Homework 4

TA: Renée Goodman, Due: Tuesday, February 6th, 2024, 11:59 PM

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Hint: You will need that the ground-state wave-function for a hydrogen-like (one-electron) atom, which is:

$$\langle \mathbf{r}|1,0,0\rangle = \frac{1}{\sqrt{4\pi}} R_{10}(r) = \frac{1}{\sqrt{\pi}}\left(\frac{Z}{a_0}\right)^{3/2} e^{-Zr/a_0} \tag{1}$$

where a_0 is the Bohr radius, Z is the number of protons in the nucleus, and where the wave-function is normalized such that:

$$\int d\mathbf{r}^3|\Psi|^2 = \int_0^\infty drr^2|R_{10}|^2 = 1 \tag{2}$$

1. (5 points) **Townsend 10.4**: Calculate the probability that an electron in the ground state of hydrogen is outside the classically allowed region.

   Solution: The classically allowed region is the region such that $KE = \frac{p^2}{2m} \geq 0$. This imparts the condition that $E_{tot} \geq V(r)$ for classical problems. Recall that the energy levels for the hydrogen atom ($Z = 1$) are given by:

   $$E_n = -\frac{1}{2n^2 a_0}\left(\frac{e^2}{4\pi\varepsilon_0}\right) \tag{3}$$

   where the Bohr radius $a_0 = \frac{4\pi\varepsilon_0\hbar^2}{me^2}$, and e is the unit charge. Therefore, electrons *outside* the classically allowed region for a Coulomb potential in the ground state ($n = 1$) will obey:

   $$-\frac{1}{2a_0}\left(\frac{e^2}{4\pi\varepsilon_0}\right) < -\frac{e^2}{4\pi\varepsilon_0 r} \tag{4}$$

   or:

   $$r > 2a_0 \tag{5}$$

   The probability, therefore, that the particle will be in this classically forbidden region, is:

   $$P(r > 2a_0) = \int dr^3|R_{10}|^2 = \frac{4}{a_0^3}\int_{2a_0}^\infty drr^2 e^{-2r/a_0} \tag{6}$$

   This integral is easily obtained by integrating by parts twice:

   $$P(r > 2a_0) = -\frac{4e^{-2r/a_0}}{a_0^3}\left[r^2\frac{a_0}{2} + 2r\frac{a_0^2}{4} + 2\frac{a_0^3}{8}\right]_{2a_0}^\infty \tag{7}$$

   Through l'Hôpital's rule we can show this expression vanishes at $r = \infty$, thus we are left with:

   $$P(r > 2a_0) = 13e^{-4} \tag{8}$$

2. (12 points) **Townsend 10.5**: What is the ground-state energy and Bohr radius for each of the following two-particle systems? What is the wavelength of the radiation emitted in the transition from the $n = 2$ state to the $n = 1$ state in each case? In what portion of the electromagnetic spectrum does this radiation reside?

Solution: The energy in Joules of a hydrogen-like atom is given by Eq. 10.34 in Townsend:

$$E_n = -\frac{\mu c^2 Z^2 \alpha^2}{2n^2} \tag{9}$$

where the fine structure constant is $\alpha = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{\hbar c} \approx 1/137$ and Z is the number of protons. The Bohr radius of such a system can be found setting the ground state energy to the Coulomb potential:

$$\frac{\mu Z^2 c^2}{2n^2} = \frac{Ze^2}{4\pi\varepsilon_0 r} \quad \Rightarrow \quad a_0^* = \frac{4\pi\varepsilon_0 \hbar^2}{\mu Z e^2} = \frac{m_e}{Z\mu} a_0 \tag{10}$$

We can find the wavelength of energy transitions using the Rydberg formula (using $\hbar = h/2\pi$, $n_2 = 2$, $n_1 = 1$):

$$\Delta E = \frac{hc}{\lambda} = -\frac{\mu\alpha^2 c^2 Z^2}{2}\left[\frac{1}{n_2^2} - \frac{1}{n_1^2}\right] \quad \Rightarrow \quad \lambda = \frac{4hc}{3|E_1|} \tag{11}$$

Using these equations we can solve for the following two-particle systems:

(a) H^2 a bound state of a deuteron and an electron

A deuteron is composed of a proton and a neutron in the nucleus. Therefore $Z = 1$ (using 10.34), and $\mu = \frac{2m_p m_e}{2m_p + m_e}$ using $m_p \approx m_n$. Thus we get:

$$E_1 = -\frac{2m_p m_e}{2m_p + m_e}\frac{\alpha^2 c^2}{2} = -13.61eV; \quad a_0^* = \frac{2m_p + m_e}{2m_p} a_0 = 1.0002a_0 \tag{12}$$

$$\lambda = \frac{4hc}{3(13.6)} = 1.22 \times 10^{-7} m \tag{13}$$

The transition is in the violet to ultraviolet range of the EM spectrum.

(b) Positronium

A positronium is composed of a positron and an electron. A positron is the anti-particle of the electron, and has the same mass as the latter ($m_{e+} = m_e$). Therefore $Z = 1$, and $\mu = \frac{m_{e+} m_e}{m_{e+} + m_e} = \frac{m_e}{2}$. Thus we get:

$$E_1 = -\frac{m_e \alpha^2 c^2}{4} = -6.8eV; \quad a_0^* = \frac{2m_e}{m_e} a_0 = 2a_0; \quad \lambda = \frac{4hc}{3(6.8)} = 2.43 \times 10^{-7} m \tag{14}$$

The transition is in the ultraviolet range of the EM spectrum.

(c) A bound state of a proton and a negative muon

A muon has mass $m_\mu = 1.88 \times 10^{-28} kg \approx \frac{m_p}{9}$. In this system we have again $Z = 1$ and $\mu = \frac{m_p(m_p/9)}{m_p/9 + m_p} = \frac{m_p}{10}$. Thus we get:

$$E_1 = -\frac{\alpha^2 c^2}{2}\frac{m_p}{10} \approx -2498.5eV; \quad a_0* = \frac{m_p}{10m_e} a_0 = 0.00545a_0; \quad \lambda = \frac{4hc}{3(2498.5)} = 6.62 \times 10^{-10} m \tag{15}$$

The transition is in the x-ray to gamma ray region of the EM spectrum.

(d) A gravitational bound state of two neutrons.

In this case, we shall modify our energy, considering instead a gravitational potential instead of a coulomb potential. This gives that:

$$\alpha_g = \frac{Gm_1m_2}{\hbar c} \tag{16}$$

and

$$a_{0g} = \frac{\hbar^2}{\mu Z}\frac{1}{Gm_1m_2} = \frac{\hbar}{\mu Zc\alpha_g} \tag{17}$$

For this system we have $\mu = \frac{m_n}{2}$ ($m_n \approx m_p$) and $Z = 1$. Thus we have:

$$E_1 = -6.68\times 10^{-69}eV; \quad a_{0g} = 8.13\times 10^{22}m; \quad \lambda = 3.59\times 10^{43}m \tag{18}$$

The transition is in the radio wave region of the EM spectrum

3. (7 points) **Townsend 10.7**: An electron is in the ground state of tritium, for which the nucleus is the isotope of hydrogen with one proton and two neutrons. A nuclear reaction instantaneously changes the nucleus into He^3, which consists of two protons and one neutron. Calculate the probability that the electron remains in the ground state of the new atom. Obtain a numerical answer.

Solution: In essence, this problem consists of an He^{3+} atom. For tritium, we have $Z = 1$, and for He^{3+} we have $Z = 2$. For both systems $\mu = \frac{3m_pm_e}{3m_p+m_e}$. We can easily obtain the two ground-state energies:

$$E_{1T} = \frac{\mu\alpha^2c^2}{2}; \quad E_{1He} = 2\mu\alpha^2c^2 \tag{19}$$

from which we can also obtain the two Bohr radii:

$$a^*_{0T} = \frac{m_e}{\mu}a_0; \quad a^*_{0He} = \frac{m_e}{2\mu}a_0 = \frac{a^*_{0T}}{2} \tag{20}$$

To find the probability that the electron remains in the ground state we must compute $|\langle\Psi_T|\Psi_{He}\rangle|^2$:

$$\langle\Psi_T|\Psi_{He}\rangle = \int dr^3\,\langle\Psi_T|r\rangle\,\langle r|\Psi_{He}\rangle \tag{21}$$

The ground state wave functions are given by (as per the hint):

$$\langle\Psi_T|r\rangle = \frac{1}{\sqrt{\pi}}\left(\frac{1}{a^*_{0T}}\right)^{3/2}e^{-r/a^*_{0T}}; \quad \langle r|\Psi_{He}\rangle = \frac{1}{\sqrt{\pi}}\left(\frac{2}{a^*_{0T}}\right)^{3/2}e^{-2r/a^*_{0T}} \tag{22}$$

Therefore:

$$\langle\Psi_T|\Psi_{He}\rangle = \frac{1}{\pi}\left(\frac{2^{3/2}}{a^{*3}_{0T}}\right)\int_0^\infty 4\pi dr r^2 e^{-\frac{3r}{a^*_{0T}}} \tag{23}$$

Again we can easily integrate by parts and evaluate:

$$\langle\Psi_T|\Psi_{He}\rangle = 4\left(\frac{2^{3/2}}{a^{*3}_{0T}}\right)\frac{2a^{*3}_{0T}}{3^3} = \frac{2^{9/2}}{3^3} \tag{24}$$

Therefore:

$$|\langle\Psi_T|\Psi_{He}\rangle|^2 = \left(\frac{2^{9/2}}{3^3}\right)^2 = 0.702 \tag{25}$$