Solution Homework # 1

1.

$$f'_j = \frac{2f_{j-1} - 3f_j + 6f_{j+1} - f_{j+2}}{6h} + \frac{h^3}{12} f_j^{(iv)} \tag{1}$$

Third order accurate..

2. (a) Using central differencing we get

$$\frac{\delta u_n v_n}{\delta x} = \frac{u_{n+1}v_{n+1} - u_{n-1}v_{n-1}}{2h} \tag{2}$$

$$\begin{aligned} u_n \frac{\delta v_n}{\delta x} + v_n \frac{\delta u_n}{\delta x} &= u_n \frac{v_{n+1} - v_{n-1}}{2h} + v_n \frac{u_{n+1} - u_{n-1}}{2h} && (3) \\ &\neq \frac{u_{n+1}v_{n+1} - u_{n-1}v_{n-1}}{2h} && (4) \end{aligned}$$

where $\overline{u}_n = (u_{n+1} + u_{n-1})/2$ is the average value of u. Thus, it is possible to express the discrete version of the first derivative above in a form that is similar to its continuous derivative, giving a consistent scheme. Using forward differencing we get

(b)

$$\begin{aligned} \frac{\delta u_n v_n}{\delta x} &= \frac{u_{n+1}v_{n+1} - u_{n-1}v_{n-1}}{2h} && (5) \\ &= \frac{u_{n+1}v_{n+1} + (u_{n-1}v_{n+1} - u_{n-1}v_{n+1}) - u_{n-1}v_{n-1}}{2h} && (6) \\ &= \frac{(u_{n+1} + u_{n-1})v_{n+1} - (v_{n+1} + v_{n-1})u_{n-1}}{2h} && (7) \\ &= \frac{(2\bar{u}_n v_{n+1}) - (2\bar{v}_n u_{n-1})}{2h} && (8) \\ &= \frac{2\bar{u}_n v_{n+1} + (-2\bar{u}_n v_{n-1} + 2\bar{u}_n v_{n-1}) - 2\bar{v}_n u_{n-1} + (2\bar{v}_n u_{n+1} - 2\bar{v}_n u_{n+1})}{2h} && (9) \\ &= 2\overline{u}_n \frac{(v_{n+1} - v_{n-1})}{2h} + 2\overline{v}_n \frac{(u_{n+1} - u_{n-1})}{2h} + \frac{(2\bar{u}_n v_{n-1} - 2\bar{v}_n u_{n+1})}{2h} && (10) \\ &= 2\overline{u}_n \frac{\delta v_n}{\delta x} + 2\overline{v}_n \frac{\delta u_n}{\delta x} + \frac{(u_{n+1}v_{n+1} - u_{n-1}v_{n-1})}{2h} && (11) \\ &= 2\overline{u}_n \frac{\delta v_n}{\delta x} + 2\overline{v}_n \frac{\delta u_n}{\delta x} - \frac{\delta(u_n v_n)}{\delta x} && (12) \\ \therefore \frac{\delta u_n v_n}{\delta x} &= \overline{u}_n \frac{\delta v_n}{\delta x} + \overline{v}_n \frac{\delta u_n}{\delta x} && (13) \end{aligned}$$

(c) Similar derivations...

3.
$$\frac{\delta^2 u_n}{\delta x^2} \approx \frac{u_{n+2} - 2u_n + u_{n-2}}{4h^2}. \tag{14}$$

Expanding u_{n+2} and u_{n-2} around u_n using Taylor series and substituting and rearranging gives

$$u_n'' = \frac{u_{n+2} - 2u_n + u_{n-2}}{4h^2} - \frac{h^2}{3} u_n'''' + ... \tag{15}$$

Thus it is a second-order formula. By calculating errors one would obtain a graph on a log-log scale that has a slope of nearly 2.

4. A general $Pade'$ type scheme (compact differences) at a boundary point $i = 0$ for the first derivative on a uniform mesh of spacing h can be written as

$$f_N' + \alpha f_{N-1}' = \frac{1}{h}\left(af_N + bf_{N-1} + cf_{N-2} + df_{N-3}\right). \tag{16}$$

Writing the Taylor table up to and including fourth order terms, and summing up columns 2 through 5 and equating to zero gives,

	f_N	hf_N'	$h^2 f_N''$	$h^3 f_N'''$	$h^4 f_N''''$	$h^5 f_N'''''$
$-hf_N'$	0	-1	0	0	0	0
$-\alpha hf_{N-1}'$	0	$-\alpha$	$+\alpha$	$-\frac{\alpha}{2}$	$+\frac{\alpha}{6}$	$-\frac{\alpha}{24}$
af_N	a	0	0	0	0	0
bf_{N-1}	b	$-b$	$\frac{b}{2}$	$\frac{-b}{6}$	$\frac{+b}{24}$	$\frac{-b}{120}$
cf_{N-2}	c	$-2c$	$\frac{c2^2}{2}$	$-c\frac{2^3}{6}$	$c\frac{2^4}{24}$	$-c\frac{2^5}{120}$
df_{N-3}	d	$-d.3$	$\frac{d3^2}{2}$	$-d\frac{3^3}{6}$	$d\frac{3^4}{24}$	$-d\frac{3^5}{120}$

$$a + b + c + d = 0 \tag{17}$$
$$b + 2c + 3d = -1 - \alpha \tag{18}$$
$$b + 4c + 9d = -2\alpha \tag{19}$$
$$b + 8c + 27d = -3\alpha \tag{20}$$

In order to make this fourth order accurate, summing the sixth column to zero gives,

$$-4\alpha = b + 16c + 81d. \tag{21}$$

Solving gives,

$$\alpha = 3, \quad a = +\frac{17}{6}, \quad b = -\frac{3}{2}, \quad c = -\frac{3}{2}, \quad d = +\frac{1}{6}. \tag{22}$$

5. Expanding all terms around x_j one obtains,

$$\frac{1}{6}f''_{j-1} + \frac{2}{3}f''_j + \frac{1}{6}f''_{j+1} - \frac{f_{j+1} - 2f_j + f_{j-1}}{h^2} = h^2/12f^{(4)}(x_j) + ... \tag{23}$$

This is a second-order accurate expression. Note that, the coefficients chosen here are different than the Pade' coefficients and hence the reduced order of accuracy. Pade scheme would give fourth order.

For boundary points, one can use the Taylor series to derive first order formulas as

$$f''_0 = \frac{f_0 - 2f_1 + f_2}{h^2} \tag{24}$$

$$f''_N = \frac{f_{N-2} - 2f_{N-1} + f_N}{h^2} \tag{25}$$

6. Formula for second derivative same as above

$$\frac{1}{6}f''(x_{i-1}) + \frac{2}{3}f''(x_i) + \frac{1}{6}f''(x_{i+1}) = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta^2} \tag{26}$$

   (a) One sided first-order formula:

$$f''_0 = \frac{f_0 - 2f_1 + f_2}{\Delta^2} - \mathcal{O}(\Delta) \tag{27}$$

$$f''_N = \frac{f_{N-2} - 2f_{N-1} + f_N}{\Delta^2} + \mathcal{O}(\Delta) \tag{28}$$

   As an example, let $N = 4$, i.e. there are total 5 points (two on the boundary) such that $i = 0, 1, 2, 3, 4$. The matrix equation for the interior points $i = 1, 2, 3$ then becomes

$$\begin{bmatrix} 1 & 0 & 0 & ... \\ 1 & 4 & 1 & 0 \\ 0 & ... & ... & 0 \\ 0 & 1 & 4 & 1 \\ ... & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f''_0 \\ f''_1 \\ f''_2 \\ .... \\ f''_{N-1} \\ f''_N \end{bmatrix} = \begin{bmatrix} (f_0 - 2f_1 + f_2)/\Delta^2 \\ (6f_0 - 12f_1 + 6f_2)/\Delta^2 \\ ... \\ (6f_{N-2} - 12f_{N-1} + 6f_N)/\Delta^2 \\ (f_{N-2} - 2f_{N-1} + f_N)/\Delta^2 \end{bmatrix} \tag{29}$$

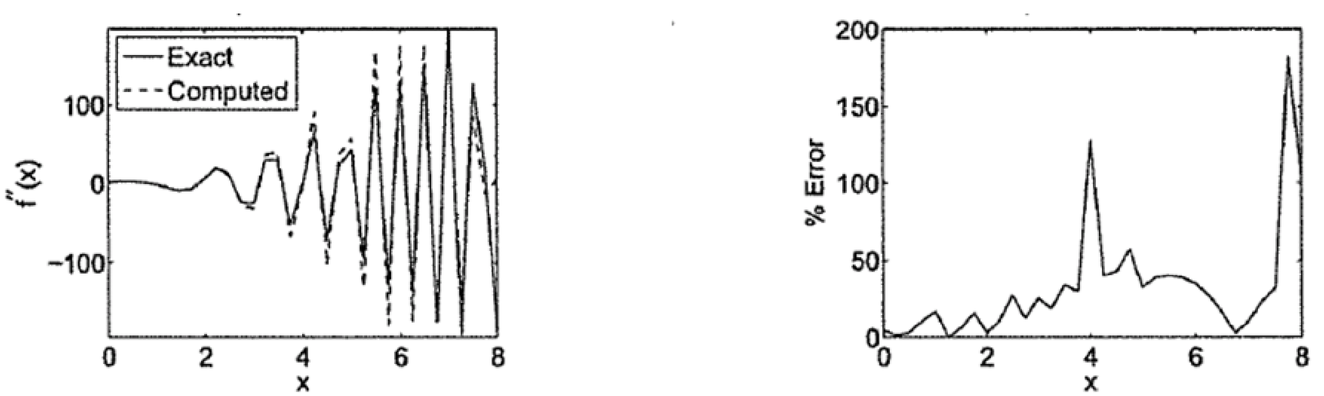
   This is a tri-diagonal system, which can be generalized to any N.

   (b) Slope of line corresponds to the accuracy of the scheme. One should obtain a straight line on the log-log curve. Then,

$$\epsilon = a\Delta^p \tag{30}$$

$$\log(\epsilon) = \log(\mathrm{a}) + \mathrm{p}\,\log(\Delta) \tag{31}$$

   Slope p corresponds to the slope and is also the accuracy of the scheme. Should be close to second order accuracy.


Exact
Computed
f''(x)
100
0
−100
0
2
4
6
8
x
200
150
100
50
% Error