This motivates us to define a superpotential $W$ as

$$W = \sqrt{S_0^2 + S_3^2} \tag{1}$$

Unfortunately, this superpotential does *not* allow one to write the BPS equations for both $\phi^0$ and $\phi^3$ as gradient flow equations. The reason for this failure is that the integrability condition required to convert the BPS equation into a gradient flow form is not satisfied; see e.g. Appendix C.2.1 of . We thus follow the strategy of to construct an approximate superpotential. Our model consists of two consistent truncations that admit flat domain walls and an exact superpotential. These are the $\phi^3 = 0, \phi^0 \neq 0$ truncation and the $\phi^0 = 0, \phi^3 \neq 0$ truncation. The corresponding flow equations are (we set $\eta = -1$ henceforth)

$$\phi^{0\prime} = -8\, \partial_{\phi^0} W|_{\phi^3=0} \qquad\qquad \phi^{3\prime} = 8\, \partial_{\phi^3} W|_{\phi^0=0} \tag{2}$$

respectively. In either truncation, the BPS equations for the warp factor and dilaton $\sigma$ can be put in the following form,

$$f' = 2\, W \qquad\qquad \sigma' = 2\, \partial_\sigma W \tag{3}$$

An important fact is that, though the gradient flow equations of do not hold exactly in the full model with $\phi^0 \neq 0, \phi^3 \neq 0$, they *do* hold up to and including $O(z^5)$. Looking at the form of the UV asymptotics of the scalar fields, one may expand the superpotential of keeping only terms contributing up to this order. This gives

$$W = \frac{1}{2} + \frac{3}{4}\sigma^2 + \frac{1}{16}(\phi^0)^2 - \frac{3}{16}(\phi^3)^2 + \frac{1}{192}(\phi^0)^4 - \frac{3}{16}(\phi^0)^2\sigma + \dots \tag{4}$$

where the dots represent terms of order $O(z^6)$. This is the approximate superpotential we will use in what follows.

### 0.0.1 Bogomolnyi trick

We now use the Bogomolnyi trick to get the finite counterterms needed to preserve supersymmetry in the case of a flat domain wall. The central idea of the Bogomolnyi trick is that for a BPS solution, the renormalized on-shell action must vanish. In order to make use of this fact, we will first want to recast the on-shell action in a simpler form. To do so, we begin by inserting. We find that

$$\mathcal{L} = -\frac{1}{4}R - 20W^2 + 2\mathcal{L}_{\text{kin}} \tag{5}$$

where we've defined

$$\mathcal{L}_{\text{kin}} = (\sigma')^2 + \frac{1}{4}\left[-(\phi^{3\prime})^2 + \cos^2\phi^3(\phi^{0\prime})^2\right] \tag{6}$$