Example: $$u' = \underbrace{\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{A} u \iff \begin{aligned} u_1' &= u_1 + 2u_2 \\ u_2' &= u_2 \\ u_3' &= 2u_3 \end{aligned}$$

Eigenvalues: $\lambda_1 = 1,\ \sigma_1 = 2$
$\lambda_2 = 2,\ \sigma_2 = 1$

Eigenvectors: $\lambda_1 = 1$: $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2 = 2$: $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$u_1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t, \quad u_2(t) = \begin{bmatrix} a_1 + b_1 t \\ a_2 + b_2 t \\ a_3 + b_3 t \end{bmatrix} e^t$$

$$u_3(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

Quick comment: If $Av = \lambda v$ and $u(t) = ve^{\lambda t}$, then
$u'(t) = \underbrace{v\lambda}_{Av} e^{\lambda t} = A(ve^{\lambda t}) = Au$, so u solves
$u' = Au$.

$$u_2' = \begin{bmatrix} a_1 + b_1 t \\ a_2 + b_2 t \\ a_3 + b_3 t \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \\ b_2 \end{bmatrix} e^t = \begin{bmatrix} (a_1 + b_1) + b_1 t \\ (a_2 + b_2) + b_2 t \\ (a_3 + b_3) + b_3 t \end{bmatrix} e^t$$

$$A u_2 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} a_1 + b_1 t \\ a_2 + b_2 t \\ a_3 + b_3 t \end{bmatrix} e^t$$

$$= \begin{bmatrix} (a_1 + 2a_2) + (b_1 + 2b_2) t \\ a_2 + b_2 t \\ 2a_3 + 2b_3 t \end{bmatrix} e^t$$

For $u_2' = Au_2$ we need

$$\begin{aligned} a_1 + b_1 &= a_1 + 2a_2 \\ a_2 + b_2 &= a_2 \\ a_3 + b_3 &= 2a_3 \end{aligned} \quad \Bigg| \quad \begin{aligned} b_1 &= b_1 + 2b_2 \\ b_2 &= b_2 \\ b_3 &= 2b_3 \end{aligned}$$

$$\Rightarrow \begin{aligned} b_1 &= 2a_2 \\ b_2 &= 0 \\ b_3 &= 0 \end{aligned} \quad \Bigg| \Rightarrow a_3 = 2a_3 \Rightarrow a_3 = 0.$$

Let a_1, a_2 be free:

$b_1 = 2a_2, \quad a_3 = b_2 = b_3 = 0.$

$$u(t) = \begin{bmatrix} a_1 + 2a_2 t \\ a_2 \\ 0 \end{bmatrix} e^t$$

$$= \left(\begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2a_2 t \\ a_2 \\ 0 \end{bmatrix} \right) e^t$$

$$= \left(a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 2t \\ 1 \\ 0 \end{bmatrix} \right) e^t$$

$$= a_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t}_{u_1(t)} + a_2 \underbrace{\begin{bmatrix} 2t \\ 1 \\ 0 \end{bmatrix} e^t}_{u_2(t)}$$

$\Rightarrow$ A fundamental system for $u' = Au$ is given by

$$u_1(t) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t, \quad u_2(t) = \begin{bmatrix} 2t \\ 1 \\ 0 \end{bmatrix} e^t, \quad u_3(t) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}.$$

Ex: $u_1' = u_1 + u_2 + 1 \qquad u_1(0) = 2$

$u_2' = -u_1 + u_2 + t, \qquad u_2(0) = \frac{1}{2}$

$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad \varphi(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$

$u' = Au + \varphi(t)$

Eigenvalues: $\det(A - \lambda I) = \det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix}$

$= (1-\lambda)^2 + 1 = 0$

$\Rightarrow (1-\lambda)^2 = -1 \Rightarrow 1-\lambda = \pm i$

$\Rightarrow \lambda = 1 \pm i$

$\lambda_1 = 1 + i, \quad v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

Check: $Av_1 = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1+i \\ -1+i \end{bmatrix}$

$= (1+i)\begin{bmatrix} 1 \\ \frac{-1+i}{1+i} \end{bmatrix} = (1+i)\begin{bmatrix} 1 \\ i \end{bmatrix} = \lambda_1 v_1$

$\frac{-1+i}{1+i} = \frac{(-1+i)(1-i)}{(1+i)(1-i)} = \frac{-1+i+i-i^2}{1^2 - i^2} = \frac{2i}{2} = i$

Similarly: $\lambda_2 = 1 - i, \; v_2 = \overline{v_1} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$

General solution of $u' = Au$ is given by

$u(t) = c_1 \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(1+i)t} + c_2 \begin{bmatrix} 1 \\ -i \end{bmatrix} e^{(1-i)t}$

$\psi(t) = \frac{1}{2}\begin{bmatrix} t \\ -(1+t) \end{bmatrix}$ is a particular solution.

Check: $\psi' = \frac{1}{2}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. $A\psi + \begin{bmatrix} 1 \\ t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} t \\ -(1+t) \end{bmatrix} + \begin{bmatrix} 1 \\ t \end{bmatrix} =$

$$= \frac{1}{2}\begin{bmatrix}-1\\-2t-1\end{bmatrix} + \begin{bmatrix}1\\t\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix} = \varphi' \checkmark$$

Now find c_1, c_2 such that initial values $u_1(0) = 2$, $u_2(0) = \frac{1}{2}$ are satisfied.
This gives a linear system of eqn. for c_1, c_2.
Solution: $c_1 = 1 - \frac{1}{2}i$, $c_2 = \frac{1}{2} - i$

$$u(t) = \frac{1}{2}\begin{bmatrix}t\\-(1+t)\end{bmatrix} + \left(1-\frac{1}{2}i\right)\begin{bmatrix}1\\i\end{bmatrix}e^{(1+i)t} + \left(\frac{1}{2}-i\right)\begin{bmatrix}1\\-i\end{bmatrix}e^{(1-i)t}$$