1 Pi by the sum of inverse squares

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

2 Pi by inverse tangent Taylor series

$$\begin{aligned}\arctan x &= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\end{aligned}$$

$$x \in [-1, 1]$$

$$\frac{\pi}{4} = \arctan 1 = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$$

3 Pi by inscribed and circumscribed polygons

3.1 Inscribed

3.1.1 Law of sines

$$\frac{\sin(\frac{\pi}{2} - \frac{\pi}{n})}{r} = \frac{\sin \frac{2\pi}{n}}{l}$$

$$= \frac{\cos \frac{\pi}{n}}{r} = \frac{2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}}{l}$$

$$l = 2r \sin \frac{\pi}{n}$$

$$P = ln = 2rn \sin \frac{\pi}{n}$$

3.1.2 Law of cosines

$$l^2 = 2r^2 - 2r^2 cos \frac{2\pi}{n} = 2r^2 \left(1 - cos \frac{2\pi}{n}\right) = 4r^2 \left(\frac{1 - cos \frac{2\pi}{n}}{2}\right) = 4r^2 \sin^2 \frac{\pi}{n}$$

$$l = 2r \sin \frac{\pi}{n} = r \sqrt{2 - 2 \cos \frac{2\pi}{n}}$$

$$P = 2nrsin \frac{\pi}{n}$$

3.1.3 Pythagorean theorem

$$r^2 = \left(\frac{l}{2}\right)^2 + h^2$$

$$h^2 = r^2 - \frac{1}{4}l^2 = r^2 - r^2\sin^2\frac{\pi}{n} = r^2\left(1 - \sin^2\frac{\pi}{n}\right) = r^2\cos^2\frac{\pi}{n}$$

$$h = r\cos\frac{\pi}{n}$$

$$A = \frac{1}{2}lhn = \frac{1}{2}\left(2r\sin\frac{\pi}{n}\right)\left(rcos\frac{\pi}{n}\right)n = nr^2\sin\frac{\pi}{n}\cos\frac{\pi}{n} = \frac{1}{2}nr^2\sin\frac{2\pi}{n}$$

3.1.4 Results

Perimeter

$$2\pi r = \lim_{n\to\infty} P = \lim_{n\to\infty} 2rn\sin\frac{\pi}{n}$$

Area

$$\pi r^2 = \lim_{n\to\infty}\frac{1}{2}nr^2\sin\frac{2\pi}{n}$$

$$\pi = \lim_{n\to\infty}\frac{1}{2}n\sin\frac{2\pi}{n} = \lim_{n\to\infty} n\sin\frac{\pi}{n}\cos\frac{\pi}{n}$$

3.2 Circumscribed

3.2.1 Law of sines

$$\frac{\sin\frac{\pi}{n}}{L} = \frac{\sin\left(\pi - \frac{2\pi}{n}\right)}{l}$$

$$\sin\left(\pi - \frac{2\pi}{n}\right) = \sin\left[2\left(\frac{\pi}{2} - \frac{\pi}{n}\right)\right] = 2\sin\left(\frac{\pi}{2} - \frac{\pi}{n}\right)\cos\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = 2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$$

$$L = \frac{l\sin\frac{\pi}{n}}{2\sin\frac{\pi}{n}\cos\frac{\pi}{n}} = \frac{1}{2}l\sec\frac{\pi}{n} = \frac{1}{2}\left(2r\sin\frac{\pi}{n}\right)\sec\frac{\pi}{n} = r\tan\frac{\pi}{n}$$

3.2.2 Law of cosines

$$l^2 = 2L^2 - 2L^2cos\frac{\pi - 2\pi}{n} = L^2\left(2 - 2\cos\left(\pi - \frac{2\pi}{n}\right)\right)$$

$$\cos\left(\pi - \frac{2\pi}{n}\right) = \cos\left(2\left(\frac{\pi}{2} - \frac{\pi}{n}\right)\right) = 1 - 2\sin^2\left(\frac{\pi}{2} - \frac{\pi}{n}\right) = 1 - 2\cos^2\frac{\pi}{2}$$

$$l^2 = L^2\left[2 - 2\left(1 - 2\cos^2\frac{\pi}{n}\right)\right] = 4L^2\cos^2\frac{\pi}{n}$$

$$l = 2L\cos\frac{\pi}{n}$$

$$2r\sin\frac{\pi}{n} = 2L\cos\pi n$$

$$L = r\tan\frac{\pi}{n}$$

3.2.3 Pythagorean theorem

$$L^2 = \left(\frac{l}{2}\right)^2 + H^2$$

$$H^2 = r^2 tan^2 \frac{\pi}{n} - r^2 \sin^2 \frac{\pi}{n} = r^2 \left(\tan^2 \frac{\pi}{n} - \sin^2 \frac{\pi}{n}\right)$$

$$tan^2 \frac{\pi}{n} - \sin^2 \frac{\pi}{n} = tan^2 \frac{\pi}{n} - \cos^2 \frac{\pi}{2} \tan \frac{\pi}{2} = \tan^2 \frac{\pi}{n} \left(1 - \cos^2 \frac{\pi}{n}\right) = \sin^2 \frac{\pi}{n} tan^2 \frac{\pi}{n}$$

$$H^2 = R^2 sin^2 \frac{\pi}{n} \tan^2 \pi n$$

$$H = r \sin \frac{\pi}{n} \tan \frac{\pi}{n}$$

3.2.4 Results

Perimeter

$$P = 2nL = 2nr \tan \frac{\pi}{n}$$

$$2\pi r = \lim_{n\to\infty} 2nr \tan \frac{\pi}{n}$$

$$\pi = \lim_{n\to\infty} n \tan \frac{\pi}{n}$$

Area

$$A = nH\frac{L}{2} + A_l$$

$$= n \left(r \sin \frac{\pi}{n} \tan \frac{\pi}{n}\right) \left(r \sin \frac{\pi}{n}\right) + A_l$$

$$= nr^2 \sin^2 \frac{\pi}{n} \tan \frac{\pi}{n} + A_l$$

$$= nr^2 \sin^2 \frac{\pi}{n} \tan \frac{\pi}{n} + nr^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n}$$

$$= nr^2 \sin \frac{\pi}{n} \left(\sin \frac{\pi}{n} \tan \frac{\pi}{n} + \cos \frac{\pi}{n}\right)$$

$$= nr^2 \tan \frac{\pi}{n} \left(\sin^2 \frac{\pi}{n} + \cos^2 \frac{\pi}{n}\right)$$

$$= nr^2 \tan \frac{\pi}{n}$$

$$\pi r^2 = \lim_{n\to\infty} nr^2 \tan \frac{\pi}{n}$$

$$\pi = \lim_{n\to\infty} n \tan \frac{\pi}{n}$$

3.3 Simplified results

$$\pi = \lim_{n\to\infty} n \sin\frac{\pi}{n}$$

$$\pi = \lim_{n\to\infty} n \sin\frac{\pi}{n} \cos\frac{\pi}{n}$$

$$\pi = \lim_{n\to\infty} n \tan\frac{\pi}{n}$$

4 Multi-dimensional spheres

$$V_1 = \int\limits_{-R}^{R} dx = 2R$$

$$V_2 = \int\limits_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dydx = \int\limits_{-R}^{R} 2\sqrt{R^2-x^2}dx = \int\limits_{-R}^{R} 2R\sin\arccos\frac{x}{R}dx$$

$$V_2 = \int\limits_{0}^{2\pi} \int\limits_{0}^{R} rdrd\theta = \int\limits_{0}^{2\pi} \frac{1}{2}R^2 d\theta = \pi R^2$$

5 Trigonometric integrals

$$\int \sin^2\theta d\theta = \int \frac{1-\cos 2\theta}{2} d\theta = \frac{1}{2}(\theta - \sin\theta\cos\theta) + C$$

$$\int \cos^2\theta d\theta = \int \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

$$\int \sin^3\theta d\theta = \int (1-\cos^2\theta)\sin\theta d\theta = \int (\sin\theta - \cos^2\theta\sin\theta) d\theta = -\cos\theta + \frac{1}{3}\cos^3\theta + C$$

$$\int \cos^3\theta d\theta = \int (1-\sin^2\theta)\cos\theta d\theta = \int (\cos\theta - \sin^2\theta\cos\theta) d\theta = \sin\theta - \frac{1}{3}\sin^3\theta + C$$

$$\int \sin^4\theta d\theta = \int \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta = \int \frac{1}{4}(1-2\cos 2\theta + \cos^2\theta) d\theta = \int \frac{1}{4}(1-2\cos 2\theta + \frac{1+\cos 4\theta}{2}) d\theta$$