P1 Affine invariance: $d(\mathbf{x};\mathbf{F}) = \mathbf{d}(\mathbf{Ax}+\mathbf{b};\mathbf{F}_{\mathbf{A},\mathbf{b}})$;

P2 Maximality at center: if F is "symmetric" around μ then $d(\mathbf{x};\mathbf{F}) \leq \mathbf{d}(\mu;\mathbf{F})$ for all $\mathbf{x}$; for a more detailed discussion on symmetry see **?**.

P3 Monotonicity: if () holds, then

$$d(\mathbf{x};\mathbf{F}) \leq \mathbf{d}(\mu+\alpha(\mathbf{x}-\mu);\mathbf{F}) \qquad \alpha \in [\mathbf{0},\mathbf{1}] \ ;$$

P4 Approaching zero: $\parallel \mathbf{x} \parallel \to \infty \Rightarrow \mathbf{d}(\mathbf{x};\mathbf{F}) \to \mathbf{0}$.

1 Gervini-Yohai depth

Here we want to show that the Gervini-Yohai depth, defined as $d_{GY}(\mathbf{t},\mathbf{F},\mathbf{G}) = \mathbf{1}-\mathbf{G}(\boldsymbol{\Delta}(\mathbf{t},\mu(\mathbf{F}),\boldsymbol{\Sigma}(\mathbf{F})))$, is a proper statistical depth function, i.e., it satisfies the four properties introduced above.

1. Affine invariance: it follows directly from the affine invariance property of the Mahalanobis distance;

2. Maximality at center: if F is elliptically symmetric around $\mu(\mathbf{F})$,

   $$d_{GY}(\mu(\mathbf{F}),\mathbf{F},\mathbf{G}) = \mathbf{1}-\mathbf{G}(\boldsymbol{\Delta}(\mu(\mathbf{F}),\mu(\mathbf{F}),\boldsymbol{\Sigma}(\mathbf{F}))) = \mathbf{1}-\mathbf{G}(\mathbf{0}).$$

   For any $\mathbf{t} \neq \mu(\mathbf{F})$ we have

   $$\begin{aligned} \Delta(\mathbf{t},\mu(\mathbf{F}),\boldsymbol{\Sigma}(\mathbf{F})) &> \mathbf{0} \\ G(\Delta(\mathbf{t},\mu(\mathbf{F}),\boldsymbol{\Sigma}(\mathbf{F}))) &\geq \mathbf{G}(\mathbf{0}) \\ 1-G(\Delta(\mathbf{t},\mu(\mathbf{F}),\boldsymbol{\Sigma}(\mathbf{F}))) &\leq \mathbf{1}-\mathbf{G}(\mathbf{0}) \\ d_{GY}(\mathbf{t},\mathbf{F},\mathbf{G}) &\leq \mathbf{d}_{\mathbf{GY}}(\mu(\mathbf{F}),\mathbf{F},\mathbf{G}), \end{aligned}$$

   when G is strictly monotone then strict inequality holds, and $\mu(\mathbf{F})$ is the unique maximizer of the Gervini-Yohai depth.

3. Monotonicity:

   $$\begin{aligned} \Delta(\mu(\mathbf{F})+\alpha(\mathbf{t}-\mu(\mathbf{F})),\mu(\mathbf{F}),\boldsymbol{\Sigma}(\mathbf{F})) &= (\alpha(\mathbf{t}-\mu(\mathbf{F})))^\top\boldsymbol{\Sigma}(\mathbf{F})^{-\mathbf{1}}(\alpha(\mathbf{t}-\mu(\mathbf{F}))) \\ &= \alpha^2(\mathbf{t}-\mu(\mathbf{F}))^\top\boldsymbol{\Sigma}(\mathbf{F})^{-\mathbf{1}}(\mathbf{t}-\mu(\mathbf{F})) \\ &= \alpha^2\Delta(\mathbf{t},\mu(\mathbf{F}),\boldsymbol{\Sigma}(\mathbf{F})) \\ &\leq \Delta(\mathbf{t},\mu(\mathbf{F}),\boldsymbol{\Sigma}(\mathbf{F})) \end{aligned}$$

   Then $d_{GY}(\mu(\mathbf{F})+\alpha(\mathbf{t}-\mu(\mathbf{F})),\mathbf{F},\mathbf{G}) \geq \mathbf{d}_{\mathbf{GY}}(\mathbf{t},\mathbf{F},\mathbf{G})$.

4. Approaching zero: if $\parallel \mathbf{t} \parallel \to \infty$ we have that $\Delta(\mathbf{t},\mu(\mathbf{F}),\boldsymbol{\Sigma}(\mathbf{F})) \to \infty$ and consequently $G(\Delta(\mathbf{t},\mu(\mathbf{F}),\boldsymbol{\Sigma}(\mathbf{F}))) \to \mathbf{1}$. Then

   $$d_{GY}(\mathbf{t},\mathbf{F},\mathbf{G}) = \mathbf{1}-\mathbf{G}(\boldsymbol{\Delta}(\mathbf{t},\mu(\mathbf{F}),\boldsymbol{\Sigma}(\mathbf{F}))) \to \mathbf{0}$$