Solutions - HW2

PHYS 457 - Winter 2024

January 2024

1 Problem 1

1.1 (a) Harmonic oscillator

The Hamiltonian for the harmonic oscillator is

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2. \tag{1}$$

The ground state energy $E_{\rm gs}$ is given by

$$E_{\rm gs} = \langle 0|H|0\rangle, \tag{2}$$

where $|0\rangle$ is the ground state. To shorten the notation, we will write $\langle A\rangle$ for the ground state expectation value $\langle 0|A|0\rangle$ of any operator A. Then, we have

$$E_{\rm gs} = \langle H\rangle = \frac{1}{2m}\langle p^2\rangle + \frac{1}{2}m\omega^2\langle x^2\rangle. \tag{3}$$

Using the uncertainty principle $\langle(\Delta p)^2\rangle\langle(\Delta x)^2\rangle = \hbar^2/4$ and $\langle(\Delta p)^2\rangle = \langle p^2\rangle$, $\langle(\Delta x)^2\rangle = \langle p^2\rangle$ (see footnote[1]) we can rewrite the equation above as

$$E_{\rm gs} = \frac{1}{2m}\frac{\hbar^2}{4\langle x^2\rangle} + \frac{1}{2}m\omega^2\langle x^2\rangle. \tag{4}$$

We have rewritten the ground-state energy in terms of a single variable (for which we don't know the value). To obtain the ground energy, we minimize the energy with respect to $\langle x^2\rangle$:

$$\frac{\partial E_{\rm gs}}{\partial\langle x^2\rangle} = -\frac{\hbar^2}{4m}\langle x^2\rangle^{-2} + m\omega^2 = 0 \implies \langle x^2\rangle = \left(\frac{\hbar^2}{4m^2\omega^2}\right)^{1/2} = \frac{\hbar}{2m\omega}. \tag{5}$$

Inserting this result in equation (4) we obtain

$$E_{\rm gs} = \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega \implies \boxed{E_{\rm gs} = \frac{1}{2}\hbar\omega}. \tag{6}$$

This is precisely what we expect, as it is the ground state energy we obtain when we solve the problem exactly.

[1] The variance in momentum is $\langle(\Delta p)^2\rangle = \langle p^2\rangle - \langle p\rangle^2$. We have $\langle p\rangle = 0$ for any stationary state (such as the ground state). For the position we also have $\langle(\Delta x)^2\rangle = \langle x^2\rangle - \langle x\rangle^2$ and $\langle x\rangle = 0$ as our potential is symmetric about $x = 0$.

1.2 (b) 1D finite square well

For a particle confined into a distance a, such that $\langle (\Delta x)^2 \rangle = a^2$, we have

$$E_{\text{kin}} = \frac{\langle p^2 \rangle}{2m} > \frac{\hbar}{8ma^2} \tag{7}$$

so that the minimal kinetic energy is $\hbar/2ma$. Above, we have used the uncertainty principle in the same way as part (a). Then, we have

$$E_{\text{kin}} > V_0 \implies \frac{\hbar^2}{8ma^2} > V_0 \implies a^2 < \frac{\hbar^2}{2mV_0} \tag{8}$$

Above, it seems like there is a minimum width for the finite well so that the kinetic energy compensates for the negative potential and the total energy of the ground state is positive (and hence, not a bound state). However, we have assume that the ground state is confined to the potential width a, which is not true. There is always a decaying part into the classically forbidden region (where $V(x) = 0$). Even in the most extreme case, as $a \to 0$, where the potential is given by a Dirac distribution $V(x) = -V_0\delta(x)$, the wavefunction has a finite decay length outside of the well.

2 Problem 2

Using $\boldsymbol{P} = \boldsymbol{p}_1 + \boldsymbol{p}_2$, $\boldsymbol{p} = (m_2\boldsymbol{p}_1 - m_1\boldsymbol{p}_2)/(m_1 + m_2)$, $M = m_1 + m_2$, and $\mu = m_1 m_2/M$, we get

$$\begin{aligned} \frac{\boldsymbol{P}^2}{M} + \frac{\boldsymbol{p}^2}{\mu} &= \frac{\boldsymbol{p}_1^2 + \boldsymbol{p}_1\boldsymbol{p}_2 + \boldsymbol{p}_2\boldsymbol{p}_1 + \boldsymbol{p}_2^2}{M} + \frac{m_2^2\boldsymbol{p}_1^2 - m_1 m_2(\boldsymbol{p}_1\boldsymbol{p}_2 + \boldsymbol{p}_2\boldsymbol{p}_1) + m_1^2\boldsymbol{p}_2^2}{M^2 m_1 m_2/M} \\ &= \boldsymbol{p}_1^2\left(\frac{1}{M} + \frac{m_2}{m_1 M}\right) + \boldsymbol{p}_2^2\left(\frac{1}{M} + \frac{m_1}{M m_2}\right) + (\boldsymbol{p}_1\boldsymbol{p}_2 + \boldsymbol{p}_2\boldsymbol{p}_1)\left(\frac{1}{M} - \frac{1}{M}\right). \end{aligned} \tag{1}$$

The third term is zero. We also have

$$\frac{1}{M} + \frac{m_2}{Mm_1} = \frac{m_1 + m_2}{Mm_1} = \frac{1}{m_1}. \tag{2}$$

Similarly, $1/M + m_1/Mm_2 = 1/m_2$. Therefore, we obtain

$$\frac{\boldsymbol{P}^2}{2M} + \frac{\boldsymbol{p}^2}{2\mu} = \frac{\boldsymbol{p}_1^2}{2m_1} + \frac{\boldsymbol{p}_2^2}{m_2}, \tag{3}$$

where we have reinstated the factor of 1/2 that we omitted in equation (1).

3 Problem 3

We have (using Einstein summation convention)

$$\begin{aligned} \boldsymbol{L}^2 &= \varepsilon_{ijk}\varepsilon_{imn} r_j p_k r_m p_n \\ &= (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}) r_j p_k r_m p_n \\ &= r_j p_k r_j p_k - r_j p_k r_k p_j \\ &= r_j r_j p_k p_k - i\hbar r_j p_j - (i\hbar r_j p_j + p_k r_k r_j p_j) \\ &= \boldsymbol{r}^2\boldsymbol{p}^2 - i\hbar\boldsymbol{r}\cdot\boldsymbol{p} - (i\hbar\boldsymbol{r}\cdot\boldsymbol{p} - 3i\hbar\boldsymbol{r}\cdot\boldsymbol{p} + r_k p_k r_j p_j) \\ &= \boldsymbol{r}^2\boldsymbol{p}^2 + i\hbar\boldsymbol{r}\cdot\boldsymbol{p} - (\boldsymbol{r}\cdot\boldsymbol{p})^2, \end{aligned} \tag{1}$$

where we have used $[r_i, p_j] = i\hbar\delta_{ij}$.

4 Problem 4

We have

$$[L_i, L_p] = \varepsilon_{ijk}\varepsilon_{mnp}\left(r_j p_k r_m p_n - r_m p_n r_j p_j\right) \tag{1}$$

Let's look at the second term inside the parenthesis:

$$\begin{aligned} r_m p_n r_j p_k &= r_m\left(-i\hbar\delta_{jn} + r_j p_n\right) p_k \\ &= -i\hbar\delta_{nj} r_m p_k + r_j r_m p_k p_n \\ &= -i\hbar\delta_{nj} r_m p_k + i\hbar\delta_{mk} r_j p_n + r_j p_k r_m p_n \end{aligned} \tag{2}$$

where we used $[r_j, p_n] = i\hbar\delta_{jn}$ in the first line and the fact that $[r_i, r_j] = [p_i, p_j] = 0$ (for any i, j). Inserting the result above in equation (1), we get

$$\begin{aligned} [L_i, L_p] &= i\hbar\varepsilon_{ijk}\varepsilon_{mnp}\left(\delta_{nj} r_m p_k - \delta_{mk} r_j p_n\right) \\ &= i\hbar\left(\varepsilon_{ijk}\varepsilon_{mjp} r_m p_k - \varepsilon_{ijk}\varepsilon_{knp} r_j p_n\right) \\ &= i\hbar\left[\left(\delta_{im}\delta_{kp} - \delta_{ip}\delta_{km}\right) r_m p_k - \left(\delta_{in}\delta_{jp} - \delta_{ip}\delta_{jn}\right) r_j p_n\right] \\ &= i\hbar\left[r_i p_p - \delta_{ip} r_m p_m - \left(r_p p_i - \delta_{ip} r_j p_j\right)\right] \\ &= i\hbar\left(r_i p_p - r_p p_i\right), \end{aligned} \tag{3}$$

which is $i\varepsilon_{ipk}L_k$. Take, for example, $i = x, p = y$. Then, we have $i\hbar\sum_{z=\{x,y,z\}}\varepsilon_{xyk}L_z = i\hbar L_z$.