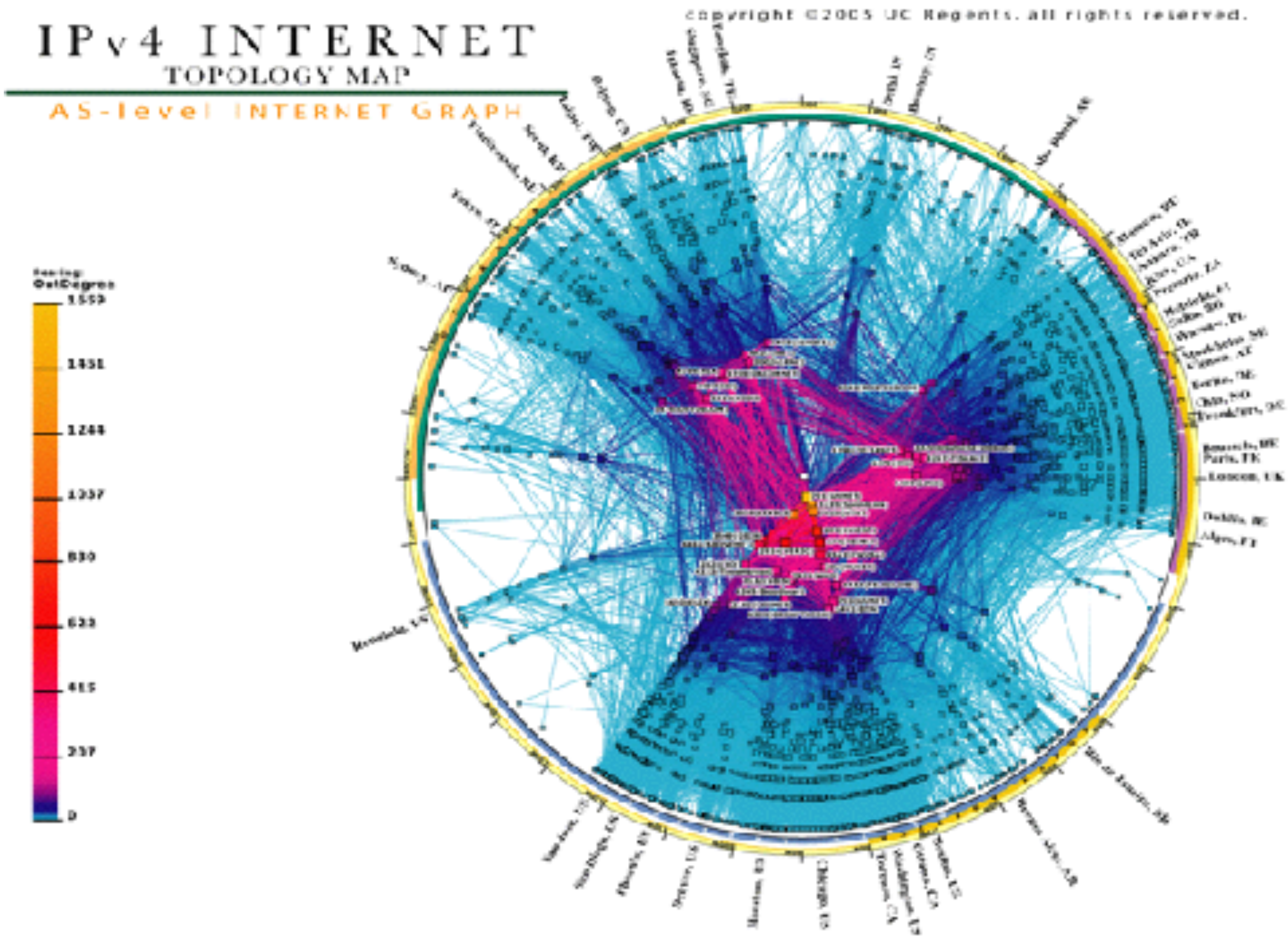
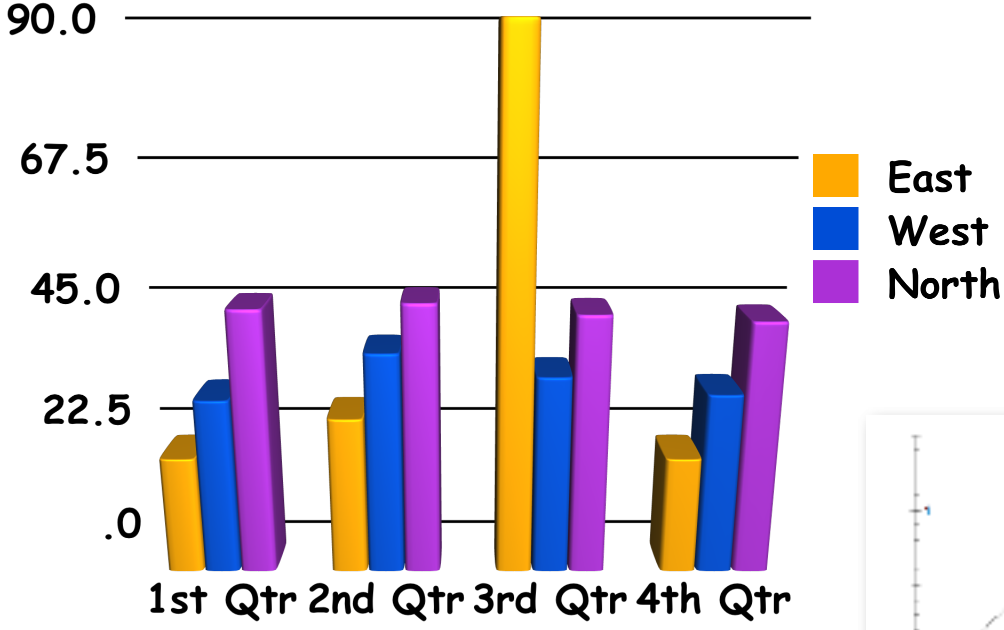


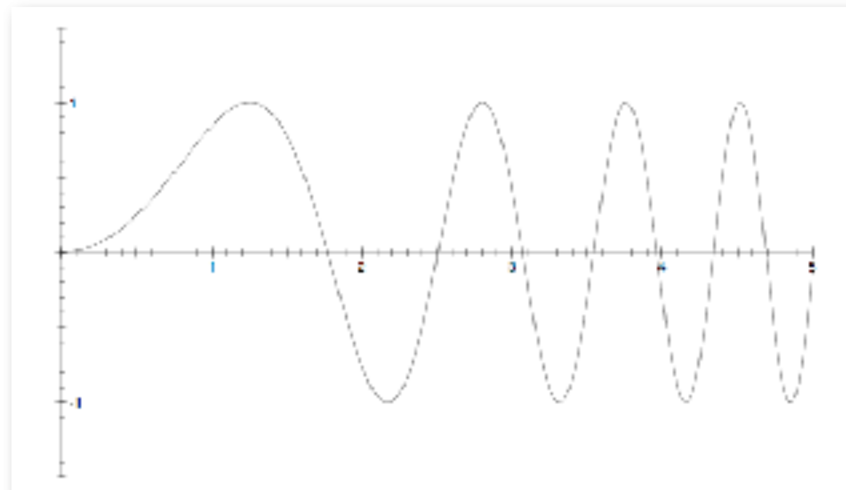
Graphs

Lecture 22

CS 2112 – Fall 2018

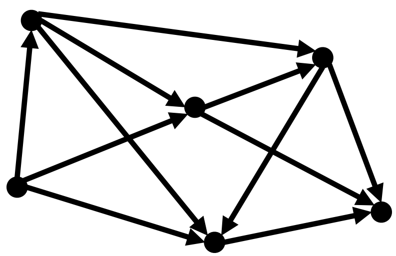
These are not Graphs

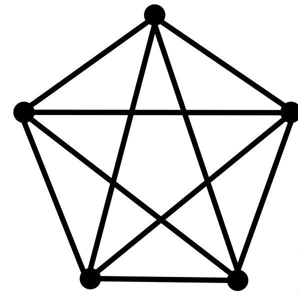




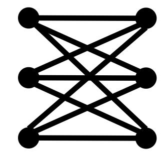
… not the kind we mean, anyway

These are Graphs

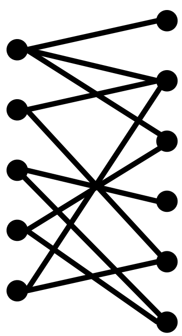


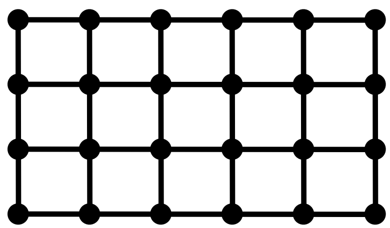


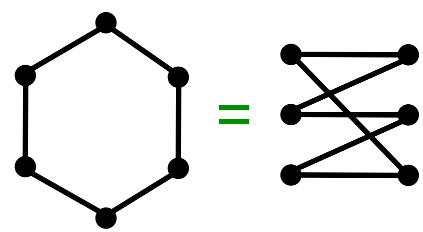
K_5



$K_{3,3}$







Applications of Graphs

- Communication networks
- Routing and shortest path problems
- Commodity distribution (flow)
- Traffic control
- Resource allocation
- Geometric modeling
- …

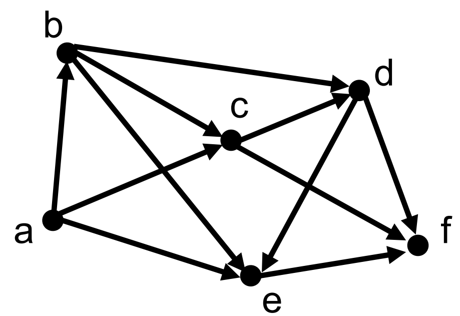
Graph Definitions

- A directed graph (or digraph) is a pair (V, E) where
  - V is a set
  - E is a set of ordered pairs (u, v) where u, v $\in$ V
    - Usually require u $\neq$ v (i.e., no self-loops)

- An element of V is called a vertex (pl. vertices) or node
- An element of E is called an edge or arc

- |V| = size of V, often denoted n
- |E| = size of E, often denoted m

Example Directed Graph (Digraph)



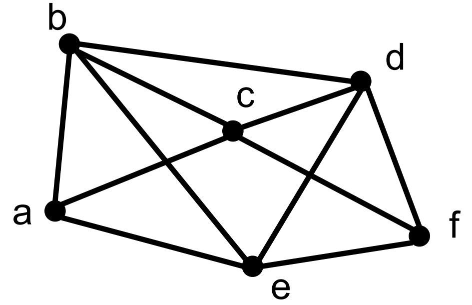
V = {a, b, c, d, e, f }
E = {(a, b), (a, c), (a, e), (b, c), (b, d), (b, e),
(c, d), (c, f), (d, e), (d, f), (e, f)}

|V| = 6, |E| = 11

Example *Undirected* Graph

An *undirected graph* is just like a directed graph, except the edges are *unordered pairs* (*sets*) {u, v}

Example:

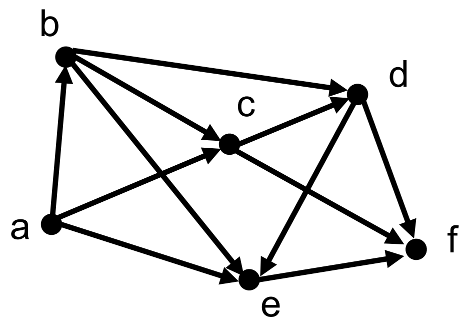


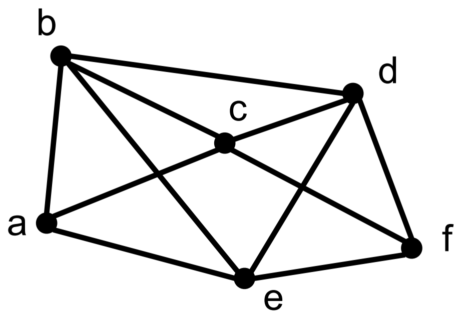
V = { a, b, c, d, e, f }

E = {{a, b}, {a, c}, {a, e}, {b, c}, {b, d}, {b, e}, {c, d}, {c, f}, {d, e}, {d, f}, {e, f}}

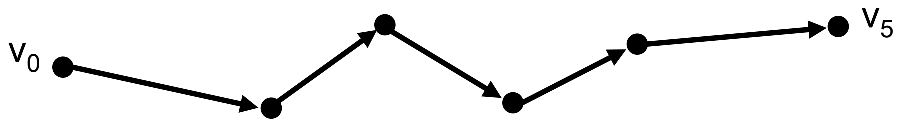
Some Graph Terminology

- Vertices u and v are called the source and sink of the directed edge (u, v), respectively
- Vertices u and v are called the endpoints of (u, v)
- Two vertices are adjacent if they are connected by an edge
- The outdegree of a vertex u in a directed graph is the number of edges for which u is the source
- The indegree of a vertex v in a directed graph is the number of edges for which v is the sink
- The degree of a vertex u in an undirected graph is the number of edges of which u is an endpoint



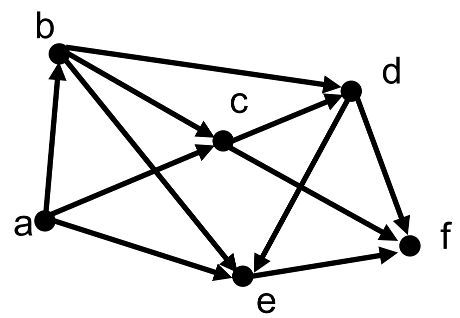


More Graph Terminology



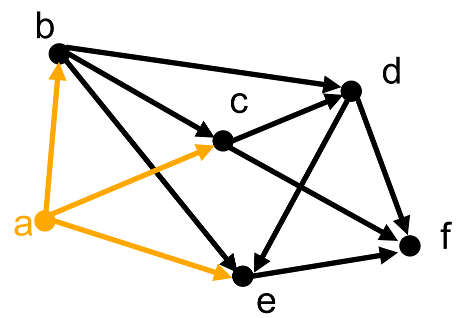
- A path is an alternating sequence $v_0, e_0, v_1, e_1, v_2, \ldots, v_n$ of vertices and edges, beginning and ending with a vertex, such that $(v_i, v_{i+1}) = e_i$, $0 \le i \le n - 1$
- The length of a path is its number of edges
  - In this example, the length is 5
  - A single vertex is a path of length 0
- A path is simple if it does not repeat any vertices
- A cycle is a path $v_0, \ldots, v_n$ with at least two edges such that $v_0 = v_n$
- A cycle is simple if it does not repeat any vertices except the first and last
- A graph is acyclic if it has no cycles
- A directed acyclic graph is called a dag

Is This a Dag?



- Intuition:
  - If it's a dag, there must be a vertex with indegree zero – why?
- This idea leads to an algorithm
  - A digraph is a dag if and only if we can iteratively delete indegree-0 vertices until the graph disappears

Is This a Dag?



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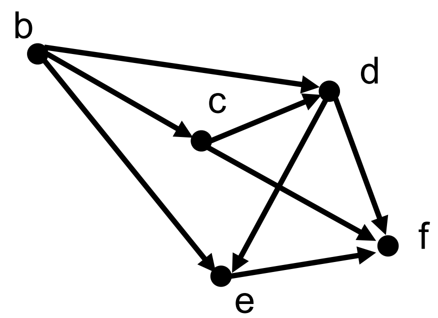

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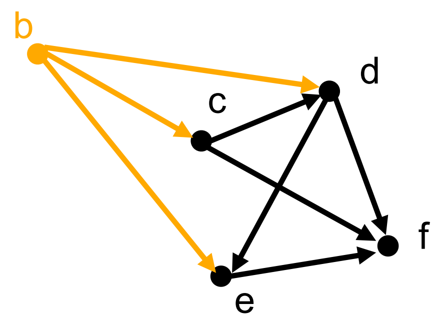

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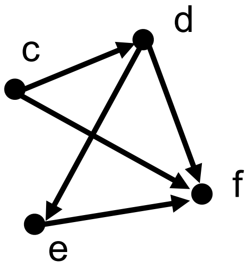

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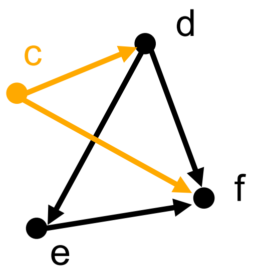

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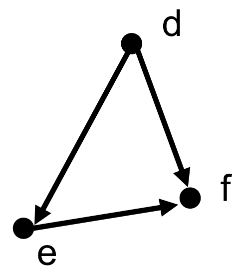

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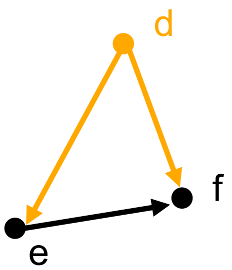

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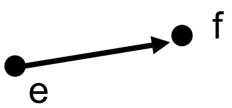

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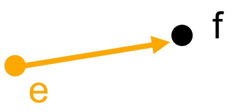

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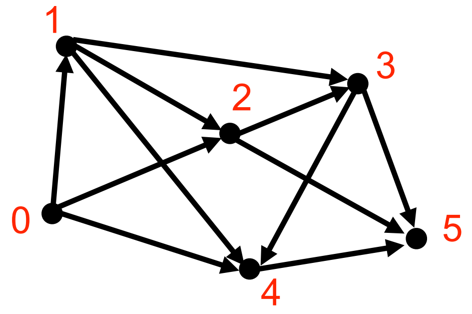
Topological Sort

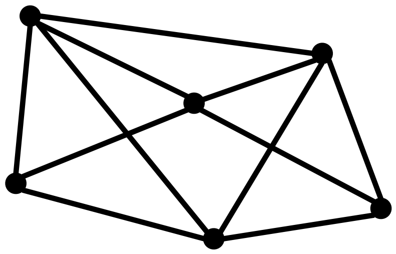
- We just computed a topological sort of the dag
  - This is a numbering of the vertices such that all edges go from lower- to higher-numbered vertices



- Useful in job scheduling with precedence constraints

Graph Coloring

- A coloring of an undirected graph is an assignment of a color to each node such that no two adjacent nodes get the same color



- Q: How many colors are needed to color this graph?

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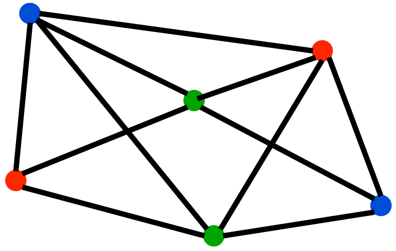
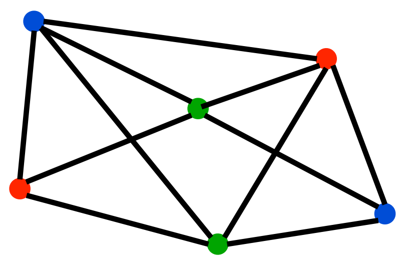


- Q: How many colors are needed to color this graph?
- A: 3

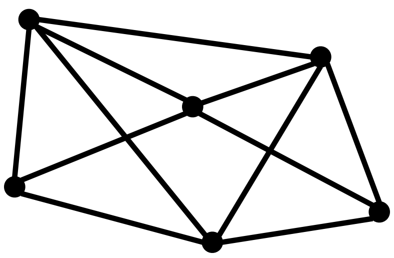
An Application of Coloring

- Vertices are jobs
- Edge (u,v) is present if jobs u and v each require access to the same shared resource, and thus cannot execute simultaneously
- Colors are time slots to schedule the jobs
- Minimum number of colors needed to color the graph = minimum number of time slots required



Planarity

- A graph is planar if it can be embedded in the plane with no edges crossing



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Planarity

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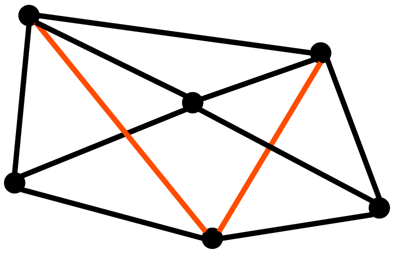

- Q: Is this graph planar?
- A: Yes

Planarity

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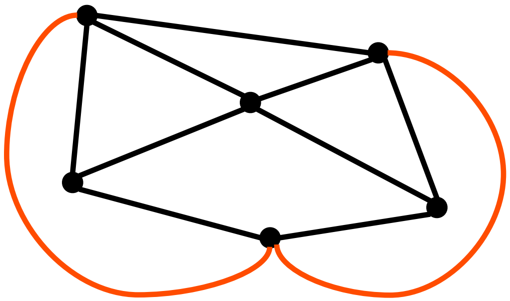
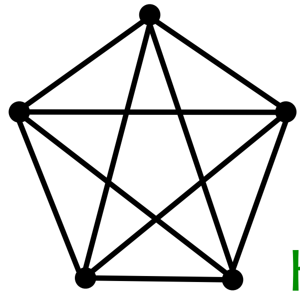


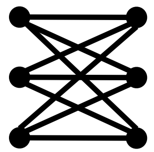
- Q: Is this graph planar?
- A: Yes

Detecting Planarity

- Kuratowski's Theorem



K_5



$K_{3,3}$

- A graph is planar if and only if it does not contain a copy of K_5 or $K_{3,3}$ (possibly with other nodes along the edges shown)

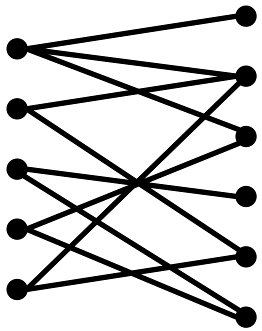
The Four-Color Theorem

Every planar graph is 4-colorable

(Appel & Haken, 1976)

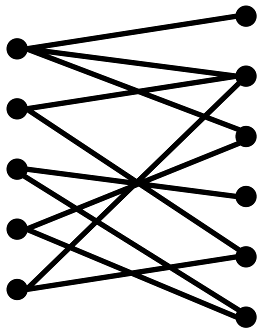
Bipartite Graphs

- A directed or undirected graph is bipartite if the vertices can be partitioned into two sets such that all edges go between the two sets

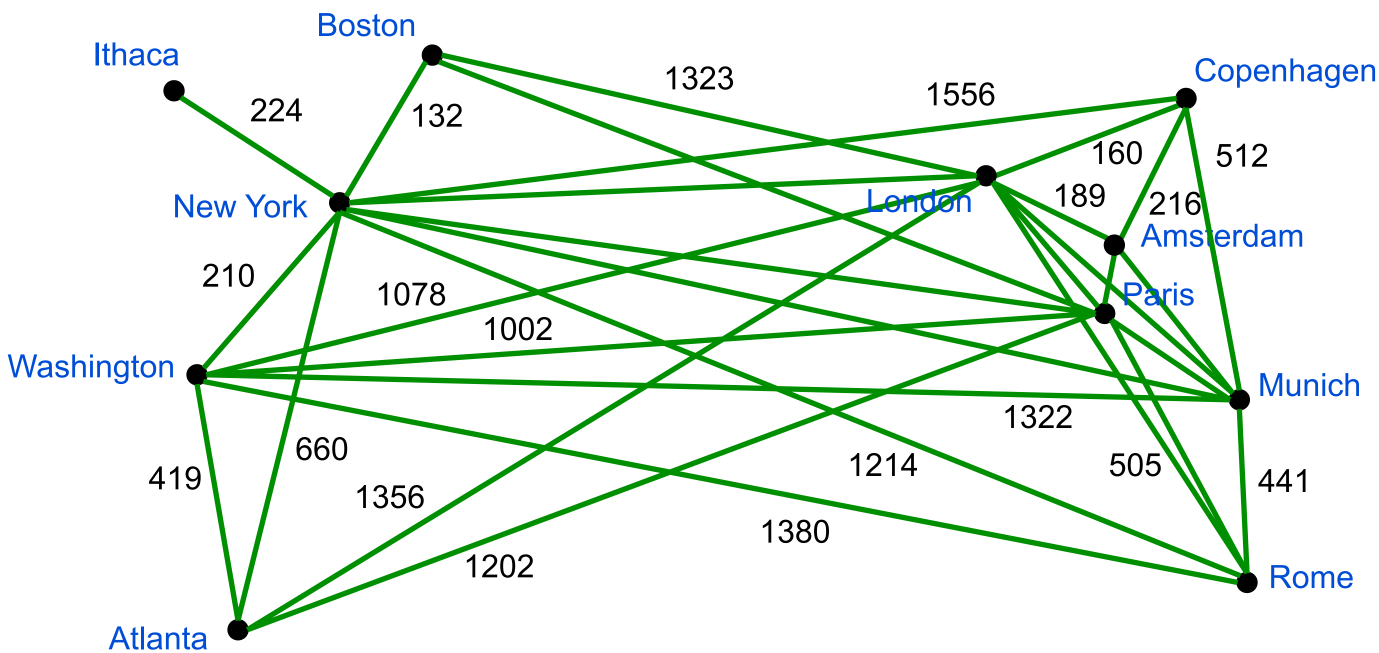


Bipartite Graphs

- The following are equivalent:
  - G is bipartite
  - G is 2-colorable
  - G has no cycles of odd length

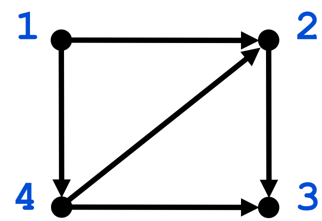


Traveling Salesperson

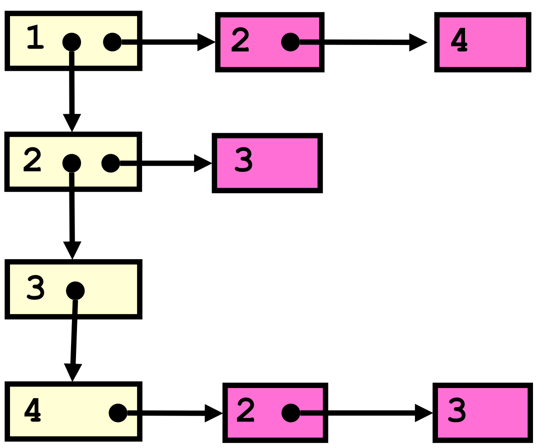


- Find a path of minimum distance that visits every city

Representations of Graphs



Adjacency List



Adjacency Matrix

	1	2	3	4
1	0	1	0	1
2	0	0	1	0
3	0	0	0	0
4	0	1	1	0

Adjacency Matrix or Adjacency List?

- Adjacency Matrix
  - Uses space $O(n^2)$
  - Can iterate over all edges in time $O(n^2)$
  - Can answer "Is there an edge from u to v?" in $O(1)$ time
  - Better for dense graphs (lots of edges) ($m \sim n^2$)

- Adjacency List
  - Uses space $O(m+n)$
  - Can iterate over all edges in time $O(m+n)$
  - Can answer "Is there an edge from u to v?" in $O(d(u))$ time
  - Better for sparse graphs (fewer edges) ($m << n^2$)

- n = number of vertices
- m = number of edges
- d(u) = outdegree of u

Conclusion

- Graphs are an extremely useful tool for modeling many different kinds of computational problems
- There are many efficient basic graph algorithms – depth-first search, breadth-first search, topological sort, …
- … and some not so efficient – traveling salesperson, coloring
- Learn about this and more in CS 4820