The expectation values of the operator $\mathcal{O}_{\phi^3}$ and the trace of the energy-momentum tensor are independent of Ω. As a check, we note that the four one-point functions satisfy the following operator relation, which is associated to the violation of conformal invariance by non-zero classical beta functions,

$$\langle T^i{}_i \rangle = -\sum_{\mathcal{O}} (d - \Delta_{\mathcal{O}})\, \phi_{\mathcal{O}}\, \langle \mathcal{O} \rangle \tag{1}$$

0.0.1 Derivative of the free energy

Following , we may now compute the derivative of F with respect to α as follows. First we note that

$$\frac{dF}{d\alpha} = \frac{dS_{\rm ren}}{d\alpha} = \lim_{\epsilon \to 0} \int d^5x \sum_{\text{fields } \Phi} \frac{\delta\left(\sqrt{\gamma}\mathcal{L}_{\rm ren}\right)}{\delta \Phi} \frac{d\Phi}{d\alpha} \bigg|_{z=\epsilon} \tag{2}$$

In our case, the terms appearing in the sum over fields are

$$\begin{aligned} \frac{\delta\left(\sqrt{\gamma}\mathcal{L}_{\rm ren}\right)}{\delta\sigma} &= \sqrt{\gamma}\,\langle O_\sigma \rangle \epsilon^3 + \dots & \qquad \frac{\delta\left(\sqrt{\gamma}\mathcal{L}_{\rm ren}\right)}{\delta\phi^0} &= \sqrt{\gamma}\,\langle O^0_\phi \rangle \epsilon^4 + \dots \\ \frac{\delta\left(\sqrt{\gamma}\mathcal{L}_{\rm ren}\right)}{\delta\phi^3} &= \sqrt{\gamma}\,\langle O^3_\phi \rangle \epsilon^3 + \dots & \qquad \frac{\delta\left(\sqrt{\gamma}\mathcal{L}_{\rm ren}\right)}{\delta\gamma^{ij}} &= \frac{1}{2}\sqrt{\gamma}\,\langle T_{ij} \rangle \epsilon^5 + \dots \end{aligned} \tag{3}$$

The dots represent terms of strictly lower order in ϵ. Furthermore, from the form of the UV asymptotic expansions , we have

$$\begin{aligned} \frac{d\sigma}{d\alpha} &= \frac{3}{4}\alpha\epsilon^2 + O(\epsilon^3) & \qquad \frac{d\phi^0}{d\alpha} &= \epsilon + O(\epsilon^3) \\ \frac{d\phi^3}{d\alpha} &= \left(1 - \alpha\frac{df_k}{d\alpha}\right) e^{-f_k}\epsilon^2 + O(\epsilon^3) & \qquad \frac{d\gamma^{ij}}{d\alpha} &= -2\frac{df_k}{d\alpha} e^{-2f_k}\epsilon^2 + O(\epsilon^2) \end{aligned} \tag{4}$$

Combining the pieces , with the results for the one-point functions in , we find that the contribution of the metric in is suppressed by ϵ^2 compared to other terms. The derivative of the free energy is then

$$\begin{aligned} \frac{dF}{d\alpha} &= \lim_{\epsilon\to 0} \int d^5x \sqrt{\gamma}\,\epsilon^5 \left[\frac{3}{2}\beta e^{-f_k} + \frac{1}{2}\beta e^{-f_k}\left(1 - \alpha\frac{df_k}{d\alpha}\right) + O(\epsilon)\right] \\ &= \mathrm{vol}_0\left(S^5\right)\, \frac{1}{2}\beta\, e^{4f_k}\left(4 - \alpha\frac{df_k}{d\alpha}\right) \end{aligned} \tag{5}$$

where $\mathrm{vol}_0(S^5) = \pi^3$ is the volume of a unit S^5. The Ω dependence in the one-point functions cancels out, consistent with the fact that F itself is independent of Ω. We thus obtain the final result

$$\frac{dF}{d\alpha} = \frac{\pi^2}{8\,G_6}\beta\, e^{4f_k}\left(4 - \alpha\frac{df_k}{d\alpha}\right) \tag{6}$$