0.0.1 Summary of first-order equations

To summarize, the first-order equations for the warp factor f and the scalars σ, ϕ^0, ϕ^3 are found to be

$$\begin{aligned} f' &= 2\left(G_0 S_0 + G_3 S_3\right) \\ \sigma' &= 2\eta\sqrt{N_0^2 + N_3^2} \\ \cos\phi^3 \left(\phi^0\right)' &= -\left(G_0 M_0 + G_3 M_3\right) \\ \left(\phi^3\right)' &= i\left(G_3 M_0 - G_0 M_3\right) \end{aligned} \tag{1}$$

Furthermore, for consistency these were required to satisfy the algebraic constraint

$$e^{-2f} = 4\left(G_0 S_0 + G_3 S_3\right)^2 - 4\left(S_0^2 + S_3^2\right) \tag{2}$$

The various functions featured in these equations were defined in and .

0.1 Numeric solutions

In order to get acceptable numerical solutions from these equations, we must choose appropriate initial conditions. It is easy to check that the following initial conditions ensure smoothness of all three scalars, as well as the vanishing of e^{2f} at the origin,

$$\begin{aligned} \phi_0^3 &= \sin^{-1}\left[\frac{1}{8\tanh\phi_0^0}\left(-3 + \sqrt{9 + 16\tanh^2\phi_0^0}\right)\right] \\ \sigma_0 &= \frac{1}{4}\log\left[\frac{\cosh\phi_0^0\left(5 + \sqrt{9 + 16\tanh^2\phi_0^0}\right)}{\sqrt{6}\sqrt{8 + \coth^2\phi_0^0\left(-3 + \sqrt{9 + 16\tanh^2\phi_0^0}\right)}}\right] \end{aligned} \tag{3}$$

We have defined for notational convenience $\phi_0^\alpha \equiv \phi^\alpha(0)$ and $\sigma_0 \equiv \sigma(0)$. For these initial conditions to be real, we must ensure that

$$|f(\phi_0^0)| \leq 1 \qquad\qquad f(\phi_0^0) \equiv \frac{1}{8\tanh\phi_0^0}\left(-3 + \sqrt{9 + 16\tanh^2\phi_0^0}\right) \tag{4}$$

Noting that

$$\lim_{\phi_0^0 \to -\infty} f(\phi_0^0) = -\frac{1}{4} \qquad\qquad \lim_{\phi_0^0 \to +\infty} f(\phi_0^0) = \frac{1}{4} \tag{5}$$

and also that $f(\phi_0^0)$ is monotonically increasing, i.e.

$$\frac{df}{d\phi_0^0} > 0 \qquad \forall \phi_0^0 \in \mathbb{R} \tag{6}$$