Reading Questions Solutions Week #4

Dark Matter Halos and Equilibria of Collisionless Systems
February 2026

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1) What is the difference between collisionless and collisional systems?

Strong gravitational interactions between individual stars dominate the evolution of a collisional system, whereas they are are negligible in a collisionless system.

2) (a) What are the relaxation time and dynamical time of a galaxy? (b) How can one argue that galaxies are collisionless, by comparing their two-body relaxation time with the age of the Universe?

(a) The relaxation time of a galaxy is the time required to change one of its star's velocity by 100%. The dynamical time of a galaxy is the crossing time for individual stars. The crossing time is $t_{\mathrm{cross}} = 2\pi R/v$, where R is the galaxy size and v is the velocity of the star. We also have that the relaxation time is $t_{\mathrm{relax}} = n_{\mathrm{relax}} t_{\mathrm{cross}}$, where $n_{\mathrm{relax}} = N/8 \ln N$, where N is the number of stars in the galaxy.

(b) For a typical galaxy, we have $N \approx 10^{11}$ and $t_{\mathrm{cross}} \approx 100$ Myr. This yields $t_{\mathrm{relax}} \approx 5 \times 10^{10}$ yr, or $\sim$5 million times the age of our Universe. This means that two-body relaxation cannot be the mechanism through which galaxies reach dynamical equilibrium; otherwise, they would be far from equilibrium today.

3) (a) What's the assumption required for dynamical equilibrium, based on the virial theorem? (b) Under this assumption, what is the relation between kinetic and potential energy?

(a) The fundamental assumption of the virial theorem is that the system is in equilibrium, e.g. that $dG/dt = 0$, where G is the virial quantity and is defined as $G = \sum_{i=1}^{N} w_i \mathbf{x}_i \cdot \mathbf{v}_i$, where w_i are arbitrary weights and $\mathbf{x}_i$ and $\mathbf{v}_i$ are the position and velocity of the i^{th} particle, respectively.

(b) With this assumption, the result of the virial theorem is that $2K + W = 0$, where K is the kinetic energy of the system and W is its gravitational potential energy.

4) What is the collisionless Boltzmann equation?

The collisionless Boltzmann equation is:

$$\frac{\partial f}{\partial t} + \dot{\mathbf{x}} \frac{\partial f}{\partial \mathbf{x}} + \dot{\mathbf{v}} \frac{\partial f}{\partial \mathbf{v}} = 0, \tag{1}$$

where $f = f(\mathbf{x}, \mathbf{v}, t)$ is the distribution function of the system. The collisionless Boltzmann equation describes the behavior of f under the influence of gravity.

5) What are the Jeans equations, and what applicability do they have for studying astronomical systems of many objects?

Jeans equations are the collisionless Boltzmann equation multiplied by powers of $\mathbf{x}$ or $\mathbf{v}$ and integrated over a part of phase-space. They are moment equations of the collisionless Boltzmann equation. They are useful in astronomy because astronomical observations are usually performed at a fixed location in the sky. Thus, multiplying the collisionless Boltzmann equation by $\mathbf{v}$ and integrating over all components of the velocity connects to observables.

6) How can one physically interpret different values of the orbital anisotropy of a galaxy?

The orbital anisotropy β is the difference in the velocity dsitribution in different directions and is defined as:

$$\beta = 1 - \frac{\sigma_\theta^2 + \sigma_\phi^2}{2\sigma_r^2}, \tag{2}$$

where σ_θ, σ_ϕ and σ_r are the velocity dispersions in each of the spherical coordinates directions. If $\beta = 0$, we have that the spread of motion in r, θ, ϕ are all the same, such that the geometry is isotropic. If $\beta \to 1$, the spread of motion in r is much greater than in θ and ϕ, in which case we say that the geometry of the system is radially biased. If $\beta \to -\infty$, then all orbits are circular since this is the limiting case where $\sigma_r \to 0$, in which case we say that the system in tangentially biased.

7) How can astronomical observations and the Jeans equation be used to constrain dark matter halos influences?

Given a certain mass model and $\beta(r)$ profile, one can calculate the solutions to Jeans equations, which connect back to velocity dispersions (i.e., observables). One can then do forward modeling: given a certain physical model for DM halo structure (e.g., NFW profile), what is the expected stellar velocity dispersion profile? And how does it compare to data? Which connects to inverse modeling: given the data for stellar velocity dispersions coming from a galaxy, how can I fit my physical model to it in order to find the most accurate physical description of the matter distribution within that galaxy?