Assignment #5

1. Jacobi, Gauss Seidel and SOR solutions.

N=11	Jacobi	GS	SOR
Iterations	436	272	100
Abs. error	0.0061	0.0028	1.35×10^{-7}
% err	2.45	1.12	$5.41 \times 10^{-}5$

N=21	Jacobi	GS	SOR
Iterations	1122	734	132
Abs. error	0.022	0.0107	.001
% err	8.91	4.27	0.42

2. Explicit Euler:

$$T_j^{(n+1)} = T_j^{(n)} - \frac{\gamma}{2}\left(T_{j+1}^{(n)} - T_{j-1}^{(n)}\right) \tag{1}$$

$$T_0^{(n+1)} = 0; \qquad T_N^{(n+1)} = T_N^{(n)} - \gamma\left(T_N^{(n)} - T_{N-1}^{(n)}\right) \tag{2}$$

Leap Frog:

first step (RK2)

$$T_0^{*} = 0; \qquad T_N^{*} = T_N^{(0)} - \frac{\gamma}{2}\left(T_N^{(0)} - T_{N-1}^{(0)}\right) \tag{3}$$

$$T_j^{*} = T_j^{(0)} - \frac{\gamma}{4}\left(T_{j+1}^{(0)} - T_{j-1}^{(0)}\right) \tag{4}$$

$$T_0^{(1)} = 0; \qquad T_N^{(1)} = T_N^{(0)} - \gamma\left(T_N^{*} - T_{N-1}^{*}\right) \tag{5}$$

$$T_j^{(1)} = T_j^{(0)} - \frac{\gamma}{2}\left(T_{j+1}^{*} - T_{j-1}^{*}\right) \tag{6}$$

$n \geq 1$

$$T_0^{*} = 0; \qquad T_N^{*} = T_N^{(0)} - \frac{\gamma}{2}\left(T_N^{(0)} - T_{N-1}^{(0)}\right) \tag{7}$$

$$T_j^{*} = T_j^{(0)} - \frac{\gamma}{4}\left(T_{j+1}^{(0)} - T_{j-1}^{(0)}\right) \tag{8}$$

$$T_0^{(n+1)} = 0; \qquad T_N^{(n+1)} = T_N^{(n-1)} - 2\gamma\left(T_N^{(n)} - T_{N-1}^{(n)}\right) \tag{9}$$

$$T_j^{(n+1)} = T_j^{(n-1)} - \gamma\left(T_{j+1}^{(n)} - T_{j-1}^{(n)}\right) \tag{10}$$

Here $\gamma = \frac{u\Delta t}{\Delta x}$ and $j = 1, ..., N-1$. Euler is unstable while Leapfrog is stable for $\gamma \leq 1$.

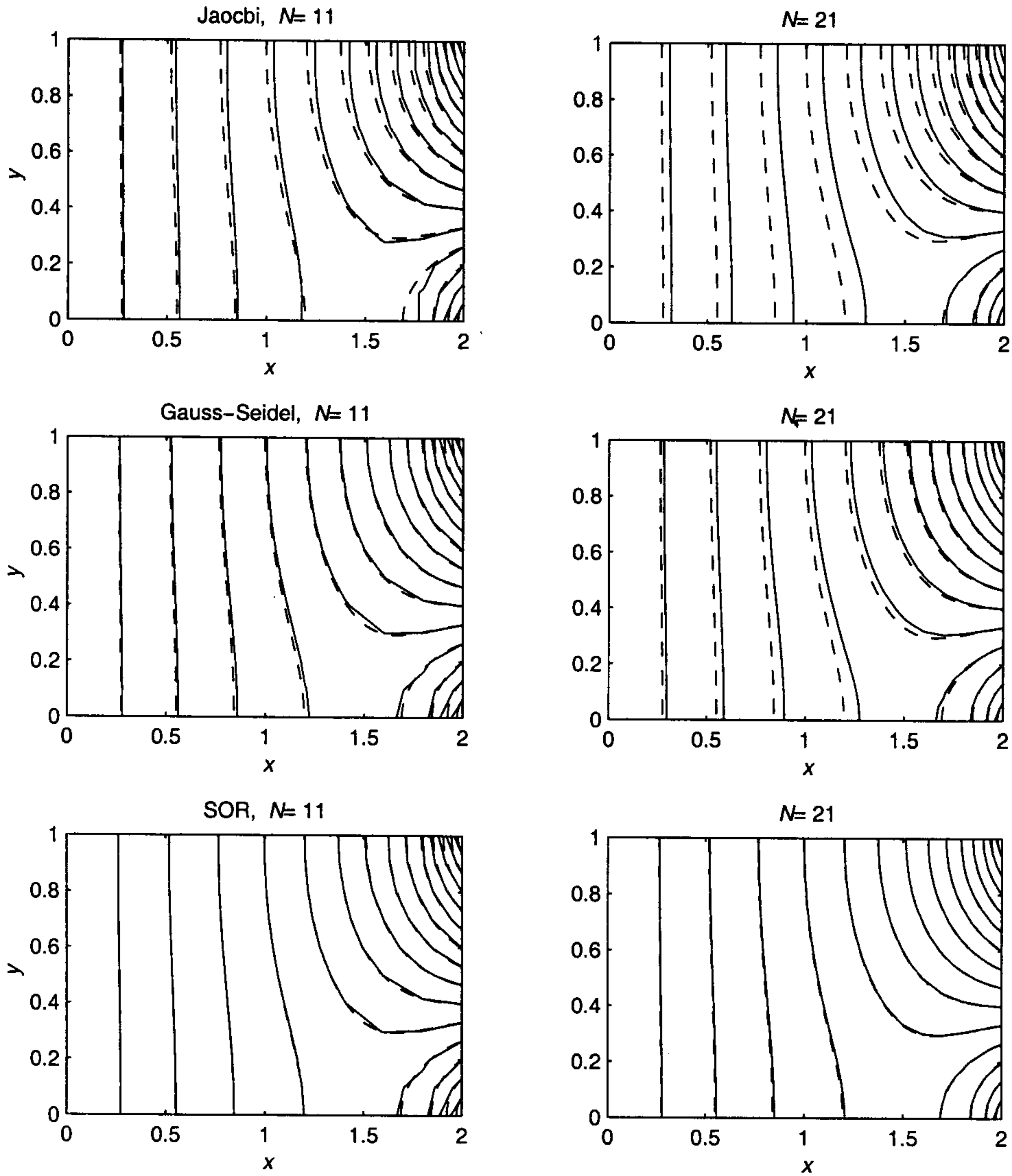

Figure 1: – – – – is exact, ——— is numerical

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$\frac{\Delta t_{max}}{\Delta x} = \frac{1}{|u_{max}} = 5.$

Lax Wendroff is second-order in time and space. Stable for $\gamma \leq 1$. Accuracy can be shown by modified equations

Modified equation for LW:

$$\begin{aligned}
\phi_t + u\phi_x &= \frac{\Delta t}{2}\left(u^2\phi_{xx} - \phi_{tt}\right) - u\frac{\Delta x^2}{6}\phi_{xxx} - \frac{\Delta t^2}{6}\phi_{ttt} + \mathcal{O}(\Delta t^3, \Delta t\Delta x^2, \Delta x^4, ...) \quad (11)\\
&= \frac{u}{6}(u^2\Delta t^2 - \Delta x^2)\phi_{xxx} + \mathcal{O}(\Delta t^3, \Delta t\Delta x^2, \Delta x^4, ...) \quad (12)
\end{aligned}$$

The last RHS was obtained by using the equation itself and replacing ϕ_{tt} in terms of ϕ_{xx} etc..

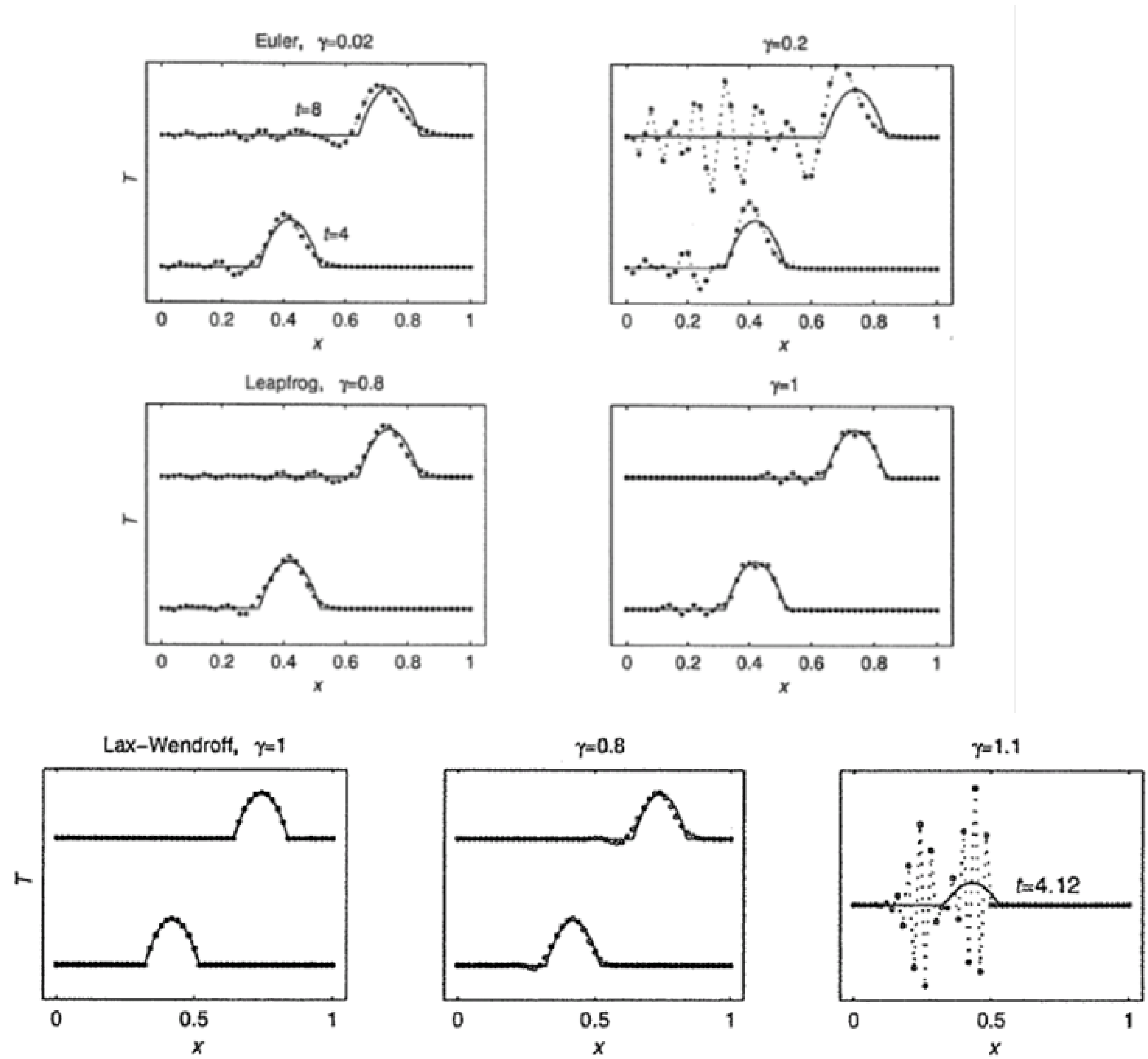

Figure 2: Pure advection: Euler, Leap Frog and Lax Wendroff.

3. Uniform discretization of the domain: $x_i = i/M$, $i = 0, 1, 2, ...M$ and $y_j = j/N$, $j = 0, 1, 2, ..., N$. Explicit Euler for time and Central for space.

   For interior points: $i = 1, 2, 3, ..., M-1$ and $j = 1, 2, 3, ..., N-1$.

$$\begin{aligned} u_{i,j}^{(n+1)} &= [1 - h(u_{i,j}^n \delta_x + v_{i,j}^n \delta_y) + \nu h(\delta_{xx} + \delta_{yy})] u_{i,j}^{(n)} && (13) \\ v_{i,j}^{(n+1)} &= [1 - h(u_{i,j}^n \delta_x + v_{i,j}^n \delta_y) + \nu h(\delta_{xx} + \delta_{yy})] v_{i,j}^{(n)} && (14) \end{aligned}$$

   for $i = 0, ..., M$:

$$u_{i,0} = u_{i,N} = \sin(2\pi x_i), v_{i,0} = 1, v_{i,N} = 0. \tag{15}$$

   for $j = 0, ..., N$:

$$u_{0,j} = u_{M,j} = 0, v_{0,j} = v_{M,j} = 1 - y_j. \tag{16}$$

   Here, δ_x, δ_y, δ_{xx}, and δ_{yy} are central difference operators in space.

   Use von-Neumann stability for maximum time-step. Assume $\sigma^n e^{ik_1 x_i} e^{ik_2 y_j}$ as solution. Take u and v in the non-linear terms as constants and equal to $\bar{u}$ and $\bar{v}$, respectively:

$$\sigma = 1 - ih(\bar{u}k_1' + \bar{v}k_2') - \nu h(k_1^2)' - \nu h(k_2^2)' \tag{17}$$

   where

$$(k_1^2)' = 2\frac{1 - \cos(k_1 \Delta_1)}{\Delta_1^2}, \quad (k_2^2)' = 2\frac{1 - \cos(k_2 \Delta_2)}{\Delta_2^2} \tag{18}$$

   and

$$(k_1)' = \frac{\sin(k_1 \Delta_1)}{\Delta_1}, \quad (k_2)' = \frac{\sin(k_2 \Delta_2)}{\Delta_2} \tag{19}$$

   For $\bar{u}$ and $\bar{v} = 1$, say, one can obtain maximum stable h.

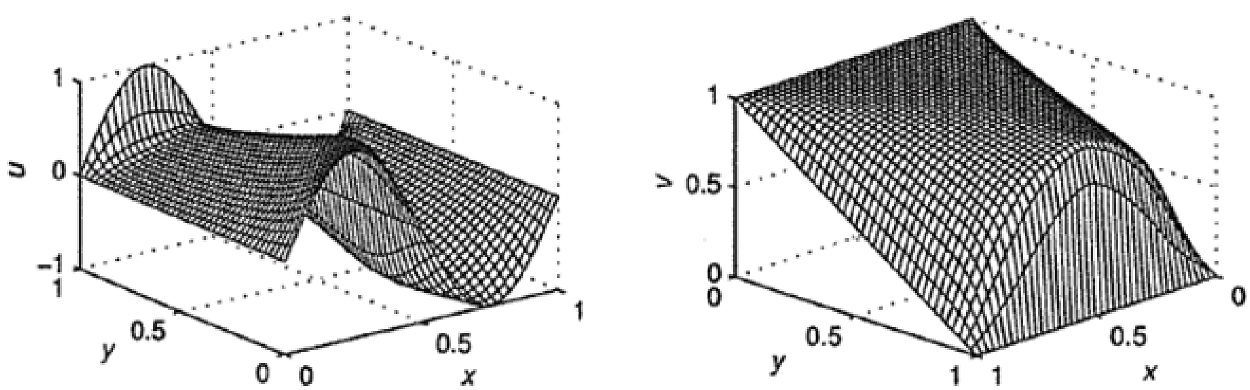

Figure 3: Nonlinear Burger's equation in 2D