$\Delta_1(q) = F^{-1}_{Y_1(1,M(1))}(q) - F^{-1}_{Y_1(0,M(0))}(q)$. The QTE can be disentangled into the direct quantile effects, denoted by $\theta_1(q,d) = F^{-1}_{Y_1(1,M(d))}(q) - F^{-1}_{Y_1(0,M(d))}(q)$, and the indirect quantile effects, denoted by $\delta_1(q,d) = F^{-1}_{Y_1(d,M(1))}(q) - F^{-1}_{Y_1(d,M(0))}(q)$. The conditional distribution function in stratum $\tau$ is $F_{Y_t(d,m)|\tau}(y) = \Pr(Y_t(d,m) \leq y|\tau)$ and the corresponding conditional quantile function is $F^{-1}_{Y_t(d,m)|\tau}(q) = \inf\{y : F_{Y_t(d,m)|\tau}(y) \geq q\}$ for $\tau \in \{a,c,d,n\}$. Using the previously described stratification framework, we define the QTE conditional on $\tau \in \{a,c,de,n\}$: $\Delta_1^\tau(q) = F^{-1}_{Y_1(1,M(1))|\tau}(q) - F^{-1}_{Y_1(0,M(0))|\tau}(q)$. The direct quantile treatment effect among never-takers equals $\Delta_1^n(q) = F^{-1}_{Y_1(1,0)|n}(q) - F^{-1}_{Y_1(0,0)|n}(q) = \theta_1^n(q)$. The direct quantile effect among always-takers equals $\Delta_1^a(q) = F^{-1}_{Y_1(1,1)|a}(q) - F^{-1}_{Y_1(0,1)|a}(q) = \theta_1^a(q)$. The total QTE among compliers equals $\Delta_1^c(q) = F^{-1}_{Y_1(1,1)|c}(q) - F^{-1}_{Y_1(0,0)|c}(q)$, the direct quantile effect among compliers equals $\theta_1^c(q,d) = F^{-1}_{Y_1(1,d)|c}(q) - F^{-1}_{Y_1(0,d)|c}(q)$, and the indirect quantile effect among compliers equals $\delta_1^c(q,d) = F^{-1}_{Y_1(d,1)|c}(q) - F^{-1}_{Y_1(d,0)|c}(q)$. Finally, we define the direct quantile treatment effects conditional on specific values $D = d$ and mediator states $M = M(d) = m$,

$$\theta_1^{d,m}(q,1) = F^{-1}_{Y_1(1,m)|D=d,M(1)=m}(q) - F^{-1}_{Y_1(0,m)|D=d,M(1)=m}(q) \text{ and}$$
$$\theta_1^{d,m}(q,0) = F^{-1}_{Y_1(1,m)|D=d,M(0)=m}(q) - F^{-1}_{Y_1(0,m)|D=d,M(0)=m}(q),$$

with the quantile function $F^{-1}_{Y_t(d,m)|D=d,M(d)=m}(q) = \inf\{y : F_{Y_t(d,m)|D=d,M(d)=m}(y) \geq q\}$ and the distribution function $F_{Y_t(d,m)|D=d,M(d)=m}(y) = \Pr(Y_t(d,m) \leq y|D = d, M(d) = m)$.

## 0.1 Observed distribution and quantile transformations

We subsequently define various functions of the observed data required for the identification results. The conditional distribution function of the observed outcome $Y_t$ conditional on treatment value $d$ and mediator state $m$, is given by $F_{Y_t|D=d,M=m}(y) = \Pr(Y_t \leq y|D = d, M = m)$ for $d, m \in \{0,1\}$. The corresponding conditional quantile