0.1 Quantile direct effect under d = 0 conditional on D = 0 and M(0) = 0

In the following, we prove that

$$\begin{aligned}\theta_1^{0,0}(q,0) &= F^{-1}_{Y_1(1,0)|D=0,M(0)=0}(q) - F^{-1}_{Y_1(0,0)|D=0,M(0)=0}(q),\\ &= F^{-1}_{Q_{10}(Y_0)|D=0,M=0}(q) - F^{-1}_{Y_1|D=0,M=0}(q).\end{aligned}$$

For this purpose, we have to show that

$$F_{Y_1(1,0)|D=0,M(0)=0}(y) = F_{Q_{10}(Y_0)|D=0,M=0}(y) \text{ and} \tag{1}$$

$$F_{Y_1(0,0)|D=0,M(0)=0}(y) = F_{Y_1|D=0,M=0}(y), \tag{2}$$

which is sufficient to show that the quantiles are also identified. We can show () using the observational rule $F_{Y_1(0,0)|D=0,M(0)=0}(y) = F_{Y_1|D=0,M=0}(y) = E[1\{Y_1 \le y\}|D = 0, M = 0]$. Using (), we obtain

$$\begin{aligned}&F_{Q_{10}(Y_0)|D=0,M=0}(y)\\ &= E[1\{Q_{10}(Y_0) \le y\}|D = 0, M = 0],\\ &= E[1\{F^{-1}_{Y_1|D=1,M=0} \circ F_{Y_0|D=1,M=0}(Y_0) \le y\}|D = 0, M = 0],\\ &= E[1\{Y_1(1,0) \le y\}|D = 0, M = 0],\\ &= F_{Y_1(1,0)|D=0,M(0)=0}(y),\end{aligned}$$

which proves ().