$\Delta_x = (\mathbf{x} - \mathbf{T_{0n}})^\top \mathbf{C_{0n}^{-1}} (\mathbf{x} - \mathbf{T_{0n}})$ the squared Mahalanobis distance of $\mathbf{x}$ using the initial location and dispersion estimates, the set can be rewritten as $C^\beta(F) = \{\mathbf{x} \in \mathbb{R}^\mathbf{p} | \mathbf{\Delta_x} > (\chi^2_\mathbf{p})^{-1}(\beta)\}$, where $(\chi^2_p)^{-1}(\beta)$ is a large quantile of a chi-squared distribution with p degrees of freedom. Now we want to show that the result given by Proposition holds for this particular case. Consider a random vector $(\mathbf{X_1}, \ldots, \mathbf{X_n}) \sim \mathbf{F_0}(\mu_\mathbf{0}, \mathbf{\Sigma_0})$ and suppose that F_0 is an elliptically symmetric distribution. Also consider a pair of location and dispersion estimators $\mathbf{T_{0n}}$ and $\mathbf{C_{0n}}$ such that $\mathbf{T_{0n}} \to \mu_\mathbf{0}$ and $\mathbf{C_{0n}} \to \mathbf{\Sigma_0}$ a.s.. Let F be a chosen reference distribution and $\hat{F}_n$ the empirical distribution function. If the reference distribution satisfies

$$\sup_{\mathbf{x} \in \mathbf{C}^\beta(\mathbf{F_0})} [d_{HS}(\mathbf{x}; \mathbf{F}) - \mathbf{d_{HS}}(\mathbf{x}; \mathbf{F_0})] < \mathbf{0}$$

where β is some large quantile of F_0, then

$$nd_n \to 0 \text{ as } n \to \infty$$

Proof. In **?**, it is proved that for i.i.d. $\mathbf{X_1}, \mathbf{X_2}, ..., \mathbf{X_n}$ with distribution F_0, as $n \to \infty$

$$\sup_{\mathbf{t} \in \mathbb{R}^\mathbf{d}} |d_{HS}(\mathbf{t}, \mathbf{F_0}) - \mathbf{d_{HS}}(\mathbf{t}, \mathbf{\hat{F}_n})| \to \mathbf{0} \text{ a.s.}$$

Note that, by the continuity of F, $F(\mathbf{T_{0n}}, \mathbf{C_{0n}}) \to \mathbf{F}(\mu_\mathbf{0}, \mathbf{\Sigma_0})$ a.s.. Hence, for each $\varepsilon > 0$ there exists n_0 such that for all $n > n_0$ we have

$$\begin{aligned}
\sup_{\mathbf{x} \in \mathbf{C}^\beta(\mathbf{F_0})} \{d_{HS}(\mathbf{x}; \mathbf{\hat{F}_n}) - \mathbf{d_{HS}}(\mathbf{x}; \mathbf{F}(\mathbf{T_{0n}}, \mathbf{C_{0n}}))\} \leq \\
\sup_{\mathbf{x} \in \mathbf{C}^\beta(\mathbf{F_0})} \{d_{HS}(\mathbf{x}; \mathbf{\hat{F}_n}) - \mathbf{d_{HS}}(\mathbf{x}; \mathbf{F_0}(\mu_\mathbf{0}, \mathbf{\Sigma_0}))\} + \\
\sup_{\mathbf{x} \in \mathbf{C}^\beta(\mathbf{F_0})} \{d_{HS}(\mathbf{x}; \mathbf{F_0}(\mu_\mathbf{0}, \mathbf{\Sigma_0})) - \mathbf{d_{HS}}(\mathbf{x}; \mathbf{F}(\mu_\mathbf{0}, \mathbf{\Sigma_0}))\} + \\
\sup_{\mathbf{x} \in \mathbf{C}^\beta(\mathbf{F_0})} \{d_{HS}(\mathbf{x}; \mathbf{F}(\mu_\mathbf{0}, \mathbf{\Sigma_0})) - \mathbf{d_{HS}}(\mathbf{x}; \mathbf{F}(\mathbf{T_{0n}}, \mathbf{C_{0n}}))\} \\
\leq \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

□

In the next example, we illustrate a univariate filter based on half-space depth that controls independently the left and the right tail of the distribution. In the univariate case, given a point x there exist only two halfspaces including it, hence the half-space depth assumes the explicit form

$$\begin{aligned}
d_{HS}(x; F) &= \min(P_F((-\infty, x]), P_F([x, \infty))) \\
&= \min(F(x), 1 - F(x) + P_F(X = x)),
\end{aligned}$$