

Ricci curvature of Markov chains on metric spaces

Yann Ollivier

Abstract

We define the Ricci curvature of Markov chains on metric spaces as a local contraction coefficient of the random walk acting on the space of probability measures equipped with a Wasserstein transportation distance. For Brownian motion on a Riemannian manifold this gives back the value of Ricci curvature of a tangent vector. Examples of positively curved spaces for this definition include the discrete cube and discrete versions of the Ornstein–Uhlenbeck process. Moreover this generalization is consistent with the Bakry–Émery Ricci curvature for Brownian motion with a drift on a Riemannian manifold.

Positive Ricci curvature is shown to imply a spectral gap, a Lévy–Gromov-like Gaussian concentration theorem and a kind of modified logarithmic Sobolev inequality. The bounds obtained are sharp in several interesting examples.

Introduction

There are numerous generalizations of the notion of a metric space with *negative sectional curvature*: manifolds with negative sectional curvature, CAT(0) and CAT(-1) spaces or δ -hyperbolic spaces are widely used in various branches of mathematics and give rise to numerous theorems. For *positive curvature* in Riemannian geometry, the right concept seems to be a lower bound on *Ricci curvature* (which is weaker than a lower bound on sectional curvature). The most basic result in this direction is the Bonnet–Myers theorem bounding the diameter of the space in function of the Ricci curvature, but let us mention Lichnerowicz’ theorem for the spectral gap of the Laplacian (Theorem 181 in [Ber03]), the Lévy–Gromov theorem for isoperimetric inequalities and concentration of measure [Gro86], or Gromov’s theorem on precompactness of the space of manifolds with given dimension, upper bound on the diameter and lower bound on the Ricci curvature.

We refer to the nice survey [Lott] for a discussion of the geometric interest of lower bounds on Ricci curvature, with further references, and the need for a generalized notion of positive Ricci curvature for metric spaces (often equipped with a measure).

There have been several generalizations of the notion of Ricci curvature. First, the study by Bakry and Émery [BE85] of hypercontractivity of diffusion processes led them to show that, when considering the Brownian motion on a manifold with an additional drift given by a tangent vector field F , the quantity $\text{Ric} - 2\nabla^{\text{sym}}F$ plays the role of a Ricci curvature for the process, as far as functional inequalities are concerned. The main example is the Ornstein–Uhlenbeck process on \mathbb{R}^N , whose invariant distribution is Gaussian, and which is positively curved in this sense.

Later, simultaneously, Sturm [Stu06], Lott and Villani [LV], and Ohta [Oht] used ideas from optimal transportation theory to define a notion of lower bound on the Ricci curvature for length spaces equipped with a measure. Their definition keeps a lot of the properties traditionally associated with positive Ricci curvature, and is compatible with the Bakry–Émery extension. However, it has two main drawbacks. First, it is infinitesimal, and in particular is meaningless for a graph. Second, the definition is rather involved and difficult to check on concrete examples. The main class of spaces for which this definition is interesting are Gromov–Hausdorff limits of manifolds of a given dimension.

Here we propose a definition of Ricci curvature for metric spaces equipped with a Markov chain or a diffusion process (which for a Riemannian manifold will typically be Brownian motion), which is hopefully simpler to check on examples. The definition is again based on optimal transportation, but in a less infinitesimal way, and can be used to define a notion of “curvature at a given scale” for a metric space. As a consequence, we can test it in discrete spaces such as graphs. Such an example is the discrete cube $\{0, 1\}^N$, which from the point of view of concentration of measure behaves very much like the sphere S^N , and is thus expected to somehow have positive curvature.

Our definition, when applied to a Riemannian manifold equipped with the Brownian motion, gives back the usual value of the Ricci curvature of a tangent vector. It is consistent with the Bakry–Émery extension, and provides a visual explanation for the contribution $-\nabla^{\text{sym}}F$ of the drift F . We are able to prove generalizations of the Bonnet–Myers theorem, of the Lichnerowicz spectral gap theorem and of the Lévy–Gromov isoperimetry theorem, as well as a kind of modified logarithmic Sobolev inequality, although with some (bounded) loss in the constants. As a by-product, we get a new proof for Gaussian concentration and the logarithmic Sobolev inequality in the Lévy–Gromov or Bakry–Émery context (though the constants are not sharp).

Related work. After having written a first version of this text, we learned that related ideas appear in several recent papers. Joulin [Jou] uses contraction of the Lipschitz constant (under the name “Wasserstein curvature”) to get a Poisson-type concentration result for continuous-time Markov chains on a countable space, at least in the bounded, one-dimensional case. Oliveira [Oli] proves that Kac’s random walk on $\mathrm{SO}(n)$ has positive Ricci curvature in our sense, which allows to improve mixing time estimates significantly. Djellout, Guillin and Wu [DGW04] use contraction of Lipschitz constants and transportation distances (without the link with Ricci curvature) in the context of dependent sequences of random variables, to get Gaussian concentration results. The link with the spectral gap appears in [Sam] (p. 94) for the particular case of graphs, and is present in the works of Chen (e.g. [CW97, Che98]).

From the discrete Markov chain point of view, the techniques presented here are just a metric version of the usual coupling method. Namely, Ricci curvature can be seen as a refined version of Dobrushin’s ergodic coefficient (see [Dob56], or e.g. section 6.7.1 in [Bré99]) using the metric structure on the underlying space.

From the Riemannian point of view, our approach boils down to contraction of the Lipschitz norm by the heat equation, which is one of the results of Bakry and Émery ([BE84, BE85], see also [ABCIGMRS00] and [RS05]). This latter property was suggested in [RS05] as a possible definition of a lower bound on Ricci curvature for diffusion operators in general spaces, though it does not provide an explicit value for Ricci curvature at a given point.

Acknowledgements. I would like to thank Vincent Beffara, Fabrice Debbasch, Alessio Figalli, Pierre Pansu, Bruno Sévennec, Romain Tessera and Cédric Villani for numerous inspiring conversations about coarse geometry and Ricci curvature, as well as Djalil Chafaï, Aldéric Joulin, Shin-ichi Ohta and Roberto Oliveira for useful remarks on the manuscript and bibliographical references. Special thanks to Pierre Py for the two points x and y .

Notation. In the paper, we use the symbol \approx to denote equality up to a multiplicative universal constant (typically 2 or 4); the symbol \sim denotes usual asymptotic equivalence. The word “distribution” is used as a synonym for “probability measure”.

1 Definitions and statements

1.1 Ricci curvature

A common framework for generalizations of Ricci curvature is that of metric measure spaces [Stu06, LV]. However, most measures appear as the invariant distribution of some process (e.g. Brownian motion on a Riemannian manifold), and it is more convenient and more general to start with a process in a metric space, as is the case in Bakry–Émery theory. See also Remark 5 below.

Here for simplicity we will mainly consider the case of a discrete-time process. Similar definitions and results can be given for continuous time (see e.g. Section 3.3.4).

DEFINITION 1 – Let (X, d) be a Polish metric space, equipped with its Borel σ -algebra.

A random walk m on X is a family of probability measures $m_x(\cdot)$ on X for each $x \in X$, satisfying the following two technical assumptions: (i) the measure m_x depends measurably on the point $x \in X$; (ii) each measure m_x has finite first moment, i.e. for some (hence any) $o \in X$ one has $\int d(o, y) dm_x(y) < \infty$.

This defines a Markov chain whose transition probability from x to y in n steps is

$$dm_x^{*n}(y) := \int_{z \in X} dm_x^{*(n-1)}(z) dm_z(y)$$

where of course $m_x^{*1} := m_x$.

Recall that a measure ν on X is *invariant* for this random walk if $d\nu(x) = \int_y d\nu(y) dm_y(x)$. It is *reversible* if moreover, the detailed balance condition $d\nu(x) dm_x(y) = d\nu(y) dm_y(x)$ holds.

This allows to define a notion of curvature as follows. Consider two very close points x, y in a Riemannian manifold, defining a tangent vector (xy) . Let w be another tangent vector at x ; let w' be the tangent vector at y obtained by parallel transport of w from x to y . Now if we follow the two geodesics issuing from x, w and y, w' , in positive curvature the geodesics will get closer, and will part away in negative curvature. Ricci curvature along (xy) is this phenomenon, averaged on all directions w at x .

So in the general case, we will measure whether following the random walk issuing from two nearby points x, y results in points that are closer than x, y were, in which case Ricci curvature will be positive, or further apart, in which case Ricci curvature will be negative. This is made precise

by the use of transportation distances between probability measures. We refer to [Vil03] for an introduction to this topic.

DEFINITION 2 – Let (X, d) be a metric space and let ν_1, ν_2 be two probability measures on X . The L^1 transportation distance between ν_1 and ν_2 is

$$\mathcal{T}_1(\nu_1, \nu_2) := \inf_{\xi \in \Pi(\nu_1, \nu_2)} \int_{(x,y) \in X \times X} d(x, y) d\xi(x, y)$$

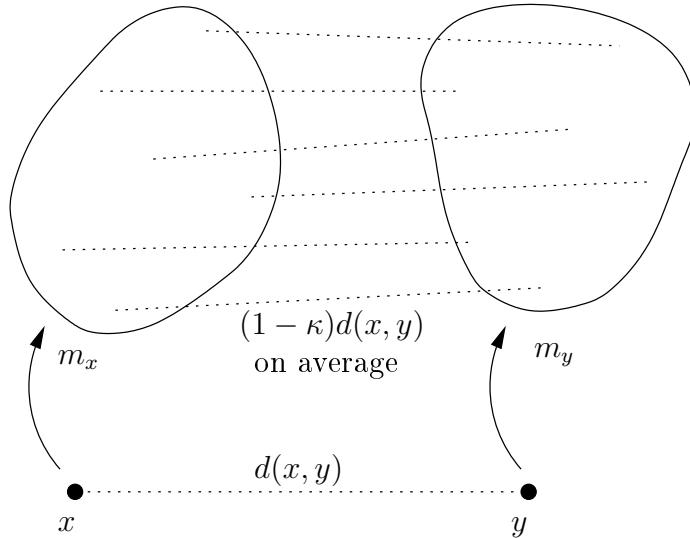
where $\Pi(\nu_1, \nu_2)$ is the set of measures on $X \times X$ projecting to ν_1 and ν_2 .

Intuitively, $d\xi(x, y)$ represents the mass that is sent from x to y , hence the constraint on the projections of ξ , ensuring that the initial measure is ν_1 and the final measure is ν_2 .

The infimum is actually attained (Theorem 1.3 in [Vil03]), but the optimal coupling is generally not unique. In what follows, it is enough to chose one such coupling.

DEFINITION 3 – Let (X, d) be a metric space with a random walk m . Let $x, y \in X$ be two distinct points. The Ricci curvature of (X, d, m) in the direction (x, y) is

$$\kappa(x, y) := 1 - \frac{\mathcal{T}_1(m_x, m_y)}{d(x, y)}$$



When (X, d) is a Riemannian manifold, if the random walk consists in randomly jumping in a ball of radius ε around x , for small ε and close enough x, y this definition captures the Ricci curvature in the direction xy (up to some factor depending on ε).

We will see below (Proposition 19) that in geodesic spaces, it is enough to know $\kappa(x, y)$ for close points x, y .

If a continuous-time Markov kernel is given, one can also define a continuous-time version of the Ricci curvature by setting

$$\kappa(x, y) := - \frac{d}{dt} \frac{\mathcal{T}_1(m_x^t, m_y^t)}{d(x, y)}$$

when this derivative exists, but for simplicity we will mainly work with the discrete-time version here. Indeed, for continuous-time Markov chains, existence of the process is already a non-trivial issue. We will sometimes use our results on concrete continuous-time examples (e.g. $M/M/\infty$ queues in section 3.3.4), but only when they appear as an obvious limit of a discrete-time approximation.

One could use the L^p transportation distance instead of the L^1 one in the definition; however, though this will result in stronger assumptions, I did not find any theorem where this would be necessary.

NOTATION – By analogy with the Riemannian case, when computing the transportation distance between measures m_x and m_y , we will think of $X \times X$ equipped with the coupling measure as a tangent space, and for $z \in X \times X$ we will write $x + z$ and $y + z$ for the two projections to X . So in this notation we have

$$\kappa(x, y) = - \frac{1}{d(x, y)} \int (d(x + z, y + z) - d(x, y)) dz$$

where implicitly dz is the optimal coupling between m_x and m_y .

1.2 Examples

EXAMPLE 4 (\mathbb{Z}^N AND \mathbb{R}^N) – Let m be the simple random walk on the graph of the grid \mathbb{Z}^N equipped with its graph metric. Then for any two points $x, y \in \mathbb{Z}^d$, the Ricci curvature along (xy) is 0.

Indeed, we can transport the measure m_x around x to the measure m_y by a translation of vector $y - x$ (and this is optimal), so that the distance between m_x and m_y is exactly that between x and y .

This example generalizes to the case of \mathbb{Z}^n or \mathbb{R}^N equipped with any translation-invariant norm and any random walk given by a translation-invariant transition kernel (consistently with [LV]). For example, the triangular tiling of the plane has 0 curvature.

REMARK 5 (RANDOM WALK AT SCALE ε) – It is easy to construct random walks on metric measure spaces. If (X, d, μ) is a metric measure

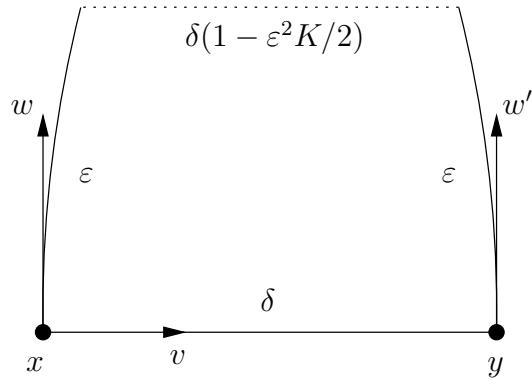
space (for example with μ the Hausdorff measure) and $\varepsilon > 0$, the *random walk at scale ε* consists in, starting at a point x , randomly jumping in the ball of radius ε around x , with probability density proportional to μ ; namely $dm_x(y) := d\mu(y)/\mu(B(x, \varepsilon))$ if $d(x, y) \leq \varepsilon$ (one can also use other functions of the distance, such as a Gaussian kernel). This allows to consider the Ricci curvature associated with this random walk.

This is what we do now on Riemannian manifolds to get back the usual Ricci curvature (up to some normalization constants), hence the terminology.

PROPOSITION 6 – Let (X, d) be a smooth complete Riemannian manifold. Let v, w be unit tangent vectors at $x \in X$. Let $\varepsilon, \delta > 0$. Let $y = \exp_x \delta v$ and let w' be the tangent vector at y obtained by parallel transport of w along the geodesic $\exp_x tv$. Then

$$d(\exp_x \varepsilon w, \exp_y \varepsilon w') = \delta \left(1 - \frac{\varepsilon^2}{2} (K(v, w) + O(\delta + \varepsilon)) \right)$$

as $(\varepsilon, \delta) \rightarrow 0$. Here $K(v, w)$ is the sectional curvature in the tangent plane (v, w) .



EXAMPLE 7 (RIEMANNIAN MANIFOLD) – Let (X, d) be a smooth complete N -dimensional Riemannian manifold. For some $\varepsilon > 0$, let the Markov chain m^ε be defined by

$$dm_x^\varepsilon(y) := \frac{1}{\text{vol}(B(x, \varepsilon))} d\text{vol}(y)$$

if $y \in B(x, \varepsilon)$, and 0 otherwise.

Let $x \in X$ and let v be a unit tangent vector at x . Let y be a point on the geodesic issuing from v , with $d(x, y)$ small enough. Then

$$\kappa(x, y) = \frac{\varepsilon^2}{2(N+2)} (\text{Ric}(v, v) + O(\varepsilon) + O(d(x, y)))$$

PROOF – This is essentially the same as Theorem 1.5 (condition (*xii*)) in [RS05], except that therein, the infimum of Ricci curvature is used instead of its value along a tangent vector. The proof is postponed to Section 8. Basically, the value of $\kappa(x, y)$ is obtained by averaging the proposition above for w in the unit ball of the tangent space at x , which provides an upper bound for κ . The lower bound requires use of the dual characterization of transportation distance (Theorem 1.14 in [Vil03]). \square

EXAMPLE 8 (DISCRETE CUBE) – Let $X = \{0, 1\}^N$ be the discrete cube equipped with the Hamming metric (each edge is of length 1). Let m be the lazy random walk on the graph X , i.e. $m_x(x) = 1/2$ and $m_x(y) = 1/2N$ if y is a neighbor of x .

Let $x, y \in X$ be neighbors. Then $\kappa(x, y) = 1/N$.

This examples generalizes to arbitrary binomial distributions (see Section 3.3.3).

Here laziness is necessary to avoid parity problems: If no laziness is introduced, points at odd distance never meet under the random walk; in this case one must consider Ricci curvature for points at even distance only.

Actually, since the discrete cube is a 1-geodesic space, one has $\kappa(x, y) \geq 1/N$ for any pair $x, y \in X$, not only neighbors (see Proposition 19).

PROOF – We can suppose that $x = 00\dots 0$ and $y = 10\dots 0$. For $z \in X$ and $1 \leq i \leq N$, let us denote by z^i the neighbor of z in which the i -th bit is switched. An optimal coupling between m_x and m_y is as follows: For $i \geq 2$, move x^i to y^i (both have mass $1/2N$ under m_x and m_y respectively). Now $m_x(x) = 1/2$ and $m_y(x) = 1/2N$, and likewise for y . To transport m_x to m_y , it is enough to move a mass $1/2 - 1/2N$ from x to y . All points are moved over a distance 1 by this coupling, except for a mass $1/2N$ which remains at x and a mass $1/2N$ which remains at y , and so the Ricci curvature is at least $1/N$.

Optimality of this coupling is obtained as follows: Consider the function $f : X \rightarrow \{0, 1\}$ which sends a point of X to its first bit. This is a 1-Lipschitz function, with $f(x) = 0$ and $f(y) = 1$. The expectations of f under m_x and m_y are $1/2N$ and $1 - 1/2N$ respectively, so that $1 - 1/N$ is a lower bound on $\mathcal{T}_1(m_x, m_y)$.

A very short but less visual proof can be obtained through the L^1 tensorization property (Proposition 26). \square

EXAMPLE 9 (ORNSTEIN–UHLENBECK PROCESS) – Let $s \geq 0, \alpha > 0$ and consider the Ornstein–Uhlenbeck process in \mathbb{R}^N given by the stochastic

differential equation

$$dX_t = -\alpha X_t dt + s dB_t$$

where B_t is a standard N -dimensional Brownian motion. The invariant distribution is Gaussian, of variance $s^2/2\alpha$.

Let $\delta t > 0$ and let the random walk m be the flow at time δt of the process. Explicitly, m_x is a Gaussian probability measure centered at $e^{-\alpha \delta t}x$, of variance $s^2(1 - e^{-\alpha \delta t})/\alpha \sim s^2 \delta t$ for small δt .

Then the Ricci curvature $\kappa(x, y)$ of this random walk is $1 - e^{-\alpha \delta t}$, for any two $x, y \in \mathbb{R}^N$.

PROOF – The transportation distance between two Gaussian distributions with the same variance is the distance between their centers, so that $\kappa(x, y) = 1 - \frac{|e^{-\alpha \delta t}x - e^{-\alpha \delta t}y|}{|x - y|}$. \square

EXAMPLE 10 (DISCRETE ORNSTEIN–UHLENBECK) – Let $X = \{-N, -N+1, \dots, N-1, N\}$ and let m be the random walk on X given by

$$m_k(k) = 1/2, \quad m_k(k+1) = 1/4 - k/4N, \quad m_k(k-1) = 1/4 + k/4N$$

which is a lazy random walk with linear drift towards 0. The binomial distribution $\frac{1}{2^{2N}} \binom{2N}{N+k}$ is reversible for this random walk.

Then, for any two neighbors x, y in X , one has $\kappa(x, y) = 1/2N$.

PROOF – Exercise. \square

EXAMPLE 11 (BAKRY–ÉMERY) – Let X be an N -dimensional Riemannian manifold and F be a tangent vector field. Consider the differential operator

$$L := \frac{1}{2}\Delta + F.\nabla$$

associated with the stochastic differential equation

$$dx_t = F dt + dB_t$$

where B_t is the Brownian motion in X . The Ricci curvature (in the Bakry–Émery sense) of this operator is $\frac{1}{2}\text{Ric} - \nabla^{\text{sym}}F$ where $\nabla^{\text{sym}}F^{ij} = \frac{1}{2}(\nabla^i F^j + \nabla^j F^i)$ is the symmetrized of ∇F .

Consider the Euler approximation scheme at time δt for this stochastic equation, which consists in following the flow of F for a time δt and then randomly jumping in a ball of radius $\sqrt{(N+2)\delta t}$.

Let $x \in X$ and let v be a unit tangent vector at x . Let y be a point on the geodesic issuing from v , with $d(x, y)$ small enough. Then

$$\kappa(x, y) = \delta t \left(\frac{1}{2}\text{Ric}(v, v) - \nabla^{\text{sym}}F(v, v) + O(d(x, y)) + O(\sqrt{\delta t}) \right)$$

PROOF – First let us explain the normalization: Jumping in a ball of radius ε generates a variance $\varepsilon^2 \frac{1}{N+2}$ in a given direction. On the other hand, the N -dimensional Brownian motion has, by definition, a variance dt per unit of time dt in any given direction, so a proper discretization at time δt requires jumping in a ball of radius $\sqrt{(N+2)\delta t}$. Also, as noted in [BE85], the generator of Brownian motion is $\frac{1}{2}\Delta$ instead of Δ , hence the $\frac{1}{2}$ factor for the Ricci part.

Now the discrete-time process begins by following the flow F for some time δt . Starting at points x and y , using elementary Euclidean geometry, it is easy to see that after this, the distance between the endpoints behaves like $d(x, y)(1 + \delta t v \cdot \nabla_v F + O(\delta t^2))$. Note that $v \cdot \nabla_v F = \nabla^{\text{sym}} F(v, v)$.

Now, just as in Example 7, randomly jumping in a ball of radius ε results in a gain of $d(x, y) \frac{\varepsilon^2}{2(N+2)} \text{Ric}(v, v)$ on transportation distances. Here $\varepsilon^2 = (N+2)\delta t$. So after the two steps, the distance between the endpoints is

$$d(x, y) \left(1 - \frac{\delta t}{2} \text{Ric}(v, v) + \delta t \nabla^{\text{sym}} F(v, v) \right)$$

as needed, up to higher-order terms. \square

Maybe the reason for the additional $-\nabla^{\text{sym}} F$ in Ricci curvature à la Bakry–Émery is made clearer in this context: it is simply the quantity by which the flow of X modifies distances between two starting points.

It is clear on this example why reversibility is not fundamental in this theory: the antisymmetric part of the force F generates an infinitesimal isometric displacement. Combining the Markov chain with an isometry of the space has no effect whatsoever on our definition.

EXAMPLE 12 (MULTINOMIAL DISTRIBUTION) – Consider the set $X = \{(x_0, x_1, \dots, x_d), x_i \in \mathbb{N}, \sum x_i = N\}$ viewed as the configuration set of N balls in $d+1$ boxes. Consider the process which consists in taking a ball at random among the N balls, removing it from its box, and putting it back at random in one of the $d+1$ boxes. More precisely, the transition probability from (x_0, \dots, x_d) to $(x_0, \dots, x_i - 1, \dots, x_j + 1, \dots, x_d)$ (with maybe $i = j$) is $x_i/N(d+1)$. The multinomial distribution $\frac{N!}{(d+1)^N \prod x_i!}$ is reversible for this Markov chain.

Equip this configuration space with the metric $d((x_i), (x'_i)) := \frac{1}{2} \sum |x_i - x'_i|$ which is the graph distance w.r.t. the moves above. Then the Ricci curvature of the Markov chain is $1/N$.

PROOF – Exercise. \square

EXAMPLE 13 (GEOMETRIC DISTRIBUTION) – Let the random walk on \mathbb{N} be defined by the transition probabilities $p_{n,n+1} = 1/3$, $p_{n+1,n} = 2/3$ and $p_{0,0} = 2/3$. This random walk is reversible with respect to the geometric measure $2^{-(n+1)}$. It is easy to check that for $n \geq 1$ one has $\kappa_{n,n+1} = 0$.

PROOF – The transition kernel is translation-invariant except at 0. \square

Section 5 contains more material about this latter example and how non-negative Ricci curvature sometimes implies exponential concentration.

EXAMPLE 14 (GEOMETRIC DISTRIBUTION, 2) – Let the random walk on \mathbb{N} be defined by the transition probabilities $p_{n,0} = \alpha$ and $p_{n,n+1} = 1 - \alpha$ for some $0 < \alpha < 1$. The geometric distribution $\alpha(1 - \alpha)^n$ is invariant (but not reversible) for this random walk. The Ricci curvature of this random walk is α .

EXAMPLE 15 (δ -HYPERBOLIC GROUPS) – Let X be the Cayley graph of a non-elementary δ -hyperbolic group with respect to some finite generating set. Let k be a large enough integer (depending on the group) and consider the random walk consisting in performing k steps of the simple random walk. Let $x, y \in X$ with $d(x, y) > 2k$. Then $\kappa(x, y) = -2k/d(x, y) + O(1/d(x, y))$.

Note that $-2k/d(x, y)$ is the smallest possible value for $\kappa(x, y)$, knowing that the steps of the random walk are bounded by k .

PROOF – For z in the ball of radius k around x , and z' in the ball of radius k around y , elementary δ -hyperbolic geometry yields $d(z, z') = d(x, y) + d(x, z) + d(y, z') - (y, z)_x - (x, z')_y$ up to some multiple of δ , where (\cdot, \cdot) denotes the Gromov product with respect to some basepoint [GH90]. Since this decomposes as the sum of a term depending on z only and a term depending on z' only, to compute the transportation distance it is enough to study the expectation of $(y, z)_x$ for z in the ball around x , and likewise for $(x, z')_y$. Knowing that balls have exponential growth, it is not difficult to see that the expectation of $(y, z)_x$ is bounded by a constant, whatever k , hence the conclusion.

The same argument applies to trees or discrete δ -hyperbolic spaces with a uniform lower bound on the exponential growth rate of balls. \square

EXAMPLE 16 (KAC'S RANDOM WALK ON ORTHOGONAL MATRICES, AFTER [OLI]) – Consider the following random walk on the set of $N \times N$ orthogonal matrices: at each step, a pair of indices $1 \leq i < j \leq N$ is selected at random, an angle $\theta \in [0; 2\pi]$ is picked at random, and a rotation of angle θ is performed in the coordinate plane i, j . Equip the set of orthogonal matrices with the Riemannian metric on $\mathrm{SO}(N)$ induced by the Hilbert-Schmidt inner product $\mathrm{Tr}(a^*b)$ on its tangent space. It is proven in a preprint

by Oliveira [Oli] that this random walk has curvature $1 - \sqrt{1 - 2/N(N-1)} \sim 1/N^2$.

This is consistent with the fact that $\mathrm{SO}(N)$ has, as a Riemannian manifold, a positive Ricci curvature in the usual sense. However, from the computational point of view, Kac's random walk above is much nicer than either the Brownian motion or the ε -scale random walk of Example 7. Oliveira uses this result to prove a new estimate $O(N^2 \ln N)$ for the mixing time of this random walk, nicely improving on previous estimates $O(N^4 \ln N)$ by Diaconis–Saloff-Coste and $O(N^{2.5} \ln N)$ by Pak–Sidenko (an easy lower bound is $\Omega(N^2)$), see [Oli].

EXAMPLE 17 (GLAUBER DYNAMICS FOR THE ISING MODEL) – Let G be a finite graph. Consider the configuration space is $X := \{-1, 1\}^G$ together with the energy function $U(S) := -\sum_{x \sim y \in G} S(x)S(y) - H \sum_x S(x)$ for $S \in X$, where $H \in \mathbb{R}$ is the external magnetic field. For some $\beta \geq 0$, equip X with the Gibbs distribution $\mu := e^{-\beta U}/Z$ where as usual $Z := \sum_S e^{-\beta U(S)}$. The distance between two states is defined as the number of vertices of G at which their value differ.

For $S \in X$ and $x \in G$, denote by S_{x+} and S_{x-} the states obtained from S by setting $S_{x+}(x) = +1$ and $S_{x-}(x) = -1$, respectively. Consider the following random walk on X (known as the Glauber dynamics): at each step, a vertex $x \in G$ is chosen at random, and a new value for $S(x)$ is picked according to local equilibrium, i.e. $S(x)$ is set to 1 or -1 with probabilities proportional to $e^{-\beta U(S_{x+})}$ and $e^{-\beta U(S_{x-})}$ respectively (note that only the neighbors of x influence the ratio of these probabilities). The Gibbs distribution is reversible for this Markov chain.

Then the Ricci curvature of this Markov chain is at least

$$\frac{1}{|G|} \left(1 - v_{\max} \frac{e^\beta - e^{-\beta}}{e^\beta + e^{-\beta}} \right)$$

where v_{\max} is the maximal valency of a vertex of G . In particular, if

$$\beta < \frac{1}{2} \ln \left(\frac{v_{\max} + 1}{v_{\max} - 1} \right)$$

then curvature is positive. Consequently, the critical β is at least this quantity.

This estimate for the critical temperature coincides exactly with the one derived in [Gri67]; actually our argument generalizes to non-constant values of the coupling J_{xy} between spins, and the positive curvature condition exactly amounts to $G(\beta) < 1$ in that paper's notation ([Gri67], Eq. (19)), or,

equivalently, to Dobrushin's criterion using a single site. For comparison, the exact value of the critical β for the Ising model on the regular infinite tree of valency v is $\frac{1}{2} \ln \left(\frac{v}{v-2} \right)$, which shows asymptotic optimality.

As shown in the rest of this paper, positive curvature implies several properties, especially, exponential convergence to the equilibrium, concentration inequalities and a modified logarithmic Sobolev inequality. I do not know how these results compare to the literature.

Since the argument presented below does not rely on exact solutions but on quantitative estimates, it is obviously not specific to the Ising model: the only property we used is that the influence of a vertex on the local equilibrium of its neighbors is bounded.

PROOF – Using Proposition 19, it is enough to bound Ricci curvature for pairs states at distance 1. Let S, S' be two states differing only at $x \in G$. We can suppose that $S(x) = -1$ and $S'(x) = 1$. Let m_S and $m_{S'}$ be the law of the step of the random walk issuing from S and S' respectively. We have to prove that the transportation distance between m_S and $m_{S'}$ is at most $1 - \frac{1}{|G|} \left(1 - v_{\max} \frac{e^\beta - e^{-\beta}}{e^\beta + e^{-\beta}} \right)$.

The measure m_S decomposes as $m_S = \frac{1}{|G|} \sum_{y \in G} m_S^y$, according to the vertex $y \in G$ which is modified by the random walk, and likewise for $m_{S'}$. To evaluate the transportation distance, we will compare m_S^y to $m_{S'}^y$.

If the step of the random walk consists in modifying the value of S at x (which occurs with probability $1/|G|$), then the resulting state has the same law for S and S' , i.e. $m_S^x = m_{S'}^x$. Thus in this case the transportation distance is 0 and the contribution to Ricci curvature is $1 \times \frac{1}{|G|}$.

If the step consists in modifying the value of S at some point y in G not adjacent to x , then the value at x does not influence local equilibrium at y , and so m_S^y and $m_{S'}^y$ are identical except at x . So in this case the distance is 1 and the contribution to Ricci curvature is 0.

Now if the step consists in modifying the value of S at some point $y \in G$ adjacent to x (which occurs with probability $v_x/|G|$ where v_x is the valency of x), then the value at x does influence the law of the new value at y , by some amount which we now evaluate. The final distance between the two laws will be this amount plus 1 (1 accounts for the difference at x), and the contribution to Ricci curvature will be negative.

Let us now evaluate this amount more precisely. Let $y \in G$ be adjacent to x . Set $a = e^{-\beta U(S_{y+})}/e^{-\beta U(S_{y-})}$. The step of the random walk consists in setting $S(y)$ to 1 with probability $\frac{a}{a+1}$, and to -1 with probability $\frac{1}{a+1}$. Setting likewise $a' = e^{-\beta U(S'_{y+})}/e^{-\beta U(S'_{y-})}$ for S' , we are left to evaluate the distance between the distributions on $\{-1, 1\}$ given by $(\frac{a}{a+1}; \frac{1}{a+1})$ and $(\frac{a'}{a'+1}; \frac{1}{a'+1})$. It is immediate to check, using the definition of the energy U , that $a' = e^{4\beta} a$.

Then, a simple computation shows that the distance between these two distributions is at most $\frac{e^\beta - e^{-\beta}}{e^\beta + e^{-\beta}}$. This value is actually achieved when y has odd valency, $H = 0$ and switching the value at x changes the majority around y . (Our argument is suboptimal here when valency is even—a more precise estimation yields the absence of a phase transition in dimension 1.)

Combining these different cases yields the desired curvature evaluation. To convert this into an evaluation of the critical β , reason as follows: Magnetization, defined as $\frac{1}{|G|} \sum_{x \in G} S(x)$, is a $\frac{1}{|G|}$ -Lipschitz function of the state. Now let μ_0 be the Gibbs measure without magnetic field, and μ_h the Gibbs measure with external magnetic field h . Use the Glauber dynamics with magnetic field h , but starting with an initial state picked under μ_0 ; Cor. 22 yields that the magnetization under μ_h is controlled by $\frac{1}{|G|} \mathcal{T}_1(\mu_0, \mu_0 * m)/\kappa$ where κ is the Ricci curvature, and $\mathcal{T}_1(\mu_0, \mu_0 * m)$ is the transportation distance between the Gibbs measure μ_0 and the measure obtained from it after one step of the Glauber dynamics with magnetic field h ; reasoning as above this transportation distance is easily bounded by $\frac{1}{|G|} \frac{e^{\beta h} - e^{-\beta h}}{e^{\beta h} + e^{-\beta h}}$, so that the derivative of the magnetization w.r.t. h stays bounded when $|G| \rightarrow \infty$. (Compare Eq. (22) in [Gri67].) \square

More examples can be found in Sections 3.3.3 (binomial and Poisson distributions), 3.3.4 ($M/M/\infty$ queues and generalizations) and 5 (geometric distributions on \mathbb{N} , exponential distributions on \mathbb{R}^N).

1.3 Overview of the results

Notation for random walks. Before presenting the main results, we need some more quantites related to the local behavior of the random walk: the *jump*, which will help control the diameter of the space, and the *spread*, which is the analogue of a diffusion constant and will help control concentration properties. Moreover, we define a notion of local dimension. The larger the dimension, the better for concentration of measure.

DEFINITION 18 (JUMP, SPREAD, DIMENSION) — *Let the jump of the random walk at x be*

$$J(x) := \mathbb{E}_{m_x} d(x, \cdot) = \mathcal{T}_1(\delta_x, m_x)$$

Let the spread of the random walk at x be

$$\sigma(x) := \left(\frac{1}{2} \iint d(y, z)^2 dm_x(y) dm_x(z) \right)^{1/2}$$

and, if ν is a invariant distribution, let

$$\sigma := \|\sigma(x)\|_{L^2(X, \nu)}$$

be the average spread.

Let also $\sigma_\infty(x) := \frac{1}{2} \operatorname{diam} \operatorname{Supp} m_x$ and $\sigma_\infty := \sup \sigma_\infty(x)$.

Let the local dimension at x be

$$n_x := \frac{\sigma(x)^2}{\sup\{\operatorname{Var}_{m_x} f, f \text{ 1-Lipschitz}\}}$$

and finally $n := \inf_x n_x$.

About this definition of dimension. Obviously $n_x \geq 1$. For the discrete-time Brownian motion on a N -dimensional Riemannian manifold, one has $n_x \approx N$ (see the end of Section 8). For the simple random walk on a graph, $n_x \approx 1$. This definition of dimension amounts to saying that in a space of dimension n , the typical variations of a (1-dimensional) Lipschitz function are $1/\sqrt{n}$ times the typical distance between two points. This is the case in the sphere S^n , in the Gaussian measure on \mathbb{R}^n , and in the discrete cube $\{0, 1\}^n$. So generally one could define the “statistical dimension” of a metric measure space (X, d, μ) by this formula i.e.

$$\operatorname{StatDim}(X, d, \mu) := \frac{\frac{1}{2} \iint d(x, y)^2 d\mu(x) d\mu(y)}{\sup\{\operatorname{Var}_\mu f, f \text{ 1-Lipschitz}\}}$$

so that for each $x \in X$ the local dimension of X at x is $n_x = \operatorname{StatDim}(X, d, m_x)$. With this definition, \mathbb{R}^N equipped with a Gaussian measure has statistical dimension N and local dimension $\approx N$, whereas the discrete cube $\{0, 1\}^N$ has statistical dimension $\approx N$ and local dimension ≈ 1 .

We now turn to the description of the main results of the paper.

Elementary properties. In Section 2 are gathered some straightforward results.

First, we prove (Proposition 19) that in an ε -geodesic space, it is enough to get a lower bound on $\kappa(x, y)$ for points x, y with $d(x, y) \leq \varepsilon$, to get a lower bound on κ for all pairs of points. This is simple yet very useful: indeed in the various graphs given above as examples, it was enough to compute the Ricci curvature for neighbors.

Second, we prove equivalent characterizations of having Ricci curvature uniformly bounded from below: A space satisfies $\kappa(x, y) \geq \kappa$ if and only if

the random walk operator is $(1 - \kappa)$ -contracting on the space of probability measures equipped with the transportation distance (Proposition 20), and if and only if the random walk operator acting on Lipschitz functions contracts the Lipschitz norm by $(1 - \kappa)$ (Proposition 28). An immediate corollary of the contracting property for probability measures is the existence of a unique invariant distribution when $\kappa > 0$.

The property of contraction of the Lipschitz norm implies, in the reversible case, that the spectral gap of the Laplacian operator associated with the random walk is at least κ ; this can be seen as a generalization of Lichnerowicz' theorem, and provides sharp estimates of the spectral gap in several examples.

In analogy with the Bonnet–Myers theorem, we prove that if Ricci curvature is bounded below by $\kappa > 0$, then the diameter of the space is at most $2 \sup_x J(x)/\kappa$ (Proposition 23). In case J is unbounded, we can evaluate instead the average distance to a given point x_0 under the invariant distribution ν (Proposition 24); namely, $\int d(x_0, y) d\nu(y) \leq J(x_0)/\kappa$. In particular we have $\int d(x, y) d\nu(x) d\nu(y) \leq 2 \inf J/\kappa$. These are L^1 versions of the Bonnet–Myers theorem rather than generalizations: from the case of manifolds one would expect $1/\sqrt{\kappa}$ instead of $1/\kappa$. Actually this L^1 version is sharp in all our examples except Riemannian manifolds; in Section 7 we investigate additional conditions for an L^2 version of the Bonnet–Myers theorem to hold.

Let us also mention two elementary constructions preserving positive curvature, namely, superposition and L^1 tensorization (Propositions 25 and 26).

Concentration results. Basically, if Ricci curvature is bounded below by $\kappa > 0$, then the invariant distribution satisfies concentration results with variance $\sigma^2/n\kappa$ (up to some constant factor). This estimate is often sharp, as discussed in Section 3.3 where we revisit some of the examples.

However, the type of concentration (Gaussian, exponential, or $1/t^2$) depends on further local assumptions: indeed, just as in the central limit theorem, positive Ricci curvature can only carry at the global scale what is already true at the local scale. Without further assumptions, one only gets that the maximal variance of a 1-Lipschitz function is at most $\sigma^2/n\kappa$, hence concentration like $\sigma^2/n\kappa t^2$ (Proposition 31). If we make the further assumption that the support of the measures m_x is uniformly bounded (i.e. $\sigma_\infty < \infty$), then we get mixed Gaussian-then-exponential concentration, with variance $\sigma^2/n\kappa$ (Theorem 32). The width of the Gaussian window depends on σ_∞ , and on the rate of variation of the spread $\sigma(x)^2$.

For the case of Riemannian manifolds, simply taking smaller and smaller steps for the random walks makes the width of the Gaussian window tend to

infinity, so that we recover Gaussian concentration as in the Lévy–Gromov or Bakry–Émery context. However, for lots of discrete examples, the Gaussian-then-exponential behavior is genuine. Examples where tails are Poisson-like (binomial distribution, $M/M/\infty$ queues) or exponential are given in Sections 3.3.3 to 3.3.5.

We also get concentration results for the finite-time distributions m_x^{*k} (Remark 33).

Log-Sobolev inequality. Using a suitable non-local notion of norm of the gradient, we are able to mimic the proof by Bakry and Émery of a logarithmic Sobolev inequality for the invariant distribution. The gradient we use (Definition 37) is $(Df)(x) := \sup_{y,z} \frac{|f(y) - f(z)|}{d(y,z)} \exp(-\lambda d(x,y) - \lambda d(x,z))$. This is a kind of “semi-local” Lipschitz constant for f . Typically the value of λ can be taken large at the “macroscopic” level; for Riemannian manifolds, taking smaller and smaller steps for the random walk allows to take $\lambda \rightarrow \infty$ so that we recover the usual gradient for smooth functions.

The inequality takes the form $\text{Ent } f \leq C \int (Df)^2 / f \, d\nu$ (Theorem 40). The main tool of the proof is the contraction relation $D(Mf) \leq (1 - \kappa/2)M(Df)$ where M is the random walk operator (Proposition 43).

That the gradient is non-local, with a maximal possible value of λ , is consistent with the possible occurrence of non-Gaussian tails.

Exponential concentration and non-negative curvature. The simplest example of a Markov chain with zero Ricci curvature is the simple random walk on \mathbb{N} or \mathbb{Z} , for which there is no invariant distribution. However, we show that if furthermore there is a “locally attracting” point, then non-negative Ricci curvature implies exponential concentration. The main examples are the geometric distribution on \mathbb{N} , and the exponential distribution $e^{-|x|}$ on \mathbb{R}^n associated with the stochastic differential equation $dX_t = dB_t - \frac{X_t}{|X_t|} dt$. In both cases we recover correct orders of magnitude.

Gromov–Hausdorff topology. One advantage of our definition is that it involves only combinations of the distance function, and no derivatives, so that it is more or less impervious to deformations of the space. In Section 6 we show that Ricci curvature is continuous for Gromov–Hausdorff convergence of metric spaces (suitably reinforced, of course, so that the random walk converges as well), so that having non-negative curvature is a closed property. We also suggest a loosened definition of Ricci curvature, requiring that $\mathcal{T}_1(m_x, m_y) \leq (1 - \kappa)d(x, y) + \delta$ instead of $\mathcal{T}_1(m_x, m_y) \leq (1 - \kappa)d(x, y)$. With this definition, positive curvature becomes an *open* property, so that a

space close to one with positive curvature has positive curvature. Properties of this loose version will be investigated in another paper.

2 Elementary properties

2.1 Geodesic spaces

The idea behind curvature is to use local properties to derive global ones. We give here a simple proposition expressing that in near-geodesic spaces, such as graphs or manifolds, it is enough to check positivity of Ricci curvature for nearby points.

PROPOSITION 19 – Suppose that (X, d) is ε -geodesic in the sense that for any two points $x, y \in X$, there exists an integer n and a sequence $x_0 = x, x_1, \dots, x_n = y$ such that $d(x_i, x_{i+1}) \leq \varepsilon$ and $d(x, y) = \sum d(x_i, x_{i+1})$.

Then, if $\kappa(x, y) \geq \kappa$ for any pair of points with $d(x, y) \leq \varepsilon$, then $\kappa(x, y) \geq \kappa$ for any pair of points $x, y \in X$.

PROOF – Since \mathcal{T}_1 is a distance, one has $\mathcal{T}_1(m_x, m_y) \leq \sum \mathcal{T}_1(m_{x_i}, m_{x_{i+1}}) \leq (1 - \kappa) \sum d(x_i, x_{i+1})$. \square

2.2 Contraction on the space of probability measures

Let $\mathcal{P}(X)$ by the space of all probability measures μ on X with finite first moment, i.e. for some (hence any) $o \in X$, $\int d(o, x) d\mu(x) < \infty$. On $\mathcal{P}(X)$, the transportation distance \mathcal{T}_1 is finite, so that it is actually a distance.

Let μ be a probability measure on X and define the measure

$$\mu * m := \int_{x \in X} d\mu(x) m_x$$

which is the image of μ by the random walk. (It may or may not belong to $\mathcal{P}(X)$.)

The following proposition also appears in [DGW04] (in the proof of Proposition 2.10) and in [Oli].

PROPOSITION 20 – Let (X, d, m) be a metric space with a random walk. Let $\kappa \in \mathbb{R}$. Then we have $\kappa(x, y) \geq \kappa$ for all $x, y \in X$, if and only if for any two probability distributions $\mu, \mu' \in \mathcal{P}(X)$ one has

$$\mathcal{T}_1(\mu * m, \mu' * m) \leq (1 - \kappa) \mathcal{T}_1(\mu, \mu')$$

Moreover in this case, if $\mu \in \mathcal{P}(X)$ then $\mu * m \in \mathcal{P}(X)$.

PROOF – First, suppose that convolution with m is contracting in \mathcal{T}_1 distance. For some $x, y \in X$, let $\mu = \delta_x$ and $\mu' = \delta_y$ be the Dirac measures at x and y . Then by definition $\delta_x * m = m_x$ and likewise for y , so that $\mathcal{T}_1(m_x, m_y) \leq (1 - \kappa)\mathcal{T}_1(\delta_x, \delta_y) = (1 - \kappa)d(x, y)$ as required.

The converse is more difficult to write than to understand. For each pair (x, y) let ξ_{xy} be a coupling (i.e. a measure on $X \times X$) between m_x and m_y witnessing for $\kappa(x, y) \geq \kappa$. According to Corollary 5.22 in [Vil], we can choose ξ_{xy} to depend measurably on the pair (x, y) . Let Ξ be a coupling between μ and μ' witnessing for $\mathcal{T}_1(\mu, \mu')$. Then $\int_{X \times X} d\Xi(x, y) \xi_{xy}$ is a coupling between $\mu * m$ and $\mu' * m$ and so

$$\begin{aligned} \mathcal{T}_1(\mu * m, \mu' * m) &\leq \int_{x,y} d(x, y) d\left\{\int_{x',y'} d\Xi(x', y') \xi_{x'y'}\right\}(x, y) \\ &= \int_{x,y,x',y'} d\Xi(x', y') d\xi_{x'y'}(x, y) d(x, y) \\ &\leq \int_{x',y'} d\Xi(x', y') d(x', y')(1 - \kappa(x', y')) \\ &\leq (1 - \kappa)\mathcal{T}_1(\mu, \mu') \end{aligned}$$

by the Fubini theorem applied to $d(x, y) d\Xi(x', y') d\xi_{x'y'}(x, y)$.

To see that in this situation $\mathcal{P}(X)$ is preserved by the random walk, fix some origin $o \in X$ and note that for any $\mu \in \mathcal{P}(X)$, the first moment of $\mu * m$ is $\mathcal{T}_1(\delta_o, \mu * m) \leq \mathcal{T}_1(\delta_o, m_o) + \mathcal{T}_1(m_o, \mu * m) \leq \mathcal{T}_1(\delta_o, m_o) + (1 - \kappa)\mathcal{T}_1(o, \mu)$. Now $\mathcal{T}_1(o, \mu) < \infty$ by assumption, and $\mathcal{T}_1(\delta_o, m_o) < \infty$ by our definition of random walks (Definition 1). \square

As an immediate consequence of this contracting property we get:

COROLLARY 21 – Suppose that $\kappa(x, y) \geq \kappa > 0$ for any two distinct $x, y \in X$. Then the random walk has a unique invariant distribution $\nu \in \mathcal{P}(X)$.

Moreover, for any probability measure $\mu \in \mathcal{P}(X)$, the sequence $\mu * m^{*n}$ tends exponentially fast to ν in \mathcal{T}_1 distance. Namely

$$\mathcal{T}_1(\mu * m^{*n}, \nu) \leq (1 - \kappa)^n \mathcal{T}_1(\mu, \nu)$$

and in particular

$$\mathcal{T}_1(m_x^{*n}, \nu) \leq (1 - \kappa)^n J(x)/\kappa$$

The last assertion follows by taking $\mu = \delta_x$ and noting that $J(x) = \mathcal{T}_1(\delta_x, m_x)$ so that $\mathcal{T}_1(\delta_x, \nu) \leq \mathcal{T}_1(\delta_x, m_x) + \mathcal{T}_1(m_x, \nu) \leq J(x) + (1 - \kappa)\mathcal{T}_1(\delta_x, \nu)$, hence $\mathcal{T}_1(\delta_x, \nu) \leq J(x)/\kappa$.

Another interesting corollary is the following, which allows to estimate the average of a Lipschitz function under the invariant measure, knowing some of its values. This is useful in concentration theorems, to get bounds not only on the deviations from the average, but on what the average actually is.

COROLLARY 22 – Suppose that $\kappa(x, y) \geq \kappa > 0$ for any two distinct $x, y \in X$. Let ν be the invariant distribution.

Let f be a 1-Lipschitz function. Then, for any distribution μ , one has $|\mathbb{E}_\nu f - \mathbb{E}_\mu f| \leq \mathcal{T}_1(\mu, \mu * m)/\kappa$.

In particular, for any $x \in X$ one has $|f(x) - \mathbb{E}_\nu f| \leq J(x)/\kappa$.

PROOF – One has $\mathcal{T}_1(\mu * m, \nu) \leq (1 - \kappa)\mathcal{T}_1(\mu, \nu)$. Since by the triangle inequality, $\mathcal{T}_1(\mu * m, \nu) \geq \mathcal{T}_1(\mu, \nu) - \mathcal{T}_1(\mu, \mu * m)$, one gets $\mathcal{T}_1(\mu, \nu) \leq \mathcal{T}_1(\mu, \mu * m)/\kappa$. Now if f is a 1-Lipschitz function, for any two distributions μ, μ' one has $|\mathbb{E}_\mu f - \mathbb{E}_{\mu'} f| \leq \mathcal{T}_1(\mu, \mu')$ hence the result.

The last assertion is simply the case when μ is the Dirac measure at x .

□

2.3 L^1 Bonnet–Myers theorems

We now give a weak analogue of the Bonnet–Myers theorem. This result shows in particular that positivity of Ricci curvature is a much stronger property than some spectral gap bound: there is no Ricci curvature analogue of a family of expanders.

PROPOSITION 23 (L^1 BONNET–MYERS) – Suppose that $\kappa(x, y) \geq \kappa > 0$ for all $x, y \in X$. Then for any $x, y \in X$ one has

$$d(x, y) \leq \frac{J(x) + J(y)}{\kappa(x, y)}$$

and in particular

$$\text{diam } X \leq \frac{2 \sup_x J(x)}{\kappa}$$

PROOF – Let $d = d(x, y)$. By assumption we have $\mathcal{T}_1(m_x, m_y) \leq d(1 - \kappa)$. By definition we have $\mathcal{T}_1(m_x, \delta_x) = J(x)$ and $\mathcal{T}_1(m_y, \delta_y) = J(y)$. So $d \leq J(x) + J(y) + d(1 - \kappa)$. □

This result is not sharp at all for Brownian motion in Riemannian manifolds (since $J \approx \varepsilon$ and $\kappa \approx \varepsilon^2 \text{Ric}/N$, it fails by a factor $1/\varepsilon$ compared to the Bonnet–Myers theorem!), but is sharp in many other examples.

For the discrete cube $X = \{0, 1\}^N$ (Example 8 above), one has $J = 1/2$ and $\kappa = 1/N$, so we get $\text{diam } X \leq N$ which is the exact value.

For the discrete Ornstein–Uhlenbeck process (Example 10 above) one has $J = 1/2$ and $\kappa = 1/2N$, so we get $\text{diam } X \leq 2N$ which once more is the exact value.

For the continuous Ornstein–Uhlenbeck process on \mathbb{R} (Example 9 with $N = 1$), the diameter is infinite, consistently with the fact that J is unbounded. If we restrict the process to some large interval $[-R; R]$ with $R \gg s/\sqrt{\alpha}$ (e.g. by reflecting the Brownian part), then $\sup J \sim \alpha R \delta t$ on this interval, and $\kappa = (1 - e^{\alpha \delta t}) \sim \alpha \delta t$ so that the diameter is bounded by $2R$, which is correct.

These examples show that one cannot replace J/κ with $J/\sqrt{\kappa}$ in this result (as could be expected from the example of Riemannian manifolds). In fact, Riemannian manifolds seem to be the only simple example where there is a diameter bound behaving like $1/\sqrt{\kappa}$. In Section 7 we investigate conditions under which an L^2 version of the Bonnet–Myers theorem holds.

In case J is not bounded, we can estimate instead the “average” diameter $\int d(x, y) d\nu(x) d\nu(y)$ under the invariant distribution ν . This estimate will prove very useful in several examples, to get bounds on the average of $\sigma(x)$ in cases where $\sigma(x)$ is unbounded but controlled by the distance to some “origin” (see e.g. Sections 3.3.4 and 3.3.5).

PROPOSITION 24 (AVERAGE L^1 BONNET–MYERS) – Suppose that $\kappa(x, y) \geq \kappa > 0$ for any two distinct $x, y \in X$. Then for any $x \in X$,

$$\int_X d(x, y) d\nu(y) \leq \frac{J(x)}{\kappa}$$

and so

$$\int_{X \times X} d(x, y) d\nu(x) d\nu(y) \leq \frac{2 \inf_x J(x)}{\kappa}$$

PROOF – The first assertion follows from Corollary 22 with $f = d(x, \cdot)$.

For the second assertion, choose an x_0 such that $J(x_0)$ is arbitrarily close to $\inf J$, and write

$$\begin{aligned} \int_{X \times X} d(y, z) d\nu(y) d\nu(z) &\leq \int_{X \times X} (d(y, x_0) + d(x_0, z)) d\nu(y) d\nu(z) \\ &= 2T_1(\delta_{x_0}, \nu) \leq 2J(x_0)/\kappa \end{aligned}$$

which ends the proof. \square

2.4 Two constructions

Here we describe two very simple constructions which trivially preserve positive curvature, namely, superposition and L^1 tensorization.

Superposition states that if we are given two random walks on the same space and construct a new one by, at each step, tossing a coin and deciding to follow either one random walk or the other, then the Ricci curvatures mix nicely.

PROPOSITION 25 (SUPERPOSITION) – *Let X be a metric space equipped with a family $(m^{(i)})$ of random walks. Suppose that for each i , the Ricci curvature of $m^{(i)}$ is at least κ_i . Let (α_i) be a family of non-negative real numbers such that $\sum \alpha_i = 1$. Define a random walk m on X by $m_x := \sum \alpha_i m_x^{(i)}$. Then the Ricci curvature of m is at least $\sum \alpha_i \kappa_i$.*

PROOF – Let $x, y \in X$ and for each i let ξ_i be a coupling between $m_x^{(i)}$ and $m_y^{(i)}$. Then $\sum \alpha_i \xi_i$ is a coupling between $\sum \alpha_i m_x^{(i)}$ and $\sum \alpha_i m_y^{(i)}$, so that

$$\begin{aligned} \mathcal{T}_1(m_x, m_y) &\leq \sum \alpha_i \mathcal{T}_1(m_x^{(i)}, m_y^{(i)}) \\ &\leq \sum \alpha_i (1 - \kappa_i) d(x, y) \\ &= \left(1 - \sum \alpha_i \kappa_i\right) d(x, y) \end{aligned}$$

Note that the coupling above, which consists in sending each $m_x^{(i)}$ to $m_y^{(i)}$, has no reason to be optimal, so that in general equality does not hold. \square

Tensorization states that if we perform a random walk in a product space by deciding at random, at each step, to move in one or the other component, then positive curvature is preserved.

PROPOSITION 26 (L^1 TENSORIZATION) – *Let (X_1, \dots, X_k) be a finite family of metric spaces equipped with a family of random walks $(m^{(1)}, \dots, m^{(k)})$. Let X be the product of the spaces X_i , equipped with the distance $\sum d_i$. Let (α_i) be a family of non-negative real numbers such that $\sum \alpha_i = 1$. Consider the random walk on X defined by*

$$m_{(x_i)} := \sum \alpha_i \delta_{x_1} \otimes \cdots \otimes m_{x_i} \otimes \cdots \otimes \delta_{x_k}$$

Suppose that for each i , the Ricci curvature of $m^{(i)}$ is at least κ_i . Then the Ricci curvature of m is at least $\inf \alpha_i \kappa_i$.

For example, this allows for a very short proof that the curvature of the lazy random walk on the discrete cube $\{0, 1\}^N$ is $1/N$ (Example 8). Indeed,

it is the N -fold product of the random walk on $\{0, 1\}$ which sends each point to the equilibrium distribution $(1/2, 1/2)$, hence is of curvature 1.

The case when some α_i is equal to 0 shows why the Ricci curvature is given by an infimum: indeed, if $\alpha_i = 0$ then the corresponding component never gets mixed, hence curvature cannot be positive (unless this component is reduced to a single point).

Here the statement is restricted to a finite product for the following technical reasons: First, to define the L^1 product of an infinite family, a basepoint has to be chosen. Second, in order for the formula above to define a random walk with finite first moment (see Definition 1), some uniform assumption on the first moments of the $m^{(i)}$ is needed.

PROOF – For $x \in X$ let $\tilde{m}_x^{(i)}$ stand for $\delta_{x_1} \otimes \cdots \otimes m_{x_i} \otimes \cdots \otimes \delta_{x_k}$.

Let $x = (x_i)$ and $y = (y_i)$ be two points in X . Then

$$\begin{aligned} \mathcal{T}_1(m_x, m_y) &\leqslant \sum \alpha_i \mathcal{T}_1(\tilde{m}_x^{(i)}, \tilde{m}_y^{(i)}) \\ &\leqslant \sum \alpha_i \left(\mathcal{T}_1(m_x^{(i)}, m_y^{(i)}) + \sum_{j \neq i} d_j(x_j, y_j) \right) \\ &\leqslant \sum \alpha_i \left((1 - \kappa_i) d_i(x_i, y_i) + \sum_{j \neq i} d_j(x_j, y_j) \right) \\ &= \sum \alpha_i \left(-\kappa_i d_i(x_i, y_i) + \sum d_j(x_j, y_j) \right) \\ &= \sum d_i(x_i, y_i) - \sum \alpha_i \kappa_i d_i(x_i, y_i) \\ &\leqslant (1 - \inf \alpha_i \kappa_i) d(x, y) \end{aligned}$$

□

2.5 Lipschitz functions and spectral gap

DEFINITION 27 (AVERAGING OPERATOR, LAPLACIAN) – For $f \in L^2(X, \nu)$ let the averaging operator M be

$$Mf(x) := \int_y f(y) dm_x(y)$$

and let $\Delta := M - \text{Id}$.

(This is the layman's convention for the sign of the Laplacian, i.e. $\Delta = \frac{d^2}{dx^2}$ on \mathbb{R} , so that on a Riemannian manifold Δ is a negative operator.)

The following proposition also appears in [DGW04] (in the proof of Proposition 2.10).

PROPOSITION 28 – *Let (X, d, m) be a random walk on a metric space. Let $\kappa \in \mathbb{R}$.*

Then the Ricci curvature of X is at least κ , if and only if, for every k -Lipschitz function $f : X \rightarrow \mathbb{R}$, the function Mf is $k(1 - \kappa)$ -Lipschitz.

PROOF – First, suppose that the Ricci curvature of X is at least κ . Then we have

$$\begin{aligned} Mf(y) - Mf(x) &= \int_z f(y+z) - f(x+z) \\ &\leq k \int_z d(x+z, y+z) \\ &= kd(x, y)(1 - \kappa(x, y)) \end{aligned}$$

Conversely, suppose that whenever f is 1-Lipschitz, Mf is $(1 - \kappa)$ -Lipschitz. The duality theorem for transportation distance (Theorem 1.14 in [Vil03]) states that

$$\begin{aligned} \mathcal{T}_1(m_x, m_y) &= \sup_{f \text{ 1-Lipschitz}} \int f \, d(m_x - m_y) \\ &= \sup_{f \text{ 1-Lipschitz}} Mf(x) - Mf(y) \\ &\leq (1 - \kappa)d(x, y) \end{aligned}$$

□

Let ν be an invariant distribution of the random walk. Consider the space $L^2(X, \nu)/\{\text{const}\}$ equipped with the norm $\|f\|_{L^2(X, \nu)/\{\text{const}\}}^2 := \|f - \int f \, d\nu\|_{L^2(X, \nu)}^2$ so that

$$\|f\|_{L^2(X, \nu)/\{\text{const}\}}^2 = \text{Var}_\nu f = \frac{1}{2} \int_{X \times X} (f(x) - f(y))^2 \, d\nu(x) \, d\nu(y)$$

The operators M and Δ are self-adjoint in $L^2(X, \nu)$ if and only if ν is reversible for the random walk.

It is easy to check, using associativity of variances, that

$$\text{Var}_\nu f = \int \text{Var}_{m_x} f \, d\nu(x) + \text{Var}_\nu Mf$$

so that $\|Mf\|_2 \leq \|f\|_2$. It is also clear that $\|Mf\|_\infty \leq \|f\|_\infty$.

Usually, spectral gap properties for Δ are expressed in the space L^2 . The proposition above only implies that the spectral radius of the operator M acting on $\text{Lip}(X)/\{\text{const}\}$ is at most $(1 - \kappa)$. In general it is not true that a bound for the spectral radius of an operator on a dense subspace of a Hilbert space implies a bound for the spectral radius on the whole space. This holds, however, when the operator is self-adjoint or when the Hilbert space is finite-dimensional.

PROPOSITION 29 – Let (X, d, m) be metric space with random walk, with invariant distribution ν . Suppose that the Ricci curvature of X is at least $\kappa > 0$ and that $\sigma < \infty$. Suppose that ν is reversible, or that X is finite.

Then the spectral radius of the averaging operator acting on $L^2(X, \nu)/\{\text{const}\}$ is at most $1 - \kappa$.

PROOF – First, if X is finite then Lipschitz functions coincide with L^2 functions, so that there is nothing to prove. So we suppose that ν is reversible, i.e. M is self-adjoint.

Let f be a k -Lipschitz function. Proposition 31 below implies that Lipschitz functions belong to L^2 and that the Lipschitz norm controls the L^2 norm. (This is where we use that $\sigma < \infty$.)

Since $M^t f$ is $k(1 - \kappa)^t$ -Lipschitz one gets $\text{Var } M^t f \leq C k^2 (1 - \kappa)^{2t}$ for some constant C so that $\lim_{t \rightarrow \infty} (\sqrt{\text{Var } M^t f})^{1/t} \leq (1 - \kappa)$. Now Lipschitz functions are dense in $L^2(X, \nu)$. Since M is bounded and self-adjoint, its spectral radius is at most $1 - \kappa$. \square

COROLLARY 30 – Let (X, d, m) be an ergodic random walk on a metric space, with invariant distribution ν . Suppose that the Ricci curvature of X is at least $\kappa > 0$ and that $\sigma < \infty$. Suppose that ν is reversible.

Then the smallest eigenvalue of $-\Delta$ on $L^2(X, \nu)/\{\text{const}\}$ is at least κ .

Moreover the following discrete Poincaré inequalities are satisfied for $f \in L^2(X, \nu)$:

$$\text{Var}_\nu f \leq \frac{1}{\kappa(2 - \kappa)} \int \text{Var}_{m_x} f \, d\nu(x)$$

and

$$\text{Var}_\nu f \leq \frac{1}{2\kappa} \iint (f(y) - f(x))^2 \, d\nu(x) \, dm_x(y)$$

PROOF – These are rewritings of the inequalities $\text{Var}_\nu Mf \leq (1 - \kappa)^2 \text{Var}_\nu f$ and $\langle f, Mf \rangle_{L^2(X, \nu)/\{\text{const}\}} \leq (1 - \kappa) \text{Var}_\nu f$, respectively. \square

The quantities $\text{Var}_{m_x} f$ and $\frac{1}{2} \int (f(y) - f(x))^2 \, dm_x(y)$ are two possible definitions of $\|\nabla f(x)\|^2$ in a discrete setting. Though the latter is more

common, the former is preferable when the support of m_x can be far away from x and cancels out the “drift”. Moreover one always has $\text{Var}_{m_x} f \leq \int (f(y) - f(x))^2 dm_x(y)$, so that the first form is generally sharper (note that since $\kappa \leq 1$ one has $1/\kappa(2-\kappa) \leq 1/\kappa$).

Reversibility is really needed here to turn an estimate of the spectral radius of M into an inequality between the norms of Mf and f , using that M is self-adjoint. When the random walk is not reversible, a version of the Poincaré inequality with a non-local gradient still holds (Theorem 40).

Let us compare this result to Lichnerowicz’ theorem in the case of the random walk at scale ε on an N -dimensional Riemannian manifold with positive Ricci curvature. The operator Δ associated with the random walk is the difference between the mean value of a function on a ball of radius ε , and its value at the center of the ball: when $\varepsilon \rightarrow 0$ this behaves like $\frac{\varepsilon^2}{2(N+2)}$ times the usual Laplacian, by taking the average on the ball of the Taylor expansion of f . Meanwhile, we saw (Example 7) that $\kappa \sim \frac{\varepsilon^2}{2(N+2)} \inf \text{Ric}$, where $\inf \text{Ric}$ is the largest K such that $\text{Ric}(v, v) \geq K$ for all unit tangent vectors v . Note that both scaling factors are the same. On the other hand the Lichnerowicz theorem states that the smallest eigenvalue of the usual Laplacian is $\frac{N}{N-1} \inf \text{Ric}$. So we miss the $\frac{N}{N-1}$ factor, but otherwise get the correct order of magnitude.

Second, let us test this corollary for the discrete cube of Example 8. In this case the eigenbase of the discrete Laplacian is well-known (characters, or Fourier/Walsh transform), and the spectral radius of the lazy random walk is exactly $1 - 1/N$. Since the Ricci curvature κ is $1/N$, the value given in the proposition is sharp.

Third, consider the Ornstein–Uhlenbeck process on \mathbb{R} , as in Example 9. Its infinitesimal generator is $L = \frac{s^2}{2} \frac{d}{dx^2} - \alpha x \frac{d}{dx}$, and the eigenfunctions are known to be $H_k(x\sqrt{\alpha/s^2})$ where H_k is the Hermite polynomial $H_k(x) := (-1)^k e^{x^2} \frac{d^k}{dx^k} e^{-x^2}$. The associated eigenvalue of L is $-n\alpha$, so that the spectral gap of L is α . Now the random walk we consider is the flow $e^{\delta t L}$ at time δt of the process (with small δt), whose eigenvalues are $e^{-n\alpha\delta t}$. So the spectral gap of the discrete Laplacian $e^{\delta t L} - \text{Id}$ is $1 - e^{-\alpha\delta t}$. Since the Ricci curvature is $1 - e^{-\alpha\delta t}$ too, the corollary is sharp again.

3 Concentration results

3.1 Variance of Lipschitz functions

We begin with the simplest kind of concentration, namely, an estimation of the variance of Lipschitz functions. Contrary to Gaussian or exponential concentration, the only assumption needed here is that the average spread σ is finite.

Since our Gaussian concentration result will yield basically the same variance $\sigma^2/n\kappa$, we discuss sharpness of this estimate in various examples in Section 3.3.

PROPOSITION 31 – *Let (X, d, m) be a random walk on a metric space, with Ricci curvature at least $\kappa > 0$. Let ν be the unique invariant distribution. Suppose that $\sigma < \infty$.*

Then the variance of a 1-Lipschitz function is at most $\frac{\sigma^2}{n\kappa(2-\kappa)} \leq \frac{\sigma^2}{n\kappa}$.

In particular, this implies that all Lipschitz functions are in $L^2/\{\text{const}\}$; especially, $\int d(x, y)^2 d\nu(x) d\nu(y)$ is finite. The fact that the Lipschitz norm controls the L^2 norm was used above in the discussion of spectral properties of the random walk operator.

PROOF – Suppose for now that f is bounded by $A \in \mathbb{R}$, so that $\text{Var } f < \infty$. We first prove that $\text{Var } M^t f$ tends to 0. Let B_r be the ball of radius r in X centered at some basepoint. Using that $M^t f$ is $(1 - \kappa)^t$ -Lipschitz on B_r and bounded by A on $X \setminus B_r$, we get $\text{Var } M^t f = \frac{1}{2} \iint (f(x) - f(y))^2 d\nu(x) d\nu(y) \leq 2(1 - \kappa)^{2t} r^2 + 2A^2 \nu(X \setminus B_r)$. Taking for example $r = 1/(1 - \kappa)^{t/2}$ ensures that $\text{Var } M^t f \rightarrow 0$.

As already mentioned, one has $\text{Var } f = \text{Var } Mf + \int \text{Var}_{m_x} f d\nu(x)$. Since $\text{Var } M^t f \rightarrow 0$, by induction we get

$$\text{Var } f = \sum_{t=0}^{\infty} \int \text{Var}_{m_x} M^t f d\nu(x)$$

Now by definition $\text{Var}_{m_x} f \leq \sigma(x)^2/n_x$. Since $M^t f$ is $(1 - \kappa)^t$ -Lipschitz, we have $\text{Var}_{m_x} M^t f \leq (1 - \kappa)^{2t} \sigma(x)^2/n_x$ so that the sum above is at most $\frac{\sigma^2}{n\kappa(2-\kappa)}$. The case of unbounded f is treated by a simple limiting argument. \square

3.2 Gaussian concentration

As mentioned above, positive Ricci curvature implies a Gaussian-then-exponential concentration theorem. The estimated variance is $\sigma^2/n\kappa$ as above, so that this is essentially a more precise version of Proposition 31, with some loss in

the constants. We will see in the discussion below (Section 3.3) that in the main examples, the order of magnitude is correct.

The fact that concentration is not Gaussian far away from the mean is genuine, as exemplified by the binomial distribution on the cube (Section 3.3.3) or $M/M/\infty$ queues (Section 3.3.4). A purely exponential behavior can be achieved in very simple examples if $\sigma_\infty(x)$ is not bounded (Example 14) or if the spread $\sigma(x)^2$ grows fast enough (Section 3.3.5). In these examples, the transition from Gaussian to non-Gaussian regime occurs roughly as predicted by the theorem.

In the case of Riemannian manifolds, simply letting the step of the random walk tend to 0 makes the width of the Gaussian window tend to infinity, so that we recover Gaussian concentration as in the Lévy–Gromov or Bakry–Émery theorems.

The width of the Gaussian window is controlled by two factors: the quantity σ_∞ , which represents the “granularity” of the process and can result in Poisson-like behavior; and the rate of variation of the spread $\sigma(x)^2$, which can result in exponential behavior. The latter phenomenon yields to the assumption that $\sigma(x)^2$ is bounded by a Lipschitz function.

THEOREM 32 – *Let (X, d, m) be an ergodic random walk on a metric space as above, with invariant distribution ν . Suppose that for any two distinct points $x, y \in X$ one has $\kappa(x, y) \geq \kappa > 0$.*

Let

$$D_x^2 := \frac{\sigma(x)^2}{n_x \kappa}$$

and

$$D^2 := \mathbb{E}_\nu D_x^2$$

Suppose that the function $x \mapsto D_x^2$ is C -Lipschitz. Set

$$t_{\max} := \frac{2D^2}{\max(2C, 3\sigma_\infty)}$$

Then for any 1-Lipschitz function f , for any $t \leq t_{\max}$ we have

$$\nu(\{x, f(x) \geq t + \mathbb{E}_\nu f\}) \leq \exp - \frac{t^2}{6D^2}$$

and for $t \geq t_{\max}$

$$\nu(\{x, f(x) \geq t + \mathbb{E}_\nu f\}) \leq \exp \left(-\frac{t_{\max}^2}{6D^2} - \frac{t - t_{\max}}{\max(2C, 3\sigma_\infty)} \right)$$

REMARK 33 – It is clear from the proof below that $\sigma(x)^2/n_x\kappa$ itself need not be Lipschitz, only bounded by some Lipschitz function. In particular, if $\sigma(x)^2$ is bounded one can always take $D^2 = \sup_x \frac{\sigma(x)^2}{n_x\kappa}$ and $C = 0$.

It might seem that, in order to estimate $\mathbb{E}_\nu D_x^2$, one needs to know in advance concentration properties for the invariant distribution ν ; however, Proposition 24 or Corollary 22 often provides sharp estimates for $\mathbb{E}_\nu D_x^2$, as we shall see in the examples.

In Section 3.3.5, we give a simple example where the Lipschitz constant of $\sigma(x)^2$ is large, resulting in exponential rather than Gaussian behavior. In Section 3.3.6 we give an example of a process with quadratic growth of $\sigma(x)^2$, and which exhibits non-exponential tails. Thus the Lipschitz assumption cannot simply be removed.

The assumption that σ_∞ is bounded can be replaced with a Gaussian-type control for the local measures m_x , which however generally results in much poorer estimates of the variance in discrete situations (see Remark 35).

PROOF – This proof is a variation on standard martingale methods for concentration (see e.g. Lemma 4.1 in [Led01]).

Let f be a 1-Lipschitz function and $\lambda \geq 0$. For any smooth function g and any real-valued random variable Y , a Taylor expansion gives $\mathbb{E}g(Y) \leq g(\mathbb{E}Y) + \frac{1}{2}(\sup g'') \text{Var } Y$, so that

$$(M e^{\lambda f})(x) \leq e^{\lambda Mf(x)} + \frac{\lambda^2 e^{\lambda(Mf(x)+2\sigma_\infty)}}{2} \text{Var}_{m_x} f$$

Take $\lambda < 1/3\sigma_\infty$ so that $e^{2\lambda\sigma_\infty} \leq 2$. By definition, $\text{Var}_{m_x} f \leq \|f\|_{\text{Lip}}^2 \sigma(x)^2/n_x$, hence

$$(M e^{\lambda f})(x) \leq e^{\lambda Mf(x)} \left(1 + \lambda^2 \frac{\sigma(x)^2}{n_x}\right) \leq e^{\lambda \left(Mf(x) + \lambda \frac{\sigma(x)^2}{n_x}\right)}$$

But since $\sigma(x)^2/n_x\kappa$ is C -Lipschitz by assumption, and since besides $Mf(x)$ is $(1 - \kappa)$ -Lipschitz by Proposition 28, the sum $Mf(x) + \lambda \frac{\sigma(x)^2}{n_x}$ is $(1 - \kappa + \lambda C\kappa)$ -Lipschitz.

From now on we take $\lambda \leq 1/2C$. We can repeat the argument, setting $f_1(x) := Mf(x) + \lambda \frac{\sigma(x)^2}{n_x}$ and using that f_1 is $(1 - \kappa/2)$ -Lipschitz. This yields

$$(M^2 e^{\lambda f})(x) \leq (M e^{\lambda f_1})(x) \leq e^{\lambda Mf_1(x) + \lambda^2 \frac{\sigma(x)^2}{n_x} (1 - \kappa/2)^2}$$

Next, Mf_1 is $(1 - \kappa)(1 - \kappa/2)$ -Lipschitz, whereas $\lambda \frac{\sigma(x)^2}{n_x} (1 - \kappa/2)^2$ is $\frac{\kappa}{2}(1 - \kappa/2)^2$ -Lipschitz. So $f_2(x) := Mf_1(x) + \lambda \frac{\sigma(x)^2}{n_x} (1 - \kappa/2)^2$ is (at least) $(1 - \kappa/2)^2$ -Lipschitz, hence

$$(M^3 e^{\lambda f})(x) \leq (M e^{\lambda f_2})(x) \leq e^{\lambda Mf_2(x) + \lambda^2 \frac{\sigma(x)^2}{n_x} (1 - \kappa/2)^4}$$

By induction, we get that $f_{k+1}(x) := Mf_k(x) + \lambda \frac{\sigma(x)^2}{n_x} (1 - \kappa/2)^{2k}$ is $(1 - \kappa/2)^{k+1}$ -Lipschitz and that $(M^k e^{\lambda f})(x) \leq e^{\lambda f_k(x)}$.

Now setting $g(x) := \frac{\sigma(x)^2}{n_x}$ and expanding f_k yields

$$f_k(x) = (M^k f)(x) + \lambda \sum_{i=1}^k (M^{k-i} g)(x) (1 - \kappa/2)^{2(i-1)}$$

so that the limit of $f_k(x)$ when $k \rightarrow \infty$ is

$$\mathbb{E}_\nu f + \lambda \sum_{i=1}^{\infty} \mathbb{E}_\nu g (1 - \kappa/2)^{2(i-1)} \leq \mathbb{E}_\nu f + \lambda \mathbb{E}_\nu g \frac{4}{3\kappa}$$

Meanwhile, $(M^k e^{\lambda f})(x)$ tends to $\mathbb{E}_\nu e^{\lambda f}$, so that

$$\mathbb{E}_\nu e^{\lambda f} \leq e^{\lambda \mathbb{E}_\nu f + \frac{4\lambda^2}{3\kappa} \mathbb{E}_\nu \frac{\sigma(x)^2}{n_x}}$$

We can conclude by a standard Chebyshev inequality argument. \square

REMARK 34 – The proof provides a similar concentration result for the finite-time measures μ_x^{*k} as well, with variance

$$D_{x,k}^2 = \sum_{i=1}^k (1 - \kappa/2)^{2(i-1)} \left(M^{k-i} \frac{\sigma(y)^2}{n_y} \right)(x)$$

and the same expression for t_{\max} .

REMARK 35 – The condition that σ_∞ is uniformly bounded can be replaced with a Gaussian-type assumption, namely that for each measure m_x there exists a number s_x such that $\mathbb{E}_{m_x} e^{\lambda f} \leq e^{\lambda^2 s_x^2 / 2} e^{\lambda \mathbb{E}_{m_x} f}$ for any 1-Lipschitz function f . Then a similar theorem holds, with $\sigma(x)^2$ replaced with s_x^2 . (When s_x^2 is constant this is Proposition 2.10 in [DGW04].) However, this is generally not well-suited to discrete settings, because when transition probabilities are small, the best s_x^2 for which such an inequality is satisfied is usually much larger than the actual variance $\sigma(x)^2$: for example, if two points x and y are at distance 1 and $m_x(y) = \varepsilon$, s_x must satisfy $e^{-1/2s_x^2} \leq \varepsilon$ hence $s_x^2 \geq 1/2 \ln(1/\varepsilon) \gg \varepsilon$. Thus making this assumption will provide extremely poor estimates of the variance D^2 when some transition probabilities are small (e.g. for binomial distributions on the discrete cube); however, when this does not occur (e.g. for the uniform distribution on the discrete cube), this assumption allows to get rid of σ_∞ , and even get genuine Gaussian concentration for all $t \in \mathbb{R}$ in the case $C = 0$.

3.3 Examples revisited

Let us test the sharpness of these estimates in some examples, beginning with the simplest ones. In each case, we gather the relevant quantities in a table. Recall that \approx denotes an equality up to a multiplicative universal constant (typically ≤ 4), while symbol \sim denotes usual asymptotic equivalence (with the correct constant).

3.3.1 Riemannian manifolds

First, let X be a N -dimensional Riemannian manifold with positive Ricci curvature. Equip this manifold with the random walk at scale $\varepsilon > 0$, as in Example 7.

Let $\inf \text{Ric}$ denote the largest $K > 0$ such that $\text{Ric}(v, v) \geq K$ for any unit tangent vector v . The relevant quantities for this random walk are as follows (see Section 8 for the proofs).

Ricci curvature	$\kappa \sim \frac{\varepsilon^2}{2(N+2)} \inf \text{Ric}$
Spread	$\sigma(x)^2 \sim \varepsilon^2 \frac{N}{N+2} \quad \forall x$
Dimension	$n \approx N$
Variance (Lévy–Gromov thm.)	$\approx 1/\inf \text{Ric}$
Gaussian variance (Thm. 32)	$D^2 \approx 1/\inf \text{Ric}$
Gaussian range	$t_{\max} \approx 1/(\varepsilon \inf \text{Ric}) \rightarrow \infty$

So, up to some (small) constants, we recover Gaussian concentration as in the Lévy–Gromov theorem.

The same applies to diffusions with a drift on a Riemannian manifold. To be consistent with the notation of Example 11, in the table above ε has to be replaced with $\sqrt{(N+2)\delta t}$, and $\inf \text{Ric}$ with $\inf (\text{Ric}(v, v) - 2\nabla^{\text{sym}} F(v, v))$ for v a unit tangent vector. (In the non-compact case, care has to be taken since the Brownian motion on the manifold may not exist, and even if it does its approximation at time δt may not converge uniformly on the manifold. In explicit examples such as the Ornstein–Uhlenbeck process, however, this is not a problem.)

3.3.2 Discrete cube

Back to the discrete cube $\{0, 1\}^N$ of Example 8, equipped with its graph distance (Hamming metric) and lazy random walk.

Ricci curvature	$\kappa = 1/N$
Spread	$\sigma(x)^2 \approx 1 \quad \forall x$
Dimension	$n \approx 1$
Gaussian variance (Thm. 32)	$D^2 \approx N$
Actual variance	$N/4$

The following simple remark allows to actually compute the small numerical constants implied in the notation \approx , and to check that Proposition 31 gives a sharp value when $N \rightarrow \infty$.

PROPOSITION 36 – *Let m be the lazy simple random walk on a locally finite graph. Then, for any vertex x one has $\sigma(x)^2/n_x \leq 1/2$.*

Applying this to the estimate of Proposition 31 for the discrete cube, one gets $\sigma^2/n\kappa(2-\kappa) \leq 1/2\kappa(2-\kappa)$ which, for $\kappa = 1/N$, yields $N/2(2-1/N) \sim N/4$. (One can actually get exactly $N/4$ by using a continuous-time random walk instead.)

PROOF – By definition $\sigma(x)^2/n_x$ is the maximal variance, under m_x , of a 1-Lipschitz function. So let f be a 1-Lipschitz function on the graph. Since variance is unvariant by adding a constant, we can assume that $f(x) = 0$. Then $|f(y)| \leq 1$ for any neighbor y of x . Since m is the lazy simple random walk, we have $m_x(x) \geq 1/2$ (with equality if there are no loops) and the mass, under m_x , of all neighbors of x is at most $1/2$. Hence $\text{Var}_{m_x} f = \mathbb{E}_{m_x} f^2 - (\mathbb{E}_{m_x} f)^2 \leq \mathbb{E}_{m_x} f^2 \leq 1/2$.

This value is actually achieved when x has an even number of neighbors and when no two distinct neighbors of x are neighbors; in this case one can take $f(x) = 0$, $f = 1$ on half the neighbors of x and $f = -1$ on the remaining neighbors of x . \square

3.3.3 Binomial distributions

The occurrence of a finite range t_{\max} for the Gaussian behavior of tails is genuine, as the following example shows.

Let $X = \{0, 1\}^N$ equipped with its Hamming metric (each edge is of length 1). Consider the following Markov chain on X : for some $0 < p < 1$, at each step, choose a bit at random among the N bits; if it is equal to 0, flip it to 1 with probability p ; if it is equal to 1, flip it to 0 with probability $1-p$. The binomial distribution $\nu((x_i)) = \prod p^{x_i}(1-p)^{1-x_i}$ is reversible for this Markov chain. The Ricci curvature of this Markov chain is $1/N$.

Let k be the number of bits of $x \in X$ which are equal to 1. Then k follows

a Markov chain on $\{0, 1, \dots, N\}$, whose transition probabilities are:

$$\begin{aligned} p_{k,k+1} &= p(1 - k/N) \\ p_{k,k-1} &= (1 - p)k/N \\ p_{k,k} &= pk/N + (1 - p)(1 - k/N) \end{aligned}$$

The binomial distribution with parameters N and p , namely $\binom{N}{k} p^k (1 - p)^{N-k}$, is reversible for this Markov chain. Moreover, the Ricci curvature of this Markov chain is $1/N$.

Now, fix some $\lambda > 0$ and consider the case $p = \lambda/N$. Let $N \rightarrow \infty$. It is well-known that the invariant distribution tends to the Poisson distribution $e^{-\lambda} \lambda^k / k!$ on \mathbb{N} .

Let us see how Theorem 32 performs on this example. The table below applies either to the full space $\{0, 1\}^N$, with k the function “number of 1’s”, or to its projection on $\{0, 1, \dots, N\}$. Note the use of Proposition 24 to estimate σ^2 , without having to resort to explicit knowledge of the invariant distribution. (All constants implied in the $O(1/N)$ notation are small and completely explicit.)

Ricci curvature	$\kappa = 1/N$
Spread	$\sigma(k)^2 = (\lambda + k)/N + O(1/N^2)$
Estimated $\mathbb{E}k$ (Prop. 24)	$\mathbb{E}k \leq J(0)/\kappa = \lambda$
Actual $\mathbb{E}k$	$\mathbb{E}k = \lambda$
Average spread	$\sigma^2 = \mathbb{E}\sigma(k)^2 = 2\lambda/N + O(1/N^2)$
Dimension	$n \geq 1$
Estimated variance (Prop. 31)	$\sigma^2/n\kappa(2 - \kappa) = \lambda + O(1/N)$
Actual variance	λ
Gaussian variance (Thm. 32)	$D^2 = 2\lambda + O(1/N)$
Lipschitz constant of D_x^2	$C = 1 + O(1/N)$
Gaussian range	$t_{\max} = 4\lambda/3$

The Poisson distribution has a roughly Gaussian behavior (with variance λ) in a range of size approximately λ around the mean; further away, it decreases like $e^{-k \ln k}$ which is not Gaussian. This is in good accordance with the theorem, and shows that the Gaussian range cannot be extended.

3.3.4 A continuous-time example: $M/M/\infty$ queues

Here we show how to apply the theorem above to a continuous-time example, the $M/M/\infty$ queue. These queues were brought to my attention by D. Chafaï.

The $M/M/\infty$ queue consists in an infinite number of ‘‘servers’’. Each server can be free (0) or busy (1). The state space consists in all sequences in $\{0, 1\}^{\mathbb{N}}$ with a finite number of 1’s. The dynamics is as follows: Fix two numbers $\lambda > 0$ and $\mu > 0$. At a rate λ per unit of time, a client arrives and the first free server becomes busy. At a rate μ per unit of time, each busy server finishes its job (independently of the others) and becomes free. The number $k \in \mathbb{N}$ of busy servers is a continuous-time Markov chain, whose transition probabilities at small times t are given by

$$\begin{aligned} p_{k,k+1}^t &= \lambda t + O(t^2) \\ p_{k,k-1}^t &= k\mu t + O(t^2) \\ p_{k,k}^t &= 1 - (\lambda + k\mu)t + O(t^2) \end{aligned}$$

If we replace λ with λ/N and μ with $1/N$, this Markov chain appears as the limit of the binomial example above. This is especially clear in the table below.

This system is often presented as a discrete analogue of an Ornstein–Uhlenbeck process, since asymptotically the drift is linear towards the origin. However, it is not symmetric around the mean, and moreover the invariant (actually reversible) distribution ν is a Poisson distribution (with parameter λ/μ), rather than a Gaussian.

In this continuous-time setting, the definition are adapted as follows: $\kappa(x, y) := -\frac{d}{dt}\mathcal{T}_1(m_x^t, m_y^t)/d(x, y)$ (as mentioned in the introduction) and $\sigma(x)^2 := \frac{1}{2}\frac{d}{dt}\iint d(y, z) dm_x^t(y) dm_x^t(z)$, where m_x^t is the law at time t of the process starting at x . It is immediate to check that the Ricci curvature of this process is μ . Proposition 31 (with $\sigma^2/2n\kappa$ instead of $\sigma^2/n\kappa(2 - \kappa)$ because both σ^2 and κ tend to 0 for the discrete-time approximation) and Theorem 32 still hold.

The relevant quantities are as follows.

Ricci curvature	$\kappa = \mu$
Spread	$\sigma(k)^2 = k\mu + \lambda$
Estimated $\mathbb{E}k$ (Prop. 24)	$\mathbb{E}k \leq J(0)/\kappa = \lambda/\mu$
Actual $\mathbb{E}k$	$\mathbb{E}k = \lambda/\mu$
Average spread	$\sigma^2 = \mathbb{E}\sigma(k)^2 = 2\lambda$
Dimension	$n \geq 1$
Estimated variance (Prop. 31)	$\sigma^2/2n\kappa = \lambda/\mu$
Actual variance	λ/μ
Gaussian variance (Thm. 32)	$D^2 = 2\lambda/\mu$
Lipschitz constant of D_x^2	$C = 1$
Gaussian range	$t_{\max} = 4\lambda/3\mu$

So once more Theorem 32 is in excellent accordance with the behavior of the random walk, whose invariant distribution is Poisson with mean and variance λ/μ .

An advantage of this approach is that it can be generalized to situations where the rates of the servers are not constant, but, say, bounded between, say, $\mu_0/10$ and $10\mu_0$. Indeed, the $M/M/\infty$ queue above can be seen as a Markov chain in the full configuration space of the servers, namely the space of all sequences over the alphabet {free, busy} containing a finite number of “busy”. It is easy to check that the Ricci curvature is still equal to μ in this configuration space. Now let us consider the case of variable rates: in this situation, the number of busy servers is generally not Markovian, so one has to work in the configuration space. If the rate of the i -th server is μ_i , the Ricci curvature is $\inf \mu_i$ in the configuration space, whereas the spread is controlled by $\sup \mu_i$. So if the rates vary in a bounded range, Ricci curvature still provides a Gaussian-type control, though an explicit description of the invariant distribution is not available.

3.3.5 An example of exponential concentration

We give here a very simple example of a Markov chain which has positive curvature but for which concentration is not Gaussian but exponential, due to large variations of the spread, resulting in a large value of C . An even simpler example, with exponential concentration due to unbounded $\sigma_\infty(x)$, was given in the introduction (Example 14).

This is a continuous-time random walk on \mathbb{N} defined as follows. Take $\alpha < \beta \in \mathbb{R}$. For $k \in \mathbb{N}$, the transition rate from k to $k + 1$ is $(k + 1)\alpha$, whereas the transition rate from $k + 1$ to k is $(k + 1)\beta$. It is immediate to check that the geometric distribution with decay α/β is reversible for this Markov chain.

The Ricci curvature of this Markov chain is easily seen to be $\beta - \alpha$. We have $\sigma(k)^2 = (k + 1)\alpha + k\beta$, so that $\sigma(k)^2$ is $(\alpha + \beta)$ -Lipschitz and $C = (\alpha + \beta)/(\beta - \alpha)$.

The expectation of k under the invariant distribution can be bounded by $J(0)/\kappa = \alpha/(\beta - \alpha)$ by Proposition 24, which is actually the exact value. So the expression above for $\sigma(k)^2$ yields $\sigma^2 = 2\alpha\beta/(\beta - \alpha)$. Consequently, the estimated variance $\sigma^2/2n\kappa$ (obtained by the continuous-time version of Proposition 31) is at most $\alpha\beta/(\beta - \alpha)^2$, which is the actual value.

Now consider the case when $\beta - \alpha$ is small. If we try to apply Theorem 32 without taking into account the variations of the spread (witnessed by the constant C), we get blatantly false results since the invariant distribution is not Gaussian at all. In the regime where $\beta - \alpha \rightarrow 0$, the width of the

Gaussian window in Theorem 32 is $D^2/C \approx \alpha/(\beta - \alpha)$. This is fine, as this is the decay distance of the invariant distribution, and in this interval both the Gaussian and geometric estimates are close to 1 anyway. But if the C factor was not included, we would get $D^2/\sigma_\infty = \alpha\beta/(\beta - \alpha)^2$, which is much larger; the invariant distribution is clearly not Gaussian on this interval.

Moreover, Theorem 32 predicts, in the exponential regime, a $\exp(-t/2C)$ behavior for concentration. Here the asymptotic behavior of the invariant distribution is $(\alpha/\beta)^t \sim (1 - 2/C)^t \sim e^{-2t/C}$ when $\beta - \alpha$ is small. So we see that (up to a constant 4) the exponential decay rate predicted by Theorem 32 is genuine.

3.3.6 Heavy tails

It is clear that a variance control alone does not imply any concentration beyond the Bienaymé-Chebyshev inequality. We now show that this is still the case even with the positive curvature assumption. Namely, in Theorem 32, neither the assumption that $\sigma(x)^2$ is Lipschitz, nor the assumption that σ_∞ is bounded, can be removed (but see Remark 35).

Heavy tails with non-Lipschitz $\sigma(x)^2$. Our next example shows that if the spread $\sigma(x)^2$ is not Lipschitz, then non-exponential tails may occur in spite of positive curvature.

Consider the continuous-time random walk on \mathbb{N} defined as follows: the transition rate from k to $k + 1$ is $a(k + 1)^2$, whereas the transition rate from k to $k - 1$ is $a(k + 1)^2 + bk$ for $k \geq 1$. Here $a, b > 0$ are fixed.

We have $\kappa = b$ and $\sigma(k)^2 = 2a(k + 1)^2 + bk$, which is obviously not Lipschitz.

This Markov chain has a reversible measure ν , which satisfies $\nu(k)/\nu(k - 1) = ak^2/(a(k + 1)^2 + bk) = 1 - \frac{1}{k}(2 + \frac{b}{a}) + O(1/k^2)$. Consequently, asymptotically $\nu(k)$ behaves like

$$\prod_{i=1}^k \left(1 - \frac{1}{i}(2 + \frac{b}{a})\right) \approx e^{-(2+b/a)\sum_{i=1}^k \frac{1}{i}} \approx k^{-(2+b/a)}$$

thus exhibiting heavy, non-exponential tails.

This shows that the Lipschitz assumption for $\sigma(x)^2$ cannot be removed, even if in this case σ_∞ is bounded by 1. It would seem reasonable to look for a systematic correspondence between the asymptotic behavior of $\sigma(x)^2$ and the behavior of tails.

Heavy tails with unbounded σ_∞ . Consider the following random walk on \mathbb{N}^* : a number k goes to 1 with probability $1 - 1/4k^2$ and to $2k$ with probability $1/4k^2$. One can check that $\kappa \geq 1/2$. These probabilities are chosen so that $\sigma(k)^2 = (2k-1)^2 \times 1/4k^2 \times (1-1/4k^2) \leq 1$, so that the variance of the invariant distribution is small. However, let us evaluate the probability that, starting at 1, the first i steps consist in doing a multiplication by 2, so that we end at 2^i ; this probability is $\prod_{j=0}^{i-1} \frac{1}{4 \cdot (2^j)^2} = 4^{-1-i(i-1)/2}$. Setting $i = \log_2 k$, we see that the invariant distribution ν satisfies

$$\nu(k) \geq \frac{\nu(1)}{4} 2^{-\log_2 k (\log_2 k - 1)}$$

for k a power of 2. This is clearly not Gaussian or exponential, though $\sigma(k)^2$ is bounded.

4 Local control and logarithmic Sobolev inequality

The estimates above (e.g. for the spectral gap) were global: we used that the averaging operator M transforms a 1-Lipschitz function into a $(1 - \kappa)$ -Lipschitz function. Now we turn to some form of control of the gradient of Mf at some point, in terms of the gradient of f at neighboring points. This is closer to classical Bakry–Émery theory, and allows to get a kind of logarithmic Sobolev inequality.

DEFINITION 37 – Choose $\lambda > 0$ and, for any function $f : X \rightarrow \mathbb{R}$, define the λ -range gradient of f by

$$(Df)(x) := \sup_{y, y' \in X} \frac{|f(y) - f(y')|}{d(y, y')} e^{-\lambda d(x, y) - \lambda d(x, y')}$$

This is a kind of “mesoscopic” Lipschitz constant of f around x . Note that if f is a smooth function on a compact Riemannian manifold, when $\lambda \rightarrow \infty$ this quantity tends to $|\nabla f(x)|$.

It is important to note that Df is 2λ -log-Lipschitz.

We will also need a control on negative curvature: In a Riemannian manifold, the Ricci curvature might be $\geq \varepsilon$ because there is a direction of curvature 1 and a direction of curvature $-1 + \varepsilon$. The next definition captures these variations.

DEFINITION 38 (UNSTABILITY) – Let

$$\kappa_+(x, y) := \frac{1}{d(x, y)} \int_z (d(x, y) - d(x+z, y+z))_+$$

and

$$\kappa_-(x, y) := \frac{1}{d(x, y)} \int_z (d(x, y) - d(x + z, y + z))_-$$

where a_+ and a_- are the positive and negative part of $a \in \mathbb{R}$, so that $\kappa(x, y) = \kappa_+(x, y) - \kappa_-(x, y)$. (The integration over z is under a coupling realizing the value of $\kappa(x, y)$.)

The unstability $U(x, y)$ is defined as

$$U(x, y) := \frac{\kappa_-(x, y)}{\kappa(x, y)} \quad \text{and} \quad U := \sup_{x, y \in X, x \neq y} U(x, y)$$

REMARK 39 – If X is ε -geodesic, then an upper bound for $U(x, y)$ with $d(x, y) \leq \varepsilon$ implies the same upper bound for U .

In most discrete examples given in the introduction (Examples 8, 10, 12, 13, 14), unstability is actually 0, meaning that the coupling between m_x and m_y never increases distances (this could be a possible definition of non-negative sectional curvature for Markov chains). In Riemannian manifolds, unstability is controlled by the largest negative sectional curvature, but this does not influence the final results since one can take arbitrarily small steps for the random walk. Interestingly, in Example 17 (Glauber dynamics), unstability depends on temperature.

Due to the use of the gradient D , the theorem below is interesting only if a reasonable estimate for Df can be obtained depending on “local” data. This is not the case when f is not λ -log-Lipschitz. This is consistent with the fact mentioned above, that Gaussian concentration of measure only occurs in a finite range, with exponential concentration afterwards, which implies that no true logarithmic Sobolev inequality can hold in general.

THEOREM 40 – Suppose that Ricci curvature is at least $\kappa > 0$. Let $\lambda \leq \frac{1}{24\sigma_\infty(1+U)}$ and consider the λ -range gradient Df . Then for any function $f : x \rightarrow \mathbb{R}$ such that $Df < \infty$, one has

$$\text{Var}_\nu f \leq \left(\sup_x \frac{4\sigma(x)^2}{\kappa n_x} \right) \int (Df)^2 d\nu$$

and for positive f ,

$$\text{Ent}_\nu f \leq \left(\sup_x \frac{4\sigma(x)^2}{\kappa n_x} \right) \int \frac{(Df)^2}{f} d\nu$$

where ν is the invariant distribution.

If moreover the random walk is reversible with respect to ν , then

$$\text{Var}_\nu f \leq \int V(x) Df(x)^2 d\nu(x)$$

and

$$\text{Ent}_\nu f \leq \int V(x) \frac{Df(x)^2}{f(x)} d\nu(x)$$

where

$$V(x) = 2 \sum_{t=0}^{\infty} (1 - \kappa/2)^{2t} M^{t+1} \left(\frac{\sigma(x)^2}{n_x} \right)$$

The form involving $V(x)$ is motivated by the fact that, for reversible diffusions in \mathbb{R}^N with non-constant diffusion coefficients, these coefficients naturally appear in the formulation of functional inequalities (see e.g. [AMTU01]). The quantity $V(x) Df(x)^2$ is to be thought of as a crude version of the Dirichlet form associated with the random walk. It would be more satisfying to obtain inequalities involving the latter (compare Corollary 30), but I could not get a version of the commutation property $DM \leq (1 - \kappa/2)MD$ involving the Dirichlet form.

REMARK 41 – If $\frac{\sigma(x)^2}{n_x \kappa}$ is C -Lipschitz (as in Theorem 32), then $V(x) \leq \frac{4}{\kappa} \int \frac{\sigma(x)^2}{n_x} d\nu(x) + 2C \frac{J(x)}{\kappa}$.

Examples. Let us compare this theorem to classical results.

In the case of a Riemannian manifold, for any smooth function f we can choose a random walk with small enough steps, so that λ can be arbitrarily large and Df arbitrarily close to $|\nabla f|$. Since moreover $\sigma(x)^2$ does not depend on x for the Brownian motion, this theorem allows to recover the logarithmic Sobolev inequality in the Bakry–Émery framework, with the correct constant up to a factor 4.

Now consider the two-point space $\{0, 1\}$, equipped with the measure $\nu(0) = 1 - p$ and $\nu(1) = p$. This is a classical space on which modified logarithmic Sobolev inequalities were introduced [BL98]. We endow this space with the Markov chain sending each point to the invariant distribution. Here we have $\sigma(x)^2 = p(1 - p)$, $n_x = 1$ and $\kappa = 1$, so that we get the inequality $\text{Ent}_\nu f \leq 4p(1 - p) \int \frac{(Df)^2}{f} d\nu$, identical to the known inequality [BL98] except for the factor 4.

Tensorizing this result provides a modified logarithmic inequality for Bernoulli and Poisson measures [BL98]. If, instead, we directly apply the theorem above to the Bernoulli measure on $\{0, 1\}^N$ or the Poisson measure

on \mathbb{N} (see Sections 3.3.3 and 3.3.4), we get slightly worse results. Indeed, consider the $M/M/\infty$ queue on \mathbb{N} , which is the limit when $N \rightarrow \infty$ of the projection on \mathbb{N} of the Markov chains on $\{0, 1\}^N$ associated with Bernoulli measures. Keeping the notation of Section 3.3.4, we get, in the continuous-time version, $\sigma(x)^2 = x\mu + \lambda$, which is not constant. So we have to use $V(x)$; Remark 41 and the formulas in Section 3.3.4 yields $V(x) \leq 8\lambda/\mu + 2(\lambda + x\mu)/\mu$ so that we get the inequality

$$\text{Ent}_\nu f \leq \frac{\lambda}{\mu} \int \frac{Df(x)^2}{f(x)} (10 + 2x\mu/\lambda) d\nu(x)$$

which is to be compared to the inequality

$$\text{Ent}_\nu f \leq \frac{\lambda}{\mu} \int \frac{D_+ f(x)^2}{f(x)} d\nu(x)$$

obtained in [BL98], with $D_+ f(x) = f(x+1) - f(x)$. So asymptotically our version is worse by a factor x . Note however that the Poisson measure satisfies $x\mu/\lambda d\nu(x) = d\nu(x-1)$, so one could say that our general, non-local notion of gradient fails to distinguish between a point and an immediate neighbor, and does not take advantage of the particular structure of a random walk on \mathbb{N} .

Proof. We now turn to the proof of Theorem 40, which is essentially a copy of the Bakry–Émery argument. The key property is Proposition 43, a commutation property between the gradient and random walk operators stating that $DM \leq (1 - \kappa/2)MD$.

LEMMA 42 – *Let A be a function on $\text{Supp } m_x$, such that $A(z) \leq e^\rho A(z')$ for any $z, z' \in \text{Supp } m_x$, with $\rho \leq \frac{1}{2(1+U)}$. Then for any $x, y \in X$ we have*

$$\int_z A(z) \frac{d(x+z, y+z)}{d(x, y)} \leq (1 - \kappa(x, y)/2) \int_z A(z)$$

and in particular

$$\int_z A(z) (d(x+z, y+z) - d(x, y)) \leq 0$$

PROOF – Set $F = \max_z A(z)$. Then

$$\int_z A(z) \frac{d(x+z, y+z)}{d(x, y)} = \int_z A(z) + F \int_z \frac{A(z)}{F} \left(\frac{d(x+z, y+z)}{d(x, y)} - 1 \right)$$

and recall that, by definition, $\kappa_-(x, y) = \int_{z, d(x+z, y+z) > d(x, y)} (d(x+z, y+z)/d(x, y) - 1)$ and $\kappa_+(x, y) = \int_{z, d(x+z, y+z) \leq d(x, y)} (1 - d(x+z, y+z)/d(x, y))$. Using that $A(z) \leq F$ on one hand and $A(z) \geq e^{-\rho}F$ on the other hand, we get

$$\int_z A(z) \frac{d(x+z, y+z)}{d(x, y)} \leq \int_z A(z) + F(\kappa_-(x, y) - e^{-\rho}\kappa_+(x, y))$$

Now, recall that by definition of U we have $\kappa_-(x, y) \leq U\kappa(x, y)$. It is not difficult to check that $\rho \leq \frac{1}{2(1+U)}$ is enough to ensure that $e^{-\rho}\kappa_+(x, y) - \kappa_-(x, y) \geq \kappa(x, y)/2$, hence

$$\begin{aligned} \int_z A(z) \frac{d(x+z, y+z)}{d(x, y)} &\leq \int_z A(z) - F\kappa(x, y)/2 \\ &\leq \int_z A(z) (1 - \kappa(x, y)/2) \end{aligned}$$

as needed. \square

PROPOSITION 43 – Suppose that the Ricci curvature is at least $\kappa > 0$, and choose some $\lambda \leq \frac{1}{24\sigma_\infty(1+U)}$. Then for any function $f : X \rightarrow \mathbb{R}$ we have

$$D(Mf)(x) \leq (1 - \kappa/2)M(Df)(x)$$

PROOF – For any $y, y' \in X$ we have

$$\begin{aligned} &\frac{|Mf(y) - Mf(y')|}{d(y, y')} e^{-\lambda(d(x, y) + d(x, y'))} \\ &\leq \int_z |f(y+z) - f(y'+z)| \frac{e^{-\lambda(d(x, y) + d(x, y'))}}{d(y, y')} \\ &\leq \int_z Df(x+z) \frac{d(y+z, y'+z)}{e^{-\lambda(d(x+z, y+z) + d(x+z, y'+z))}} \frac{e^{-\lambda(d(x, y) + d(x, y'))}}{d(y, y')} \\ &= \int_z A(z)B(z) \frac{d(y+z, y'+z)}{d(y, y')} \end{aligned}$$

where $A(z) = Df(x+z)$ and $B(z) = e^{\lambda(d(x+z, y+z) - d(x, y) + d(x+z, y'+z) - d(x, y'))}$.

For any z we have $(1 - \kappa(x, y))d(x, y) - 4\sigma_\infty \leq d(x+z, y+z) \leq (1 - \kappa(x, y))d(x, y) + 4\sigma_\infty$ and likewise for y' , so that B varies by a factor at most $e^{8\lambda\sigma_\infty}$. Likewise, since Df is 2λ -log-Lipschitz, A varies by a factor at most $e^{4\lambda\sigma_\infty}$. So the quantity $A(z)B(z)$ varies by at most $e^{12\lambda\sigma_\infty}$.

So if $\lambda \leq \frac{1}{24\sigma_\infty(1+U)}$, we can apply Lemma 42 and get

$$\int_z A(z)B(z) \frac{d(y+z, y'+z)}{d(y, y')} \leq (1 - \kappa/2) \int_z A(z)B(z)$$

Now we have $\int_z A(z)B(z) = \int_z A(z) + \int_z A(z)(B(z)-1)$. Unwinding $B(z)$ and using that $e^a - 1 \leq ae^a$ for any $a \in \mathbb{R}$, we get

$$\begin{aligned} \int_z A(z)(B(z)-1) &\leq \\ \lambda \int_z A(z)B(z) (d(x+z, y+z) - d(x, y) + d(x+z, y'+z) - d(x, y')) & \end{aligned}$$

which is non-positive by Lemma 42. Hence $\int_z A(z)B(z) \leq \int_z A(z)$, which ends the proof. \square

Let ν be the invariant distribution. Let f be a positive function with $\int f d\nu = 1$. We know that

$$\begin{aligned} \text{Ent } f &= \int_x Mf(x) \left(\text{Ent}_{m_x} \frac{f}{Mf(x)} \right) d\nu(x) + \text{Ent } Mf \\ &= \sum_{t \geq 0} \int_x M^{t+1}f(x) \left(\text{Ent}_{m_x} \frac{M^t f}{M^{t+1}f(x)} \right) d\nu(x) \end{aligned}$$

and similarly

$$\text{Var } f = \sum_{t \geq 0} \int_x \text{Var}_{m_x} M^t f d\nu(x)$$

Now for any $y, z \in \text{Supp } m_x$ we have $|f(y) - f(z)| \leq Df(y)d(y, z)e^{\lambda d(y, z)}$. Since Df is 2λ -log-Lipschitz, we have $Df(y) \leq e^{4\lambda\sigma_\infty} M(Df)(x)$, so that $|f(y) - f(z)| \leq d(y, z) M(Df)(x) e^{6\lambda\sigma_\infty}$, i.e. f is $M(Df)(x) e^{6\lambda\sigma_\infty}$ -Lipschitz. Consequently

$$\text{Var}_{m_x} f \leq \frac{2(M(Df)(x))^2 \sigma(x)^2}{n_x}$$

and, using that $a \log a \leq a^2 - a$, we get that $\text{Ent}_{m_x} \frac{f}{Mf(x)} \leq \frac{1}{Mf(x)^2} \text{Var}_{m_x} f$ so

$$\text{Ent}_{m_x} \frac{f}{Mf(x)} \leq \frac{2(M(Df)(x))^2 \sigma(x)^2}{n_x Mf(x)^2}$$

Thus

$$\text{Var } f \leq 2 \sum_{t \geq 0} \int_x \frac{\sigma(x)^2}{n_x} (M(DM^t f)(x))^2 d\nu(x)$$

and

$$\text{Ent } f \leq 2 \sum_{t \geq 0} \int_x \frac{\sigma(x)^2}{n_x} \frac{(M(DM^t f)(x))^2}{M^{t+1} f(x)} d\nu(x)$$

By Proposition 43, we have $(DM^t f)(y) \leq (1 - \kappa/2)^t M^t(Df)(y)$, so that

$$\text{Var } f \leq 2 \sum_{t \geq 0} \int_x \frac{\sigma(x)^2}{n_x} (M^{t+1} Df(x))^2 (1 - \kappa/2)^{2t} d\nu(x)$$

and

$$\text{Ent } f \leq 2 \sum_{t \geq 0} \int_x \frac{\sigma(x)^2}{n_x} \frac{(M^{t+1} Df(x))^2}{M^{t+1} f(x)} (1 - \kappa/2)^{2t} d\nu(x)$$

Now since the norm of M acting on $L^2(\nu)$ is at most 1, we have

$$\begin{aligned} \text{Var } f &\leq 2 \sup_x \frac{\sigma(x)^2}{n_x} \sum_{t \geq 0} (1 - \kappa/2)^{2t} \int_x (M^{t+1} Df(x))^2 d\nu(x) \\ &\leq \frac{4}{\kappa} \sup_x \frac{\sigma(x)^2}{n_x} \int_x (Df(x))^2 d\nu(x) \end{aligned}$$

For the entropy of f , the Cauchy–Schwarz inequality yields

$$(M^{t+1} Df(x))^2 = \left(M^{t+1} \left(\frac{Df}{\sqrt{f}} \cdot \sqrt{f} \right)(x) \right)^2 \leq M^{t+1} \left(\frac{(Df)^2}{f} \right)(x) M^{t+1} f(x)$$

so that finally

$$\begin{aligned} \text{Ent } f &\leq 2 \sum_{t \geq 0} \int_x \frac{\sigma(x)^2}{n_x} M^{t+1} \left(\frac{(Df)^2}{f} \right)(x) (1 - \kappa/2)^{2t} d\nu(x) \\ &\leq \frac{4}{\kappa} \sup_x \frac{\sigma(x)^2}{n_x} \int_x \frac{(Df(x))^2}{f(x)} d\nu(x) \end{aligned}$$

5 Exponential concentration in non-negative curvature

We have seen that positive Ricci curvature implies a kind of Gaussian concentration. We now show that non-negative Ricci curvature and the existence of an “attracting point” imply exponential concentration.

The basic example to keep in mind is the following. Let \mathbb{N} be the set of non-negative integers equipped with its standard distance. Let $0 < p < 1$

and let the nearest-neighbor random walk on \mathbb{N} that goes to the left with probability p ; explicitly $m_k = p\delta_{k-1} + (1-p)\delta_{k+1}$ for $k \geq 1$, and $m_0 = p\delta_0 + (1-p)\delta_1$.

Since for $k \geq 1$ the transition kernel is translation-invariant, it is immediate to check that $\kappa(k, k+1) = 0$. Besides, $\kappa(0, 1) = p$. There exists a invariant distribution if and only if $p > 1/2$, and it satisfies exponential concentration with characteristic decay distance $1/\log(p/(1-p))$. For $p = 1/2 + \varepsilon$ with small ε this behaves like $1/4\varepsilon$.

Geometrically, what entails exponential concentration in this example is the fact that, for $p > 1/2$, the point 0 “pulls” its neighbor, and the pulling is transmitted by non-negative Ricci curvature. We now formalize this situation in the following theorem.

THEOREM 44 – Let $(X, d, (m_x))$ be a metric space with random walk. Suppose that for some $o \in X$ and $r > 0$ one has:

- $\kappa(x, y) \geq 0$ for all $x, y \in X$,
- for all $x \in X$ with $r \leq d(o, x) < 2r$, one has $\mathcal{T}_1(m_x, \delta_o) < d(x, o)$,
- X is r -geodesic,
- There exists $s > 0$ such that each measure m_x satisfies the Gaussian-type Laplace transform inequality

$$\int e^{\lambda f} dm_x \leq e^{\lambda^2 s^2 / 2} e^{\lambda \int f dm_x}$$

for any $\lambda > 0$ and any 1-Lipschitz function $f : \text{Supp } m_x \rightarrow \mathbb{R}$.

Set $\rho = \inf\{d(x, o) - \mathcal{T}_1(m_x, \delta_o), r \leq d(o, x) < 2r\}$ and assume $\rho > 0$.

Then there exists a invariant distribution for the random walk. Moreover, setting $D = s^2/\rho$ and $m = r + 2s^2/\rho + \rho(1 + J(o)^2/4s^2)$, for any invariant distribution ν we have

$$\int e^{d(x,o)/D} d\nu(x) \leq (4 + J(o)^2/s^2) e^{m/D}$$

and so for any 1-Lipschitz function $f : X \rightarrow \mathbb{R}$ and $t \geq 0$ we have

$$\Pr(|f - f(o)| \geq t + m) \leq (8 + 2J(o)^2/s^2) e^{-t/D}$$

So we get exponential concentration with characteristic decay distance s^2/ρ .

Note that the last assumption is satisfied with $s = 2\sigma_\infty$ thanks to Proposition 1.16 in [Led01].

Before proceeding to the proof, let us show how this applies to the geometric distribution above on \mathbb{N} . We take of course $o = 0$ and $r = 1$. We can take $s = 2\sigma_\infty = 2$. Now there is only one point x with $r \leq d(o, x) < 2r$, which is $x = 1$. It satisfies $m_1 = p\delta_0 + (1-p)\delta_2$, so that $\mathcal{T}_1(m_1, \delta_0) = 2(1-p)$, which is smaller than $d(0, 1) = 1$ if and only if $p < 1/2$ as was to be expected. So we can take $\rho = 1 - 2(1-p) = 2p - 1$. We get exponential concentration with characteristic distance $4/(2p - 1)$. When p is very close to 1 this is not so good (because the discretization is too coarse), but when p is close to $1/2$ this is within a factor 2 of the optimal value.

Another example is the stochastic differential equation $dX_t = S dB_t - \alpha \frac{X_t}{|X_t|} dt$ on \mathbb{R}^n , for which $\exp(-|x|\alpha/S^2)$ is a reversible measure. Consider the Euler scheme at time δt for this stochastic differential equation. Taking $r = nS^2/\alpha$ yields that $\rho \geq \alpha\delta t/2$ after some simple computation. Since we have $s^2 = S^2\delta t$ for Gaussian measures at time δt , we get exponential concentration with characteristic decay distance $2S^2/\alpha$, which is correct up to a factor 2. The additive constant in the deviation inequality is $m = r + \rho(1 + J(o)^2/4s^2) + 2s^2/\rho$ which is equal to $(n+4)S^2/\alpha + O(\delta t)$ (note that $J(o)^2 \approx s^2$), which is the correct order of magnitude for the average distance to 0 in dimension n .

If $\kappa > 0$ in some large enough ball around o , then the invariant distribution is unique. However, this is not true in general: for example, start with the random walk on \mathbb{N} above with a geometric invariant distribution; now consider the disjoint union $\mathbb{N} \cup (\mathbb{N} + \frac{1}{2})$ where we keep the same random walk on \mathbb{N} and the same walk translated by $\frac{1}{2}$ on $\mathbb{N} + \frac{1}{2}$: clearly there are two disjoint invariant distributions, however, curvature is non-negative and the assumptions of the theorem are satisfied with $r = 1$ and $o = 0$.

PROOF OF THE THEOREM –

Let us first prove a lemma which shows how non-negative curvature transmits the “pulling”.

LEMMA 45 – Let $x \in X$ with $d(x, o) \geq r$. Then $\mathcal{T}_1(m_x, o) \leq d(x, o) - \rho$.

PROOF – If $d(o, x) < 2r$ then this is one of the assumptions. So we suppose that $d(o, x) \geq 2r$.

Since X is r -geodesic, let $o = y_0, y_1, y_2, \dots, y_n = x$ be a sequence of points with $d(y_i, y_{i+1}) \leq r$ and $\sum d(y_i, y_{i+1}) = d(o, x)$. We can assume that $d(o, y_2) > r$ (otherwise, remove y_1). Set $z = y_1$ if $d(o, y_1) = r$ and $z = y_2$ if

$d(o, y_1) < r$, so that $r \leq d(o, z) < 2r$. Now

$$\begin{aligned}\mathcal{T}_1(\delta_o, m_x) &\leq \mathcal{T}_1(\delta_o, m_z) + \mathcal{T}_1(m_z, m_x) \\ &\leq d(o, z) - \rho + d(z, x)\end{aligned}$$

since $\kappa(z, x) \geq 0$. The conclusion follows from the fact that $d(o, x) = d(o, z) + d(z, x)$. \square

We are now ready to prove the theorem. The idea is to consider the function $e^{\lambda d(x,o)}$. For points far away from the origin, since under the random walk the average distance to the origin decreases by ρ by the previous lemma, we expect the function to be multiplied by $e^{-\lambda\rho}$ under the random walk operator. Close to the origin, the evolution of the function is controlled by the variance s^2 and the jump $J(o)$ of the origin. Since the integral of the function is preserved by the random walk operator, and it is multiplied by a quantity < 1 far away, this shows that the weight of faraway points cannot be too large.

More precisely, we need to tamper a little bit with what happens around the origin. Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be defined by $\varphi(x) = 0$ if $x < r$; $\varphi(x) = (x - r)^2 / kr$ if $r \leq x < r(\frac{k}{2} + 1)$ and $\varphi(x) = x - r - kr/4$ if $x \geq r(\frac{k}{2} + 1)$, for some $k > 0$ to be chosen later. Note that φ is a 1-Lipschitz function and that $\varphi'' \leq 2/kr$.

If Y is any random variable with values in \mathbb{R}_+ , we have

$$\mathbb{E}\varphi(Y) \leq \varphi(\mathbb{E}Y) + \frac{1}{2} \text{Var } Y \sup \varphi'' \leq \varphi(\mathbb{E}Y) + \frac{1}{kr} \text{Var } Y$$

Now choose some $\lambda > 0$ and consider the function $f : X \rightarrow \mathbb{R}$ defined by $f(x) = e^{\lambda\varphi(d(o,x))}$. Note that $\varphi(d(o,x))$ is 1-Lipschitz, so that by the Laplace transform assumption we have

$$Mf(x) \leq e^{\lambda^2 s^2 / 2} e^{\lambda M\varphi(d(o,x))}$$

The Laplace transform assumption implies that the variance under m_x of any 1-Lipschitz function is at most s^2 . So by the remark above, we have

$$M\varphi(d(o,x)) \leq \varphi(\mathcal{T}_1(m_x, \delta_o)) + \frac{s^2}{kr}$$

so that finally

$$Mf(x) \leq e^{\lambda^2 s^2 / 2 + \lambda s^2 / kr} e^{\lambda \mathcal{T}_1(m_x, \delta_o)}$$

So for any x with $d(o, x) \geq r$, we get

$$Mf(x) \leq e^{\lambda^2 s^2 / 2 + \lambda s^2 / kr} e^{\lambda \varphi(d(x,o)) - \rho}$$

If $d(x, o) \geq r(\frac{k}{2} + 1) + \rho$ then $\varphi(d(x, o) - \rho) = \varphi(d(x, o)) - \rho$ so that

$$Mf(x) \leq e^{\lambda^2 s^2 / 2 + \lambda s^2 / kr - \lambda \rho} f(x)$$

If $r \leq d(x, o) < r(\frac{k}{2} + 1) + \rho$, then $\varphi(d(x, o) - \rho) \leq \varphi(d(x, o))$ so that

$$Mf(x) \leq e^{\lambda^2 s^2 / 2 + \lambda s^2 / kr} f(x)$$

If, finally, $d(x, o) < r$, then use non-negative curvature to write $\mathcal{T}_1(m_x, \delta_o) \leq \mathcal{T}_1(m_x, m_o) + J(o) \leq d(x, o) + J(o)$ so that $\varphi(\mathcal{T}_1(m_x, \delta_o)) \leq \varphi(r + J(o)) = J(o)^2 / kr$ and

$$Mf(x) \leq e^{\lambda^2 s^2 / 2 + \lambda s^2 / kr + \lambda J(o)^2 / kr} f(x)$$

Let ν be a probability measure such that $\int f d\nu < \infty$. Let $X' = \{x \in X, d(x, o) < r(\frac{k}{2} + 1)\}$ and $X'' = X \setminus X'$. Set $A(\nu) = \int_{X'} f d\nu$ and $B(\nu) = \int_{X''} f d\nu$. We have shown that

$$\begin{aligned} \int f d(\nu * m) &= \int Mf d\nu = \int_{X'} Mf d\nu + \int_{X''} Mf d\nu \\ &\leq e^{\lambda^2 s^2 / 2 + \lambda s^2 / kr + \lambda J(o)^2 / kr} \int_{X'} f d\nu + e^{\lambda^2 s^2 / 2 + \lambda s^2 / kr - \lambda \rho} \int_{X''} f d\nu \end{aligned}$$

so that

$$A(\nu * m) + B(\nu * m) \leq \alpha A(\nu) + \beta B(\nu)$$

with $\alpha = e^{\lambda^2 s^2 / 2 + \lambda s^2 / kr + \lambda J(o)^2 / kr}$ and $\beta = e^{\lambda^2 s^2 / 2 + \lambda s^2 / kr - \lambda \rho}$.

Choose λ small enough and k large enough (see below) so that $\beta < 1$. Using that $A(\nu) \leq e^{\lambda kr / 4}$ for any measure ν , we get $\alpha A(\nu) + \beta B(\nu) \leq (\alpha - \beta)e^{\lambda kr / 4} + \beta(A(\nu) + B(\nu))$. In particular, if $A(\nu) + B(\nu) \leq \frac{(\alpha - \beta)e^{\lambda r}}{1 - \beta}$, we get $\alpha A(\nu) + \beta B(\nu) \leq \frac{(\alpha - \beta)e^{\lambda kr / 4}}{1 - \beta}$. So setting $R = \frac{(\alpha - \beta)e^{\lambda kr / 4}}{1 - \beta}$, we have just shown that the set C of probability measures ν such that $\int f d\nu \leq R$ is invariant under the random walk.

Moreover, if $A(\nu) + B(\nu) > R$ then $\alpha A(\nu) + \beta B(\nu) < A(\nu) + B(\nu)$. Hence, if ν is a invariant distribution, necessarily $\nu \in C$. This, together with an evaluation of R given below, provides the bound for $\int f d\nu$ stated in the theorem.

We now turn to existence of a invariant distribution. First, C is obviously closed and convex. Moreover, C is tight: indeed if K is a compact, say included in a ball of radius a around o , then for any $\nu \in C$ we have $\nu(X \setminus K) \leq Re^{-\lambda a}$. So by Prokhorov's theorem, C is compact in the weak convergence topology. So C is compact convex in the topological vector space of all (signed) Borel measures on X , and is invariant by the random walk operator,

which is an affine map. By the Markov–Kakutani theorem (Theorem I.3.3.1 in [GD03]), it has a fixed point.

Let us finally evaluate R . We have

$$\begin{aligned} R &= \frac{\alpha/\beta - 1}{1/\beta - 1} e^{\lambda kr/4} \\ &= \frac{e^{\lambda J(o)^2/kr + \lambda\rho} - 1}{e^{\lambda\rho - \lambda s^2/kr - \lambda^2 s^2/2} - 1} e^{\lambda kr/4} \\ &\leq \frac{\rho + J(o)^2/kr}{\rho - s^2/kr - \lambda s^2/2} e^{\lambda J(o)^2/kr + \lambda\rho + \lambda kr/4} \end{aligned}$$

using $e^a - 1 \leq ae^a$ and $e^a - 1 \geq a$.

Now take $\lambda = \rho/s^2$ and $k = 4s^2/r\rho$. This yields

$$R \leq (4 + J(o)^2/s^2) e^{\lambda(s^2/\rho + \rho(1+J(o)^2/4s^2))}$$

Let ν be some invariant distribution. Since $d(x, o) \leq \varphi(d(x, o) + r(1 + k/4))$ we have $\int e^{\lambda d(x, o)} d\nu \leq e^{\lambda r(1+k/4)} \int f d\nu \leq Re^{\lambda r(1+k/4)}$ hence the result in the theorem. \square

6 Ricci curvature and Gromov–Hausdorff topology

We introduce here a Gromov–Hausdorff-like topology for metric spaces equipped with a random walk. Two spaces are close in this topology if they are close in the Gromov–Hausdorff topology and if moreover, the measures issuing from each point x are (uniformly) close in the L^1 transportation distance. More precisely:

DEFINITION 46 – Let $(X, (m_x)_{x \in X})$ and $(Y, (m_y)_{y \in Y})$ be two metric spaces equipped with a random walk. For $e > 0$, we say that these spaces are e -close if there exists a metric space Z and two isometric embeddings $f_X : X \hookrightarrow Z$, $f_Y : Y \hookrightarrow Z$ such that the Hausdorff distance between $f_X(X)$ and $f_Y(Y)$ is at most e , and, moreover, for any $x \in X$, there exists $y \in Y$ such that $d_Z(f_X(x), f_Y(y)) \leq e$ and the L^1 transportation distance between the pushforward measures $f_X(m_X)$ and $f_Y(m_Y)$ is at most $2e$, and likewise for any $y \in Y$.

The Ricci curvature is a continuous function in this topology. Namely, a limit of spaces with Ricci curvature at least κ has Ricci curvature at least κ .

Below, we will relax the definition of Ricci curvature so as to allow any variation at small scale; withthis perturbed definition, having Ricci curvature

greater than κ will become an *open* property. In particular, any space close to a space with positive Ricci curvature will have positive Ricci curvature in this perturbed sense.

PROPOSITION 47 – Let $(X^n, (m_x^N)_{x \in X^n})$ be a sequence of metric spaces with random walk, converging to a metric space with random walk $(X, (m_x)_{x \in X})$. Let x, y be two distinct points in X and let $(x^N, y^N) \in (X^N, Y^N)$ be a sequence of pairs of points converging to (x, y) . Then $\kappa(x^N, y^N) \rightarrow \kappa(x, y)$.

In particular, if all spaces X^N have Ricci curvature at least κ , then so does X .

In order for positive curvature to be an open property in some topology à la Gromov–Hausdorff, one needs a rougher behavior at small scales. This is achieved as follows.

DEFINITION 48 – Let (X, d) be a metric space equipped with a random walk m . Let $\delta \geq 0$. The Ricci curvature up to δ along $x, y \in X$ is

$$\kappa^\delta(x, y) := 1 - \frac{(\mathcal{T}_1(m_x, m_y) - \delta)_+}{d(x, y)}$$

i.e. it is the largest $\kappa \leq 1$ for which one has

$$\mathcal{T}_1(m_x, m_y) \leq (1 - \kappa)d(x, y) + \delta$$

With this definition, the following is easy.

PROPOSITION 49 – Let $(X, (m_x))$ be a metric space with random walk with Ricci curvature at least κ up to $\delta \geq 0$. Let $\delta' > 0$. Then there exists a neighborhood \mathcal{V}_X of X such that any space $Y \in \mathcal{V}_X$ has Ricci curvature at least κ up to $\delta + \delta'$.

Consequently, the property “having curvature at least κ for some $\delta \geq 0$ ” is open.

7 L^2 Bonnet–Myers theorems

As seen in Section 2.3, it is generally not possible to give a bound for the diameter of a positively curved space involving the square root of curvature, because of such simple counterexamples as the discrete cube. Here we describe additional conditions which provide such a bound in two different types of situation.

We first give a bound similar to the Bonnet–Myers one, but on the *average* distance between two points rather than the diameter; it holds when there is an “attractive point” and is relevant for examples such as the Ornstein–Uhlenbeck process (Example 9) or its discrete analogue (Example 10).

Next, we give a direct generalization of the genuine Bonnet–Myers theorem for Riemannian manifolds. Actually, the only example where a Bonnet–Myers theorem holds seems to be the ordinary Brownian motion on a Riemannian manifold. Despite this lack of further examples, we found it interesting to provide an axiomatization of the Bonnet–Myers theorem in our language. This is done by reinforcing the positive curvature assumption, which compares the transportation distance between the measures issuing from two points x and y at a given time, by requiring a transportation distance inequality between the measures issuing from two given points at *different* times.

7.1 Average L^2 Bonnet–Myers

We now describe a Bonnet–Myers-like estimate on the average distance between two points, provided there is some “attractive point”. This is rather similar to Theorem 44 in non-negative curvature.

PROPOSITION 50 (AVERAGE L^2 BONNET–MYERS) – *Let $(X, d, (m_x))$ be a metric space with random walk, with Ricci curvature at least $\kappa > 0$. Suppose that for some $o \in X$ and $r \geq 0$, one has*

$$\int d(o, y) dm_x(y) \leq d(o, x)$$

for any $x \in X$ with $r \leq d(o, x) < 2r$, and that moreover X is r -geodesic.

Then

$$\int d(o, x) d\nu(x) \leq \sqrt{\frac{1}{\kappa} \int \frac{\sigma(x)^2}{n_x} d\nu(x) + 5r}$$

where as usual ν is the invariant distribution.

Note that the assumption $\int d(o, y) dm_x(y) \leq d(o, x)$ cannot hold for x in some ball around o unless o is a fixed point. This is why the assumption is restricted to an annulus.

As in the Gaussian concentration theorem (Theorem 32), in case $\sigma(x)^2$ is Lipschitz, Corollary 22 may provide a useful bound on $\int \frac{\sigma(x)^2}{n_x} d\nu(x)$ in terms of its value at some point.

As a first example, consider the discrete Ornstein–Uhlenbeck process of Example 10, which is the Markov chain on $\{-N, \dots, N\}$ given by the transition probabilities $p_{k,k} = 1/2$, $p_{k,k+1} = 1/4 - k/4N$ and $p_{k,k-1} = 1/4 + k/4N$;

the Ricci curvature is $\kappa = 1/2N$, and the invariant distribution is the binomial $\binom{2N}{N+k}$. This example is interesting because the diameter is $2N$ (as is the bound provided by Proposition 23), whereas the average distance between two points is $\approx \sqrt{N}$. It is immediate to check 0 is attractive, namely that $o = 0$ and $r = 1$ fulfill the assumptions. Since $\sigma(x)^2 \approx 1$ and $\kappa \approx 1/N$, the proposition recovers the correct order of magnitude for distance to the origin.

Our next example is the Ornstein–Uhlenbeck process $dX_t = -\alpha X_t dt + s dB_t$ on \mathbb{R}^N (Example 9). Here it is clear that 0 is attractive in some sense, so $o = 0$ is a natural choice. The invariant distribution is a Gaussian of variance s^2/α ; under this distribution the average distance to 0 is $\approx \sqrt{Ns^2/\alpha}$.

At small time τ , a point $x \in \mathbb{R}^N$ is sent to a Gaussian centered at $(1 - \alpha\tau)x$, of variance τs^2 . The average quadratic distance to the origin under this Gaussian is $(1 - \alpha\tau)^2 d(0, x)^2 + Ns^2\tau + o(\tau)$ by a simple computation. If $d(0, x)^2 > Ns^2/2\alpha$ this is less than $d(0, x)^2$, so that we can take $r = \sqrt{Ns^2/2\alpha}$. Considering the random walk discretized at time τ we have we have $\kappa \sim \alpha\tau$, $\sigma(x)^2 \sim Ns^2\tau$ and $n_x \approx N$. So in the proposition above, the first term is $\approx \sqrt{s^2/\alpha}$, whereas the second term is $5r \approx \sqrt{Ns^2/\alpha}$, which is thus dominant. So the proposition gives the correct order of magnitude; in this precise case, the first term in the proposition reflects concentration of measure (which is dimension-independent for Gaussians), whereas it is the second term $5r$ which carries the correct dependency on dimension for the average distance to the origin.

PROOF – Let $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $\varphi(x) = 0$ if $x \leq 2r$, and $\varphi(x) = (x - 2r)^2$ otherwise. Note that for any real-valued random variable Y , we have

$$\mathbb{E}\varphi(Y) \leq \varphi(\mathbb{E}Y) + \frac{1}{2} \operatorname{Var} Y \sup \varphi'' = \varphi(\mathbb{E}Y) + \operatorname{Var} Y$$

Now let $f : X \rightarrow \mathbb{R}$ be defined by $f(x) = \varphi(d(o, x))$. We are going to show that

$$Mf(x) \leq (1 - \kappa)^2 f(x) + \frac{\sigma(x)^2}{n_x} + 9r^2$$

for all $x \in X$. Since $\int f d\nu = \int Mf d\nu$, we will get $\int f d\nu \leq (1 - \kappa)^2 \int f d\nu + \int \frac{\sigma(x)^2}{n_x} d\nu + 9r^2$ which easily implies the result.

First, suppose that $r \leq d(o, x) < 2r$. We have $f(x) = 0$. Now $\int d(o, y) dm_x(y)$ is at most $d(o, y)$ by assumption. Using the bound above for φ , together with the definition of $\sigma(x)^2$ and n_x , we get

$$Mf(x) = \int \varphi(d(o, y)) dm_x(y) \leq \varphi \left(\int d(o, y) dm_x(y) \right) + \frac{\sigma(x)^2}{n_x} = \frac{\sigma(x)^2}{n_x}$$

since $\int d(o, y) dm_x(y) \leq 2r$ by assumption.

Second, suppose that $d(x, o) \geq 2r$. Using that X is r -geodesic, we can find a point x' such that $d(o, x) = d(o, x') + d(x', x)$ and $r \leq d(o, x') < 2r$ (take the second point in a sequence joining o to x). Now we have

$$\begin{aligned} \int d(o, y) dm_x(y) &= \mathcal{T}_1(\delta_o, m_x) \\ &\leq \mathcal{T}_1(\delta_o, m_{x'}) + \mathcal{T}_1(m_{x'}, m_x) \\ &\leq \mathcal{T}_1(\delta_o, m_{x'}) + (1 - \kappa)d(x', x) \\ &= \int d(o, y) dm_{x'}(y) + (1 - \kappa)d(x', x) \\ &\leq d(o, x') + (1 - \kappa)d(x', x) \leq (1 - \kappa)d(o, x) + 2\kappa r \end{aligned}$$

and as above, this implies

$$\begin{aligned} Mf(x) &\leq \varphi \left(\int d(o, y) dm_x(y) \right) + \frac{\sigma(x)^2}{n_x} \\ &\leq ((1 - \kappa)d(o, x) + 2\kappa r - 2r)^2 + \frac{\sigma(x)^2}{n_x} \\ &= (1 - \kappa)^2 \varphi(d(o, x)) + \frac{\sigma(x)^2}{n_x} \end{aligned}$$

as needed.

The last case to consider is $d(o, x) < r$. In this case we have

$$\begin{aligned} \int d(o, y) dm_x(y) &= \mathcal{T}_1(\delta_o, m_x) \\ &\leq \mathcal{T}_1(\delta_o, m_o) + \mathcal{T}_1(m_o, m_x) = J(o) + \mathcal{T}_1(m_o, m_x) \\ &\leq J(o) + (1 - \kappa)d(o, x) \leq J(o) + r \end{aligned}$$

So we need to bound $J(o)$. If X is included in the ball of radius r around o , the result trivially holds, so that we can assume that there exists a point x with $d(o, x) \geq r$. Since X is r -geodesic we can assume that $d(o, x) < 2r$ as well. Now $J(o) = \mathcal{T}_1(m_o, \delta_o) \leq \mathcal{T}_1(m_o, m_x) + \mathcal{T}_1(m_x, \delta_o) \leq (1 - \kappa)d(o, x) + \mathcal{T}_1(m_x, \delta_o) \leq (1 - \kappa)d(o, x) + d(o, x)$ by assumption, so that $J(o) \leq 4r$.

Plugging this into the above, for $d(o, x) < r$ we get $\int d(o, y) dm_x(y) \leq 5r$ so that $\varphi(\int d(o, y) dm_x(y)) \leq 9r^2$ hence $Mf(x) \leq 9r^2 + \frac{\sigma(x)^2}{n_x}$.

Combining the results, we get that whatever $x \in X$

$$Mf(x) \leq (1 - \kappa)^2 f(x) + \frac{\sigma(x)^2}{n_x} + 9r^2$$

as needed. \square

7.2 Strong L^2 Bonnet–Myers

As mentioned above, positive Ricci curvature alone does not imply a $1/\sqrt{\kappa}$ -like diameter control, because of such simple counter-examples as the discrete cube or the Ornstein–Uhlenbeck process. We now extract a property satisfied by the ordinary Brownian motion on Riemannian manifolds (without drift), which guarantees a genuine Bonnet–Myers theorem. Of course, this is of limited interest since the only available example is Riemannian manifolds, but nevertheless we found it interesting to find a sufficient condition expressed in our present language.

Our definition of Ricci curvature controls the transportation distance between the measures issuing from two points x and x' at a given time t . The condition we will now use controls the transportation distance between the measures issuing from two points at two *different* times. It is based on what holds for Gaussian measures in \mathbb{R}^N . For any $x, x' \in \mathbb{R}^N$ and $t, t' > 0$, let m_x^{*t} and $m_{x'}^{*t'}$ be the laws of the standard Brownian motion issuing from x at time t and from x' at time t' , respectively. It is easy to check that the L^2 transportation distance between these two measures is

$$\mathcal{T}_2(m_x^{*t}, m_{x'}^{*t'})^2 = d(x, x')^2 + N(\sqrt{t} - \sqrt{t'})^2$$

hence

$$\mathcal{T}_1(m_x^{*t}, m_{x'}^{*t'}) \leq d(x, x') + \frac{N(\sqrt{t} - \sqrt{t'})^2}{2d(x, x')}$$

The important feature here is that, when t' tends to t , the second term is of second order in $t' - t$. This is no more the case if we add a drift term to the diffusion.

We now take this inequality as an assumption and use it to mimick the traditional proof of the Bonnet–Myers theorem. Here, for simplicity of notation we suppose that we are given a continuous-time Markov chain; however, the proof uses only a finite number of different values of t , so that discretization is possible (this is important in Riemannian manifolds, because the heat kernel is positive on the whole manifold at any positive time, and there is no simple control on it far away from the initial point; taking a discrete approximation with bounded steps solves this problem).

PROPOSITION 51 (STRONG L^2 BONNET–MYERS) – *Let X be a metric space equipped with a continuous-time random walk m^{*t} . Assume that X is ε -geodesic, and that there exists constants $\kappa > 0, C \geq 0$ such that for any two small enough t, t' , for any $x, x' \in X$ with $\varepsilon \leq d(x, x') \leq 2\varepsilon$ one has*

$$\mathcal{T}_1(m_x^{*t}, m_{x'}^{*t'}) \leq e^{-\kappa \inf(t, t')} d(x, x') + \frac{C(\sqrt{t} - \sqrt{t'})^2}{2d(x, x')}$$

with $\kappa > 0$. Assume moreover that $\varepsilon \leq \frac{1}{2}\sqrt{C/2\kappa}$.

Then

$$\text{diam } X \leq \pi \sqrt{\frac{C}{2\kappa}} \left(1 + \frac{4\varepsilon}{\sqrt{C/2\kappa}} \right)$$

When $t = t'$, the assumption reduces to $\mathcal{T}_1(m_x^{*t}, m_{x'}^{*t}) \leq e^{-\kappa t} d(x, x')$, which is just the continuous-time version of the positive curvature assumption. The constant C plays the role of a diffusion constant, and is equal to N for (a discrete approximation of) Brownian motion on a Riemannian manifold. We restrict the assumption to $d(x, x') \geq \varepsilon$ to avoid divergence problems for $\frac{C(\sqrt{t}-\sqrt{t'})^2}{2d(x, x')}$ when $x' \rightarrow x$.

For the Brownian motion on an N -dimensional Riemannian manifold, we can take $\kappa = \frac{1}{2} \inf \text{Ric}$ by Bakry-Émery theory (the $\frac{1}{2}$ is due to the fact that the infinitesimal generator of Brownian motion is $\frac{1}{2}\Delta$), and $C = N$ as in \mathbb{R}^N . So we get the usual Bonnet-Myers theorem, up to a factor \sqrt{N} instead of $\sqrt{N-1}$ (similarly to our spectral gap estimate in comparison with the Lichnerowicz theorem), but with the correct constant π .

PROOF – Let $x, x' \in X$. Since X is ε -geodesic, we can find a sequence $x = x_0, x_1, \dots, x_{k-1}, x_k = x'$ of points in X with $d(x_i, x_{i+1}) \leq \varepsilon$ and $\sum d(x_i, x_{i+1}) = d(x_0, x_k)$. By taking a subsequence (denoted x_i again), we can assume that $\varepsilon \leq d(x_i, x_{i+1}) \leq 2\varepsilon$ instead.

Set $t_i = \eta \sin \left(\frac{\pi d(x, x_i)}{d(x, x')} \right)^2$ for some (small) value of η to be chosen later. Now, since $t_0 = t_k = 0$ we have

$$\begin{aligned} d(x, x') &= \mathcal{T}_1(\delta_x, \delta_{x'}) \leq \sum \mathcal{T}_1(m_{x_i}^{*t_i}, m_{x_{i+1}}^{*t_{i+1}}) \\ &\leq \sum e^{-\kappa \inf(t_i, t_{i+1})} d(x_i, x_{i+1}) + \frac{C(\sqrt{t_{i+1}} - \sqrt{t_i})^2}{2d(x_i, x_{i+1})} \end{aligned}$$

by assumption. Now, for $a < b$ we have $\sin b - \sin a = 2 \sin \frac{b-a}{2} \cos \frac{a+b}{2} \leq (b-a) \cos \frac{a+b}{2}$ so that

$$\frac{C(\sqrt{t_{i+1}} - \sqrt{t_i})^2}{2d(x_i, x_{i+1})} \leq \frac{C\eta\pi^2 d(x_i, x_{i+1})}{2d(x, x')^2} \cos^2 \left(\pi \frac{d(x, x_i) + d(x, x_{i+1})}{2d(x, x')} \right)$$

Besides, if η is small enough, one has $e^{-\kappa \inf(t_i, t_{i+1})} = 1 - \kappa \inf(t_i, t_{i+1}) + O(\eta^2)$. So we get

$$\begin{aligned} d(x, x') &\leq \sum d(x_i, x_{i+1}) - \kappa \inf(t_i, t_{i+1}) d(x_i, x_{i+1}) \\ &\quad + \frac{C\eta\pi^2 d(x_i, x_{i+1})}{2d(x, x')^2} \cos^2 \left(\pi \frac{d(x, x_i) + d(x, x_{i+1})}{2d(x, x')} \right) + O(\eta^2) \end{aligned}$$

Now the terms $\sum d(x_i, x_{i+1}) \cos^2\left(\pi \frac{d(x, x_i) + d(x, x_{i+1})}{2d(x, x')}\right)$ and $\sum \inf(t_i, t_{i+1})d(x_i, x_{i+1})$ are close to the integrals $d(x, x') \int_0^1 \cos^2(\pi u) du$ and $d(x, x')\eta \int_0^1 \sin^2(\pi u) du$ respectively; the relative error in the Riemann sum is easily bounded by $\pi\varepsilon/d(x, x')$ so that

$$\begin{aligned} d(x, x') &\leq d(x, x') - \kappa \eta d(x, x') \left(\frac{1}{2} - \frac{\pi\varepsilon}{d(x, x')} \right) \\ &\quad + \frac{C\eta\pi^2}{2d(x, x')^2} d(x, x') \left(\frac{1}{2} + \frac{\pi\varepsilon}{d(x, x')} \right) + O(\eta^2) \end{aligned}$$

hence, taking η small enough,

$$d(x, x')^2 \leq \frac{C\pi^2}{2\kappa} \frac{1 + 2\pi\varepsilon/d(x, x')}{1 - 2\pi\varepsilon/d(x, x')}$$

so that either $d(x, x') \leq \pi\sqrt{C/2\kappa}$, or $2\pi\varepsilon/d(x, x') \leq 2\pi\varepsilon/\pi\sqrt{C/2\kappa} \leq 1/2$ by the assumption that ε is small, in which case we use $(1+a)/(1-a) \leq 1+4a$ for $a \leq 1/2$, hence the conclusion. \square

8 Transportation distance in Riemannian manifolds

Here we give the proofs of Proposition 6 and of the statements of Example 7 and Section 3.3.1.

We begin with Proposition 6 and evaluation of the Ricci curvature of the random walk at scale ε .

Let X be a smooth N -dimensional Riemannian manifold and let $x \in X$. Let v, w be unit tangent vectors at x . Let $\delta, \varepsilon > 0$ small enough. Let $y = \exp_x(\delta v)$. Let $x' = \exp_x(\varepsilon w)$ and $y' = \exp_y(\varepsilon w')$ where w' is the tangent vector at y obtained by parallel transport of w along the geodesic $t \mapsto \exp_x(tv)$. The first claim is that $d(x', y') = \delta \left(1 - \frac{\varepsilon^2}{2} K(v, w) + O(\delta\varepsilon^2 + \varepsilon^3) \right)$.

We suppose for simplicity that w and w' are orthogonal to v .

We will work in cylindrical coordinates along the geodesic $t \mapsto \exp_x(tv)$. Let $v_t = \frac{d}{dt} \exp_x(tv)$ be the speed of this geodesic. Let E_t be the orthogonal of v_t in the tangent space at $\exp_x(tv)$. Each point z in some neighborhood of x can be uniquely written as $\exp_{\exp_x(\tau(z)v)}(\varepsilon\zeta(z))$ for some $\tau(z) \in \mathbb{R}$ and $\zeta(z) \in E_{\tau(z)}$.

Consider the function f equal to the distance of a point to $\exp_x(E_0)$ (taken in some small enough neighborhood of x), equipped with a $-$ sign if

the point is not on the same side of E_0 as y . Clearly f is 1-Lipschitz, so that $d(x', y') \geq f(y') - f(x')$.

The distance from $\exp_x(E_0)$ to y' is realized by some geodesic γ starting at some point of $\exp_x(E_0)$ and ending at y . If δ and ε are small enough, this geodesic is arbitrarily close to the Euclidean situation so that the coordinate τ is strictly increasing along γ . Let us parametrize γ using the coordinate τ , so that $\tau(\gamma(t)) = t$. Let also $w_t = \zeta(\gamma(t)) \in E_t$. In particular, $w_\delta = w'$.

Now by definition we have $\gamma(t) = \exp_{\exp_x(tv)}(\varepsilon w_t)$. Considering the family of geodesics $s \mapsto \exp_{\exp_x(tv)}(sw_t)$ and applying the Jacobi equation yields

$$\left| \frac{d\gamma(t)}{dt} \right|^2 = |v_t|^2 + 2\varepsilon \langle v_t, \dot{w}_t \rangle + \varepsilon^2 |\dot{w}_t|^2 - \varepsilon^2 \langle R(w_t, v_t)w_t, v_t \rangle + O(\varepsilon^3)$$

where $\dot{w}_t = \frac{d}{dt}w_t$. But since by definition $w_t \in E_t$, we have $\langle v_t, \dot{w}_t \rangle = 0$. Since moreover $|v_t| = 1$ we get

$$\left| \frac{d\gamma(t)}{dt} \right| = 1 + \frac{\varepsilon^2}{2} |\dot{w}_t|^2 - \frac{\varepsilon^2}{2} \langle R(w_t, v_t)w_t, v_t \rangle + O(\varepsilon^3)$$

which is always greater than $1 - \frac{\varepsilon^2}{2} \langle R(w_t, v_t)w_t, v_t \rangle + O(\varepsilon^3)$. Integrating from $t = 0$ to $t = \delta$ and using that $\langle R(w_t, v_t)w_t, v_t \rangle = K(w, v) + O(\delta)$ yields that the length of the geodesic is

$$\delta \left(1 - \frac{\varepsilon^2}{2} K(v, w) + O(\varepsilon^3) + O(\varepsilon^2 \delta) \right)$$

so that the distance from x' to y' is at least this quantity. But this value is achieved for $\dot{w}_t = 0$, in which case $\gamma(0) = x'$ by definition, so this is exactly $d(x', y')$. This proves Proposition 6.

Let us now prove the statement of Example 7. Let μ_0, μ_1 be the uniform probability measures on the balls of radius ε centered at x and y respectively. We have to prove that

$$\mathcal{T}_1(\mu_0, \mu_1) = d(x, y) \left(1 - \frac{\varepsilon^2}{2(N+2)} \text{Ric}(v, v) \right)$$

up to higher-order terms.

Let μ'_0, μ'_1 be the images under the exponential map, of the uniform probability measures on the balls of radius ε in the tangent spaces at x and y' respectively. So μ'_0 is a measure having density $1 + O(\varepsilon^2)$ w.r.t. μ_0 , and likewise for μ'_1 .

If we average Proposition 6 over w in the ball of radius ε in the tangent space at x , we get that

$$\mathcal{T}_1(\mu'_0, \mu'_1) \leq d(x, y) \left(1 - \frac{\varepsilon^2}{2(N+2)} \text{Ric}(v, v) \right)$$

up to higher-order terms, since the coupling by parallel transport realizes this value. Indeed, $\text{Ric}(v, v)$ is the sum of $K(v, w)$ for w in an orthonormal basis of the tangent space at x . Consequently, the average of $K(v, w)$ on the unit sphere is $\frac{1}{N} \text{Ric}(v, v)$. Averaging on the ball instead of the sphere yields an $\frac{1}{N+2}$ factor instead.

Now the density of μ'_0, μ'_1 with respect to μ_0, μ_1 is $1 + O(\varepsilon^2)$. Moreover the $O(\varepsilon^2)$ terms decompose as the sum of an $O(d(x, y)\varepsilon^2)$ term and an $O(\varepsilon^2)$ term which is the same for μ'_0 and μ'_1 (indeed, μ'_0 and μ'_1 coincide when $x = y$). Plugging this in the estimate above, we get the inequality for $\mathcal{T}_1(\mu_0, \mu_1)$ up to higher-order terms.

The converse inequality is proven as follows: if f is any 1-Lipschitz function, the L^1 transportation distance between measures μ_0 and μ_1 is at least the difference of the integrals of f under μ_0 and μ_1 (and actually, a clever choice of f realizes this transportation distance, see Theorem 1.14 in [Vil03]). Arguments similar to the above for integrating under μ_0 and μ_1 , applied to the function f above equal to the distance of a point to the set $\exp_x(E_0)$, yield the desired inequality.

Finally, let us briefly sketch the proofs of the other statements of Section 3.3.1, namely, evaluation of the spread and local dimension (Definition 18). Up to a multiplicative factor $O(1 + \varepsilon)$, these can be computed in the Euclidean space.

A simple computation shows that the expectation of the square distance of two points taken at random in a ball of radius ε is $\varepsilon^2 \frac{2N}{N+2}$, hence the value $\varepsilon^2 \frac{N}{N+2}$ for the spread.

To evaluate the local dimension (Definition 18), we have to bound the maximal variance of a 1-Lipschitz function on a ball of radius ε . We will prove that the local dimension n_x is comprised between $N - 1$ and N . A projection to a coordinate axis provides a function with variance $\frac{\varepsilon^2}{N+2}$, so that local dimension is at most N . For the other bound, let f be a 1-Lipschitz function on the ball and let us compute an upper bound for its variance. Take $\varepsilon = 1$ for simplicity. Write the ball of radius 1 as the union of the spheres S_r of radii $r \leq 1$. Let $v(r)$ be the variance of f restricted to the sphere S_r , and let $a(r)$ be the average of f on S_r . Then associativity of variances gives

$$\text{Var } f = \int_{r=0}^1 v(r) d\mu(r) + \text{Var}_\mu a(r)$$

where μ is the measure on the interval $[0; 1]$ given by $\frac{r^{N-1}}{Z} dr$ with $Z = \int_{r=0}^1 r^{N-1} dr = \frac{1}{N}$.

Since the variance of a 1-Lipschitz function on the $(N - 1)$ -dimensional unit sphere is at most $\frac{1}{N}$, we have $v(r) \leq \frac{r^2}{N}$ so that $\int_{r=0}^1 v(r) d\mu(r) \leq \frac{1}{N+2}$. To evaluate the second term, note that $a(r)$ is again 1-Lipschitz as a function of r , so that $\text{Var}_\mu a(r) = \frac{1}{2} \iint (a(r) - a(r'))^2 d\mu(r)d\mu(r')$ is at most $\frac{1}{2} \iint (r - r')^2 d\mu(r)d\mu(r') = \frac{N}{(N+1)^2(N+2)}$. So finally

$$\text{Var } f \leq \frac{1}{N+2} + \frac{N}{(N+1)^2(N+2)}$$

so that the local dimension n_x is bounded below by $\frac{N(N+1)^2}{N^2+3N+1} \geq N - 1$.

References

- [ABCFGMRS00] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, G. Scheffer, *Sur les inégalités de Sobolev logarithmiques*, Panoramas et Synthèses **10**, Société Mathématique de France (2000).
- [AMTU01] A. Arnold, P. Markowich, G. Toscani, A. Unterreiter, *On convex Sobolev inequalities and the rate of convergence to equilibrium for Fokker-Planck type equations*, Comm. Partial Differential Equations **26** (2001), n° 1-2, 43–100.
- [Ber03] M. Berger, *A panoramic view of Riemannian geometry*, Springer, Berlin (2003).
- [BE84] D. Bakry, M. Émery, *Hypercontractivité de semi-groupes de diffusion*, C. R. Acad. Sci. Paris Sér. I Math. **299** (1984), n° 15, 775–778.
- [BE85] D. Bakry, M. Émery, *Diffusions hypercontractives*, Séminaire de probabilités, XIX, 1983/84. Lecture Notes in Math. **1123**, Springer, Berlin (1985), 177–206.
- [BL98] S. Bobkov, M. Ledoux, *On modified logarithmic Sobolev inequalities for Bernoulli and Poisson measures*, J. Funct. Anal. **156** (1998), n° 2, 347–365.
- [Bré99] P. Brémaud, *Markov chains*, Texts in Applied Mathematics **31**, Springer, New York (1999).

- [Che98] M.-F. Chen, *Trilogy of couplings and general formulas for lower bound of spectral gap*, in *Probability towards 2000 (New York, 1995)*, Lecture Notes in Statist. **128**, Springer, New York (1998), 123–136.
- [CW97] M.-F. Chen, F.-Y. Wang, *Estimation of spectral gap for elliptic operators*, Trans. Amer. Math. Soc. **349** (1997), n° 3, 1239–1267.
- [Dob56] R. L. Dobrušin, *On the condition of the central limit theorem for inhomogeneous Markov chains* (Russian), Dokl. Akad. Nauk SSSR (N.S.) **108** (1956), 1004–1006.
- [DGW04] H. Djellout, A. Guillin, L. Wu, *Transportation cost-information inequalities and applications to random dynamical systems and diffusions*, Ann. Prob. **32** (2004), n° 3B, 2702–2732.
- [GD03] A. Granas, J. Dugundji, *Fixed point theory*, Springer Monographs in Mathematics, Springer, New York (2003).
- [GH90] É. Ghys, P. de la Harpe, *Sur les groupes hyperboliques d'après Mikhael Gromov*, Progress in Math. **83**, Birkhäuser (1990).
- [Gri67] R. B. Griffiths, *Correlations in Ising ferromagnets III*, Commun. Math. Phys. **6** (1967), 121–127.
- [Gro86] M. Gromov, in V. Milman, G. Schechtman, *Asymptotic theory of finite dimensional normed spaces*, Lecture Notes in Mathematics **1200**, Springer, Berlin (1986).
- [Jou] A. Joulin, *Poisson-type deviation inequalities for curved continuous time Markov chains*, preprint.
- [Led01] M. Ledoux, *The concentration of measure phenomenon*, Mathematical Surveys and Monographs **89**, AMS (2001).
- [Lott] J. Lott, *Optimal transport and Ricci curvature for metric-measure spaces*, expository manuscript.
- [LV] J. Lott, C. Villani, *Ricci curvature for metric-measure spaces via optimal transport*, preprint.
- [Oht] S.-i. Ohta, *On the measure contraction property of metric measure spaces*, preprint.

- [Oli] R. I. Oliveira, *On the convergence to equilibrium of Kac's random walk on matrices*, preprint, [arXiv:0705.2253](https://arxiv.org/abs/0705.2253)
- [RS05] M.-K. von Renesse, K.-T. Sturm, *Transport inequalities, gradient estimates, and Ricci curvature*, Comm. Pure Appl. Math. **68** (2005), 923–940.
- [Sam] M. D. Sammer, *Aspects of mass transportation in discrete concentration inequalities*, PhD thesis, Georgia institute of technology, 2005, etd.gatech.edu/theses/available/etd-04112005-163457/unrestricted/sammer_marcus_d_200505_phd.pdf
- [Stu06] K.-T. Sturm, *On the geometry of metric measure spaces*, Acta Math. **196** (2006), n°1, 65–177.
- [Vil03] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics **58**, AMS (2003).
- [Vil] C. Villani, *Optimal transport, old and new*, July 12, 2007 version, www.umpa.ens-lyon.fr/~cvillani/Cedrif/B07B.StFlour.pdf