Analysis I - Chapter 2

Theorems, Postulates, Definitions & Remarks

2.1 - Limits of Sequences

Definition 2.1. A sequence of real numbers $\{x_n\}$ is said to converge to a real number $a \in \mathbb{R}$ if and only if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ (which in general depends on ε) such that

$$n \geq N \quad implies \quad |x_n - a| < \varepsilon.$$

Remark 2.4. A sequence can have at most one limit.

Definition 2.5. By a subsequence of a sequence $\{x_n\}_{n\in\mathbb{N}}$, we shall mean a sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$, where each $n_k \in \mathbb{N}$ and $n_1 < n_2 < ....$

Remark 2.6. If $\{x_n\}_{n\in\mathbb{N}}$ converges to a and $\{x_{n_k}\}_{k\in\mathbb{N}}$ is any subsequence of $\{x_n\}_{n\in\mathbb{N}}$, then x_{n_k} converges to a as $k \to \infty$.

Definition 2.7. Let $\{x_n\}$ be a sequence of real numbers.
i. The sequence $\{x_n\}$ is said to be bounded above if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded above.
ii. The sequence $\{x_n\}$ is said to be bounded below if and only if the set $\{x_n : n \in \mathbb{N}\}$ is bounded below.
iii. $\{x_n\}$ is said to be bounded if and only if it is bounded both above and below.

Theorem 2.8. Every convergent sequence is bounded.

2.2 - Limit Theorems

Theorem 2.9 (**SQUEEZE THEOREM**)**.** Suppose that $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ are real sequences.
i. If $x_n \to a$ and $y_n \to a$ (the SAME a) as $n \to \infty$, and if there is an $N_0 \in \mathbb{N}$ such that

$$x_n \leq w_n \leq y_n \quad for \quad n \geq N_0,$$

then $w_n \to a$ as $n \to \infty$.
ii. If $x_n \to 0$ as $n \to \infty$ and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.

Theorem 2.11. Let $E \subset \mathbb{R}$. If E has a finite supremum (respectively, a finite infimum), then there is a sequence $x_n \in E$ such that $x_n \to sup\, E$ (respectively, a sequence $y_n \in E$ such that $y_n \to inf\ E$) as $n \to \infty$.

Theorem 2.12. Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences and that $\alpha \in \mathbb{R}$. If $\{x_n\}$ and $\{y_n\}$ are convergent, then
i.

$$\lim_{n\to\infty} (x_n + y_n) = \lim_{n\to\infty} x_n + \lim_{n\to\infty} y_n,$$

ii.

$$\lim_{n\to\infty} (\alpha x_n) = \alpha \lim_{n\to\infty} x_n,$$

iii.

$$\lim_{n\to\infty} (x_n y_n) = (\lim_{n\to\infty} x_n)(\lim_{n\to\infty} y_n),$$

If, in addition, $y_n \neq 0$ and $\lim_{n\to\infty} y_n \neq 0$, then
iv.

$$\lim_{n\to\infty} \frac{x_n}{y_n} = \frac{\lim_{n\to\infty} x_n}{\lim_{n\to\infty} y_n}$$

(In particular, all these limits exist.)

Definition 2.14. Let $\{x_n\}$ be a sequence of real numbers.
i. $\{x_n\}$ is said to diverge to $+\infty$ (notation: $x_n \to +\infty$ as $n \to \infty$ or $lim_{n\to\infty} x_n = +\infty$) if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad implies \quad x_n > M.$$

ii. $\{x_n\}$ is said to diverge to $-\infty$ (notation: $x_n \to -\infty$ as $n \to \infty$ or $lim_{n\to\infty} x_n = -\infty$) if and only if for each $M \in \mathbb{R}$ there is an $N \in \mathbb{N}$ such that

$$n \geq N \quad implies \quad x_n < M.$$

Theorem 2.15. Suppose that $\{x_n\}$ and $\{y_n\}$ are real sequences such that $x_n \to +\infty$ (respectively, $x_n \to -\infty$) as $n \to \infty$.
i. If y_n is bounded below (respectively, y_n is bounded above), then

$$\lim_{n\to\infty} (x_n + y_n) = +\infty \quad (respectively \quad \lim_{n\to\infty} (x_n + y_n) = -\infty).$$

ii. If $\alpha > 0$, then

$$\lim_{n\to\infty} (\alpha x_n) = +\infty \quad (respectively \quad \lim_{n\to\infty} (\alpha x_n) = -\infty).$$

iii. If $y_n > M_0$ for some $M_0 > 0$ and all $n \in \mathbb{N}$, then

$$\lim_{n\to\infty} (x_n y_n) = +\infty \quad (respectively \quad \lim_{n\to\infty} (x_n y_n) = -\infty).$$

iv. If $\{y_n\}$ is bounded and $x_n \neq 0$, then

$$\lim_{n\to\infty} \frac{y_n}{x_n} = 0.$$

Corollary 2.16. Let $\{x_n\}$, $\{y_n\}$ be real sequences and α, x, y be extended real numbers. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then

$$\lim_{n\to\infty} (x_n + y_n) = x + y$$

provided that the right side is not of the form $\infty - \infty$, and

$$\lim_{n\to\infty} (\alpha x_n) = \alpha x, \quad \lim_{n\to\infty} (x_n y_n) = xy$$

provided that none of these products is of the form $0 \cdot \pm\infty$.

Theorem 2.17 (**COMPARISON THEOREM**)**.** Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences. If there is an $N_0 \in \mathbb{N}$ such that

$$x_n \leq y_n \quad for \quad n \geq N_0,$$

then

$$\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n.$$

In particular, if $x_n \in [a, b]$ converges to some point c, then point c must belong to $[a, b]$.

2.3 - BOLZANO-WEIERSTRASS THEOREM

Definition 2.18. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers.
i. $\{x_n\}$ is said to be increasing (respectively, strictly increasing) if and only if $x_1 \leq x_2 \leq ...$ (respectively $x_1 < x_2 < ...$).
ii. $\{x_n\}$ is said to be decreasing (respectively, strictly decreasing) if and only if $x_1 \geq x_2 \geq ...$ (respectively $x_1 > x_2 > ...$).
iii. $\{x_n\}$ is said to be monotone if and only if it is either increasing or decreasing.
(Some authors call decreasing sequences nonincreasing and increasing sequences nondecreasing.)

Theorem 2.19 (**MONOTONE CONVERGENCE THEOREM**)**.** If $\{x_n\}$ is increasing and bounded above, or if $\{x_n\}$ is decreasing and bounded below, then $\{x_n\}$ converges to a finite limit.

Definition 2.22. A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be nested if and only if

$$I_1 \supseteq I_2 \supseteq ....$$

Theorem 2.23 (**NESTED INTERVAL PROPERTY**). If $\{I_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty closed bounded intervals, then $E := \bigcap_{n=1}^{\infty} I_n$ is nonempty. Moreover, if the lengths of these intervals satisfy $|I_n| \to 0$ as $n \to \infty$, then E is a single point.

Remark 2.24. The Nested Interval Property might not hold if "closed" is omitted.

Remark 2.25. The Nested Interval Property might not hold if "bounded" is omitted.

Theorem 2.26 (**BOLZANO-WEIERSTRASS THEOREM**). Every bounded sequence of real numbers has a convergent subsequence.

2.4 - CAUCHY SEQUENCES

Definition 2.27. A sequence of points $x_n \in \mathbb{R}$ is said to be Cauchy (in $\mathbb{R}$) if and only if for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that

$$n, m \geq N \quad \textit{imply} \quad |x_n - x_m| < \varepsilon.$$

Remark 2.28. If $\{x_n\}$ is convergent, then $\{x_n\}$ is Cauchy.

Theorem 2.29 (**CAUCHY**). Let $\{x_n\}$ be a sequence of real numbers. Then $\{x_n\}$ is Cauchy if and only if $\{x_n\}$ converges (to some point a in $\mathbb{R}$).

Remark 2.31. A sequence that satisfies $x_{n+1} - x_n \to 0$ is not necessarily Cauchy.

Wade, W. R. (2009). *An Introduction to Analysis* (4th ed.). Prentice Hall.