Numerical Solution of Ordinary Differential Equations
(MTH 452/552)

Homework due Wednesday, January 25, 2017

Problem 1. (20 points). Let u be a solution of the differential equation $u' = f(t,u)$ with $u: \; R \to \mathbf{R}$ and $f : \mathbf{R}^2 \to \mathbf{R}$. Use the chain rule and the differential equation to show that

a) $u''(t) = f_t + f\,f_u$

b) $u'''(t) = f_{tt} \,+\, 2f f_{tu} \,+\, f^2 f_{uu} + f_u(f_t + f f_u)$

Here and in the next problem the abbreviated notation f, f_t, etc. means $f(t,u(t))$, $f_t(t,u(t))$, etc. .

Problem 2. (20 points). Consider the single differential equation $u' = f(t,u)$ and the numerical method

$$\begin{aligned} U^{n+1} &= U^n + \frac{\Delta t}{2}(K_1 + K_2) \\ K_1 &= f(t_n, U^n) \\ K_2 &= f\left(t_n + \Delta t,\, U^n + \frac{\Delta t}{2}\,(K_1 + K_2)\right) \end{aligned}$$

Let u denote the solution of the differential equation that satisfies $u(t_n) = U^n$. Assume that f and u are sufficiently often differentiable and use Taylor's theorem to show that

a) $K_2 \;=\; f + O(\Delta t) \;=\; f + \Delta t(f_t + f f_u) + O(\Delta t^2)$

b) $U^{n+1} = u(t_n + \Delta t) + O(\Delta t^3)$.

c) Based on part b), , what order of convergence (as $\Delta t \to 0$) would you expect if this method is used to solve an initial value problem on an interval $[t_0, t_1]$? State the reason for your answer.

Problem 3. (20 points). Consider two bodies of masses $\mu = 0.012277471$ and $\hat{\mu} = 1 - \mu$ in planar motion, and a third body of negligible mass moving in the same plane. The motion is governed by the equations

$$\begin{aligned} u_1'' &= u_1 + 2u_2' - \hat{\mu}\frac{u_1 + \mu}{D_1} - \mu\frac{u_1 - \hat{\mu}}{D_2} \\ u_2'' &= u_2 - 2u_1' - \hat{\mu}\frac{u_2}{D_1} - \mu\frac{u_2}{D_2} \\ D_1 &= \left((u_1 + \mu)^2 + u_2^2\right)^{3/2} \\ D_2 &= \left((u_1 - \hat{\mu})^2 + u_2^2\right)^{3/2} \end{aligned} \tag{1}$$

For the initial conditions

$$u_1(0) = 0.994, \quad u_2(0) = 0, \quad u_1'(0) = 0$$
$$u_2' = -2.00158510637908252240537862224$$

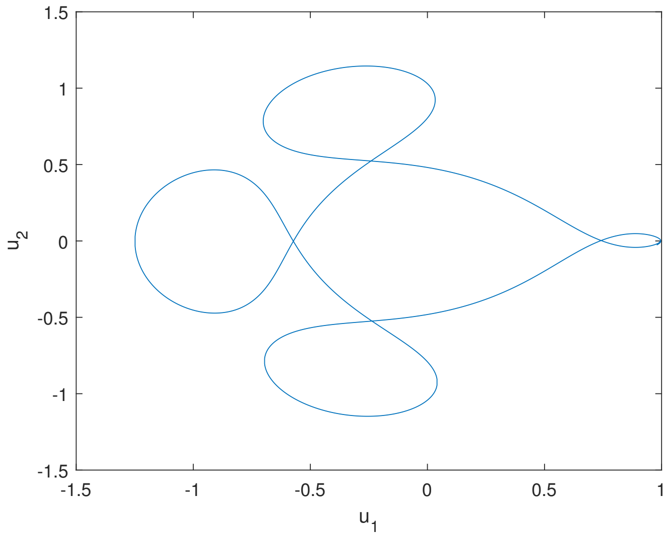

Figure 1: Astronomical orbit

the solution is periodic with period < 17.1. The orbit is sketched in Figure 1.

a) Use the Euler method to solve this initial value problem for the time interval $[0, 17.1]$. How small does Δt need to be in order for the orbit to appear qualitatively correct? Plot the orbit for this Δt. Also, use this same value of Δt to try to compute the solution on the larger time interval $[0, 34.3]$ (corresponding to two revolutions) and again plot the resulting orbit. Does this experiment indicate some limitations of the numerical method used?