1. Adiabatic winds

In these notes, we work out the solutions for an adiabatic wind. We assume that the gas obeys $P = K\rho^\gamma$, with given constant values of the adiabatic index γ and entropy prefactor K. We derive the analytic solution first and then compare with numerical results from `Athena++`.

1.1. Sonic point

Consider spherically-symmetric, steady, radial flow away from a star of mass M and radius R. The continuity equation is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho v\right) = 0 \tag{1}$$

which shows that $r^2\rho v$ is constant throughout the flow. It is convenient to write this in terms of the mass loss rate

$$\dot{M} = 4\pi r^2 \rho v \tag{2}$$

(which has units of g s^{-1}). The momentum equation is

$$\rho v\frac{dv}{dr} = -\frac{dP}{dr} - \rho\frac{GM}{r^2}, \tag{3}$$

where we assume that the gravitational acceleration is dominated by the mass of the star M, i.e. the self-gravity of the flow itself is negligible.

For an adiabatic flow satisfying $P = K\rho^\gamma$ with K constant, we can write $dP = c^2 d\rho$ where $c^2 = \gamma P/\rho = \gamma K\rho^{\gamma-1}$ is the adiabatic sound speed. The momentum equation is therefore

$$v\frac{dv}{dr} = -c^2\frac{d\ln\rho}{dr} - \frac{GM}{r^2}, \tag{4}$$

where we've divided through by ρ. Similarly, since $r^2\rho v$ is constant, we can rewrite the continuity equation as

$$\frac{d\ln\rho}{dr} = -\frac{2}{r} - \frac{d\ln v}{dr}. \tag{5}$$

Using this to eliminate the density gradient from the momentum equation, we obtain

$$\left(v - \frac{c^2}{v}\right)\frac{dv}{dr} = \frac{2c^2}{r} - \frac{GM}{r^2}. \tag{6}$$

or

$$\left(1 - \frac{v^2}{c^2}\right)\frac{d\ln v}{d\ln r} = 2\left(\frac{GM}{2c^2 r} - 1\right). \tag{7}$$

This shows that if the flow makes a transition from subsonic to supersonic, it must happen at a radius r_s given by

$$\boxed{r_s = \frac{GM}{2c_s^2}} \tag{8}$$

where $c_s = c(r_s)$, the sound speed at $r = r_s$. This defines the location of the **sonic point**.

1.2. Energy

Since

$$c^2 d\ln\rho = \gamma K\rho^{\gamma-2}d\rho = \frac{\gamma}{\gamma-1}d\left(K\rho^{\gamma-1}\right) = \frac{\gamma}{\gamma-1}d\left(\frac{P}{\rho}\right), \tag{9}$$

equation (4) can be written

$$\frac{d}{dr}\left(\frac{1}{2}v^2 + \frac{\gamma}{\gamma-1}\frac{P}{\rho} - \frac{GM}{r}\right) = 0 \tag{10}$$

showing that there is another constant of the flow, the energy

$$\boxed{E = \frac{1}{2}v^2 + \frac{c^2}{\gamma-1} - \frac{GM}{r}}. \tag{11}$$

Note that the middle term in equation (10) is the enthalpy

$$U + \frac{P}{\rho} = \frac{1}{\gamma-1}\frac{P}{\rho} + \frac{P}{\rho} = \frac{\gamma}{\gamma-1}\frac{P}{\rho}, \tag{12}$$

since the internal energy per unit mass of an ideal gas is given by $P = (\gamma-1)\rho U$.

1.3. Solution in terms of quantities at the sonic point

By evaluating $\dot{M}$ and E at the sonic point, we can write down a solution for the flow that depends only on the conditions at the sonic point. Starting with $\dot{M}$,

$$\dot{M} = 4\pi r^2\rho v = 4\pi r_s^2\rho_s c_s = \pi(GM)^2\frac{\rho_s}{c_s^3} \tag{13}$$

(where a subscript s indicates that the quantity is evaluated at $r = r_s$), we find

$$\boxed{\dot{M} = \frac{\pi(GM)^2}{(\gamma K)^{1/(\gamma-1)}}c_s^{\frac{5-3\gamma}{\gamma-1}}.} \tag{14}$$

The energy is

$$E = \frac{1}{2}v^2 + \frac{c^2}{\gamma - 1} - \frac{GM}{r} = \frac{1}{2}c_s^2 + \frac{c_s^2}{\gamma - 1} - \frac{GM}{r_s}, \tag{15}$$

which simplifies to

$$\boxed{E = \frac{c_s^2}{2}\left(\frac{5 - 3\gamma}{\gamma - 1}\right)}. \tag{16}$$

With the values of $\dot{M}$ and E determined, equation (15) can be used together with the relation

$$v(r) = \frac{\dot{M}}{4\pi r^2 \rho(r)} \tag{17}$$

to determine the profiles $v(r)$, $\rho(r)$ and $c(r)$.

1.4. Determining the sonic point location and sound speed at the sonic point

The previous section gives the solution for the wind in terms of the sound speed at the sonic point c_s, or equivalently the sonic point location r_s or density ρ_s. We can determine the sonic point location using the conditions at the stellar surface. Evaluating E at $r = R$, where the sound speed and velocity are $c_\star$ and $v_\star$ respectively, gives

$$E = \frac{c_s^2}{2}\left(\frac{5 - 3\gamma}{\gamma - 1}\right) = \frac{1}{2}v_\star^2 + \frac{c_\star^2}{\gamma - 1} - \frac{GM}{R}. \tag{18}$$

For flows that start with a small Mach number, we can neglect the $v_\star$ term relative to the $c_\star$ term. Setting $v_\star = 0$ gives the closed form solution

$$E = \frac{c_s^2}{2}\left(\frac{5 - 3\gamma}{\gamma - 1}\right) = \frac{GM}{R}\left(\frac{1}{\gamma - 1}\frac{c_\star^2 R}{GM} - 1\right). \tag{19}$$

Writing the Jeans escape parameter as

$$\lambda_\star = \frac{\rho_\star GM}{P_\star R} = \frac{\gamma GM}{c_\star^2 R} = \frac{GM}{R}\frac{1}{K\rho_\star^{\gamma-1}}, \tag{20}$$

this is

$$E = \frac{c_s^2}{2}\left(\frac{5 - 3\gamma}{\gamma - 1}\right) = \frac{GM}{R}\left(\frac{\lambda_{\rm crit}}{\lambda_\star} - 1\right) \tag{21}$$

where

$$\lambda_{\rm crit} = \frac{\gamma}{\gamma - 1}. \tag{22}$$

The energy is positive, implying an outflow is possible, when $\lambda_\star < \lambda_{\rm crit}$.

The approximation that $v_\star \ll c_\star$ is good for flows that are close to isothermal $\gamma \approx 1$, but becomes less good for larger γ. In that case, we can keep the $v_\star$ term, but rewrite it in terms of c_s:

$$4\pi R^2 \rho_\star v_\star = 4\pi r_s^2 \rho_s c_s \Rightarrow v_\star^2 = c_s^2 \left(\frac{r_s}{R}\right)^4 \left(\frac{\rho_s}{\rho_\star}\right)^2 . \tag{23}$$

Defining $\tilde{E} = E/(GM/R)$, the updated version of equation (21) is

$$\tilde{E} = \frac{\lambda_{\rm crit}}{\lambda_\star} - 1 + \frac{(2/\alpha)^\alpha}{32} \tilde{E}^\alpha \left(\frac{\lambda_\star}{\gamma}\right)^{2/(\gamma-1)} \tag{24}$$

with $\alpha = (5-3\gamma)/(\gamma-1)$. Once $\tilde{E}$ has been determined, then c_s follows from

$$c_s^2 = c_\star^2 \left(\frac{2\tilde{E}}{\alpha}\right)\left(\frac{\lambda_\star}{\gamma}\right). \tag{25}$$

This shows that even when we include the $v_\star$ term, the conditions at the sonic point are determined by the single parameter $\lambda_\star$.

As an example, it is interesting to look at the case $\gamma = 7/5 = 1.4$ because then $\alpha = 2$ and we get a quadratic equation

$$0 = \frac{3125\lambda_\star^5}{537824}\tilde{E}^2 - \tilde{E} + \frac{7}{2\lambda_\star} - 1 \tag{26}$$

where we have inserted the value $\lambda_{\rm crit} = 7/2$. As $\lambda_\star \to 7/2$, the solution has a small value of $\tilde{E}$ and the quadratic term can be neglected, we fall back to the previous solution where we assumed $v_\star = 0$. This corresponds to a situation where the sonic point lies well above the stellar surface. However, as $\lambda_\star$ becomes smaller, $\tilde{E}$ increases and the quadratic term becomes important. Both roots of the quadratic converge on the value $\tilde{E} = 1/2$ when $\lambda_\star = 2\gamma = 14/5$. From the definition of $\lambda_\star$, we see that this corresponds to $r_s = R$.

More generally,

$$\frac{r_s}{R} = \frac{GM}{2c_s^2 R} = \frac{GM}{2R}\frac{\alpha}{2E} \tag{27}$$

(since $E = \alpha c_s^2/2$ from eq. [21]), so we have $r_s = R$ when $\tilde{E} = \alpha/4$. Substituting $\tilde{E} = \alpha/4$ into equation (24) gives

$$\frac{\alpha}{4} + 1 - \frac{\lambda_{\rm crit}}{\lambda_\star} = \frac{1}{2^{\alpha+5}}\left(\frac{\lambda_\star}{\gamma}\right)^{2/(\gamma-1)}, \tag{28}$$

which is satisfied when $\lambda_\star = 2\gamma$. So we have the general behavior that r_s/R starts at large values for $\lambda_\star$ just below $\lambda_{\rm crit} = \gamma/(\gamma-1)$, and then moves closer to the star as $\lambda_\star$ decreases. The sonic point reaches the stellar radius for $\lambda_\star = 2\gamma$.

Therefore, transonic solutions require

$$\boxed{2\gamma < \lambda_\star < \lambda_{\rm crit} = \frac{\gamma}{\gamma - 1}.} \tag{29}$$

This range shrinks to zero at $\gamma = 3/2$, so transonic solutions occur only for $1 < \gamma < 3/2$. It is largest for $\gamma \to 1$, in which case $\lambda_{\rm crit}$ diverges and there is always a wind solution no matter how small $c_\star$ becomes.

The value of the Mach number at the surface can be obtained from equation (18) and using the definition of $\lambda_\star$, which gives

$$\mathcal{M}_\star^2 = \frac{2\lambda_\star}{\gamma}\left[\tilde{E} - \left(\frac{\lambda_{\rm crit}}{\lambda_\star} - 1\right)\right]. \tag{30}$$

Note that using equation (21) for $\tilde{E}$ gives $\mathcal{M}_\star = 0$, consistent with the approximation made in deriving equation (21). Using equation (24), we can rewrite this as

$$\mathcal{M}_\star^2 = \frac{1}{16}\left(\frac{2\tilde{E}}{\alpha}\right)^{\alpha}\left(\frac{\lambda_\star}{\gamma}\right)^{(\gamma+1)/(\gamma-1)} = \frac{1}{16}\left(\frac{c_s}{c_\star}\right)^{2\alpha}\left(\frac{\lambda_\star}{\gamma}\right)^{4}. \tag{31}$$

1.5. Hydrostatic solutions at large $\lambda_\star$

We can get some further insight into the value of $\lambda_{\rm crit}$ by looking for hydrostatic solutions. With $P = K\rho^\gamma$, hydrostatic balance is

$$\frac{dP}{dr} = \gamma K \rho^{\gamma-1}\frac{d\rho}{dr} = -\rho\frac{GM}{r^2}. \tag{32}$$

Integrating this from the stellar surface outwards,

$$\int_{\rho_\star}^{\rho} \gamma K \rho^{\gamma-2} d\rho = -\int_R^r \frac{GM}{r^2} dr \tag{33}$$

$$\frac{\gamma K}{\gamma - 1}\left(\rho^{\gamma-1} - \rho_\star^{\gamma-1}\right) = GM\left(\frac{1}{r} - \frac{1}{R}\right) \tag{34}$$

$$\left(\frac{\rho}{\rho_\star}\right)^{\gamma-1} - 1 = \frac{\lambda_\star}{\lambda_{\rm crit}}\left(\frac{R}{r} - 1\right), \tag{35}$$

or

$$\left(\frac{\rho}{\rho_\star}\right)^{\gamma-1} = \frac{\lambda_\star}{\lambda_{\rm crit}}\left(\frac{R}{r} - \frac{R}{r_{\rm max}}\right), \tag{36}$$

where the maximum extent of the atmosphere, the radius where the density drops to zero, is given by

$$r_{\rm max} = \frac{R}{1-\lambda_{\rm crit}/\lambda_\star}. \tag{37}$$

Therefore we see that there is a hydrostatic solution as long as $\lambda_\star > \lambda_{\rm crit}$. As $\lambda_\star$ approaches $\lambda_{\rm crit}$ from above, the extent of the atmosphere $r_{\rm max} \rightarrow \infty$. For even smaller values of $\lambda_\star$, we transition into the wind solutions discussed earlier. As $\gamma \rightarrow 1$ and $\lambda_{\rm crit}$ diverges, we lose the hydrostatic solutions. This is consistent with the density profile of an isothermal atmosphere, which falls off exponentially. In contrast to a polytropic atmosphere, an isothermal atmosphere does not have a finite thickness.

How do we interpret $\lambda_\star = \lambda_{\rm crit}$? Using the definition of $\lambda_\star$, this corresponds to

$$\gamma\frac{GM}{c_\star^2 R} = \frac{\gamma}{\gamma-1} \tag{38}$$

or

$$\frac{GM}{R} = \frac{c_\star^2}{\gamma-1}, \tag{39}$$

i.e. the specific enthalpy of the gas is equal to the gravitational binding energy at the surface of the star. A wind solution requires that the enthalpy be large enough to provide the gravitational energy required to take mass out to infinity, this allows $E \rightarrow v^2/2$ to be positive at infinity.

An atmosphere with $\gamma > 3/2$ will develop a wind even when it still has a finite extent. For these values of γ, we have $\gamma_{\rm crit} < 2\gamma$, so a transonic wind is not possible. Instead, once $\lambda_\star$ decreases below 2γ, i.e. the sound speed at the stellar surface satisfies

$$c_\star^2 > \frac{GM}{2R} \tag{40}$$

then a supersonic wind forms. The radius of the atmosphere at this point is (substituting $\lambda_\star = 2\gamma$ into the expression for $r_{\rm max}$)

$$r_{\rm wind} = \frac{2(\gamma-1)R}{2\gamma-3}. \tag{41}$$

(Note that $r_{\rm wind} \rightarrow \infty$ for $\gamma = 3/2$).

1.6. Numerical solutions from `Athena++`

We ran adiabatic wind models in `Athena++`. We choose units by setting the inner edge of the numerical grid at $r = 1$, ie. we measure distance in units of the stellar radius, and we choose $GM = 1$ and set $\rho_\star = 1$ at the surface of the star. We used 1024 logarithmically-spaced grid points from $r = 1$ to $r = 100$. At the inner boundary, we assume hydrostatic balance for the ghost cells, and $dv/dr = 0$. At the outer boundary, we set $dv/dr = 0$ and $d(r^2\rho)/dr = 0$. To establish the wind, we initialize the region $R < r < 1.1R$ with density $\rho_\star$, and density 10^{-14} on the rest of the grid, zero velocities everywhere. This results in rapid ejection of a shell and then the solution evolves to the steady-state.

Given the choice of units, we calculate analytic predictions as follows. With $\rho_\star = 1$, we have $c_\star^2 = \gamma K$, and $\lambda_\star = 1/K$. We then solve equations (24) and (25) to find $\tilde{E}$ ($= E$ in code units) and c_s. The accretion rate and velocity at the surface are

$$\dot{M} = \frac{\pi}{(\gamma K)^{1/(\gamma-1)}} c_s^\alpha, \qquad v_\star = \frac{c_s^\alpha}{4} (\gamma K)^{-\frac{3+\alpha}{2}}. \tag{42}$$

Figures 1 and 2 show the results from a set of models with $\gamma = 1.4$ and $\gamma = 1.1$ respectively. We find that for $K < K_{\rm crit}$, the numerical solution matches well with the hydrostatic profile as expected. The sonic point approaches the stellar surface as $K \to 1/2\gamma$.

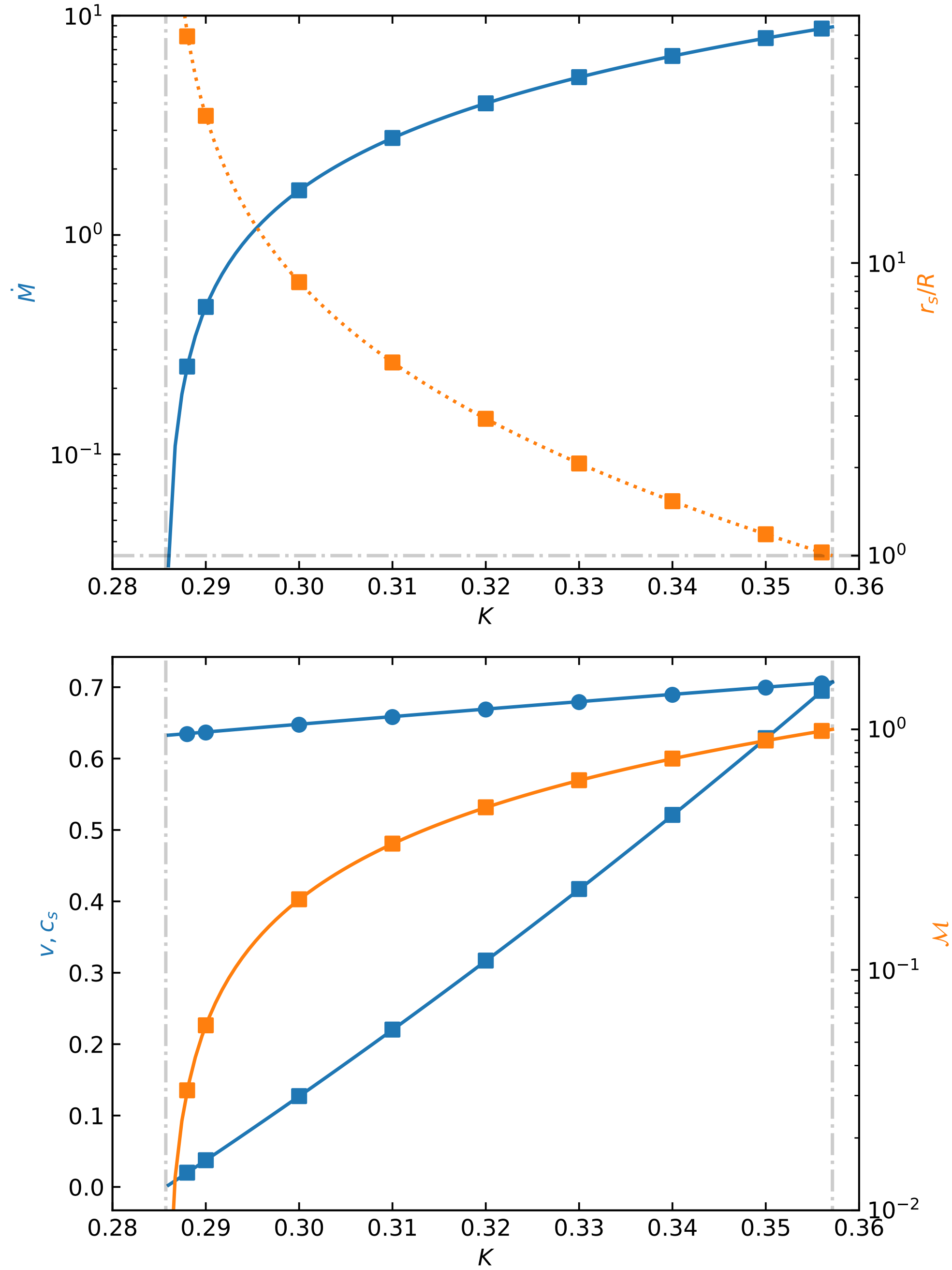

Fig. 1.— Properties of the wind as a function of K for transonic solutions with $\gamma = 7/5$. Lines are the analytic solution for the adiabatic wind, and the symbols are the values measured from the numerical simulations. The dot-dashed lines show $r_s = R$ (horizontal line), $K = (\gamma-1)/\gamma$ (left vertical line) and $K = 1/2\gamma$ (right vertical line).

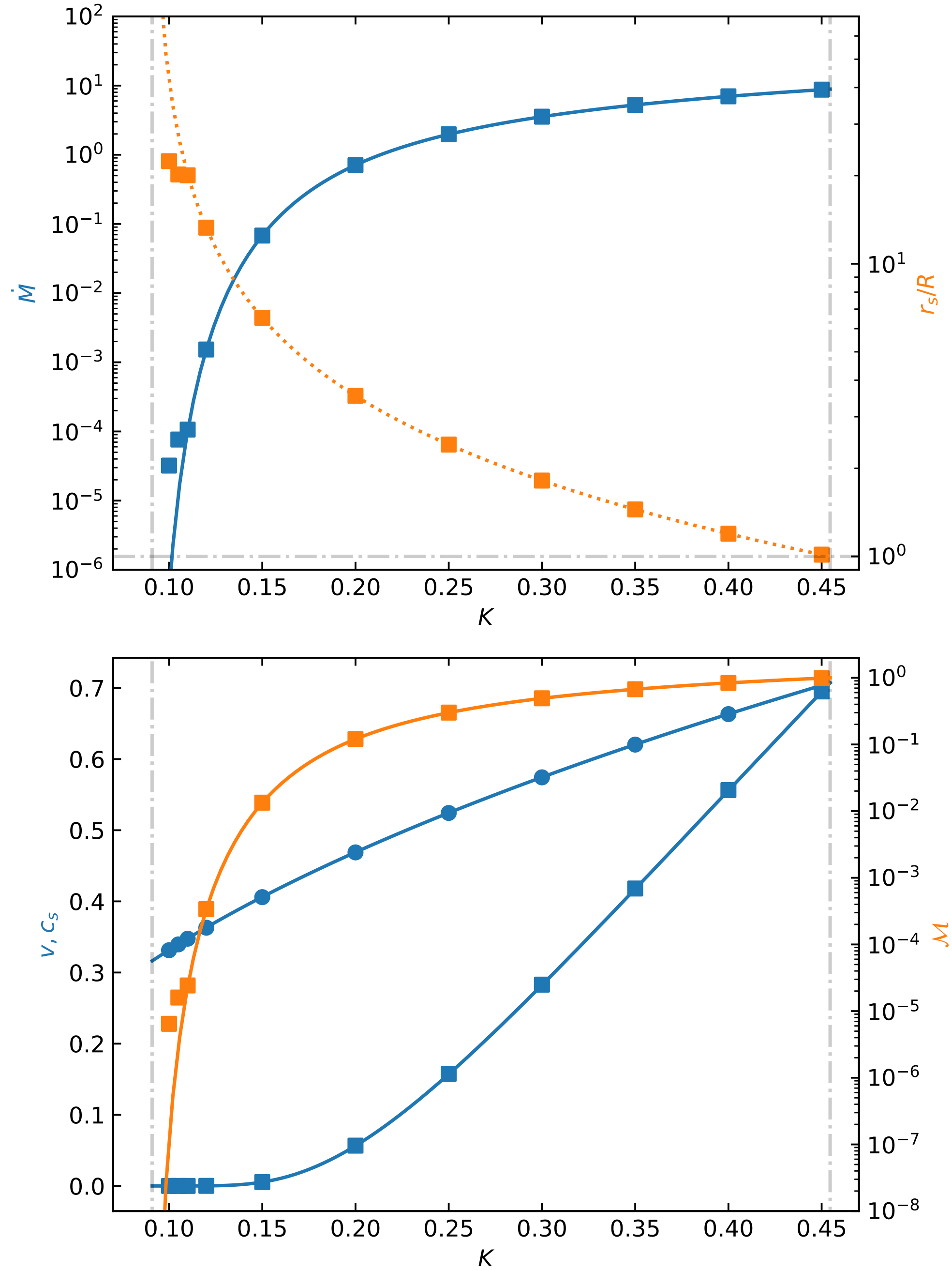

Fig. 2.— Same as Figure 1 but for $\gamma = 1.1$.