Practice Prob. #1

1. $u = y + 2x$
$v = x - 2y$
or $V = (y+2x)\hat{i} + (x-2y)\hat{j}$

a. $$\frac{DV}{Dt} = \frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\overbrace{\frac{\partial V}{\partial z}}^{0}$$

$$= 0 + (y+2x)[2\hat{i} + 1\hat{j}] + (x-2y)[1\hat{i} - 2\hat{j}]$$

$$= [2(y+2x) + (x-2y)]\hat{i} + [(y+2x) - 2(x-2y)]\hat{j}$$

$$5x\hat{i} - 5y\hat{j}$$

b. $\nabla P = \rho\frac{DV}{Dt} - \gamma\nabla h$ (neglect grav. term here).

$$\nabla P = -\rho(5x\hat{i} - 5y\hat{j})$$

c. $P = -\rho\frac{V_s^2}{2} + C$

$V_s^2 = V\cdot V$
$= (y+2x)^2 + (x-2y)^2$
$= y^2 + 4xy + 4x^2 + x^2 - 4xy + 4y^2$
$= 5x^2 + 5y^2$

$$\frac{\partial P}{\partial x} = -\frac{\rho}{2}\frac{\partial V_s^2}{\partial x}$$

$$= -\frac{\rho}{2}10x = -\rho 5x$$

$$\frac{\partial P}{\partial y} = -\frac{\rho}{2}\frac{\partial V_s^2}{\partial y} = -\frac{\rho}{2}10y = -\rho 5y$$

d. $\frac{\partial v}{\partial x} \stackrel{?}{=} \frac{\partial u}{\partial y}$: $1 \stackrel{?}{=} 1$ yes.

e. $u = \frac{\partial\psi}{\partial y} = y + 2x$ $\quad v = -\frac{\partial\psi}{\partial x} = x - 2y$

$\psi = \frac{y^2}{2} + 2xy + f(x)$ & $\psi = -\frac{x^2}{2} + 2xy + f(y)$

soln. to both: $\psi = \frac{-x^2}{2} + 2xy + \frac{y^2}{2} + C$

f. $\frac{\partial u}{\partial x} \stackrel{?}{=} -\frac{\partial v}{\partial y}$: $2 = -(-2)$ yes.

2. $\phi = y + x^2 - y^2$

a. $u = -\frac{\partial \phi}{\partial x} = -2x = \frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \phi}{\partial y} = 2y - 1 = -\frac{\partial \psi}{\partial x}$

integrate both:

$\psi = -2xy + f(x)$

$\psi = -2xy + f(y) + x$

$\Rightarrow \psi = -2xy + x + C$

b. irrotational?

$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(2y-1) - \frac{\partial}{\partial y}(-2x) = 0 - 0 = 0$

(yes)

c. $u = -2x$

$v = 2y - 1$

3. $u = c\left[1 - \frac{y^2}{a^2}\right]$

(a) $\zeta_3 = -\frac{\partial u}{\partial x_2} = -\frac{\partial u}{\partial y} = -2c\, \frac{y}{a^2}$

(b) $\frac{\partial \psi}{\partial y} = u = c(1 - \frac{y^2}{a^2}) \rightarrow \psi = cy - \frac{c}{3a^2}y^3 + f(x)$

$\frac{\partial \psi}{\partial x} = v = 0$ so $f(x) = C_1$ (const.)

(c) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ or $\frac{\partial u}{\partial x} = 0 \Rightarrow$ <u>yes</u>

(d) $u = -\frac{\partial \phi}{\partial x} \rightarrow \phi = x(c - \frac{cy^2}{a^2}) + f(y)$

$v = -\frac{\partial \phi}{\partial y} = 0 \rightarrow \phi = f(x)$ so $f(y) = C_2$

but both eqns. can't be satisfied so ϕ doesn't exist

4. $\nabla^2 \psi \overset{?}{=} 0$ irrotational flow: $\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = 0$

$u_2 = -\frac{\partial \psi}{\partial x_1} \quad u_1 = \frac{\partial \psi}{\partial x_2}$

so $\frac{\partial}{\partial x_1}\left(-\frac{\partial \psi}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial \psi}{\partial x_2}\right) = 0$

$-\left(\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2}\right) = 0$ #

(linear eqn.)

5. $\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = 0$; so $+\frac{\partial}{\partial x_1}\left(\frac{\partial \phi}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial \phi}{\partial x_2}\right) = 0$ #

if also irrotational $\Big\}$ $\frac{\partial}{\partial x_1}\left(\frac{\partial \phi}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial \phi}{\partial x_1}\right) \overset{?}{=} 0$ this is always true

6. $\rho \frac{Du_i}{Dt} = -\frac{\partial}{\partial x_i}(P + \gamma h) + \mu \frac{\partial u_i}{\partial x_j \partial x_j} = \rho \frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$

7. $a_k = b c_i d_{ki}$ (sum over i, left with k as free index so valid). b = scalar

$a_x = b(c_x d_{xx} + c_y d_{xy} + c_z d_{xz})$ (here $k = x$ component and sum over i)

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8. $V = 3y\,i + 2x\,j$ so $u = 3y$

$v = 2x$

(i) $\frac{\partial \psi}{\partial y} = 3y$

$-\frac{\partial \psi}{\partial x} = 2x$ } integrate each and find $\psi = \frac{3}{2}y^2 - x^2 + C$ (scalar)

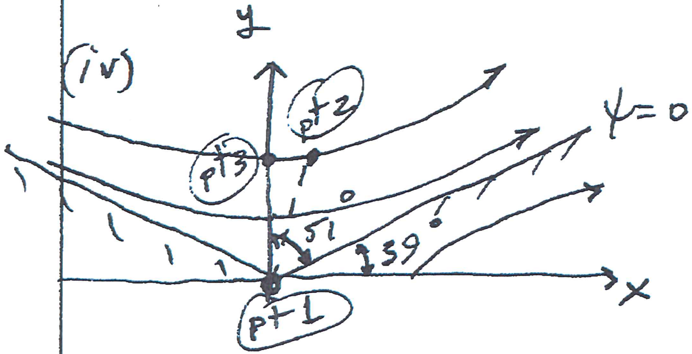
(ii) $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \stackrel{?}{=} 0 = 0 + 0$ yes

(iii) $\psi(x=0, y=0) = 0$ so set $C = 0$

at $x = 1$ $y = 2$: $\psi = \frac{3}{2}(2^2) - 1^2$

$= 5$

(iv)



note that this flow could be for a deflection of a wall 39° from horizontal as in a "corner flow"

(v) pick pts. (0,0) and (1,2) to find $\dot{v}$

$\dot{v} = \Delta\psi = 5 - 0 = 5$ (units based on units of V given)

Note that this is same $\dot{v}$ between pts ① → ③ shown above → can you find pt. 3 (same ψ as pt. 2 but $x = 0$).

(vi) $\frac{\partial \phi}{\partial x} = -u = -3y$ $\phi = -3yx + f(y)$ } can't be

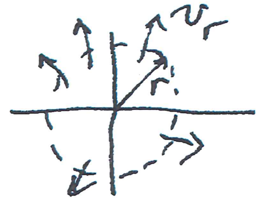
$\frac{\partial \phi}{\partial y} = -v = -2x$ $\phi = -2xy + f(x)$ } satisfied

note: $\zeta_3 = -1$ so rotational !!

9. Source strength $= Q\ (m^3/s\text{-}m) =$ constant

(i) at some value r the flow rate through a circle must be $Q = 2\pi r v_r$

where v_r is the radial velocity.



so $v_r = \dfrac{Q}{2\pi r} = \dfrac{c}{r}$ #

Streamfunction lines:

$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{c}{r} \quad \text{or} \quad \frac{\partial \psi}{\partial \theta} = c$$

so $\psi = c\theta$

→ radial lines along constant value of θ.

10. if $\psi = \psi_1 + \psi_2$ show $u = u_1 + u_2$
$v = v_1 + v_2$

pretty simple to say:

$$u = \frac{\partial \psi}{\partial y} \; ; \quad \frac{\partial \psi_1}{\partial y} = u_1 \;\; \& \;\; \frac{\partial \psi_2}{\partial y} = u_2$$

take $\frac{\partial}{\partial y}$ of ψ eqn. above:

$$\frac{\partial \psi}{\partial y} = \frac{\partial \psi_1}{\partial y} + \frac{\partial \psi_2}{\partial y}$$

$$u = u_1 + u_2 \quad \#$$

Same for v.