AN INTRODUCTION TO THE FINITE VOLUME METHOD USING THE ELASTIC WAVE EQUATION *

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Abstract. An introductory description of finite volume methods for solving partial differential equations is discussed below. Additionally, apadtive mesh refinment is disscussed in some detail.

Key words. Finite Volume, ClawPack, Elastic Wave Equation

1. Introduction.

Numerical methods for solving partial differential equations (PDEs) arise across all scientific fields. From modeling future predictions of global sea-level rise to modeling heat flow through a rod, numerical solution to PDEs at the heart of modelling physical phenomena. The finite difference method, which approximates derivatives using the Taylor expansion, is a common starting point for numerical solutions to PDEs. By using the the Taylor expansion ordinary differential equations or partial PDEs can be reduced to a series of algebraic statements for which matrix methods can be used to solve efficiently.

While the finite difference methods is powerful staring point it has it's limitations, one of which is dealing with discontinues. Due to failure of the finite difference method and other numerical methods to deal with discontinuities, the finite volume method arises. The finite volume method is commonly used to solve hyperbolic partial differential equations, due to their susceptibility to developing shock waves, but can be used to solve all types of PDEs. At the heart of the finite volume method is the *Riemann Problem*, an initial value problem with piecewise constant initial conditions which naturally arises due the discretized domain which the equations are solved on.

We begin with a introduction to the finite volume method as motivated by the one dimensional advection equation. We then derive first and second order finite volume approximations, and discuss higher order methods in the context of `ClawPack`. In order to test these numerical methods we investigate the one dimensional elastic wave equation. First, we derive an analytical solutions for the homogeneous (*i.e.* constant coefficient) equation that we use to validate our numerical methods. We then use the validated numerical methods to investigate the heterogeneous form of the elastic wave equation. This report concludes with conclusions drawn from the numerical experiments and outlines the next steps in order to use the finite volume method to investigate a real scientific question.

2. Conservation Laws.

We being by deriving conservation laws based on based on physical intuition of fluid flow. We consider a fluid flow such as dye tracer in a river or a supraglacial stream with a positive advection speed ($\bar{a}$), that is the fluid is flowing from left to right. Our derivation closely follows that of [1] who in turn closely follow [2]. The total mass of the quantity in question (i.e. concentration of the tracer) in a given unit volume (in the one dimensional case just length) is

$$\int_{x_l}^{x_r} q(x,t)\,dx \tag{2.1}$$

where x_l and x_r are the left and right cell boundaries of unit volume respectively [1]. Under the assumption there is no source term (*e.g.* rainfall), the only changes in time

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will be through flux across the left or right cell cell boundaries. Therefore,

$$\frac{\partial}{\partial t}\int_{x_l}^{x_r} q(x,t)\,dx = F_l(t) - F_r(t) \tag{2.2}$$

where $F_i(t)$ are mass fluxes rates across the cell boundaries [1]. Equation 2.2 represents the integral form of our conservation law [2]. The power of the finite volume method lies in the fact that Equation 2.2 still holds at discontinuities where PDEs is no longer valid [1]. Therefore, the finite volume method is able to accurately approximate solutions at discontinuities (given a high enough order approximation) while other numerical methods like the finite difference method will fail to produce accurate results. If we rewrite the fluxes F_i as functions of $q(x,t)$ such that

$$F \to f(q(x,t)) \tag{2.3}$$

we can write our the change in unit volume with time as

$$\frac{\partial}{\partial t}\int_{x_l}^{x_r} q(x,t)\,dx = f(q(x_l,t)) - f(q(x_r,t)). \tag{2.4}$$

Under the assumption that f and q are sufficiently smooth, the equation above can be rewritten as

$$\frac{\partial}{\partial t}\int_{x_l}^{x_r} q(x,t)\,dx = -\int_{x_l}^{x_r} \frac{\partial}{\partial x} f(q(x,t))\,dx \tag{2.5}$$

using the definition of a definite integral [2]. Further simplification leads to

$$\int_{x_l}^{x_r} \left[\frac{\partial}{\partial t} q(x,t) + \frac{\partial}{\partial x} f(q(x,t))\right]\,dx = 0. \tag{2.6}$$

Since the definite integral of Equation 2.6 evaluated from x_l to x_r is equal to zero, the quantity being integrated must be identically zero [2]. This results in

$$\frac{\partial}{\partial t} q(x,t) - \frac{\partial}{\partial x} f(q(x,t)) = 0 \tag{2.7}$$

known as the differential form of our conservation law [2]. The conservation law above can be written in form the advection equation as

$$\frac{\partial}{\partial t} q(x,t) - \bar{a}\frac{\partial}{\partial x} q(x,t) = 0 \tag{2.8}$$

for a homogeneous material or as

$$\frac{\partial}{\partial t} q(x,t) - \frac{\partial}{\partial x} A(x,t) q(x,t) = 0 \tag{2.9}$$

for a heterogeneous material. In the case of the homogeneous material the advection velocity is constant, while the advection velocity varies in space and time for the heterogeneous material. Both scenarios will be investigated below.

3. Finite Volume. We now derive the finite volume method for one dimensional conservation laws on a numerical grid where the i-th grid cell, $\mathcal{C}_i$ has cell interfaces $x_{i-1/2}, x_{i+1/2}$. We can approximate the value of the solution field $q(x,t)$ by the average quantity Q_i^n within a given grid cell as

$$Q_i^n = \frac{1}{dx}\int_{\mathcal{C}_i} q(x,t)\,dx \approx \frac{1}{\Delta x}\int_{\mathcal{C}_i} q(x,t)\,dx \tag{3.1}$$

were the subscript denotes the grid cell, the superscript denotes the time integration step and Δx is the size of the gird cell [2, 1]. We use the above definition of average quantity along with the integral form of our conservation law (Equation (2.2)) to derive a time intergration algorithm. By integrating Equation (2.4) from t_n to t_{n+1} we then get

$$\int_{x_l}^{x_r} q(x,t_{n+1})\,dx - \int_{x_l}^{x_r} q(x,t_n)\,dx = \int_{t_n}^{t_{n+1}} f(q(x_l,t))\,dt - \int_{t_n}^{t_{n+1}} f(q(x_r,t))\,dt \tag{3.2}$$

In order for our intergrated equation to match the cell avergared form derived above we devide all terms by Δx and then solve for $q(x,t_{n+1})$ ending up with

$$\begin{aligned}\frac{1}{\Delta x}\int_{x_l}^{x_r} q(x,t_{n+1})dx &= \frac{1}{\Delta x}\int_{x_l}^{x_r} q(x,t_n)dx \\ &\quad - \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} f(q(x_r,t))dt - \int_{t_n}^{t_{n+1}} f(q(x_l,t))dt\right].\end{aligned} \tag{3.3}$$

Under the assumption we do not have exact form of $q(x_i,t)$ we generally cannot exactly compute the time integration on the right hand side of the equation above [2]. The equation above does allude to the development of numerical solution of the form

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[F_{i+1/2}^n - F_{i-1/2}^n\right] \tag{3.4}$$

where $F_{i\pm1/2}$ is an approximation of the average flux [2] of the form

$$F_{i\pm1/2} \approx \frac{1}{\Delta t}\int_n^{n+1} f(q(x_{i\pm1/2},t))\,. \tag{3.5}$$

Under the assumption that $F_{i\pm1/2}$ can be approximated by just using the flux at adjacent cells we can state the numerical flux as

$$F_{i+1/2}^n = \mathcal{F}\left(Q_{i+1}, Q_i\right) \tag{3.6}$$

The simplification of just using neighboring cell to approximate the average flux is appropriate for our purposes given that waves propagate at finite speeds in hyperbolic systems [2]. We explicitly demonstrate the elastic wave equation to be hyperbolic in the following section. This formulation of the numerical flux produces a numerical method of the form

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left[\mathcal{F}(Q_{i+1},Q_i) - \mathcal{F}(Q_i,Q_{i-1})\right]. \tag{3.7}$$

How we choose to calculate the numerical flux $\mathcal{F}$ determines the order of accuracy of finite volume method [2].

For example we have the centered, second order Lax-Wendroff method

$$Q_i^{n+1} = Q_i^n - \frac{a\Delta t}{2\Delta x}\left[\Delta Q_{i+1} - \Delta Q_{i-1}\right] + \frac{1}{2}\left(\frac{a\Delta t}{\Delta x}\right)^2 \left(Q_{i+1}^n - 2Q_i^n + Q_{i-1}^n\right) \tag{3.8}$$

which is equivalent to the Lax-Wendroff method for the finite difference method [1]. None the less the Lax-Wendroff is also a finite volume scheme, in the case above using downwind slopes [1], were the numerical fluxes are

$$\begin{aligned} F_{i-1/2}^n &= \frac{1}{2}a(Q_{i-1}^n + Q_i^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}a^2(Q_i^n + Q_{i-1}^n) \\ F_{i+1/2}^n &= \frac{1}{2}a(Q_{i+1}^n + Q_i^n) - \frac{1}{2}\frac{\Delta t}{\Delta x}a^2(Q_{i+1}^n + Q_i^n)\,. \end{aligned} \tag{3.9}$$

The statement of upwind method will follow below after we have derived analytical solution to the homogeneous equation, which makes the Riemann problem more evident.

4. Homogeneous Elastic Wave Equation. In order to test our derived numerical methods we investigate the one dimensional elastic wave equation propagating through a homogeneous medium. The source free form of the elastic wave equation can be written as a coupled system of equations

$$\begin{aligned} \frac{\partial}{\partial t}\sigma - \mu\frac{\partial}{\partial x}v = 0 \\ \frac{\partial}{\partial t}v - \frac{1}{\rho}\frac{\partial}{\partial x}\sigma = 0, \end{aligned} \tag{4.1}$$

where $\sigma = \sigma_{xy}$ is the shear stress component of the stress tensor, v is the transverse velocity, ρ is the density, and μ is the bulk modulus [1]. Written in matrix-vector notation our elastic wave eqaution becomes

$$\partial_t Q + A\partial_x Q = 0 \tag{4.2}$$

where

$$Q = \begin{bmatrix}\sigma \\ v\end{bmatrix}, \quad A = \begin{bmatrix} 0 & -\mu \\ -1/\rho & 0\end{bmatrix}.$$

Equation 4.2 resembles the one dimensional advection equation (Eqaution 2.8) except we have a coupled system of $m = 2$ equations. In order to make use of the numerical methods outlined above we need to decouple the system of equations. In order to do so we find the eigendecomposition of A such that

$$A = X\Lambda X^{-1} \tag{4.3}$$

where $X \in \mathbb{R}^{m\times m}$ is the matrix of eigenvectors of A and $\Lambda \in \mathbb{R}^{m\times m}$ is the diagonal matrix with the eigenvalues of A along the diagonal. Given that the eigendecomposition exists this linear system is hyperbolic [2]. By replacing A by it's eigendecomposition and multiplying both sides of the equation by X^{-1} Equation 4.2 can be simplified to

$$\partial_t W + \Lambda\partial_x W = 0 \tag{4.4}$$

where $W = X^{-1}Q$ is the solution vector, comprised of the characteristic variables $w_{1,2}$ [1, 2]. Since Λ is diagonal we now left with $m = 2$ decoupled advection equations [2] for the characteristic variables in W such that

$$\partial_t w_p + \lambda_p \partial_x w_p \text{ for } p = 1, 2. \tag{4.5}$$

We have m waves traveling at characteristic speeds (λ_p) [2].

We find the eigenvalues of A to be $\lambda_{1,2} = \pm\sqrt{\mu/\rho} = \pm C$, which by definition is the shear velocity [1]. With these eigenvalues we determine our eigenvectors to be

$$x_1 = \begin{bmatrix} Z \\ 0 \end{bmatrix} \ , \ x_2 = \begin{bmatrix} -Z \\ 0 \end{bmatrix} \tag{4.6}$$

where $Z = \rho c$ which by definition is the seismic impedance [1]. We now have a matrix of eigenvectors such that

$$X = \begin{bmatrix} Z & -Z \\ 1 & 1 \end{bmatrix} \ , \ X^{-1} = \frac{1}{2Z}\begin{bmatrix} 1 & Z \\ -1 & Z \end{bmatrix}. \tag{4.7}$$

Now that we have our eigendecomposition of A we can find an analytical solution to Equations 4.1 that can be used to validate our numerical methods against. The simplest generic solution to Equations 4.1 is of the from $w_{1,2} = w^0_{1,2}(x,t)$ where $w^0_{1,2}$ are waveforms being advected [2, 1]. From Equation 4.4 we can solve for the characteristic variables

$$\begin{aligned} W = X^{-1}Q &= \frac{1}{2Z}\begin{bmatrix} 1 & Z \\ -1 & Z \end{bmatrix}\begin{bmatrix} \sigma^0(x \pm Ct) \\ v^0(x \pm Ct) \end{bmatrix} \\ &= \frac{1}{2Z}\begin{bmatrix} \sigma^0(x + Ct) + Zv^0(x + Ct) \\ -\sigma^0(x - Ct) + Zv^0(x - Ct) \end{bmatrix} \end{aligned} \tag{4.8}$$

based on our initial waveforms being advected. We then relate the characteristic variables to stress (σ) and velocity (v) by

$$Q = XW = \frac{1}{2Z}\begin{bmatrix} Z & -Z \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \sigma^0(x + Ct) + Zv^0(x + Ct) \\ -\sigma^0(x - Ct) + Zv^0(x - Ct) \end{bmatrix}. \tag{4.9}$$

The product of this expression $Q = XW$ produces the analytical solution

$$\begin{aligned} \sigma(x,t) &= \frac{1}{2}(\sigma^0(x + Ct) + \sigma^0(x - Ct)) + \frac{Z}{2}(v^0(x + Ct) - v^0(x - Ct)) \\ v(x,t) &= \frac{1}{2Z}(\sigma^0(x + Ct) - \sigma^0(x - Ct)) + \frac{1}{2}(v^0(x + Ct) + v^0(x - Ct)). \end{aligned} \tag{4.10}$$

We now have an analytical solution to validate our numerical results against. Our analytical solution (Equation 4.10) can be written as

$$Q(x,t) = \sum_{p=1}^{m} w_p(x,t) x_p \tag{4.11}$$

where x_p is an eigenvector and $w_p(x,t)$ is the coefficient of the eigen vector [1, 2]. Therefore, we can think of our analytical solution as a sum of eigenvectors x_p of strength $w_p(x,t)$ being a advected at a velocity λ_p [2]. From our analytical solution we see that we have an initial wave $w_p(x,0)$ advected at speed λ_p along the characteristic curve $X(t) = x_0 + \lambda_p t$ [2]. Given that we have a constant coefficient in Equation 4.2, our characteristic curves are strait lines. The assumptions used to derive Equation 4.10 only hold for a constant coefficient (homogeneous material) system of equations.

Variable	Description	Value	units
ρ	Density	2500	kg/m^3
C	Transverse velocity	2500	m/s
L	Length of spatial domain	10	km
nx	Number of girdcells	1000	
	Initial Condition Parameters		
γ		5×10^{-6}	
x_0	Initial position	2	km

TABLE 1
Caption

5. Numerical Experiments.

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