(HANDOUT)

Material Derivative in Cylindrical Coordinates

We want to find an expression for DV/Dt in cylindrical coordiantes (and streamwise coordinates). Here we will neglect the z coordinate and only use r,θ since z doesn't differ from Cartesian Coordinates.

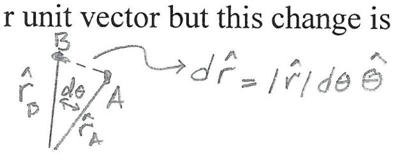
Given a vector $V = u_r\hat{r} + u_\theta\hat{\theta}$

In general we write: $\dfrac{DV}{Dt} = \dfrac{\partial V}{\partial t} + u_r\dfrac{\partial V}{\partial r} + \dfrac{u_\theta}{r}\dfrac{\partial V}{\partial \theta}$

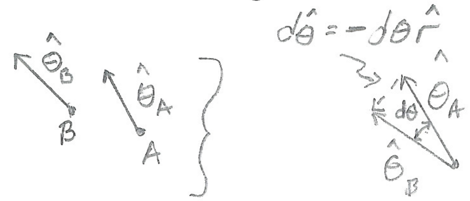
But if we have a change in θ (change in angular position) then we can show that $\hat{r}$ and $\hat{\theta}$ are not constant (remember that these are unit vecotrs, their magnitudes do not change but their direction may change.

So we write: $\dfrac{\partial V}{\partial \theta} = \dfrac{\partial}{\partial \theta}\left(u_r\hat{r} + u_\theta\hat{\theta}\right) = \hat{r}\dfrac{\partial u_r}{\partial \theta} + u_r\dfrac{\partial \hat{r}}{\partial \theta} + \hat{\theta}\dfrac{\partial u_\theta}{\partial \theta} + u_\theta\dfrac{\partial \hat{\theta}}{\partial \theta}$

Consider a blob of fluid moving from pt A to pt B a small distance that results in moving a small $d\theta$. In doing this the unit vector in r will change as : $d\hat{r} = |\hat{r}|d\theta\hat{\theta}$ which is $= d\theta\hat{\theta}$ which is a change of the r unit vector but this change is in the theta direction. So $\dfrac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$



Similarly the change of the unit vector in the theta direction is $d\hat{\theta} = -\hat{r}d\theta$. This is negative because if the change in the unit vector is counterclockwise the change in the r unit vector is in the negative r direction. So $\dfrac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$



With this we can go back to the Material Derivative expression:

$$\frac{DV}{Dt} = \left(\frac{\partial u_r}{\partial t} + u_r\frac{\partial u_r}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta} - \frac{u_\theta^2}{r}\right)\hat{r} + \left(\frac{\partial u_\theta}{\partial t} + u_r\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r u_\theta}{r}\right)\hat{\theta}$$

This applies to Cylindrical Coordinates, but also to streamwise coordinates where the unit vectors may change direction as you move along the coordinate. The net result is additional "acceleration terms" that account for these changes.