Delta function and its evolution

A toy model for jet physics

define a jet function $J(z, \mu)$

at leading order, it's given by a delta-function

$$J(z,\mu) = \delta(1-z)$$

If one goes to higher order, one can show such a function satisfies the following evolution equation

$$\frac{\partial}{\partial \ln\mu^2} J(z,\mu) = \frac{\alpha_s(\mu)}{2\pi} \int_z^1 \frac{dx}{x} P(x) \, J\left(\frac{z}{x}, \mu\right)$$

where $$P(x) = C_F \left[\frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \right]$$

Here we have a "+" function, it has the following meaning

$$\int_0^1 dz \frac{1}{(1-z)_+} f(z) \equiv \int_0^1 dz \frac{f(z)-f(1)}{1-z}$$

If one has a different integration limits from $[0,1]$, then we have to do

$$\text{Integral} = \int_a^1 dz \frac{1}{(1-z)_+} f(z) = \left[\int_0^1 - \int_0^a\right] dz \frac{1}{(1-z)_+} f(z)$$

$$= \underbrace{\int_0^1 dz \frac{1}{(1-z)_+} f(z)}_{\text{I}} - \underbrace{\int_0^a dz \frac{1}{(1-z)_+} f(z)}_{\text{II}}$$

$$\text{I} = \int_0^1 dz \frac{f(z)-f(1)}{(1-z)}$$

$$\mathbb{I} = \int_0^a dz \, \frac{1}{(1-z)_+} f(z)$$

$\Downarrow$ Since integration limits $0 < z < a$ and thus "$z = 1$" singularity is not in this region, we can simply neglect "+" function

$$= \int_0^a dz \, \frac{1}{(1-z)} f(z)$$

now $\quad I = \int_0^1 dz \, \frac{1}{(1-z)} [f(z) - f(1)]$

$$= \int_a^1 dz \, \frac{1}{(1-z)} [f(z) - f(1)] + \underbrace{\int_0^a dz \, \frac{1}{(1-z)} [f(z) - f(1)]}_{I_2}$$

note: $\quad I_2 = \int_0^a dz \, \frac{1}{(1-z)} f(z) - \int_0^a dz \, \frac{1}{(1-z)} f(1)$

$$= \int_0^a dz \, \frac{1}{(1-z)} f(z) - \ln \frac{1}{(1-z)} \Big|_0^a f(1)$$

$$= \int_0^a dz \, \frac{1}{(1-z)} f(z) + \ln(1-a) * f(1)$$

$\Rightarrow \quad I = \int_a^1 dz \, \frac{f(z) - f(1)}{(1-z)} + \int_0^a dz \, \frac{f(z)}{1-z} + \ln(1-a) * f(1)$

on the other hand, $\quad \mathbb{I} = \int_0^a dz \, \frac{f(z)}{(1-z)}$

then: $\quad$ Integral $= I - \mathbb{I} = \int_a^1 dz \, \frac{f(z) - f(1)}{1-z} + \ln(1-a) * f(1) \qquad (*)$

Now the task is to solve this evolution equation

$$\frac{\partial}{\partial \ln \mu^2} J(z, \mu) = \frac{\alpha_s(\mu)}{2\pi} \int_z^1 \frac{dx}{x} P(x)\, J\left(\frac{z}{x}, \mu\right)$$

we can choose the initial condition

$$J(z, \mu_0) = \delta(1-z)$$

from this, Please give the numerical form for $J(z, \mu)$ at an arbitrary "μ"

to get started: you could choose $\mu_0 = 10$ GeV

then choose to get the solution $J(z,\mu)$ for

$\mu = 100$ GeV

Here: $\alpha_s(\mu)$ is the strong coupling constant, for simplicity, let's just choose $\alpha_s(\mu) = 0.1$ (even though $\alpha_s(\mu)$ actualy varies with respect to μ)

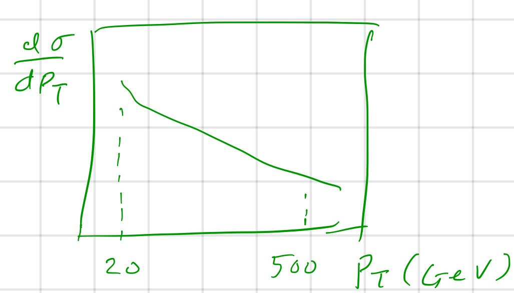
once you obtain a table for $J(z, \mu)$, we can use it to Compute jet cross section! we'll also do a toy model

$$\frac{d\sigma}{dp_T} = \int_{z_{min}}^1 dz \frac{z^2}{p_T^4} J(z, \mu = p_T)$$

with $z_{min} = \frac{2 p_T}{\sqrt{s}}$

Choose $\sqrt{s} = 13$ TeV

eventually, I'll need a plot like



The way to tackle the problem

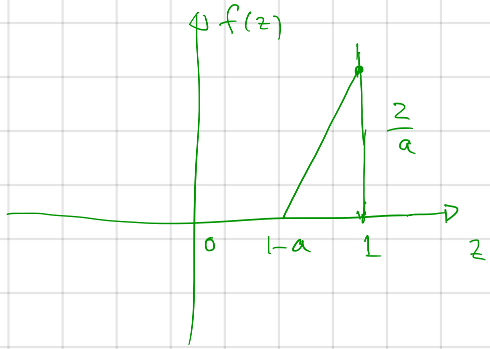
It's not easy to deal with delta function, since they're NOT a smooth function

$$\delta(1-z) = \begin{cases} 0 & \text{if } z \neq 1 \\ \infty & \text{if } z = 1 \end{cases} \quad \text{and} \quad \int_0^1 dz\, \delta(1-z) = 1$$

How could one deal with this ?

one possible way is the following

Choose



Choose a very small "a"

$$f(z) = \begin{cases} 0 & z < 1-a \\ \frac{2}{a^2}\left[z-(1-a)\right] & z \geq 1-a \end{cases}$$

then $f(z) \xrightarrow{z \to 1} \frac{2}{a^2}[a] = \frac{2}{a} \to \infty$ for $a \to 0$

$f(z) \xrightarrow{z < 1-a} 0$

at the same time $\int_0^1 dz\, f(z) = \int_0^{1-a} dz * 0$

$$+ \int_{1-a}^{1} dz\, \frac{2}{a^2}\left[z-(1-a)\right]$$

$$= 0 + \frac{2}{a^2}\left[\frac{1}{2}z^2 - (1-a)z\right]_{1-a}^{1}$$

$$= \frac{2}{a^2}\left[\frac{1}{2}\left(1-(1-a)^2\right) - (1-a)\left(1-(1-a)\right)\right]$$

$$= \frac{2}{a^2}\left[\frac{1}{2} - \frac{1}{2}(1-a)^2 - (1-a) + (1-a)^2\right]$$

$$= \frac{2}{a^2} * \frac{a^2}{2} = 1$$

Satisfies the normalization

Thus one could use the above $f(z)$ to solve the evolution equation!

The hope is that the cross section is independent of the value "$\underline{\underline{a}}$"!