Math 53: Multivariable Calculus Notes

1 Parametric Curves, Polar Coordinates

1. **Parametric Equations**

   (a) Many curves cannot be expressed in the form $y = f(x)$ or $x = f(y)$. Sometimes we can express x and y as functions of a third variable t, called a **parameter.** The **parametric equations** $x = f(t)$ and $y = g(t)$ trace out a curve as t varies. If we restrict $a \leq t \leq b$, then the curve has initial point $(f(a), g(a))$ and terminal point $(f(b), g(b))$.

   (b) Note that familiar functions $y = f(x)$ can be written in parametric form as $x = t$ and $y = f(t)$. We can attempt to eliminate the parameter t to recover a Cartesian equation of the curve.

   (c) The parametric equations $x = a\cos t$, $y = b\sin t$, $0 \leq t \leq 2\pi$ sketch out the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in a counterclockwise direction.

   (d) The curve traced out by a point on the circumference of a circle as the circle rolls along a straight line is called a **cycloid**. Parametric equations for the cycloid if the point starts at the origin on a circle of radius r are $x = r(\theta - \sin\theta)$ and $y = r(1 - \cos\theta)$ for $\theta \in \mathbb{R}$; one arch is $0 \leq \theta \leq 2\pi$.

2. **Calculus of Parametric Curves**

   (a) **Derivatives:** By the chain rule, for $dx/dt \neq 0$,

   $$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

   (b) **Second derivative:** We also have

   $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$$

   (c) In computing tangents, recall point-slope form of a line: $y - y_1 = m(x - x_1)$.

   (d) Horizontal tangents occur when the numerator is 0; vertical tangents occur when the denominator is 0; use L'Hospitals rule when both the numerator and denominator are 0.

   (e) **Areas:** If a curve is sketched out by parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, then the area under the curve is

   $$A = \int_a^b y\,dx = \int_\alpha^\beta g(t)f'(t)\,dt$$

   where $a = f(\alpha)$ and $b = f(\beta)$. There is an analogous formula when integrating with respect to y:

   $$A = \int_c^d x\,dy = \int_\alpha^\beta g'(t)f(t)\,dt$$

   where $c = g(\alpha)$ and $d = g(\beta)$ In other words, set up integral with respect to x or y, then replace dx (or dy) with $\frac{dx}{dt}dt$ (or $\frac{dy}{dt}dt$) and the limits of integration with values of t that produce those limits.

   (f) **Arc length:** If a curve C is described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$, and C is traversed exactly once as t increases from α to β, then the length of C is

   $$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

   (g) **Surface Area:** As before, a curve C is described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$, $g(t) \geq 0$, and C is traversed exactly once as t increases from α to β. If C is rotated about an axis, the area of the resulting surface is

   $$S = \int_\alpha^\beta 2\pi y\,ds \qquad \text{rotation about x-axis}$$
   $$S = \int_\alpha^\beta 2\pi x\,ds \qquad \text{rotation about y-axis}$$

   where

   $$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

3. **Polar Coordinates**

   (a) The **polar coordinate system** can be more convenient than Cartesian coordinates for some purposes. There exists a **pole** O, or origin, and a ray starting at O called the **polar axis**.

   (b) Every point P has coordinates (r, θ), where r is the distance from P to O and θ is the angle between the polar axis and the line OP. Note that $(0, \theta)$ represents O for any angle θ.

   (c) We use the convention that $(-r, \theta)$ is the point $(r, \theta + \pi)$.

   (d) The relationship between polar and Cartesian coordinates is given by

   $$x = r\cos\theta \qquad y = r\sin\theta$$

   If x and y are known, we have the relation:

   $$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

(e) The graph of a polar equation $r = f(\theta)$ consists of all points that have a polar representation (r, θ) whose coordinates satisfy the equation. In sketching a polar curve, it is sometimes helpful to graph $r = f(\theta)$ in Cartesian coordinates to read off values of r as θ increases.

(f) A polar curve can be converted into Cartesian coordinates by manipulating the equation into groups of $r\cos$ and $r\sin$ terms, and substituting x and y into them, respectively.

(g) If the polar equation is unchanged when θ is replaced with $-\theta$, the curve is symmetric about the polar axis.

(h) $r = \sin\theta$ is a circle with radius $\frac{1}{2}$ and center $(0, \frac{1}{2})$, while $r = \cos\theta$ is a circle with radius $\frac{1}{2}$ and center $(\frac{1}{2}, 0)$. The graphs of $r = \sin c\theta$ and $r = \cos c\theta$ both look like flowers with c petals if c is odd and flowers with $2c$ petals if c is even.

(i) **Derivatives:** For $r = f(\theta)$, regard θ as a parameter to get $x = f(\theta)\cos\theta$ and $y = f(\theta)\sin\theta$. Then

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$$

(j) **Areas:** The area of a region whose boundary is given by a polar equation $r = f(\theta)$:

$$A = \frac{1}{2}\int_a^b f(\theta)^2\, d\theta$$

The area of the region bounded above by $f(\theta)$ and below by $g(\theta)$ is

$$A = \frac{1}{2}\int_a^b (f(\theta)^2 - g(\theta)^2)\, d\theta$$

(k) **Arc Length:** The length of a curve with polar equation $r = f(\theta)$, $a \le \theta \le b$ is

$$L = \int_a^b \sqrt{f(\theta)^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

2 Vectors and the Geometry of Space

1. **Three-Dimensional Coordinates**

(a) In two-dimensional geometry, the graph of an equation involving x and y is a curve in $\mathbb{R}^2$. In three-dimensional geometry, an equation in x, y, z is a **surface** in $\mathbb{R}^3$.

(b) The distance between two points in three dimensions is $\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$.

(c) The equation of a sphere with center $C(h, k, l)$ and radius r is

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$$

2. **Vectors (in $\mathbb{R}^n$)**

(a) Note we we will use the notation $\langle a_1, \ldots, a_n\rangle$ to denote a vector in $\mathbb{R}^n$ (not an inner product).

(b) Parallelogram Law: If $\vec{u}$ and $\vec{v}$ are vectors positioned so that the initial point of $\vec{v}$ is at the terminal point of $\vec{u}$, then $\vec{v} + \vec{u}$ is the vector from the initial point of $\vec{u}$ to the terminal point of $\vec{v}$.

(c) The **magnitude** of vector $\vec{v} = \langle a_1, a_2, a_3\rangle$ is the length:

$$\|\vec{v}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

(d) The standard basis vectors of $\mathbb{R}^3$:

$$\vec{i} = \langle 1, 0, 0\rangle \qquad \vec{j} = \langle 0, 1, 0\rangle \qquad \vec{k} = \langle 0, 0, 1\rangle$$

Sometimes vectors are written as linear combinations of $\vec{i}, \vec{j}, \vec{k}$.

(e) The distance between vectors $\vec{v}$ and $\vec{u}$ is $\|\vec{v} - \vec{u}\|$.

(f) The **dot product** (the standard inner product) of vectors $\vec{u}$ and $\vec{v} \in \mathbb{R}^n$ is defined as:

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \ldots u_n v_n$$

Properties:

$$\begin{aligned} \vec{u} \cdot \vec{v} &= \vec{v} \cdot \vec{u} \\ (c\vec{u}) \cdot \vec{v} &= c(\vec{u} \cdot \vec{v}) \\ (\vec{u} + \vec{x}) \cdot \vec{v} &= \vec{u} \cdot \vec{v} + \vec{x} \cdot \vec{v} \end{aligned}$$

If θ is the angle between $\vec{v}$ and $\vec{u}$, we have

$$\vec{u} \cdot \vec{v} = \|\vec{u}\|\,\|\vec{v}\|\cos\theta$$

Two vectors $\vec{u}$ and $\vec{v}$ are **orthogonal** iff $\vec{u} \cdot \vec{v} = 0$.

(g) The projection of a vector $\vec{y}$ onto another vector $\vec{x}$ is given by

$$\text{proj}_{\vec{x}}\vec{y} = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|^2}\vec{x}$$

The scalar projection of $\vec{y}$ onto $\vec{x}$ is given by

$$\text{comp}_{\vec{x}}\vec{y} = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\|}$$

(h) The **cross product** of $\vec{u}$ and $\vec{v}$, denoted $\vec{u} \times \vec{v}$, is given by the determinant:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. Properties:

$$\begin{aligned} \vec{u} \times \vec{v} &= -\vec{v} \times \vec{u} \\ (c\vec{u}) \times \vec{v} &= c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v}) \\ (\vec{u} + \vec{v}) \times \vec{z} &= \vec{u} \times \vec{z} + \vec{v} \times \vec{z} \end{aligned}$$

(i) If θ is the angle between $\vec{u}$ and $\vec{v}$, we have

$$\|\vec{u} \times \vec{v}\| = \|\vec{u}\| \, \|\vec{v}\| \sin\theta$$

Thus $\vec{u} \times \vec{v} = \vec{0}$ iff $\vec{u}$ and $\vec{v}$ are parallel. We can interpret $\|\vec{u} \times \vec{v}\|$ as the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

(j) The product $\vec{u} \cdot (\vec{v} \times \vec{z})$ is known as the **scalar triple product** of vectors $\vec{u}, \vec{v}, \vec{z}$:

$$\vec{u} \cdot (\vec{v} \times \vec{z}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$$

The volume of the parallelepiped determined by the vectors $\vec{u}, \vec{v}, \vec{z}$ is $\|\vec{u} \cdot (\vec{v} \times \vec{z})\|$.

3. **Equations of Lines and Planes**

(a) An equation of the line through the point (x_0, y_0, z_0) parallel to the direction vector $\vec{v} = \langle a, b, c\rangle$ is $\vec{r} = \vec{r_0} + t\vec{v}$. In parametric form,

$$x = x_0 + at \qquad y = y_0 + bt \qquad z = z_0 + ct$$

(b) **Skew lines** are lines in 3D space that are not parallel and do not intersect. Lines are parallel if their corresponding direction vectors are proportional. Lines intersect if there exist values of t and s such that x, y, z parameters are all equal.

(c) A **plane** in space is completely determined by a point and a vector orthogonal to the plane.

(d) A scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $\vec{n} = \langle a, b, c\rangle$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

We can collect terms to obtain a linear equation of the form $ax + by + cz + d = 0$, where $\langle a, b, c\rangle$ is the normal vector.

(e) To find the plane passing through points P, Q, R, we can find $\vec{PQ}$ and $\vec{PR}$ to obtain two vectors on the plane (recall the vector from A to B is $B - A$). Take their cross product to obtain a normal vector.

(f) Two planes are parallel if their normal vectors are parallel. If they are not, then they intersect.

(g) To find the line of intersection between two planes, find a point on that line (e.g., take $z = 0$ and solve for x, y) and take the cross product of the normal vectors to obtain a direction vector of the line.

(h) The angle between planes is given by the acute angle between their normal vectors (use definition of dot product).

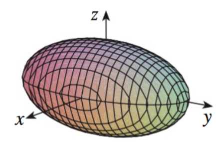
(i) The distance between the point $P_1(x_1, y_1, z_1)$ and the plane $ax + by + cz + d = 0$ is the scalar projection of P_1 onto the normal vector of the plane:

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

4. **Quadric Surfaces**

(a) A **quadric** surface is the graph of a degree-2 equation in x, y, z. If one of the variables is missing, the surface is cylindrical.

Ellipsoid

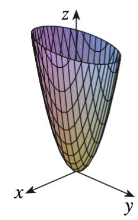


$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

All traces are ellipses.

If $a = b = c$, the ellipsoid is a sphere.

Elliptic Paraboloid



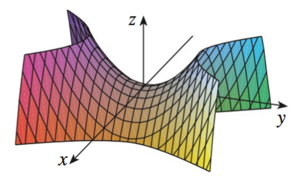
$$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses.

Vertical traces are parabolas.

The variable raised to the first power indicates the axis of the paraboloid.

Hyperbolic Paraboloid



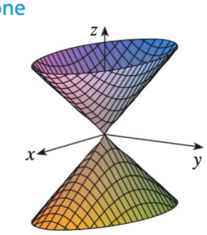
$$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

Horizontal traces are hyperbolas.

Vertical traces are parabolas.

The case where $c < 0$ is illustrated.

Cone

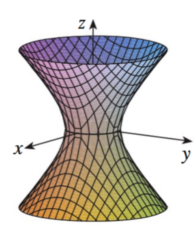


$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Horizontal traces are ellipses.

Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.

Hyperboloid of One Sheet



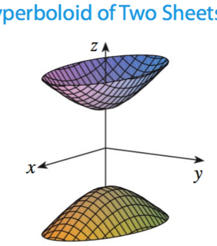
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Horizontal traces are ellipses.

Vertical traces are hyperbolas.

The axis of symmetry corresponds to the variable whose coefficient is negative.

Hyperboloid of Two Sheets



$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$.

Vertical traces are hyperbolas.

The two minus signs indicate two sheets.

(b) To sketch the graph of a surface, it is helpful to draw **traces**, or cross-sections parallel to the coordinate planes (take x, y or z to be 0 and consider the resulting shape).

3 Vector Functions

1. **Vector Functions**

(a) A vector-valued function maps real numbers to vectors. We will study functions $\vec{r} : \mathbb{R} \mapsto \mathbb{R}^3$:

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$$

The domain of $\vec{r}$ consists of all values of t for which the component functions are defined. As t varies throughout an interval, $\vec{r}$ defines a **space curve** traced out by the tip of the vector $\vec{r}(t)$.

(b) We say a space curve given by $\vec{r}(t)$ is **smooth** if $\vec{r'}$ is continuous and $\vec{r'}(t) \neq \vec{0}$ for all t.

(c) To determine a vector function for the curve of intersection of two surfaces: Amounts to finding x, y, z such that the equations of two surfaces are satisfied. Try to parametrize a cylinder with two variables (with cos and sin) and solve for third variable using other equation.

(d) Derivatives:

$$\vec{r'}(t) = \langle f'(t), g'(t), h'(t) \rangle$$

The tangent line to a space curve at point P is defined to be the line through P parallel to the tangent vector $\vec{r'}(t)$.

(e) More facts about space curves (e.g. arc length) will be explored later with line integrals.

4 Partial Derivatives

1. **Functions of Several Variables**

(a) A function f of two variables assigns a real number $f(x, y)$ to each ordered pair x, y from a set D, the domain.

(b) The graph of f is the set of all points in $\mathbb{R}^3$ such that $z = f(x, y)$ and $x, y \in D$.

(c) The **level curves** of f are the curves with equations $f(x, y) = k$ for some constant k. Sketching level curves at various values of k can help visualize the graph of the function.

2. **Limits**

(a) We say that the limit of $f(x, y)$ as (x, y) approaches (a, b) is L, written

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that if $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \epsilon$. In other words, the distance between $f(x, y)$ and L can be made arbitrarily small by making the distance from (x, y) and (a, b) sufficiently small (but not 0).

(b) The **squeeze theorem** can be used to prove a limit exists. If $f(x, y) \leq g(x, y) \leq h(x, y)$ for all (x, y) near (a, b), and

$$\lim_{(x,y)\to(a,b)} f(x, y) = L = \lim_{(x,y)\to(a,b)} h(x, y)$$

then

$$\lim_{(x,y)\to(a,b)} g(x, y) = L$$

This is done by trapping a subset of the function within a finite range, transforming all sides of the inequality to obtain the original function, then taking the limit of the inequality. Recall that $\lim_{x\to 0} \frac{sinx}{x} = 1$.

(c) If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ *does not exist*. Some simple paths to try include $x = a, y = b$, or $y = mx$.

(d) Another method for computing limits is to convert to polar coordinates; take $x = r\cos\theta$ and $y = \sin\theta$, and $\lim_{(x,y)\to(0,0)}$ becomes $\lim_{r\to 0}$. Can then apply L'Hospital's rule.

(e) $f(x, y)$ is **continuous** at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$. All polynomials and rational functions (ratios of polynomials) of two variables are continuous *on their domain*, so limits can be found by direct substitution.

3. **Partial Derivatives**

(a) The **partial derivative** of f with respect to x, denoted f_x or $\partial f/\partial x$, is the derivative of f with all other variables fixed:

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

(b) Implicit differentiation: To compute $\frac{\partial z}{\partial x}$, differentiate implicity with respect to x, treating y as a constant.

(c) The notation f_{xyz} means first differentiate with respect to x, then y, then z. By **Clairaut's theorem**, if f_{xy} and f_{yx} exist and are continuous on the domain of f, then $f_{xy} = f_{yx}$.

(d) Partial differential equations arise in a wide variety of phenomena in physics. For example, Laplace's equation in three-dimensions is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

If $u(x, y, z)$ represents electric or gravitational potential at (x, y, z), then u satisfies Lapace's equation.

4. **Tangent Planes and Linear Approximation**

(a) An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

(b) The tangent plane at point P is the plane that most closely approximates the surface S near P. Solving the plane equation for z yields the linearization at P, denoted $L(x, y)$.

(c) If the partial derivatives f_x and f_y exist and are continuous at (a, b), then f is **differentiable** at (a, b).

(d) For a differentiable function $z = f(x, y)$, the **differential** dz, defined by

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

represents the change in height of the tangent plane at a particular point when x and y change by an amount dx and dy, respectively. dz is an approximation for Δz for small changes.

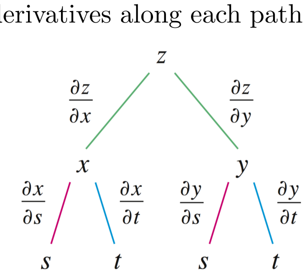
5. **The Chain Rule**

(a) If z is a function of several variables, which are in turn functions of other variables, then z is a composite function that can be differentiated according to the chain rule.

(b) **Chain rule:** Suppose u is a differentiable function of n variables $x_1, \ldots, x_n$ and each x_i is a differentiable function of m variables $t_1, \ldots t_m$. Then u is a function of $t_1, \ldots t_m$ and

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \ldots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$$

(c) It is helpful to draw the tree diagram of a function to visualize the chain rule: draw edges to indicate function-of relationships and write the corresponding partial derivative on each branch. Then to compute $\partial z/\partial s$, for example, sum the product of partial derivatives along each path from z to s.



(d) Second order partial derivative example: consider the function described above in the diagram. Then

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)$$

Computing this will involve applications of the product rule and chain rule.

6. **Directional Derivatives and Gradients**

(a) The partial derivative of f with respect to x yields the rate of change of f in the direction of the x-axis. We now consider a new kind of derivative indicating the rate of change in any direction.

(b) The **directional derivative** of f at (x, y) in the direction of the unit vector $\vec{u} = \langle a, b\rangle$ is

$$D_u f(x, y) = \lim_{h \to 0} \frac{f(x + ha, y + hb) - f(x, y)}{h}$$

Hence f_x and f_y are just special cases of the directional derivative for $\vec{i}$ and $\vec{j}$, respectively.

(c) If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = \langle a, b\rangle$ and

$$D_u f(x, y) = f_x(x, y)a + f_y(x, y)b$$

Note that if $\vec{u}$ makes an angle θ with the postive x-axis, then $\vec{u} = \langle \cos\theta, \sin\theta\rangle$.

(d) If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f:

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$

Note that $D_u f(x, y) = \nabla f(x, y) \cdot \vec{u}$, i.e., the scalar projection of the gradient vector onto $\vec{u}$.

(e) More generally, if f is a differentiable function of n variables, its gradient is the vector whose components are the n partial derivatives of f. The gradient vector points in the direction of steepest ascent.

(f) Let f be a differentiable function of n variables. The maximum value of the directional derivative $D_u f(\vec{x})$ is $\|\nabla f(\vec{x})\|$, which occurs when $\vec{u}$ has the same direction as $\nabla f(\vec{x})$.

(g) If S is a surface with equation $F(x, y, z) = k$, the gradient vector at $P(x_0, y_0, z_0)$, $\nabla F(x_0, y_0, z_0)$, is orthogonal to the tangent vector of any curve on S that passes through P.

(h) **Tangent planes to level surfaces**: Suppose S is a surface with equation $F(x, y, z) = k$. Let $P(x_0, y_0, z_0)$ be a point on S. The tangent plane to the level surface S at P has normal vector $\nabla F(x_0, y_0, z_0)$, and its equation is given by

$$\begin{aligned} F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) \\ + F_z(x_0, y_0, z_0)(z - z_0) = 0 \end{aligned}$$

Note that this is a generalization of tangent planes; the earlier discussion was for the special case where S is of the form $z = f(x, y)$.

(i) To find the angle of intersection of a vector function and surface: find point(s) of intersection, then calculate angle θ between tangent vector to curve and normal vector of tangent plane to surface. The angle of intersection is $90° - \theta$.

7. **Maximum and Minimum Values**

(a) A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in some disk with center (a, b). If the inequality holds for all (x, y) in the domain of f, then f has an **absolute maximum** at (a, b). Local/absolute minimums are defined similarly.

(b) If f has a local minimum or maximum at (a, b), and the first-order partial derivatives of f exist there, then $\nabla f(a, b) = \vec{0}$.

(c) (a, b) is called a **critical point** of f if $\nabla f(a, b) = \vec{0}$ or one of the partial derivatives does not exist. Note that a critical point might not be a maximum or minimum value (called **extrema**).

(d) Solving $f_x = 0$ and $f_y = 0$: if f_x and f_y are functions of one variable where f_x has a roots and f_y has b roots, then there are ab critical points to consider. Otherwise, solve for one variable to find the corresponding values of the other variable.

(e) Second derivative test: The **Hessian** matrix describes the local curvature of a function of many variables. If f is a function of n variables, and all second partial derivatives of f exist and are continuous over the domain, then the Hessian is

$$\begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

For functions of two variables, the determinant of the Hessian can be used to obtain information. Suppose (a, b) is a critical point of f and the second partial derivatives of f are continuous at (a, b). Let D be the determinant of the Hessian, that is,

$$D = f_{xx}(a, b) f_{yy}(a, b) - f_{xy}(a, b)^2$$

Then

- If $D > 0$ and $f_{xx} > 0$, then $f(a, b)$ is a local minimum.
- If $D > 0$ and $f_{xx} < 0$, then $f(a, b)$ is a local maximum.
- If $D < 0$, then $f(a, b)$ is a *saddle* point of f.

Note that if $D = 0$, the test is inconclusive.

(f) Optimization problems can be approached by writing an expression for the quantity we seek to minimize/maximize, using constraints to write some variables in terms of others, then finding critical points of the function.

(g) Minimizing distance is equivalent to minimizing the square of distance, which is often simpler. This is because distance is stictly increasing.

(h) If f is continuous on a closed, bounded set D in $\mathbb{R}^2$, then f attains an absolute maximum and absolute minimum value at some points in D. To find such values, consider the values of f at the critical points in D *and* the extreme values of f on the boundary of D.

(i) To find extreme values on a boundary, parametrize f according to the equation of the boundary and find the critical points. If the boundary is a rectangle, for example, consider all four boundaries *and* the corners. For more complicated boundaries, use the method of lagrange multipliers with the boundary equation as the constraint.

8. **Lagrange Multipliers**

(a) The method of Lagrange multipliers is a strategy for finding the local maxima and minima of a function subject to constraints.

(b) The intuition is as follows: the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ must occur where g tangentially touches a level curve of f, i.e., their gradient vectors are parallel. So if f has an extreme value at $P(x_0, y_0, z_0)$, and $\nabla g(x_0, y_0, z_0) \neq \vec{0}$, then there is a scalar λ such that $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$. λ is called a **Lagrange multiplier.**

(c) Summary of the method: To find the extrema of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ (assuming the values exist and $\nabla g \neq \vec{0}$):

i. Find all values of x, y, z and λ such that

$$\begin{aligned} \nabla f(x, y, z) &= \lambda \nabla g(x, y, z) \\ g(x, y, z) &= k \end{aligned}$$

ii. Evaluate f at all points that result. The largest of these is the maximum value of f and the smallest of these is the minimum value.

In general, for a function of n variables, the first step amounts to solving a system of $n+1$ equations in $n+1$ unknowns. Pay attention not to accidentally divide by zero; some ingenuity is required. Try breaking the system into cases where $\lambda = 0, \lambda \neq 0$, exactly one of x, y, z is 0, exactly two of x, y, z are 0, etc.

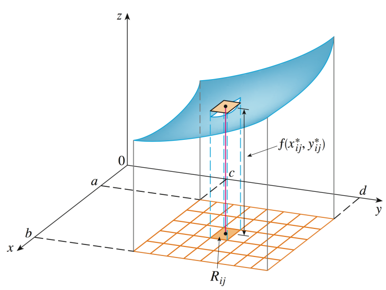
(d) Multiple constraints: to find the extrema of $f(x,y,z)$ subject to $g(x,y,z) = k$ and $h(x,y,z) = c$, modify step (i) to find all values of x, y, z, λ and μ such that

$$\begin{aligned} \nabla f(x,y,z) &= \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) &= k \\ h(x,y,z) &= c \end{aligned}$$

5 Multiple Integrals

1. **Double Integrals over Rectangles**

(a) Let R be a closed rectangle: $R = [a,b] \times [c,d]$. Divide the interval $[a,b]$ into m sub-intervals of width $\Delta x = (b-a)/m$ and $[c,d]$ into n sub-intervals of width $\Delta y = (d-c)/n$, so that we have mn sub-rectangles R_{ij} each with area $\Delta A = \Delta x \Delta y$. Choose a sample point (x_{ij}^*, y_{ij}^*) in each R_{ij}.



The **double integral** of f over the rectangle R is defined as a double Riemann sum:

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$$

If this limit exists, then f is *integrable*. Note that all continuous functions are integrable.

(b) It is of course impractical to compute double integrals from the definition, so we decompose double integrals into **iterated integrals**. Suppose $f(x,y)$ is integrable on $R = [a,b] \times [c,d]$. Then the notation

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

means perform partial integration with respect to y from $y = c$ to $y = d$, then integrate the result with respect to x from $x = a$ to $x = b$.

(c) Fubini's theorem: If f is continuous on the closed rectangle $R = [a,b] \times [c,d]$, then

$$\begin{aligned} \iint_R f(x,y)\,dA &= \int_a^b \int_c^d f(x,y)\,dy\,dx \\ &= \int_c^d \int_a^b f(x,y)\,dx\,dy \end{aligned}$$

Hence it is wise to choose the order of integration that results in the simplest integrands.

(d) If $f(x,y)$ can be factored into a function of x and a function of y only, that is, $f(x,y) = g(x)h(y)$, then the double integral can be written as the product of single integrals:

$$\iint_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$$

This important result holds for triple integrals too.

(e) The **average value** of $f(x,y)$ defined on a rectangle R is

$$f_{avg} = \frac{1}{A_R} \iint_R f(x,y)\,dA$$

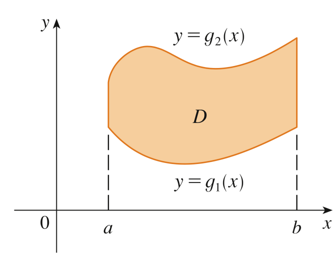
where A_R is the area of R.

2. **Double Integrals over General Regions**

(a) Now we consider the case when we wish to integrate not just over rectangles but regions D of a more general shape.

(b) A plane region D is said to be **type 1** if it lies between the curves of two continuous functions of x, that is,

$$D = \{(x,y) \mid a \le x \le b, g_1(x) \le y \le g_2(x)\}$$

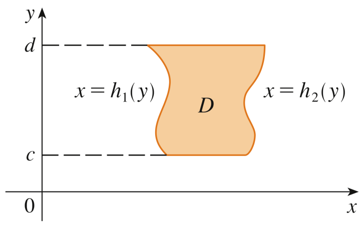


If f is continuous on a type 1 region D, then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(c) A plane region D is said to be **type 2** if it lies between the curves of two continuous functions of y, that is,

$$D = \{(x,y) \mid c \le y \le d, h_1(y) \le x \le h_2(y)\}$$



If f is continuous on a type 2 region D, then

$$\iint_D f(x,y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

(d) A region can both type 1 and type 2, so choose the simpler description. A region can also be neither type 1 nor type 2; in such a case, try to decompose the region into a union of type 1 and type 2 regions.

(e) Changing the order of integration now requires extracting the region D and providing an alternate description of it in terms of the other variable.

(f) Properties of Double Integrals:

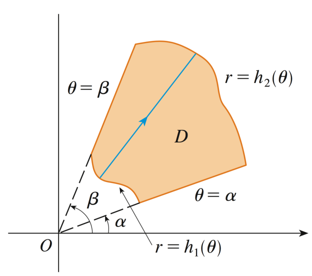
- $\iint_D [f(x,y) + g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$
- $\iint_D cf(x,y)\,dA = c\iint_D f(x,y)\,dA$
- If $\alpha \le f(x,y) \le \beta$ for all $(x,y) \in D$, then $\alpha A_D \le \iint_D f(x,y)\,dA \le \beta A_D$

(g) Calculating the double integral of the constant 1 over a region D yields the area of D.

3. **Double Integrals in Polar Coordinates**

(a) Sometimes a region can be more easily described using polar coordinates. Consider a general polar region D:

$$D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$$



If f is continuous on a polar region D, then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r\,dr\,d\theta$$

(b) Convert a double integral in Cartesian coordinates into a double integral in Polar coordinates by taking $x = r\cos\theta$, $y = r\sin\theta$ and determining the appropriate limits of integration.

4. **Applications of Double Integrals**

(a) **Density and Mass**
A **lamina** is a two-dimensional planar closed surface with mass and density. Let $\mathcal{L}$ be a lamina of uniform density bounded by the curves $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$, where $f(x) \ge g(x)$ and $b \ge a$. The **centroid**, or center of mass, of $\mathcal{L}$ is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx$$
$$\bar{y} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)^2 - g(x)^2]\,dx$$

and A is the area of the lamina, i.e. $A = \int_a^b [f(x) - g(x)]\,dx$.

(b) Now we consider a lamina of variable density. Suppose a lamina occupies a region D in the plane and its density (mass per unit area) at a point (x,y) is given by $\rho(x,y)$, where ρ is a continuous function on D. Then the **total mass m of the lamina** is

$$m = \iint_D \rho(x,y)\,dA$$

The **center of mass** of the lamina is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\rho(x,y)\,dA$$
$$\bar{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\rho(x,y)\,dA$$

The quantities M_x and M_y are called the moment about the x-axis and moment about the y-axis, respectively.

(c) **Continuous Probability**
Recall the **probability density function** (PDF) of a continuous random variable X is a real-valued function $f(x)$ such that $f(x) \ge 0$ and $\int_{\mathbb{R}} f(x)\,dx = 1$. We define the probability that X lies in some interval $[a,b]$ as

$$P(X \in [a,b]) = \int_a^b f(x)\,dx$$

(d) Also recall a random variable X is *exponentially distributed* if its PDF is of the form

$$f(x) = \lambda e^{-\lambda x}$$

so that $E(X) = \frac{1}{\lambda}$ and $\mathrm{Var}(X) = \frac{1}{\lambda^2}$. A random variable X is *normally distributed* if its PDF is of the form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

so that $E(X) = \mu$ and $\mathrm{Var}(X) = \sigma^2$.

(e) Now we consider two continuous random variables X and Y. The **joint density function** of X and Y is a function $f(x,y)$ such that $f(x,y) \ge 0$ and

$\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. We define the probability that X and Y lie in a region D as

$$P((X,Y) \in D) = \iint_D f(x,y)\,dA$$

If X and Y are random variables with PDFs f_X and f_Y, respectively, then X and Y are **independent** iff their joint density function is a product of their individual PDFs:

$$f(x,y) = f_X(x) f_Y(y)$$

(f) If X and Y are random variables with joint density function f, define the expected values of X and Y to be

$$E(X) = \iint_{\mathbb{R}^2} x f(x,y)\,dA$$
$$E(Y) = \iint_{\mathbb{R}^2} y f(x,y)\,dA$$

5. **Triple Integrals**

(a) Analogous to double integrals, we define triple integrals for functions of three variables as a triple Riemann sum over a box.

(b) If $f(x,y,z)$ is continuous on the box $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

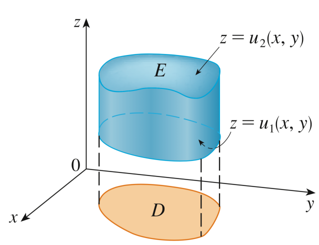
By Fubini's theorem, there are six possible orderings that yield the same result.

(c) As before, now we consider triple integrals over a general bounded region E, which is now a solid.

(d) A solid region E is said to be **type 1** if it lies between the graphs of two continuous functions of x and y, that is,

$$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$$

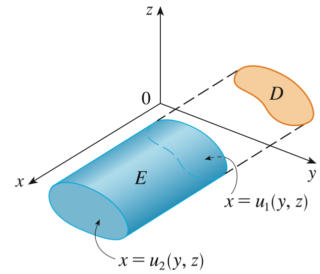
where D is the projection of E onto the xy-plane.



In this case, we have

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right] dA$$

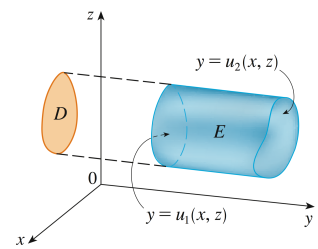
(e) Similarly, a **type 2** solid region E is of the form



We have

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right] dA$$

(f) Lastly, a **type 3** solid region E is of the form



We have

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right] dA$$

(g) Each of these three cases involves a double integral. In order to evaluate the outer double integral, we must classify the projection D as a type 1 plane region or type 2 plane region, and then proceed as before. It is wise to draw a diagram of the solid E as well as its projection D in computing triple integrals.

(h) The corresponding interpretation of a triple integral is the hypervolume of a four-dimensional object, which is not very useful. However, calculating the triple integral of the constant 1 over a region E yields the volume of E.

(i) Suppose a solid object occupies a region E and its density (mass per unit volume) at a point (x,y,z) is given by $\rho(x,y,z)$. The total mass m of the solid is

$$m = \iiint_E \rho(x,y,z)\,dV$$

Its moments about the three coordinate planes are

$$M_{yz} = \iiint_E x\rho(x,y,z)\,dV$$
$$M_{xz} = \iiint_E y\rho(x,y,z)\,dV$$
$$M_{xy} = \iiint_E z\rho(x,y,z)\,dV$$

The center of mass is located at $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{M_{yz}}{m} \qquad \bar{y} = \frac{M_{xz}}{m} \qquad \bar{z} = \frac{M_{xy}}{m}$$

If the density is constant, the center of mass is called the centroid of E.

6. **Triple Integrals in Cylindrical Coordinates**

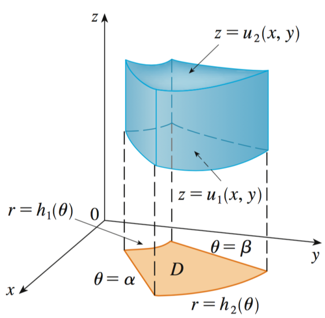
   (a) In the **cylindrical coordinate system**, a point P in three-dimensional space is represented by the ordered triple (r, θ, z) where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.

   (b) The relationship between rectangular and cylindrical coordinates is given by

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{z} \qquad z = z$$

   (c) Suppose E is a type 1 region whose projection D onto the xy-plane is conveniently described in polar coordinates.



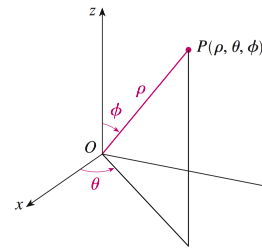
Combining results from earlier, we have

$$\iiint_E f(x, y, z)\, dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta, r\sin\theta)}^{u_2(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z) r\, dz\, dr\, d\theta$$

   (d) It is worthwhile to convert a triple integral from rectangular to cylindrical coordinates if E is a solid region easily described in cylindrical coordinates (e.g., symmetrical about z-axis), and especially when $f(x, y, z)$ involves the expression $x^2 + y^2$.

7. **Triple Integrals in Spherical Coordinates**

   (a) In the **spherical coordinate system**, a point P in three-dimensional space is represented by (ρ, θ, ϕ), where ρ is the distance from P to the origin O, θ is the same as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and line segment OP. Note that $\rho \geq 0$ and $0 \leq \phi \leq \pi$.



   (b) The relationship between rectangular and spherical coordinates is given by

$$x = \rho\sin\phi\cos\theta \qquad y = \rho\sin\phi\sin\theta \qquad z = \rho\cos\phi$$

$$\rho^2 = x^2 + y^2 + z^2$$

   (c) We have

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi$$

   where E is a spherical wedge given by

$$E = \{(\rho, \theta, \phi) \mid a \leq \rho \leq b, \alpha \leq \theta \leq \beta, c \leq \phi \leq d\}$$

8. **Change of Variables in Multiple Integrals**

   (a) In single-variable calculus we often used a change of variable (or u-substitution) to simplify an integral.

   (b) Now we consider more generally a change of variables given by a transformation T from the uv-plane to the xy-plane, where x and y are related to u and v by

$$x = g(u, v) \qquad y = h(u, v)$$

   The goal is generally to transform a region R in xy-coordinates into a region in uv-coordinates that is easier to describe.

   (c) The **Jacobian** of the transformation T given by $x = g(u, v)$ and $y = h(u, v)$ is the determinant

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

   (d) *Change of variables in a double integral*: Let T be a one-to-one transformation whose Jacobian is nonzero that maps a region S in the uv-plane to a region R in the xy-plane. Let f be a continuous function on R, and that R and S are type 1 or type 2 plane regions. Then

$$\iint_R f(x, y)\, dA = \iint_S f(g, h) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

(e) In other words, we convert an integral in x and y into an integral in u and v by expressing x and y in terms of u and v and writing $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$. Observe that in the single-variable case, the Jacobian is $\frac{dx}{du}$, which yields the familiar formula for u-substitution.

(f) When the region of integration R is too difficult to describe, construct a transformation T from the xy-plane to a region S in the uv-plane such that the images of the boundaries of R under T are simpler. Use S to determine the new limits of integration and complete the change of variable to evaluate the integral.

(g) The formula for double integration in polar coordinates is just a special case of the above, which results from using the transformation T from the $r\theta$-plane to the xy-plane given by $x = r\cos\theta$ and $y = r\sin\theta$.

(h) Now we consider triple integrals. Let T be a transformation that maps a region S in uvw-space onto a region R in xyz-space by means of

$$x = g(u,v,w) \qquad y = h(u,v,w) \qquad z = k(u,v,w)$$

The Jacobian of T is the 3×3 determinant

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

(i) *Change of variables in a triple integral*: Using similar hypotheses to those for double integrals, we have

$$\iiint_R f(x,y,z)\,dV = \iiint_S f(g,h,k)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

where g, h, k are the functions of u, v, w defined earlier.

(j) The formulas for triple integration in cylindrical and spherical coordinates are just special cases of the above, using the appropriate relations.

6 Vector Calculus

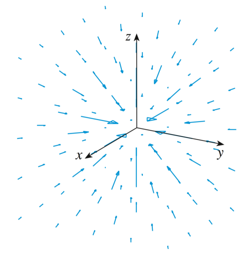
1. **Vector Fields**

(a) Let D be a subset of $\mathbb{R}^2$ and E be a subset of $\mathbb{R}^3$. A **vector field** on $\mathbb{R}^2$ is a function $\vec{F}$ that assigns to each point $(x,y) \in D$ a vector $\vec{F}(x,y)$. Similarly, a vector field on $\mathbb{R}^3$ is a function $\vec{F}$ that assigns to each point $(x,y,z) \in E$ a vector $\vec{F}(x,y,z)$.

(b) $\vec{F}$ can be written in terms of its component functions P, Q (or P, Q, R in $\mathbb{R}^3$) as follows:

$$\vec{F}(x,y) = P(x,y)\vec{i} + Q(x,y)\vec{j} = \langle P(x,y), Q(x,y)\rangle$$

(c) A vector field on D is usually visualized by drawing the vector $\vec{F}(x,y)$ starting at the point (x,y) for a few representable points in D.



(d) A common vector field in physics is a **velocity field** $\vec{V}$, which assigns a velocity vector $\vec{V}(x,y,z)$ to each point (x,y,z) in a certain domain.

(e) Suppose an object of mass M is located at the origin and an object on mass m is located at $\vec{x} = \langle x,y,z\rangle$. By Newton's Law of Gravitation, the gravitational force acting on the object at $\vec{x}$ is

$$\vec{F}(\vec{x}) = -\frac{mMG}{\|\vec{x}\|^3}\vec{x}$$

$\vec{F}$ is called the **gravitational field**.

(f) Suppose an electric charge Q is located at the origin. By Coulomb's Law, the electric force per unit charge $\vec{E}(\vec{x})$ exerted on a charge q located at $\vec{x} = \langle x,y,z\rangle$ is

$$\vec{E}(\vec{x}) = \frac{\epsilon Q}{\|\vec{x}\|^3}\vec{x}$$

$\vec{E}$ is called the **electric field** of Q.

(g) The gradient ∇f of a scalar function f of n variables is a vector field on $\mathbb{R}^n$, and is called a **gradient vector field**.

(h) A vector field $\vec{F}$ is called a **conservative vector field** if it the gradient of some scalar function, that is, there exists a function f such that $\vec{F} = \nabla f$. In this case f is called a **potential function** for $\vec{F}$.

(i) For example, a potential function for the gravitational field is

$$f(x,y,z) = \frac{mMG}{\sqrt{x^2+y^2+z^2}}$$

2. **Line Integrals**

(a) We now define an integral similar to a single integral, but over a curve instead of an interval.

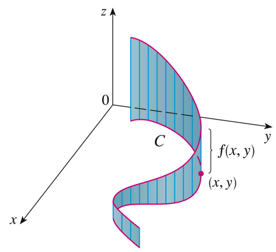
(b) **Line Integrals for plane curves**:
Suppose C is a smooth curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

or in vector form, $\vec{r}(t) = \langle x(t), y(t)\rangle$. If f is a function of two variables defined on C, the **line**

integral of f along C with respect to arc length is

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$



(c) The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

(d) Just as with single integrals, we can interpret the line integral of a non-negative function as an area.

(e) If $\rho(x,y)$ represents the linear density at a point (x,y) of a wire shaped like a curve C, the mass m of the wire is

$$m = \int_C \rho(x,y)\,ds$$

The center of mass of the wire is $(\bar{x},\bar{y})$ where

$$\bar{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds \qquad \bar{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds$$

(f) Two other line integrals are obtained by replacing ds with dx and dy:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\;x'(t)\,dt$$
$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\;y'(t)\,dt$$

They are called the line integrals of f along C with respect to x and y, respectively.

(g) If the curve C is a line segment, it is useful to remember the vector representation of a line segment starting at $\vec{r}_0$ and ending at $\vec{r}_1$ is

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \qquad 0 \le t \le 1$$

(h) Note that a given parametrization determines an **orientation** of a curve C, with positive direction corresponding to increasing values of t. Reversing the orientation has the effect of negating the integral when integrating with respect to x or y (but does not affect the integral when integrating with respect to arc length).

(i) **Line Integrals for space curves:**
Now suppose C is a smooth space curve given by the parametric equations

$$x = x(t) \qquad y = y(t) \qquad z = z(t) \qquad a \le t \le b$$

If f is a function of three variables continuous on some region containing C, then

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\,ds$$

where

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

(j) Line integrals along C with respect to x, y or z can also be defined in a similar manner to line integrals in the plane.

(k) In the case $f(x,y,z) = 1$,

$$\int_C ds = L$$

where L is the length of the curve C.

(l) **Line Integrals for vector fields:**
Let $\vec{F}$ be a continuous vector field defined on a smooth curve C given by a vector function $\vec{r}(t), a \le t \le b$. The line integral of $\vec{F}$ along C is

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}'(t)\,dt$$

(m) Suppose the vector field $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{j}$. We have the following connection between line integrals of vector fields and line integrals of scalar fields:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz$$

(n) If $\vec{F}$ is a continuous force field on $\mathbb{R}^3$, such as the gravitational or electric field, we can interpret this integral as the work done by this force in moving a particle along a smooth curve C. The work done is negative if the field impedes movement along the curve, and positive otherwise.

3. **Fundamental Theorem for Line Integrals**

(a) The fundamental theorem of calculus provides a method of evaluating definite integrals by way of a function's anti-derivative. The following theorem can be regarded as a more general version of it for line integrals.

(b) Let C be a smooth curve given by $\vec{r}(t), a \le t \le b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C. Then

$$\int_C \nabla f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

(c) We say the line integral $\int_C \vec{F} \cdot d\vec{r}$ is **independent of path** if $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for any two paths C_1 and C_2 that share the same initial and terminal points.

(d) The theorem implies that line integrals of conservative vector fields are independent of path. Thus we can evaluate the line integral of a conservative vector field simply by knowing the value of a potential function f at the endpoints of C.

(e) A curve C is **closed** if its terminal point coincides with its initial point. A **simple curve** is a curve that does not intersect itself between endpoints. A **simply-connected** region D is a connected region D such that every simple closed curve in D encloses only points that are in D (intuitively, this means D is not disjoint and does not contain holes).

(f) $\int_C \vec{F} \cdot d\vec{r}$ is independent of path iff $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path C in the domain of $\vec{F}$.

(g) In fact, the *only* vector fields that are independent of path are conservative. Suppose $\vec{F}$ is a vector field that is continuous on an open (i.e., does not include boundaries) connected region D. If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D, then $\vec{F}$ is a conservative vector field on D.

(h) Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field on an open simply-connected region D. If P and Q have continuous first-order partial derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

then $\vec{F}$ is conservative (this follows from Clairaut's theorem).

(i) Suppose we know $\vec{F} = P\vec{i} + Q\vec{j}$ is conservative. To obtain a potential function f, we can perform partial integration:

$$f(x, y) = \int P(x, y)\, dx \quad \text{or} \quad f(x, y) = \int Q(x, y)\, dy$$

Choose whichever is easier to evaluate. Note that the "constant of integration" will actually be a function of the opposite variable, $h(y)$ or $h(x)$. Differentiating f with respect to the appropriate variable and comparing the result to P or Q allows us to solve for $h'(y)$ (or $h'(x)$); then partially integrate to obtain h.

(j) The method for finding a potential function is much the same for a vector field in $\mathbb{R}^3$. Integrate with respect to x, y or z to obtain f; the constant of integration is a function of the other two variables, say $g(y, z)$. Compute f_y and compare with Q to solve for $g_y(y, z)$, then partially integrate with respect to y to obtain $g(y, z)$ (which will involve a constant term $h(z)$). Lastly compute f_z to solve for $h(z)$, as before.

(k) Conservative vector fields appear naturally in physics as they represent forces of physical systems in which energy is conserved (i.e., the sum of kinetic and potential energy is constant).

4. **Green's Theorem**

(a) Green's theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C.

(b) A simple closed curve C has a **positive orientation** if it is traced out in a counter-clockwise direction. In other words, if C is given by vector function $\vec{r}(t)$, then D is always on the left when we traverse C as t increases.

(c) Let $\vec{F} = P\vec{i} + Q\vec{j}$ be a vector field. Let C be a positively oriented, piecewise smooth, simple, closed curve and let D be the region bounded by C. If P and Q have continuous first order partial derivatives on an open region containing D, then

$$\oint_C P\, dx + Q\, dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

(d) $\oint_C$ denotes a line integral over a curve C that satisfies the condition of Green's Theorem.

(e) Green's Theorem can be used to simplify a line integral over a curve with a lengthy parametrization (e.g., a rectangle).

(f) An application of Green's Theorem using the reverse direction is calculating areas. Since the area A of a plane region D is $\iint_D 1\, dA$, choose P and Q such that $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$. Some possibilities for (P, Q) include $(0, x)$, $(-y, 0)$, $(-\frac{1}{2}y, \frac{1}{2}x)$. Thus

$$A = \oint_C x\, dy = -\oint y\, dx = \frac{1}{2} \oint_C x\, dy - y\, dx$$

5. **Curl and Divergence**

(a) Now we define two operations on vector fields with important applications in physics.

(b) Let $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ be a vector field on $\mathbb{R}^3$. The **curl** of $\vec{F}$ is the vector field on $\mathbb{R}^3$ is given by

$$\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \vec{k}$$

(c) Define the vector differential *operator* ∇ ("del") as

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$$

It operates on a scalar function f to produce the gradient of f. Thus we can write

$$\text{curl } \vec{F} = \nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

(d) If $\vec{F}$ is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\vec{F} = \vec{0}$, then $\vec{F}$ is a conservative vector field.

(e) If $\vec{F}$ represents the velocity field in fluid flow, then particles near (x, y, z) tend to rotate about the axis that points in the direction of curl $\vec{F}(x, y, z)$. The magnitude of this vector is a measure of how quickly the particles move about the axis. If curl $\vec{F} = \vec{0}$ at a point P, $\vec{F}$ is called irrotational at P.

(f) Let $\vec{F} = P\vec{i}+Q\vec{j}+R\vec{k}$ be a vector field on $\mathbb{R}^3$. The **divergence** of $\vec{F}$ is the function of three variables defined by

$$\text{div } \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

In operator notation, we can write

$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

(g) We have the following relationship between curl and the divergence:

$$\text{div curl } \vec{F} = 0$$

(h) If $\vec{F}$ represents the velocity field in fluid flow, then div $\vec{F}(x, y, z)$ is the net rate of change of the mass of the fluid flowing from the point (x, y, z) per unit volume. If div $\vec{F} = 0$, then $\vec{F}$ is said to be incompressible. If div $\vec{F}(P) > 0$, the net flow is outward near P and P is called a source. If div $\vec{F}(P) < 0$, the net flow is inward near P and P is called a sink.

(i) Another differential operator arises in computing the divergence of a gradient vector field ∇f. We have

$$\text{div } \nabla f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

The operator $\nabla^2 = \nabla \cdot \nabla$ is called the **Laplace operator**. Observe that Laplace's equation can be written $\nabla^2 f = 0$.

(j) We can also express Green's theorem in terms of curl and divergence. We have

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_D (\text{curl } \vec{F}) \cdot \vec{k}\, dA$$

where $\vec{k}$ is the standard unit vector in the z direction.

(k) If the curve C is given by the vector equation $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$, $a \le t \le b$, the outward unit normal vector to C is given by

$$\vec{n}(t) = \frac{y'(t)}{\|\vec{r'}(t)\|}\vec{i} - \frac{x'(t)}{\|\vec{r'}(t)\|}\vec{j}$$

We have

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_D \text{div } \vec{F}(x, y)\, dA$$

6. **Parametric Surfaces**

(a) In much the same way we describe a space curve by a vector function $\vec{r}(t)$ of t, we can describe a surface by a vector function $\vec{r}(u, v)$ of u and v.

(b) Suppose $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ is a vector-valued function on a region D in the uv-plane. The set of all points (x, y, z) such that $\vec{r}(u, v) = \langle x, y, z\rangle$ as (u, v) varies throughout D is called a **parametric surface**.

(c) Setting u or v constant in the vector function $\vec{r}(u, v)$ correspond to vertical and horizontal lines in the uv-plane and are called **grid curves**.

(d) The ellipsoid $ax^2 + by^2 + cz^2 = d^2$ can be parametrized as

$$\vec{r}(\phi, \theta) = \left\langle \frac{d}{\sqrt{a}} \sin\phi\cos\theta, \frac{d}{\sqrt{b}} \sin\phi\sin\theta, \frac{d}{\sqrt{c}} \cos\phi \right\rangle$$

for $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$.

(e) A surface given as the graph of a function $z = f(x, y)$ can always be regarded as a parametric surface by taking x and y as parameters and writing the parametric equation for z as $z = f(x, y)$.

(f) The surface obtained by rotating the curve $y = f(x)$, $a \le x \le b$, $f(x) \ge 0$ about the x-axis can be parametrized by $x = x$, $y = f(x)\cos\theta$, $z = f(x)\sin\theta$ for $a \le x \le b$, $0 \le \theta \le 2\pi$.

(g) Let S be the surface traced out by vector function $\vec{r}(u, v)$. Let $\vec{r}_u$ and $\vec{r}_v$ denote the partial derivatives of $\vec{r}(u, v)$ with respect to u and v, respectively (which are vector-valued functions). If $\vec{r}_u \times \vec{r}_v \ne \vec{0}$ (i.e., S is smooth), $\vec{r}_u \times \vec{r}_v$ is a normal vector to the tangent plane to S.

(h) If a smooth parametric surface S is given by vector equation $\vec{r}(u, v)$ for $(u, v) \in D$ and S is covered just once as (u, v) ranges throughout the parameter domain D, then the **surface area** of S is

$$SA = \iint_D \|\vec{r}_u \times \vec{r}_v\|\, dA$$

(i) In the special case of a surface S with equation $z = f(x, y)$ for $(x, y) \in D$, the surface area formula becomes

$$SA = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

7. **Surface Integrals**

(a) Surface integrals are the double integral analog of the line integral; we integrate a function of three variables along a surface.

(b) If a smooth parametric surface S is given by vector equation $\vec{r}(u,v)$ for $(u,v) \in D$ and S is covered just once as (u,v) ranges throughout the parameter domain D, the **surface integral** of f over the surface S is

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\|\vec{r}_u \times \vec{r}_v\|\,dA$$

(c) Observe that in the case $f(x,y,z) = 1$, $\iint_D dS$ is the surface area of S.

(d) If a thin sheet with density function $\rho(x,y,z)$ has the shape of a surface S, the mass of the sheet is

$$m = \iint_S \rho(x,y,z)\,dS$$

The center of mass is $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{1}{m}\iint_S x\rho(x,y,z)\,dS$$

with $\bar{y}$ and $\bar{z}$ defined similarly.

(e) In the case S is a surface with equation $z = g(x,y)$, this formula becomes

$$\iint_S f(x,y,z)\,dS$$
$$= \iint_D f(x,y,g(x,y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

Similar formulas apply if it is more convenient to express S as a function of y and z or x and z.

(f) **Surface Integrals for vector fields**
An **orientable** surface S is two-sided (distinction is needed as a Mobius strip is one-sided). If it is possible to choose a unit normal vector $\vec{n}$ at every point (x,y,z) so that $\vec{n}$ varies continuously over S, then S is called an **oriented surface**. The choice of $\vec{n}$ provides S with an **orientation**; there are two possible orientations from using $\vec{n}_1$ or $\vec{n}_2 = -\vec{n}_1$.

(g) For a closed surface, that is, a surface that is the boundary of a solid region E, the **positive orientation** is the one for which the normal vectors point outward from E, and inward-pointing normals give the **negative orientation**.

(h) If $\vec{F}$ is a continuous vector field defined on an oriented surface S with unit normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over S is

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{n}\,dS$$

This integral is also called the **flux** of $\vec{F}$ across S.

(i) If S is given by vector function $\vec{r}(u,v)$, we have

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA$$

In the case S is given by the graph $z = g(x,y)$, then

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\,dA$$

which assumes the upward orientation of S; for the downward orientation multiply the integrand by -1. Similar formulas apply if S is given by $y = h(x,z)$ or $x = k(y,z)$.

(j) If $\vec{F}$ is a velocity field, the surface integral can be interpreted as the flow rate of fluid through S. Over an electric field, the surface integral is called the electric flux. By Gauss's Law, the net charge enclosed by a closed surface S in an electric field $\vec{E}$ is

$$Q = \epsilon_0 \iint_S \vec{E}\cdot d\vec{S}$$

where ϵ_0 is the permittivity of free space.

8. **Stokes' Theorem**

(a) Stokes' Theorem is a higher-dimensional version of Green's Theorem. It relates a surface integral over a surface S to a line integral around the boundary curve of S.

(b) Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous partial deriviatives on an open region in $\mathbb{R}^3$ that contains S. Then

$$\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl } \vec{F}\cdot d\vec{S}$$

Note that the positively oriented boundary curve of S is often written as ∂S instead of C.

(c) In the special case where S is flat and lies in the xy-plane with upward orientation, the unit normal vector is $\vec{k}$, the surface integral becomes a double integral, and Stokes' Theorem becomes the curl vector version of Green's Theorem given earlier.

9. **The Divergence Theorem**

(a) Let E be a simple solid region (i.e., E is simultaneously type 1, 2, and 3) and let S be the boundary surface of E with positive outward orientation. Let $\vec{F}$ be a vector field whose component functions have continuous partial derivatives. Then

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \text{div } \vec{F}\,dV$$

So the flux of $\vec{F}$ across the boundary surface of E is equal to the triple integral of the divergence of $\vec{F}$ over E.

(b) In practice, this theorem is useful for converting a difficult surface integral into a simpler triple integral.

7 Appendix

1. **Trigonometry**

   (a)

$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \tan\theta = \frac{\text{opp}}{\text{adj}}$$

   (b)

Angle	Sin	Cos	Tan	Cot	Sec	Csc
0	0	1	0	undef	1	undef
$\pi/6$	$1/2$	$\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$2/\sqrt{3}$	2
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\pi/3$	$\sqrt{3}/2$	$1/2$	$\sqrt{3}$	$1/\sqrt{3}$	2	$2/\sqrt{3}$
$\pi/2$	1	0	undef	0	undef	1
$2\pi/3$	$\sqrt{3}/2$	$-1/2$	$-\sqrt{3}$	$-1/\sqrt{3}$	-2	$2/\sqrt{3}$
$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
$5\pi/6$	$1/2$	$-\sqrt{3}/2$	$-1/\sqrt{3}$	$-\sqrt{3}$	$-2/\sqrt{3}$	2
π	0	-1	0	undef	-1	undef
$7\pi/6$	$-1/2$	$-\sqrt{3}/2$	$1/\sqrt{3}$	$\sqrt{3}$	$-2/\sqrt{3}$	-2
$5\pi/4$	$-\sqrt{2}/2$	$-\sqrt{2}/2$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
$4\pi/3$	$-\sqrt{3}/2$	$-1/2$	$\sqrt{3}$	$1/\sqrt{3}$	-2	$-2/\sqrt{3}$
$3\pi/2$	-1	0	undef	0	undef	-1
$5\pi/3$	$-\sqrt{3}/2$	$1/2$	$-\sqrt{3}$	$-1/\sqrt{3}$	2	$-2/\sqrt{3}$
$7\pi/4$	$-\sqrt{2}/2$	$\sqrt{2}/2$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
$11\pi/6$	$-1/2$	$\sqrt{3}/2$	$-1/\sqrt{3}$	$-\sqrt{3}$	$2/\sqrt{3}$	-2

   (c)

$$\sin^2\theta + \cos^2\theta = 1 \qquad \sin 2\theta = 2\sin\theta\cos\theta$$

$$1 + \tan^2\theta = \sec^2\theta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta \qquad = 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

   (d)

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \sin -\theta = -\sin\theta$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \cos -\theta = \cos\theta$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta} \qquad \tan -\theta = -\tan\theta$$

   (e)

$$\sin\alpha+\beta = \sin\alpha\cos\beta + \sin\beta\cos\alpha$$

$$\cos\alpha+\beta = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

2. **Derivatives**

   (a) Product Rule: $(f(x)g(x))' = f'(x)g(x)+f(x)g'(x)$

   (b) Quotient Rule: $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2}$

   (c) Chain Rule: $(f(g(x)))' = f'(g(x))g'(x)$

   (d) A function is concave up (convex) if $f''(x) > 0$ and concave down if $f''(x) < 0$.

   (e) Implicit Differentiation: y is defined *implicitly* as a function of x. Differentiate both sides of the equation with respect to x; y becomes $\frac{dy}{dx}$.

   (f) Common Derivatives:

$$\frac{d}{dx}(x) = 1 \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x \qquad \frac{d}{dx}(a^x) = a^x \ln(a)$$

$$\frac{d}{dx}(\sin x) = \cos x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x \qquad \frac{d}{dx}(e^x) = e^x$$

$$\frac{d}{dx}(\cos x) = -\sin x \qquad \frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\ln(x)) = \frac{1}{x},\ x>0$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(\ln|x|) = \frac{1}{x},\ x\neq 0$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2} \qquad \frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln a},\ x>0$$

3. **Integrals**
   Common Integrals:

$$\int k\,dx = kx + c \qquad \int \cos u\,du = \sin u + c \qquad \int \tan u\,du = \ln|\sec u| + c$$

$$\int x^n\,dx = \tfrac{1}{n+1}x^{n+1} + c, n \neq -1 \qquad \int \sin u\,du = -\cos u + c \qquad \int \sec u\,du = \ln|\sec u + \tan u| + c$$

$$\int x^{-1}\,dx = \int \tfrac{1}{x}\,dx = \ln|x| + c \qquad \int \sec^2 u\,du = \tan u + c \qquad \int \tfrac{1}{a^2+u^2}\,du = \tfrac{1}{a}\tan^{-1}\left(\tfrac{u}{a}\right) + c$$

$$\int \tfrac{1}{ax+b}\,dx = \tfrac{1}{a}\ln|ax+b| + c \qquad \int \sec u \tan u\,du = \sec u + c \qquad \int \tfrac{1}{\sqrt{a^2-u^2}}\,du = \sin^{-1}\left(\tfrac{u}{a}\right) + c$$

$$\int \ln u\,du = u\ln(u) - u + c \qquad \int \csc u \cot u\,du = -\csc u + c$$

$$\int e^u\,du = e^u + c \qquad \int \csc^2 u\,du = -\cot u + c$$

   (a) Arc length: The arc length of a curve of the form $y = f(x)$, $a \leq x \leq b$ is given by

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx$$

   (b) U-Substitution: To compute $\int f(g(x))g'(x)\,dx$, let $u = g(x)$ and $du = g'(x)\,dx$. Then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

   Compute the anti-derivative and then re-substitute u.

   (c) Integration by Parts:

$$\int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du$$

4. **Miscellaneous**

   (a) L'Hospital's rule: If we have

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0} \text{ or } \frac{\pm\infty}{\pm\infty}$$

   then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

   (b) Completing the square:

$$x^2 + bx = 0 \to x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2$$

$$\to \left(x + \frac{b}{2}\right)^2 = \left(\frac{b}{2}\right)^2$$