# Homework 1

TA: Renée Goodman, Due: Tuesday, Jan. 16th, 2024, 11:59 PM

January 17, 2024

1. (10 points) **Townsend 8.1**: Use the free-particle propagator 8.9 in section 8.3 to determine how the Gaussian position-space wave packet 6.59 evolves in time. Check your result by comparing with Townsend 6.76. A useful formula is:

$$\int_{-\infty}^{\infty} dx e^{-(\alpha x^2+\beta x)} = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha} \quad ; \quad Re(\alpha) > 0 \tag{1}$$

*Solution*:
The free particle propagator given by 8.9 in Townsend is:

$$\langle x', t' | x_0, t_0 \rangle = \sqrt{\frac{m}{2\pi\hbar i(t'-t_0)}} e^{im(x'-x_0)^2/2\hbar(t'-t_0)} \tag{2}$$

and the normalized Gaussian position-space wave packet is given by 6.62 in Townsend:

$$\langle x|\psi\rangle = \psi(x) = \frac{1}{\sqrt{a\sqrt{\pi}}} e^{-x^2/2a^2} \tag{3}$$

We can use Eq. 8.3 in Townsend to determine how an arbitrary state propagates in time, namely:

$$\langle x'|\psi(t')\rangle = \int_{-\infty}^{\infty} dx_0 \, \langle x', t'|x_0 . t_0\rangle \, \langle x_0|\psi(t_0)\rangle \tag{4}$$

Putting together Eq.'s 2 & 3 into 4:

$$\langle x'|\psi(t')\rangle = \frac{1}{\sqrt{a\sqrt{\pi}}} \int_{-\infty}^{\infty} dx_0 \sqrt{\frac{m}{2\pi\hbar i(t'-t_0)}} e^{im(x'-x_0)^2/2\hbar(t'-t_0)} e^{-x_0^2/2a^2} \tag{5}$$

We can expand the exponent to get:

$$\langle x'|\psi(t')\rangle = \frac{1}{\sqrt{a\sqrt{\pi}}} \sqrt{\frac{m}{2\pi\hbar i(t'-t_0)}} \int_{-\infty}^{\infty} dx_0 \exp\left\{-\left[\left(\frac{1}{2a^2} - \frac{im}{2\hbar(t'-t_0)}\right)x_0^2 + \frac{x' im}{\hbar(t'-t_0)}x_0 - \frac{imx'^2}{2\hbar(t'-t_0)}\right]\right\} \tag{6}$$

Using the given relation we have:

$$\langle x'|\psi(t')\rangle = \frac{1}{\sqrt{a\sqrt{\pi}}} \sqrt{\frac{m}{2\pi\hbar i(t'-t_0)}} \sqrt{\frac{\pi}{\frac{1}{2a^2} - \frac{im}{2\hbar(t'-t_0)}}} \exp\left\{\frac{imx'^2}{2\hbar(t'-t_0)} + \frac{\left(\frac{x' im}{\hbar(t'-t_0)}\right)^2}{4\left[\frac{1}{2a^2} - \frac{im}{2\hbar(t'-t_0)}\right]}\right\} \tag{7}$$

Simplifying:

$$\langle x'|\psi(t')\rangle = \frac{1}{\sqrt{\sqrt{\pi}\left[1 + \frac{i\hbar}{ma^2}(t'-t_0)\right]}} e^{-x^2/2a^2\left[1+\frac{i\hbar}{ma^2}(t'-t_0)\right]} \tag{8}$$

Comparing to Eq. 6.76 in Townsend which is given by:

$$\psi(x,t) = \frac{1}{\sqrt{\sqrt{\pi}\left[a + (i\hbar t/ma)\right]}} e^{-x^2/2a^2[1+(i\hbar t/ma^2)]} \tag{9}$$

we find we get the appropriate result.

2. (10 points) **Townsend 8.3**: Determine, up to an overall multiplicative function of time, the transition amplitude, or propagator, for the harmonic oscillator using path integrals. See Feynman and Hibbs, *Path Integrals and Quantum Mechanics*, Sections 3.5 and 3.6. Comment on the form of your answer. Recall for a quandratic potential, the propagater is $\propto e^{iS_c/\hbar}$, so you just need to determine $S_c$, the action corresponding to the classical path $x_c(t)$. You could start with the general solution for motion of an oscillator $x(t) = A\sin(\omega t) + B\cos(\omega t)$.

*Solution*:
Recall that the action corresponding to a classical path is given by:

$$S_c = \int dt \mathcal{L}_c \tag{10}$$

where $\mathcal{L}_c$ is the classical Lagrangian defined as the difference between kinetic and potential energy. For a simple harmonic oscillator following the path $x(t) = A\sin(\omega t) + B\cos(\omega t)$, we have a Lagrangian given by:

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 \tag{11}$$

(also given in Feynman and Hibbs, Eq. 3.58). We know using the Feynman integral we expect a kernel of the form:

$$K(b,a) = e^{iS_c/\hbar}F(t_b,t_a) \tag{12}$$

thus we just need to compute

$$S_{cl} = \frac{m}{2}\int_{t_a}^{t_b} dt\dot{x}^2 - \omega^2 x^2 \tag{13}$$

Computing the first term by parts we get:

$$\int_{t_a}^{t_b} dt\dot{x}^2 = [x\dot{x}]_0^T - \int_0^T x\ddot{x}dt \tag{14}$$

noting that from Newton's laws we get $\ddot{x} = -\omega^2 x$. The total action is then:

$$S_{cl} = \frac{m}{2}\left[[x\dot{x}]_0^T + \omega^2\int_0^T xxdt - \omega^2\int_0^T x^2 dt\right] = \frac{m}{2}[x\dot{x}]_0^T \tag{15}$$

For a path $[x_a, x_b]$ follwoing the path $x(t)$, we can establish the boundary conditions $x(0) = x_a$ and $x(T) = x_b$ to determine that:

$$A = \frac{x_b - x_a\cos(\omega T)}{\sin(\omega T)}, \qquad B = x_a \tag{16}$$

Furthermore we have:

$$x(T) = A\sin(\omega T) + B\cos(\omega T), \qquad \dot{x}(T) = A\omega\cos(\omega T) - B\omega\sin(\omega T) \tag{17}$$

$$x(0) = B, \qquad \dot{x}(0) = A\omega \tag{18}$$

Putting everything together:

$$S_{cl} = \frac{m\omega}{2\sin(\omega T)}\left[(x_b^2 + x_a^2)\cos^2(\omega T) - 2x_a x_b\right] \tag{19}$$

$$K(b,a) \propto e^{\frac{im\omega}{2\hbar\sin(\omega T)}\left[(x_b^2+x_a^2)\cos^2(\omega T) - 2x_a x_b\right]} \tag{20}$$

3. (5 points) **Townsend 8.4**: Estimate the size of the action for free neutrons with $\lambda = 1.419$Å traversing a distance of 10 cm.

*Solution*: For a free neutron we have the Lagrangian which is purely kinetic energy.

$$\mathcal{L} = \frac{1}{2}mv^2 = \frac{p^2}{2m} \tag{21}$$

where $p = h/\lambda$. Using that $S \approx \mathcal{L}\Delta t$:

$$p \approx m\frac{\Delta x}{\Delta t} \quad \rightarrow \quad \Delta t \approx m\frac{\Delta x}{p} = \frac{ml\lambda}{h} \tag{22}$$

$$S \approx \frac{hl}{2\lambda} \approx 3 \times 10^{-19} \tag{23}$$

4. (6 points) **Townsend 8.5**: For which of the following does classical mechanics give an adequate description of motion? Explain.

   (a) (4 points) An electron with a speed $v/c = 1/137$, which is typical in the ground state of the hydrogen atom, traversing a distance of 0.5Å, which is a characteristic size of the atom.

   *Solution*: We found that for a free particle, $S = \frac{mx^2}{2t} = \frac{m}{2}v \cdot x$. Since $m$ and $v$ are the same quantum mechanically, only $x$ is different. Small deviations $\epsilon$ lead to $S = S_{cl}(1 + \frac{\epsilon^2}{3})$ around the classical path and $S = S_o(1 + \frac{\epsilon}{2})$ around another path. Therefore the classical path loses coherence when $S_{cl}\epsilon^2/3\hbar \approx 2\pi$ and the non-classical path for $S_o\epsilon/2\hbar \approx 2\pi$. When it comes to comparing different situations, a good measure is:

$$\left|\frac{S_{cl} - S_o}{\hbar}\right| = \frac{mv}{6\hbar}x' \tag{24}$$

   For $x' = 0.5$Å, we have

$$\left|\frac{S_{cl} - S_o}{\hbar}\right| \approx 0.05 \tag{25}$$

   In this case, even non-classical paths have very similar action to clasical action so they should be taken into account. Indeed for problems of electrons in atoms we need QM.

   (b) (2 points) An electron with the same speed as in (a) traversing a distance of 1cm

   *Solution*: Using the same formalism as in (a) we get:

$$\left|\frac{S_{cl} - S_o}{\hbar}\right| \approx 10^7 \tag{26}$$

   thus classical mechanics suffices.