Class 1: Modelling with seasonality and the frequency domain

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Learning outcomes

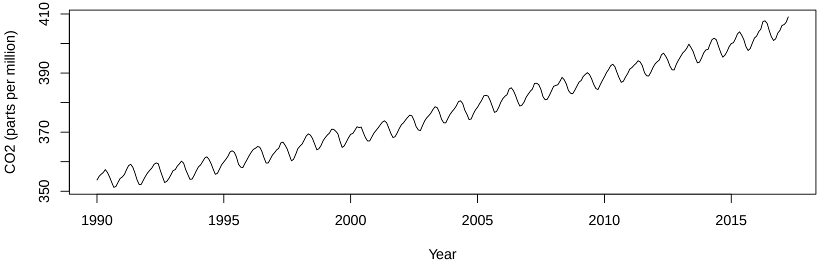
- Understand how to fit seasonal models in `forecast` and JAGS
- Understand seasonal differencing and sARIMA models
- Know the difference between time and frequency domain models and be able to implement a basic Fourier model

Seasonal time series

- So far we haven't covered how to deal with data that are *seasonal* in nature
- These data generally fall into two categories:
  1. Data where we know the frequency or frequencies (e.g. monthly data on a yearly cycle, frequency = 12)
  2. Data where we want to estimate the frequencies (e.g. climate time series, animal populations, etc)
- The former are easier, and there are many techniques for inducing seasonal behaviour
- The latter are much more interesting. The ACF and PACF can help, but we can usually do much better by creating a *power spectrum*

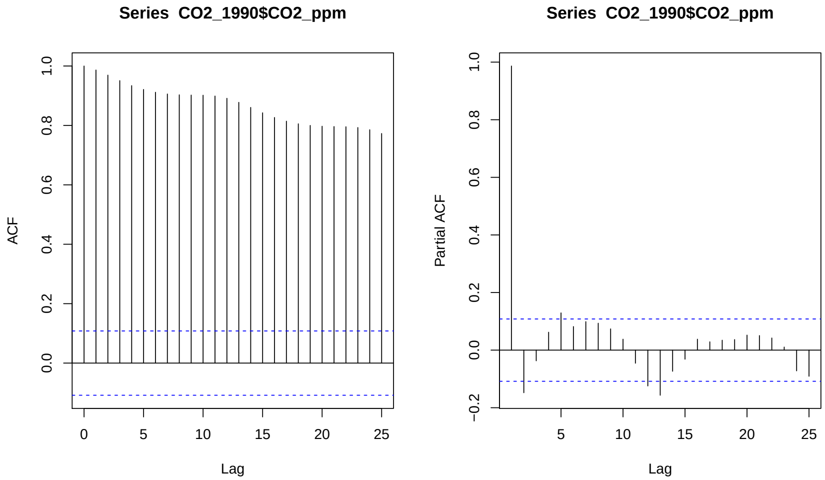
An example seasonal series

```
CO2 = read.csv(file = '../../data/CO2.csv',
               na.strings = -99.99)
CO2$date = as.Date(paste(CO2$year, CO2$month ,'01',
                         sep = '-'))
CO2_1990 = CO2[CO2$year >= 1990, ]
with(CO2_1990, plot(date, CO2_ppm, type = 'l',
                    ylab = 'CO2 (parts per million)',
                    xlab = 'Year'), las = 1)
```



ACF and PACF

```r
par(mfrow = c(1, 2))
acf(CO2_1990$CO2_ppm)
pacf(CO2_1990$CO2_ppm)
```



Seasonal time series 1: including seasonality as a covariate

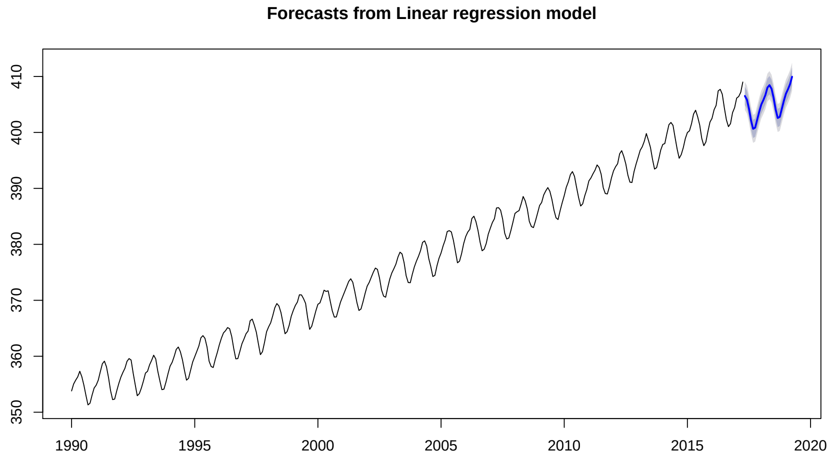
- The simplest way is to include month as a covariate in a regression type model

```
CO2_1990$mfac = model.matrix(~ as.factor(CO2_1990$month) - 1)
colnames(CO2_1990$mfac) = month.abb
lm(CO2_ppm ~ year + mfac, data = CO2_1990)
```

```
##
## Call:
## lm(formula = CO2_ppm ~ year + mfac, data = CO2_1990)
##
## Coefficients:
## (Intercept)         year      mfacJan
##  -3501.5897       1.9362      -0.6749
##     mfacFeb      mfacMar      mfacApr
##      0.1222       1.0272       2.3590
##     mfacMay      mfacJun      mfacJul
##      2.7967       2.1367       0.5126
##     mfacAug      mfacSep      mfacOct
##     -1.5704      -3.0774      -2.8770
```

Forecasts

```
CO2_ts = ts(CO2_1990$CO2_ppm, frequency = 12,
            start = c(1990, 1))
s_model_1 = tslm(CO2_ts ~ trend + season)
plot(forecast(s_model_1, h = 24))
```



What is the time series model doing here?

- This is just a regression model, so that:

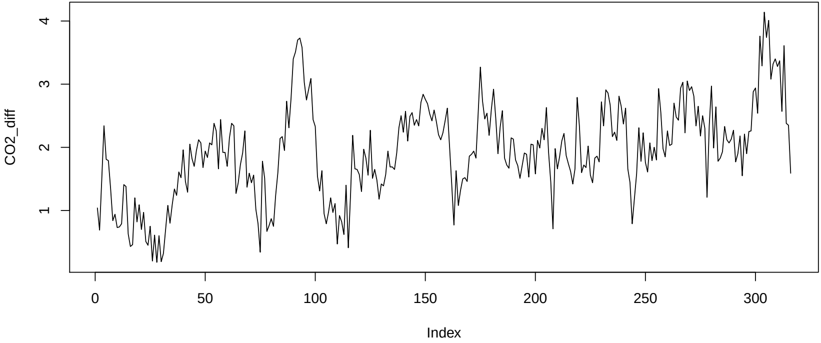
$$y_t = \alpha + \beta_t \text{year}_t + \gamma_{1t}\text{Feb}_t + \gamma_{2t}\text{Mar}_t + \ldots + \gamma_{11,t}\text{Dec}_t + \epsilon_t$$

- You can do this using `lm` or using `forecast`'s special function for linear regression forecasting `tslm`
- The `tslm` function is clever because it can automatically create the seasonal indicator variables
- Remember that when dealing with indicator variables you have to drop one factor level for the model to fit

Seasonal time series 2: seasonal differencing

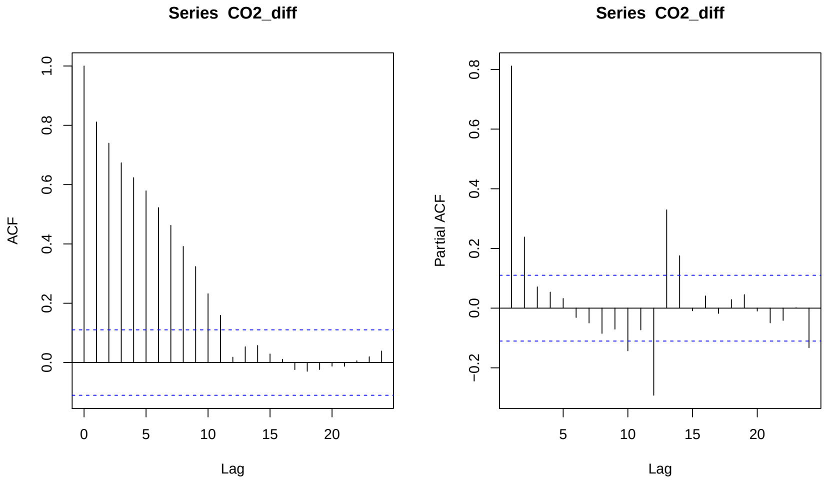
- We have already met methods which difference the data (possibly multiple times) at lag 1
- We can alternatively create a seasonal difference by differencing every e.g. 12th observation

```
CO2_diff = diff(CO2_1990$CO2_ppm, lag = 12)
plot(CO2_diff, type = 'l')
```



Differenced acf and pacf

```
par(mfrow = c(1, 2))
acf(CO2_diff, na.action = na.pass)
pacf(CO2_diff, na.action = na.pass)
```



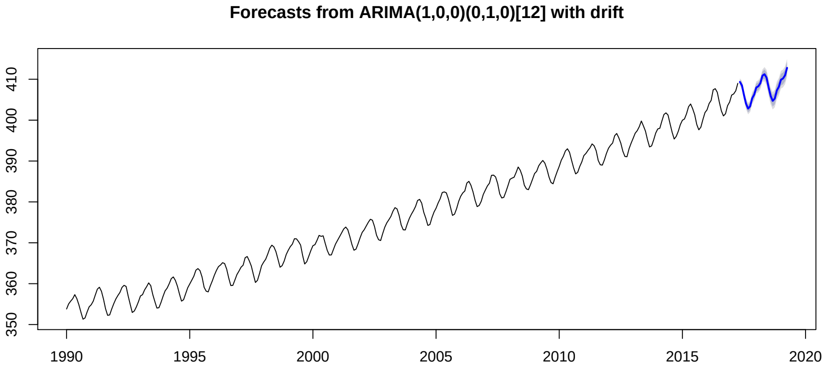
Fit an ARIMA model with a seasonal difference

```r
CO2_1990_ts = ts(CO2_1990$CO2_ppm, frequency = 12,
                 start =c(1990, 1))
Arima(CO2_1990_ts, order = c(1, 0, 0),
      seasonal = c(0, 1, 0),
      include.drift = TRUE)
```

```
## Series: CO2_1990_ts
## ARIMA(1,0,0)(0,1,0)[12] with drift
##
## Coefficients:
##          ar1   drift
##       0.8129  0.1588
## s.e.  0.0325  0.0107
##
## sigma^2 estimated as 0.1943:  log likelihood=-185.12
## AIC=376.23   AICc=376.31   BIC=387.5
```

Forecasts from seasonally differenced series

```
s_model_2 = Arima(CO2_1990_ts, order = c(1, 0, 0),
                  seasonal = c(0, 1, 0), include.drift = TR
plot(forecast(s_model_2, h = 24))
```



- Pretty good. Might be able to do better with some richer models

A full seasonal arima model

- We previously met the ARIMA specification where:

$$\text{diff}^d(y_t) = \text{constant} + \text{AR terms} + \text{MA terms} + \text{error}$$

- We can extend this to include seasonal differencing and *seasonal AR and MA* terms to create a seasonal ARIMA or sARIMA model
- For example:

$$y_t - y_{t-12} = \alpha + \beta y_{t-1} + \gamma y_{t-12} + \epsilon_t$$

- This is a sARIMA$(1,0,0)(1,1,0)_{12}$ model

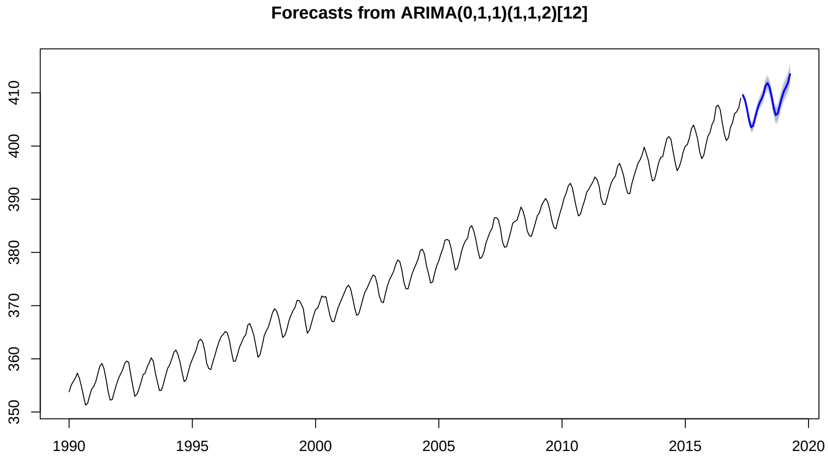
Fitting sARIMA models in `forecast`

```r
auto.arima(CO2_1990_ts)
```

```
## Series: CO2_1990_ts
## ARIMA(0,1,1)(1,1,2)[12]
##
## Coefficients:
##           ma1      sar1      sma1      sma2
##       -0.3888   -0.7684   -0.1020   -0.6482
## s.e.   0.0582    0.4837    0.4891    0.4246
##
## sigma^2 estimated as 0.1137:  log likelihood=-103.27
## AIC=216.55   AICc=216.74   BIC=235.31
```

Plotting forecasts

```
s_model_3 = auto.arima(CO2_1990_ts)
plot(forecast(s_model_3, h = 24))
```



A simple sARIMA model with JAGS

```
model_code = '
model
{
  # Likelihood
  for (t in (s+1):T) {
    y[t] ~ dnorm(mu[t], sigma^-2)
    mu[t] <- alpha + beta * y[t-1] + gamma * y[t-s]
  }

  # Priors
  alpha ~ dnorm(0, 10^-2)
  beta ~ dnorm(0, 10^-2)
  gamma ~ dnorm(0, 10^-2)
  sigma ~ dunif(0, 100)
}
'
```

Fitting a sARIMA$(1,0,0)(1,0,0)_{12}$ model in JAGS

```
s_model_4 = jags(data = list(y = CO2_ts, s = 12,
                              T = length(CO2_ts)),
                 parameters.to.save = c('alpha', 'beta',
                                        'gamma', 'sigma'),
                 model.file=textConnection(model_code))
```

```
print(s_model_4)
```

```
## Inference for Bugs model at "5", fit using jags,
##  3 chains, each with 2000 iterations (first 1000 discarded)
##  n.sims = 3000 iterations saved
##          mu.vect sd.vect    2.5%     25%     50%
## alpha     -6.387   0.873  -8.098  -6.969  -6.386
## beta       0.189   0.024   0.144   0.173   0.189
## gamma      0.832   0.024   0.783   0.816   0.832
## sigma      0.599   0.024   0.554   0.582   0.599
## deviance 571.705   2.890 568.132 569.615 571.055
##              75%   97.5%  Rhat n.eff
## alpha     -5.790  -4.689 1.001  3000
## beta       0.205   0.238 1.002  1600
## gamma      0.848   0.879 1.002  1600
## sigma      0.614   0.648 1.001  3000
## deviance 573.074 579.233 1.002  1800
##
## For each parameter, n.eff is a crude measure of effective sample size,
## and Rhat is the potential scale reduction factor (at convergence, Rhat=1).
##
## DIC info (using the rule, pD = var(deviance)/2)
## pD = 4.2 and DIC = 575.9
## DIC is an estimate of expected predictive error (lower deviance is better).
```

Multiple seasonality

- Very occasionally you come across multiple seasonality models
- For example you might have hourly data over several months with both hourly and monthly seasonality
- `forecast` has a special function for creating multiple series time series: `msts`

```
x = msts(taylor, seasonal.periods=c(48,336),
         start=2000+22/52)
```

- The above is half-hourly data so has period 48 hours and 336 hours, i.e. weekly ($336/48 = 7$)
- `forecast` has some special functions (notably `tbats`) for modelling multi seasonality data

Frequency estimation

Methods for estimating frequencies

- The most common way to estimate the frequencies in a time series is to decompose it in a *Fourier Series*
- We write:

$$y_t = \alpha + \sum_{k=1}^{K} \left[\beta_k \sin(2\pi t f_k) + \gamma_k \cos(2\pi t f_k)\right] + \epsilon_t$$

- Each one of the terms inside the sum is called a *harmonic*. We decompose the series into a sum of sine and cosine waves rather than with AR and MA components
- Each sine/cosine pair has its own frequency f_k. If the corresponding coefficients β_k and γ_k are large we might believe this frequency is important

Estimating frequencies via a Fourier model

- It's certainly possible to fit the model in the previous slide in JAGS, as it's just a linear regression model with clever explanatory variables
- However, it can be quite slow to fit and, if the number of frequencies K is high, or the frequencies are close together, it can struggle to converge
- More commonly, people repeatedly fit the simpler model:

$$y_t = \alpha + \beta \sin(2\pi t f_k) + \gamma \cos(2\pi t f_k) + \epsilon_t$$

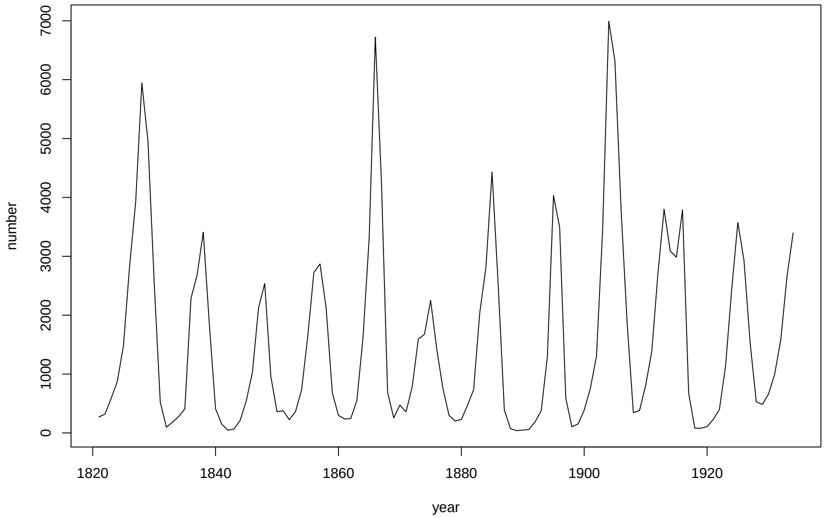
  for lots of different values of f_k. Then calculate the *power spectrum* as $P(f_k) = \frac{\beta^2+\gamma^2}{2}$. Large values of the power spectrum indicate important frequencies
- It's much faster to do this outside of JAGS, using other methods, but we will stick to JAGS

JAGS code for a Fourier model

```
model_code = '
model
{
  # Likelihood
  for (t in 1:T) {
    y[t] ~ dnorm(mu[t], sigma^-2)
    mu[t] <- alpha + beta * cos(2*pi*t*f_k) +
                gamma * sin(2*pi*t*f_k )
  }
  P = (pow(beta, 2) + pow(gamma, 2)) / 2

  # Priors
  alpha ~ dnorm(0, 10^-2)
  beta ~ dnorm(0, 10^-2)
  gamma ~ dnorm(0, 10^-2)
  sigma ~ dunif(0, 100)
}
'
```

Example: the Lynx data

```
lynx = read.csv('../../data/lynx.csv')
plot(lynx, type = 'l')
```

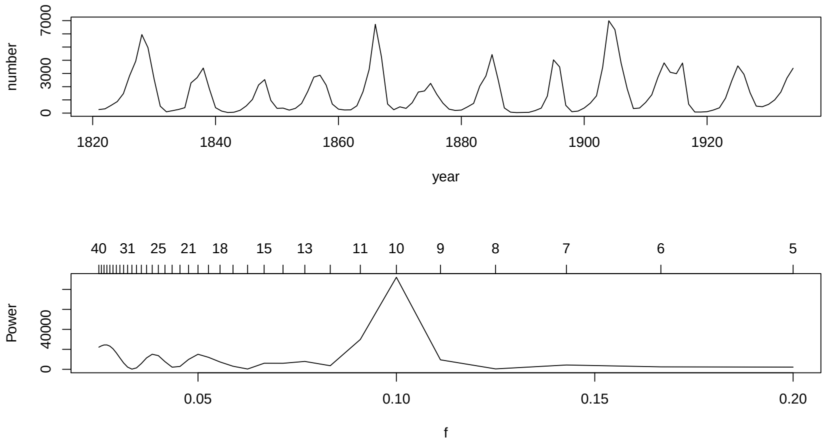


Code to run the JAGS model repeatedly

```
periods = 5:40
K = length(periods)
f = 1/periods
Power = rep(NA,K)

for (k in 1:K) {
  curr_model_data = list(y = as.vector(lynx[,2]),
                         T = nrow(lynx),
                         f_k = f[k],
                         pi = pi)

  model_run = jags(data = curr_model_data,
                   parameters.to.save = "P",
                   model.file=textConnection(model_code))

  Power[k] = mean(model_run$BUGSoutput$sims.list$P)
}
```

Plotting the periodogram

```
par(mfrow = c(2, 1))
plot(lynx, type = 'l')
plot(f, Power, type='l')
axis(side = 3, at = f, labels = periods)
```



Bayesian vs traditional frequency analysis

- For quick and dirty analysis, there is no need to run the full Bayesian model, the R function `periodogram` in the TSA package will do the job, or `findfrequency` in `forecast` which is even simpler
- However, the big advantage (as always with Bayes) is that we can also plot the uncertainty in the periodogram, or combine the Fourier model with other modelling ideas (e.g. ARIMA)
- There are much fancier versions of frequency models out there (e.g. Wavelets, or frequency selection models) which can also be fitted in JAGS but require a bit more time and effort
- These Fourier models work for continuous time series too

Summary

- We now know how to fit models for seasonal data via seasonal factors, seasonal differencing, and sARIMA models
- We can fit these using `forecast` or JAGS
- We've seen a basic Fourier model for estimating frequencies via the Bayesian periodogram