An Intuitionistic Version of Computation Tree Logic

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Abstract. Model checking plays a central role in the formal verification of dynamic systems. However, classical temporal logics such as CTL assume a complete and absolute perspective on truth, which limits their expressiveness in contexts where partial information and constructivity are essential. In this paper, we propose Intuitionistic Computation Tree Logic (ICTL), a novel temporal logic that integrates the semantics of CTL with the constructive principles of intuitionistic logic. ICTL adopts a birelational semantic framework which models the interaction between temporal evolution and the growth of knowledge over time. We formally define the syntax and semantics of ICTL, analyze its foundational properties, and show that key fixed-point characterizations of classical CTL hold in this setting. We then present an efficient model checking algorithm for the novel logic, maintaining the same PTIME complexity of classical CTL. ICTL is particularly well-suited for verifying properties in systems where traceability, information growth, and constructive reasoning are crucial, such as medical monitoring, geophysical data analysis, and database auditing. We validate our theoretical contributions through the extension of VITAMIN, a verification tool for Multi-Agent Systems, to support ICTL. Benchmarks confirm the practical tractability of ICTL model checking, reinforcing its potential for constructive temporal verification in real-world settings.

Keywords: Formal Methods · Intuitionistic Logic · Knowledge Representation

1 Introduction

Formal methods for temporal reasoning are central to the specification and verification of reactive systems [13,24,16]. Over the past decades, temporal logics such as Linear Temporal Logic (LTL) and Computation Tree Logic (CTL) [12] have become standard tools for reasoning about the correctness of programs and systems that evolve over time. These logics are typically interpreted over Kripke structures and rely on classical logic, where every proposition is either true or false in each state, reflecting the assumption of complete and static information.

While this assumption suits many classical verification tasks, it is inadequate in scenarios where system observations or guarantees rely on information that evolves progressively, or is not completely accessible at once. For example, consider the requirement that a database write operation must persist eventually; ensuring such persistence involves reasoning over a potentially uncertain or unfolding computation where the truth of a property can only be constructed over time.

Intuitionistic Logic. In modal and temporal extensions of classical propositional logic, the law of the excluded middle $\varphi \vee \neg\varphi$ is valid. From an information-based perspective, this means that these logics can only represent complete information: every formula φ is either true or false in a model. The assumption of complete information is, however, inadequate when it comes to representing the information available to real-world scenarios. To represent the development of imperfect or fallible information over time, it turns out that constructive logics are useful as base logics for temporal reasoning [15,28,14]. Intuitionistic logic (IL) [5,8,21,22] is a subsystem of classical logic which historically arose out of intuitionism school developed in the early 1900s whose main intent was to formulate a more constructive foundation for mathematics. In this setting, the notion of truth for a formula is procedural and depends on the ability to know or prove it. As a result, in IL, the law of the excluded middle $\varphi \vee \neg\varphi$ does not hold in general, i.e., it is not always possible to have knowledge (i.e., prove or verify) φ or its negation. There have been several successful attempts to create semantics for IL such as Beth's tableaux [19], topological and algebraic models [25], and Kripke models [20]. The best known semantics is based on Kripke models [20] where the accessibility relation is a partial order over the set of states or worlds which models knowledge or information accumulation. Intuitively, a model describes a process of investigation where an external agent learn progressively and procedurally from the system by moving from less informative states to more informative ones. Thus, the truth of a formula at a state s depends upon the states s' which are reachable from s (in epistemic logic terminology, the states s' represent the information set associated with s). Intuitionistic extensions of modal logics have been explored in [18,23,26], where semantics is based on birelational Kripke models with two accessibility relations: the intuitionistic information partial order and the modal relation. Moreover, several studies have explored the development of intuitionistic versions of temporal frameworks such as Linear Temporal Logic (LTL) [4,3] and Computation Tree Logic (CTL) [11]. Additionally, in the area of nonmonotonic reasoning, IL has played an important role within the well-known Answering Set Programming paradigm (ASP) [7] leading to temporal extensions of ASP [9,6] that are supported by intuitionistic temporal logics like the temporal logic of here and there [4]. These contributions highlight a growing interest in integrating IL into temporal contexts.

Contribution. In this paper, we build upon the preliminary definition of Intuitionistic CTL (ICTL) introduced in [11], extending and refining the framework to enhance its expressiveness and semantic robustness. Specifically, we propose a more disciplined class of Kripke models, introducing an additional structural condition that ensures the validity of fixed-point equivalences for universal temporal operators. This refinement is essential to preserving the constructive interpretation of universally quantified path formulas, an aspect only partially addressed in the original formulation. ICTL is an extension of classical CTL that incorporates propositional IL to model a specific form of partial information. One of its core features is the ability to reason about progressive information acquisition, the process by which a system or external verifier incrementally constructs knowledge about its state over time. This is particularly relevant in verification tasks where temporal properties must be constructively established through the evolution of system behaviors, rather than assumed from a globally fixed state. By enriching Kripke models with a partial order over states to capture the accumulation of observable information, ICTL enables a dynamic semantics in which truth is not statically assigned but may emerge over time as evidence is gathered. Unlike classical CTL, where each state encodes a complete and final snapshot, ICTL supports a more nuanced and realistic modeling of incomplete or evolving observations. This makes it especially suited for reasoning about constructive guarantees, such as eventual consistency or write persistence, where the verification process itself unfolds temporally. As a result, our version of ICTL guarantees that the truth of a universal temporal formula (e.g., $A\top\mathsf{U}\varphi$) can be derived constructively, reflecting the intuitionistic semantics of information growth. Moreover, this refinement allows us to design a simpler and more efficient model checking algorithm, as it avoids some of the complications arising from non-monotonicity or lack of fixed-point preservation in prior approaches. The key insight is to interpret the temporal model through a Kripke structure enriched with a partial order that tracks the progression of information: the system does not instantaneously know the truth value of a formula, but acquires it progressively as computation unfolds. ICTL is thus suitable for reasoning in settings where the validation of a specification requires constructive evidence, such as persistence, convergence, or stabilization properties. We formally define the syntax and semantics of ICTL and analyze its fundamental properties. We formally define the syntax and semantics of ICTL, and furthermore, we provide key properties that underpin the structure of the logic, ensuring that strategies and decision-making processes are not only well-defined but also computationally tractable. To this end, we present an algorithm for model checking ICTL and prove its PTIME-completeness, making it computationally equivalent to CTL. This demonstrates that constructive temporal reasoning can be computationally tractable while providing a richer semantic basis for verification under incomplete knowledge. We support our theoretical contributions with a set of benchmarks, evaluating the efficiency of our approach through an application scenario involving write-persistence in a database system, highlighting how ICTL provides a practical and effective means of modeling observational guarantees in systems where state visibility evolves over time, in a computationally feasible manner.

2 Preliminaries

We fix a finite non-empty set AP of atomic propositions. For a word (or sequence) w over some alphabet, $|w|$ denotes the length of w (we set $|w| = \infty$ if w is infinite) and for each $0 \leq i < |w|$, $w(i)$ is the $(i+1)^{th}$ letter of w.

2.1 Intuitionistic Propositional Logic

We first recall *Intuitionistic Propositional Logic* (IPL for short) and its standard Kripke semantics. In IPL, the truth of a formula φ is understood as φ is provable. The set of IPL formulas φ over AP is inductively defined as follows:

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi$$

where $\bot$ is the falsehood symbol and $p \in \mathsf{AP}$. Negation of φ is defined as $\neg\varphi \stackrel{\text{def}}{=} \varphi \rightarrow \bot$. In the Kripke semantics, IPL formulas are interpreted over *Intuitionistic Kripke structures* (IKS) which are tuples $\mathcal{K} = \langle S, S_I, \preceq, \mathrm{V}\rangle$, where S is a set of states or worlds, $S_I \subseteq S$ is the set of initial states, $\preceq$ is a partial order over S, [4] and $\mathrm{V} : S \mapsto 2^{\mathsf{AP}}$ is a *propositional valuation* that assigns to each state s the set of propositions holding at s. Moreover, we require that the valuation V satisfies the *monotonicity condition*, that is: for all states $s, t \in S$, if $s \preceq t$ then $\mathrm{V}(s) \subseteq \mathrm{V}(t)$. Intuitively, states represent partial information and $s \preceq s'$ means that information increases in moving from s to s'. Regarding the meaning of atomic propositions, they represent assertions or facts, as in the classical setting. However, in the intuitionistic framework, the truth value of a proposition p at a state s may be undetermined. Specifically, if $p \in \mathrm{V}(s)$, then the truth value of p at s is true. If $p \notin \mathrm{V}(s)$, then the truth value of p at s is not necessarily false. According to the intuitionistic interpretation of negation, p is false at s iff for every refinement s' of s (that is, $s \preceq s'$), $p \notin \mathrm{V}(s)$.

Semantics of IPL. For a state s of $\mathcal{K}$ and a formula φ, the satisfaction relation $\mathcal{K}, s \models \varphi$ is inductively defined as follows:

$$
\begin{array}{ll}
\mathcal{K}, s \not\models \bot & \\
\mathcal{K}, s \models p & \Leftrightarrow p \in \mathrm{V}(s) \\
\mathcal{K}, s \models \varphi_1 \rightarrow \varphi_2 & \Leftrightarrow \text{for all states } t \in S \text{ such that } s \preceq t\text{:} \\
 & \qquad \mathcal{K}, t \models \varphi_1 \text{ implies } \mathcal{K}, t \models \varphi_2 \\
\mathcal{K}, s \models \varphi_1 \wedge \varphi_2 & \Leftrightarrow \mathcal{K}, s \models \varphi_1 \text{ and } \mathcal{K}, s \models \varphi_2 \\
\mathcal{K}, s \models \varphi_1 \vee \varphi_2 & \Leftrightarrow \mathcal{K}, s \models \varphi_1 \text{ or } \mathcal{K}, s \models \varphi_2
\end{array}
$$

Note that $\varphi_1 \rightarrow \varphi_2$ is checked at all the states greater or equal to the current state s. Moreover, $\mathcal{K}, s \models \neg\varphi$ if and only if for all states t such that $s \preceq t$, $\mathcal{K}, t \not\models \varphi$. Intuitionistic semantics has the feature that for any formula φ and states $s \preceq t$ of a Kripke model, if $\mathcal{K}, s \models \varphi$, then $\mathcal{K}, t \models \varphi$ holds as well; that is, *truth is monotone* (with respect to $\preceq$).

[4] A partial order over S is a reflexive, transitive and antisymmetric binary relation over S.

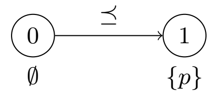

Fig. 1. Counter model $\mathcal{K}$ for the excluded middle principle

Due to intuitionistic semantics of implication, $\mathcal{K}, s \models \neg\varphi$ implies $\mathcal{K}, s \not\models \varphi$ but not the opposite. In other words, $\mathcal{K}, s \not\models \varphi$ just means that φ is not provable in s, but this does not implies that $\neg\varphi$ is provable in s. Indeed, IPL calculus has one axiom less than classical one, namely it does not contain the law of the excluded middle: $\varphi \vee \neg\varphi$. Alternatively, to obtain the classical propositional calculus, one can add the double negation elimination axiom ($\neg\neg\varphi \rightarrow \varphi$) to the intuitionistic calculus. Figure 1 illustrates a *counterexample model* $\mathcal{K}$ for the formula $p \vee \neg p$. The formula is not satisfied at state 0. Indeed, $\mathcal{K}, 0 \not\models p$ because $p \notin \mathrm{V}(0)$ and $\mathcal{K}, 0 \not\models \neg p$ because $\mathcal{K}, 1 \models p$. Hence, the formulas $p \vee \neg p$ and $\neg\neg p \rightarrow p$ are not valid. A formula φ is *valid* if $\mathcal{K}, s \models \varphi$ for all IKS $\mathcal{K}$ and states s of $\mathcal{K}$. A formula φ is *satisfiable* (*valid*) if $\mathcal{K}, s \models \varphi$ for some (all) IKS $\mathcal{K}$ and state s. Due negation semantics, validity of a formula φ does not correspond to unsatisfiability of $\neg\varphi$. In fact, while the set of IPL satisfiable formulas equals the set Sat of classically satisfiable formulas, this correspondence does not extend to validity. As a matter of fact, checking validity in IPL is Pspace-complete [27].

3 Intuitionistic CTL

We now introduce our formulation of Intuitionistic *CTL* (ICTL) by revisiting and refining the definitions from [11], addressing previously unresolved aspects and completing the semantic framework to support constructive temporal reasoning, and provide a characterization of the birelational frames that ensure monotonicity of the satisfaction relation. In Subsection 3.1, we illustrate the novel framework by modelling the verification of eventual and persistent storage guarantees in a database system, where the success of a write must be established through constructive observation of system executions. Moreover, in Subsection 3.2 we address expressiveness issues, and give denotational characterizations of the ICTL semantics.

Syntax of ICTL. The syntax of ICTL coincides with that of standard CTL but with emphasis on the use of the implication connective. Formally, for the given set AP of atomic propositions, the syntax of *state formulas* φ and *path formulas* ψ over AP is inductively defined as follows:

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \mathsf{A}\psi \mid \mathsf{E}\psi$$
$$\psi ::= \bigcirc\varphi \mid \varphi \mathsf{U} \varphi \mid \varphi \mathsf{R} \varphi$$

where $p \in \mathsf{AP}$, $\bigcirc$, U, and R are the standard "next", "until", and "release" temporal modalities, respectively, and A and E are the existential and universal path quantifiers, respectively. We also exploit the double implication $\leftrightarrow$ defined as follows: $\varphi \leftrightarrow \varphi' \stackrel{\text{def}}{=} (\varphi \to \varphi') \wedge (\varphi' \to \varphi)$. Formula $\mathsf{A}\psi$ expresses that for every path from the current state, the property ψ holds, while formula $\mathsf{E}\psi$ requires that there exists a path starting at the current state such that the property ψ holds. Formulas of ICTL are all and only the state formulas. The size $|\varphi|$ of a formula φ is the number of distinct (path or state) subformulas of φ.

We also consider the existential fragment $\text{ICTL}_\exists$ and the universal fragment $\text{ICTL}_\forall$ of ICTL, obtained by disallowing the universal path quantifiers for $\text{ICTL}_\exists$ and the existential path quantifiers for $\text{ICTL}_\forall$, respectively. Note that under the semantics of CTL, the two fragments are equivalent since the path quantifier A can be expressed in terms of E, and vice versa. However, as we will see in Subsection 3.2, this does not hold for the intuitionistic semantics of ICTL. Similarly, the dual modalities U and R are not interdefinable in ICTL.

ICTL formulas are interpreted over *Birelational* models which extend classical Kripke structures by a partial order over the set of states.

Definition 1. *A* Birelational *Frame (BF for short) is a tuple $\mathcal{F} = \langle S, S_I, \longrightarrow, \preceq \rangle$ where S and S_I are defined as for IKS, $\longrightarrow \subseteq S \times S$ is a temporal transition relation which is* left-total *(that is for every s there is s' s.t. $s \longrightarrow s'$), and $\preceq$ is a partial order over S. A path π of $\mathcal{F}$ is an infinite sequence of states $\pi = s_0, s_1, \ldots$ such that $s_i \longrightarrow s_{i+1}$ for each $i \geq 0$. A finite path of $\mathcal{F}$ is a non-empty prefix of some path.*

We say that $\mathcal{F}$ is well-behaved *if the partial order $\preceq$ satisfies the following two additional conditions for all states $s, s' \in S$:*

(C_1) *if $s \preceq s'$ and $s \longrightarrow t$ for some state $t \in S$, then there is a state $t' \in S$ such that $s' \longrightarrow t'$ and $t \preceq t'$ (see fig. 2);*
(C_2) *if $s \preceq s'$ and $s' \longrightarrow t'$ for some state $t' \in S$, then there is a state $t \in S$ such that $s \longrightarrow t$ and $t \preceq t'$ (see fig. 3).*

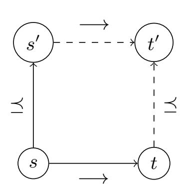

Fig. 2. Condition C_1

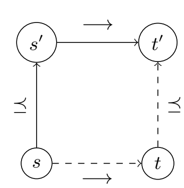

Fig. 3. Condition C_2

A Birelational *Model (BM for short) is a tuple $\mathcal{M} = \langle S, S_I, \longrightarrow, \preceq, V \rangle$ consisting of a BF $\langle S, S_I, \longrightarrow, \preceq \rangle$ and a valuation $V : S \mapsto 2^{AP}$ satisfying the* monotonicity condition*, that is: for all $s, t \in S$, if $s \preceq t$ then $V(s) \subseteq V(t)$.*

A BM is *well-behaved* if the underlying frame is well-behaved. Conditions C_1 and C_2 formalize the interplay between the partial order $\preceq$ (which models information accumulation of the agents) and the time passage induced by agent

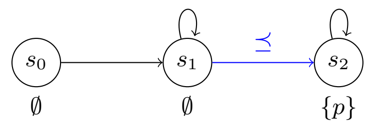

Fig. 4. Well-behaved BM satisfying at state s_0 the formula $\neg\langle\langle \mathsf{Ag} \rangle\rangle \bigcirc p \wedge \neg\langle\langle \mathsf{Ag} \rangle\rangle \bigcirc \neg p$. For simplicity, we report only the minimal $\preceq$-edges which generate the partial order $\preceq$.

actions, regulating the dynamic of information change which is induced by time passage. Intuitively, in a well-behaved BM $\mathcal{M}$, condition C_1 ensures that the partial order $\preceq$ behaves like an *alternating-time simulation* over $\mathcal{M}$ [2], and similarly for condition C_2 over the inverse $\succeq$ of $\preceq$: if a state s' is at least as informative as s, then for each A-decision d_A available at s, there is an A-decision available at s' that produces a set of outcomes at least as informative as those produced at s (and vice versa). As we will see, the well-behaved property is the minimal requirement for ensuring *truth monotonicity* of ICTL, a crucial semantics constraint in the intuitionistic setting: in moving from a state to a more informative one, the truth of the statements is preserved. It is worth noting that by [2], checking the well-behaved requirement for *finite* BM can be done in polynomial time. Moreover, the BM models in which each state has exactly one successor correspond to the linear frames introduced in [3] for intuitionistic LTL (ILTL). In particular, for the subclass of linear frames, the additional conditions C_1 and C_2 coincide and are equivalent to the property of *forward confluence* as defined in [3] (*expanding frames*). It is worth emphasizing that the variant of ILTL we consider is the one interpreted over expanding frames [3] as our framework only requires forward confluence.

Semantics of ICTL. The semantics of ICTL differs from that of standard CTL by the intuitionistic interpretation of implication $\rightarrow$. Let $\mathcal{M} = \langle S, S_I, \longrightarrow, \preceq, \mathrm{V}\rangle$ be a BM, $s \in S$, and π a path of $\mathcal{M}$. For a path formula ψ and a state formula φ, the satisfaction relations $\mathcal{M}, s \models \varphi$ and $\mathcal{M}, \pi \models \psi$ are inductively defined as follows, where we omit the base cases and the cases for $\vee$, $\wedge$, and the temporal modalities which are standard:

$$
\begin{array}{ll}
\mathcal{M}, s \not\models \bot & \\
\mathcal{M}, s \models p & \Leftrightarrow p \in \mathrm{V}(s) \\
\mathcal{M}, s \models \varphi_1 \rightarrow \varphi_2 & \Leftrightarrow \text{for all states } t \in S \text{ such that } s \preceq t\text{:} \\
& \quad\ \text{if } \mathcal{M}, t \models \varphi_1 \text{ then } \mathcal{M}, t \models \varphi_2 \\
\mathcal{M}, s \models \mathsf{A}\psi & \Leftrightarrow \text{for all paths } \pi \text{ starting at } s,\ \mathcal{M}, \pi \models \psi \\
\mathcal{M}, s \models \mathsf{E}\psi & \Leftrightarrow \text{for some path } \pi \text{ starting at } s,\ \mathcal{M}, \pi \models \psi
\end{array}
$$

Moreover, we write $\mathcal{M}, s \models_{\mathrm{CTL}} \varphi$ to mean that the Kripke model (KM for short) embedded into $\mathcal{M}$, i.e., the tuple $\langle S, S_I, \longrightarrow, \mathrm{V}\rangle$, satisfies φ at state s under the standard CTL semantics. We observe that unlike IPL versus propositional logic, ICTL satisfiability *does not correspond* to CTL satisfiability.

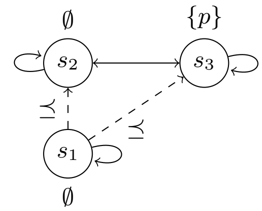

Fig. 5. Countermodel for non-validity of dualities

Proposition 1. *Every ICTL-formula φ which is satisfiable under the CTL-semantics is also ICTL-satisfiable. However, there are ICTL-satisfiable formulas which are unsatisfiable under the standard CTL semantics.*

Proof. Since BF can be seen as well-behaved BM whose partial order $\preceq$ is the identity, the first part of Proposition 1 trivially holds. For the second part, we consider the formula $\varphi \stackrel{\text{def}}{=} \neg E \bigcirc p \wedge \neg E \bigcirc \neg p$. This formula is unsatisfiable under the CTL-semantics. On the other hand, the state s_0 of the well-behaved BM $\mathcal{M}$ illustrated in Figure 4 satisfies φ under the ICTL-semantics (note that $\mathcal{M}, s_1 \not\models p$ and $\mathcal{M}, s_1 \not\models \neg p$).

Due to the intuitionistic interpretation of negation, standard dualities which allow to express a temporal modality or a path quantifier in term of its dual are not valid formulas in ICTL.

Proposition 2. *The following ICTL formulas are not valid:*

$$\begin{array}{l}
\mathsf{A} \bigcirc \varphi \leftrightarrow \neg \mathsf{E} \bigcirc \neg \varphi; \\
\mathsf{E} \bigcirc \varphi \leftrightarrow \neg \mathsf{A} \bigcirc \neg \varphi; \\
\mathsf{A}(\varphi_1 \mathsf{U} \varphi_2) \leftrightarrow \neg \mathsf{E}(\neg \varphi_1 \mathsf{R} \neg \varphi_2); \\
\\
\mathsf{E}(\varphi_1 \mathsf{U} \varphi_2) \leftrightarrow \neg \mathsf{A}(\neg \varphi_1 \mathsf{R} \neg \varphi_2); \\
\mathsf{A}(\varphi_1 \mathsf{R} \varphi_2) \leftrightarrow \neg \mathsf{E}(\neg \varphi_1 \mathsf{U} \neg \varphi_2); \\
\mathsf{E}(\varphi_1 \mathsf{R} \varphi_2) \leftrightarrow \neg \mathsf{A}(\neg \varphi_1 \mathsf{U} \neg \varphi_2).
\end{array}$$

Proof. Let us consider the BM $\mathcal{M}$ illustrated in Figure 5, We have that $\mathcal{M}, s_1 \models \neg \mathsf{A} \bigcirc p$ and $\mathcal{M}, s_1 \not\models \mathsf{E} \bigcirc \neg p$. Hence, $\neg \mathsf{A} \bigcirc p \rightarrow \mathsf{E} \bigcirc \neg p$ is not satisfied at state s_1. Analogously, $\neg \mathsf{A}(\top \mathsf{U} p) \rightarrow \mathsf{E}(\bot \mathsf{R} \neg p)$ is not satisfied at state s_1. The remaining cases are similar. □

Monotonicity of truth of formulas. A BF $\mathcal{F} = \langle S, S_I, \longrightarrow, \preceq \rangle$ is *ICTL-monotonic* if for all propositional valuations $\mathrm{V} : S \mapsto 2^{\mathsf{AP}}$ such that $(\mathcal{F}, \mathrm{V})$ is a BM (i.e., V satisfies the monotonicity condition), the following holds for all states s, t and ICTL formulas φ: if $\mathcal{M}, s \models \varphi$ and $s \preceq t$ then $\mathcal{M}, t \models \varphi$. The notions of ICTL$_\exists$-monotonicity and ICTL$_\forall$-monotonicity are similar. In the following, we show that conditions C_1 and C_2 in Definition 1 are necessary and sufficient conditions on the partial order $\preceq$ of a BF for ensuring ICTL-monotonicity.

Let $\mathcal{F} = \langle S, S_I, \longrightarrow, \preceq \rangle$ be a BF. For two paths π and π' of $\mathcal{F}$, we write $\pi \preceq \pi'$ to mean that $\pi(i) \preceq \pi'(i)$ for all $i \geq 0$. Moreover, for two finite paths ρ and ρ', we write $\rho \preceq \rho'$ to mean that $|\rho| = |\rho'|$ and $\rho(i) \preceq \rho'(i)$ for all $0 \leq i < |\rho|$.

Lemma 1. *Let $\mathcal{F}$ be a BF satisfying condition C_1 and $s \preceq s'$. Then, for each path π of $\mathcal{F}$ starting from s, there is a path π' of $\mathcal{F}$ starting from s' such that $\pi \preceq \pi'$.*

Proof. Let $\pi = s_0, s_1, s_2, \ldots$ be a path with $s_0 = s$ and for all $i \geq 0$, let π_i be the prefix of π of length $i+1$. For each $i \geq 0$, we construct a finite path π'_i starting from s' of length $i+1$ such that $\pi_i \preceq \pi'_i$ and in case $i > 0$, π'_i is of the form $\pi'_i = \pi'_{i-1} \cdot s'_i$ for some state s'_i. Hence, the infinite sequence $\pi' = s'_0, s'_1, s'_2, \ldots$ with $s'_0 = s'$ is an infinite path starting from s' such that $\pi \preceq \pi'$. For the base case ($i = 0$), we set $\pi'_0 = s'$, and being $s \preceq s'$, the result trivially follows. Now, let $i > 0$ and π'_{i-1} be a finite path of length i starting from s and leading to some state s'_{i-1} such that $\pi_{i-1} \preceq \pi'_{i-1}$. Since the structure $\mathcal{F}$ satisfies condition C_1, $s_{i-1} \preceq s'_{i-1}$ and $s_{i-1} \longrightarrow s_i$, there must be a state s'_i such that $s'_{i-1} \longrightarrow s'_i$ and $s_i \preceq s'_i$. We set $\pi'_i := \pi'_{i-1} \cdot s'_i$, and the result follows. □

For BF satisfying condition C_2 in Definition 1, we obtain a result similar to Lemma 1.

Lemma 2. *Let $\mathcal{F}$ be a BF satisfying condition C_2 and $s \preceq s'$. Then, for each path π' of $\mathcal{F}$ starting from s', there is a path π of $\mathcal{F}$ starting from s such that $\pi \preceq \pi'$.*

By exploiting Lemmata 1 and 2, we show that conditions C_1 and C_2 characterize the ICTL-monotonic BF.

Theorem 1. *Let $\mathcal{F}$ be a BF. Then:*

1. *$\mathcal{F}$ satisfies condition C_1* iff *$\mathcal{F}$ is ICTL$_\exists$-monotonic.*
2. *$\mathcal{F}$ satisfies condition C_2* iff *$\mathcal{F}$ is ICTL$_\forall$-monotonic.*
3. *$\mathcal{F}$ is well-behaved* iff *$\mathcal{F}$ is ICTL-monotonic.*

Proof. We prove Property (1). The proof of Property (2) is similar, and Property (3) easily follows from Properties (1) and (2). For the *left to right implication* of Property (1), assume that $\mathcal{F}$ satisfies condition C_1. Let $\mathcal{M} = \langle S, S_I, \longrightarrow, \preceq, \mathrm{V} \rangle$ be a BM whose underlying frame $\langle S, S_I, \longrightarrow, \preceq \rangle$ is $\mathcal{F}$, φ an ICTL$_\exists$ formula, $s \preceq s'$, and $\mathcal{M}, s \models \varphi$. We prove by structural induction on φ that $\mathcal{M}, s' \models \varphi$. The cases where φ is an atomic proposition directly follows from the monotonicity condition on the propositional valuation V, while the cases where the root operator of φ is a Boolean connective directly follow from the induction hypothesis. Thus since φ is an ICTL$_\exists$ formula, it remains to consider the case where φ is of the form $\mathsf{E}\psi$ for some path formula ψ. Since $\mathcal{M}, s \models \mathsf{E}\psi$, there exists a path π starting from s such that $\mathcal{M}, \pi \models \psi$. Being $s \preceq s'$, by Lemma 1, there exists a path π' starting from s' such that $\pi(j) \preceq \pi'(j)$ for all $j \geq 0$. Assume that ψ is of the form $\varphi_1 \mathsf{U} \varphi_2$ (the cases where ψ is of the form $\bigcirc\varphi_1$ or $\varphi_1 \mathsf{R} \varphi_2$ are similar). We show that $\mathcal{M}, \pi' \models \varphi_1 \mathsf{U} \varphi_2$. Hence, the result follows. Since $\mathcal{M}, \pi \models \varphi_1 \mathsf{U} \varphi_2$, there is $i \geq 0$ such that $\mathcal{M}, \pi(i) \models \varphi_2$ and $\mathcal{M}, \pi(k) \models \varphi_1$ for all $0 \leq k < i$. Thus, by the induction hypothesis, $\mathcal{M}, \pi'(i) \models \varphi_2$ and $\mathcal{M}, \pi'(k) \models \varphi_1$ for all $0 \leq k < i$. This means that $\mathcal{M}, \pi' \models \varphi_1 \mathsf{U} \varphi_2$ and we are done.

The *right to left implication* of Property (1) is proved by contrapositive. Assume that $\mathcal{F} = \langle S, S_I, \longrightarrow, \preceq \rangle$ does not satisfy condition C_1. We prove that $\mathcal{F}$ is not ICTL$_\exists$-monotonic. By hypothesis there must be three states s, s' and t with $s \preceq s'$ and $s \longrightarrow t$ such that for each state t', if $s' \longrightarrow t'$ then $t \not\preceq t'$. Let $p \in \mathsf{AP}$ and $\mathrm{V} : S \mapsto 2^{\mathsf{AP}}$ be the valuation defined as follows for all states $u \in S$: $\mathrm{V}(u) = \{p\}$ if $t \preceq u$, and $\mathrm{V}(u) = \emptyset$ otherwise. We observe that V satisfies the monotonicity condition w.r.t. $\preceq$. Indeed, let $u \preceq u'$. If $\mathrm{V}(u) = \emptyset$, then $\mathrm{V}(u) \subseteq \mathrm{V}(u')$. Otherwise, $t \preceq u$ and by transitivity of $\preceq$, $t \preceq u'$ holds as well. Hence, $\mathrm{V}(u') = \{p\}$, and the result follows. Thus, the tuple $\mathcal{M} = \langle S, S_I, \longrightarrow, \preceq, \mathrm{V} \rangle$ is a BM with embedded frame $\mathcal{F}$. By construction, $\mathrm{V}(t) = \{p\}$ and $s \longrightarrow t$. Hence, $\mathcal{M}, s \models \mathsf{E} \bigcirc p$. Thus, since $s \preceq s'$, in order to show that $\mathcal{F}$ is not ICTL$_\exists$-monotonic, we prove that $\mathcal{M}, s' \not\models \mathsf{E} \bigcirc p$. Indeed, for each t' such that $s' \longrightarrow t'$, we have that $t \not\preceq t'$, hence, $\mathrm{V}(t') = \emptyset$. This means that $\mathcal{M}, s' \not\models E \bigcirc p$, and we are done. □

3.1 Example: Traceable and Persistent Database Transaction

We illustrate the ICTL framework with a verification scenario involving the constructive guarantee of data persistence in a database system. The goal is to ensure that once a write is successfully performed, its effect becomes permanently observable and cannot be erased or invalidated. To this end, we model a system where users perform read and write operations over successive observation steps i, each including: an initial idle state $init_i$, a waiting state $wait_i$, and a successful write state $write_i$. Each state includes atomic propositions: w_{ij} (user i writes data j), r_i (user i requests a write), and c_j (data j is erased). Observations accumulate knowledge over time, formalized by a partial order $\preceq$ such that $write_i \preceq init_{i+1}$, guaranteeing monotonic information growth across steps. We capture the persistence property via the ICTL formula $\varphi = \mathsf{A}[\bot \mathsf{R}(r_i \to \mathsf{E}(\top \mathsf{U}\, w_{ij}))] \to \neg c_j$, which states that whenever a write request occurs, the data will eventually be written and never lost or overwritten. Crucially, in the intuitionistic setting, this implication is constructive: it ensures that evidence of w_{ij} persists in all future refinements, and that c_j remains false in all informational extensions. Figure 6 shows the birelational model BM $\mathcal{M}$ instantiated with three observation steps. This structure forms the foundation for scaling to more realistic scenarios, including centralized and distributed systems with replicated databases and coordinated consistency mechanisms.

3.2 Properties of ICTL

In this section, we first establish some expressiveness results about ICTL. Then, we provide least and greatest fix-point characterizations of the temporal quantifiers. Note that these results are independent of the well-behaveness assumption on BF which characterizes truth monotonicity in ICTL. However, without monotonicity, ICTL would be not an intuitionistic logic, i.e., it would be not conservative over IPL.

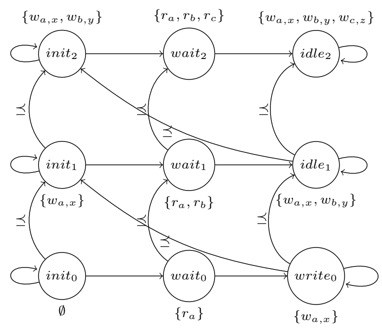

Fig. 6. BM for Database traceability problem with 3 observations

For a formula φ and a BM $\mathcal{M}$ with set of states S, we denote by $[\![\varphi]\!]_{\mathcal{M}}$ the set of $\mathcal{M}$-states satisfying φ that is $[\![\varphi]\!]_{\mathcal{M}} = \{s \in S \mid \mathcal{M}, s \models \varphi\}$. We simply write $[\![\varphi]\!]$ if $\mathcal{M}$ is clear from the context. Two formulas φ_1 and φ_2 are *equivalent*, written $\varphi_1 \equiv \varphi_2$, if for each BM $\mathcal{M}$, $[\![\varphi_1]\!]_{\mathcal{M}} = [\![\varphi_2]\!]_{\mathcal{M}}$. The following result is straightforward.

Proposition 3. *For all formulas φ_1 and φ_2, $\varphi_1 \equiv \varphi_2$* iff *the formula $\varphi_1 \leftrightarrow \varphi_2$ is valid.*

Expressiveness issues. We establish the following results, where ICTL$_{\mathsf{U}}$ and ICTL$_{\mathsf{R}}$ are the fragments of ICTL obtained by disallowing the release modality for ICTL$_{\mathsf{U}}$ and the until modality for ICTL$_{\mathsf{R}}$, respectively.

Theorem 2. *ICTL$_\exists$ and ICTL$_\forall$ are expressively incomparable, and ICTL$_{\mathsf{U}}$ and ICTL$_{\mathsf{R}}$ are expressively incomparable.*

Proof. The proof for ICTL$_\exists$ and ICTL$_\forall$ is by contraposition. Assume that either ICTL$_\exists$ is subsumed by ICTL$_\forall$ or ICTL$_\forall$ is subsumed by ICTL$_\exists$. We examine the first case (the second case being similar). Hence, for each ICTL$_\exists$ formula, there exists an equivalent ICTL$_\forall$ formula. Let us consider a *BF* $\mathcal{F}$ with partial order $\preceq$ which does not satisfy condition C_1 in Definition 1, but satisfies condition C_2 (obviously, such frames exist). By Theorem 1, $\mathcal{F}$ is not ICTL$_\exists$-monotonic. Hence, there is a BM $\mathcal{M}$ whose frame is $\mathcal{F}$, two states s and s' with $s \preceq s'$, and an ICTL$_\exists$ formula $\varphi_\exists$ such that $\mathcal{M}, s \models \varphi_\exists$ and $\mathcal{M}, s' \not\models \varphi_\exists$. By hypothesis, there exists an ICTL$_\forall$ formula $\varphi_\forall$ such that $\varphi_\forall \equiv \varphi_\exists$. Hence, $\mathcal{M}, s \models \varphi_\forall$ and $\mathcal{M}, s' \not\models \varphi_\forall$. Since $\mathcal{F}$ satisfies C_2, by Theorem 1, $\mathcal{F}$ is ICTL$_\forall$-monotonic. Hence, we deduce that $\mathcal{M}, s' \models \varphi_\forall$, and we reach a contradiction. For the second part of the theorem, it is known [3] that modalities U and R are both necessary for ensuring the full expressivity of intuitionistic LTL (ILTL). Since the models of ILTL correspond to BM where each state has exactly one successor, and over these models, ICTL coincides with ILTL, the result follows. $\square$

Fix-point characterizations of path quantifiers. Let $\mathcal{M} = \langle S, S_I, \longrightarrow, \preceq, \mathrm{V} \rangle$ be a BM. Given $X \subseteq S$, we denote by $X^{\mathbf{c}}$ its complement, i.e., the set $S \setminus X$, and by $X^{\uparrow}$ the set of states $s \in S$ such that for each state s' with $s \preceq s'$, $s' \in X$. Note that $X^{\uparrow} \subseteq X$. Moreover, we write $\mathrm{Pre}^{\exists}(X)$ for the set of states s such that *some successor* of s is in X: $pre^{\exists}(X) \stackrel{\mathrm{def}}{=} \{s \in S \mid \text{there is } s' \in X \text{ such that } s \longrightarrow s'\}$. Finally, we denote by $\mathrm{Pre}^{\forall}(X)$ the set of states s such that *all successors* of s are in X: $pre^{\forall}(X) \stackrel{\mathrm{def}}{=} \{s \in S \mid \text{for all } s' \in S: \text{ if } s \longrightarrow s' \text{ then } s' \in X\}$. The following two propositions easily follows from the semantics of ICTL.

Proposition 4. *Given a BM $\mathcal{M}$ with set of states S, the following holds:*
- $[\![\varphi_1 \rightarrow \varphi_2]\!] = ([\![\varphi_1]\!]^{\mathbf{c}} \cup [\![\varphi_2]\!])^{\uparrow}$.
- $[\![\mathsf{E} \bigcirc \varphi]\!] = Pre^{\exists}([\![\varphi]\!])$.
- $[\![\mathsf{A} \bigcirc \varphi]\!] = Pre^{\forall}([\![\varphi]\!])$.
- $(Pre^{\forall}([\![X]\!]))^{\mathbf{c}} = Pre^{\exists}([\![X]\!]^{\mathbf{c}})$ *for each* $X \subseteq S$.

Proposition 5. *The following equivalences hold:*
- $\mathsf{E}(\varphi_1 \mathsf{U} \varphi_2) \equiv \varphi_2 \vee (\varphi_1 \wedge \mathsf{E} \bigcirc (\mathsf{E}(\varphi_1 \mathsf{U} \varphi_2)))$.
- $\mathsf{E}(\varphi_1 \mathsf{R} \varphi_2) \equiv \varphi_2 \wedge (\varphi_1 \vee \mathsf{E} \bigcirc (\mathsf{E}(\varphi_1 \mathsf{R} \varphi_2)))$.
- $\mathsf{A}(\varphi_1 \mathsf{U} \varphi_2) \equiv \varphi_2 \vee (\varphi_1 \wedge \mathsf{A} \bigcirc (\mathsf{A}(\varphi_1 \mathsf{U} \varphi_2)))$.
- $\mathsf{A}(\varphi_1 \mathsf{R} \varphi_2) \equiv \varphi_2 \wedge (\varphi_1 \vee \mathsf{A} \bigcirc (\mathsf{A}(\varphi_1 \mathsf{R} \varphi_2)))$.

Let $\mathcal{M}$ be a BM with set of states S and g be a monotonic function $2^S \mapsto 2^S$. The *dual* $\widetilde{g}$ of g is the function $2^S \mapsto 2^S$ defined as follows for each $X \subseteq S$: $\widetilde{g}(X) \stackrel{\mathrm{def}}{=} (g(X^{\mathbf{c}}))^{\mathbf{c}}$. Note that $\widetilde{g}$ is monotonic too. Recall that by Tarski theorem, both g and $\widetilde{g}$ have the least and the greatest fixpoints. Moreover, the following holds.

Proposition 6. *Let $\mathcal{M}$ be a BM with set of states S and $g : 2^S \mapsto 2^S$. If g is monotonic, then the greatest fixpoint (resp., least fixpoint) of g coincides with the complement of the least fixpoint (resp., greatest fixpoint) of $\widetilde{g}$.*

For all formulas φ_1 and φ_2 and BM $\mathcal{M}$ with set of states S, we consider the following monotonic functions $2^S \mapsto 2^S$ defined as follows for each $X \subseteq S$:
- $\mathsf{EU}_{\varphi_1,\varphi_2}(X) \stackrel{\mathrm{def}}{=} [\![\varphi_2]\!] \cup ([\![\varphi_1]\!] \cap \mathrm{Pre}^{\exists}(X))$
- $\mathsf{ER}_{\varphi_1,\varphi_2}(X) \stackrel{\mathrm{def}}{=} [\![\varphi_2]\!] \cap ([\![\varphi_1]\!] \cup \mathrm{Pre}^{\exists}(X))$
- $\mathsf{AU}_{\varphi_1,\varphi_2}(X) \stackrel{\mathrm{def}}{=} [\![\varphi_2]\!] \cup ([\![\varphi_1]\!] \cap \mathrm{Pre}^{\forall}(X))$
- $\mathsf{AR}_{\varphi_1,\varphi_2}(X) \stackrel{\mathrm{def}}{=} [\![\varphi_2]\!] \cap ([\![\varphi_1]\!] \cup \mathrm{Pre}^{\forall}(X))$

We deduce the following fix-point characterizations of the path quantifiers.

Theorem 3. *For all formulas φ_1 and φ_2 and every BM $\mathcal{M}$, the following holds:*
1. $[\![\mathsf{E}(\varphi_1 \mathsf{U} \varphi_2)]\!]$ *is the least fix-point of* $\mathsf{EU}_{\varphi_1,\varphi_2}$;
2. $[\![\mathsf{E}(\varphi_1 \mathsf{R} \varphi_2)]\!]$ *is the greatest fix-point of* $\mathsf{ER}_{\varphi_1,\varphi_2}$;
3. $[\![\mathsf{A}(\varphi_1 \mathsf{U} \varphi_2)]\!]$ *is the least fix-point of* $\mathsf{AU}_{\varphi_1,\varphi_2}$;
4. $[\![\mathsf{A}(\varphi_1 \mathsf{R} \varphi_2)]\!]$ *is the greatest fix-point of* $\mathsf{AR}_{\varphi_1,\varphi_2}$.

Proof. We prove 1 and 4, the proof of 2 and 3 being similar.

1. Let $X = [\![\mathsf{E}(\varphi_1 \mathsf{U} \varphi_2)]\!]$. In virtue of Propositions 4 and 5, it is clear that X is a fix-point of $\mathsf{EU}_{\varphi_1,\varphi_2}$. Let Y be another fix-point of the latter function. We show that $X \subseteq Y$. Suppose $X \neq \emptyset$ (otherwise the result is trivial) and let $x \in X$. By definition, there is a path π starting at x such that for some $j \geq 0$: $\pi(j) \in [\![\varphi_2]\!]$ and $\pi(i) \in [\![\varphi_1]\!]$ for each $0 \leq i < j$. Suppose that $\pi(j)$ is the first element of π that belongs to $[\![\varphi_2]\!]$. Since $\pi(j) \in [\![\varphi_2]\!]$, we obtain that $\pi(j) \in Y = [\![\varphi_2]\!] \cup ([\![\varphi_1]\!] \cap \mathrm{Pre}^{\exists}(Y))$. To conclude, we show that for each $0 < k \leq j$, $\pi(k) \in Y$ implies $\pi(k-1) \in Y$. Thus, suppose that $\pi(k) \in Y$. Since $\pi(k-1) \longrightarrow \pi(k)$, we obtain that $\pi(k-1) \in \mathrm{Pre}^{\exists}(Y)$, and since $\pi(k-1) \in [\![\varphi_1]\!]$, we conclude that $\pi(k-1) \in Y$ as desired.

4. Let $X = [\![\mathsf{A}(\varphi_1 \mathsf{R} \varphi_2)]\!]$. We introduce an additional temporal modality $\mathsf{U}^{\mathbf{c}}$ which is the intuitionistic dual of R:

$$\mathcal{M}, \pi \models \varphi_1 \mathsf{U}^{\mathbf{c}} \varphi_2 \Leftrightarrow \text{there is } i \geq 0 \text{ such that } \mathcal{M}, \pi(i) \not\models \varphi_2 \text{ and}$$
$$\mathcal{M}, \pi(k) \not\models \varphi_1 \text{ for all } 0 \leq k < i$$

By the semantics, $[\![\mathsf{A}(\varphi_1 \mathsf{R} \varphi_2)]\!] = [\![\mathsf{E}(\varphi_1 \mathsf{U}^{\mathbf{c}} \varphi_2)]\!]^{\mathbf{c}}$. Let us consider the monotonic function $\mathsf{EU}^{\mathbf{c}}_{\varphi_1,\varphi_2} : 2^S \to 2^S$ defined as follows for each $X \subseteq S$:

$$\mathsf{EU}^{\mathbf{c}}_{\varphi_1,\varphi_2}(X) \stackrel{\text{def}}{=} [\![\varphi_2]\!]^{\mathbf{c}} \cup ([\![\varphi_1]\!]^{\mathbf{c}} \cap \mathrm{Pre}^{\exists}(X)).$$

By proceeding as in the proof of Property 1, we deduce that $[\![\mathsf{E}(\varphi_1 \mathsf{U}^{\mathbf{c}} \varphi_2)]\!]$ is the least fixpoint of $\mathsf{EU}^{\mathbf{c}}_{\varphi_1,\varphi_2}$. By Proposition 4, $\mathsf{EU}^{\mathbf{c}}_{\varphi_1,\varphi_2}$ is the dual of the mapping $\mathsf{AR}_{\varphi_1,\varphi_2}$. Thus, since $[\![\mathsf{A}(\varphi_1 \mathsf{R} \varphi_2)]\!] = [\![\mathsf{E}(\varphi_1 \mathsf{U}^{\mathbf{c}} \varphi_2)]\!]^{\mathbf{c}}$, by Proposition 6, the result follows, i.e., $[\![\mathsf{A}(\varphi_1 \mathsf{R} \varphi_2)]\!]$ is the greatest fix-point of $\mathsf{AR}_{\varphi_1,\varphi_2}$. $\square$

4 ICTL Model Checking

We address the model-checking problem for ICTL, that is, checking for given *finite* BM $\mathcal{M}$ and ICTL formula φ, whether $\mathcal{M}, s \models \varphi$ for each initial state s. By applying the results of Section 3.2, we show that ICTL model checking has the same complexity as standard CTL model checking.

Theorem 4. *ICTL model checking is* PTIME*-complete and can be solved in time* $O(|\mathcal{M}|^2 \cdot |\Phi|)$ *for given finite BM* $\mathcal{M}$ *and formula* Φ*.*

Proof. PTIME-hardness follows from PTIME-hardness of CTL model checking [12] and the fact that checking that a finite KS $\mathcal{M}$ is an CTL-model of a formula φ reduces to checking that the BM extension of $\mathcal{M}$ with the identity partial order is an ICTL-model of φ. For the matching upper bound, we consider the model checking algorithm illustrated in Figure 7 which computes for a given finite BM $\mathcal{M}$ and ICTL formula Φ, the set of states $[\![\Phi]\!]$ which satisfy Φ. The algorithm computes the denotations $[\![\varphi]\!]$ of the subformulas φ of Φ by a bottom-up traversal of the parse tree of the formula Φ in accordance with Proposition 4 and Theorem 3. In particular, the subformulas of Φ are ordered in a queue $\mathrm{Sub}(\Phi)$,

```
for each φ ∈ Sub(Φ)
    case φ = p: ⟦φ⟧ := {s|p ∈ V(s)};
    case φ = φ1 ∨ φ2 : ⟦φ⟧ := ⟦φ1⟧ ∪ ⟦φ2⟧;
    case φ = φ1 ∧ φ2 : ⟦φ⟧ := ⟦φ1⟧ ∩ ⟦φ2⟧;
    case φ = φ1→φ2 : ⟦φ⟧ := (⟦φ1⟧^c ∪ ⟦φ2⟧)^↑;
    case φ = E ◯ φ1 : ⟦φ⟧ := Pre^∃(⟦φ1⟧);
    case φ = A ◯ φ1 : ⟦φ⟧ := Pre^∀(⟦φ1⟧);
    case φ = Eφ1Uφ2 :
        Q1 := ∅; Q2 := ⟦φ1⟧; Q3 := ⟦φ2⟧;
        while Q3 ⊈ Q1 do Q1 := Q1 ∪ Q3; Q3 := Pre^∃(Q1) ∩ Q2;
        ⟦φ⟧ := Q1;
    case φ = Aφ1Uφ2 : as in the previous case but
            we use Pre^∀ instead of Pre^∃;
    case φ = Eφ1Rφ2 :
        Q1 := S; Q2 := ⟦φ1⟧; Q3 := ⟦φ2⟧;
        while Q1 ⊈ Q3 do Q1 := Q1 ∩ Q3; Q3 := Pre^∃(Q1) ∪ Q2;
        ⟦φ⟧ := Q1;
    case φ = Aφ1Rφ2 :as in the previous case but
            we use Pre^∀ instead of Pre^∃;
return ⟦Φ⟧;
```

Fig. 7. Algorithm for ICTL model checking

where a subformula φ_1 precedes φ_2 iff $|\varphi_1| \leq |\varphi_2|$. Each while loop requires at most $O(|S|)$ iterations, and the time spent by a while loop is at most $O(|\mathcal{M}|)$, as the computation of pre-images can be implemented in such a way that each transition of the model is visited exactly once during the various iterations of the loop. Note that the computation of the upward closure $X^{\uparrow}$ of a subset X of S requires quadratic time in $|X|$. Each subformula is processed exactly once. Hence, the algorithm runs in time $O(|\mathcal{M}|^2 \cdot |\Phi|)$.

Implementation & Benchmark. To evaluate the performance of our proposed ICTL model checking algorithm, we extended the VITAMIN model checker [17] to support the specific semantics of intuitionistic CTL. The implementation adheres to privacy and copyright constraints while ensuring reproducibility of our results. The data representation follows the schema introduced in [17], and the model structure is based on the setting illustrated in Figure 6.

Table 1. Average model checking time for ICTL over 10 runs

#	$States_{n\times m}$	N_{States}	$\overline{T_{\mathrm{MC}}}$ (s)
1	10 × 10	100	0.0036
2	20 × 20	400	0.0201
3	50 × 50	2500	0.2115
4	80 × 80	6400	1.0212
5	90 × 90	8100	1.6683
6	100 × 100	10000	2.5088
7	150 × 150	22500	10.592
8	180 × 180	32400	24.227
9	200 × 200	40000	31.499
10	230 × 230	52900	54.788

A model generation algorithm was developed to generalize over a variable number m of db operations and n observations, enabling scalability and adaptability of our approach. The experimental evaluation was conducted using the model $\mathcal{M}$ and the ICTL formulas φ described in Section 3.1. All experiments were run on a desktop machine equipped with an *Intel i7 12700K processor and 64GB of RAM*, providing a consistent and controlled testing environment. Table 1 reports the average execution time (in seconds) across 10 independent runs for increasing model sizes, achieved by varying the number of observations and transactions. The results show that our ICTL model checking algorithm exhibits efficient scalability: as the number of states grows, execution times remain stable without significant degradation. Our implementation of the model checking process, depicted in Figure 7, accepts both the model and the logical formula as inputs and systematically performs the checks by reproducing the testing procedure described. This ensures that the experiments can be reliably replicated under similar conditions, fostering transparency and facilitating the validation of our findings.

5 Conclusion

In this work, we introduced Intuitionistic CTL (ICTL), an extension of classical CTL that incorporates the principles of intuitionistic logic. This framework provides a richer and more flexible formalism for reasoning about systems with incomplete or evolving knowledge. While classical CTL assumes that every proposition is either definitively true or false in a given state, thus presuming complete information, ICTL relaxes this assumption, allowing for the representation of partial knowledge and constructively acquired truths. This makes it especially suited to applications involving information refinement, sensing, and dynamic knowledge acquisition, where certainty is often deferred or progressively built. Looking ahead, we plan to deepen our understanding of ICTL by developing a complete axiomatization of the logic. Furthermore, we aim to generalize the intuitionistic approach beyond CTL, extending it to more expressive temporal logics such as CTL*. In parallel, we are also investigating an intuitionistic variant of Alternating-time Temporal Logic (ATL) [1], with the goal of capturing strategic abilities under dynamic and constructively represented knowledge.

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