Generalized Integral Eqn. (sec. 12.3)

<u>x mom. eqn.</u> $\quad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \underbrace{-\frac{1}{\rho}\frac{dP}{dx}}_{} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}$ (2D steady)

$$= U\frac{dU}{dx} \text{ from Euler's Eqn.}$$

Integrate across boundary layer $y = 0 \to \delta$ for each term.

Look at second term on left: $\int_0^\delta v\frac{\partial u}{\partial y}\,dy$

Use conservation of mass to get v at any y value:

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \longrightarrow v = -\int_0^y \frac{\partial u}{\partial x}\,dy$$

So term becomes: $-\int_0^\delta \frac{\partial u}{\partial y}\left(\int_0^y \frac{\partial u}{\partial x}\,dy\right)dy$

Need to integrate by parts: $\int_1^2 b\,da = ab\Big|_1^2 - \int_1^2 a\,db$

let $b = \int_0^y \frac{\partial u}{\partial x}\,dy$ so $db = \frac{\partial u}{\partial x}\,dy$ and $da = \frac{\partial u}{\partial y}\,dy = \partial u$

and $a = u$

So term is: $-\left[\left(u\int_0^y \frac{\partial u}{\partial x}\,dy\right)\Big|_0^\delta - \int_0^\delta u\frac{\partial u}{\partial x}\,dy\right]$

which $= -U\int_0^\delta \frac{\partial u}{\partial x}\,dy + \int_0^\delta u\frac{\partial u}{\partial x}\,dy \quad$ (note: $u(\delta) = U$, $u(0) = 0$)

Now plug into original eqn.

$$\text{(A)} \quad \int_0^\delta u\frac{\partial u}{\partial x}\,dy - U\int_0^\delta \frac{\partial u}{\partial x}\,dy + \int_0^\delta u\frac{\partial u}{\partial x}\,dy = \int_0^\delta U\frac{dU}{dx}\,dy + \int_0^\delta \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2}\,dy$$

Last term can be written as:

$$\int_0^\delta \frac{1}{\rho}\frac{\partial \tau}{\partial y}\,dy = \frac{1}{\rho}\left[\tau(y=\delta) - \tau(y=0)\right] = -\frac{\tau_w}{\rho} \quad (\tau \text{ at } y=\delta = 0)$$

Combine 1st + 3rd terms on left side of (A)

<u>or</u> $\int_0^\delta 2u\frac{\partial u}{\partial x}\,dy = \int_0^\delta \frac{\partial(u\,u)}{\partial x}\,dy$

Rewrite 2nd term of (A) $\rightarrow \int_0^\delta \left(-\frac{\partial(uU)}{\partial x} + u\frac{\partial U}{\partial x}\right)dy$
(bring U inside integral)

Now we can combine stuff.

$$\int_0^\delta \left(\frac{\partial}{\partial x}(u \cdot u) - \frac{\partial (uU)}{\partial x} + u\frac{\partial U}{\partial x} - U\frac{dU}{dx}\right) dy = -\frac{\tau_w}{\rho}$$

regroup:

$$\int_0^\delta \frac{\partial}{\partial x}\left(u(u-U)\right) dy + \int_0^\delta \frac{dU}{dx}(u-U)\, dy = -\tau_w/\rho$$

- exchange order of integration + derivative in 1st term

also change sign of all terms

$$\frac{\partial}{\partial x}\underbrace{\int_0^\delta u(U-u)\, dy}_{=U^2\delta_2} + \frac{dU}{dx}\underbrace{\int (U-u)\, dy}_{=U\delta_1} = \tau_w/\rho$$

Soooo...

$$\boxed{\frac{\partial}{\partial x}\left(U^2\delta_2\right) + \delta_1 U\frac{dU}{dx} = \tau_w/\rho} \xrightarrow{\text{Flat Plate}} 2\frac{d\delta_2}{dx} = \frac{\tau_w}{\frac{1}{2}\rho U^2}$$

to solve need to know U, δ_2 + δ_1 each as $f(x)$

then can find τ_w.

note: if $U = \text{const}\ (= U_\infty)$ — like flat plate

$$U^2\frac{d\delta_2}{dx} + 0 = \tau_w/\rho$$

or $C_f = 2\frac{d\delta_2}{dx}$ (same result we got for flat plate)

where $C_f = \frac{\tau_w}{\frac{1}{2}\rho U^2}$

Integral Solution Method

Procedure:

1. assume "similarity" vel. profile:

$$f = \frac{u}{U} = f(\underbrace{y/\delta}_{``\eta"})$$

2. calc. $\left(\delta_2/\delta\right)$ based on integral

3. calc. $f'(0)$ based on vel. profile

4. calc. $C = \left(\frac{2f'(0)}{(\delta_2/\delta)}\right)^{1/2}$ $\quad f'(0) = \left.\frac{df}{d\eta}\right|_{\eta=0}$

5. calc. $c_f = \frac{\delta_2}{\delta}\frac{C}{Re_x^{1/2}}$

$$C_f = 2c_f \text{ (at } x = L)$$

6. $\tau_w = c_f(\rho U_\infty^2/2) = f(x)$

$$F_D = \left(C_f \frac{\rho U_\infty^2}{2} wL\right)$$