Week 11: Linear mixed models & autocorrelation

ANTH 674: Research Design & Analysis in Anthropology

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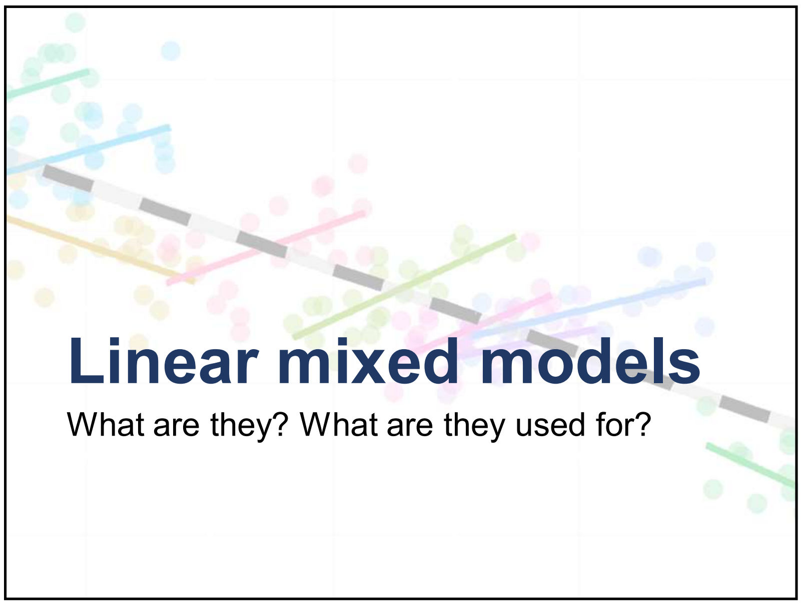
Lecture outline

1. Linear mixed models
   1. What are they? What are they used for?
   2. Variance partitioning w/ crossed or nested designs
2. Measuring & dealing with autocorrelation
   1. Correlograms
   2. Association between two variables
      1. Cross-correlograms
      2. Generalized least squares



Linear mixed models

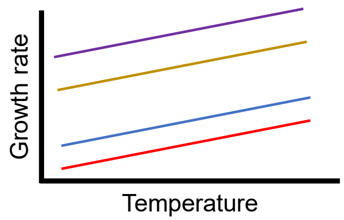
What are they? What are they used for?

What is a linear mixed model?

- A linear model with **<u>fixed</u>** AND **<u>random</u>** factors/effects
- I've always found the distinction between the two factors confusing, so let's go through them carefully

Fixed factors

- Measured levels are the only ones of interest
- E.g., want to know how intercept of growth rate ~ temperature changes w/ **different levels of nitrogen** (i.e., ANCOVA)
- Thus far, we have only looked at fixed factors

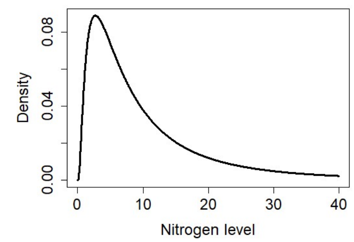


0N +1N
+2N +3N

5

Random factors

- Measured levels are a random, representative sample of some population of levels
- E.g., want to infer something about population of intercepts (i.e., SD of intercepts across levels)



0N +1N
+2N +3N

6

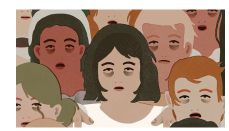
Ultimately, depends on research question!

- **Fixed factor**: estimate how intercept and/or slope changes ~ different N levels (i.e., ANCOVA)
- **Random factor**: estimate SD of population of intercepts and/or slopes across N levels

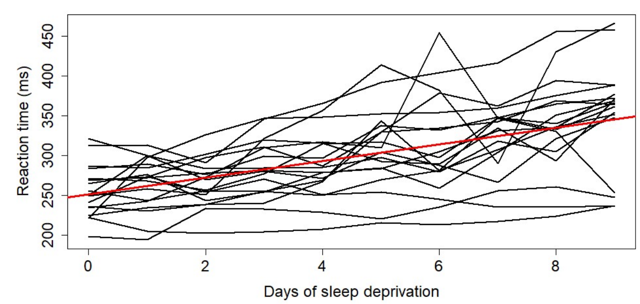
0N +1N +2N +3N

7

For example…



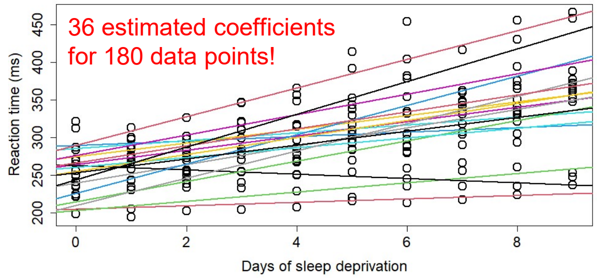
- Study measured reaction times of 18 subjects each day after full night's sleep & then 9 days of sleep deprivation (180 total observations)



8

Fixed factor approach

- Estimates intercept & slope of `reaction ~ days` for each subject (factor level)
- ANCOVA: `lm(reaction ~ days * subject)`

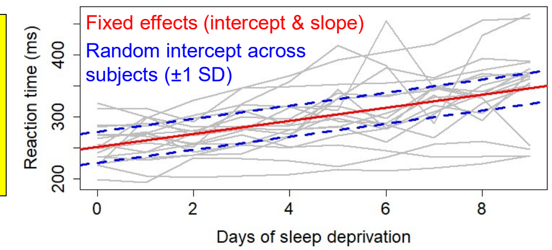


9

Random factor approach

- Estimates an intercept & slope for ALL observations (fixed effects)
- **Main difference**: estimates variation (e.g., SD) across levels for slope and/or intercept (random effects)

- **Intercept for each subject** = fixed intercept + random effect
- Random effect ~ $N(0, \sigma^2)$



10

Random factor approach

- Estimation done via maximum likelihood

```
Random effects:
              StdDev   Corr
(Intercept) 24.740241 (Intr)
Days         5.922103 0.066
Residual    25.591843

Fixed effects: Reaction ~ Days
               Value Std.Error  DF  t-value p-value
(Intercept) 251.40510  6.824516 161 36.83853       0
Days         10.46729  1.545783 161  6.77151       0
```

Coefficient estimates same as `lm(reaction ~ days)`!

Can only calculate P-values in simple mixed models (e.g., balanced design, no crossed random effects)

11

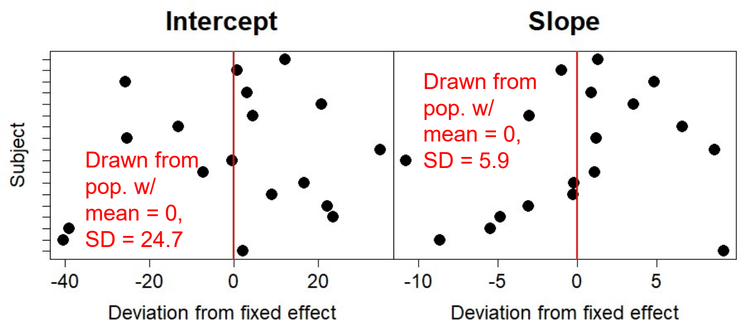
Why is this important?

- A big goal in statistics is to understand what causes variation in the DV
  - e.g., R^2 in linear models w/ ≥1 IV(s)
- By estimating SD of intercept and/or slope, you are attributing variation in these coefficients to levels w/in your random factor
- Beginning to understand what is causing variation in your DV → modeled IVs (and their coefficients) and variation around coefficients

12

Random factor approach

- Can get intercept & slope for each subject (**NOT** estimated parameters!)



13

Random factor approach

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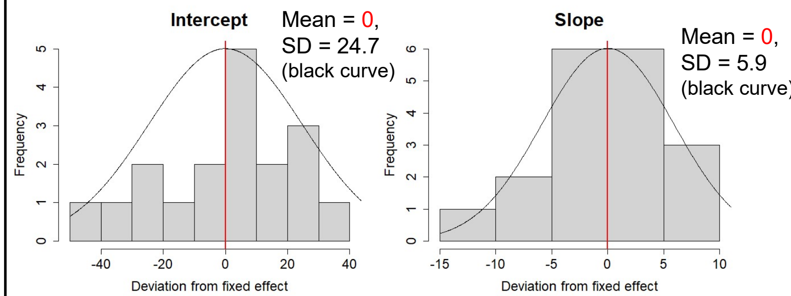
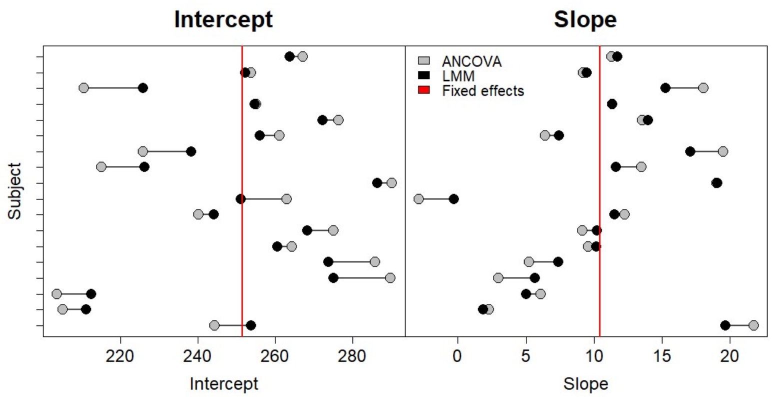
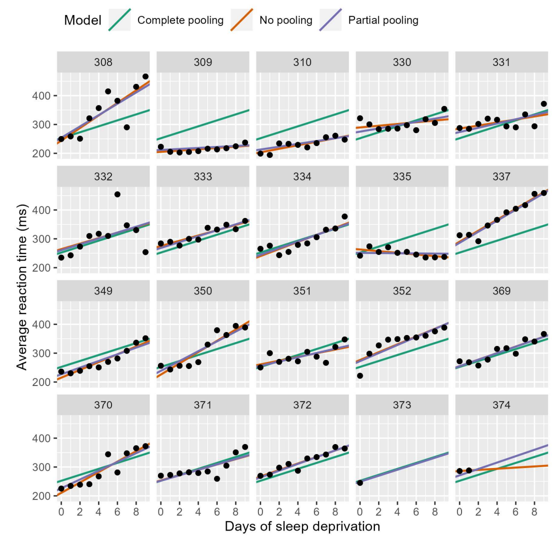


14

Compared to ANCOVA



15



Complete pooling
One intercept & slope for all data

No pooling
ANCOVA w/ diff. intercepts & slopes

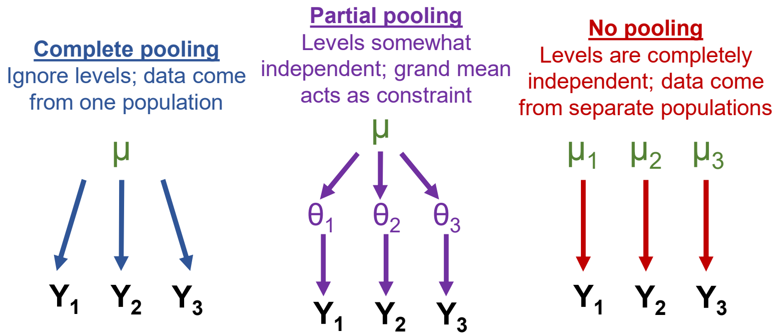
Partial pooling
LMM shrinkage

https://www.tjmahr.com/plotting-partial-pooling-in-mixed-effects-models/

16

Another way of looking at it

- Imagine we are estimating the mean of some dataset w/ diff. levels (= intercept-only model)



*Each **Y** is a different level, (e.g., plot, individual, block) w/ replicate observations

17

Shrinkage or regularization

- In ANCOVA, each line is fit to data for one factor level **only**
- In LMM, levels are drawn from a population, so they are expected to share characteristics (w/ a distribution mean = fixed effect)
  - This is how you can predict for new factor levels w/ LMM (can't do this w/ ANCOVA)
- If a level has few observations, it borrows more information from other levels and is pulled more towards the average (fixed effect)

18

Fixed vs. random factors

- **Fixed factors**
  - Interested in how intercept and slope changes ***independently** for each subject*
  - Potentially need to estimate *A LOT* of coefficients
- **Random factors**
  - Interested in the SD of intercepts and slopes *across all **possible** subjects*
  - Estimates fixed effects AND SDs
  - Shrinks intercepts & slopes towards means

https://dynamicecology.wordpress.com/2015/11/04/is-it-a-fixed-or-random-effect/

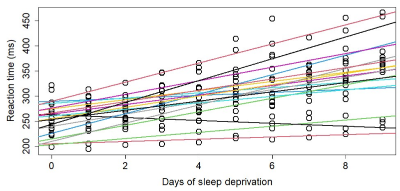
19

Questions?

20

Reasons for doing LMM

1. Factor levels are random sample from larger population → interested in estimating SD of population of intercepts/slopes
2. Ratio of data points to # coefficients in ANCOVA too small → overfitting (only need to estimate SDs in LMM)



36 estimated coefficients for 180 data points!

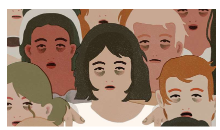
21

Reasons for doing LMM

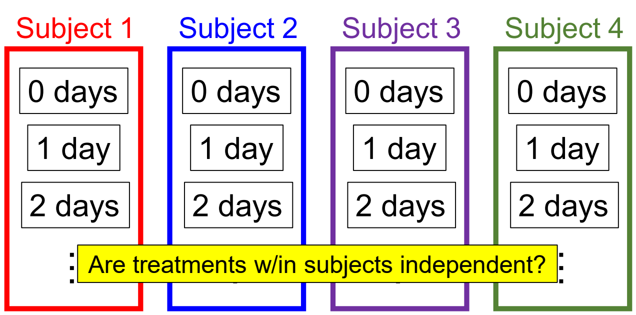
1. Factor levels are random sample from larger population → interested in estimating SD of population of intercepts/slopes
2. Ratio of data points to # coefficients in ANCOVA too small → overfitting (only need to estimate SDs in LMM)
3. Shrink intercepts and slopes towards mean
4. Want to account for non-independence in data using a blocking factor (very popular reason)

22

E.g., sleep study



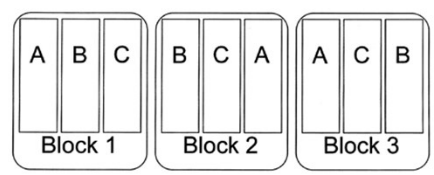
- Each subject is treated as a block, each w/ treatments of different days of sleep deprivation (AKA **repeated measures design**)



23

Analyzing block design

- Can do ANCOVA, but problematic if lots of levels & not enough data
- Also, many times blocks are not of interest (i.e., don't care how intercept/slope change w/ block)
- **Can use LMM & include block as a random factor to account for it**



24

Random factor rules

- Random factor **<u>MUST</u>** be categorical (i.e., it must be a factor w/ levels)
  - A continuous IV can be entered into model as covariate (only one extra estimated coefficient)
- Need at least five levels (would you calculate SD of a sample w/ < 5 data points?)
  - Otherwise, just make factor a fixed effect

25

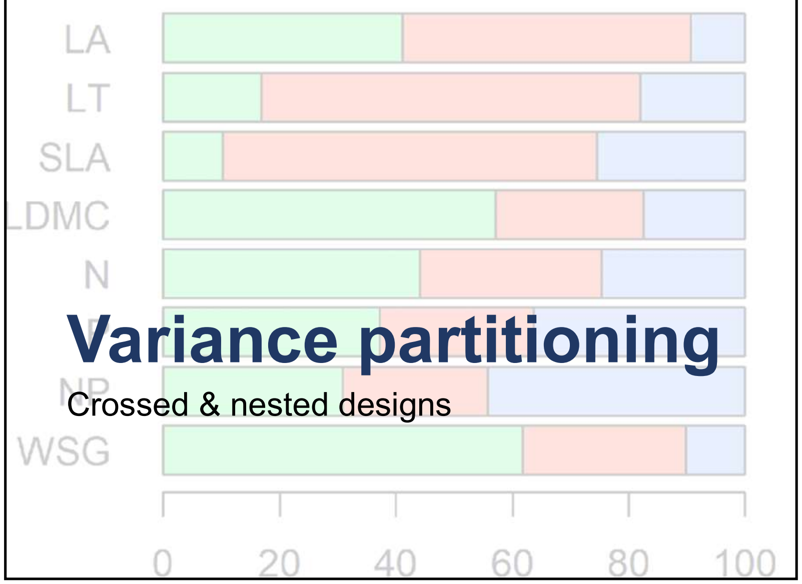
Generalized linear mixed models (GLMMs)

- Mixed model w/ non-normally distributed errors (cf. generalized linear models)
- E.g., presence/absence of lions ~ wildebeest population density w/ plot as blocking factor

26

Questions?

27



Variance partitioning

Crossed & nested designs

28

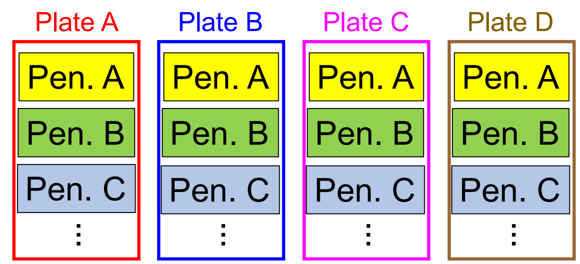
Variance partitioning

- Can partition DV variance **not accounted for by fixed effects** among multiple random factors
- Or, can fit an intercept-only model to partition DV variance among random factors

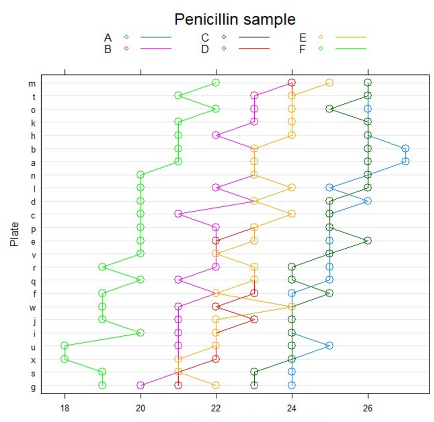
Multiple random factors

1. **Crossed random factors**: each level from a factor is w/ each level from the other

- E.g., how penicillin inhibiting growth of *Bacillus* varies across plates & penicillin samples



Crossed random factors

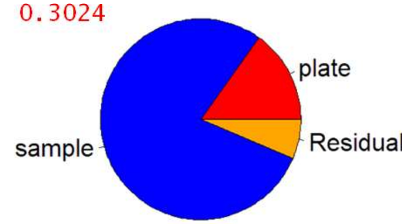


- Fully crossed! Every combination of `sample` and `plate` is represented
- Unreplicated and completely balanced (only one observation per `sample/plate` combo)

Crossed random factors

- Intercept-only LMM w/ `plate` & `sample` as crossed random effects
- `Random effects:`

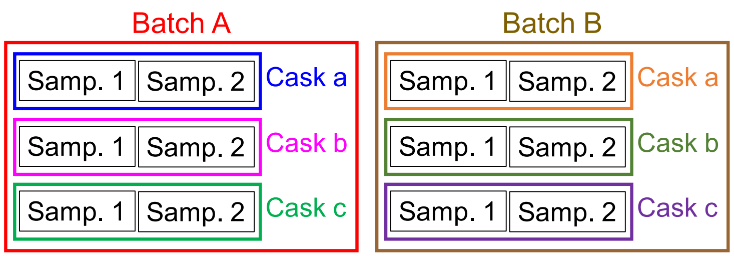
```
Groups   Name        Variance
plate    (Intercept) 0.7169
sample   (Intercept) 3.7311
Residual             0.3024
```



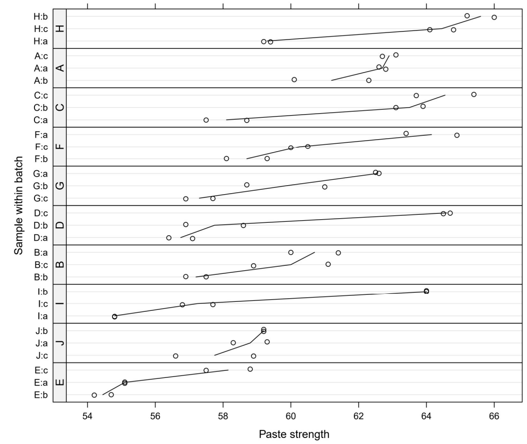
Multiple random factors

2. **Nested random factors**: factors are nested within each other

- E.g., strength of chemical paste, where two samples were taken from each cask from each batch



Nested random factors

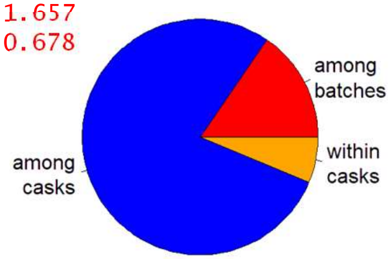


- 10 batches (capital letters)
- 3 casks per batch (lower case)
- Two samples per cask

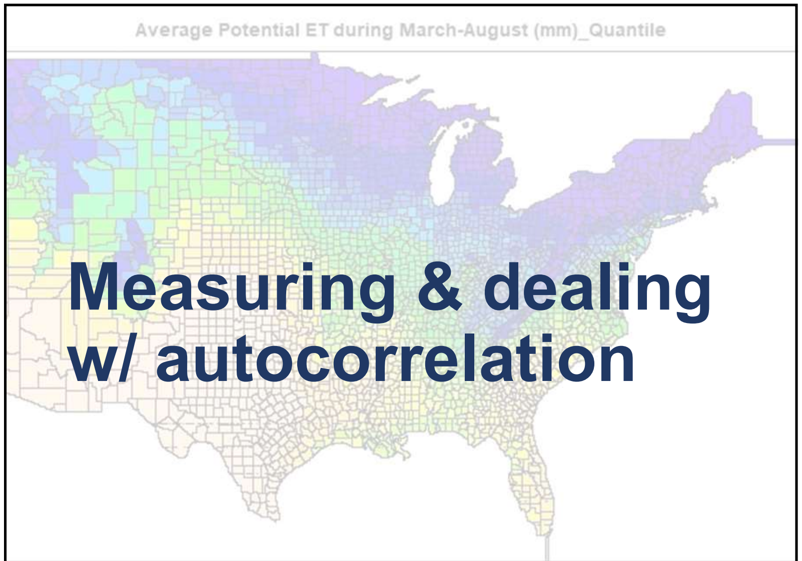
Nested random factors

- Intercept-only LMM w/ `cask` nested within `batch`
- Random effects:

```
Groups   Name         Variance
cask     (Intercept)  8.434
batch    (Intercept)  1.657
Residual              0.678
```



Questions?

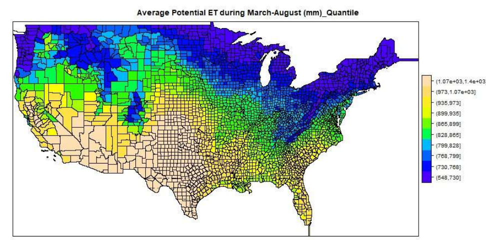


Measuring & dealing w/ autocorrelation

37

Non-independent data

- Data across space, time, or a phylogeny will almost always be non-independent
- Data closer in space/time/relatedness will be more similar



38

Three components

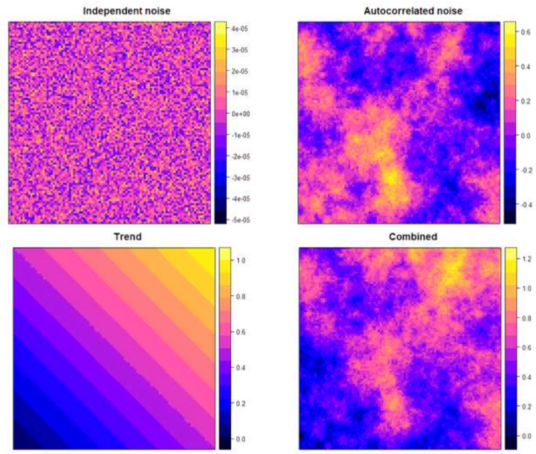
- Patterns in space & time (& phylogeny) can be decomposed into three components
- $z(d) = \mu(d) + \eta(d) + \varepsilon$, where (d) means component is a function of distance
  - i.e., pattern = trend + autocorrelated (red) noise + independent (white) noise
  - Temporal data can also include a cyclical component too (e.g., monthly, seasonality, orbital cycles)

39

Simulated example for <u>time</u>

40

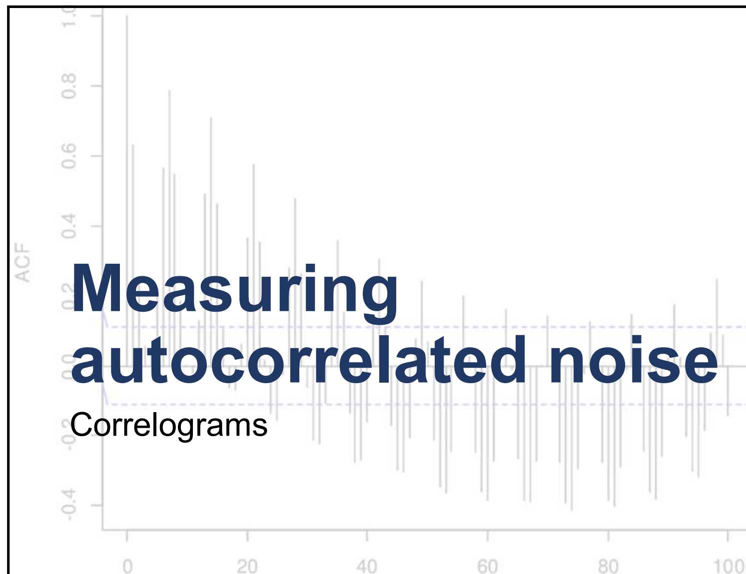
Simulated example for <u>space</u>



Autocorrelated noise: nuisance or interesting?

- **<u>Nuisance:</u>** non-independence violates one of the assumptions of linear models (increases Type I error rate)
  - Must correct for this (e.g., w/ blocking)!
- **<u>Interesting:</u>** How does strength of autocorrelation vary as a function of distance in space/time/phylogeny?
  - May tell us something about the underlying process generating autocorrelation!

Questions?



Measuring autocorrelated noise

Correlograms

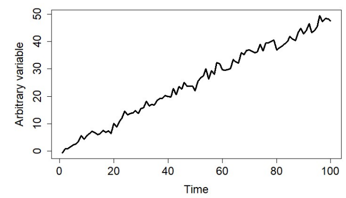
Detrending

- If temporal or spatial data are strongly trended, need to first remove trend (i.e., **detrending**) to focus on autocorrelated noise

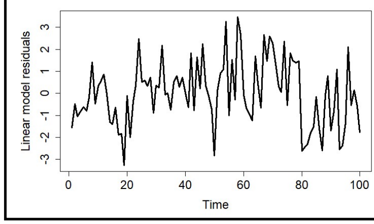
1. Can fit a trend to data and take the residuals
2. Calculate first differences: $x_i' = x_{i+1} - x_i$

45

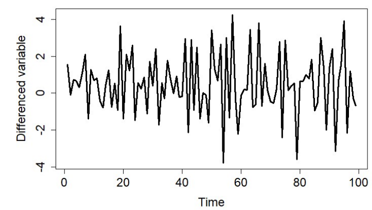
Example w/ temporal data



`lm(x~time)$residuals`



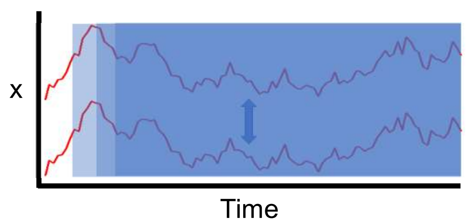
`diff(x)`



46

Autocorrelation function

- Quantifies how time series is correlated with itself at different lags
- E.g., `c(1, 2, 3, 4, 5)`
  - time lag = 1, `c(1, 2, 3, 4)` vs. `c(2, 3, 4, 5)`
  - time lag = 2, `c(1, 2, 3)` vs. `c(3, 4, 5)`



47

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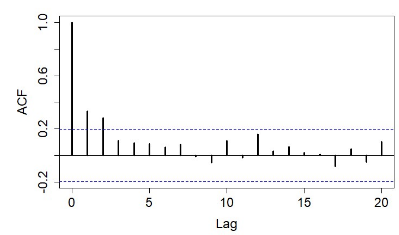
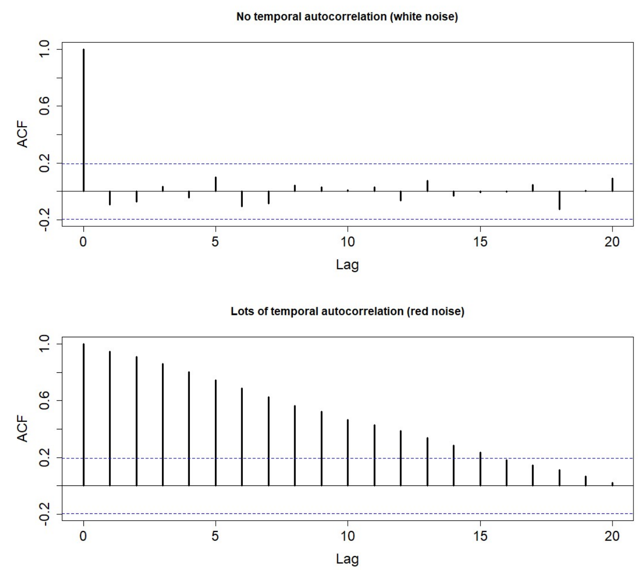


`acf(lm(x~time)$residuals)`

- Dashed blue lines delineate null region (assuming white noise)
- Be careful of multiple comparisons!

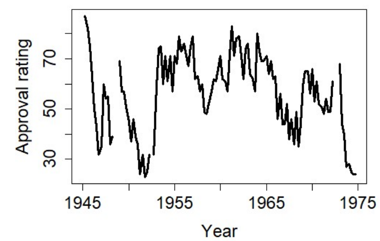
48



49

A real example

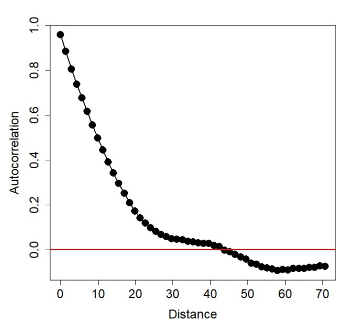
- Quarterly approval rating of US president from 1945 to 1974
- Not strongly trended, so no need to detrend
- Can begin to hypothesize what's generating AC!

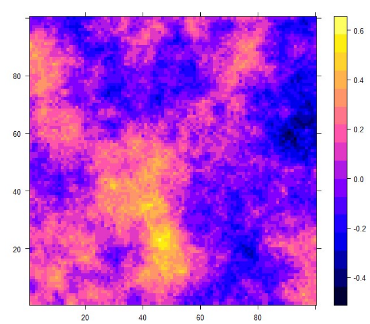


50

Spatial correlogram

- Quantifies correlation between pairs of points as you increase the distance (lag) between them

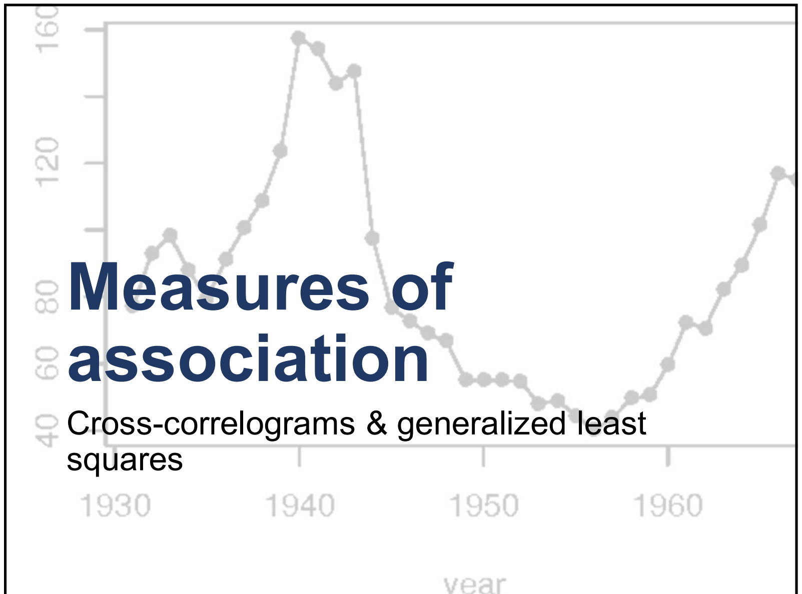




51

Questions?

52



Measures of association

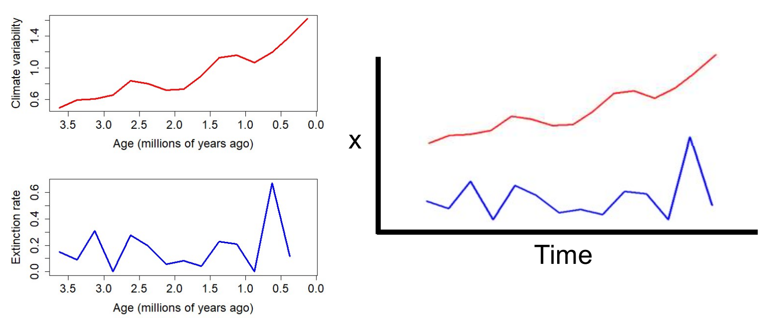
Cross-correlograms & generalized least squares

Comparing two variables

1. **Cross-correlograms**: looks at how two variables are correlated at different lags
   - Perhaps one variable responds to another only after a certain amount of time/distance
   - As w/ correlation, symmetry between variables
2. **Generalized least squares**: models a DV as a function of one or more IVs, while explicitly modeling errors as non-independent
   - Good if variables are obviously DV or IV, or if you want to do prediction (as w/ linear models)

Cross-correlation function

- E.g., climate variability vs. mammal extinction rates in Plio-Pleistocene eastern Africa



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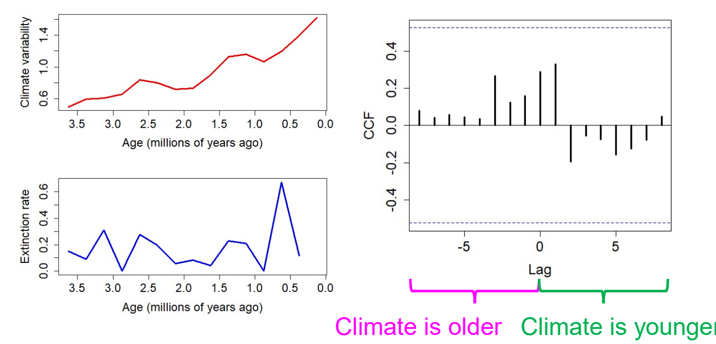
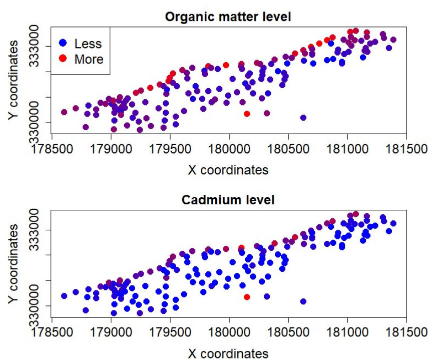
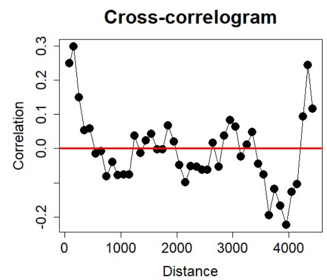


Spatial cross-correlogram

- E.g., soil organic matter & cadmium level in floodplain of Meuse River, Netherlands





Data binned into distance classes

57

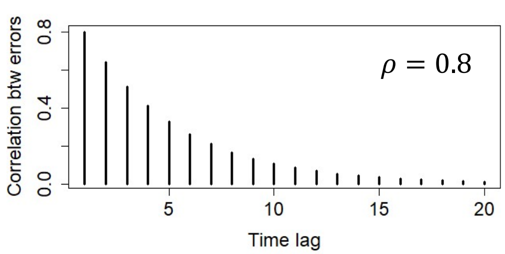
Generalized least squares

- Relaxes GLM assumption of independent errors
- GLS estimates extra parameters to model non-independent errors
- Parameters estimated using maximum likelihood

58

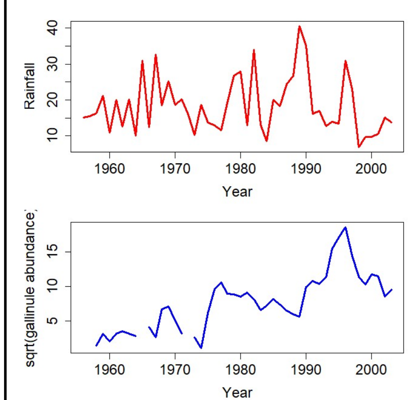
Temporal data

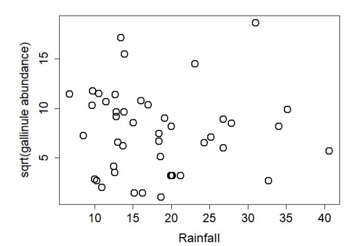
- AR1 autocorrelation error structure is popular
- Correlation between errors one time step apart estimated as ρ (usually positive), two steps apart is ρ^2, three steps is ρ^3, etc.



59

E.g., Hawaiian gallinule





60

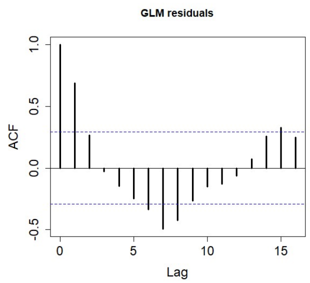
`sqrt(gallinule)~Rainfall+Year`

Coefficients are very similar!

```
lm coefficients:
              Estimate   Pr(>|t|)
(Intercept) -4.777e+02  1.26e-10
Rainfall     8.604e-04  0.986
Year         2.450e-01  8.15e-11
```

```
GLS coefficients:
                  Value  p-value
(Intercept) -436.4326   0.0030
Rainfall      -0.0098   0.7649
Year           0.2241   0.0026
```

$\rho = 0.77$

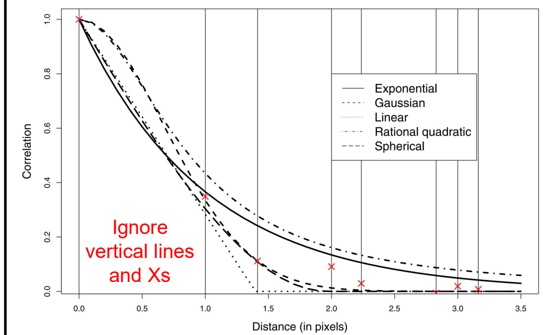


Autocorrelation in residuals accounted for by GLS error model (AR1)

61

Spatial data

- Non-independence between errors modeled as a function of distance between them

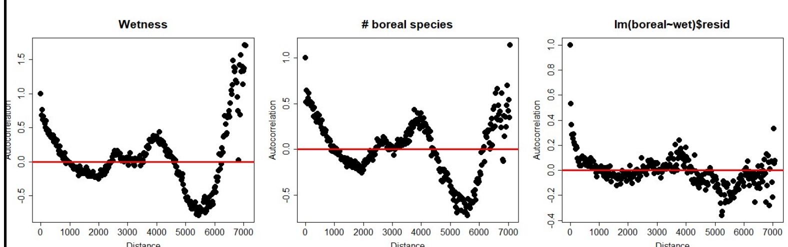


- Five common models
- Different parameterizations will slightly change shape of each model

62

One last note

- If an IV captures autocorrelation in DV, then there will be no autocorrelation remaining in residuals (GLM assumptions not violated!)



63

Questions?

64

Summary

- Use LMMs (random effects) if:
  1. Factor levels are random, representative samples from some larger population and are interested in estimating SD of population of intercepts/slopes
  2. Have too few data (prevent overfitting w/ ANCOVA)
  3. Want to account for non-independence w/ blocking (and have too few data)
  4. Interested in shrinkage of estimates
  5. You want to partition variance in DV
- Correlograms quantify autocorrelation structure, which is interesting in itself!
- Cross-correlograms & GLS quantify association btw variables in time/space/phylogeny (but remember, OLS estimates are unbiased!)

Statistics vignette

Let's play a game…

Rules

1. Pick an outcome of three coin flips, e.g., HHH
2. I pick my own outcome
3. If your outcome appears first in *n* flips, you win

https://plus.maths.org/content/os/issue55/features/nishiyama/index

What's the secret?

- This is a game called "Penney's game"

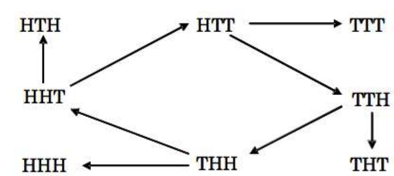
You: HHH

Reverse second flip

T

Move to front

Me: THHH



A's choice	B's choice	Odds in favor of B
HHH	THH	7 to 1
HHT	THH	3 to 1
HTH	HHT	2 to 1
HTT	HHT	2 to 1
THH	TTH	2 to 1
THT	TTH	2 to 1
TTH	HTT	3 to 1
TTT	HTT	7 to 1

Why does it work? (an example)

<u>You</u>: HHH

<u>Me</u>: THH

You win only if HHH comes up in first three flips:
HHH…

Otherwise, T *must* precede any HHH, and I win:
…THHH…

https://www.youtube.com/watch?v=Sa9jLWKrX0c