Homework # 4

ME526/NSE526

Due: November 15

1. You are encouraged to work in a group of up to 2 students and submit one solution per student.
2. Your solution must be clearly legible. Illegible work may not be graded and returned without any points. Although not necessary, you may type your work.
3. All problems must be solved. However, all problems may not be graded. A random sample of problems will be selected for grading.
4. If you are required to write a computer program, attach your code with several comment statements on the code wherever possible.

1. Consider the undamped pendulum problem

$$y'' + \omega^2 y = 0, \quad y(0) = 1, \quad y'(0) = 0; \quad \omega = 4. \tag{1}$$

   (a) Using the Leap Frog scheme (with Forward Euler for the first time-step), compute the solution of the above problem for $0 \le t \le 6$ with $\Delta t = 0.15$. Plot your solution on the same plot as the exact solution and compare. Perform systematic time-step refinement study (for at least 3 other time-steps) and plot the error versus time-step on a log-log curve to identify the order of accuracy of the scheme.

   (b) Solve the above problem using Central Differencing scheme in time and using $\Delta t = 0.15$. Compare the solution with Leap Frog's solution and exact solution on the same plot.

   (c) Explain the above predicted solutions (Leap Frog and Central) by conducting an analysis to find the errors (magnitude and phase) in solution. You can do this using the stability analyses and looking at the amplification factors.

   (d) It is suggested to use a mixed scheme wherein we alternate between central differencing (trapezoidal) and Leap Frog. So the idea is to use *two* steps of Central Differencing followed by one step of Leap Frog Scheme and then repeat this (Note, since we are using Central Differencing first, there is no issue of starting scheme). Using this mixed method, solve the above problem with $\Delta t = 0.15$. Compare your solution with the above two schemes and also the exact solution. What is rationale behind such a scheme? Is this mixed scheme stable?

2. **Blasius Boundary Layer by Shooting Method** A laminar boundary layer on a flat plate

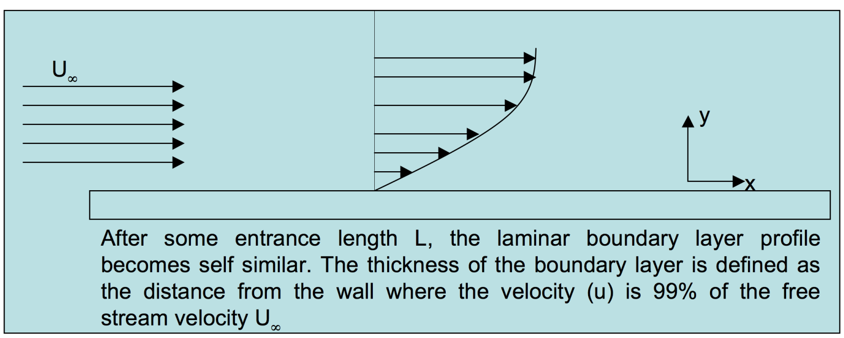

Figure 1: Schematic of the forces acting on a simple pendulum of mass m and length l.

is self-similar and is governed by

$$f''' + ff'' = 0 \tag{2}$$

where $f = f(\eta)$ and $\eta = \frac{y}{x}\sqrt{Re}$ is the similarity variable with Re being the Reynolds number at position x, $Re = Ux/\nu$. The variable f and its derivatives are proportional to fluid mechanical quantities: $f \ \alpha \ \Psi$, the stream function; $f' = u/U$, where u is the fluid velocity and U the free stream velocity; and $f'' \ \alpha \ \tau$, the shear stress. Boundary conditions for the equations are derived from the physical boundary conditions on the fluid: 'no-slip' at the wall and free stream conditions at large distances from the wall. They are summarized as:

$$f'(0) = f(0) = 0; \quad f'(\infty) = 1. \tag{3}$$

Solve for f and its derivatives in the boundary layer. Define your domain from $\eta = 0$ to $\eta = 10(\approx \infty)$. In shooting method, use a convergence criteria such that your predicted value for the quantity you are shooting for is accurate within 10 digits.

(a) Using shooting method, clearly formulate the problem and indicate the algorithm you will use specifically for this problem. Clearly indicate the boundary conditions and what variable you are shooting for.

(b) Using shooting method evaluate your solutions over the domain. Use Forward Euler (or ode45) for solving the ODEs. Use $\Delta\eta = 0.01$ and $\Delta\eta = 0.005$ for your computations and compare your results (note you will have to use these step sizes for ode45 too). Plot $f,\ f',\ f''$ as a function of η for both grid sizes.

(c) Indicate how many iterations were required for convergence.

(d) What is the shear stress at the wall (i.e. evaluate f'' at the wall)? What is the boundary layer thickness in terms of η.

3. Consider the heat transfer equation in one-dimension for steady conduction:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + 10 = 0; \quad k = 2 + 10x^2 \tag{4}$$

where, $0 \le x \le 10$, with the boundary conditions, $T(0) = 100$, and $T(10) = 500$.

(a) Rewriting the equation as,

$$k\left(\frac{d^2T}{dx^2}\right) + \frac{dk}{dx}\frac{dT}{dx} + 10 = 0 \tag{5}$$

and directly evaluting dk/dx, write down an appropriate finite difference approximation using central differencing. Write down the system of equations into a matrix vector form (you can do this for a sample of say 5 total points). For $\Delta x = 1$, solve the above system of equations and plot $T(x)$.

(b) Another way to discretize the original equation is by assuming $k(x)T' = g(x)$ and approximating $g'(x)$ at half-way between the two points,

$$g'(x_j) = \frac{g(x_{j+1/2}) - g(x_{j-1/2})}{\Delta x}. \tag{6}$$

Then substituing for $g(x) = kT'$ and approximating $k(x)dT/dx$ at points half-way between the two grid points, using a centered approximation:

$$k(x_{j+1/2})T'(x_{j+1/2}) = k_{j+1/2}\left(\frac{T_{j+1} - T_j}{\Delta x}\right) \tag{7}$$

and similar equation for $k(x_{j-1/2})T'(x_{j-1/2})$.

Derive a finite-difference approximation using the above procedure. Write down the system of equations into a matrix vector form (you can do this for a sample of say 5 total points). For $\Delta x = 1$, solve the above system of equations and plot $T(x)$.

(c) Compare the solutions obtained using the above two procedures to the exact solution. Do you see any issues in terms of accuracy? What happens if you coarsen the grid or refined the grid?