0.1 Quantile direct effect on the always-takers

We prove that

$$\theta_1^a(q) = F^{-1}_{Y_1(1,1)|\tau=a}(q) - F^{-1}_{Y_1(0,1)|\tau=a}(q),$$
$$= F^{-1}_{Q_{11}(Y_0)|D=0,M=1}(q) - F^{-1}_{Y_1|D=0,M=1}(q).$$

This requires showing that

$$F_{Y_1(1,1)|\tau=a}(y) = F_{Q_{11}(Y_0)|D=0,M=1}(y) \text{ and} \tag{1}$$
$$F_{Y_1(0,1)|\tau=a}(y) = F_{Y_1|D=0,M=1}(y). \tag{2}$$

Under Assumptions 7 and 8,

$$\begin{aligned} F_{Y_t|D=0,M=1}(y) &= E[1\{Y_t \leq y\}|D=0,M=1] \\ &\overset{A7,A8}{=} E[1\{Y_t(0,1) \leq y\}|\tau=a] \\ &= F_{Y_t(0,1)|\tau=a,}(y). \end{aligned} \tag{3}$$

which proves (). From (), we have

$$F_{Q_{11}(Y_0)|D=0,M=1}(y) = F_{Y_1(1,1)|D=0,M(0)=1}(y) = E[1\{Y_1(1,1) \leq y\}|D=0,M(0)=1].$$

Under Assumption 7 and 8,

$$\begin{aligned} E[1\{Y_1(1,1) \leq y\}|D=0,M(0)=1] &\overset{A7,A8}{=} E[1\{Y_1(1,1) \leq y\}|\tau=a] \\ &= F_{Y_1(1,1)|\tau=a}(y), \end{aligned} \tag{4}$$

which proves.