

Ricci Flow on Binary Trees

Andrew Ely

February 16, 2026

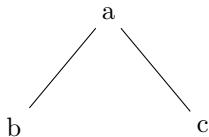
I use “the weighted graphs paper” to refer to the paper “Ollivier Ricci-Flow On Weighted Graphs” by Shuliang Bai, Yong Lin, Linyuan Lu, Zhiyu Wang, and Shing-Tung Yau.

I use “the trees paper” to refer to the paper “On The Ricci Flow On Trees” by Shuliang Bai, Bobo Hua, Yong Lin, and Shuang Liu.

The trees paper seems to provide some theorems that apply to any general tree under unnormalized flow, while under normalized flow they only prove convergence for a special class of trees called caterpillar trees. The weighted graphs paper proves convergence for star graphs, and path graphs. The theorems that apply to general trees under unnormalized flow seem to not make claims about convergence guarantees in normalized flow. So, it seems we can still look at different families of trees and try to prove convergence under normalized flow, which has not been done yet based on my understanding of the trees paper.

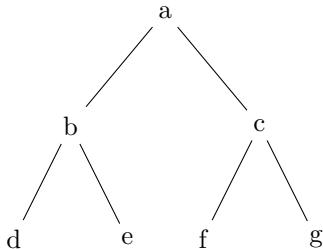
Considering complete binary trees, where l denotes the level of the tree.

For $l = 1$, our tree is:



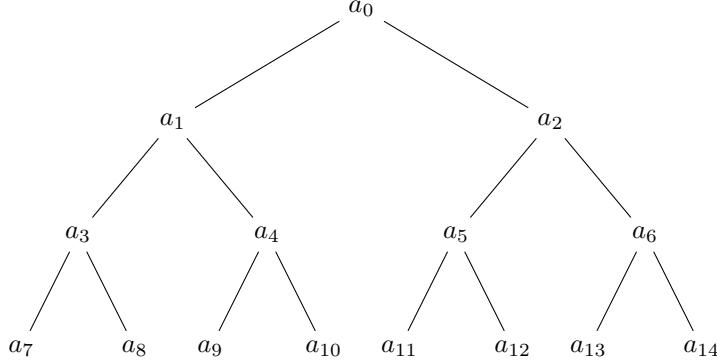
Which is a path of length two, which we know converges from the weighted graph paper.

For $l = 2$ our tree is:



Which is a caterpillar tree if we take $d - b - a - c - g$ as the spine, which we know converges from the trees paper.

For $l = 3$ our tree is:



But considering an $l = 3$ complete binary tree will involve an ODE system of 14 equations, which will be hard to solve without clever insight. So before attempting $l = 3$, even though $l = 2$ is a caterpillar tree and already solved by the trees paper, maybe we can examine the $l = 2$ tree in some kind of way that allows for generalizing when adding more levels.

Continuing with $l = 2$ based on the weighted graphs papers formulas:

$$\mu_x^\alpha(y) = \begin{cases} \alpha & \text{if } y = x \\ (1 - \alpha) \frac{\gamma(w_{xy})}{\sum_{z \sim x} \gamma(w_{xz})} & \text{if } y \sim x \\ 0 & \text{otherwise} \end{cases}$$

Gives the mass distribution for a given vertex.

$$V = \{a, b, c, d, e, f, g\}$$

$$\mu = \{\mu_a^\alpha, \mu_b^\alpha, \mu_c^\alpha, \mu_d^\alpha, \mu_e^\alpha, \mu_f^\alpha, \mu_g^\alpha\}$$

Where a is our root node, b, c are our internal nodes (symmetric), and d, e, f, g are our leaf nodes (symmetric), we get the following mass distributions:

$$\mu_a^\alpha(v) = \begin{cases} \alpha & \text{if } v = a \\ (1 - \alpha) \frac{\gamma(w_{ab})}{\gamma(w_{ab}) + \gamma(w_{ac})} & \text{if } v = b \\ (1 - \alpha) \frac{\gamma(w_{ac})}{\gamma(w_{ab}) + \gamma(w_{ac})} & \text{if } v = c \end{cases}$$

$$\mu_b^\alpha(v) = \begin{cases} \alpha & \text{if } v = b \\ (1 - \alpha) \frac{\gamma(w_{ab})}{\gamma(w_{ab}) + \gamma(w_{bd}) + \gamma(w_{be})} & \text{if } v = a \\ (1 - \alpha) \frac{\gamma(w_{bd})}{\gamma(w_{ab}) + \gamma(w_{bd}) + \gamma(w_{be})} & \text{if } v = d \\ (1 - \alpha) \frac{\gamma(w_{be})}{\gamma(w_{ab}) + \gamma(w_{bd}) + \gamma(w_{be})} & \text{if } v = e \end{cases}$$

$$\mu_c^\alpha(v) = \begin{cases} \alpha & \text{if } v = c \\ (1 - \alpha) \frac{\gamma(w_{ac})}{\gamma(w_{ac}) + \gamma(w_{cf}) + \gamma(w_{cg})} & \text{if } v = a \\ (1 - \alpha) \frac{\gamma(w_{cf})}{\gamma(w_{ac}) + \gamma(w_{cf}) + \gamma(w_{cg})} & \text{if } v = f \\ (1 - \alpha) \frac{\gamma(w_{cg})}{\gamma(w_{ac}) + \gamma(w_{cf}) + \gamma(w_{cg})} & \text{if } v = g \end{cases}$$

$$\mu_d^\alpha(v) = \begin{cases} \alpha & \text{if } v = d \\ (1 - \alpha) & \text{if } v = b \end{cases} \quad \mu_e^\alpha(v) = \begin{cases} \alpha & \text{if } v = e \\ (1 - \alpha) & \text{if } v = b \end{cases}$$

$$\mu_f^\alpha(v) = \begin{cases} \alpha & \text{if } v = f \\ (1 - \alpha) & \text{if } v = c \end{cases} \quad \mu_g^\alpha(v) = \begin{cases} \alpha & \text{if } v = g \\ (1 - \alpha) & \text{if } v = c \end{cases}$$

Our Ricci Curvature for an edge at any given time is given by the Ollivier-Lin-Lu-Yau formula:

$$k(x, y) = \inf_{\substack{f \in Lip(1) \\ \nabla_{yx} f = 1}} \nabla_{xy} \Delta f$$

$$f \in Lip(1)$$

$$\nabla_{yx} f = 1$$

Where we pick f for edge $u - v$ given by 2.1 from the trees paper:

$$f(x) = \begin{cases} w_{ux}(t) + w_{uv}(t) & \text{if } x \sim u, x \neq v \\ w_{uv}(t) & \text{if } x = u \\ 0 & \text{if } x = v \\ -w_{vy}(t) & \text{if } y \sim v, y \neq u \\ 0 & \text{otherwise} \end{cases}$$

Which guarantees Lipchitz-1 condition holds at any given time. We also need:

$$\Delta f(v) = \frac{1}{\sum_{y \sim v} \gamma(w_{vy})} \sum_{y \sim v} \gamma(w_{vy})(f(y) - f(v))$$

$$\nabla_{xy} \Delta f = \frac{\Delta f(x) - \Delta f(y)}{w_{xy}}$$

We have 6 edges, so we will need 6 curvatures k:

$$k = \{k(a, b), k(b, d), k(b, e), k(a, c), k(c, f), k(c, g)\}.$$

Let u_i denote vertex u 's $\mu_u^\alpha(v)$ term that is multiplied by $(1 - \alpha)$ when $v = i$.

for $k(b, d)$:

$$\Delta f(d) = -w_{bd}$$

$$\begin{aligned}\Delta f(b) &= \frac{\gamma(w_{ab})(-w_{ab}) + \gamma(w_{bd})(w_{bd}) + \gamma(w_{be})(-w_{be})}{\gamma(w_{ab}) + \gamma(w_{bd}) + \gamma(w_{be})} \\ &= -b_a(w_{ab}) + b_d(w_{bd}) - b_e(w_{be})\end{aligned}$$

$$\begin{aligned}\nabla_{bd}\Delta f &= \frac{-b_a(w_{ab}) + b_d(w_{bd}) - b_e(w_{be}) + w_{bd}}{w_{bd}} \\ &= 1 + b_d - \frac{b_a(w_{ab})}{w_{bd}} - \frac{b_e(w_{be})}{w_{bd}}\end{aligned}$$

The following (other 3) leaf edges are similar to $k(b, d)$:

for $k(b, e)$:

$$\begin{aligned}\nabla_{be}\Delta f &= \frac{-b_a(w_{ab}) + b_d(w_{bd}) - b_e(w_{be}) + w_{bd}}{w_{bd}} \\ &= 1 + b_e - \frac{b_a(w_{ab})}{w_{be}} - \frac{b_d(w_{bd})}{w_{be}}\end{aligned}$$

for $k(c, f)$:

$$\begin{aligned}\nabla_{cf}\Delta f &= \frac{-c_a(w_{ac}) + c_f(w_{cf}) - c_g(w_{cg}) + w_{cf}}{w_{cf}} \\ &= 1 + c_f - \frac{c_a(w_{ac})}{w_{cf}} - \frac{c_g(w_{cg})}{w_{cf}}\end{aligned}$$

for $k(c, g)$:

$$\begin{aligned}\nabla_{cg}\Delta f &= \frac{-c_a(w_{ac}) + c_f(w_{cf}) - c_g(w_{cg}) + w_{cf}}{w_{cg}} \\ &= 1 + c_g - \frac{c_a(w_{ac})}{w_{cg}} - \frac{c_f(w_{cf})}{w_{cg}}\end{aligned}$$

The inner edges $k(a, c)$ and $k(a, b)$ are different from leaf edges:

for $k(a, c)$:

$$\begin{aligned}\nabla_{ac}\Delta f &= \frac{-a_b(w_{ab}) + a_c(w_{ac}) + c_a(w_{ac}) - c_f(w_{cf}) - c_g(w_{cg})}{w_{ac}} \\ &= a_c + c_a - \frac{c_f(w_{cf})}{w_{ac}} - \frac{c_g(w_{cg})}{w_{ac}} - \frac{a_b(w_{ab})}{w_{ac}}\end{aligned}$$

for $k(a, b)$:

$$\begin{aligned}\nabla_{ab} \Delta f &= \frac{a_b(w_{ab}) - a_c(w_{ac}) + b_a(w_{ab}) - b_d(w_{bd}) - b_e(w_{be})}{w_{ab}} \\ &= a_b + b_a - \frac{a_c(w_{ac})}{w_{ab}} - \frac{b_d(w_{bd})}{w_{ab}} - \frac{b_e(w_{be})}{w_{ab}}\end{aligned}$$

Now that we have all our curvatures, equation (13) from the weighted graphs paper gives our unnormalized flow:

$$\begin{cases} \frac{dw_e}{dt} = -k_e w_e & ((13) \text{ from weighted graphs paper}) \\ X(0) = X_0 \end{cases}$$

We get the following ODEs:

$$\begin{aligned}\frac{dw_{bd}}{dt} &= -k_{bd} w_{bd} = -w_{bd} - b_d(w_{bd}) + b_a(w_{ab}) + b_e(w_{be}) \\ \frac{dw_{be}}{dt} &= -k_{be} w_{be} = -w_{be} - b_e(w_{be}) + b_a(w_{ab}) + b_d(w_{bd}) \\ \frac{dw_{cf}}{dt} &= -k_{cf} w_{cf} = -w_{cf} - c_f(w_{cf}) + c_a(w_{ac}) + c_g(w_{cg}) \\ \frac{dw_{cg}}{dt} &= -k_{cg} w_{cg} = -w_{cg} - c_g(w_{cg}) + c_a(w_{ac}) + c_f(w_{cf}) \\ \frac{dw_{ac}}{dt} &= -k_{ac} w_{ac} = -a_c(w_{ac}) - c_a(w_{ac}) + c_f(w_{cf}) + c_g(w_{cg}) + a_b(w_{ab}) \\ \frac{dw_{ab}}{dt} &= -k_{ab} w_{ab} = -a_b(w_{ab}) - b_a(w_{ab}) + a_c(w_{ac}) + b_d(w_{bd}) + b_e(w_{be})\end{aligned}$$

Equation (9) from the weighted graphs paper gives our normalized flow:

$$\begin{cases} \frac{dw_e}{dt} = -k_e w_e + w_e \sum_{h \in E(G)} k_h w_h & ((9) \text{ from weighted graphs paper}) \\ X(0) = X_0 \end{cases}$$

Where the $X(0)$ case refers to the initial edge weights.

We get the following ODEs:

The sum of all 6 curvatures multiplied by edge weights:

$$\sum_{h \in E(G)} k_h w_h = w_{bd}(1 - b_d) + w_{be}(1 - b_e) + w_{cf}(1 - c_f) + w_{cg}(1 - w_{cg}) - c_a(w_{ac}) - b_a(w_{ab})$$

$$\frac{dw_{bd}}{dt} =$$

$$\begin{aligned}-k_{bd} w_{bd} + w_{bd} \sum_{h \in E(G)} k_h w_h &= -w_{bd} - b_d(w_{bd}) + b_a(w_{ab}) + b_e(w_{be}) + w_{bd}^2(1 - b_d) \\ &\quad + w_{bd} w_{be}(1 - b_e) + w_{bd} w_{cf}(1 - c_f) + w_{bd} w_{cg}(1 - w_{cg}) \\ &\quad - w_{bd} c_a(w_{ac}) - w_{bd} b_a(w_{ab})\end{aligned}$$

$$\frac{dw_{be}}{dt} =$$

$$\begin{aligned} -k_{be}w_{be} + w_{be} \sum_{h \in E(G)} k_h w_h &= -w_{be} - b_e(w_{be}) + b_a(w_{ab}) + b_d(w_{bd}) + w_{be}w_{bd}(1 - b_d) \\ &\quad + w_{be}^2(1 - b_e) + w_{be}w_{cf}(1 - c_f) + w_{be}w_{cg}(1 - w_{cg}) \\ &\quad - w_{be}c_a(w_{ac}) - w_{be}b_a(w_{ab}) \end{aligned}$$

$$\frac{dw_{cf}}{dt} =$$

$$\begin{aligned} -k_{cf}w_{cf} + w_{cf} \sum_{h \in E(G)} k_h w_h &= -w_{cf} - c_f(w_{cf}) + c_a(w_{ac}) + c_g(w_{cg}) + w_{cf}w_{bd}(1 - b_d) \\ &\quad + w_{cf}w_{be}(1 - b_e) + w_{cf}^2(1 - c_f) + w_{cf}w_{cg}(1 - w_{cg}) \\ &\quad - w_{cf}c_a(w_{ac}) - w_{cf}b_a(w_{ab}) \end{aligned}$$

$$\frac{dw_{cg}}{dt} =$$

$$\begin{aligned} -k_{cg}w_{cg} + w_{cg} \sum_{h \in E(G)} k_h w_h &= -w_{cg} - c_g(w_{cg}) + c_a(w_{ac}) + c_f(w_{cf}) + w_{cg}w_{bd}(1 - b_d) \\ &\quad + w_{cg}w_{be}(1 - b_e) + w_{cg}w_{cf}(1 - c_f) + w_{cg}^2(1 - w_{cg}) \\ &\quad - w_{cg}c_a(w_{ac}) - w_{cg}b_a(w_{ab}) \end{aligned}$$

$$\frac{dw_{ac}}{dt} =$$

$$\begin{aligned} -k_{ac}w_{ac} + w_{ac} \sum_{h \in E(G)} k_h w_h &= -a_c(w_{ac}) - c_a(w_{ac}) + c_f(w_{cf}) + c_g(w_{cg}) + a_b(w_{ab}) \\ &\quad + w_{ac}w_{bd}(1 - b_d) + w_{ac}w_{be}(1 - b_e) + w_{ac}w_{cf}(1 - c_f) \\ &\quad + w_{ac}w_{cg}(1 - w_{cg}) - c_a(w_{ac}^2) - b_a(w_{ab})(w_{ac}) \end{aligned}$$

$$\frac{dw_{ab}}{dt} =$$

$$\begin{aligned} -k_{ab}w_{ab} + w_{ab} \sum_{h \in E(G)} k_h w_h &= -a_b(w_{ab}) - b_a(w_{ab}) + a_c(w_{ac}) + b_d(w_{bd}) + b_e(w_{be}) \\ &\quad + w_{ab}w_{bd}(1 - b_d) + w_{ab}w_{be}(1 - b_e) + w_{ab}w_{cf}(1 - c_f) \\ &\quad + w_{ab}w_{cg}(1 - w_{cg}) - c_a(w_{ac})(w_{ab}) - b_a(w_{ab}^2) \end{aligned}$$

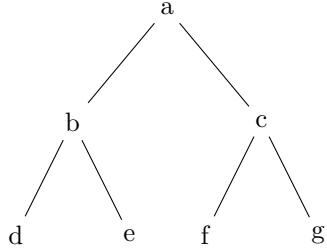
Neither the normalized or unnormalized ODEs seem reasonable to solve directly. But we know from the trees paper that for general graphs, convergence is guaranteed under unnormalized flow (Theorem 1), with the caveat that an internal edge may ‘converge’ to infinity. Also, since the complete binary tree of level 2 is a caterpillar tree, it is also guaranteed to converge to a metric with constant curvature zero under normalized flow (Theorem 2). So we can’t find the solution to the ODEs directly, and we already know the results anyway. Advancing to a level 3 complete binary

tree will only result in much harder ODEs to solve, and since we are stuck trying to find the direct solution for level 2 binary tree, it is unlikely that this approach will scale to the level 3 complete binary tree, or that a generalization to general complete binary trees will fall out naturally.

However, there are formulas in the trees paper that actually seem to simplify the long $k(u, v)$ and ODEs derived, which may make the complete level 2 binary tree actually easy to solve. If this is the case, then advancing to level 3 complete binary tree may also be easy to solve. (section 2.2, equation (5), critically). It is worth revisiting complete binary trees, and even when moving onto non-complete binary trees, we should utilize more of the trees papers' formulas, not just the direct naive expansions like we did above.

Reconsidering $l = 2$ using the trees papers approach.

Our tree is:



We know the following gives an optimal Lipschitz function for our curvatures.

$$f(x) = \begin{cases} w_{ux}(t) + w_{uv}(t) & \text{if } x \sim u, x \neq v \\ w_{uv}(t) & \text{if } x = u \\ 0 & \text{if } x = v \\ -w_{vy}(t) & \text{if } y \sim v, y \neq u \\ 0 & \text{otherwise} \end{cases}$$

We also have for any vertex u , let

$$D_u = \sum_{u \sim v} \gamma(w_{uv})$$

Where we take $\gamma(x) = \frac{1}{x}$ as recommended throughout the papers.

$$D_a = \frac{1}{w_{ab}} + \frac{1}{w_{ac}} \quad D_b = \frac{1}{w_{ab}} + \frac{1}{w_{bd}} + \frac{1}{w_{be}} \quad D_c = \frac{1}{w_{ca}} + \frac{1}{w_{cf}} + \frac{1}{w_{cg}}$$

$$D_d = \frac{1}{w_{bd}} \quad D_e = \frac{1}{w_{be}} \quad D_f = \frac{1}{w_{cf}} \quad D_g = \frac{1}{w_{cg}}$$

The curvature on edge uv can be calculated as follows:

$$k_{uv} = \frac{2-d_u}{w_{uv}D_u} + \frac{2-d_v}{w_{uv}D_v}$$

$$k_{ab} = -\frac{1}{w_{ab}D_b} \quad k_{ac} = -\frac{1}{w_{ac}D_c} \quad k_{bd} = 1 - \frac{1}{w_{bd}D_b}$$

$$k_{be} = 1 - \frac{1}{w_{be}D_b} \quad k_{cf} = 1 - \frac{1}{w_{cf}D_c} \quad k_{cg} = 1 - \frac{1}{w_{cg}D_c}$$

Unnormalized flow is given by:

$$\begin{cases} \frac{dw_e}{dt} = -k_e w_e \\ X(0) = X_0 \end{cases} \quad ((13) \text{ from weighted graphs paper})$$

$$\frac{dw_{ab}}{dt} = D_b \quad \frac{dw_{ac}}{dt} = D_c \quad \frac{dw_{bd}}{dt} = D_b - w_{ab}$$

$$\frac{dw_{be}}{dt} = D_b - w_{be} \quad \frac{dw_{cf}}{dt} = D_c - w_{cf} \quad \frac{dw_{cg}}{dt} = D_c - w_{cg}$$

Normalized flow is given by:

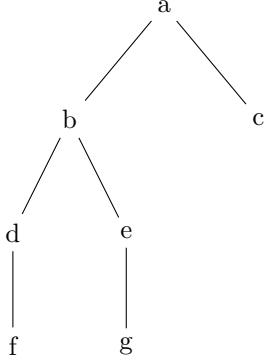
$$\begin{cases} \frac{dw_e}{dt} = -k_e w_e + w_e \sum_{h \in E(G)} k_h w_h \\ X(0) = X_0 \end{cases} \quad ((9) \text{ from weighted graphs paper})$$

We know:

$$\begin{aligned} \sum_{h \in E(G)} k_h w_h &= \sum_{u \in V(G)} \frac{(2-d_u)d_u}{D_u} \\ \sum_{h \in E(G)} k_h w_h &= w_{bd} + w_{be} + w_{cf} + w_{cg} - \frac{3}{D_b} - \frac{3}{D_c} \\ \frac{dw_{ab}}{dt} &= D_b + w_{ab}(w_{bd} + w_{be} + w_{cf} + w_{cg} - \frac{3}{D_b} - \frac{3}{D_c}) \\ \frac{dw_{ac}}{dt} &= D_c + w_{ac}(w_{bd} + w_{be} + w_{cf} + w_{cg} - \frac{3}{D_b} - \frac{3}{D_c}) \\ \frac{dw_{bd}}{dt} &= D_b - w_{ab} + w_{bd}(w_{bd} + w_{be} + w_{cf} + w_{cg} - \frac{3}{D_b} - \frac{3}{D_c}) \\ \frac{dw_{be}}{dt} &= D_b - w_{be} + w_{be}(w_{bd} + w_{be} + w_{cf} + w_{cg} - \frac{3}{D_b} - \frac{3}{D_c}) \\ \frac{dw_{cf}}{dt} &= D_c - w_{cf} + w_{cf}(w_{bd} + w_{be} + w_{cf} + w_{cg} - \frac{3}{D_b} - \frac{3}{D_c}) \\ \frac{dw_{cg}}{dt} &= D_c - w_{cg} + w_{cg}(w_{bd} + w_{be} + w_{cf} + w_{cg} - \frac{3}{D_b} - \frac{3}{D_c}) \end{aligned}$$

*We should find a solution to the unnormalized ODEs and verify that it agrees with the Trees Paper. We should also find a solution to the normalized ODEs and verify that it agrees with the Trees paper. We should also be able to verify the relationship suspected by equation (3) in the Trees Paper between the **limit** of the unnormalized flow and normalized flow.*

Considering non-complete binary trees, the simplest one that is not a path, caterpillar, or star graph seems to be:



This seems like it might be no harder than our complete binary tree of level 2, because we still have 6 edges and 7 vertices. We will call this above tree S . But we will lose symmetry so that might cause challenges. But it seems to be the first "novel" graph outside of the domain of the papers. So even though we were stuck on the ODEs for the level 2 complete binary tree, solving the ODEs for this graph S may be worth it because it will give actual new results. Importantly, from the trees paper, Theorem 1 says that this graph will converge under unnormalized flow, but there is no claim for its convergence under normalized flow.

The best direction seems to be to answer:

Under **normalized** Ricci Flow, does S converge to a constant metric.

First we consider unnormalized flow, which we can normalize after solving the ODE system.

The D_u variables are:

$$\begin{aligned} D_a &= \frac{1}{w_{ab}} + \frac{1}{w_{ac}} & D_b &= \frac{1}{w_{ab}} + \frac{1}{w_{bd}} + \frac{1}{w_{be}} & D_c &= \frac{1}{w_{ac}} \\ D_d &= \frac{1}{w_{bd}} + \frac{1}{w_{df}} & D_e &= \frac{1}{w_{be}} + \frac{1}{w_{eg}} & D_f &= \frac{1}{w_{df}} & D_g &= \frac{1}{w_{eg}} \end{aligned}$$

The curvatures on edges are:

$$\begin{aligned} k_{ab} &= -\frac{1}{w_{ab}D_b} & k_{bd} &= -\frac{1}{w_{bd}D_b} & k_{be} &= -\frac{1}{w_{be}D_b} \\ k_{ac} &= 1 & k_{df} &= 1 & k_{eg} &= 1 \end{aligned}$$

Unnormalized flow is given by:

$$\begin{cases} \frac{dw_e}{dt} = -k_e w_e \\ X(0) = X_0 \end{cases} \quad ((13) \text{ from weighted graphs paper})$$

$$\begin{aligned}\frac{dw_{ab}}{dt} &= \frac{1}{D_b} & \frac{dw_{bd}}{dt} &= \frac{1}{D_b} & \frac{dw_{be}}{dt} &= \frac{1}{D_b} \\ \frac{dw_{ac}}{dt} &= -w_{ac} & \frac{dw_{df}}{dt} &= -w_{df} & \frac{dw_{eg}}{dt} &= -w_{eg}\end{aligned}$$

$$w_{ac}(t) = w_{ac}(0)e^{-t} \quad w_{df}(t) = w_{df}(0)e^{-t} \quad w_{eg}(t) = w_{eg}(0)e^{-t}$$

Which means these leaf edges all converge to 0.

$$\text{For } \frac{dw_{ab}}{dt} = \frac{dw_{bd}}{dt} = \frac{dw_{be}}{dt} = \frac{1}{D_b} = \frac{1}{\frac{1}{w_{ab}} + \frac{1}{w_{bd}} + \frac{1}{w_{be}}}$$

Assuming $w_{ab}(t) = w_{bd}(t) = w_{be}(t)$ based on symmetry,

$$\frac{dw_{be}}{dt} = \frac{1}{3}w_{be}$$

yields

$$w_{be}(t) = w_{be}(0)e^{t/3}$$

So,

$$w_{ab}(t) = w_{ab}(0)e^{t/3} \quad w_{bd}(t) = w_{bd}(0)e^{t/3} \quad w_{be}(t) = w_{be}(0)e^{t/3}$$

Which means the internal edge weights grow exponentially to infinity. Which agrees with the trees papers Theorem 1, and more importantly Lemma 8, which says there must be at least one internal edge that grows to infinity under unnormalized flow.

Normalized flow is given by:

$$\begin{cases} \frac{dw_e}{dt} = -k_e w_e + w_e \sum_{h \in E(G)} k_h w_h \\ X(0) = X_0 \end{cases} \quad ((9) \text{ from weighted graphs paper})$$

$$\begin{aligned}\sum_{h \in E(G)} k_h w_h &= w_{ac} + w_{df} + w_{eg} - \frac{3}{D_b} \\ \frac{dw_{ab}}{dt} &= \frac{1}{D_b} + w_{ab}(w_{ac} + w_{df} + w_{eg} - \frac{3}{D_b}) & \frac{dw_{bd}}{dt} &= \frac{1}{D_b} + w_{bd}(w_{ac} + w_{df} + w_{eg} - \frac{3}{D_b}) \\ \frac{dw_{be}}{dt} &= \frac{1}{D_b} + w_{be}(w_{ac} + w_{df} + w_{eg} - \frac{3}{D_b}) & \frac{dw_{ac}}{dt} &= -w_{ac} + w_{ac}(w_{ac} + w_{df} + w_{eg} - \frac{3}{D_b}) \\ \frac{dw_{df}}{dt} &= -w_{df} + w_{df}(w_{ac} + w_{df} + w_{eg} - \frac{3}{D_b}) & \frac{dw_{eg}}{dt} &= -w_{eg} + w_{eg}(w_{ac} + w_{df} + w_{eg} - \frac{3}{D_b})\end{aligned}$$

Alternatively, we can use the known relation between unnormalized and normalized weights:

$$w_e(t) = \frac{\hat{w}_e(t)}{\sum_{h \in E} \hat{w}_h(t)} \quad \hat{w} := \text{unnormalized edge weight}, w := \text{normalized edge weight}.$$

$$\hat{w}_{ac}(t) = w_{ac}(0)e^{-t} \quad \hat{w}_{df}(t) = w_{df}(0)e^{-t} \quad \hat{w}_{eg}(t) = w_{eg}(0)e^{-t}$$

$$\hat{w}_{ab}(t) = w_{ab}(0)e^{t/3} \quad \hat{w}_{bd}(t) = w_{bd}(0)e^{t/3} \quad \hat{w}_{be}(t) = w_{be}(0)e^{t/3}$$

$$w_{ac}(t) = \frac{w_{ac}(0)e^{-t}}{w_{ac}(0)e^{-t} + w_{df}(0)e^{-t} + w_{eg}(0)e^{-t} + w_{ab}(0)e^{t/3} + w_{bd}(0)e^{t/3} + w_{be}(0)e^{t/3}} \rightarrow 0$$

$$w_{ab}(t) = \frac{w_{ab}(0)e^{t/3}}{w_{ac}(0)e^{-t} + w_{df}(0)e^{-t} + w_{eg}(0)e^{-t} + w_{ab}(0)e^{t/3} + w_{bd}(0)e^{t/3} + w_{be}(0)e^{t/3}} \rightarrow \frac{1}{3}$$

So, under normalized flow the leaf edges w_{ac}, w_{df}, w_{eg} converge to 0, and the internal edges w_{ab}, w_{bd}, w_{be} converge to $\frac{1}{3}$.

Which suggests that under normalized Ricci Flow, tree S converges to a constant metric tree of 0 curvature.

However, there are doubts that we can just use equation 3 to go from unnormalized to normalized this way. So, we can try plugging in the 0 and $\frac{1}{3}$ values into our ODE system for normalized flow and see if it solves the system.

Let $\hat{w}_e(t)$ be the unnormalized edge weight of edge e at time t .

Let $S(t) = \sum_{h \in E(G)} \hat{w}_h(t)$ be the sum of all unnormalized edge weights at time t .

The unnormalized flow is given by:

$$\frac{d\hat{w}_e}{dt} - k_e \hat{w}_e$$

Our conjectured normalized edge weights were given by: $w_e(t) = \frac{\hat{w}_e(t)}{S(t)}$

Differentiating $w_e = \frac{\hat{w}_e}{S}$ we get:

$$\frac{dw_e}{dt} = \frac{\hat{w}'_e S - \hat{w}_e S'}{S^2}$$

And since for unnormalized flow, $\hat{w}'_e = -k_e \hat{w}_e$ we get:

$$S' = \sum_h \hat{w}'_h = \sum_h (-k_h \hat{w}_h) = - \sum_h k_h \hat{w}_h$$

So, plugging these values in, we get:

$$\frac{dw_e}{dt} = \frac{(-k_e \hat{w}_e)S - \hat{w}_e(-\sum_h k_h \hat{w}_h)}{S^2} = \frac{-k_e \hat{w}_e S + \hat{w}_e \sum_h k_h \hat{w}_h}{S^2} = \hat{w}_e \cdot \frac{-k_e S + \sum_h k_h \hat{w}_h}{S^2}$$

But $\hat{w}_e = w_e S$ and $\hat{w}_h = w_h S$ so,

$$\sum_h k_h \hat{w}_h = S \sum_h k_h w_h$$

So, plugging these values in we get:

$$\frac{dw_e}{dt} = \hat{w}_e \cdot \frac{-k_e S + \sum_h k_h \hat{w}_h}{S^2} = w_e S \cdot \frac{-k_e S + S \sum_h k_h w_h}{S^2} = w_e (-k_e + \sum_h k_h w_h) = -k_e w_e + w_e \sum_{h \in E(G)} k_h w_h$$

Which is exactly our normalized flow ODE. Therefore, if for any unnormalized edge weight, $\hat{w}_e(t)$ solves the unnormalized ODE, then the normalized edge weight, $w_e(t)$ solves the normalized ODE.

To see this more clearly, we will show directly for our specific conjectured formulas:

$$\hat{w}_{ac}(t) = w_{ac}(0)e^{-t} \quad w_{ac}(t) = \frac{w_{ac}(0)e^{-t}}{w_{ac}(0)e^{-t} + w_{df}(0)e^{-t} + w_{eg}(0)e^{-t} + w_{ab}(0)e^{t/3} + w_{bd}(0)e^{t/3} + w_{be}(0)e^{t/3}}$$

$$\hat{w}_{ab}(t) = w_{ab}(0)e^{t/3} \quad w_{ab}(t) = \frac{w_{ab}(0)e^{t/3}}{w_{ac}(0)e^{-t} + w_{df}(0)e^{-t} + w_{eg}(0)e^{-t} + w_{ab}(0)e^{t/3} + w_{bd}(0)e^{t/3} + w_{be}(0)e^{t/3}}$$

Where the other edge and other leaves are symmetrical. Note that there is a potential notation confusion that the unnormalized edge weights are defined as w hats with no hat w meaning initial edge weight, and then we also use no hat w notation for defining normalized edge weight, but they do not mean the same thing.

Let $S(t) = \hat{w}_{ac} + \hat{w}_{df} + \hat{w}_{eg} + \hat{w}_{ab} + \hat{w}_{bd} + \hat{w}_{be}$

Let $A = w_{ac}(0) + w_{df}(0) + w_{eg}(0)$

Let $B = w_{ab}(0) + w_{bd}(0) + w_{be}(0)$

Then $S(t) = Ae^{-t} + Be^{t/3}$

Now we introduce normalized edge weights:

$$w_e(t) = \frac{\hat{w}_e(t)}{S(t)}$$

So using our S ,

$$w_{ac}(t) = \frac{w_{ac}(0)e^{-t}}{Ae^{-t} + Be^{t/3}} \quad w_{ab}(t) = \frac{w_{ab}(0)e^{t/3}}{Ae^{-t} + Be^{t/3}}$$

Differentiating $w_{ac}(t)$:

The normalized weight is:

$$w_{ac}(t) = \frac{\hat{w}_{ac}(t)}{S(t)}$$

Applying the quotient rule gives

$$\frac{dw_{ac}}{dt} = \frac{\hat{w}'_{ac} S - \hat{w}_{ac} S'}{S^2}$$

For the unnormalized flow,

$$\hat{w}'_{ac} = -\hat{w}_{ac}$$

And we compute $S'(t)$ as,

$$S'(t) = -Ae^{-t} + \frac{1}{3}Be^{t/3}$$

Plugging into $\frac{dw_{ac}}{dt}$ we get,

$$\frac{dw_{ac}}{dt} = \frac{\hat{w}'_{ac} S - \hat{w}_{ac} S'}{S^2} = \frac{(-\hat{w}_{ac}) S - \hat{w}_{ac} S'}{S^2} = -\frac{\hat{w}_{ac}}{S} + \hat{w}_{ac} \left(-\frac{S'}{S^2} \right) = -w_{ac} + w_{ac} \left(-\frac{S'}{S} \right)$$

Rewriting $-S'/S$ in terms of normalized weights,

$$-\frac{S'}{S} = \frac{Ae^{-t} - \frac{1}{3}Be^{t/3}}{Ae^{-t} + Be^{t/3}}$$

Since

$$w_{ac} + w_{df} + w_{eg} = \frac{Ae^{-t}}{S} \quad w_{ab} + w_{bd} + w_{be} = \frac{Be^{t/3}}{S}$$

We get:

$$-\frac{S'}{S} = (w_{ac} + w_{df} + w_{eg}) - \frac{1}{3}(w_{ab} + w_{bd} + w_{be})$$

And since

$$D_b = \frac{1}{w_{ab}} + \frac{1}{w_{bd}} + \frac{1}{w_{be}}$$

And

$$\sum_{h \in E(G)} k_h w_h = w_{ac} + w_{df} + w_{eg} - \frac{3}{D_b}$$

Then,

$$-\frac{S'}{S} = \sum_h k_h w_h$$

Plugging this into our earlier expression:

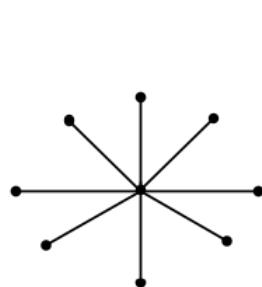
$$\frac{dw_{ac}}{dt} = -w_{ac} + w_{ac} \sum_h k_h w_h$$

Since $k_{ac} = 1$, this is exactly

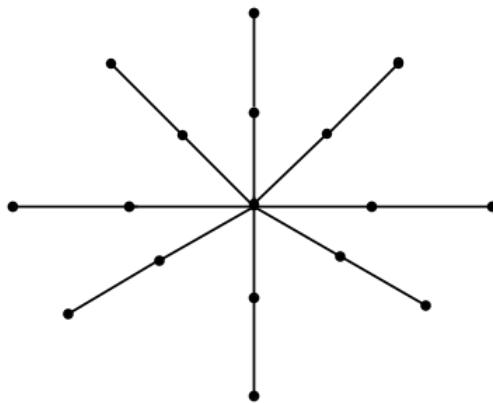
$$\frac{dw_{ac}}{dt} = -k_{ac} w_{ac} + w_{ac} \sum_h k_h w_h$$

which is exactly the normalized flow ODE for the edge w_{ac} .

The specific graph above, besides being a binary tree, it also has the form of a “level 2” star graph. The weighted graphs paper proved convergence for a star graph, which is a graph with one central node and all other nodes connected only to that central node. Considering a level 2 star graph as just a standard star graph except all non-central nodes have exactly one additional leaf nodes, we are looking at graphs like this:



Star Graph



Level 2 Star Graph

Convergence of Level 2 star graph.

Let $T = (V, E)$ be a level 2 star graph, which is a tree with one center vertex such that:

For all vertices $v \in V$,

If v is the center vertex, then v is adjacent to exactly $\frac{|V|-1}{2}$ internal vertices.

Let v_c refer to the one center vertex.

if v is an internal vertex, then v is adjacent to the center vertex, and exactly one leaf vertex.

Let v_i refer to an internal vertex.

if v is a leaf vertex, the v is adjacent to exactly one internal vertex.

Let v_j refer to a leaf vertex.

Let d_u denote the degree of vertex u .

For vertices $u, v \in V$, we write $u \sim v$ if $\{u, v\} \in E$.

Let $\hat{w}_{uv}(t)$ denote the unnormalized edge weight between vertex u and v at time t .

Let $\frac{|V|-1}{2} = m$ denote the number of internal vertices.

We also have for any vertex u , let

$$D_u = \sum_{u \sim v} \gamma(\hat{w}_{uv}(t)) \quad \text{and} \quad \gamma(x) = \frac{1}{x}$$

Since the center vertex v_c is adjacent to all the internal v_i vertices,

$$D_c = \sum_{i=1}^m \frac{1}{\hat{w}_{ci}(t)}$$

Since the internal vertices v_i are adjacent to the center vertex v_c and exactly one leaf vertex v_j ,

$$D_i = \frac{1}{\hat{w}_{ci}(t)} + \frac{1}{\hat{w}_{ij}(t)}$$

where $\hat{w}_{ij}(t)$ is the edge between the internal vertex v_i and its leaf node v_j .

Since leaf vertices v_j are adjacent to only a single internal vertex v_i ,

$$D_j = \frac{1}{\hat{w}_{ij}(t)}$$

where $\hat{w}_{ij}(t)$ is the edge between the leaf node v_j and its internal vertex v_i .

The curvature on edge uv can be calculated as follows:

$$k_{uv} = \frac{2 - d_u}{\hat{w}_{uv}(t)D_u} + \frac{2 - d_v}{\hat{w}_{uv}(t)D_v}$$

For edges ci between internal vertices and the center vertex,

Since $d_i = 2$ and $d_c = \frac{|V|-1}{2} = m$

$$k_{ic}(t) = \frac{2 - d_i}{\hat{w}_{ci}(t)D_i} + \frac{2 - d_c}{\hat{w}_{ci}(t)D_c} = \frac{2 - d_c}{\hat{w}_{ci}(t)D_c} = \frac{2 - m}{\hat{w}_{ci}(t)D_c}$$

For edges ij between internal vertices and leaf vertices,

Since $d_i = 2$ and $d_j = 1$ and $D_j = \frac{1}{\hat{w}_{ij}(t)}$

$$k_{ij}(t) = \frac{2 - d_i}{\hat{w}_{ij}(t)D_i} + \frac{2 - d_j}{\hat{w}_{ij}(t)D_j} = \frac{1}{\hat{w}_{ij}(t)D_j} = 1$$

Unnormalized flow is given by:

$$\frac{dw_{uv}}{dt} = -k_{uv}w_{uv} \quad (13) \text{ from weighted graphs paper}$$

For edges ie between internal vertices and leaf vertices,

$$\frac{d\hat{w}_{ij}}{dt} = -k_{ij}\hat{w}_{ij} = -1\hat{w}_{ij} = -\hat{w}_{ij}$$

Since the ij equations only depend on their own \hat{w}_{ij} term, for all ij edges we get:

$$\hat{w}_{ij}(t) = \hat{w}_{ij}(0)e^{-t}$$

For edges ci between internal vertices and the center vertex,

$$\frac{d\hat{w}_{ci}}{dt} = -k_{ci}\hat{w}_{ci} = -\frac{2-m}{\hat{w}_{ci}D_c}\hat{w}_{ci} = \frac{m-2}{D_c}$$

Since each \hat{w}_{ci} has the same derivative,

$$\frac{d\hat{w}_{ci}}{dt} - \frac{d\hat{w}_{cn}}{dt} = 0 \quad \text{for all internal vertices } n \neq i.$$

So, the pairwise difference between internal edge weights is constant for all time t ,

$$\hat{w}_{ci}(t) - \hat{w}_{cn}(t) = \hat{w}_{ci}(0) - \hat{w}_{cn}(0)$$

for some single scalar function $u(t)$ and constants $\Delta_i = \hat{w}_{ci}(0) - \hat{w}_{cn}(0)$ determined by the initial metric, let

$$\hat{w}_{ci}(t) = u(t) + \Delta_i$$

Define $u(t)$ using any one internal vertex, say $i = 1$,

$$u(t) := \hat{w}_{c1}(t).$$

Then for any internal vertex i ,

$$\hat{w}_{ci}(t) = u(t) + \Delta_i$$

Now rewrite D_c as

$$D_c = \sum_i^m \frac{1}{\hat{w}_{ci}(t)} = \sum_i^m \frac{1}{u(t) + \Delta_i}$$

Now rewrite $\frac{d\hat{w}_{c1}}{dt}$ as

$$\frac{d\hat{w}_{c1}}{dt} = \frac{du}{dt} = \frac{m-2}{D_c(u)} = \frac{m-2}{\sum_i^m \frac{1}{u+\Delta_i}}$$

Since this is a single scalar ODE in $u(t)$, we can write:

$$dt = \frac{1}{m-2} \left(\sum_i^m \frac{1}{u + \Delta_i} \right) du$$

Integrating:

$$\int dt = \frac{1}{m-2} \sum_i^m \int \frac{1}{u + \Delta_i} du$$

So,

$$t + C = \frac{1}{m-2} \sum_i^m \log |u + \Delta_i|$$

And since $u(0) = \hat{w}_{c1}(0)$, using $t = 0$ to fix C :

$$\sum_i^m \log(u(t) + \Delta_i) = \sum_i^m \log(u(0) + \Delta_i) + (m-2)t$$

Exponentiate:

$$\prod_i^m (u(t) + \Delta_i) = \left[\prod_i^m (u(0) + \Delta_i) \right] e^{(m-2)t}$$

Since $u(t) + \Delta_i = \hat{w}_{ci}(t)$ and $u(0) + \Delta_i = \hat{w}_{ci}(0)$,

$$\prod_i^m \hat{w}_{ci}(t) = \left[\prod_i^m \hat{w}_{ci}(0) \right] e^{(m-2)t}$$

If the internal edges are initially the same weights, they will always remain the same weights, then all $\Delta_i = 0$, and $w_{ci}(t) = w_{cn}(t) = w_{ci}(0)$ for all internal vertices $n \neq i$, then for m internal vertices:

$$D_c(t) = \sum_i^m \frac{1}{w_{ci}(t)} = \frac{m}{w_{ci}(t)}$$

And the ODE becomes:

$$\frac{d\hat{w}_{ci}}{dt} = \frac{m-2}{D_c} = \frac{m-2}{m} \hat{w}_{ci}$$

So,

$$\hat{w}_{ci} = \hat{w}_{ci}(0) e^{\frac{m-2}{m} t}$$

Unnormalized flow to normalized flow

let $w_{uv}(t)$ denote the normalized edge weight of edge uv at time t .

Using the relation between normalized and unnormalized weights:

$$w_{uv}(t) = \frac{\hat{w}_{uv}(t)}{\sum_{u,v \in E} \hat{w}_{uv}(t)}$$

Let S be the sum of all unnormalized edge weights,

$$S := \sum_{u,v \in E} \hat{w}_{uv}(t)$$

Since for all internal-leaf edges we have:

$$\hat{w}_{ij}(t) = \hat{w}_{ij}(0) e^{-t}$$

and for all center-internal edges we have:

$$\hat{w}_{ci} = \hat{w}_{ci}(0) e^{\frac{m-2}{m} t}$$

Then we can rewrite S as:

$$S = \left[\sum_i^m \hat{w}_{ij}(0)e^{-t} \right] + \left[\sum_i^m \hat{w}_{ci}(0)e^{\frac{m-2}{m}t} \right]$$

And since we assumed all internal-center edges are equal,

$$\left[\sum_i^m \hat{w}_{ci}(0)e^{\frac{m-2}{m}t} \right] = m(\hat{w}_{ci}(0)e^{\frac{m-2}{m}t})$$

So we can rewrite S,

$$S = \left[\sum_i^m \hat{w}_{ij}(0)e^{-t} \right] + m(\hat{w}_{ci}(0)e^{\frac{m-2}{m}t})$$

Our normalized flow w_{ij} for edges between leafs and internal vertices is:

$$w_{ij}(t) = \frac{\hat{w}_{ij}}{S} = \frac{\hat{w}_{ij}(0)e^{-t}}{S}$$

And since the numerator goes to 0 as $t \rightarrow \infty$,

$$w_{ij} \rightarrow 0$$

The normalized internal-leaf edges converge to 0.

Our normalized flow w_{ci} for edges between internal vertices and the center vertex is:

$$w_{ci}(t) = \frac{\hat{w}_{ci}}{S} = \frac{\hat{w}_{ci}(0)e^{\frac{m-2}{m}t}}{S} = \frac{\hat{w}_{ci}(0)e^{\frac{m-2}{m}t}}{\left[\sum_i^m \hat{w}_{ij}(0)e^{-t} \right] + m(\hat{w}_{ci}(0)e^{\frac{m-2}{m}t})}$$

And since $\left[\sum_i^m \hat{w}_{ij}(0)e^{-t} \right] \rightarrow 0$ as $t \rightarrow \infty$,

$$w_{ci} \rightarrow \frac{1}{m}$$

The normalized internal-center edges converge to one divided by the number of internal vertices.

Therefore, under normalized Ricci Flow, any level-2 star graph with initially equal internal-center edges converges to a star graph with all leaf edges equal to 0 and internal-center edges equal to one divided by the number of internal-center edges. The curvature is constant since all the positive weight edges are the same.

For non-equal initial internal-center edges

The sum of all unmoralized edge weights $S(t)$ is:

$$S(t) = \sum_i^m \hat{w}_{ij}(0)e^{-t} + \sum_i^m (u(t) + \Delta_i) = \sum_i^m \hat{w}_{ij}(0)e^{-t} + mu(t) + \sum_i \Delta_i$$

and since as $t \rightarrow \infty$

$$\sum_i^m \hat{w}_{ij}(0)e^{-t} \rightarrow 0$$

and $\sum_i \Delta_i$ is a constant as $t \rightarrow \infty$,

$$S(t) \rightarrow mu(t)$$

leading to our normalized center-internal edges:

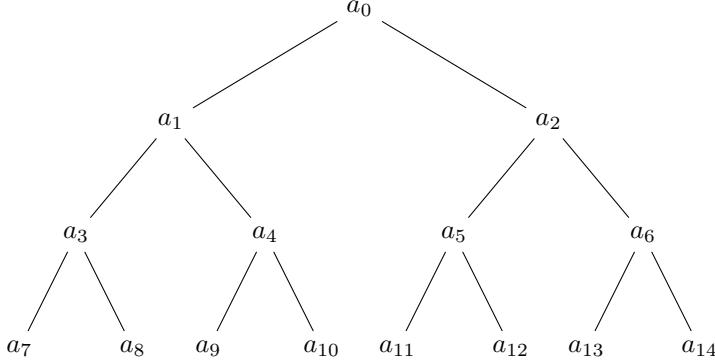
$$w_{ci}(t) = \frac{\hat{w}_{ci}(t)}{S(t)} = \frac{u(t) + \Delta_i}{mu(t) + \sum_i \Delta_i + a} \quad \text{for constant } a.$$

Divide numerator and denominator by $u(t)$:

$$w_{ci}(t) = \frac{1 + \Delta_i/u(t)}{m + (\sum_i \Delta_i)/u(t)} \rightarrow \frac{1}{m}$$

Therefore, under normalized Ricci Flow, any level-2 star graph with m branches converges to a star graph where leaf edges are weight 0 and internal edges are weight $\frac{1}{m}$, resulting in constant negative curvature of $\frac{2-m}{m}$.

Considering complete binary tree level 3



For $n \in \{0, 1, 2, 3, \dots, 14\}$

Let $w_{an,L}$ refer to the edge between node an and its left child

Let $w_{an,R}$ refer to the edge between node an and its right child

Let $w_{an,P}$ refer to the edge between node an and its parent

For any vertex u , let

$$D_u = \sum_{u \sim v} \gamma(w_{uv}) \quad \gamma(x) = \frac{1}{x}$$

for our root node a_0 ,

$$D_{a0} = \frac{1}{w_{a0,L}} + \frac{1}{w_{a0,R}}$$

for our leaf nodes denoted by aj for $j \in \{7, 8, 9, 10, 11, 12, 13, 14\}$,

$$D_{aj} = \frac{1}{w_{aj,P}}$$

for our internal degree 3 nodes denoted by ai for $i \in \{1, 2, 3, 4, 5, 6\}$

$$D_{ai} = \frac{1}{w_{ai,L}} + \frac{1}{w_{ai,R}} + \frac{1}{w_{ai,P}}$$

The curvature on edge uv can be calculated as follows:

$$k_{uv} = \frac{2 - d_u}{w_{uv}D_u} + \frac{2 - d_v}{w_{uv}D_v}$$

for our root node $a0$'s edges,

$$k_{a0,a1} = \frac{-1}{w_{a0,a1}D_{a1}}$$

for our leaf nodes aj edges,

$$k_{a3,a7} = \frac{-1}{w_{a3,a7}D_{a3}} + \frac{1}{w_{a3,a7}D_{a7}} = \frac{-1}{w_{a3,a7}D_{a3}} + \frac{1}{w_{a3,a7}\left(\frac{1}{w_{a3,a7}}\right)} = \frac{-1}{w_{a3,a7}D_{a3}} + 1$$

for our edges between two internal a_i nodes,

$$k_{a1,a3} = \frac{-1}{w_{a1,a3}D_{a1}} + \frac{-1}{w_{a1,a3}D_{a3}}$$

Where all other edges fall into one of the three above types, symmetrically.

Unnormalized flow is given by:

$$\begin{cases} \frac{dw_e}{dt} = -k_e w_e \\ X(0) = X_0 \end{cases} \quad (13) \text{ from weighted graphs paper}$$

for our root node $a0$'s edges,

$$\frac{dw_{a0,a1}}{dt} = -\left(\frac{-1}{w_{a0,a1}D_{a1}}\right) \cdot w_{a0,a1} = \frac{1}{D_{a1}}$$

for our leaf nodes aj edges,

$$\frac{dw_{a3,a7}}{dt} = -\left(\frac{-1}{w_{a3,a7}D_{a3}} + 1\right) \cdot w_{a3,a7} = \frac{1}{D_{a3}} - w_{a3,a7}$$

for our edges between two internal a_i nodes,

$$\frac{dw_{a1,a3}}{dt} = -\left(\frac{-1}{w_{a1,a3}D_{a1}} + \frac{-1}{w_{a1,a3}D_{a3}}\right) \cdot w_{a1,a3} = \frac{1}{D_{a1}} + \frac{1}{D_{a3}}$$

Positive constant curvature trees

Our definition for constant curvature is “ if the non-zero edge weights are all the same, that curvature is the constant curvature”. In order for this constant curvature to be positive, we need the above equation to be positive for each non-zero edge weight. However, the equation seems to be very restrictive to allowing positive values at all, so we will see in what situations can we even have positive curvature. We suspect that positive constant curvature requires very specific structure.

SEE PAGE 8 OF WEIGHTED GRAPHS PAPER – WEIGHTS ARE ALWAYS POSITIVE

If initial edge weights are non zero, and at some finite time T all edge weights are strictly greater than 0 and stay strictly greater (as stated in the paper), then we can specifically compute the curvature at this time T following the below cases. The curvatures at this time T using the positive edge weights in the computation will provide a certain structure for most of the curvatures, if not at the limit, at least at time T and finite time $t >> T$. Specifically, this should lead to the conclusion that internal edges can never have positive curvature.

INTERMEDIATE FACTS WE WILL USE BUT SHOULD ALSO VERIFY FROM THE TREES PAPER PAGE 3-4

For a leaf e and an internal edge f incident to the same vertex, the internal edge's weight remains larger than the leaf's weight for large time. *A stronger, seemingly implied and used fact would be that their ratio tends to zero, i.e. the internal edge weight grows far larger than the leaf's weight.*

$$w_e(t) < w_f(t) \quad t >> 1$$

The weights of two leaf edges e and g incident to the same vertex evolve such that their initial order is persevered over time, their difference tends to zero and their ratio converges to 1 as t goes to infinity.

$$(w_e - w_g)(t) = (w_e(0) - w_g(0))e^{-t}$$

$$\lim_{t \rightarrow \infty} \frac{w_e(t)}{w_g(t)} = 1$$

For what values of $d_u, d_v, w_{uv}, D_u, D_v$ does the following hold.

$$k_{uv} = \frac{2-d_u}{w_{uv}D_u} + \frac{2-d_v}{w_{uv}D_v} > 0$$

where

$$\begin{aligned} d_u, d_v &\in \mathbb{Z} \quad d_u, d_v \geq 1 \\ w_{uv} &\in \mathbb{R} \quad 0 < w_{uv} \leq 1 \\ D_u &= \sum_{j \sim u} \frac{1}{w_{uj}} \quad D_v = \sum_{j \sim v} \frac{1}{w_{vj}} \end{aligned}$$

Since edges are undirected, symmetrical cases need not be repeated. For example, $d_u = 1, d_v = 2$ is the same as $d_u = 2, d_v = 1$ since our equation for k_{uv} accounts for both directions. All possible cases can be listed as follows.

$$(1) \quad d_u \leq 3, \quad d_v \leq 3$$

$$d_u = 1, \quad d_v = 1$$

$$k_{uv} = \frac{2-1}{w_{uv}(\frac{1}{w_{uv}})} + \frac{2-1}{w_{uv}(\frac{1}{w_{uv}})} = 2$$

$$d_u = 1, \quad d_v = 2$$

$$k_{uv} = \frac{2-1}{w_{uv}(\frac{1}{w_{uv}})} + \frac{2-2}{w_{uv}D_v} = 1$$

$$d_u = 1, \quad d_v = 3$$

$$k_{uv} = \frac{2-1}{w_{uv}(\frac{1}{w_{uv}})} + \frac{2-3}{w_{uv}D_v} = 1 - \frac{1}{w_{uv}(\frac{1}{w_{uv}} + \frac{1}{w_{av}} + \frac{1}{w_{ab}})} = 1 - \frac{1}{1 + \frac{w_{uv}}{w_{av}} + \frac{w_{uv}}{w_{bv}}} > 0 \quad \text{iff} \quad \frac{w_{uv}}{w_{av}} + \frac{w_{uv}}{w_{bv}} \notin [-1, 0]$$

Where w_{av}, w_{bv} are the other two neighbors of v .

$$d_u = 2, \quad d_v = 2$$

$$k_{uv} = \frac{2-2}{w_{uv}D_u} + \frac{2-2}{w_{uv}D_v} = 0$$

$$d_u = 2, \quad d_v = 3$$

$$k_{uv} = \frac{2-2}{w_{uv}D_u} + \frac{2-3}{w_{uv}D_v} = \frac{-1}{1 + \frac{w_{uv}}{w_{av}} + \frac{w_{uv}}{w_{ab}}} > 0 \quad \text{iff} \quad \frac{w_{uv}}{w_{av}} + \frac{w_{uv}}{w_{ab}} < -1$$

$$d_u = 3, \quad d_v = 3$$

$$k_{uv} = \frac{2-3}{w_{uv}D_u} + \frac{2-3}{w_{uv}D_v} = \frac{-1}{w_{uv}D_u} - \frac{1}{w_{uv}D_v} = \frac{-1}{1 + \frac{w_{uv}}{w_{cu}} + \frac{w_{uv}}{w_{du}}} - \frac{1}{1 + \frac{w_{uv}}{w_{av}} + \frac{w_{uv}}{w_{bv}}} > 0$$

(2) $d_u \leq 3$, $d_v \geq 4$ (we already covered $d_v = 3$ above)

$d_u = 1$, $d_v \geq 4$

$$k_{uv} = \frac{2-1}{w_{uv}D_u} + \frac{2-d_v}{w_{uv}D_v} = \frac{1}{w_{uv}(\frac{1}{w_{uv}})} + \frac{2-d_v}{w_{uv}D_v} = 1 + \frac{2-d_v}{w_{uv}D_v} > 0 \quad \text{iff} \quad \frac{2-d_v}{w_{uv}D_v} > -1$$

$d_u = 2$, $d_v \geq 4$

$$k_{uv} = \frac{2-2}{w_{uv}D_u} + \frac{2-d_v}{w_{uv}D_v} = \frac{2-d_v}{w_{uv}D_v} > 0$$

$d_u = 3$, $d_v \geq 4$

$$k_{uv} = \frac{2-3}{w_{uv}D_u} + \frac{2-d_v}{w_{uv}D_v} = \frac{-1}{w_{uv}D_u} + \frac{2-d_v}{w_{uv}D_v} > 0$$

(3) $d_u \geq 4$, $d_v \geq 4$ (we already covered $d_u = d_v = 3$)

$$k_{uv} = \frac{2-d_u}{w_{uv}D_u} + \frac{2-d_v}{w_{uv}D_v} > 0$$

Of all the above possible cases, the following are leaf edges:

$d_u = 1$, $d_v = 1$

Which is always curvature 2. The only possible tree that has this edge is a tree with only two vertices and the one edge. This tree trivially has positive constant curvature.

$d_u = 1$, $d_v = 2$

Which is always curvature 1.

$$\frac{d\hat{w}_{ij}}{dt} = -k_{ij}\hat{w}_{ij} = -1\hat{w}_{ij} = -\hat{w}_{ij}$$

$$\dot{w}_{ij}(t) = \hat{w}_{ij}(0)e^{-t}$$

So the unnormalized edge weight converges to 0, So the normalized edge weight must converge to 0. So, the presence of this edge will not 'contribute' to positive constant curvature.

$$d_u = 1 \quad , \quad d_v = 3$$

For large enough finite time, this will always be positive curvature strictly less than 1.

$$d_u = 1 \quad , \quad d_v \geq 4$$

Which can either be positive, negative, or 0 curvature.

The following are internal edges:

$$d_u = 2 \quad , \quad d_v = 2$$

Which always have curvature 0.

$$d_u = 2 \quad , \quad d_v = 3$$

Which for large enough finite time T will never be positive.

$$d_u = 3 \quad , \quad d_v = 3$$

Which for large enough finite time T will never be positive.

$$d_u = 2 \quad , \quad d_v \geq 4$$

Which for large enough finite time T will never be positive.

$$d_u = 3 \quad , \quad d_v \geq 4$$

Which for large enough finite time T will never be positive.

$$d_u \geq 4 \quad , \quad d_v \geq 4$$

Which for large enough finite time T will never be positive.

Every tree except the trivial two node one edge tree must have at least one internal edge. It is impossible for an internal edge to have positive curvature. That means for a tree to have positive constant curvature, all internal edge weights must go to weight 0. We must show that there must exist an internal edge weight that has non zero weight in the limit, to show that positive curvature trees are impossible.

Constant curvature at time t not in the limit

We know that a tree converges to constant curvature 0 if and only if it is a caterpillar tree. A question remains: at any time $0 < t < \infty$ can a non-caterpillar tree have constant 0 curvature.

We can show that a leaf edge never reaches 0 curvature in finite time, and that a leaf edge never reaches 0 weight in finite time. Then since a tree must have at least one leaf, it will be impossible for finite time constant 0 curvature.

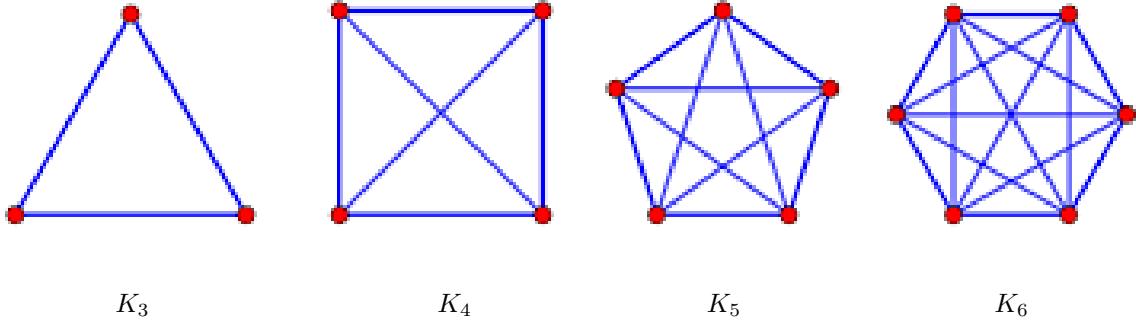
Other gamma functions / definitions of curvature / other variations from the papers

General, Cyclic graphs (not trees)

Considering complete graphs

A trivial graph structure for Discrete Ricci Flow is a complete graph with all equal initial edge weights. The Ricci Flow evolves the curvature and edge weights equally due to the symmetrical structure, making convergence automatic. An analysis of this trivial case can demonstrate how analyzing convergence on general graphs can be done for other non trivial cases.

A complete graph $K_n = (V, E)$ is a graph with n vertices, and every pair of vertices (u, v) has an edge between them. These are examples of a complete graph:



$w_{u,v}(t)$ refers to the edge weight between nodes u, v at time t . $d_{u,v}(t)$ refers to the shortest distance between nodes u, v at time t . With initial edge weights all equal, since there is a direct path between all pairs of nodes, $d_{u,v}(0) = w_{u,v}(0)$ for all edges. Due to symmetry, all edges will remain equal for all time. Let $w(t)$ refer to the edge weight of all edges at time t . Hence we have $d_{u,v}(t) = w_{u,v}(t) = w(t)$ for all edges, for all finite time.

The α -Ricci curvature k_α is defined to be

$$k_\alpha(x, y) = 1 - \frac{W(\mu_x^\alpha, \mu_y^\alpha)}{d_{x,y}}$$

The probability distribution μ_x^α is defined as

$$\mu_x^\alpha(y) = \begin{cases} \alpha & \text{if } y = x \\ (1 - \alpha) \frac{\gamma(w_{x,y})}{\sum_{z \sim x} \gamma(w_{x,z})} & \text{if } y \sim x \\ 0 & \text{otherwise} \end{cases}$$

For simplicity, we will take $\gamma(x) = x$ and $\alpha = 0$. Since an edge of weight $w(t)$ exists between all pairs of nodes, the probability distribution simplifies to

$$\mu_x^\alpha(y) = 0 \quad \text{if } y=x \quad \mu_x^\alpha(y) = \frac{w(t)}{(n-1)w(t)} = \frac{1}{n-1} \quad \text{otherwise} \quad (1)$$

Due to symmetry and no dependence on t , all probability distributions for all time will be the same form, the above distribution.

The Wasserstein function is defined as

$$W(\mu_1, \mu_2) = \inf_A \sum_{x,y \in V} A(x,y)d(x,y)$$

Where the infimum is taken over all probability distributions A between μ_1, μ_2 . Any feasible coupling gives an upper bound for the Wasserstein function. A coupling A is a matrix defined between two probability distributions μ_u and μ_v where $A(x,y)$ is an entry of the matrix that satisfies

$$\sum_{y \in V} A(x,y) = \mu_u(x) \quad \text{and} \quad \sum_{x \in V} A(x,y) = \mu_v(y)$$

Consider the coupling A as

$$A(x,y) = \begin{cases} \frac{1}{n-1} & \text{if } x = y, x \neq u, v \\ \frac{1}{n-1} & \text{if } x = v, y = u \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \sum_{y \in V} A(x,y) &= \mu_1(x) = 0 \quad \text{if } x = u \\ \sum_{y \in V} A(x,y) &= \mu_1(x) = \frac{1}{n-1} \quad \text{if } x \neq u \\ \sum_{x \in V} A(x,y) &= \mu_2(y) = 0 \quad \text{if } y = v \\ \sum_{x \in V} A(x,y) &= \mu_2(y) = \frac{1}{n-1} \quad \text{if } y \neq v \end{aligned}$$

So this coupling is feasible, and the Wasserstein function becomes

$$W(\mu_1, \mu_2) \leq \inf_A \sum_{x,y \in V} A(x,y)d(x,y) = \frac{w(t)}{n-1}$$

Since the diagonal entries that get mass have 0 cost, and the only non diagonal entry that gets mass is entry (v, u) with cost $w(t)$.

To ensure optimality, we consider the dual problem. The Wasserstein function can be written in dual form as

$$W(\mu_1, \mu_2) = \sup_f \sum_{x \in V} f(x)[\mu_1(x) - \mu_2(x)]$$

Where the supremum is taken over all 1-Lipschitz functions f . $f(x)$ is defined for all nodes $x \in V$ across a given μ_u, μ_v distribution. Any 1-Lipschitz function gives a lower bound for the Wasserstein function. The optimal 1-Lipschitz function f that obtains the supremum is referred to as the optimal Kantorovich potential function.

f is Lipschitz-1 if for all x, y it satisfies

$$|f(x) - f(y)| \leq d_{x,y}$$

Consider f as

$$g(x) = w(t) \quad \text{if } x = v, \quad g(x) = 0 \quad \text{otherwise}$$

when $v = x$ or $v = y$, $|g(x) - g(y)| = w(t) = d(t)$

when $v \neq x, v \neq y$, $|g(x) - g(y)| = 0 \leq d(t)$

$g(x)$ is 1-Lipschitz, which is feasible for the dual problem.

The Wasserstein function becomes,

$$W(\mu_1, \mu_2) = \sum_{x \in V} g(x)[\mu_1(x) - \mu_2(x)] = \frac{w}{n-1}$$

Since $g(x) = 0$ except when $x = v$. This shows $g(x)$ is the optimal Kantorovich function since we have equality with the primal problem. The Wasserstein function can be expressed as $\frac{w}{n-1}$.

Our curvature becomes,

$$k_\alpha(x, y) = 1 - \frac{W(\mu_x^\alpha, \mu_y^\alpha)}{d_{x,y}} = 1 - \frac{\frac{w}{n-1}}{w} = \frac{n-2}{n-1}$$

So our curvature only depends on the number of nodes. For all time, the curvature remains the same. We converge to constant positive curvature.

Our normalized flow is given by

$$\frac{\partial w(t)}{\partial t} = -k(t)w(t) + w(t) \sum_{h \in E(G)} k_h w_h(t)$$

Since k and w are the same for all edges, and we normalize weights by ensuring $\sum_{h \in E(G)} w_h = 1$ which implies $|E|w(t) = 1$, we have

$$\frac{\partial w(t)}{\partial t} = -k(t)w(t) + w(t) [|E|w(t)k(t)] = -k(t)w(t) + k(t)w(t) = 0$$

Curvature is constant for all time, we trivially converge to constant positive curvature $\frac{n-2}{n-1}$ with edge weights $w = \frac{1}{|E|}$ for all time.