0.1 Quantile direct effect under $\mathbf{d = 1}$ conditional on $\mathbf{D = 1}$ and $\mathbf{M(1) = 0}$

In the following, we prove that

$$\begin{aligned}\theta_1^{1,0}(q,1) &= F^{-1}_{Y_1(1,0)|D=1,M(1)=0}(q) - F^{-1}_{Y_1(0,0)|D=1,M(1)=0}(q), \\ &= F^{-1}_{Y_1|D=1,M=0}(q) - F^{-1}_{Q_{00}(Y_0)|D=1,M=0}(q).\end{aligned}$$

For this purpose, we have to show that

$$F_{Y_1(1,0)|D=1,M(1)=0}(y) = F_{Y_1|D=1,M=0}(y) \text{ and} \tag{1}$$

$$F_{Y_1(0,0)|D=1,M(1)=0}(y) = F_{Q_{00}(Y_0)|D=1,M=0}(y), \tag{2}$$

which is sufficient to show that the quantiles are also identified. We can show () using the observational rule $F_{Y_1(1,0)|D=1,M(1)=0}(y) = F_{Y_1|D=1,M=0}(y) = E[1\{Y_1 \leq y\}|D = 1, M = 0]$, with $1\{\cdot\}$ being the indicator function. Using (), we obtain

$$\begin{aligned}&F_{Q_{00}(Y_0)|D=1,M=0}(y) \\ &= E[1\{Q_{00}(Y_0) \leq y\}|D = 1, M = 0], \\ = E[1\{F^{-1}_{Y_1|D=0,M=0} \circ &F_{Y_0|D=0,M=0}(Y_0) \leq y\}|D = 1, M = 0], \\ &= E[1\{Y_1(0,0) \leq y\}|D = 1, M = 0], \\ &= F_{Y_1(0,0)|D=1,M(1)=0}(y),\end{aligned} \tag{3}$$

which proves ().

0.2 Average direct effect under $\mathbf{d = 0}$ conditional on $\mathbf{D = 0}$ and $\mathbf{M(0) = 0}$

In the following, we show that $\theta_1^{0,0}(0) = E[Y_1(1,0) - Y_1(0,0)|D = 0, M(0) = 0] = E[Q_{10}(Y_0) - Y_1|D = 0, M = 0]$. Using the observational rule, we obtain $E[Y_1(0,0)|D = 0, M(0) = 0] = E[Y_1|D = 0, M = 0]$. Accordingly, we have to show