An Examination of Partitions of Loops of Order 5 That Result in a Schur Ring-Like Structure

Andrew Hatch

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We begin by defining a quasigroup. (Definition from Introduction to Abstract Algebra by Jonathan D.H. Smith)

Definition 1. ***Quasigroup.*** *Let a given set Q be closed under a multiplication $x \cdot y$ or xy of its elements x, y. Suppose that when the equation*

$$x \cdot y = z$$

holds for elements x, y, z of Q, then knowledge of any two of x, y, z specifies the third uniquely. In this case, the structure $(Q, \cdot)$ or Q is said to be a quasigroup.

Quasigroups differ from groups in that they are not necessarily associative, but they can be. Quasigroups have a form of cancellation or division that acts similarly to the associative law. Thus, any group G forms a quasigroup.

Due to the form of division shown above in Definition 1, the Cayley Table of a quasigroup will always be a Latin square. That is, each element appears exactly once in any given row or column. In fact, a finite nonempty set will form a quasigroup if and only if its Cayley table is a Latin square. Thus, we can determine whether a set forms a quasigroup under a given operation simply by looking at the Cayley table.

Quasigroups can also have a weaker form of associativity called power associativity.

Definition 2. ***Power associativity.*** *A quasigroup $(Q, \cdot)$ is power associative if for an element a in Q,*

$$a(a(aa)) = a(aa)a = (aa)(aa) = ((aa)a)a$$

In other words, a^n for some $n \in \mathbb{N}$ is well-defined.

Elements of a quasigroup can have distinct left and right orders. These orders are denoted as (m, n). We next define the (m, n)-order of an element in a quasigroup. (Definition from "On orders of elements in quasigroups" by Victor Shcherbacov)

(Link to source)

Definition 3. ***(m,n)-orders of elements.*** *An element a of a quasigroup $(Q, \cdot)$ has the order (m,n) (or element a is an (m,n)-element) if there exist natural numbers m,n such that $L_a^m = R_a^n = e$ and the element a is not the (m_1, n_1)-element for any integers m_1, n_1 such that $1 \leq m_1 < m$, $1 \leq n_1 < n$.*

A quasigroup that also has an identity element is called a loop. The focus of this project is examining the loops of order 5. There are 5 loops of order 5 up to isomorphism. The Cayley tables of these loops are:

	1	a	b	c	d
1	1	a	b	c	d
a	a	1	c	d	b
b	b	d	1	a	c
c	c	b	d	1	a
d	d	c	a	b	1

	1	a	b	c	d
1	1	a	b	c	d
a	a	1	d	b	c
b	b	d	c	a	1
c	c	b	1	d	a
d	d	c	a	1	b

	1	a	b	c	d
1	1	a	b	c	d
a	a	b	c	d	1
b	b	c	d	1	a
c	c	d	1	a	b
d	d	1	a	b	c

	1	a	b	c	d
1	1	a	b	c	d
a	a	1	c	d	b
b	b	c	d	1	a
c	c	d	a	b	1
d	d	b	1	a	c

	1	a	b	c	d
1	1	a	b	c	d
a	a	b	1	d	c
b	b	c	d	1	a
c	c	d	a	b	1
d	d	1	c	a	b

We can create Cayley tables of partitions of the elements of these loops that result in a Schur Ring structure (that is, each entry in the Cayley table is a linear combination of its own elements).

We begin with the first listed loop.

	1	a	b	c	d
1	1	a	b	c	d
a	a	1	c	d	b
b	b	d	1	a	c
c	c	b	d	1	a
d	d	c	a	b	1

The following partitions successfully result in an S-Ring structure for this loop:

$$\{\{1\},\{a\},\{b\},\{c\},\{d\}\}$$
$$\{\{1\},\{a+b+c+d\}\}$$
$$\{\{1\},\{a\},\{b+c+d\}\}$$
$$\{\{1\},\{b\},\{a+c+d\}\}$$
$$\{\{1\},\{c\},\{a+b+d\}\}$$
$$\{\{1\},\{d\},\{a+b+c\}\}$$
$$\{\{1\},\{a+b\},\{c+d\}\}$$
$$\{\{1\},\{a+c\},\{b+d\}\}$$
$$\{\{1\},\{a+d\},\{b+c\}\}$$

The resulting Cayley table of the first partition, the identity partition, is, of course, the loop itself, as it is for all of the following loops.

We will take a closer look at the structure of this loop. It is not commutative, as shown by

$$ab = c \neq d = ba.$$

It also is not associative. Consider

$$(ac)b = db = a \neq b = ad = a(cb).$$

However, the loop does have a form of associativity. Since each element has order (2,2), this loop is power associative. For example, $a^2a = 1a = a = a1 = aa^2$. While power associativity is a weaker condition than associativity, it does provide some benefit. It means that an element raised to some power, such as c^5, is well-defined. Any element raised to an odd power will result in the element itself and raised to an even power will give the identity.

We next move on to examine the properties of the partitions of this loop. The resulting Cayley table of the second partition is:

	1	a+b+c+d
1	1	a+b+c+d
a+b+c+d	a+b+c+d	4(1) + 3(a+b+c+d)

Since the Cayley table is symmetric, it is commutative. Also, it will be shown later that the partition $\{\{1\}\{a+b+c+d\}\}$ results in an identical Cayley table for all five of the loops.

This partition is also associative.

$$(a+b+c+d)^2(a+b+c+d) = (4(1)+3(a+b+c+d))(a+b+c+d)$$
$$= 4(a+b+c+d)+3(4(1)+3(a+b+c+d)) = 13(a+b+c+d)+12(1)$$
$$(a+b+c+d)(a+b+c+d)^2 = (a+b+c+d)(4(1)+3(a+b+c+d))$$
$$= 4(a+b+c+d)+3(4(1)+3(a+b+c+d)) = 13(a+b+c+d)+12(1)$$

Thus,

$$(a+b+c+d)^2(a+b+c+d) = (a+b+c+d)(a+b+c+d)^2 = (a+b+c+d)^3,$$

so $(aa)a = a(aa)$ for the single non-identity element in the partition. Thus, the partition is associative.

The following partitions that contain a single non-identity element by itself successfully form an S-ring structure because that single element is its own inverse. Otherwise, the partition does not work. Since each element in this loop is its own inverse, all partitions that group a single element by itself and group the other three elements together work to form an S-ring structure. The overall structure of these Cayley tables is the same for each loop in which they work. And since each of these tables is symmetric, they are commutative.

The resulting Cayley table of the third partition is:

	1	a	b+c+d
1	1	a	b+c+d
a	a	1	b+c+d
b+c+d	b+c+d	b+c+d	3(1)+3(a)+b+c+d

Let us test the associativity of this partition.

$$(aa)a = (1)a = a = a(1) = a(aa)$$

$$(a(b+c+d))(a) = (b+c+d)(a) = b+c+d = (a)(b+c+d) = (a)((b+c+d)a)$$

$$((b+c+d)(a))(b+c+d) = (b+c+d)(b+c+d) = 3(1)+3(a)+b+c+d$$
$$= (b+c+d)(b+c+d) = (b+c+d)((a)(b+c+d))$$

$$((b+c+d)(b+c+d))(b+c+d) = (3(1)+3(a)+b+c+d)(b+c+d)$$
$$= 3(b+c+d)+3(b+c+d)+3(1)+3(a)+(b+c+d)$$
$$= 7(b+c+d)+3(1)+3(a)$$

$$(b+c+d)((b+c+d)(b+c+d)) = (b+c+d)(3(1)+3(a)+b+c+d)$$
$$= 3(b+c+d)+3(b+c+d)+3(1)+3(a)+(b+c+d)$$
$$= 7(b+c+d)+3(1)+3(a)$$

Thus, since $(ab)a = a(ba)$ for all a, b in the partition, the partition is associative. A similar process can be done with each of the remaining partitions that contain a single non-identity element by itself and the other three elements grouped together.

The resulting Cayley table of the fourth partition is:

	1	b	a+c+d
1	1	b	a+c+d
b	b	1	a+c+d
a+c+d	a+c+d	a+c+d	3(1)+3(b)+a+c+d

The resulting Cayley table of the fifth partition is:

	1	c	a+b+d
1	1	c	a+b+d
c	c	1	a+b+d
a+b+d	a+b+d	a+b+d	3+3(c)+a+b+d

The resulting Cayley table of the sixth partition is:

	1	d	a+b+c
1	1	d	a+b+c
d	d	1	a+b+c
a+b+c	a+b+c	a+b+c	3(1)+3(d)+a+b+c

We next move to the partitions that group elements in pairs of two. These partitions also result in a symmetric Cayley table and are therefore commutative.

The resulting Cayley table of the seventh partition is:

	1	a+b	c+d
1	1	a+b	c+d
a+b	a+b	2(1)+c+d	a+b+c+d
c+d	c+d	a+b+c+d	2(1)+a+b

$$((a+b)(a+b))(a+b) = (2(1)+(c+d))(a+b) = 2(a+b)+a+b+c+d = 3(a+b)+(c+d)$$

$$(a+b)((a+b)(a+b)) = (a+b)(2(1)+(c+d)) = 2(a+b)+a+b+c+d = 3(a+b)+(c+d)$$

$$((a+b)(c+d))(a+b) = ((a+b)+(c+d))(a+b)$$
$$= 2(1)+(c+d)+(a+b)+(c+d) = 2(1)+(a+b)+2(c+d)$$

$$(a+b)((c+d)(a+b)) = (a+b)((a+b)+(c+d))$$
$$= 2(1)+(c+d)+(a+b)+(c+d) = 2(1)+(a+b)+2(c+d)$$

$$((c+d)(a+b))(c+d) = ((a+b)+(c+d))(c+d)$$
$$= (a+b)+(c+d)+2(1)+(a+b) = 2(1)+2(a+b)+(c+d)$$

$$(c+d)((a+b)(c+d)) = (c+d)((a+b)+(c+d))$$
$$= (a+b)+(c+d)+2(1)+(a+b) = 2(1)+2(a+b)+(c+d)$$

Thus, since $(ab)a = a(ba)$ for all a, b in the partition, the partition is associative. A similar process can be done with each of the remaining partitions that group the elements in pairs of two.

The resulting Cayley table of the eighth partition is:

	1	a+c	b+d
1	1	a+c	b+d
a+c	a+c	2(1)+b+d	a+b+c+d
b+d	b+d	a+b+c+d	2(1)+a+c

The resulting Cayley table of the ninth partition is:

	1	a+d	b+c
1	1	a+d	b+c
a+d	a+d	2(1)+b+c	a+b+c+d
b+c	b+c	a+b+c+d	2(1)+a+d

Next, we have the loop

	1	a	b	c	d
1	1	a	b	c	d
a	a	1	d	b	c
b	b	d	c	a	1
c	c	b	1	d	a
d	d	c	a	1	b

This loop has far fewer partitions that work:

$$\{\{1\},\{a\},\{b\},\{c\},\{d\}\}$$
$$\{\{1\},\{a+b+c+d\}\}$$
$$\{\{1\},\{a\},\{b+c+d\}\}$$

The resulting Cayley table of the first partition is, once again, the loop itself. The Cayley table is not symmetric and the loop is thus non-commutative, as shown by

$$bc = a \neq 1 = cb$$

The loop is also non-associative. Consider

$$(ab)d = dd = b \neq a = a1 = a(bd)$$

Additionally, this loop is not power associative. This is evidenced by the fact that a is the only element of order (2,2).

Consider

$$d(d(dd)) = d(db) = da = c \neq d = 1d = (bd)d = ((dd)d)d$$

Thus, d^4 is not well-defined since the left and right operations do not agree.

Next, let us find the (m,n) order of each element in the loop.

$$(aa) = 1, \text{ so } |a| = 2$$
$$(bb)b = (cb) = 1, \text{ so } L_b = 3$$
$$b(bb(bb)) = b(cc) = bd = 1, \text{ so } R_b = 5$$

Thus, b is a $(3,5)$-order element.

$$(cc)c = dc = 1, \text{ so } L_c = 3$$
$$c(cc(cc)) = c(dd) = cb = 1, \text{ so } R_c = 5$$

Thus, c is a $(3,5)$-order element.

$$(dd)d = bd = 1, \text{ so } L_d = 3$$
$$d(dd(dd)) = d(bb) = dc = 1, \text{ so } R_d = 5$$

Thus, d is a $(3,5)$-order element.

Let us next move on to the other partitions of the loop. The resulting Cayley table of the second partition is:

	1	a+b+c+d
1	1	a+b+c+d
a+b+c+d	a+b+c+d	4(1) + 3(a+b+c+d)

The resulting Cayley table of the third partition is:

	1	a	b+c+d
1	1	a	b+c+d
a	a	1	b+c+d
b+c+d	b+c+d	b+c+d	3(1) + 3(a) + b+c+d

The next loop we will examine is:

	1	a	b	c	d
1	1	a	b	c	d
a	a	b	c	d	1
b	b	c	d	1	a
c	c	d	1	a	b
d	d	1	a	b	c

The successful partitions of this loop are:

$$\{\{1\}, \{a\}, \{b\}, \{c\}, \{d\}\}$$
$$\{\{1\}, \{a + b + c + d\}\}$$
$$\{\{1\}, \{a + d\}, \{b + c\}\}$$

This loop is $(Z_5, +)$. Since it is a cyclic group, it is abelian. It is therefore associative and commutative.

This loop has no elements of order 2. Since it is associative and commutative, the left and right orders will be the same. We proceed to find the order of each element.

$$(((aa)a)a)a = ((ba)a)a = (ca)a = da = 1$$
$$a(a(a(aa))) = a(a(ab)) = a(ac) = ad = 1$$

Thus, a is a $(5,5)$-order element.

$$(((bb)b)b)b = ((db)b)b = (ab)b = cb = 1$$

Thus, b is a $(5,5)$-order element.

$$(((cc)c)c)c = ((ac)c)c = (dc)c = bc = 1$$

Thus, c is a $(5,5)$-order element.

$$(((dd)d)d)d = ((cd)d)d = (bd)d = ad = 1$$

Thus, d is a $(5,5)$-order element.

The resulting Cayley table of the second partition is:

	1	a+b+c+d
1	1	a+b+c+d
a+b+c+d	a+b+c+d	4(1)+3(a+b+c+d)

The resulting Cayley table of the third partition is:

	1	a+d	b+c
1	1	a+d	b+c
a+d	a+d	2(1)+b+c	a+b+c+d
b+c	b+c	a+b+c+d	2(1)+a+d

The fourth loop is:

	1	a	b	c	d
1	1	a	b	c	d
a	a	1	c	d	b
b	b	c	d	1	a
c	c	d	a	b	1
d	d	b	1	a	c

The successful partitions of this loop are:

$$\{\{1\},\{a\},\{b\},\{c\},\{d\}\}$$
$$\{\{1\},\{a+b+c+d\}\}$$
$$\{\{1\},\{a\},\{b+c+d\}\}$$

This loop's Cayley table is not symmetric, so it is not commutative. Consider

$$bc = 1 \neq a = cb.$$

Additionally, this Cayley table is not associative. Consider

$$a(bc) = a1 = a \neq b = cc = (ab)c.$$

This loop is also not power associative. Consider

$$(bb)b = db = 1 \neq a = bd = b(bb)$$

The resulting Cayley table of the second partition is:

	1	a+b+c+d
1	1	a+b+c+d
a+b+c+d	a+b+c+d	4(1)+3(a+b+c+d)

The resulting Cayley table of the third partition is:

	1	a	b+c+d
1	1	a	b+c+d
a	a	1	b+c+d
b+c+d	b+c+d	b+c+d	3(1)+3(a)+b+c+d

The fifth loop is:

	1	a	b	c	d
1	1	a	b	c	d
a	a	b	1	d	c
b	b	c	d	1	a
c	c	d	a	b	1
d	d	1	c	a	b

The successful partitions of this loop are:

$$\{\{1\},\{a\},\{b\},\{c\},\{d\}\}$$
$$\{\{1\},\{a+b+c+d\}\}$$
$$\{\{1\},\{a+c\},\{b+d\}\}$$

This loop is not commutative because the Cayley table is not symmetric. Consider

$$ab = 1 \neq c = ba.$$

It is also not associative. Consider

$$(ab)c = 1c = c \neq a = a1 = a(bc)$$

This loop is also not power associative. Consider

$$(aa)a = ba = c \neq 1 = ab = a(aa)$$

Thus, a^n is not well-defined for the elements of this loop.

The (m,n)-orders of the different elements are as follows:

$$(((aa)a)a)a = ((ba)a)a = (ca)a = da = 1$$

$$a(aa) = ab = 1$$

Thus, a is a $(5,3)$-order element.

$$(((bb)b)b)b = ((db)b)b = (cb)b = ab = 1$$
$$b(b(b(bb))) = b(b(bd)) = b(ba) = bc = 1$$

Thus, b is a $(5,5)$-order element.

$$(cc)c = bc = 1$$
$$c(c(c(cc))) = c(c(cb)) = c(ca) = cd = 1$$

Thus, c is a $(3,5)$-order element.

$$(((dd)d)d)d = ((bd)d)d = (ad)d = cd = 1$$
$$d(d(d(dd))) = d(d(db)) = d(dc) = da = 1$$

Thus, d is a $(5,5)$-order element.

The resulting Cayley table of the second partition is:

	1	a+b+c+d
1	1	a+b+c+d
a+b+c+d	a+b+c+d	4(1)+3(a+b+c+d)

The resulting Cayley table of the third partition is:

	1	a+c	b+d
1	1	a+c	b+d
a+c	a+c	2(b+d)	2(1)+a+c
b+d	b+d	2(1)+a+c	a+c+b+d

This Cayley table has a slightly different structure than the same partition for the other loops.

$$((a+c)(a+c))(a+c) = (2(b+d))(a+c) = 2(2(1)+(a+c)) = 4(1)+2(a+c)$$

$$(a+c)((a+c)(a+c)) = (a+c)(2(b+d)) = 2(2(1)+(a+c)) = 4(1)+2(a+c)$$

$$((a+c)(b+d))(a+c) = (2(1)+(a+c))(a+c) = 2(a+c)+2(b+d)$$
$$(a+c)((b+d)(a+c)) = (a+c)(2(1)+(a+c)) = 2(a+c)+2(b+d)$$

$$((b+d)(a+c))(b+d) = (2(1)+(a+c))(b+d) = 2(b+d)+2(1)+(a+c)$$
$$(b+d)((a+c)(b+d)) = (b+d)(2(1)+(a+c)) = 2(b+d)+2(1)+(a+c)$$

$$((b+d)(b+d))(b+d) = ((a+c)+(b+d))(b+d)$$
$$= 2(1)+(a+c)+(a+c)+(b+d) = 2(1)+2(a+c)+(b+d)$$

$$(b+d)((b+d)(b+d)) = (b+d)((a+c)+(b+d))$$
$$= 2(1)+(a+c)+(a+c)+(b+d) = 2(1)+2(a+c)+(b+d)$$

Since $(ab)a = a(ba)$ for all a, b in the partition, the partition is associative.

There are many similarities between the structures of the different partitions for the different loops. In fact, the partitions largely result in an identical Cayley table, regardless of which loop is being examined. While the loops themselves are usually neither commutative nor associative, the partitions generally are both commutative and associative. While this project has only made an introductory examination of the properties of these partitions, there is certainly more to discover about them.