Analysis I - Chapter 1

Theorems, Postulates, Definitions & Remarks

1.2 - Ordered Field Axioms

Postulate 1 (FIELD AXIOMS). There are functions $+$ and $\cdot$, defined on $\mathbb{R}^2 := \mathbb{R} \times \mathbb{R}$, which satisfy the following properties for every $a, b, c \in \mathbb{R}$:
Closure Properties. $a + b$ and $a \cdot b$ belong to $\mathbb{R}$.
Associative Properties. $a + (b + c) = (a + b) + c$ and $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
Commutative Properties. $a + b = b + a$ and $a \cdot b = b \cdot a$.
Distributive Law. $a \cdot (b + c) = a \cdot b + a \cdot c$.
Existence of the Additive Identity. There is a unique element $0 \in \mathbb{R}$ such that $0 + a = a$ for all $a \in \mathbb{R}$.
Existence of the Multiplicative Identity. There is a unique element $1 \in \mathbb{R}$ such that $1 \neq 0$ and $1 \cdot a = a$ for all $a \in \mathbb{R}$.
Existence of the Additive Inverses. For every $x \in \mathbb{R}$ there is a unique element $-x \in \mathbb{R}$ such that $x + (-x) = 0$.
Existence of the Multiplicative Inverses. For every $x \in \mathbb{R} \setminus \{0\}$ there is a unique element $x^{-1} \in \mathbb{R}$ such that $x \cdot x^{-1} = 1$.

Postulate 2 (ORDER AXIOMS). There is a relation $<$ on $\mathbb{R} \times \mathbb{R}$ that has the following properties:
Trichotomy Property. Given $a, b \in \mathbb{R}$, one and only one of the following statements holds:

$$a < b, \quad b < a, \quad \textit{or} \quad a = b.$$

Transitive Property. For $a, b, c \in \mathbb{R}$,

$$a < b \ \textit{ and } \ b < c \ \textit{ imply } \ a < c.$$

The Additive Property. For $a, b, c \in \mathbb{R}$,

$$a < b \ \textit{ and } \ c \in \mathbb{R} \ \textit{ imply } \ a + c < b + c.$$

The Multiplicative Properties. For $a, b, c \in \mathbb{R}$,

$$a < b \ \textit{ and } \ c > 0 \ \textit{ imply } \ ac < bc$$

and

$$a < b \ \textit{ and } \ c < 0 \ \textit{ imply } \ bc < ac.$$

Remark 1.1. We will assume that the sets $\mathbb{N}$ and $\mathbb{Z}$ satisfy the following properties.
i. If $n, m \in \mathbb{Z}$, then $n+m$, $n-m$, and mn belong to $\mathbb{Z}$.
ii. If $n \in \mathbb{Z}$, then $n \in \mathbb{N}$ if and only if $n \geq 1$.
iii. There is no $n \in \mathbb{Z}$ that satisfies $0 < n < 1$.

Definition 1.4. The absolute value of a number $a \in \mathbb{R}$ is the number

$$|a| := \begin{cases} a & a \geq 0 \\ -a & a < 0. \end{cases}$$

Remark 1.5. The absolute value is multiplicative; that is, $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.

Theorem 1.6 (FUNDAMENTAL THEOREM OF ABSOLUTE VALUES). Let $a \in \mathbb{R}$ and $M \geq 0$. Then $|a| \leq M$ if and only if $-M \leq a \leq M$.

Theorem 1.7. The absolute value satisfies the following three properties.
i.[POSITIVE DEFINITE]. For all $a \in \mathbb{R}$, $|a| \geq 0$ with $|a| = 0$ if and only if $a = 0$.
ii.[SYMMETRIC]. For all $a, b \in \mathbb{R}$, $|a-b| = |b-a|$.
iii.[TRIANGLE INEQUALITIES]. For all $a, b \in \mathbb{R}$,

$$|a+b| \leq |a| + |b| \quad \textit{and} \quad \Big||a| - |b|\Big| \leq |a-b|.$$

Theorem 1.9. Let $x, y, a \in \mathbb{R}$.
i. $x < y + \varepsilon$ for all $\varepsilon > 0$ if and only if $x \leq y$.
ii. $x > y + \varepsilon$ for all $\varepsilon > 0$ if and only if $x \geq y$.
iii. $|a| < \varepsilon$ for all $\varepsilon > 0$ if and only if $a = 0$.

1.3 - Completeness Axiom

Definition 1.10. Let $E \subset \mathbb{R}$ be nonempty.
i. The set E is said to be bounded above if and only if there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in E$ in which case M is called an upper bound of E.
ii. A number s is called a supremum of the set E if and only if s is an upper bound of E and $s \leq M$ for all upper bounds M of E. (In this case, we shall say that E has a finite supremum s and write $s = sup\, E$).

Remark 1.12. If a set has one upper bound, it has infinitely many upper bounds.

Remark 1.13. If a set has a supremum, then it has only one supremum.

Theorem 1.14 (**APPROXIMATION PROPERTY FOR SUPREMA**). If E has a finite supremum and $\varepsilon > 0$ is any positive number, then there is a point $a \in E$ such that

$$sup\, E - \varepsilon < a \leq sup\, E.$$

Theorem 1.15. If $E \subset \mathbb{Z}$ has a supremum, then $sup\, E \in E$. In particular, if the supremum of a set, which contains only integers, exists, that supremum must be an integer.

Postulate 3 (**COMPLETENESS AXIOM**). If E is a nonempty subset of $\mathbb{R}$ that is bounded above, then E has a finite supremum.

Theorem 1.16 (**ARCHIMEDEAN PRINCIPLE**). Given real numbers a and b, with $a > 0$, there is an integer $n \in \mathbb{N}$ such that $b < na$.

Theorem 1.18 (**DENSITY OF RATIONALS**). If $a, b \in \mathbb{R}$ satisfy $a < b$, then there is a $q \in \mathbb{Q}$ such that $a < q < b$.

Definition 1.19. Let $E \subset \mathbb{R}$ be nonempty.
i. The set E is said to be bounded below if and only if there is an $m \in \mathbb{R}$ such that $a \geq m$ for all $a \in E$ in which case m is called a lower bound of E.
ii. A number t is called an infimum of the set E if and only if t is a lower bound of E and $t \geq m$ for all lower bounds m of E. In this case, we shall say that E has an infimum t and write $t = inf\, E$.
iii. E is said to be bounded if and only if it is bounded both above and below.

Theorem 1.20 (**REFLECTION PRINCIPLE**). Let $E \subseteq \mathbb{R}$ be nonempty.
i. E has a supremum if and only if $-E$ has an infimum, in which case

$$inf(-E) = -sup(E).$$

ii. E has an infimum if and only if $-E$ has a supremum, in which case

$$sup(-E) = -inf(E).$$

Theorem 1.21 (**MONOTONE PROPERTY**). Suppose $A \subseteq B$ are nonempty subsets of $\mathbb{R}$.
i. If B has a supremum, then $sup\, A \leq sup\, B$.
ii. If B has an infimum, then $inf\, A \geq inf\, B$.

1.4 - Mathematical Induction

Theorem 1.22 (**WELL-ORDERING PRINCIPLE**)**.** If E is a nonempty subset of $\mathbb{N}$, then E has a least element (ie, E has a finite infimum and $inf\ E \in E$).

Theorem 1.23. Suppose for each $n \in \mathbb{N}$ that $A(n)$ is a proposition (ie, a verbal statement or formula) which satisfies the following properties:
i. $A(1)$ is true.
ii. For every $n \in \mathbb{N}$ for which $A(n)$ is true, $A(n+1)$ is also true.
Then $A(n)$ is true for all $n \in \mathbb{N}$.

Lemma 1.25. If $n, k \in \mathbb{N}$ and $1 \leq k \leq n$, then

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}.$$

Theorem 1.26 (**BINOMIAL FORMULA**)**.** If $a, b \in \mathbb{R}$, $n \in \mathbb{N}$, and 0^0 is interpreted to be 1, then

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k.$$

Remark 1.27. If $x > 1$ and $x \notin \mathbb{N}$, then there is an $n \in \mathbb{N}$ such that $n < x < n+1$.

Remark 1.28. If $n \in \mathbb{N}$ is not a perfect square (ie, if there is no $m \in \mathbb{N}$ such that $n = m^2$), then $\sqrt{n}$ is irrational.