UNIVERSITÀ DEGLI STUDI DI PADOVA

Dynamical Systems

Double Gyre

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Outline

1. First problem: Integrable double gyre
   - Statement of the exercise
   - Recall some theory of 2D incompressible flows
   - Solution
2. Second problem: Non integrable double gyre
   - Recall of CLE and FLI
   - Solution
   - Regular Vs Chaotic
3. Problem 3
   - Statement
   - Solution

Exercise

Let us define the integrable double gyre by

$$\psi(x, y) = -\sin(\pi x) sin(\pi y), \quad (x, y) \in [0, 2] \times [0, 1]. \tag{1}$$

Exercise: Study the dynamics of the double gyre. Precisely:

I. Write the equations of motion of the double gyre.
II. Compute all the equlibrium points.
III. Compute the invariant lines defined by: $x = const$ or $y = const$.
IV. Find the elliptic equilibria.
V. Represent the phase portrait of the system.

Two dimensional incompressible fluids

- Trajectory of a particle is the solution of the 2 dof ODE

$$\dot{x} = X(x, y, t), \quad \dot{y} = Y(x, y, t). \tag{2}$$

- Icompressibility of the flow means $\nabla \cdot \underline{X} = 0$.
- Flows of incompressible fluid are icompressible ($\rho(\underline{x}) = \underline{const} \implies \nabla \cdot \underline{X} = 0$.)
- Density $\rho(\underline{x}, t)$ is a constant of motion for incompressible flows ($\nabla \cdot \underline{X} = 0 \implies \frac{d}{dt}\rho(\underline{x}(t), t) = 0$)

Stream function

The condition $\nabla \cdot \underline{X} = 0$ is immediately satisfied for

$$X = \dot{x} = \frac{\partial \psi}{\partial y} \quad Y = \dot{y} = -\frac{\partial \psi}{\partial x}. \tag{3}$$

- ψ is an Hamiltonian function for the system.
- If $\psi = \psi(x, y)$, autonomous, ψ is a first integral. The motion is integrable, and it's called *Stationary*.

$$\frac{d}{dt}\psi(\underline{x}(t), t) = \nabla\psi \cdot \underline{X} + \frac{\partial \psi}{\partial t} \tag{4}$$

Solution 1

1. Equations of motion:

$$\begin{aligned} \dot{x} &= \frac{\partial \psi}{\partial y} = -\pi \sin(\pi x) \cos(\pi y) \\ \dot{y} &= -\frac{\partial \psi}{\partial x} = \pi \cos(\pi x) \sin(\pi y) \end{aligned} \tag{5}$$

2. Equilibrium points

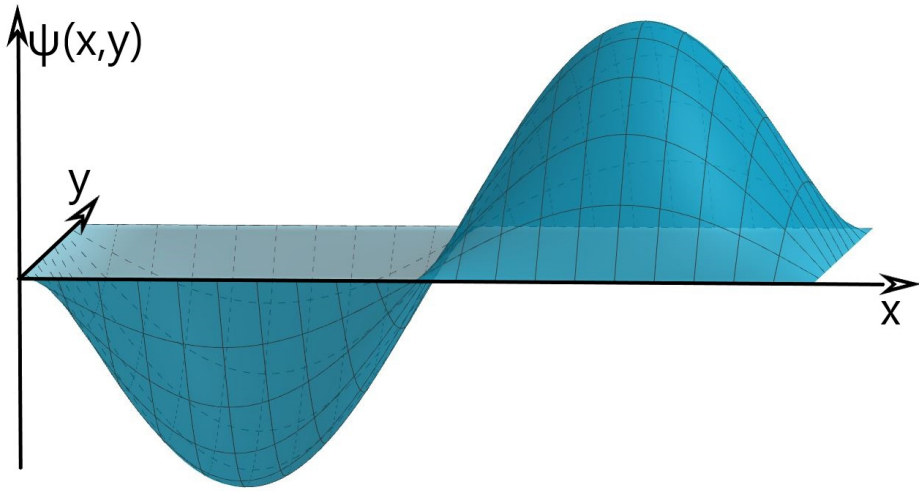
$$\dot{x} = 0, \quad \dot{y} = 0 \tag{6}$$

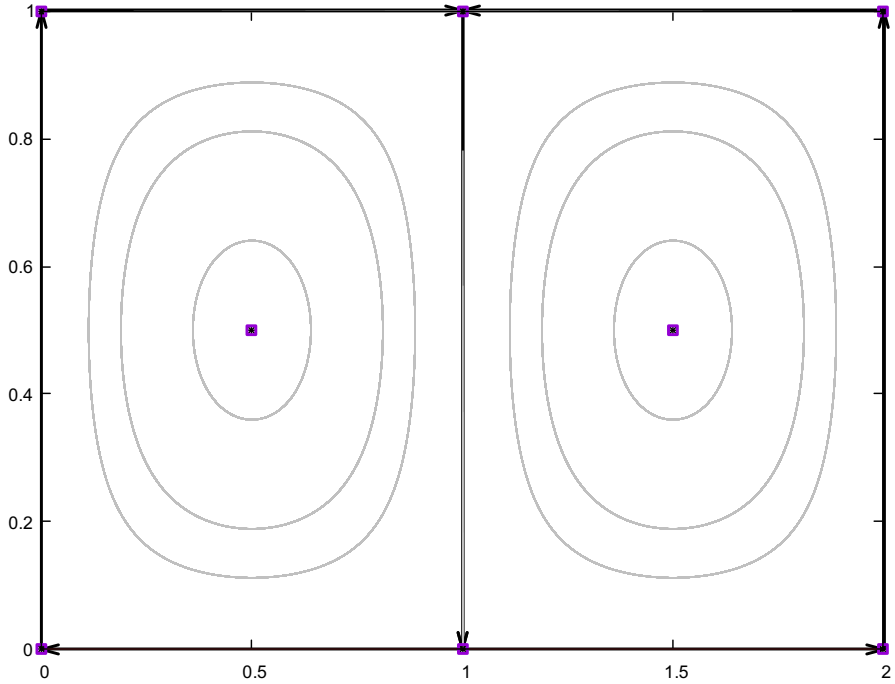
3. Check invariant lines

$$\begin{aligned} \dot{y}(x, y_*) &= \pi \cos(\pi x) \sin(\pi y_*) = 0 \\ \dot{x}(x_*, y) &= -\pi \sin(\pi x_*) \cos(\pi y) = 0 \end{aligned} \tag{7}$$

Solution 2

- Analyze Jacobian matrix at $(\frac{1}{2}, \frac{1}{2})$, $(\frac{3}{2}, \frac{1}{2})$ or look at the graph of $\psi(x, y)$. From the graph of the first integral is easy to identify the elliptic equilibria.
- The level curves are closed, particle doesn't move from one quadrant to another.





Time dependent double gyre

The stream function is now non autonomous

$$\begin{aligned} \psi(x,y,t) &= -\sin(\pi f(x,t))\sin(\pi y) \\ f(x,t) &= \epsilon \sin(\omega t)x^2 + (1 - 2\epsilon(\sin(\omega t))x, \end{aligned} \tag{8}$$

with $\omega = 2\pi$ frequency of f, and ϵ small parameter of a perturbation.

- ψ is no more a first integral

$$\frac{d}{dt}\psi(x(t),y(t),t) = \nabla\psi \cdot \underline{X} + \frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial t} \ \ (\neq 0 \ a.e.) \tag{9}$$

Exercise 2

Consider the time dependent double gyre.

1. Write the variational equations of the system.
2. For $\epsilon = 0.001$ represent the phase portrait of the Poincarè section at time t=1.
3. Select three orbits and compute their FLI for $T \in [0, 1000]$ for the initial tangent vector ($\underline{v}(0) = (1, 1)$).

Brief recall of the variational dynamics

Separation of two initial conditions using the tangent map applied to the initial difference of the two points (tangent vector).

- If $\phi(t, \underline{x})$ denotes the flow of a differential equation $\dot{\underline{x}} = \underline{X}(\underline{x})$, the idea is that

$$\underline{v}(t) = \phi(t, \underline{x}'') - \phi(t, \underline{x}') = D\phi(t, \underline{x}')(\underline{x}'' - \underline{x}') + \mathcal{O}(||\underline{x}'' - \underline{x}'||^2), \quad (10)$$

and extending the reasoning, we have

$$\dot{\underline{v}}(t) = JX(\phi(t, \underline{x}'))(\underline{v}(t)) \quad (11)$$

$$JX_{i,k} = \frac{\partial X_i}{\partial x_k}.$$

Variational equation

For any initial condition $\underline{x}_0$, the time evolution $\underline{v}_t$ of the initial tangent vector $\underline{v}_0$, is the solution of

$$\dot{\underline{v}} = JX(\phi(t, \underline{x}_0))\underline{v}. \tag{12}$$

Numerically we solve

$$\begin{cases} \dot{\underline{x}} = \underline{X}(\underline{x}) \\ \dot{\underline{v}} = JX(\underline{x})\underline{v} \end{cases} \tag{13}$$

CLE Vs FLI

CLE

$\chi(\underline{x}_0, \underline{v}_0) = \lim_{t\to\infty} \frac{1}{t} \ln ||\underline{v}(t)||$

- Exact asymptotic growth ($\chi(\underline{x}_0, \underline{v}_0) \leq 0$ regular, $\chi(\underline{x}_0, \underline{v}_0) > 0$ chaotic).
- Independent on norm of $\underline{v}_0$.
- Constant of motion, $\chi(\underline{x}_0, \underline{v}_0) = \chi(\underline{x}_t, \underline{v}_t)$.
- For a.e. initial $\underline{v}_0$ the CLE is equal to the largest one.
- Time scale of exponential separation

$$T_L = \frac{1}{\max\limits_{\underline{v}_0 \in T_{\underline{x}_0} D \setminus 0} \chi(\underline{x}_0, \underline{v}_0)}.$$

FLI

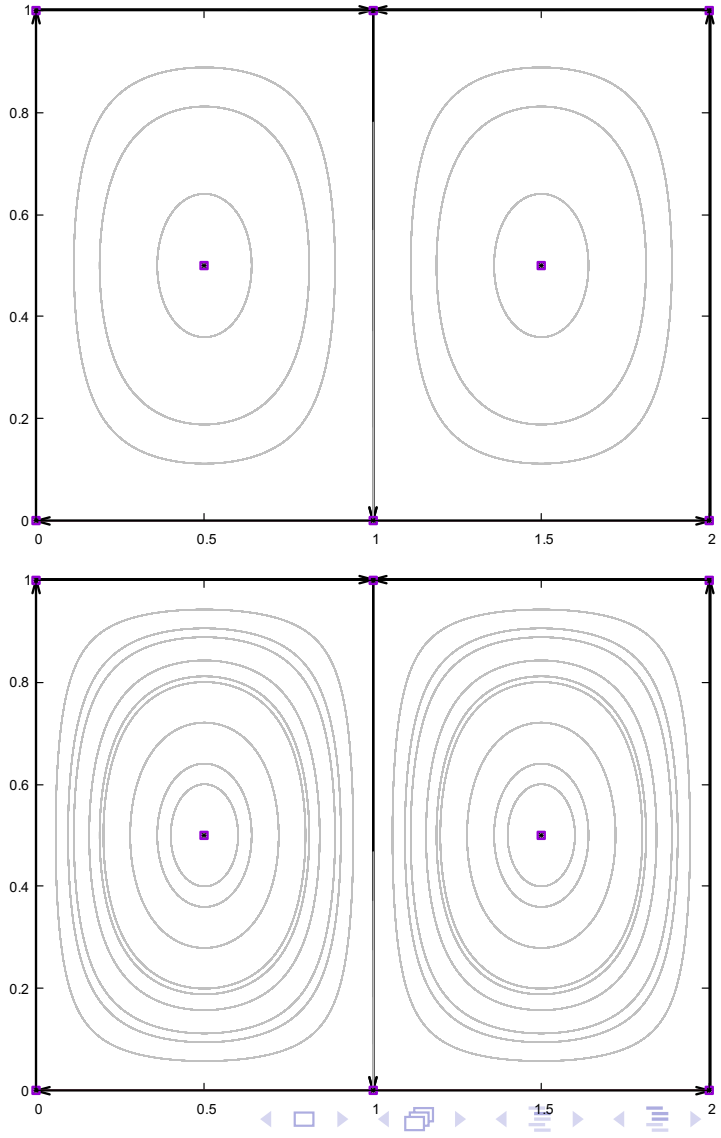
$FLI(\underline{x}_0, \underline{v}_0; T) = \max\limits_{t\in[0,T]} \log ||\underline{v}(t)||$

- FLI/T for $T \to \infty$ converges to $\chi(\underline{x}_0, \underline{v}_0)$.
- Discriminate chaotic/regular in much smaller times.
- Localize separatrices
- $||\underline{v}(t)|| \sim e^{\lambda t} \implies$ FLI grows linearly.
- $||\underline{v}(t)|| \sim t \implies$ FLI growth is logarithmic.

Solution: Check integrator

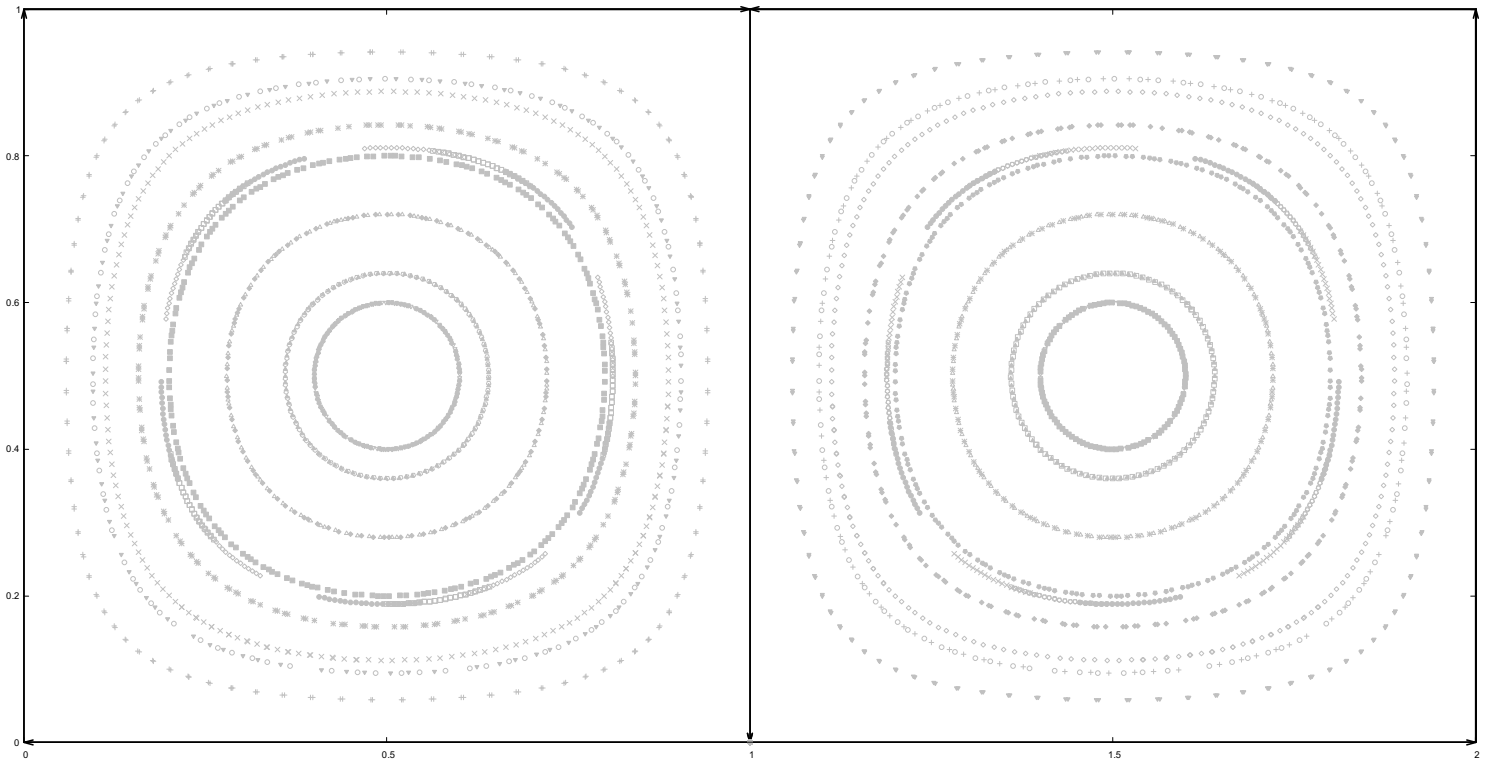
Check solution in integrable case ($\epsilon = 0$). Conservation of the first integral, $\psi(x, y)$.

- Double precision$+RK4 + \Delta t = 10^{-4}$ max error in 1000s $\sim 10^{-13}$
- Double precision$+RK6 + \Delta t = 10^{-3}$ max error in 1000s $\sim 10^{-14}$
- Quadruple precision$+RK6 + \Delta t = 10^{-4}$ max error in 1000s $\sim 10^{-24}$



Check Solution: Poincarè section

Poincarè section for the integrable case



Solution: Variational equations

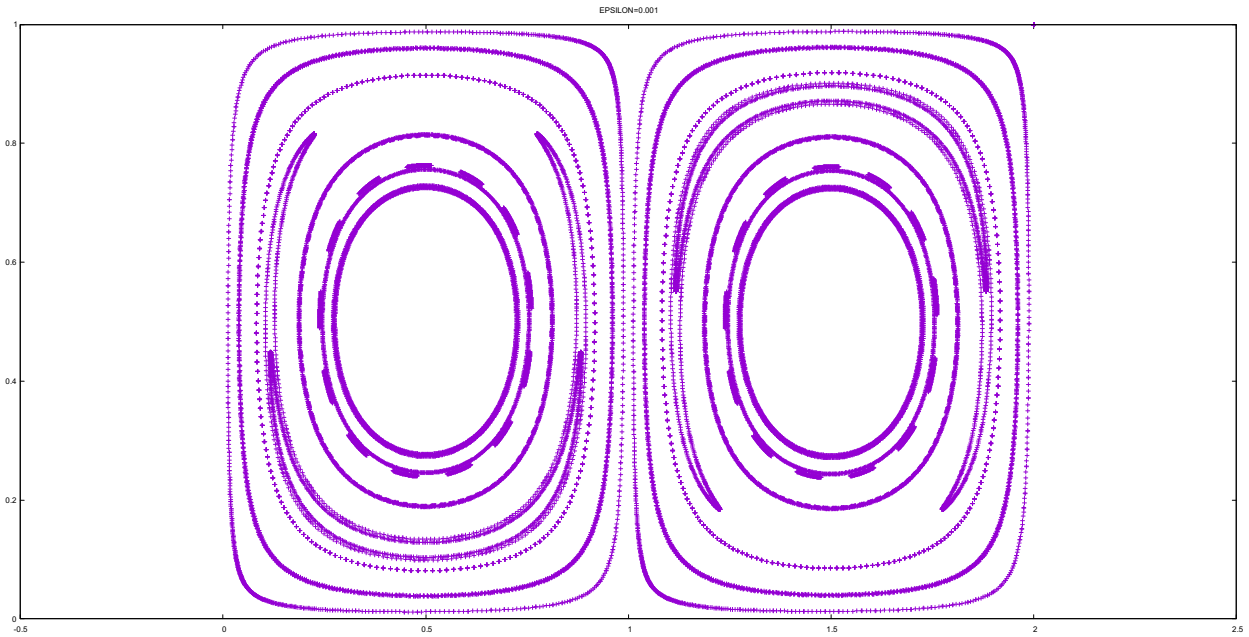
Recall that the stream function is

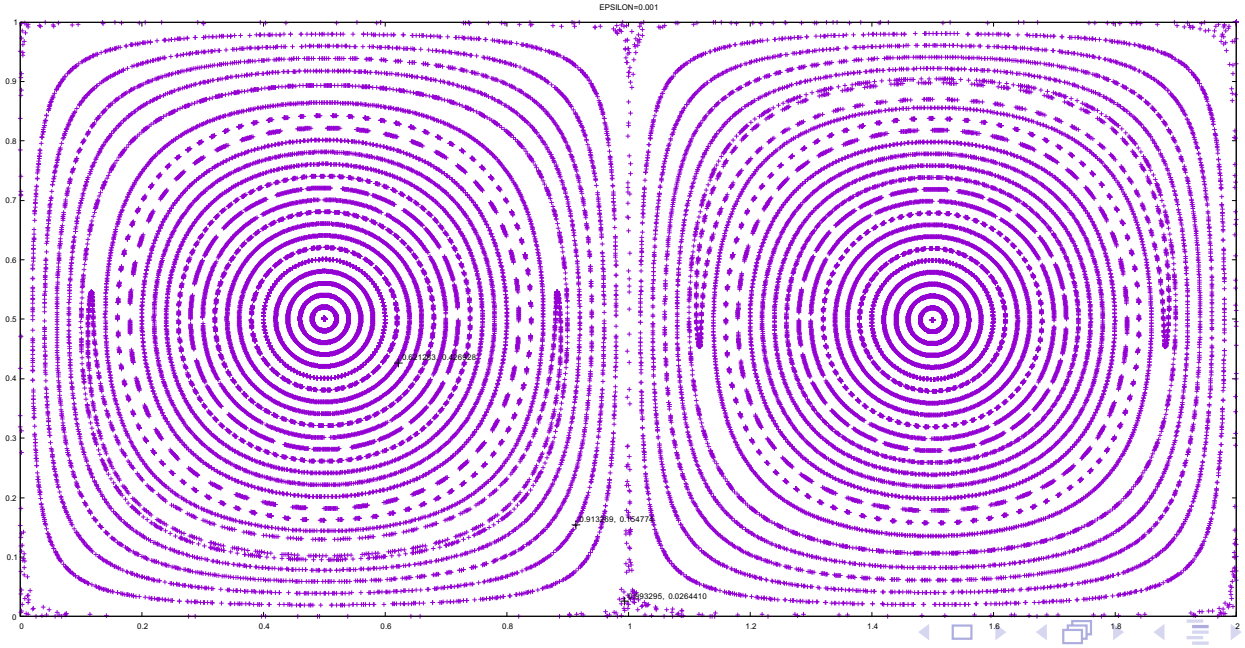
$$\begin{aligned} \psi(x,y,t) &= -\sin(\pi f(x,t))\sin(\pi y) \\ f(x,t) &= \epsilon \sin(\omega t)x^2 + (1 - 2\epsilon[\sin(\omega t))x], \end{aligned} \tag{14}$$

with $\omega = 2\pi$ frequency of f.
The system to be solved is then

$$\begin{cases} \dot{x} = \dfrac{\partial \psi}{\partial y} \\ \dot{y} = -\dfrac{\partial \psi}{\partial x} \\ \dot{\tau} = 1 \\ \dot{v}_x = \dfrac{\partial X}{\partial x} v_x + \dfrac{\partial X}{\partial y} v_y \\ \dot{v}_x = \dfrac{\partial Y}{\partial x} v_x + \dfrac{\partial Y}{\partial y} v_y \end{cases} \tag{15}$$

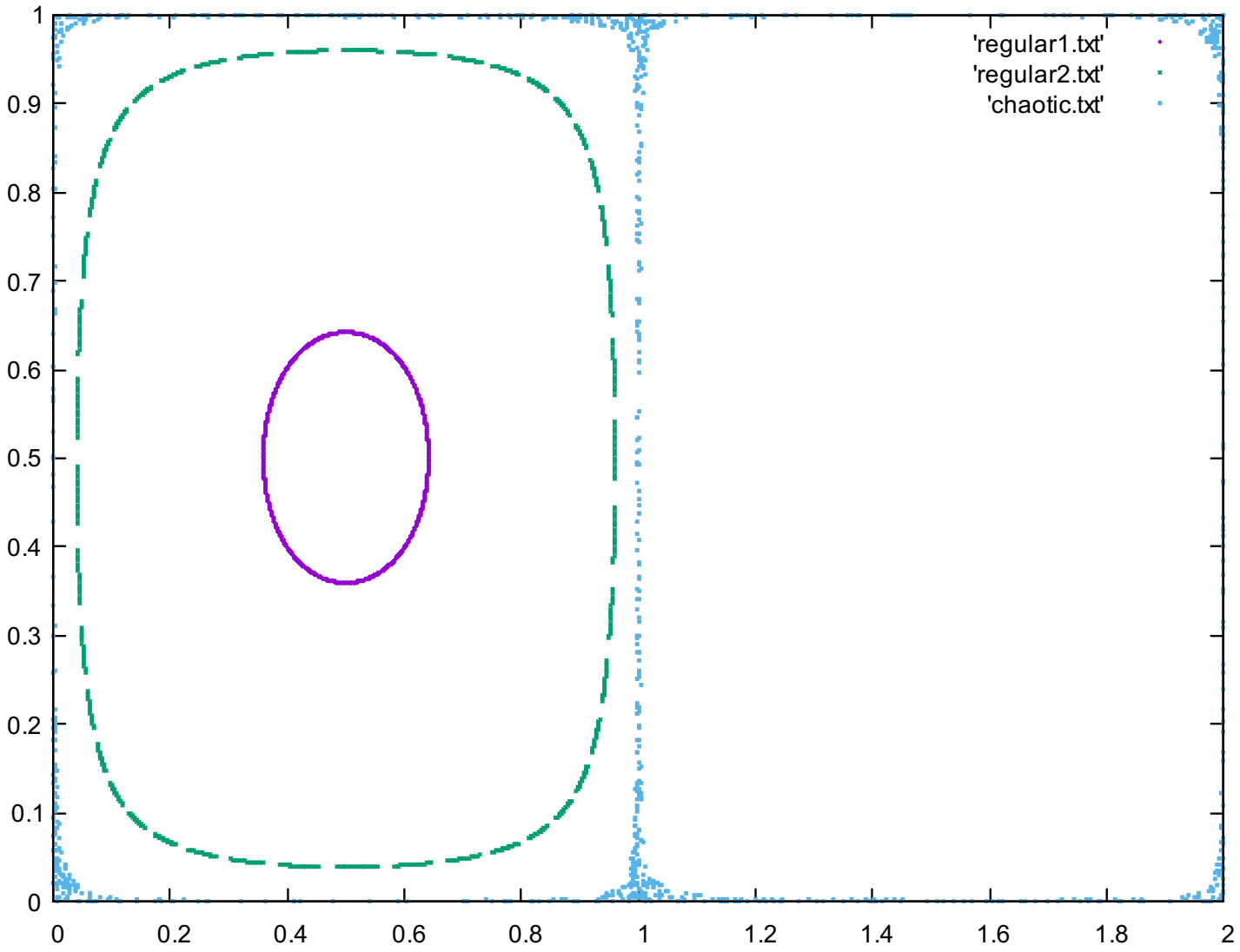
Solution: Poincarè section





Solution: Regular Vs Chaotic behaviour

The flow for $\epsilon \neq 0$ is no more restricted to the quadrant where it started.



Solution: FLI Vs Time

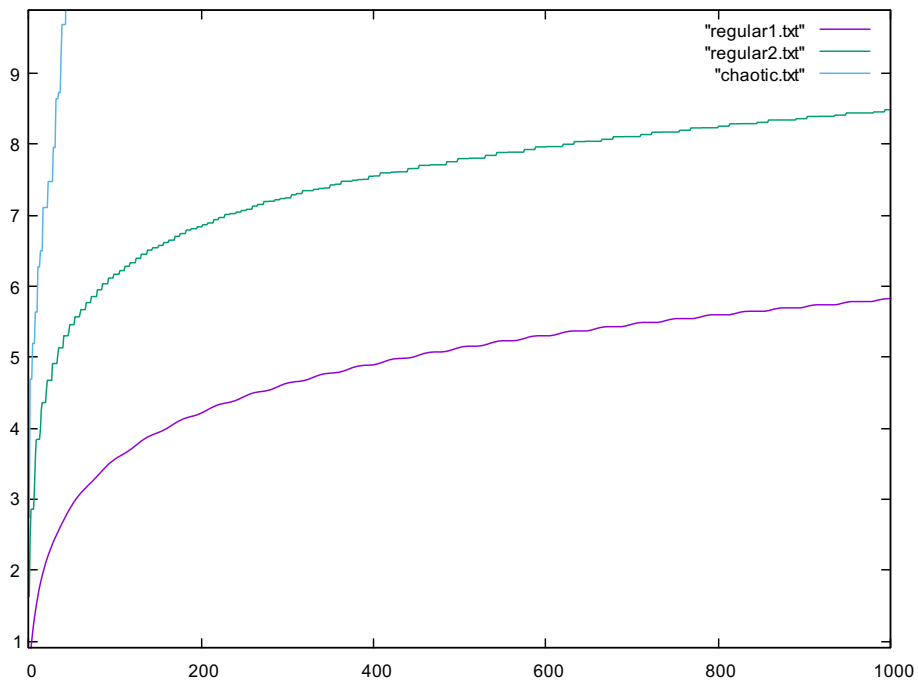


Figure: FLI Vs time

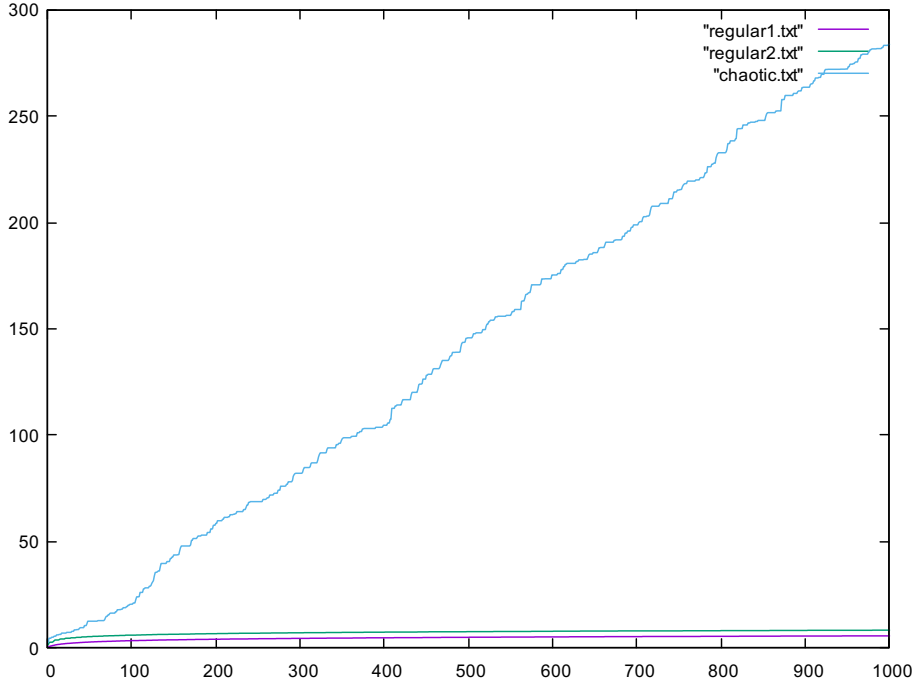


Figure: FLI Vs time

Solution: CLE Vs Time

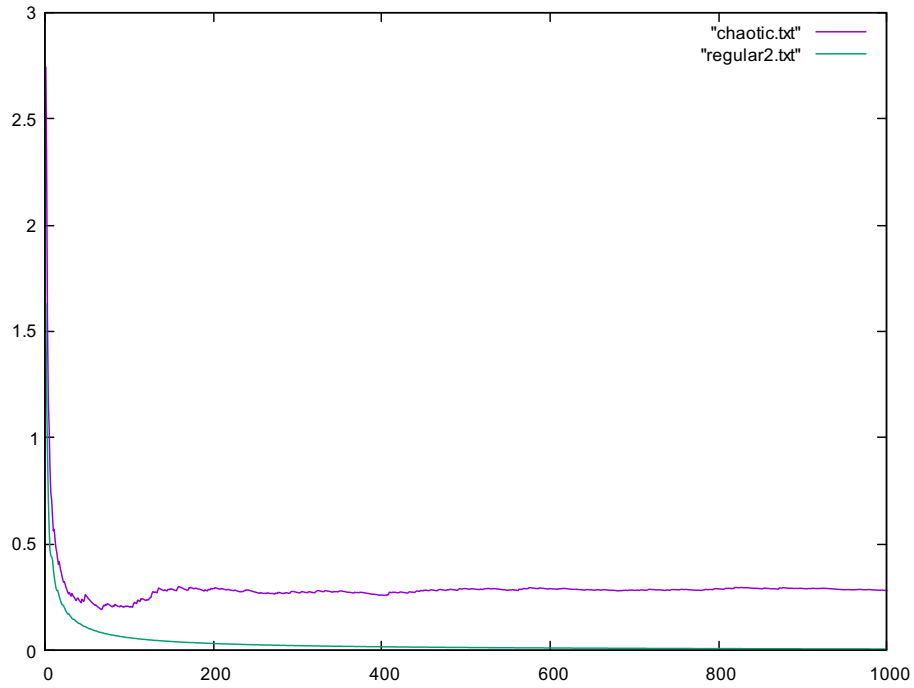


Figure: CLE$\sim$ FLI/T Vs time

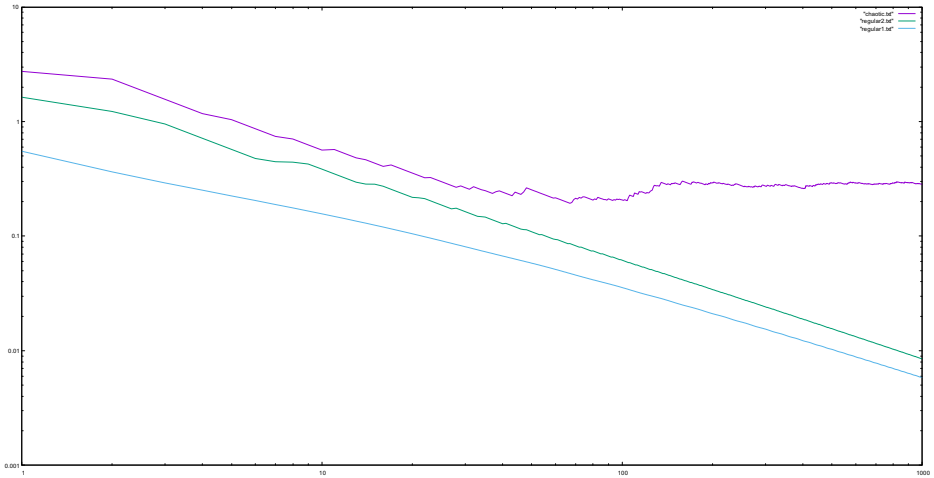


Figure: CLE$\sim$ FLI/T Vs time. Log-Log scale

Choice of initial tangent vector

Almost all initial tangent vectors provides maximum growth of $||\underline{v}(t)||$

- Initial vector must have a non zero component on the maximal expansion direction.
- $FLI(\underline{x}, \mathcal{B}; T) = \max_{t \in [0,T]} \max_j \log ||e_j(t)||$, $e_j \in \mathcal{B}$, $j = 1, ..., n$, orthonormal basis.

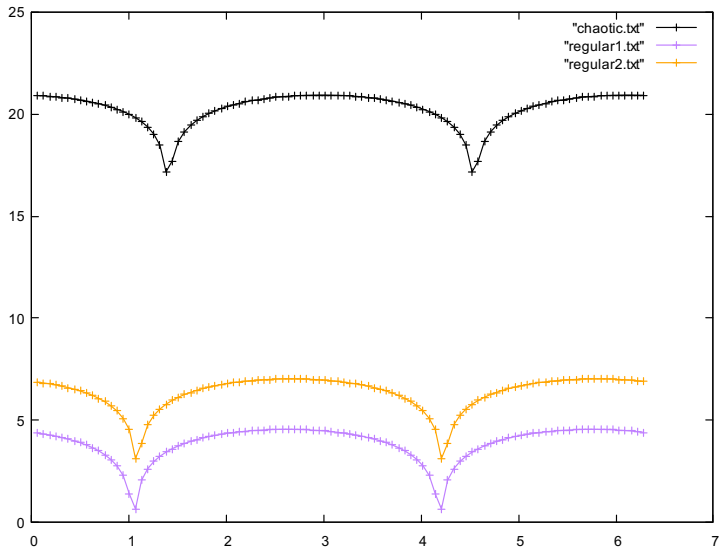


Figure: FLI Vs α at T=100 for $\underline{v}_0 = (\cos\alpha, \sin\alpha)$

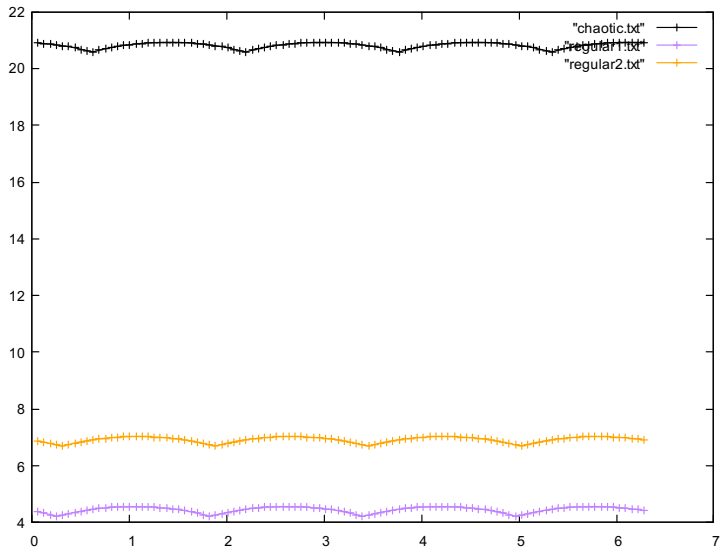


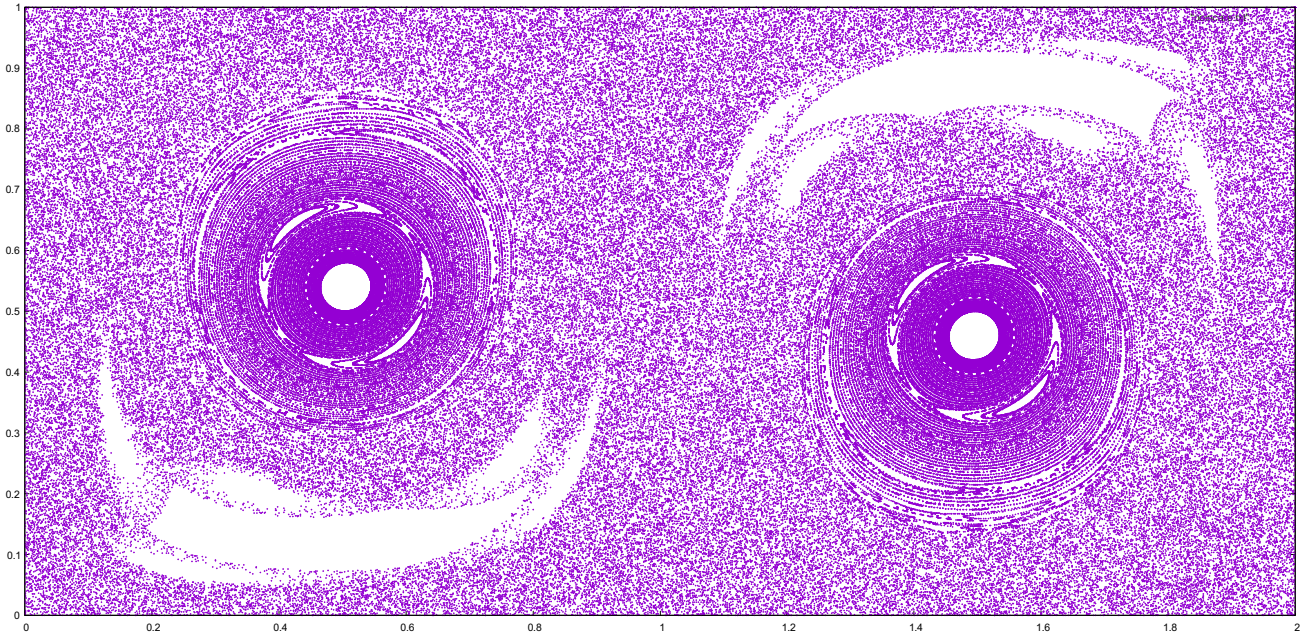
Figure: FLI Vs α at T=100 for the basis $\mathcal{B} = \{\underline{e}_1(0) = (\cos\alpha, \sin\alpha),\ \underline{e}_1(0) = (-\sin\alpha, \cos\alpha)\}$

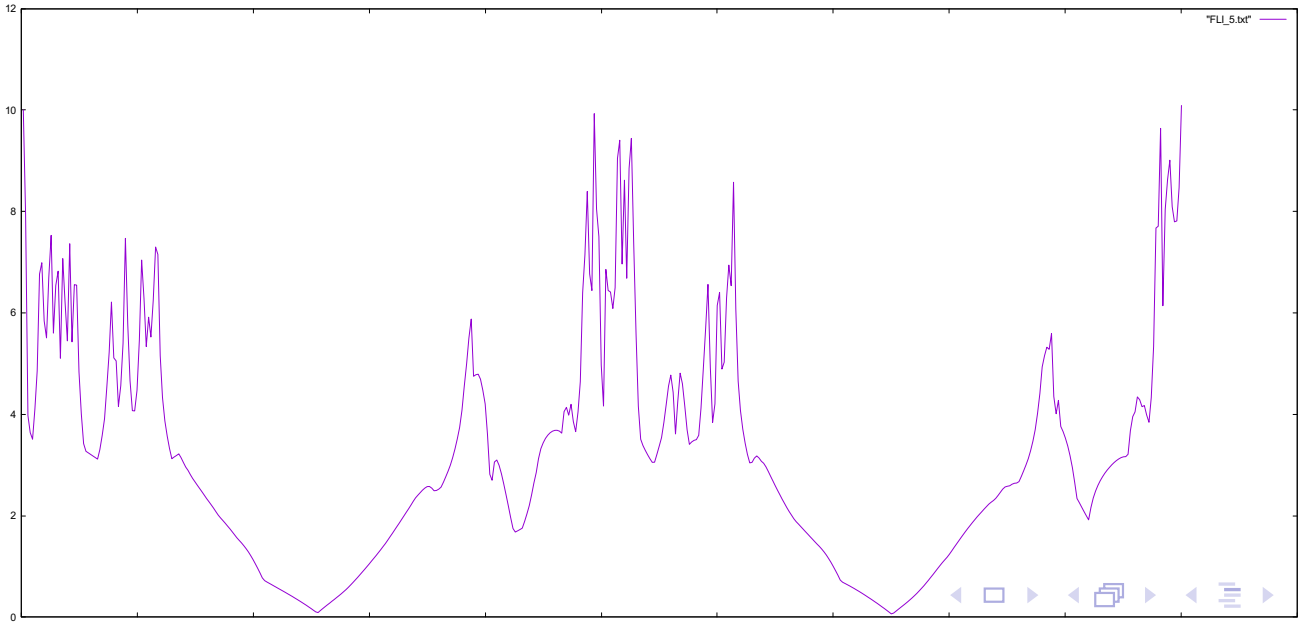
Problem

Exercise Compute the FLI for the time-dependent double gyre with $\epsilon = 0.05$, a one dimensional grid 500 of initial conditions with $y(0) = 1/2, x(0) \in [0, 2]$. Choose T suitably large ti clearly distinguish the regular orbits from the chaotic ones. Repeat the computation on a grid of 500×500 initial conditions on the phase-space, by using the integration time selected above. Repeat the computation again by zooming in some interesting part of the phase space.

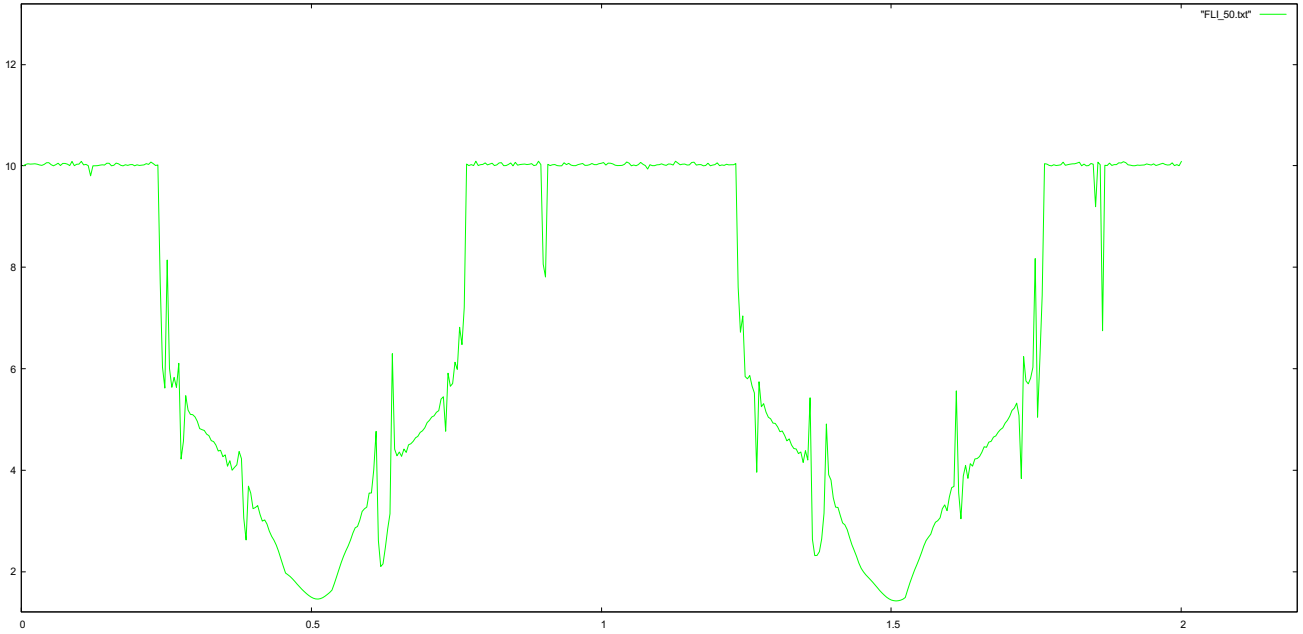
- We don't know a practical way to choose T.
- Repeat until the picture doesn't change for larger times.

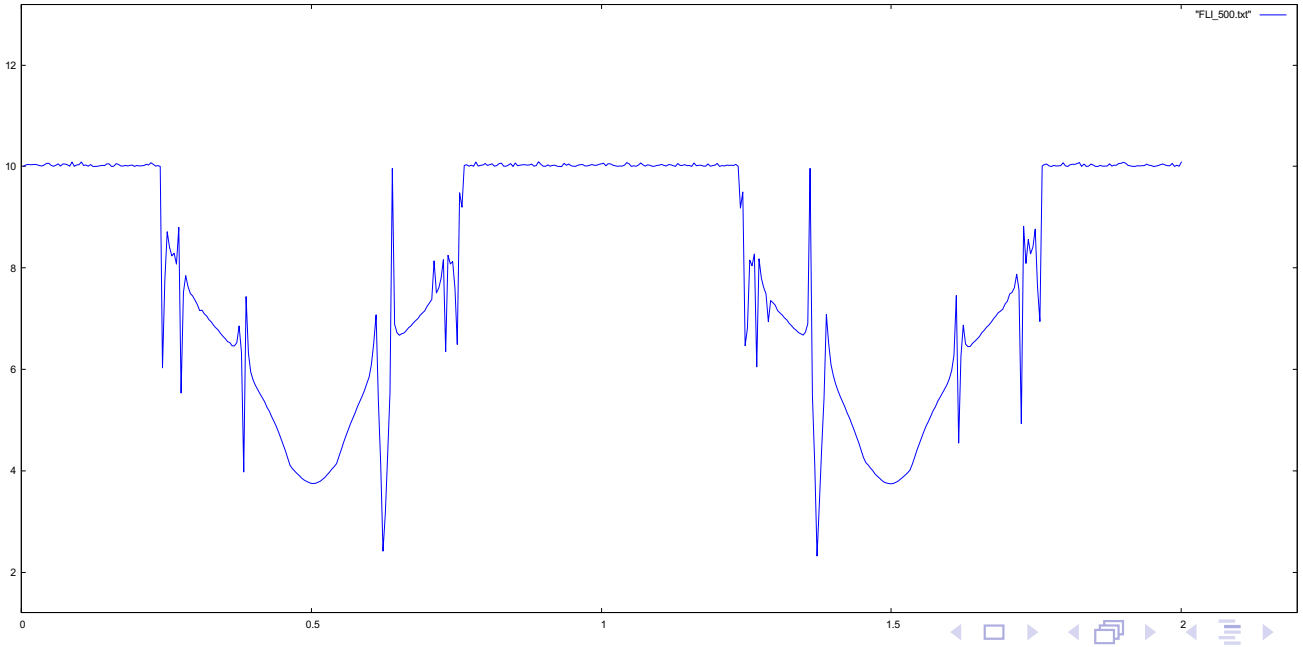
Solution:Poincarè, 500 orbits, T=500. FLI at T=5



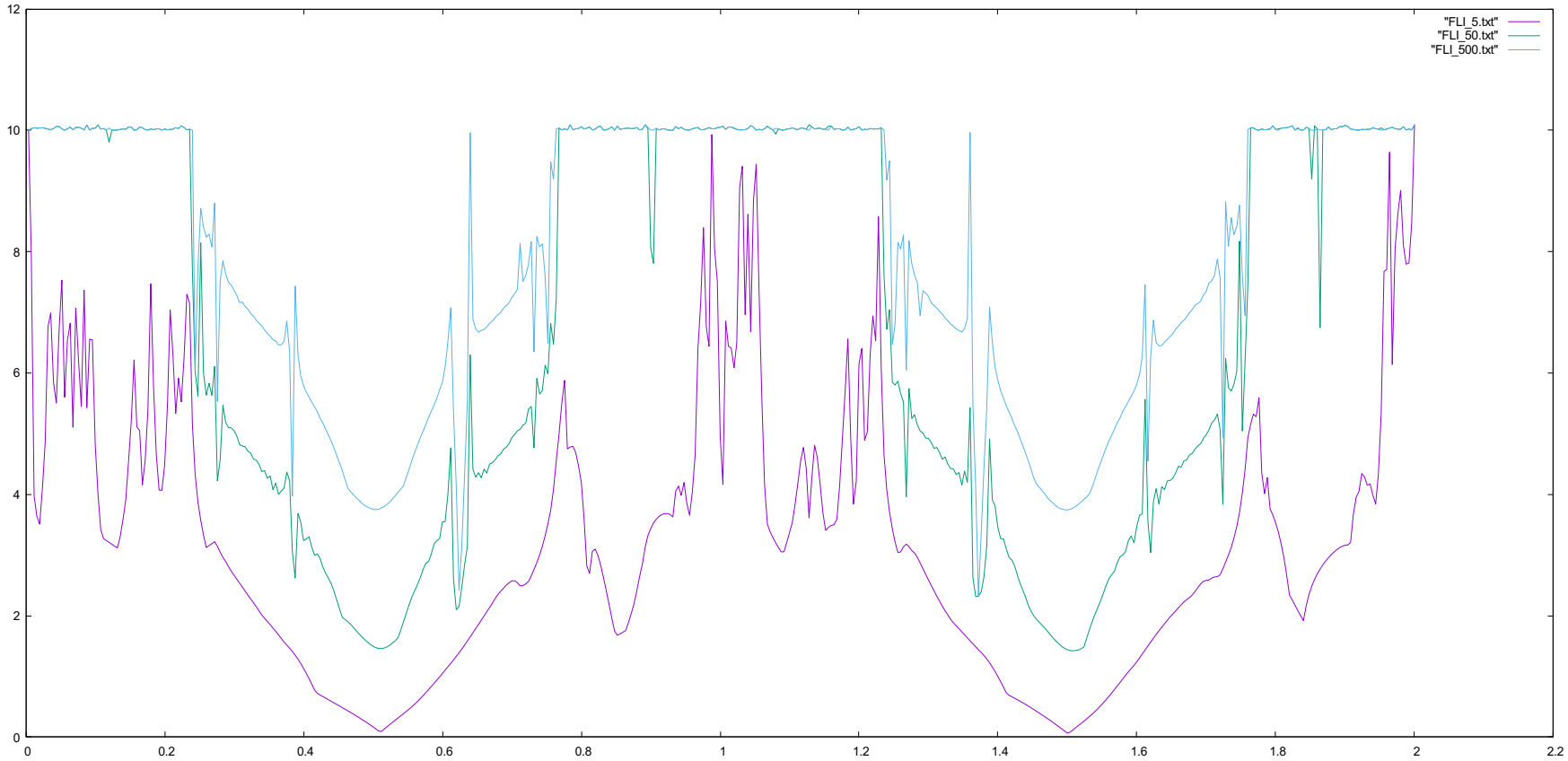


Solution: FLI for T=50, 500



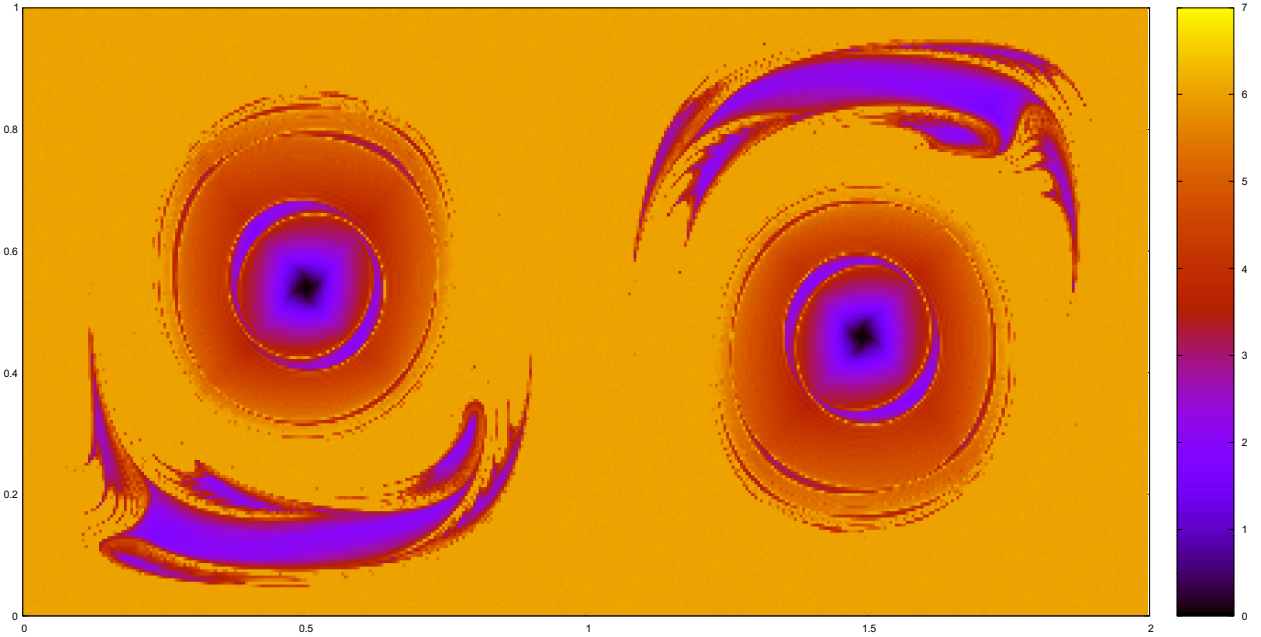


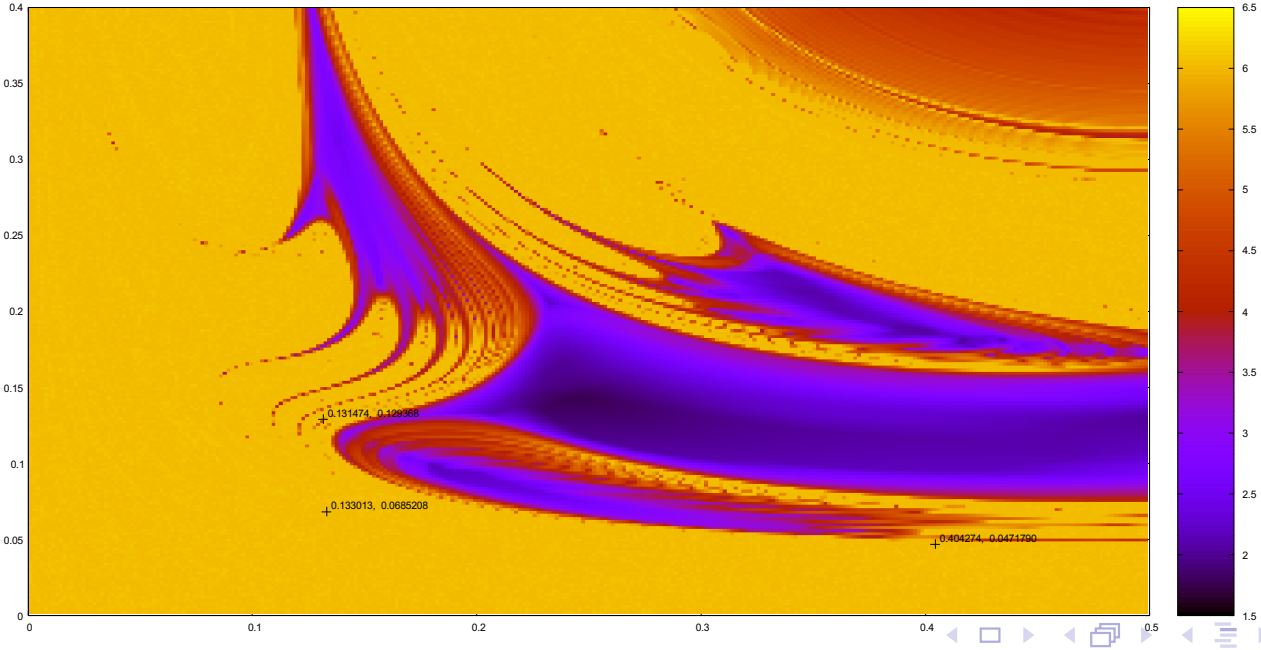
Solution: FLI for T=500, comparisons



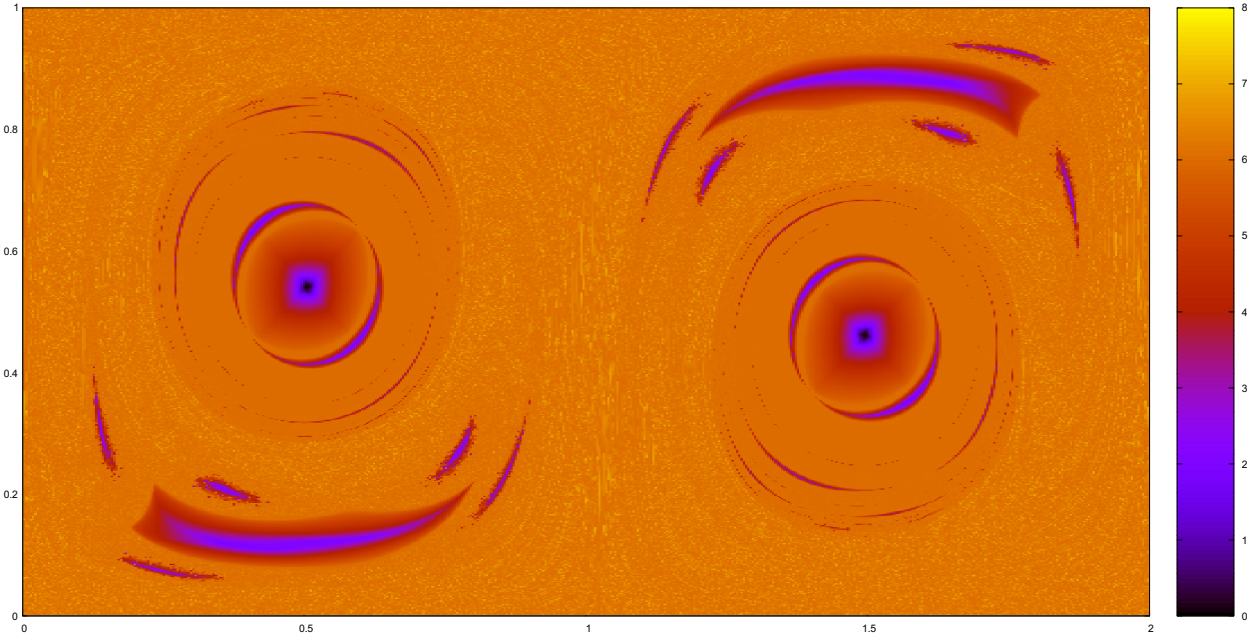
For regular orbits $FLI \sim \log const * t = \log const + \log t$

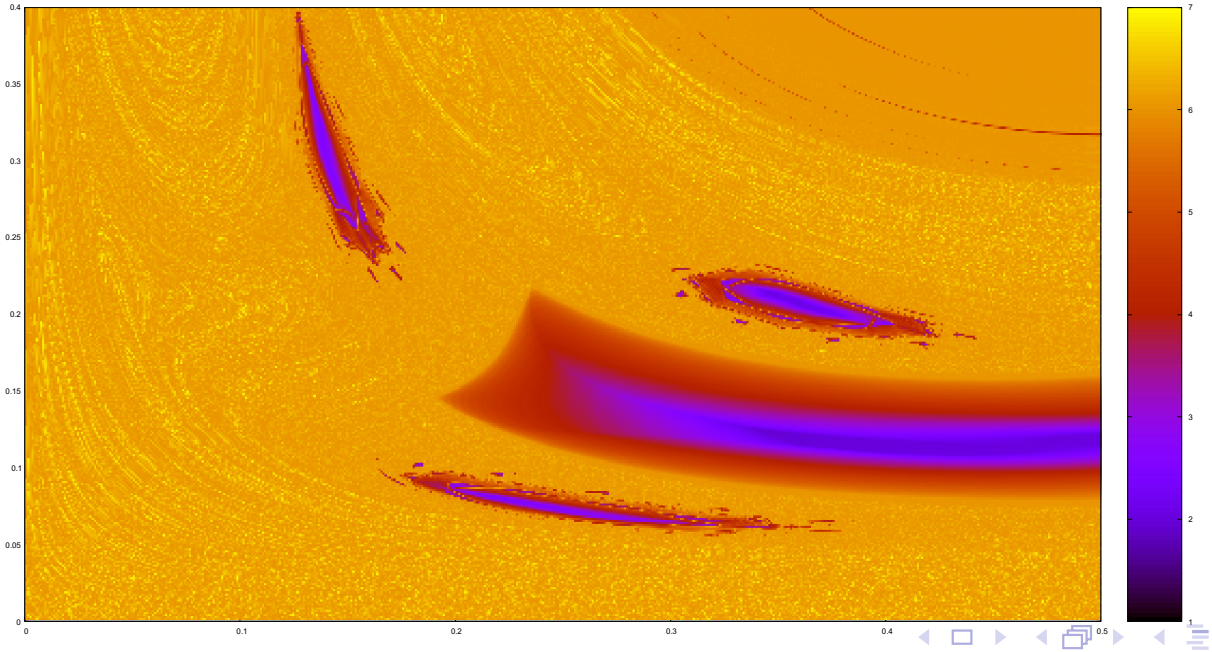
FLI in a 2D grid for T=500, RK4 with $\Delta t = 1e-4$





Solution: FLI in a 2D grid with RK6, $\Delta t = 1e-3$





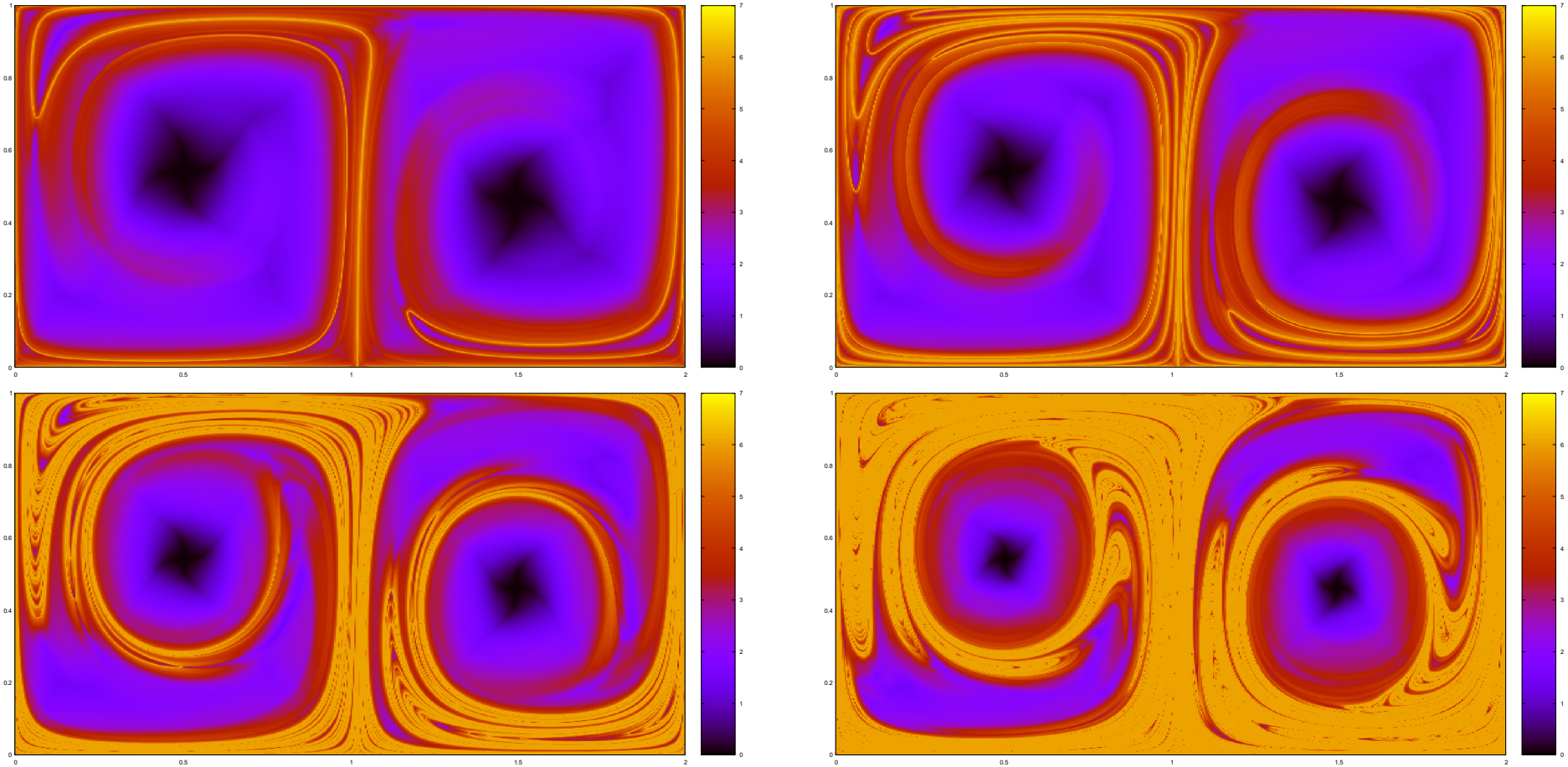
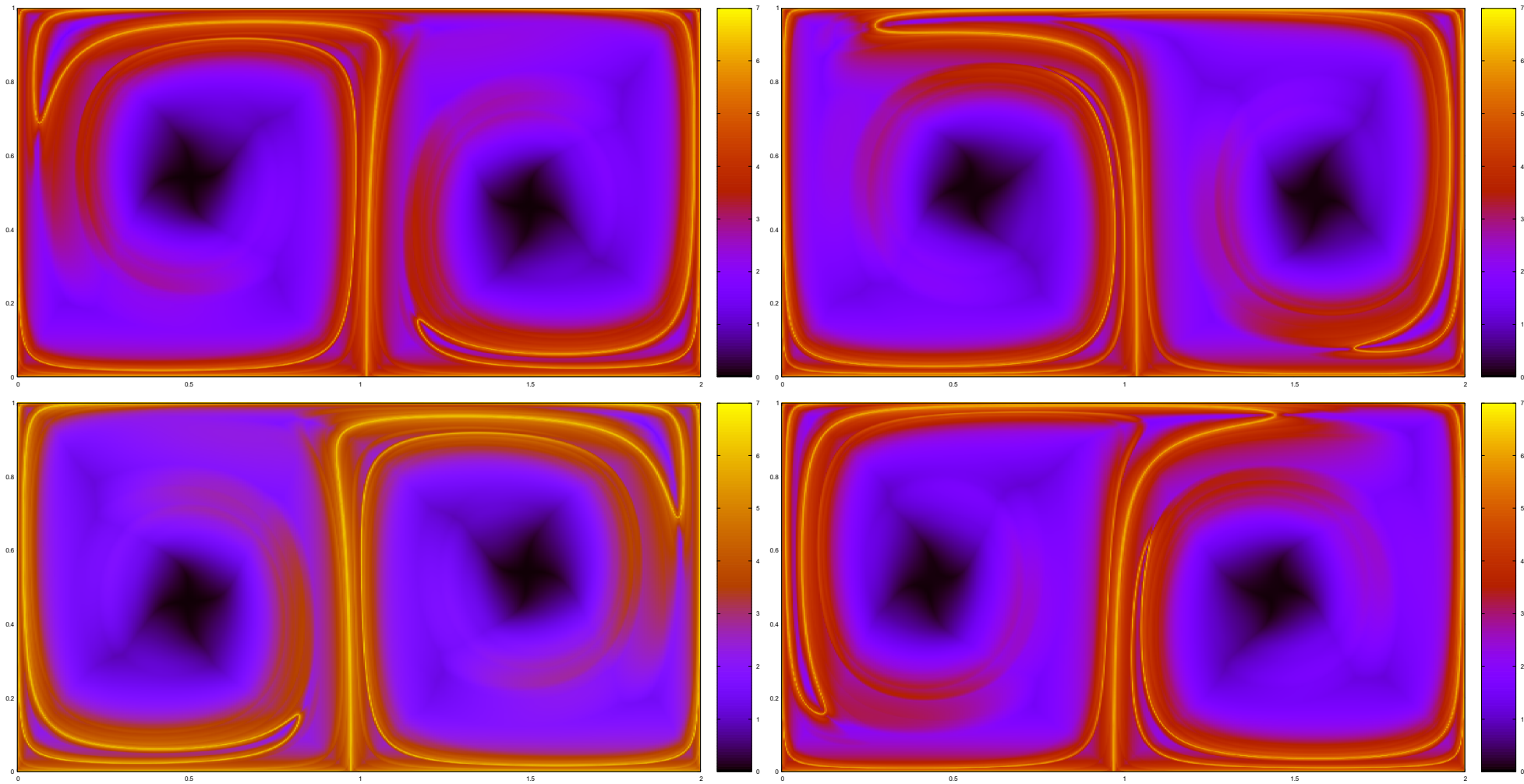
Comparison between numerical precision

In a chaotic system every small change in the initial data and in the algorithms used for the integration could develop in a completely different result for each specific orbit.

- The contour graphs obtained with RK4 and RK6 and different time steps have small differences.
- What we can really deduce from the graphs are the region of stability/ instability, but not the precise characterization for each initial condition.

Lagrangian Coherent Structures

The FLI is able to detect stable/unstable manifolds of an equilibrium/periodic orbit, if the growth of the tangent vectors is larger closer to it,



LCS: Stable manifold, T_0 =0, 0.2, 0.5, 0.8