Worksheet 4, Oct 1, 2020

Norms and condition numbers

1. Compute $\|A\|_\infty$ and $\|A\|_1$ for the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 7 & 2 & 3 & 5 \\ 2 & -4 & 3 & 8 \\ -3 & 5 & 3 & 1 \end{bmatrix}$$

2. Find $\|A\|_2$ for the matrix

$$A = \begin{bmatrix} 1 & \varepsilon \\ \varepsilon & 1 \end{bmatrix},$$

where $\varepsilon \in (0,1)$ (Hint: for symmetric matrices A, the eigenvalues of A^TA are simply the squares of the eigenvalues of A).

3. Continued from previous item: Suppose that you have two systems

$$\begin{aligned} x_1 + \varepsilon x_2 &= b_1 \\ \varepsilon x_1 + x_2 &= b_2 \end{aligned} \qquad \text{and} \qquad \begin{aligned} \tilde{x}_1 + \varepsilon \tilde{x}_2 &= \tilde{b}_1 \\ \varepsilon \tilde{x}_1 + \tilde{x}_2 &= \tilde{b}_2 \end{aligned}$$

where $\tilde{\boldsymbol{b}} = (\tilde{b}_1, \tilde{b}_2)^T$ is approximately equal to $\boldsymbol{b} = (b_1, b_2)^T$, with a 5% relative error, that is $\frac{\|\tilde{\boldsymbol{b}} - \boldsymbol{b}\|_2}{\|\boldsymbol{b}\|_2} \leq 0.05$. Find an upper bound for the relative error $\frac{\|\tilde{\boldsymbol{x}} - \boldsymbol{x}\|_2}{\|\boldsymbol{x}\|_2}$ where $\tilde{\boldsymbol{x}} = (\tilde{x}_1, \tilde{x}_2)^T$ and $\boldsymbol{x} = (x_1, x_2)^T$. This upper bound will depend on ε.

4. Show that for symmetric positive definite (i.e., all eigenvalues are positive) matrices $A \in \mathbb{R}^{n\times n}$, the 2-norm condition number can also be computed as the ratio between the largest and the smallest eigenvalue of A, i.e.: $\kappa_2(A) = \lambda_{\max}/\lambda_{\min}$. Hint: Think about what the largest eigenvalue of A^{-1} is.

5. The Hilbert matrix $H \in \mathbb{R}^{n\times n}$ is a matrix with entries

$$h_{ij} = \frac{1}{i+j-1}.$$

Using MATLAB or Python, compute the 2-norm based condition numbers for $n = 3, 5, 10, 20, 25$. Let's consider a relative right hand side perturbation $\delta\boldsymbol{b}$ of a linear system with $\|\delta\boldsymbol{b}\|_2/\|\boldsymbol{b}\|_2 \approx 10^{-15}$. Write down the corresponding bounds $\|\delta\boldsymbol{x}\|_2/\|\boldsymbol{x}\|_2$ from the theory we discussed in class.

Now, let's compute the actual error. Use the right hand side vector with entries $b_i = \sum_{j=1}^n (j/(i+j-1))$ chosen such that the solution vector has entries $x_i = i$. Now, Compute the numerical solutions[1] $\boldsymbol{x}$, then re-compute $\boldsymbol{b} = H\boldsymbol{x}$ and compare the relative right hand side error and the relative error in the solutions. How much are these better than the estimates you got from the condition number?

[1] Note that all these computations contain tiny errors due to the final precision of computer computations.