Finite-Difference Method for BVP

MAT425

March 28th 2017

FDM for linear BVP

Review:

$$\begin{cases} y'' = p(x)y' + q(x)y + r(x) \\ y(a) = \alpha \ , \ y(b) = \beta. \end{cases}$$

Discretization: $\Delta x = \frac{b-a}{N+1}, \quad x_i = a + i\Delta x$

$$-(1 + \frac{p_i}{2}\Delta x)y_{i-1} + (2 + q_i\Delta x^2)y_i - (1 - \frac{p_i}{2}\Delta x)y_{i+1} = -\Delta x^2 r_i$$

Linear system:

$$AY = b, \qquad \text{with } b = \begin{pmatrix} \vdots \\ -\Delta x^2 r(x_i) \\ \vdots \end{pmatrix} + \begin{pmatrix} (1 + \frac{p(x_1)}{2}\Delta x)\alpha \\ 0 \\ (1 - \frac{p(x_N)}{2}\Delta x)\beta \end{pmatrix}$$

FDM for linear BVP

Exercises: Consider the BVP

$$\begin{cases} y'' = -\frac{1}{x+1}y' + 2y + (1-x^2)\mathrm{e}^{-x} \\ y(0) = 2\ ,\ y(1) = 2. \end{cases}$$

- Use the finite difference method to solve this problem.
  Hint: finish the routine 'FDM_linear_exo.m'
- Find the accuracy of the method.

FDM for non-linear BVP

Review:

$$\begin{cases} y'' = f(x, y, y') \\ y(a) = \alpha \ , \ y(b) = \beta. \end{cases}$$

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$$-y_{i-1} + 2y_i - y_{i+1} + \Delta x^2 f(x_i, y_i) = 0$$

Non-Linear system:

$$LY + \Delta x^2 F(x, Y) - \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix} = 0, \qquad ,$$

with $L = diag([-1, 2, -1])$, $F(x, Y) = diag(f(x_i, y_i))$.

FDM for non-linear BVP

Newton's method: Consider

$$J(Y) = LY + \Delta x^2 F(x, Y) - \begin{pmatrix} \alpha \\ 0 \\ \beta \end{pmatrix}.$$

Apply the following algorithm:

- initiate $Y_0 = [y_1 \dots y_N]'$
- update using the following code:

$$Y_{n+1} = Y_n - [DJ(Y_n)]^{-1} J(Y_n).$$

Remark: here $DJ(Y) = L + \Delta x^2 \cdot diag(\frac{\partial f}{\partial y_i}(x_i, y_i))$.

FDM for non-linear BVP

Exercises: Consider the BVP

$$\begin{cases} y'' = y^2 \\ y(0) = 1\ ,\ y(1) = 2. \end{cases}$$

- Use the finite difference method to solve this problem.
  Hint: finish the routine 'FDM_nonlinear_exo.m'
- Test different initial condition Y_0.

BVP and asymptotic expansions

Consider the linear BVP with $\varepsilon > 0$:

$$\begin{cases} \varepsilon y'' + y' = 1 \\ y(0) = y(1) = 0. \end{cases}$$

- For different ε, find the solutions denoted y_ε.
  Can you guess what is the limit solution $\lim_{\varepsilon \to 0^+} y_\varepsilon$?

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  Can you guess what is the limit solution $\lim_{\varepsilon \to 0^+} y_\varepsilon$?
- Try to 'justify' the result by passing to the limit $\varepsilon \to 0$ in the ODE.
- **Maximum principle**: show (analytically) that the solution $y_\varepsilon \leq 0$.
  Deduce what will be the solution $\lim_{\varepsilon \to 0^-} y_\varepsilon$.