1 Proof of Theorem 2

1.1 Average direct effect under $\mathbf{d = 0}$ conditional on $\mathbf{D = 0}$ and $\mathbf{M(0) = 1}$

In the following, we show that $\theta_1^{0,1}(0) = E[Y_1(1,1) - Y_1(0,1)|D = 0, M(0) = 1] = E[Q_{11}(Y_0) - Y_1|D = 0, M = 1]$. Using the observational rule, we obtain $E[Y_1(0,1)|D = 0, M(0) = 1] = E[Y_1|D = 0, M = 1]$. Accordingly, we have to show that $E[Y_1(1,1)|D = 0, M(0) = 1] = E[Q_{11}(Y_0)|D = 0, M = 1]$ to finish the proof. Under Assumptions 1 and 5a, the conditional potential outcome distribution function equals

$$\begin{aligned} F_{Y_t(d,0)|D=1,M=0}(y) &\overset{A1}{=} \Pr(h(d,m,t,U) \le y|D = 0, M = 1, T = t), \\ &= \Pr(U \le h^{-1}(d,m,t;y)|D = 0, M = 1, T = t), \\ &\overset{A5a}{=} \Pr(U \le h^{-1}(d,m,t;y)|D = 0, M = 1), \\ &= F_{U|01}(h^{-1}(d,m,t;y)), \end{aligned} \tag{1}$$

for $d, d' \in \{0,1\}$. We use these quantities in the following. First, evaluating $F_{Y_1(1,1)|D=0,M=1}(y)$ at $h(1,1,1,u)$ gives

$$F_{Y_1(1,1)|D=0,M=1}(h(1,1,1,u)) = F_{U|01}(h^{-1}(1,1,1;h(1,1,1,u))) = F_{U|01}(u).$$

Applying $F^{-1}_{Y_1(1,1)|D=0,M=1}(q)$ to both sides, we have

$$h(1,1,1,u) = F^{-1}_{Y_1(1,1)|D=0,M=1}(F_{U|01}(u)). \tag{2}$$

Second, for $F_{Y_0(1,1)|D=0,M=1}(y)$ we have

$$F^{-1}_{U|01}(F_{Y_0(1,1)|D=0,M=1}(y)) = h^{-1}(1,1,0;y). \tag{3}$$