Class 2: Stochastic volatility models and heteroskedasticity

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Learning outcomes

- Learn how to model changing variance in a time series
- Understand how to fit ARCH, GARCH and SVM models in JAGS
- Know how to compare and plot the output from these models

General principles of models for changing variance

- So far we have looked at models where the mean changes but the variance is constant:

$$y_t \sim N(\mu_t, \sigma^2)$$

- In this module we look at methods where instead:

$$y_t \sim N(\alpha, \sigma_t^2)$$

- These are:
    - Autoregressive Conditional Heteroskedasticity (ARCH)
    - Generalised Autoregressive Conditional Heteroskedasticity (GARCH)
    - Stochastic Volatility Models (SVM)
- They follow the same principles as ARIMA, but work on the standard deviations or variances instead of the mean
- `forecast` doesn't include any of these models so we'll use JAGS. There are other R packages to fit these models

Extension 1: ARCH

- An ARCH(1) Model has the form:

$$\sigma_t^2 = \gamma_1 + \gamma_2 \epsilon_{t-1}^2$$

where ϵ_t is the residual, just like an MA model
- Note that $\epsilon_t = y_t - \alpha$ so the above can be re-written as:

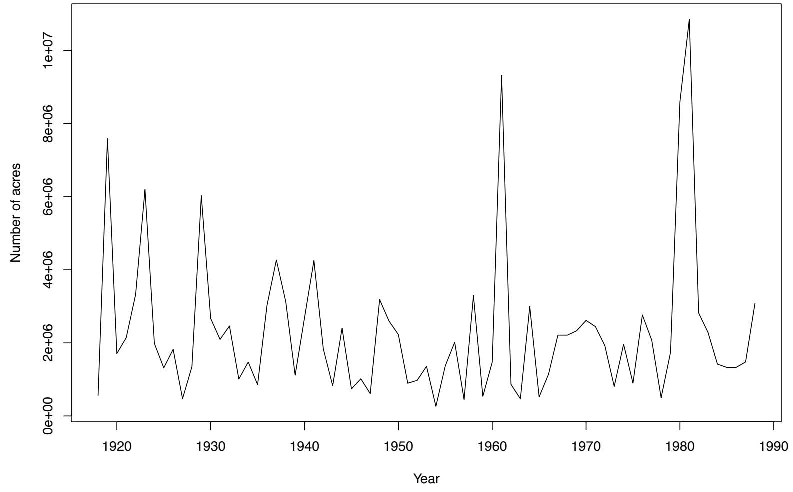
$$\sigma_t^2 = \gamma_1 + \gamma_2 (y_{t-1} - \alpha)^2$$

- The variance at time t thus depends on the previous value of the series (hence the autoregressive in the name)
- The residual needs to be squared to keep the variance positive.
- The parameters γ_1 and γ_2 also need to be positive, and usually $\gamma_1 \sim U(0, 1)$

JAGS code for ARCH models

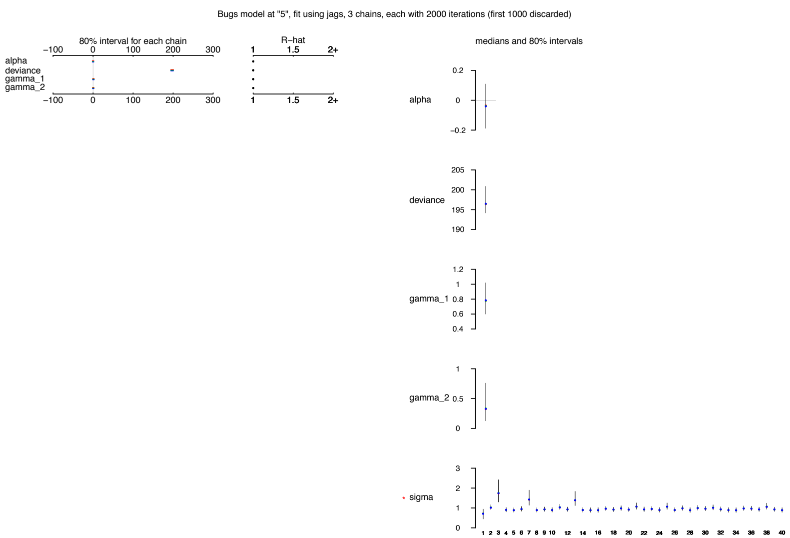
```
model_code = '
model
{
  # Likelihood
  for (t in 1:T) {
    y[t] ~ dnorm(alpha, sigma[t]^-2)
  }
  sigma[1] ~ dunif(0, 1)
  for(t in 2:T) {
    sigma[t] <- sqrt(gamma_1 + gamma_2 * pow(y[t-1] - alpha, 2))
  }

  # Priors
  alpha ~ dnorm(0.0, 100^-2)
  gamma_1 ~ dunif(0, 100)
  gamma_2 ~ dunif(0, 100)
}
'
```


Reminder: forest fires data



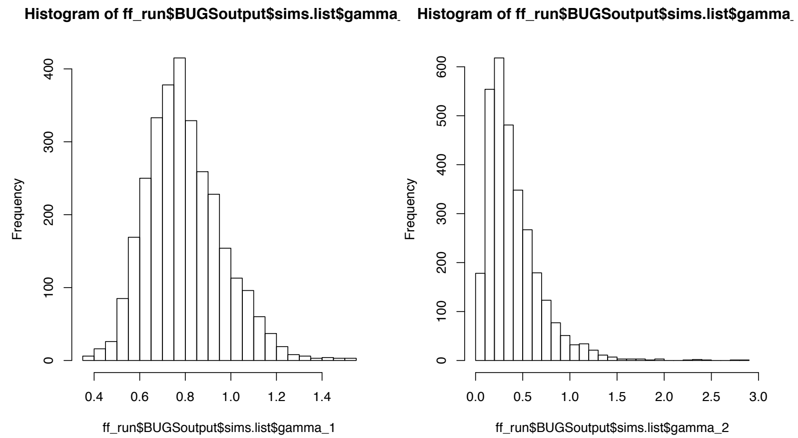
ARCH(1) applied to forest fires data

```
plot(ff_run)
```

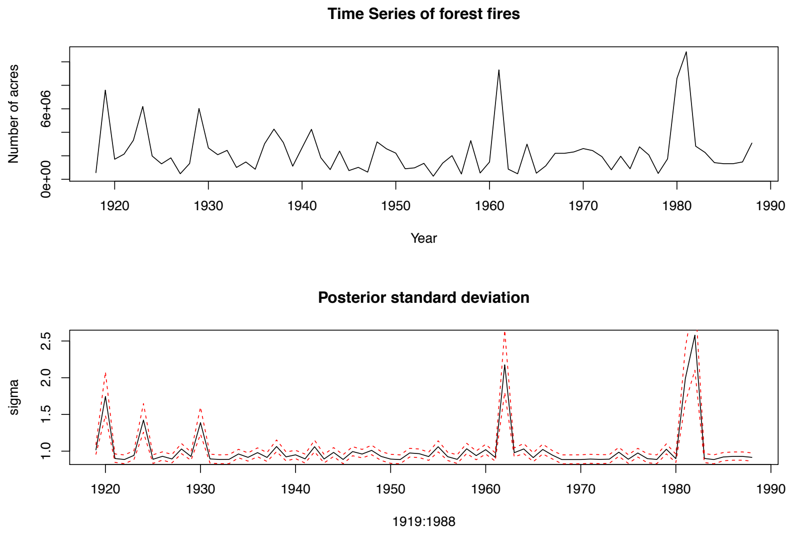


Plot the ARCH parameters

```
par(mfrow=c(1,2))
hist(ff_run$BUGSoutput$sims.list$gamma_1, breaks=30)
hist(ff_run$BUGSoutput$sims.list$gamma_2, breaks=30)
```



Plot the posterior standard deviations



From ARCH to GARCH

- The Generalised ARCH model works by simply adding the previous value of the variance, as well as the previous value of the observation
- The GARCH(1,1) model thus has:

$$\sigma_t^2 = \gamma_1 + \gamma_2(y_{t-1} - \alpha)^2 + \gamma_3\sigma_{t-1}^2$$

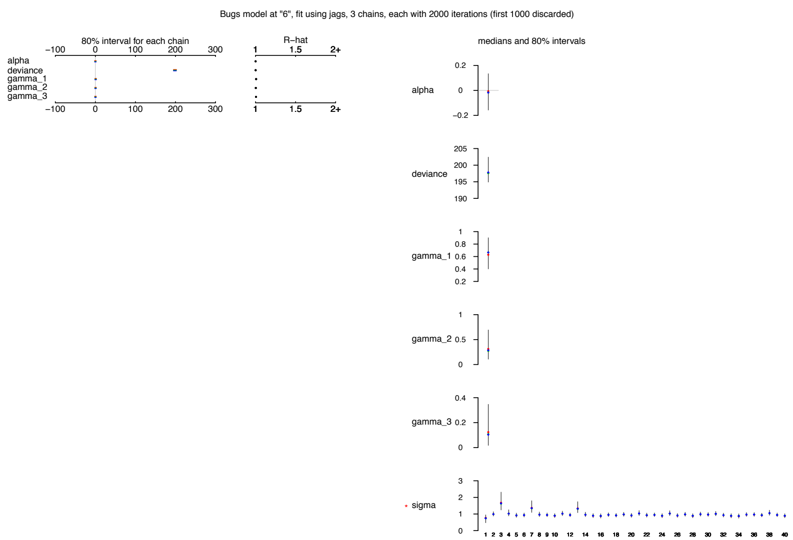
- There are, as always, complicated restrictions on the parameters, though like the stationarity conditions in ARIMA models we can relax this assumption and see if the data support it
- It's conceptually easy to extend to general GARCH(p,q) models which add in extra previous lags

Example of using the GARCH(1,1) model

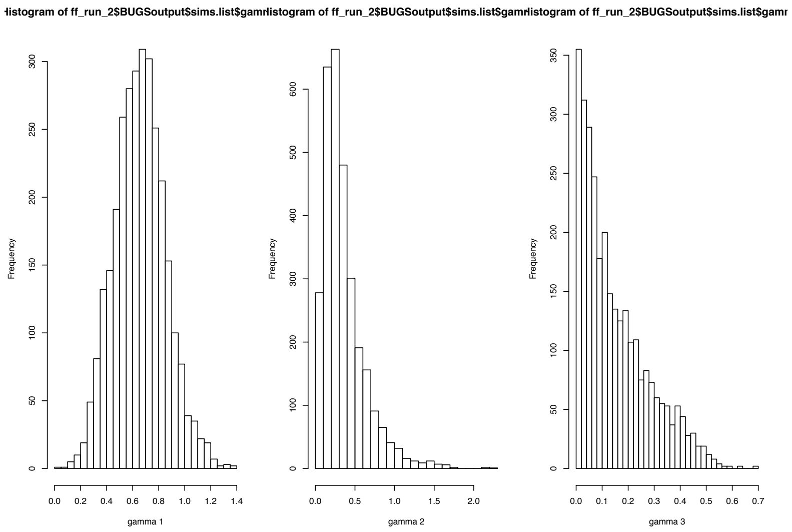
```
model_code = '
model
{
  # Likelihood
  for (t in 1:T) {
    y[t] ~ dnorm(alpha, sigma[t]^-2)
  }
  sigma[1] ~ dunif(0,1)
  for(t in 2:T) {
    sigma[t] <- sqrt(gamma_1 + gamma_2 * pow(y[t-1] - alpha, 2)
                      + gamma_3 * pow(sigma[t-1], 2))
  }
  # Priors
  alpha ~ dnorm(0, 10^-2)
  gamma_1 ~ dunif(0, 10)
  gamma_2 ~ dunif(0, 10)
  gamma_3 ~ dunif(0, 10)
}
'
```


Using the forest fire data again

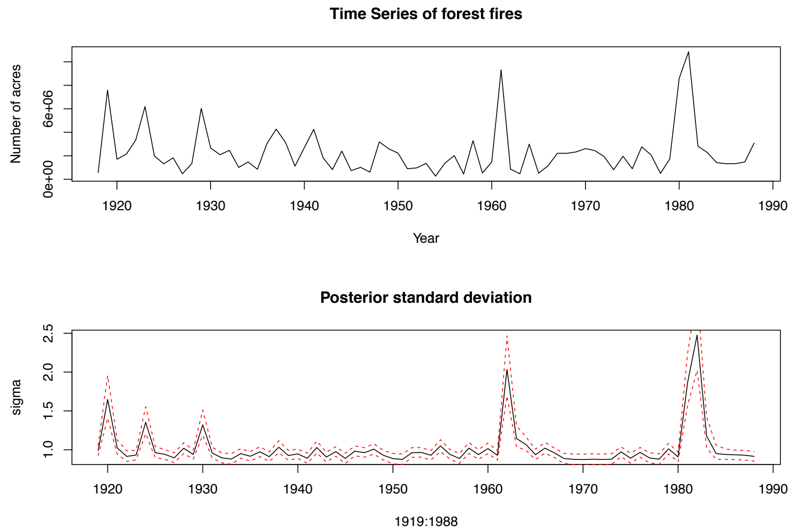
```
plot(ff_run_2)
```



Looking at the GARCH parameters



Posterior standard deviations over time



Compare with DIC

```r
with(r_1, print(c(DIC, pD)))
```

```
## [1] 200.937773   3.879226
```

```r
with(r_2, print(c(DIC, pD)))
```

```
## [1] 203.046212   4.746636
```

- Suggests not much difference between the models

Stochastic Volatility Modelling

- Both ARCH and GARCH propose a deterministic relationship for the current variance parameter
- By contrast a Stochastic Volatility Model (SVM) models the variance as its own *stochastic process*
- SVMs, ARCH and GARCH are all closely linked if you go into the bowels of the theory
- The general model structure is often written as:

$$y_t \sim N(\alpha, \exp(h_t))$$

$$h_t \sim N(\mu + \phi h_{t-1}, \sigma^2)$$

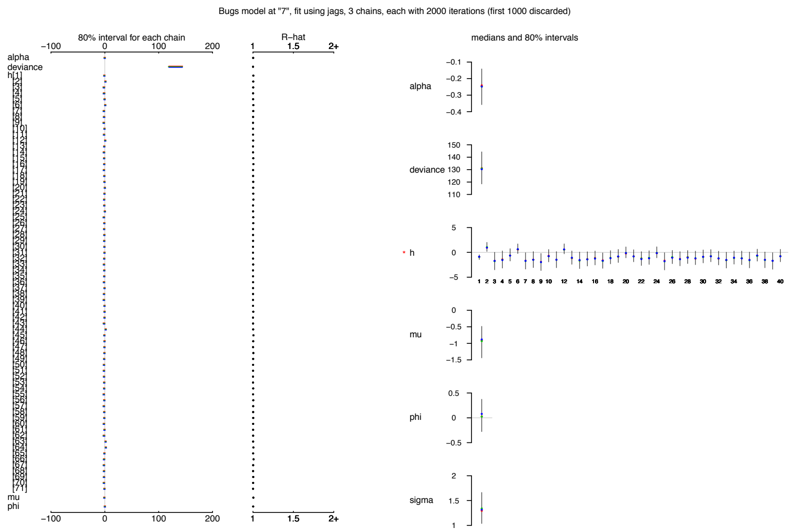
- You can think of an SVM being like a GLM but with a log link on the variance parameter

JAGS code for the SVM model

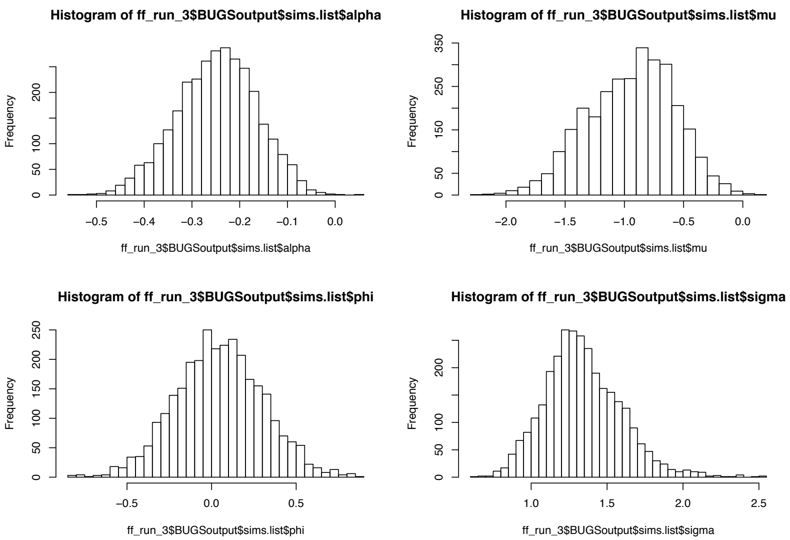
```
model_code = '
model
{
  # Likelihood
  for (t in 1:T) {
    y[t] ~ dnorm(alpha, sigma_h[t]^-2)
    sigma_h[t] <- sqrt(exp(h[t]))
  }
  h[1] <- mu
  for(t in 2:T) {
    h[t] ~ dnorm(mu + phi * h[t-1], sigma^-2)
  }

  # Priors
  alpha ~ dnorm(0, 100^-2)
  mu ~ dnorm(0, 100^-2)
  phi ~ dunif(-1, 1)
  sigma ~ dunif(0,100)
}
'
```


Example of SVMs and comparison of DIC

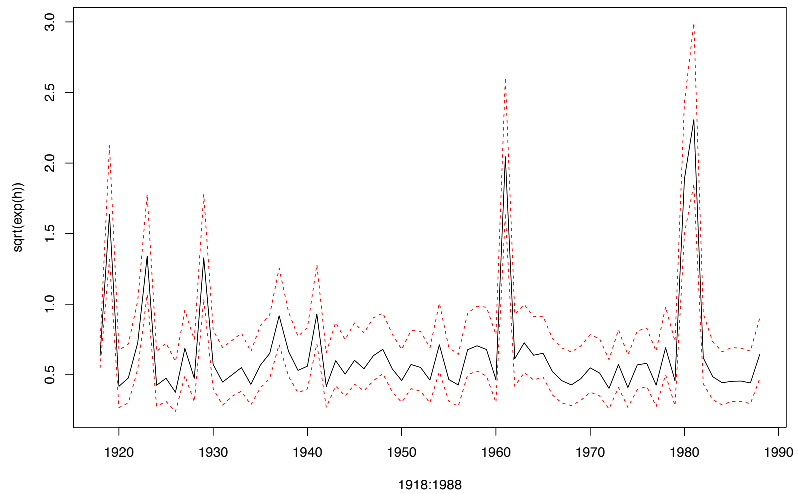
```
plot(ff_run_3)
```



Look at all the parameters



Plot of $\sqrt{\exp(h)}$



Comparison with previous models

```
with(r_1, print(c(DIC, pD)))
```

```
## [1] 200.937773   3.879226
```

```
with(r_2, print(c(DIC, pD)))
```

```
## [1] 203.046212   4.746636
```

```
with(r_3, print(c(DIC, pD)))
```

```
## [1] 183.8367  52.7872
```

Much better fit, despite many extra parameters due to h!

Summary

- We know that ARCH extends the ARIMA idea into the variance using the previous values of the series
- We know that GARCH extends ARCH with previous values of the variance too
- We know that SVMs give the variance its own stochastic process
- We can combine these new models with all the techniques we have previously learnt