Elliptic/Parabolic PDE

MAT425

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FDM for elliptic PDE

Review:

$$\begin{cases} \Delta u = f & \text{on } \Omega = [a,b]\times[c,d] \\ u = g & \text{at } \partial\Omega. \end{cases}$$

FDM: Notation $u_{i,j} = u(x_i, y_j)$ with $x_i = a + i\Delta x$, $y_j = c + j\Delta y$
3 ingredients:

- Discretization: $\partial_x^2 u = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{\Delta x^2} + O(\Delta x^2)$,
- Vector form: $u_\ell = u_{i,j}$ with $\ell = i + (m-1-j)\cdot(n-1)$
- System: $A\mathbf{u} = \mathbf{b}$ with ($\lambda = \Delta x^2/\Delta y^2$)

$$\begin{aligned} A(\ell,\ell') &= \begin{cases} -2-2\lambda & \text{if } \ell' = \ell \\ 1 & \text{if } \ell' = \ell \pm 1 \\ \lambda & \text{if } \ell' = \ell \pm n-1 \\ 0 & \text{otherwise} \end{cases} \\ b(\ell) &= \Delta x^2 \cdot f_{i,j} + \quad {\color{red}BC} \end{aligned}$$

Direct Vs Iterative method

- **Direct method**:

$$\mathbf{A}\mathbf{u} = \mathbf{b} \quad \Rightarrow \quad \mathbf{u} = A\backslash\mathbf{b} \qquad (LU\ decomp)$$

+ Exact
- Need a matrix inversion

- **Iterative method**: Decomposition: $A = D + R$ with D diagonal matrix

$$\mathbf{u}^{n+1} = D^{-1}\big(b - R\mathbf{u}^{n}\big).$$

+ No matrix inversion
- Iterative process

Direct Vs Iterative method

Task 1

- Implement the direct method.
  Hint: finish the routine 'elliptic_PDE_exo.m'

Task 2

- Implement the iterative method.
  Hint: finish the routine 'elliptic_PDE_iterative_exo.m'
- Compare the computation time of the direct and iterative method to solve elliptic problem.
  Try with several choices of Δx, Δy.

Algorithm iterative method

Algorithm:

- Initialization: $\mathbf{u} = 0$, $\mathbf{b}$, $TOL = 10^{-10}$, $error = \|A\mathbf{u} - b\|$
- Loop:

```
while error>TOL
  for i=1:(n-1)
    for j=1:(m-1)
      u_new(i,j) = -1/(2+2ld)*(b(i,j) - u(i+1,j)
                - u(i-1,j) - ld*u(i,j+1) - ld*u(i,j-1))
    end
  end
  % update
  u = u_new
  error = =||Au-b||
end
```

FDM for Parabolic PDE

Review: Fix a length $\ell > 0$ and a coefficient of diffusion D.

$$\begin{cases} \partial_t u = D\partial_x^2 u & \text{for } 0 < x < \ell,\ t > 0 \\ u(x,0) = u_0(x) & \text{for } 0 < x < \ell,\ t = 0 \\ u(0,t) = u(\ell,t) & \text{for } t \geq 0. \end{cases}$$

FDM: forward difference

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = D\,\frac{u_{i-1}^n - 2u_i^n + u_{i+1}^n}{\Delta x^2}.$$

Thus, denoting $\lambda = D\frac{\Delta t}{\Delta x^2}$

$$u_i^{n+1} = (1-2\lambda)u_i^n + \lambda(u_{i-1}^n + u_{i+1}^n).$$

FDM for parabolic PDE

Task 2

- Implement the FDM for the parabolic PDE.
  Solve the PDE for $D = 2$, $\ell = 10$ and $u_0(x)$ given by:

  $$u_0(x) = \begin{cases} 1 & \text{if } 4 \leq x \leq 6 \\ 0 & \text{otherwise} \end{cases}$$

  Plot the solution at different times $t = 1, 2, \ldots$
- Play with different Δt, Δx.

Task 3 [extra]
Suppose we fix the boundary condition $u(0, t) = 1$ and $u(\ell, t) = 0$.
What is happening when $t \to \infty$?