Sample Review Questions

1. Define and explain the meaning of (with proper example wherever necessary)
   (a) Ordinary differential equation (ODE)
   (b) Order of an ODE
   (c) Linear and Non-linear ODE
   (d) Homogeneous and In-homogeneous ODE
   (e) Stability of a numerical scheme
   (f) Accuracy of a numerical scheme

2. Consider the following equations:

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \tag{1}$$

$$\frac{\partial \phi}{\partial t} = \alpha\frac{\partial^2 \phi}{\partial x^2} \tag{2}$$

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = \alpha\frac{\partial^2 \phi}{\partial x^2} \tag{3}$$

   where c and α are constants. Identify the type of each pde (i.e. elliptic, parabolic, hyperbolic). Make use of proper definitions.

3. Find the modified wavenumbers for
   (a)
$$\frac{1}{6}f'_{j+1} + \frac{2}{3}f'_j + \frac{1}{6}f'_{j-1} = \frac{f_{j+1} - f_{j-1}}{2\Delta x} \tag{4}$$
   (b)
$$\frac{1}{12}f''_{j+1} + \frac{10}{12}f''_j + \frac{1}{12}f''_{j-1} = \frac{f_{j+1} - 2f_j + f_{j-1}}{\Delta^2} \tag{5}$$

4. Consider the following non-linear ordinary differential equation:

$$\frac{dy}{dt} + y(1-y) = 0; \qquad y(0) = 1/2 \tag{6}$$

   and its numerical solution by the trapezoidal method[1]
   (a) Write down the finite difference approximation to the governing equation. Derive the order of accuracy of the scheme. (Note: The order of accuracy should be given for the difference formula $y' = (y_{n+1} - y_n)/\Delta t$. .
   (b) Apply time-linearization to the non-linear terms. Derive a formula for y_{n+1} which depends on the solution at time level n and the time-step Δt.
   (c) Derive an expression for the amplification factor for the linearized scheme. Identify the constraints on the scheme for stability.

[1] Trapezoidal method applied to $y' = f(y,t)$ gives a fda: $\frac{y_{n+1}-y_n}{\Delta t} = \frac{1}{2}(f(y_{n+1}, t_{n+1}) + f(y_n, t_n))$

(d) What is the order of accuracy of the linearized scheme?

5. Consider the following parabolic PDE:

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} \tag{7}$$

Applying the Du-Fort Frankel Scheme for this equation gives the following finite-difference approximation:

$$\frac{\phi_j^{n+1} - \phi_j^{n-1}}{2\Delta t} = \frac{\alpha}{\Delta x^2}\left[\phi_{j+1}^{n} - \phi_j^{n+1} - \phi_j^{n-1} + \phi_{j-1}^{n}\right] \tag{8}$$

(a) Perform von-Neumann stability analysis of the scheme. Clearly indicate all constraints for stability. Identify (through proper mathematical arguments) whether the scheme is conditionally stable, unconditionally stable, or unconditionally unstable.
[Hint: For stability one seeks solutions of the form $\phi_j^n = \sigma^n e^{ikx_j}$].

(b) Derive the modified equation for the Du-Fort Frankel scheme. Define consistency of a scheme and identify (through proper mathematical arguments) whether the scheme is consistent or inconsistent. What are the implications?

6. Consider the following parabolic PDE:

$$\frac{\partial \phi}{\partial t} + c\frac{\partial \phi}{\partial x} = 0 \tag{9}$$

(a) Using forward-Euler and centered differencing, perform von-Neumann stability analysis of the scheme. Clearly indicate all constraints for stability. Identify through proper mathematical arguments whether the scheme is stable, unconditionally stable or unstable.

(b) Using Crank-Nicolson and centered differencing, repeat the above item

(c) If $c > 0$, use forward Euler and backward differencing (for spatial derivatives) and repeat the above.

7. Consider the equation,

$$y' + (2 - 0.01x^2 + 0.1x)y = 0; \quad y(0) = 1, \quad 0 \le x \le 10. \tag{10}$$

Using logical mathematical steps, identify if $\Delta x = 1$ will result in a stable or unstable solution over the entire x domain, when the above equation is discretized using the second-order Runge-Kutta scheme.

8. Consider $y' = -y$, $y(0) = 1$. It is proposed to use the forward Euler scheme with a time-step of 2. Will the Forward Euler scheme result in a stable solution with this time-step?

9. Find (i) the finite difference formula and (ii) the complete form of the leading order truncation term for the second derivative ($f''(x)$) at point x_j using sequential application of forward and backward differencing for the first derivatives. That is, consider the second derivative as

$$f''_j = \left.\frac{\delta^2 f}{\delta x^2}\right|_j = \left.\frac{\delta}{\delta x}\left(\frac{\delta}{\delta x} f\right)\right|_j = \left.\frac{\delta}{\delta x}\right|^{\text{BE}} \left[\left.\frac{\delta}{\delta x}\right|^{\text{FE}} [f_j]\right]. \tag{11}$$

What will be the finite difference formula if you interchange the order of forward and backward differencing?

10. Consider the following coupled algebraic equations:

$$T_1 = T_2 - 1 \tag{12}$$

$$T_2 = 2.5T_1 - 0.5 \tag{13}$$

(a) Find the exact solution for T_1 and T_2.

(b) Verify whether the system of equations satisfies Scarborough criterion for convergence of solution using iterative methods. If not, rearrange the equations such that the criterion is satisfied (clearly indicate what the criterion is).

(c) Set up an iterative solution procedure for the above system of equations using

i. Gauss-Seidel

ii. Successive Over Relaxation ($\alpha = 1.3$)

Create a table of *Iteration number,* T_1 *and* T_2 for different iterations. Find out the number of iterations to reach a converged solution within a tolerance of 10^{-2}. Clearly indicate your solution for all intermediate iterations and how you are deciding whether the solution has converged or not.