1. $y'' - w^2 y$; $y(0) = 1$; $y'(0) = 0$; $w = 4$

Initial value problem. Convert to 2 first-order ODEs

$$y' = v \qquad Y = \begin{bmatrix} y \\ v \end{bmatrix}$$
$$v' = -w^2 y$$

$$\therefore \; Y' = AY \; ; \quad A = \begin{bmatrix} 0 & 1 \\ -w^2 & 0 \end{bmatrix}$$

(a) Leap frog.

$$\boxed{Y_{n+1} = Y_{n-1} + 2\Delta t A Y_n}$$

explicit scheme
needs starter scheme
for $t=0$
2nd-order

(b) central diff

$$Y_{n+1} = Y_n + \Delta t \left[\frac{\overbrace{A Y_n + A Y_{n+1}}^{\text{Crank Nicholson}}}{2} \right]$$

$$\therefore \left[I - A\frac{\Delta t}{2} \right] Y_{n+1} = \left[I + A\frac{\Delta t}{2} \right] Y_n$$

or
$$\boxed{Y_{n+1} = \left[I - A\frac{\Delta t}{2} \right]^{-1} \left[I + A\frac{\Delta t}{2} \right] Y_n}$$

fully implicit scheme
2nd order

(c) One can analyze C.D. & L.F. using a single equation. Note that $eig(A) = \pm i\omega$

$\therefore \quad z' = \pm i\omega z$ is model eq.

consider $\quad z' = i\omega z$

L.F: $\quad z_{n+1} = z_{n-1} + 2i\omega\Delta t \, z_n$

$$\therefore \quad \frac{z_{n+1}}{z_n} = \frac{z_{n-1}}{z_n} + 2i\omega\Delta t$$

$$\therefore \quad \sigma = \frac{1}{\sigma} + 2i\omega\Delta t$$

$$\therefore \quad \sigma^2 - 2i\omega\Delta t\,\sigma - 1 = 0$$

$$\therefore \quad \sigma_{1,2} = \frac{2i\omega\Delta t \pm \sqrt{4 - 4\omega^2\Delta t^2}}{2}$$

$$\therefore \quad \boxed{\sigma_{1,2} = i\omega\Delta t \pm \sqrt{1-\omega^2\Delta t^2}}$$

If $\boxed{\omega\Delta t \leq 1}$; we get complex eigenvalues.
such that $|\sigma_{1,2}| = 1 \quad \therefore$ unconditionally stable!

Also notice that there is no amplitude error!
But a phase error is present

For $+i\omega$ root

$$\phi^{L.F.} = \tan^{-1}\left(\frac{\omega\Delta t}{\sqrt{1-\omega^2\Delta t^2}}\right)$$

$$\phi^{exact} = \omega\Delta t$$

phase error

$$-\phi^{L.F} + \phi^{exact} = -\tan^{-1}\left(\frac{\omega\Delta t}{\sqrt{1-\omega^2\Delta t^2}}\right) + \omega\Delta t$$

For. C.D.

$$z' = \pm i\omega z$$

$$z_{n+1} = z_n \pm i\omega\Delta t\left(\frac{z_{n+1}+z_n}{2}\right)$$

$$\therefore z_{n+1}\left(1 \mp \frac{\omega\Delta t}{2}\right) = z_n\left(1 \pm i\frac{\omega\Delta t}{2}\right)$$

$$\boxed{\therefore \sigma = \frac{z_{n+1}}{z_n} = \frac{1 \pm i\omega\Delta t/2}{1 \mp i\omega\Delta t/2}}$$

always stable. Also $|\sigma| = 1$ $\therefore$ No amplitude error. Phase error is present

consider $+i\omega$ root

$$\phi^{C.D.} = \frac{1 + i\frac{\omega\Delta t}{2}}{1 - i\frac{\omega\Delta t}{2}} = e^{i(\phi_1 - \phi_2)}$$

$$\phi_1 = \tan^{-1}\left(\frac{\frac{\omega\Delta t}{2}}{1}\right)$$

$$\phi_2 = \tan^{-1}\left(-\frac{\omega\Delta t}{2}\right)$$

$$\therefore \phi^{C.D.} = \phi_1 - \phi_2 = 2\tan^{-1}\left(\frac{\omega\Delta t}{2}\right)$$

TOPS 35500

$$\therefore \quad \phi^{L.F} = \tan^{-1}\left(\frac{\omega\Delta t}{\sqrt{1-\omega^2\Delta t^2}}\right) = \quad .$$

$$\phi^{C.D.} = 2\tan^{-1}\left(\frac{\omega\Delta t}{2}\right) = 2\left[\frac{\omega\Delta t}{2} - \frac{(\omega\Delta t)^3}{24} + \cdots\right]$$

$$\phi^{exact} = \omega\Delta t$$

$$\boxed{\phi^{C.D.}_{error} = \phi^{exact} - \phi^{C.D.} = \omega\Delta t - \omega\Delta t + \frac{\omega^3\Delta t^3}{12} - \cdots \simeq \frac{\omega^3\Delta t^3}{12}}$$

likewise

$$\boxed{\phi^{L.F} = \phi^{exact} - \phi^{C.D.} \simeq \frac{-\omega^3\Delta t^3}{6}}$$

Notice that L.F. has twice the phase error as C.D. and they are in opposite directions!

$\therefore$ Use 2 steps of C.D. and 1 step of L.F. This eliminates the phase error in the sol^n.

<u>Unfortunately the combined scheme is not stable!</u>

3a

$$K \frac{d^2T}{dx^2} + \frac{dk}{dx}\frac{dT}{dx} + 10 = 0$$

$$K = 2 + 10x^2$$

$$\therefore \frac{dk}{dx} = 20x.$$

$$\therefore K \frac{d^2T}{dx^2} + 20x \frac{dT}{dx} + 10 = 0$$

Evaluate @ x_j

$$\therefore K_j \frac{T_{j+1} - 2T_j + T_{j-1}}{\Delta x^2} + 20x_j \frac{T_{j+1} - \cancel{2}T_{j-1}}{2\Delta x} + 10 = 0$$

$$\therefore (2 + 10x_j^2) \frac{T_{j+1} - 2T_j + T_{j-1}}{\Delta x^2} + 20x_j \frac{T_{j+1} - T_{j-1}}{2\Delta x} + 10 = 0$$

$$\left(\frac{K_j}{\Delta x^2} - \frac{10x_j}{\Delta x}\right) T_{j-1} - \frac{2K_j}{\Delta x^2} + \left(\frac{K_j}{\Delta x^2} + \frac{10x_j}{\Delta x}\right) T_{j+1} = -10$$

For certain large Δx the coeff can become negative!

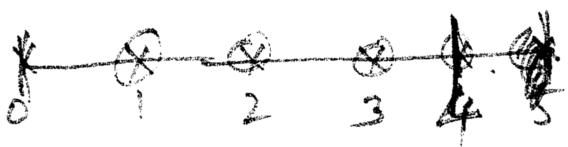
This will have issues with conservation

(b) $\dfrac{d}{dx}\left(K\dfrac{dT}{dx}\right) + 10 = 0$

$$\therefore\quad K_{j+\frac{1}{2}}\left(\frac{T_{j+1}-T_j}{\Delta x}\right) - K_{j-\frac{1}{2}}\left(\frac{T_j - T_{j-1}}{\Delta x}\right) + 10 = 0$$

$$\therefore\quad K_{j-\frac{1}{2}}\,T_{j-1} - \left(K_{j+\frac{1}{2}} + K_{j-\frac{1}{2}}\right)T_j + K_{j+\frac{1}{2}}\,T_{j+1} = -10\,\Delta x$$

$$\therefore\quad \left(2+10x^2_{j-\frac{1}{2}}\right)T_{j-1} - \left(4+10x^2_{j-\frac{1}{2}}+10x^2_{j+\frac{1}{2}}\right)T_j + \left(2+10x^2_{j+\frac{1}{2}}\right)T_{j+1} = -10\,\Delta x$$



$j = 1 \rightarrow 3$

$$\begin{bmatrix} -\left(K_{\frac{3}{2}}+K_{\frac{1}{2}}\right) & K_{\frac{3}{2}} & 0 \\ K_{\frac{3}{2}} & -\left(K_{\frac{3}{2}}+K_{\frac{5}{2}}\right) & K_{\frac{5}{2}} \\ 0 & K_{\frac{5}{2}} & -\left(K_{\frac{5}{2}}+K_{\frac{7}{2}}\right) \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} -10\,\Delta x - K_{\frac{1}{2}}T_0 \\ -10\,\Delta x \\ -10\,\Delta x - K_{\frac{7}{2}}T_4 \end{bmatrix}$$