Numerical Solution of Ordinary Differential Equations
(MTH 452/552)
Some practice problems for the final

following problems are not to be turned in but are relevant for the final. Additional practice problems are given by the midterm, the midterm practice problems, and the homework problems.

1. For both parts of this problem you may use a calculator but not Matlab.

   a) Perform two steps of the Newton method for the equation $f(s) = 2 - s^2 = 0$. Start with $s^{(0)} = 1$. Observe that this gives a method to compute $\sqrt{2}$ that requires only additions, multiplications, and divisions. How many correct digits does $s^{(2)}$ have?

   b) Consider the non-linear system of equations

$$\begin{aligned} x_1^2 + x_1 x_2 &= 6 \\ x_1 x_2^2 + x_2^3 &= 3 \end{aligned}$$

   Rewrite the system in the form $F(x_1, x_2) = 0$ and perform one step of Newton's method with starting value $x^{(0)} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

2. **a)** For $s \in \mathbf{R}$ Find the solution of the IVP

$$w'' - w = 0, \qquad w(0) = 0, \;\; w'(0) = s.$$

   b) Perform by hand one step of the shooting method for the boundary value problem (BVP)

$$w'' - w = 0, \qquad w(0) = 0, \;\; w(1) = 1.$$

   Use the starting values $w(0) = 0, w'(0) = 1$. How accurate is the answer? What happens if the starting value for $w'(0)$ is changed, say $w'(0) = s$ for some $s \in \mathbf{R}$?

3. Consider the method

$$U^{n+1} = U^n + \Delta t\, f\left(t_n + \frac{\Delta t}{2},\, U^n + \frac{\Delta t}{2} f(t_n, U^n)\right) \tag{1}$$

   and the implicit Euler method

$$U^{n+1} = U^n + \Delta t\, f(t_{n+1}, U^{n+1}). \tag{2}$$

   a) For both methods determine the stability function and whether or not the method is A-stable and/or isometry preserving.

b) Which of the two methods can be expected to perform better on the problem $u' = 1+u^2$, $u(0) = 0$, $0 \le t \le 1$? Justify your answer with an appropriate mathematical analysis of the methods.

c) Which of the two methods can be expected to perform better on the problem

$$u' = \begin{bmatrix} -2001 & 1001 \\ -2002 & 1002 \end{bmatrix} u, \quad u(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad 0 \le t \le 1?$$

Justify your answer with an appropriate mathematical analysis of the methods.

4. At the bottom of page 126 our textbook states that consistency of a Runge-Kutta method requires the condition $\sum_j b_j = 1$ as well as the additional conditions $\sum_j a_{ij} = c_i$, $i = 1, \ldots, r$. Investigate whether or not the additional conditions are indeed necessary for consistency. Either prove that they are necessary or give an example of a consistent RK method that fails to satisfy at least one of these conditions.