We can write

$$\mathfrak{O}_{\mathfrak{I}_D}^{\mathfrak{I}_B,\left(1,0;(\widetilde{A}\widetilde{B}/\widetilde{D}\widetilde{C})^{\frac{1}{2}}\right)}\mathfrak{F}(u)=\frac{\prod_{l=1}^{4}(1-a_l u)}{(1-u^2)\left(1-qu^2\right)}(\mathfrak{F}(qu)-\mathfrak{F}(u))$$
$$+\frac{\prod_{j=1}^{4}\left(1-a_j u^{-1}\right)}{(1-u^{-2})\left(1-qu^{-2}\right)}\left(\mathfrak{F}\left(qu^{-1}\right)-\mathfrak{F}(u)\right)+E_{\mathfrak{I}_D}^{\mathfrak{I}_B,\left(1,0;(\widetilde{A}\widetilde{B}/\widetilde{D}\widetilde{C})^{\frac{1}{2}}\right)}\mathfrak{F}(u),$$
$$a_1=q^{\frac{1}{2}}t\widetilde{A}^{-1},\qquad a_2=q^{\frac{1}{2}}t\widetilde{C}^{-1},\qquad a_3=q^{\frac{1}{2}}t^{-1}\widetilde{A}^{-1},\qquad a_4=q^{\frac{1}{2}}t^{-1}\widetilde{C}^{-1}.$$

The first two terms define the rank one Koornwinder operator and we have an additional constant term. The eigenfunctions are the Askey–Wilson polynomials. In general these polynomials have four independent parameters $a_l$ but in our case they are restricted to $a_1a_4=a_2a_3$. The constant term is

$$E_{\mathfrak{I}_D}^{\mathfrak{I}_B,(1,0;\left(\widetilde{A}\widetilde{B}/\widetilde{D}\widetilde{C})^{\frac{1}{2}}\right)}=\left[-\frac{q^3t^2}{(\widetilde{A}\widetilde{C})^2}-q^2t^2\left(1-\frac{\widetilde{B}}{\widetilde{C}\widetilde{D}\widetilde{A}}-\frac{1}{\widetilde{C}^2}-\frac{1+t^{-2}}{\widetilde{A}\widetilde{C}}\right)\right.$$
$$\left.-q\left(-\frac{\widetilde{B}}{\widetilde{C}^3\widetilde{A}\widetilde{D}}+\frac{\widetilde{B}}{\widetilde{C}^2\widetilde{D}}+\frac{\widetilde{B}}{\widetilde{A}\widetilde{D}\widetilde{C}}+\frac{1}{\widetilde{C}^2}+\frac{\widetilde{B}t^2}{\widetilde{C}^2\widetilde{D}}\right)+\frac{\widetilde{B}\widetilde{A}}{\widetilde{D}\widetilde{C}}\right].$$

Let us take the limit of the three punctured sphere. We give details of the computation in Appendix with the final result given here (taking $\lim_{p\to 0}$ of the right-hand side as in (??))

$$T_{\mathfrak{I}_B,\mathfrak{I}_C,\mathfrak{I}_D}(w,u,v)=\frac{1}{(qp\frac{1}{ABCD}t^{-2};q)}\left(qpt^2\frac{1}{ABCD};q\right)^2\left(qp\frac{1}{BACD};q\right)\quad(1)$$
$$\times\frac{1}{\left(\sqrt{qp}\frac{t}{AC}v^{\pm1},\sqrt{qp}\frac{t}{DB}v^{\pm1};q\right)}\frac{1}{\left(\sqrt{qp}\frac{t}{AB}w^{\pm1},\sqrt{qp}\frac{t}{DC}w^{\pm1};q\right)}\frac{1}{\left(\sqrt{qp}\frac{t}{BC}u^{\pm1},\sqrt{qp}\frac{t}{AD}u^{\pm1};q\right)}.$$

The index factorizes. In particular on general grounds we expect the index of the three punctured sphere to be

$$\frac{1}{\left(\sqrt{qp}\frac{t}{AC}v^{\pm1},\sqrt{qp}\frac{t}{DB}v^{\pm1};q\right)}\frac{1}{\left(\sqrt{qp}\frac{t}{AB}w^{\pm1},\sqrt{qp}\frac{t}{DC}w^{\pm1};q\right)}$$
$$\times\frac{1}{\left(\sqrt{qp}\frac{t}{BC}u^{\pm1},\sqrt{qp}\frac{t}{AD}u^{\pm1};q\right)}\sum_\lambda C_\lambda\psi_\lambda(u)\psi_\lambda(w)\psi_\lambda(v),$$

where the sum is over all the eigenvalues of the Koornwinder operator and $\psi_\lambda(z)$ are Askey–Wilson polynomials. What we have shown above is that