = u^2 v; \qquad \frac{dv}{dt} = uv^2 \tag{8}$$

with $u(0) = 10$ and $v(0) = 1$ where u and v are functions of time t.

Consider a linearized trapezoidal scheme $y' = f(t, y)$ is given as $\frac{y^{n+1}-y^n}{\Delta t} = \frac{f^L(t^n,y^n)+f^L(t^{n+1},y^{n+1})}{2}$, where f^L is the linearized function f. For example, in going from t_0 to t_1 (the first time step), expanding f around u_0, t_0, f^L can be written as:

$$\begin{aligned}
f_1^L(u, v, t) &= f(u_0, v_0, t_0) + \left.\frac{\partial f}{\partial u}\right|_{u_0,v_0,t_0}(u - u_0) + \left.\frac{\partial f}{\partial v}\right|_{u_0,v_0,t_0}(v - v_0) + \text{higher order terms} && (9)\\
&= u_0^2 v_0 + (2u_0 v_0)(u - u_0) + (u_0^2 \cdot 1)(v - v_0) + \text{higher order terms} && (10)\\
&\approx [(2u_0 v_0)u] + [(u_0^2)v] + [(u_0^2 v_0 - 2u_0^2 v_0 - u_0^2 v_0)] && (11)\\
&= [(2u_0 v_0)u] + [(u_0^2)v] - [(2u_0^2 v_0)] && (12)
\end{aligned}$$

and

$$\begin{aligned}
f_2^L(u, v, t) &= f(u_0, v_0, t_0) + \left.\frac{\partial f}{\partial u}\right|_{u_0,v_0,t_0}(u - u_0) + \left.\frac{\partial f}{\partial v}\right|_{u_0,v_0,t_0}(v - v_0) + \text{higher order terms} && (13)\\
&= u_0 v_0^2 + (1 \cdot v_0^2)(u - u_0) + (2u_0 v_0)(v - v_0) + +\text{higher order terms} && (14)\\
&\approx [(v_0^2)u] + [(2u_0 v_0)v] + [(u_0 v_0^2 - 2u_0 v_0^2 - u_0 v_0^2)] && (15)\\
&= [(v_0^2)u] + [(2u_0 v_0)v] - [(2u_0 v_0^2)] && (16)
\end{aligned}$$

where u_0 and v_0 are known constants. Then the linearized trapezoidal scheme can be written as:

$$\begin{aligned}
\frac{u_{n+1} - u_n}{\delta t} &= \frac{{f_1^L}^n + {f_1^L}^{n+1}}{2} && (17)\\
&= (2u_n v_n)\frac{(u_n + u_{n+1})}{2} + u_n^2 \frac{(v_n + v_{n+1})}{2} - 2u_n^2 v_n && (18)
\end{aligned}$$

and

$$\begin{aligned}
\frac{v_{n+1} - v_n}{\delta t} &= \frac{{f_2^L}^n + {f_2^L}^{n+1}}{2} && (19)\\
&= v_n^2 \frac{(u_n + u_{n+1})}{2} + (2u_n v_n)\frac{(v_n + v_{n+1})}{2} + -2u_n v_n^2 && (20)
\end{aligned}$$

where u_n and v_n are known values of u and v at time level t_n. If you notice closely the above two equations are linear in u^{n+1} and v^{n+1}, respectively, which are the unknowns we are trying to find.