Mathematical Methods for Engineers

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Chapter 2

Differential Equations

Equations containing derivatives are called ‘differential equations.’ Differential equations are classified as (i) Ordinary Differential Equation (ODE), or (ii) Partial Differential Equation (PDE). This distinction is based on how many independent variables there are in a given problem. For example, Let ϕ be the unknown function for which we are writing a differential equation. Here ϕ is a dependent variable. Typically, in most engineering problems we will use (x, y, z, t) as the independent variables. Here (x, y, z) are the space co-ordinates and t is time. If, for a particular problem, ϕ depends on only one other variable, say t, then one can write derivatives of ϕ only with respect to t, giving rise to ordinary derivative and an ordinary differential equation. If ϕ depends on two or more independent variables, say $\phi(x, t)$, then one can write its derivatives with either x or t, keeping the other variable constant. This gives rise to a partial derivative and subsequently a partial differential equation. Whether it is an ODE or a PDE, the *order* of the differential equation is the order of the highest derivative that appears in the equation.

It is possible to have more than one dependent variable in a given problem. For example, in order to model the trajectory of a baseball, one needs to track the position of the baseball's center (x_c, y_c, z_c), its linear velocity components (u_c, v_c, w_c), and its angular velocity components $(\omega_{x,c}, \omega_{y,c}, \omega_{z,c})$. These nine variables are functions of time (t), but their solution is also linked to each other. In this case, we have nine dependent variables and one independent variable. By using laws of physics, one can write nine ordinary differential equations, one for each dependent variable, solutions of which are interdependent. This is called a *system of ordinary differential equations.* Similarly, one can also have multiple dependent and multiple independent variables in a given problem. This will result in a *system of coupled partial differential equations.* We will look at examples of each. A single higher-order differential equation, can be converted to a system of coupled, first-order differential equations, but introducing additional *dependent* variables. Likewise, a system of n coupled differential equations with n different dependent variables, can be converted to an n^{th}-order differential equation of a single variable by eliminating the remaining dependent variables through substitution.

To solve differential equations for a specific problem, we also need some conditions specified. Depending upon what types of conditions are specified, one gets (i) an initial value problem, or a (ii) boundary value problem. An initial value problem is the one where conditions for all dependent variables are specified at a same single value of the independent variable. For example, if time, t, is the independent variable, and we specify all conditions for the nine dependent variables in the baseball problem at $t = 0$, then this is an *initial value problem.* If for a problem, the conditions are specified for at least two different locations of the independent variable, then it is called a *boundary value problem.* For example, consider a long, slender steel rod, a skewer, used to heat up marshmallows. Then, at one of the rod, we have an insulating material (such as a wooden handle), and at the other end the marshmallows are heated up by raising their temperature. Thus, two different conditions are specified at two different locations along the rod, making this a boundary value problem of heat transfer.

2.1 Examples of Differential Equations

2.1.1 Free Fall

Consider a spherical ball falling freely under gravity (see figure 2.1). Motion of the ball in the y direction is governed by Newton's law, $ma = \sum F$, where m is the mass of the ball a is acceleration in the vertical direction and $\sum F$ is the summation of all forces acting on the ball in the vertical direction.

The acceleration of the ball is given as $a = \frac{dv}{dt}$, where v is velocity in the vertical direction, t is time, and v is a function of time. Here $g = 9.8 m/s^2$ is the gravitational acceleration. Then, $m\frac{dv}{dt} = F_D - mg$. The drag force on the ball depends on the size of the ball, its velocity, and fluid properties such as viscosity. We need a "model" to represent the drag force. Based on Fluid Mechanics, this drag force can be modeled as $F_D = \alpha v$; where α depends on the drag coefficient C_D of the ball. (α has units of kg/s).

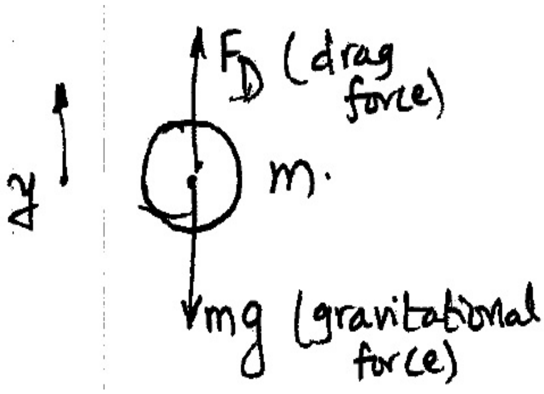


Figure 2.1: Freely falling ball.

$$m\frac{dv}{dt} = \alpha v - mg. \tag{2.1.1}$$

Here, m and α are the parameters of this problem. Then,

$$\frac{dv}{dt} = \frac{\alpha}{m}v - g \tag{2.1.2}$$

The above equation has one independent variable (t) and one dependent variable (v), two parameters (α and m) and one constant (g). In addition, we need to specify what is the initial velocity of the ball. Let us assume that the ball starts from rest, that is it is released with zero initial velocity ($v = 0$ at $t = 0$). This is an 'ordinary differential equation'. This is also an 'initial value problem' (IVP) because we only need the initial conditions to find $v(t)$. The independent variable t is called a *one-way variable*, as the information of the initial condition only propagates in one direction, forward in time. By the nature of the problem, the future solution depends on what happened in the past, but the past solution is not going to be affected by what is going to happen in the future.

2.1.2 Unsteady Heat Conduction

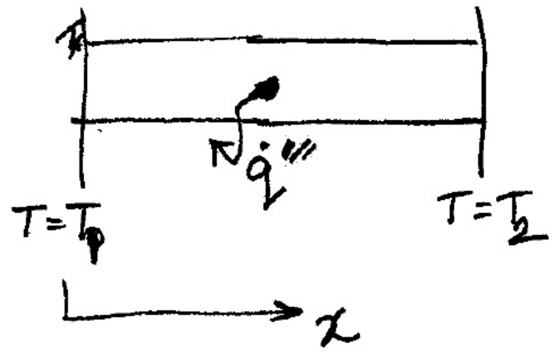


Figure 2.2: Unsteady heat conduction.

Consider a long, thin, cylindrical, steel rod (figure 2.2). Let us assume that, initially ($t = 0$), the rod is at uniform temperature equal to the surrounding temperature ($T = T_{atm}$ at $t = 0$ and at all x locations). At $t = 0$, one end of the rod ($x = 0$) is heated to a temperature of $T_1 = 100°$C and the other end ($x = L = 1$ m) is cooled to $T_2 = 0°$C. Because of the two different temperatures, heat is going to be conducted through the metal rod of conductivity, k. The rate of heat transfer is also going to depend on the property of the steel rod, such as density ρ, specific heat c_p. In addition, heat can also be lost to the surroundings through convection and radiation. Let us say the heat lost to the surroundings is given by $\dot{q}'''$, which will be function of the surrounding temperature, the heat transfer coefficients, among others. The goal is to find how the temperature of the rod changes with space (x) and time (t). This is given by the following partial differential equation for $T(x,t)$,

$$\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + \dot{q}''', \tag{2.1.3}$$

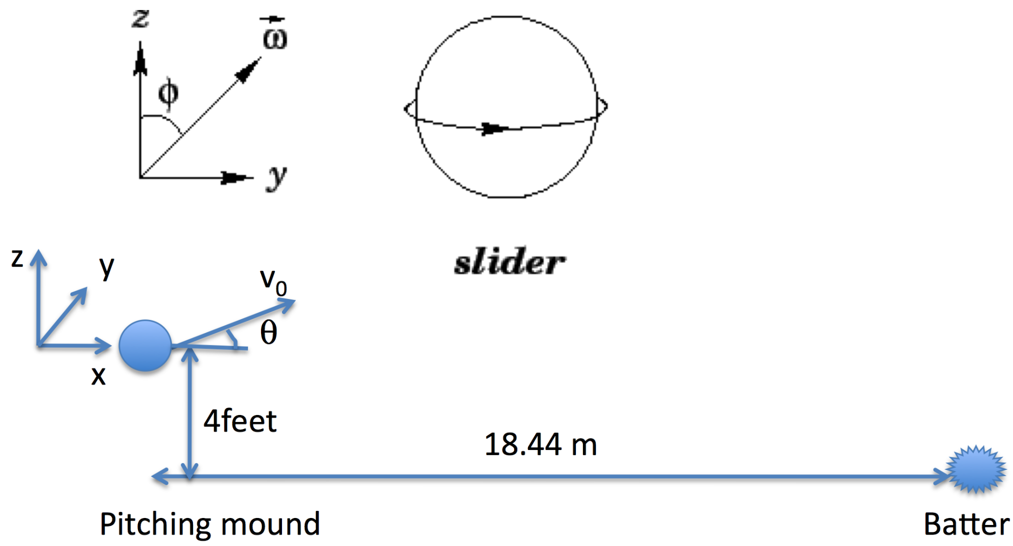
with the conditions

$$\begin{aligned} T(x,0) &= T_{atm} \text{ (initial condition)} && (2.1.4)\\ T(0,t>0) &= T_1 \text{ (boundary condition)} && (2.1.5)\\ T(L,t>0) &= T_2 \text{ (boundary condition)} && (2.1.6) \end{aligned}$$

Notice that, the PDE is first-order in time and second-order in space. Again, time here behaves as a one-way co-ordinate. So the information of the temperature conditions set at $t = 0$ propagates forward in time. This is an initial value problem in time. But, for space x, two different conditions are given for $x = 0$ and $x = L$. This is a boundary value problem in x. Information related to the end temperatures of the rod is going to propagate from both ends of the rod inward. Here, space x is a two-way co-ordinate.

2.1.3 Dynamics of a Baseball

Baseball dynamics depends on the ball's seam, the air drag, the initial rotational and linear speeds and the orientation of the ball. Consider CC Sabathia (a left-handed pitcher) pitching a 'slider' to Kevin Youkilis (right-handed batter) in ALCS (see attached figure). Assuming that x measures displacement from the pitcher to the batter, z measures vertical displacement of the ball ($+z$ is upward displacement direction (against gravity)), and y measures horizontal displacement ($+y$) corresponds to the displacement to the hitter's right-hand side), v_0 is the initial speed of the ball, θ is the angle made by the ball's trajectory with the x axis (see figure), $\vec{\omega}$ is the angular velocity vector written as $\vec{\omega} = \omega(0, \sin(\phi), \cos(\phi))$, where ϕ is the angle made by the angular velocity vector with the vertical axis (z). For a slider, $\phi = 0$. Let the distance between the pitcher and the batter be 18.44 m. The initial rotational spin is $|\omega| = 1800$ rpm with $\phi = 0°$. Since CC is a left handed pitcher, the spin will be in the reverse direction, and so use $\omega = -1800$ rpm. The trajectory and the speed of the baseball is governed by the following system of equations:



$$\frac{dx}{dt} = v_x \tag{2.1.7}$$

$$\frac{dy}{dt} = v_y \tag{2.1.8}$$

$$\frac{dz}{dt} = v_z \tag{2.1.9}$$

$$\frac{dv_x}{dt} = -F(v)vv_x + B\omega(v_z\sin(\phi) - v_y\cos(\phi)) \tag{2.1.10}$$

$$\frac{dv_y}{dt} = -F(v)vv_y + B\omega v_x\cos(\phi) \tag{2.1.11}$$

$$\frac{dv_z}{dt} = -g - F(v)vv_z + B\omega v_x\sin(\phi) \tag{2.1.12}$$

where v_x, v_y, and v_z are the velocity components in the x, y, and z directions, respectively. Also, v corresponds to the speed of the ball $v = \sqrt{v_x^2 + v_y^2 + v_z^2}$, $g = 9.81$ is the gravitational acceleration, and $F(v)$ is the drag force experienced by the baseball which is given as,

$$F(v) = 0.0039 + \frac{0.0058}{1 + \exp[(v - 35)/5]}, \tag{2.1.13}$$

in SI units. B in the above expression results from a force called **Magnus Force**, a force due to rotation of the ball in a shearing flow that causes the ball to curve during its forward flight. Use $B = 4.1 \times 10^{-4}$ (note that B is dimensionless quantity).

Let the following be the initial conditions for the above system of equations:

$$x(t = 0) = 0 \tag{2.1.14}$$

$$y(t = 0) = 0 \tag{2.1.15}$$

$$z(t = 0) = 0 \tag{2.1.16}$$

$$v_x(t = 0) = v_0\cos(\theta) \tag{2.1.17}$$

$$v_y(t = 0) = 0 \tag{2.1.18}$$

$$v_z(t = 0) = v_0\sin(\theta), \tag{2.1.19}$$

where $v_0 = 85$ mph is the initial speed, and $\theta = 1°$.

This is a complex system of coupled ODEs, with time, t, as the independent variable, and six dependent variables x, y, z, v_x, v_y, v_z. It is an initial value problem and any initial conditions for the ball affect its trajectory in the future. This is a realistic conceptual model for the baseball trajectory and if solved, can yield sufficiently accurate predictions of its trajectory.

2.2 Incompressible Fluid Flow

Direct numerical simulation is a technique used by researchers to understand problems involving fluid flow problems. For example, turbulent flow in pipe bends, flow over a micro-air vehicle or a drone, reacting flow inside a gas-turbine engine, among others. Below, are the equations for an incompressible flow (constant density), a coupled system of four partial differential equations for the four unknowns: the three velocity components $u(x, y, z, t)$, $v(w, x, y, z, t)$, $w(x, y, z, t)$, and pressure $p(x, y, z, t)$. These equations are basically conservation laws (of mass and momentum), written for a general three-dimensional problem in Cartesian coordinates.

The continuity, conservation of mass, equation is given as,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{2.2.1}$$

The three momentum equations are,

$$\rho \left\{ \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right\} = -\frac{\partial p}{\partial x} + \mu \left\{ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right\} + \rho g_x \tag{2.2.2}$$

$$\rho \left\{ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right\} = -\frac{\partial p}{\partial y} + \mu \left\{ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right\} + \rho g_y \tag{2.2.3}$$

$$\rho \left\{ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right\} = -\frac{\partial p}{\partial z} + \mu \left\{ \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right\} + \rho g_z. \tag{2.2.4}$$

The momentum equations above are second-order differential equations. They are non-homogeneous, due to the pressure and the gravitational acceleration term. In addition, these are non-linear equations because of the advection terms involving product of velocity and derivatives of velocity.

2.3 Linear and Non-linear Equations

The differential equations can be classified as linear or non-linear differential equations. If each term in a differential equation appears as a 'linear' function of the *dependent variable* and it's derivatives, then the differential equation is 'linear'. Else it is a non-linear equation.

Consider the following general equation,

$$a_0(t)\frac{d^n y}{dt^n} + a_1(t)\frac{d^{n-1}y}{dt^{n-1}} + ... + a_n(t)y = g(t). \tag{2.3.1}$$

Here the superscripts n, $n-1$ etc. denote the n^{th} and $(n-1)^{\text{th}}$ derivatives, respectively. To find out whether the above equation is linear or non-linear, first identify the dependent and independent variables. Here t is the independent variable and y is the dependent variable ($y(t)$). Looking at each term in the equation, it is clear that the dependent variable y and its derivatives appear in a linear form. The coefficients $a_0, a_1, ..., a_n$ and $g(t)$ only depend on the 'independent' variable. These coefficients could then be anything; zero, constant, or linear or non-linear functions of t (e.g. $a_0(t) = t^2$). Still, the above equation will be a linear equation, as long as the coefficients only depend on the independent variable. If, however, the coefficients are also functions of the dependent variable, e.g. let $a_0(t) = yt^n$, then we have at least one term in the equation $a_0 \frac{d^n y}{dt^n} = yt^n \frac{d^n y}{dt^n}$, where we have product of y and its n^{th} derivative. This is a non-linear term and the equation would become a non-linear equation. This is not because we have t^n, but because of product of dependent variable y and its higher derivative.

Let us consider some more examples below.

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\theta = 0 \longrightarrow \text{linear} \tag{2.3.2}$$

$$\frac{d^2\theta}{dt^2} + \frac{g}{L}\sin(\theta) = 0 \longrightarrow \text{non-linear.} \tag{2.3.3}$$

The two equations above are for $\theta(t)$. The first equation is linear as each term appears as a linear function of θ. The second equation contains $\sin(\theta)$, and is a non-linear equation.

Consider the following equations for $y(t)$.

$$t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + y = \sin(t) \longrightarrow \text{linear} \tag{2.3.4}$$

$$\frac{d^3y}{dt^3} + 2e^t\frac{d^2y}{dt^2} + y\frac{dy}{dt} = t^4 \longrightarrow \text{non-linear .} \tag{2.3.5}$$

The first equation above is linear, even though we have a term containing $\sin(t)$. Since t is an independent variable, this is a linear equation. The second equation, however, is non-linear because of the product $y\frac{dy}{dt}$.

2.4 Homogeneous and Non-homogeneous Differential Equation

A homogeneous differential equation is the one in which each term of the equation involves the dependent variable or one of it's derivatives. A non-homogeneous differential equation contains additional terms, known as non-homogeneous terms, source terms or forcing functions, which do not involve the dependent variable. The non-homogeneous terms can be constants or functions of the independent variable. Consider the example of a spring mass system from mechanical vibrations

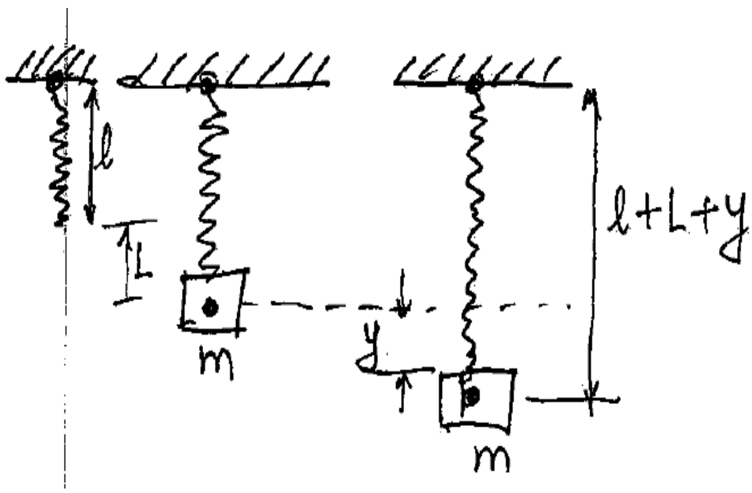


Figure 2.3: A spring mass system.

as shown in figure 2.3. A mass
of m is attached to a spring of free length ℓ. The spring stiffness is k. Due to the attached mass, the spring displaces by a distance of L. The mass is then pulled in the direction of gravity and released. We are interested in the displacement y of the center of the mass which is governed by,

$$m\frac{d^2y}{dt^2} + ky = 0 \tag{2.4.1}$$
$$mg - kL = 0 \text{ (equilibrium state)} \tag{2.4.2}$$
$$k = \frac{mg}{L} \tag{2.4.3}$$

In the above equation each term contains the dependent variable y, and hence it is a second order, linear, homogeneous ODE.

Consider the example of forced vibrations. If we apply a periodic (or any other force) to the spring-mass system, one obtains a non-homogeneous differential equation. Let the Force is $F_0cos(\omega t)$, where F_0 & ω are positive constants representing the amplitude and frequency of forcing functions, be applied to the spring. The ODE then becomes,

$$m\frac{d^2y}{dt^2} + ky = F_0\cos(\omega t). \tag{2.4.4}$$

The term on the right hand side does not contain any dependent variable and is called the non-homogeneous term or source term. The above ODE is non-homogeneous, second-order and linear.

2.5 Order Reduction of Differential Equations

Higher order differential equations can always be converted to lower order equations, specifically first-order equations, by introducing additional dependent variables and constructing a system of coupled, first order differential equations. For an n^{th}-order differential equation, we can obtain a system of n coupled, first order differential equations. After converting the higher-order equation to a system of equations, one also has to convert all the conditions for the higher-order equation to the new variables. A system of n equations should have n conditions to completely solve the problem. Consider the angular motion of a pendulum (see figure 2.4) is given by the following differential equation,

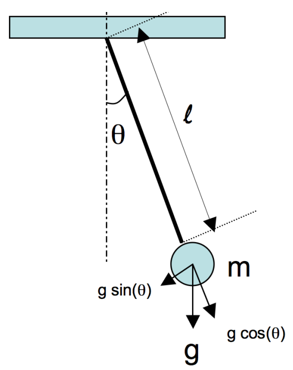


Figure 2.4: Motion of a pendulum.

$$\ddot{\theta} + \omega^2\sin(\theta) = 0, \tag{2.5.1}$$

where θ is the angle the pendulum string makes with the vertical axis, $\omega = \sqrt{g/\ell}$ where g is gravitational acceleration and ℓ is the length of the string. At $t = 0$, $\theta = \pi/6$ and $\dot{\theta} = 0$. This is a second-order, homogeneous, non-linear differential equation. One can convert that into two first-order, coupled set of differential equations. In order to convert into two equations (because the order of the original equation is 2), we need two dependent variables. One of the variables is θ as before. If we introduce a new variable, say u, which is the rate of change of the angle, θ, then we have,

$$u = \dot{\theta}. \tag{2.5.2}$$

One can find a differential equation for u, by using the original equation.

$$\ddot{\theta} = \frac{d}{dt}(\dot{\theta}) = \frac{du}{dt} \tag{2.5.3}$$

$$\therefore \frac{du}{dt} = \ddot{\theta} = -\omega^2 \sin(\theta). \tag{2.5.4}$$

Then the system of two first order differential equations in the two dependent variables θ and u is,

$$\dot{\theta} = u \tag{2.5.5}$$

$$\dot{u} = -\omega^2 \sin(\theta). \tag{2.5.6}$$

Notice that the solution of θ depends on u, and solution of u depends on θ. This is a coupled, set of first-order differential equations. Since there are two equations with two variables, we need to re-write the conditions as well. The initial conditions are

$$\theta = \frac{\pi}{6}, \quad \dot{\theta} = u = 0, \text{ at } t = 0. \tag{2.5.7}$$

The system of the first order equations would have to be solved in a coupled fashion. Similarly, a system of differential equations can be converted to a single, higher order differential equation, by eliminating some of the variables from the system. For example, if we substitute for $u = \dot{\theta}$ in equation 2.5.6, one can eliminate u and only get an equation in θ, which will be our original equation. As we will see later, there are certain numerical methods that require us to convert the higher-order equation, into a system of first-order equation to make the solution procedure easier.

2.6 Summary

Differential Equations can be classified as in several different ways. General classifications include,

1. Ordinary and partial differential equations
2. Linear and non-linear
3. Homogeneous and non-homogeneous

Being able to classify the differential equations into above categories is critical as different solution techniques/tools are used to solve different types of problems. If there are multiple dependent variables and a single independent variable, one obtains a system of coupled ordinary differential equation. If there are multiple dependent and multiple independent variables, one obtains a system of coupled partial differential equation.

To solve differential equations for a specific problem, one needs to specify certain conditions on the dependent variable. The order of the differential equation, given by the order of the highest derivative in the equation, decides how many conditions are needed to completely solve the problem. A second-order ordinary differential equation will require two conditions. Based on the location of the independent variable at which the boundary conditions are specified, the problems are classified as

1. Initial value problem
2. Boundary value problem.

By definition, a boundary value problem requires conditions given at two different locations for an independent variable. Thus, a boundary value problem has to be at least a second or higher order differential equation.

Depending upon the nature of the boundary conditions used, the independent variable can be thought of as 'one-way' or 'two-way' co-ordinate. For example, time t is a one-way co-ordinate. That means, information always propagates only in one direction of time (usually forward). For such independent variable, conditions are usually given at a single time, giving rise to an initial value problem. If conditions are specified at different locations of an independent variable, then that is a 'two-way' co-ordinate. That is, information propagates from both sides of the space co-ordinate. Space typically behaves as a two-way co-ordinate. However, in certain problems, both space and time behave as one-way co-ordinates. As we will discuss later, these give rise to a special form of partial differential equations, called hyperbolic equations.