2 Springs

We already covered this in class but the solution boils down to a simple fourth degree linear homogenous diff eq.

$ma = F = -kx$

Spring 1 constant k_1, mass of M_1, position u_1. Spring 2 constant k_2, mass of M_2, position u_2.

A: $M_1u_1'' = -k_1u_1 - k_2(-(u_2 - u_1))$
B: $M_2u_2'' = -k_2(u_2 - u_1)$

A: $M_1u_1'' + (k_1 + k_2)u_1 = k_2u_2$

A: $u_2 = \frac{1}{k_2}(M_1u_1'' + (k_1 + k_2)u_1)$
A": $u_2'' = \frac{1}{k_2}(M_1u_1'''' + (k_1 + k_2)u_1'')$

B—A, A": $\frac{M_2}{k_2}(M_1u_1'''' + (k_1 + k_2)u_1'') = -(M_1u_1'' + (k_1 + k_2)u_1) + k_2u_1$

$(\frac{M_1M_2}{k_2})u_1'''' + (\frac{M_2(k_1+k_2)+M_1k_2}{k_2})u_1'' + (k_1)u_1 = 0$

$\frac{M_1M_2}{k_2}r^4 + (M_1 + M_2 + M_2\frac{k_1}{k_2})r^2 + k_1r = 0$

3 Springs

Similarly, we can create equations for a 3 spring problem. We end up with a 6th degree equation this time.

$M_3u_3'' = -k_3(u_3 - u_2)$
$M_2u_2'' = -k_2(u_2 - u_1) - k_3(-(u_3 - u_2))$
$M_1u_1'' = -k_1u_1 - k_2(-(u_2 - u_1))$

$M_1u_1'' + (k_1 + k_2)u_1 = k_2u_2$
$u_2 = \frac{1}{k_2}(M_1u_1'' + (k_1 + k_2)u_1)$
$u_2'' = \frac{1}{k_2}(M_1u_1'''' + (k_1 + k_2)u_1'')$

$M_2u_2'' = -k_2(u_2 - u_1) - k_3(-(u_3 - u_2))$

$M_2u_2'' + (k_2 + k_3)u_2 - k_2u_1 = k_3u_3$

$\frac{M_2}{k_2}(M_1u_1'''' + (k_1 + k_2)u_1'') + \frac{(k_2+k_3)}{k_2}(M_1u_1'' + (k_1 + k_2)u_1) - k_2u_1 = k_3u_3$

$(\frac{M_1 M_2}{k_2})u_1'''' + (\frac{(k_1+k_2)+M_2(k_2+k_3)}{k_2})u_1'' + (\frac{(k_1+k_2)(k_2+k_3)-k_2^2}{k_2})u_1 = k_3 u_3$

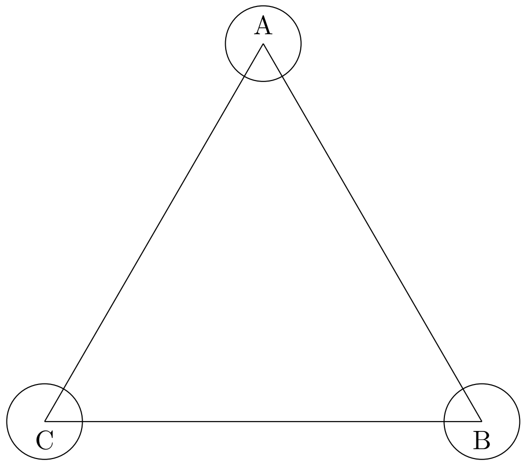
$u_3 = \frac{1}{k_2 k_3}(M_1 M_2 u_1'''' + ((k_1+k_2)+M_2(k_2+k_3))u_1'' + ((k_1+k_2)(k_2+k_3)-k_2^2)u_1)$
$u_3'' = \frac{1}{k_2 k_3}(M_1 M_2 u_1'''''' + ((k_1+k_2)+M_2(k_2+k_3))u_1'''' + ((k_1+k_2)(k_2+k_3)-k_2^2)u_1'')$

$M_3 u_3'' + k_3 u_3 = k_3 u_2$

$\frac{M_3}{k_2 k_3}(M_1 M_2 u_1'''''' + ((k_1+k_2)+M_2(k_2+k_3))u_1'''' + ((k_1+k_2)(k_2+k_3)-k_2^2)u_1'') + \frac{1}{k_2}(M_1 M_2 u_1'''' + ((k_1+k_2)+M_2(k_2+k_3))u_1'' + ((k_1+k_2)(k_2+k_3)-k_2^2)u_1) - \frac{k_3}{k_2}(M_1 u_1'' + (k_1+k_2)u_1) = 0$

$(\frac{M_1 M_2 M_3}{k_2 k_3})u_1'''''' + (\frac{M_3}{k_2 k_3}((k_1+k_2)+M_2(k_2+k_3)) + \frac{M_1 M_2}{k_2})u_1'''' + (\frac{M_3}{k_2 k_3}((k_1+k_2)(k_2+k_3)-k_2^2) + \frac{1}{k_2}((k_1+k_2)+M_2(k_2+k_3)) + \frac{k_3}{k_2}M_1)u_1'' + (\frac{1}{k_2}((k_1+k_2)(k_2+k_3)-k_2^2) + \frac{k_3}{k_2}(k_1+k_2))u_1 = 0$

$(\frac{M_1 M_2 M_3}{k_2 k_3})x^6 + (\frac{M_3}{k_2 k_3}((k_1+k_2)+M_2(k_2+k_3)) + \frac{M_1 M_2}{k_2})x^4 + (\frac{M_3}{k_2 k_3}((k_1+k_2)(k_2+k_3)-k_2^2) + \frac{1}{k_2}((k_1+k_2)+M_2(k_2+k_3)) + \frac{k_3}{k_2}M_1)x^2 + (\frac{1}{k_2}((k_1+k_2)(k_2+k_3)-k_2^2) + \frac{k_3}{k_2}(k_1+k_2)) = 0$



Three springs in triangle

We can write equations for the position of each mass at t. We will need the length and the angle of each spring to find the force at any time.

$x_a(t), y_a(t)$
$x_b(t), y_b(t)$
$x_c(t), y_c(t)$

$L_a(t) = \sqrt{(x_b(t) - x_c(t))^2 + (y_b(t) - y_c(t))^2}$
$L_b(t) = \sqrt{(x_a(t) - x_c(t))^2 + (y_a(t) - y_c(t))^2}$
$L_c(t) = \sqrt{(x_b(t) - x_a(t))^2 + (y_b(t) - y_a(t))^2}$

$\theta_a(t) = \tan^{-1}(\ )$

Law of cosines:
$\theta_c(t) = \cos^{-1}(\frac{-(L_c(t)^2 - L_a(t)^2 - L_b(t)^2)}{2L_a(t)L_b(t)})$

$M_a x_a'' = \cos(\theta_b(t)) \cdot (-k_b(L_b(t) - L_b(0))) + \cos(\theta_c(t)) \cdot (-k_c(L_c(t) - L_c(0)))$
$M_a y_a'' = \sin(\theta_b(t)) \cdot (-k_b(L_b(t) - L_b(0))) + \sin(\theta_c(t)) \cdot (-k_c(L_c(t) - L_c(0)))$

This may be solvable but it would be very tough to do.

A much easier way to do this is by using Lagrangian Mechanics. Lagrangian Mechanics has to do with the Lagrangian, defined as:

$\mathscr{L} = T - V$

Where T is kinetic energy and V is potential energy.

The principle of least action states that systems that the physical functions are the stationaries of the Lagrangian. What does this mean in practice? This means that we can use the Euler-Lagrange equation from Calculus of variations to solve for the physical states.

$\frac{\partial \mathscr{L}}{\partial q} - \frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}} = 0$

Where q is a function of t. This is analogous to taking the first derivative and setting it equal to one. We can apply this to a simple spring mass problem to see how this works.

Our Lagrangian is equal to kinetic minus potential:

$\mathscr{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$

Where x(t) is the linear displacement of the mass. Applying the Euler-Lagrange equation with respect to x we get:

$-kx - \frac{d}{dt}m\dot{x} = 0$
$m\ddot{x} = -kx$

This is Hooke's law which we know is the right answer.

Now we can try applying this to our three spring problem to get a system of equations describing the motion of the masses. We'll start by writing our Lagrangian. We will use 6 independent functions of time $x_1, y_1, x_2, y_2, x_3, y_3$ to describe the positions of the masses, 6 constants $m_1, m_2, m_3, k_1, k_2, k_3$ and we will also define 6 auxiliary functions $L_1, L_2, L_3, v_1, v_2, v_3$ to make the math a bit easier. v_1 is the speed of mass 1. L_1 describes the length of the spring between mass 2 and 3, and k_1 is the spring constant of that spring. L_{1_0} is the equilibrium length of spring 1. Our auxiliary functions are defined as:

$L_1 = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2} = \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2}$
$L_2 = \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2} = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$
$L_3 = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
$v_1 = \sqrt{\dot{x_1}^2 + \dot{y_1}^2}$
$v_2 = \sqrt{\dot{x_2}^2 + \dot{y_2}^2}$
$v_3 = \sqrt{\dot{x_3}^2 + \dot{y_3}^2}$

Here's the non-zero partials of these functions and their squares that we will need to use in the Euler-Lagrange equation:

$\frac{\partial L_1}{\partial x_2} = \frac{x_2 - x_3}{L_1}$

$\frac{\partial L_1}{\partial x_3} = \frac{x_3 - x_2}{L_1}$

$\frac{\partial L_1^2}{\partial x_2} = \frac{\partial}{\partial x_2}((x_2 - x_3)^2 + (y_2 - y_3)^2) = 2(x_2 - x_3)$

$\frac{\partial L_1^2}{\partial x_3} = \frac{\partial}{\partial x_3}((x_3 - x_2)^2 + (y_3 - y_2)^2) = 2(x_3 - x_2)$

$\frac{\partial v_1}{\partial \dot{x}_1} = \frac{\dot{x}_1}{v_1}$

$\frac{\partial v_1^2}{\partial \dot{x}_1} = \frac{\partial}{\partial \dot{x}_1}(\dot{x}_1^2 + \dot{y}_1^2) = 2\dot{x}_1$

Now we can write our lagrangian:

$\frac{1}{2}(m_1 v_1^2 + m_2 v_2^2 + m_3 v_3^2 - k_1(L_1 - L_{1_0})^2 + k_2(L_2 - L_{2_0})^2 + k_3(L_3 - L_{3_0})^2)$

We can get one equation by using the Euler-Lagrange equation with x_1.

$\frac{\partial}{\partial x_1}(\frac{1}{2}(k_2(L_2 - L_{2_0})^2 + k_3(L_3 - L_{3_0})^2)) - \frac{d}{dx_1}(\frac{\partial}{\partial \dot{x}_1}\frac{1}{2}(m_1 v_1^2)) = 0$

Before tackling this whole evaluation, let's analyze a small but important chunk.

$\frac{\partial}{\partial x}(L_2 - L_{2_0})^2 = \frac{\partial}{\partial x}(L_2^2 - 2L_{2_0} \cdot L_2 + L_{2_0}^2) = 2(x_1 - x_3) - 2\frac{L_{2_0}}{L_2}(x_1 - x_3) = 2((x_1 - x_3)(1 - \frac{L_{2_0}}{L_2})$

Using this and the equation for the partial of V_1 that we derived earlier, we now have the tools to evaluate this expression:

$\frac{1}{2}(2k_2((x_1 - x_3)(1 - \frac{L_{2_0}}{L_2}) + 2k_3((x_1 - x_2)(1 - \frac{L_{3_0}}{L_3})) - \frac{d}{dx_1}(\frac{1}{2}(2m_1\dot{x}_1)) = 0$

$k_2(x_1 - x_3)(1 - \frac{L_{2_0}}{L_2}) + k_3(x_1 - x_2)(1 - \frac{L_{3_0}}{L_3}) - m_1\ddot{x}_1 = 0$

On the next page we have the entire system of equations.

Now when we do this for all six variables we get this system of equations:

$$\ddot{x}_1 = \frac{1}{m_1}(k_2(x_1 - x_3)(1 - \frac{L_{2_0}}{L_2}) + k_3(x_1 - x_2)(1 - \frac{L_{3_0}}{L_3}))$$

$$\ddot{y}_1 = \frac{1}{m_1}(k_2(y_1 - y_3)(1 - \frac{L_{2_0}}{L_2}) + k_3(y_1 - y_2)(1 - \frac{L_{3_0}}{L_3}))$$

$$\ddot{x}_2 = \frac{1}{m_2}(k_1(x_2 - x_3)(1 - \frac{L_{1_0}}{L_1}) + k_3(x_2 - x_1)(1 - \frac{L_{3_0}}{L_3}))$$

$$\ddot{y}_2 = \frac{1}{m_2}(k_1(y_2 - y_3)(1 - \frac{L_{1_0}}{L_1}) + k_3(y_2 - y_1)(1 - \frac{L_{3_0}}{L_3}))$$

$$\ddot{x}_3 = \frac{1}{m_3}(k_1(x_3 - x_2)(1 - \frac{L_{1_0}}{L_1}) + k_2(x_3 - x_1)(1 - \frac{L_{2_0}}{L_2}))$$

$$\ddot{y}_3 = \frac{1}{m_3}(k_1(y_3 - y_2)(1 - \frac{L_{1_0}}{L_1}) + k_2(y_3 - y_1)(1 - \frac{L_{2_0}}{L_2}))$$

This is a very messy equation. I won't solve it here but I may eventually try to completely solve it. My first steps in trying to solve it would be recognizing that the $(x_1 - x_3)(1 - \frac{L_{2_0}}{L_2})$ term can be distributed to be $(x_1 - x_3) - L_{2_0}\frac{(x_1 - x_3)}{L_2}$. The latter term can be rewritten like so:

$$((\frac{(x_1-x_3)}{L_2})^2)^{\frac{1}{2}} = \sqrt{\frac{(x_1-x_3)^2}{(x_1-x_3)^2+(y_1-y_3)^2}}$$

I would also try to use the same technique applied to the linear spring equations where we took two derivatives of one equation and plugged in the equation for the second derivative of our other variables.