SLD-Resolution Reduction of Second-Order Horn Fragments

Sophie Tourret **Andrew Cropper**

Max Planck Institute for Informatics
University of Oxford

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SLD-Resolution Reduction of Second-Order Horn Fragments

Prolog-based systems

- on Horn clauses
- using SLD-resolution (Selective Linear Definite)
  - sound and refutationally complete on Horn clauses
  - without factorization of literals
  - here duplication of literals is forbidden (implies loss of refutational completeness)

SLD-Resolution Reduction of Second-Order Horn Fragments

Prolog-based Inductive Logic Programming (ILP) systems

Input

({Parent_of(Ana,Ben),Parent_of(Ben,Carol)},Grand_parent_of(Ana,Carol))
({Parent_of(Arthur,Betty),Parent_of(Betty,Charlie)},Grand_parent_of(Arthur,Charlie))

$\downarrow$

ILP

$\downarrow$

Output

Grand_parent_of(A,C) $\leftarrow$ Parent_of(A,B),Parent_of(B,C)

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Metarules

$$\vdots$$
$$P_0(A,C) \leftarrow P_1(A,B), P_2(B,C)$$
$$P_3(A) \leftarrow P_4(A), P_5(A)$$
$$P_0(A,B) \leftarrow P_1(A,B), P_3(B)$$
$$\vdots$$

ILP

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Second-Order Horn Fragment $\mathcal{H}$

$P_0(A) \leftarrow P_1(A, B), P_2(C, C)$ [not interesting]

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2-Connected Fragment $\mathcal{H}^{2c}$

$P_0(A, C) \leftarrow P_1(A, B), P_2(B, C), P_4(A)$ [very interesting]

SLD-Resolution Reduction of Second-Order Horn Fragments

What is the best set of metarules to use?

- Describes the desired fragment completely.
- Does not take too much memory.
- Allows for an efficient exploration of the search space

Can we reduce a fragment to a finite subset with these properties?

First Idea: Entailment Reduction

[Cropper, Muggleton, ILP'14]

$$
\begin{aligned}
C_1 &= P_0(A,B) \leftarrow P_1(A,B) \\
C_2 &= P_0(A,B) \leftarrow P_1(A,B), P_2(A) \\
C_3 &= P_0(A,B) \leftarrow P_1(A,B), P_3(A,B) \\
C_4 &= P_0(A,B) \leftarrow P_1(A,B), P_3(A,B), P_4(A,B)
\end{aligned}
$$

$$\{C_1\} \models \{C_1, C_2, C_3, C_4\}$$

Loss of completeness

$$C_1 \nvdash_{\mathrm{SLD}} C_2, C_3, C_4$$

Better Idea: Derivation Reduction

$$
\begin{aligned}
C_1 &= P_0(A, B) \leftarrow P_1(A, B) \\
C_2 &= P_0(A, B) \leftarrow P_1(A, B), P_2(A) \\
C_3 &= P_0(A, B) \leftarrow P_1(A, B), P_3(A, B) \\
C_4 &= P_0(A, B) \leftarrow P_1(A, B), P_3(A, B), P_4(A, B)
\end{aligned}
$$

$$\{C_1, C_2, C_3\} \vdash_{\mathrm{SLD}} \{C_1, C_2, C_3, C_4\}$$

This problem is undecidable!

Can this be done for the fragments of interest?

Reduction of Connected Fragments

Given a fragment $\mathcal{F}$, the fragment $\mathcal{F}_{a,b}$ is such that:

- a is the maximal arity of the predicates,
- b is the maximal number of literals in the body of clauses,
- ∞ means unbounded.
- $\mathcal{F}$ is reducible from $\mathcal{F}'$ if any clause in $\mathcal{F}$ can be derived using SLD-resolution from clauses in $\mathcal{F}'$.

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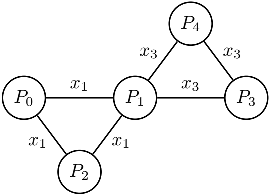
The fragment $\mathcal{H}^c$ is reducible to $\mathcal{H}^c_{\infty,2}$.

Reduction of the Connected Fragment $\mathcal{H}^{c}_{2,\infty}$

$$P_0(x_1, x_2) \leftarrow P_1(x_3, x_1), P_2(x_1), P_3(x_3), P_4(x_3)$$



Reduction of the Connected Fragment $\mathcal{H}^c_{2,\infty}$

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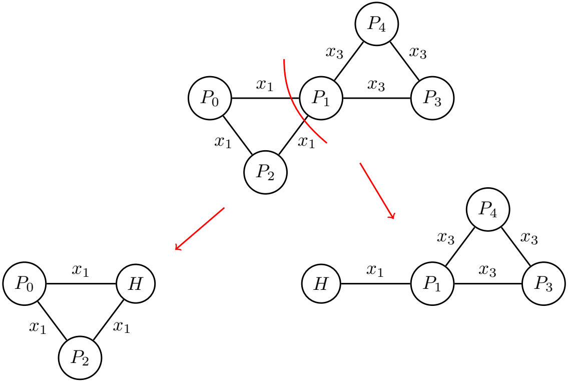

$$P_0(x_1, x_2) \leftarrow P_2(x_1), H(x_1) \qquad H(x_1) \leftarrow P_1(x_3, x_1), P_3(x_3), P_4(x_3)$$

Proof Idea of the Reduction of Connected Fragments

find a spanning tree where two adjacent vertices have at most a outgoing edges [here $a = 2$]

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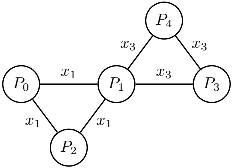
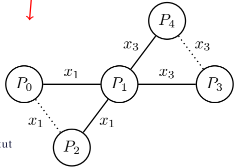
find a <u>spanning tree</u> where two adjacent vertices have at most *a* outgoing edges [here $a = 2$]

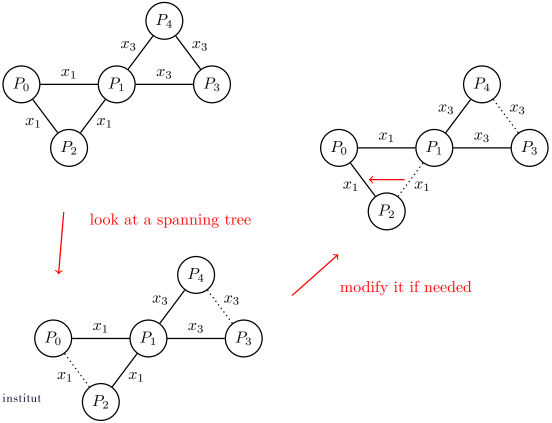


look at a spanning tree



Proof Idea of the Reduction of Connected Fragments

find a spanning tree where two adjacent vertices have at most a outgoing edges [here $a = 2$]

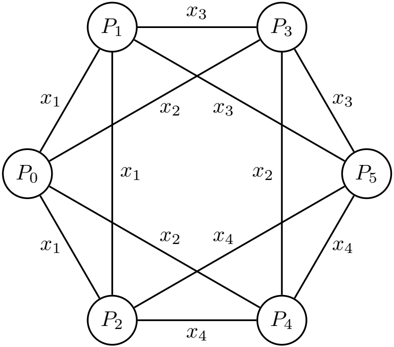


Reduction of the 2-Connected Fragment $\mathcal{H}^{2c}_{2,\infty}$

~~Reduction~~ of the 2-Connected Fragment $\mathcal{H}^{2c}_{2,\infty}$

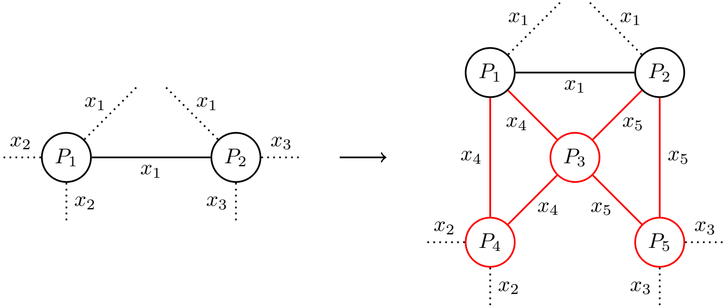
Not possible

Counter-example for $\mathcal{H}^{2c}_{2,5}$



$$P_0(x_1, x_2) \leftarrow P_1(x_1, x_3), P_2(x_1, x_4), P_3(x_2, x_3), P_4(x_2, x_4), P_5(x_3, x_4)$$

Counter-example for $\mathcal{H}^{2c}_{2,\infty}$



This transformation preserves irreducibility while increasing the size of the clause.

Summary

	SLD-resolution
connected ($\mathcal{H}^c$)	$\mathcal{H}^c_{\infty,2}$
2-connected ($\mathcal{H}^{2c}_{2,\infty}$)	NO

Summary

	SLD-resolution	resolution
connected ($\mathcal{H}^c$)	$\mathcal{H}^c_{\infty,2}$	$\mathcal{H}^c_{\infty,2}$
2-connected ($\mathcal{H}^{2c}_{2,\infty}$)	NO	$\mathcal{H}^{2c}_{2,2}$

Counter-measures for the 2-Connected Fragment $\mathcal{H}^{2c}_{2,\infty}$

- Use standard resolution
- Allow a restricted use of triadic predicates
- Add irreducible clauses dynamically
- ... ?