Lecture 8 ARIMA Models

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Introduction of correlation

- Recall that we write time series x_t in the simple additive format

$$x_t = s_t + v_t \tag{1}$$

  where s_t denotes some unknown signal and v_t denotes a time series that may be white or correlated over time.

- In the *trend stationary* model, the process has stationary behavior around a trend:

$$x_t = \mu_t + y_t \tag{2}$$

  where x_t are the observations, μ_t denotes the trend, and y_t is a stationary process.

- We could model trend μ_t using a linear model $\mu_t = \beta_0 + \beta_1 t$.

Introduction of correlation

- Classical regression models, developed for the static case, only allow the dependent variable to be influenced by current values of the independent variables, which is insufficient.
- In the time series case, it is desirable to allow the dependent variable to be influenced by the past values of the independent variables and possibly by its own past values.
- The introduction of correlation may be generated through lagged linear relations.
- This leads to proposing the *autoregressive (AR)* and *autoregressive moving average (ARMA)* models (Whittle 1951).
- Adding nonstationary models to the mix leads to the *autoregressive integrated moving average (ARIMA)* model (Box and Jenkins 1970).

Table of Contents

Autoregressive Models

- Autoregressive models are based on the idea that the current value of the series, x_t, can be explained as a function of p past values, $x_{t-1}, x_{t-2}, \cdots, x_{t-p}$.
- An **autoregressive model** of order p, denoted as **AR**(p), is of the form

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t \tag{3}$$

where x_t is stationary, $w_t \sim wn(0, \sigma_w^2)$, and $\phi_1, \cdots, \phi_p$ are constants ($\phi_p \neq 0$).
- If the mean, μ, of x_t is not zero, we replace x_t by $x_t - \mu$ and write

$$x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t \tag{4}$$

where $\alpha = \mu(1 - \phi_1 - \cdots - \phi_p)$.
- Introducing the **autoregressive operator**, we write

$$\phi(B)x_t = w_t, \quad \phi(B) = 1 - \sum_{i=1}^{p} \phi_i B^i \tag{5}$$

Autoregressive Models

- Consider the AR(1) model

$$x_t = \phi x_{t-1} + w_t \tag{6}$$

- we could use backward substitution to get

$$x_t = \phi x_{t-1} + w_t = \phi(\phi x_{t-2} + w_{t-1}) + w_t = \cdots = \phi^k x_{t-k} + \sum_{j=0}^{k-1} \phi^j w_{t-j} \tag{7}$$

- Assuming $|\phi| < 1$ and $\sup_t \mathrm{Var}(x_t) < \infty$, we get the following linear process

$$x_t = \sum_{j=0}^{\infty} \phi^j w_{t-j} \tag{8}$$

- What is the autocovariance? Autocorrelation function (ACF)?

Autoregressive Models

- First $\mathrm{E}(x_t) = \sum_{j=0}^{\infty} \phi^j \mathrm{E}(w_{t-j}) = 0$. Second, the autocovariance

$$\gamma(h) = \mathrm{Cov}(x_{t+h}, x_t) = \sigma_w^2 \sum_{j=0}^{\infty} \phi^{h+j}\phi^j = \frac{\sigma_w^2 \phi^h}{1-\phi^2}, \quad h \geq 0 \tag{9}$$

- Then the ACF of an AR(1) is

$$\rho(h) = \frac{\gamma(h)}{\gamma(0)} = \phi^h, \quad h \geq 0 \tag{10}$$

- Note that $\rho(h)$ satisfies the recursion

$$\rho(h) = \phi\rho(h-1), \quad h = 1, 2, \cdots \tag{11}$$

Autoregressive Models

- Now if $\phi = 1$, Is $x_t = x_{t-1} + w_t$ stationary?
- What if $|\phi| > 1$? Such processes are called explosive because the values of the time series quickly become large in magnitude.
- However, using the forward substitution we get

$$\begin{aligned} x_t &= \phi^{-1}x_{t+1} - \phi^{-1}w_{t+1} = \phi^{-1}(\phi^{-1}x_{t+2} - \phi^{-1}w_{t+2}) - \phi^{-1}w_{t+1} \\ &= \cdots = \phi^{-k}x_{t+k} + \sum_{j=1}^{k-1}\phi^{-j}w_{t+j} \end{aligned}$$

- Under the same assumption, we have the process in terms of its future

$$x_t = -\sum_{j=1}^{\infty}\phi^{-j}w_{t+j} \tag{12}$$

- When a process does not depend on the future, we say it is *causal*.

Autoregressive Models

Example

For the non-causal stationary process

$$x_t = \phi x_{t-1} + w_t, \quad |\phi| > 1 \tag{13}$$

and $w_t \overset{iid}{\sim} N(0, \sigma_w^2)$. *What is the autocovariance? ACF?*

Autoregressive Models

- To express AR(1) in linear process, we could also consider matching coefficients

$$\phi(B)x_t = w_t, \quad \phi(B) = 1 - \phi B \tag{14}$$

- We could write

$$x_t = \psi(B)w_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} \tag{15}$$

- Then we have $\phi(B)\psi(B) = 1$, which implies

$$\psi_1 = \phi, \quad \psi_j = \psi_{j-1}\phi \tag{16}$$

  and it yields $\psi_j = \phi^j$.

- Another way to obtain this result is by the following series expansion

$$\phi^{-1}(z) = \frac{1}{1 - \phi z} = \sum_{j=0}^{\infty} \phi^j z^j, \quad |z| \leq 1 \tag{17}$$

Moving Average Models

- Alternative to the autoregressive representation, x_t can be a linear combination of *white noise* $\{w_t\}$.
- The **moving average model** of order q, or **MA**(q), is defined

$$x_t = w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q} \tag{18}$$

where $w_t \sim wn(0, \sigma_w^2)$ and $\theta_1, \cdots, \theta_q(\theta_q \neq 0)$ are paremeters

- Introducing the **moving average operator**, we can write

$$x_t = \theta(B)w_t, \quad \theta(B) = \sum_{i=0}^{q} \theta_i B^i \tag{19}$$

where B is the backward operator such that $B^i w_t = w_{t-i}$.

Moving Average Models

- Consider the MA(1) model

$$x_t = w_t + \theta w_{t-1} \tag{20}$$

- Then $\mathrm{E}(x_t) = 0$, and the autocovariance

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma_w^2, & h = 0 \\ \theta\sigma_w^2, & h = 1 \\ 0, & h > 1 \end{cases} \tag{21}$$

  and the ACF is

$$\rho(h) = \begin{cases} \frac{\theta}{1+\theta^2}, & h = 1 \\ 0, & h > 1 \end{cases} \tag{22}$$

- But how do we distinguish between

$$x_t = w_t + \frac{1}{5} w_{t-1}, \ w_t \overset{iid}{\sim} N(0, 25) \quad vs. \quad y_t = v_t + 5 v_{t-1}, \ v_t \overset{iid}{\sim} N(0, 1)? \tag{23}$$

Moving Average Models

- We will choose the model with an infinite AR representation. Such a process is called an *invertible* process.
- We reverse the roles of x_t and w_t:

$$w_t = -\theta w_{t-1} + x_t \tag{24}$$

which has an infinite AR representation when $|\theta| < 1$:

$$w_t = \sum_{j=0}^{\infty} (-\theta)^j x_{t-j} \tag{25}$$

- In general, we write MA process as $w_t = \pi(B) x_t$, where $\pi(B) = \theta^{-1}(B)$.
- For MA(1), if $|\theta| < 1$, we have

$$\pi(B) = \theta^{-1}(B) = (1 + \theta B)^{-1} = \sum_{j=0}^{\infty} (-\theta)^j B^j \tag{26}$$

Autoregressive Moving Average Models

- A time series $\{x_t; t = 0, \pm 1, \cdots\}$ is **ARMA**(p, q) if it is stationary and

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q} \quad (27)$$

  where $w_t \sim wn(0, \sigma_w^2)$, and $\phi_p \neq 0, \theta_q \neq 0$, and $\sigma_w^2 > 0$.
- The parameters p and q are called the autoregressive and the moving average orders, respectively.
- If x_t has a nonzero mean μ, we set $\alpha = \mu(1 - \phi_1 - \cdots - \phi_p)$ and have

$$x_t = \alpha + \phi_1 x_{t-1} + \phi_2 x_{t-2} + \cdots + \phi_p x_{t-p} + w_t + \theta_1 w_{t-1} + \cdots + \theta_q w_{t-q} \quad (28)$$

- With autoregressive and moving average operators, **ARMA**(p, q) model is:

$$\phi(B) x_t = \theta(B) w_t \quad (29)$$

Autoregressive Moving Average Models

- There are following problems for **ARMA**(p, q)
  1. parameter redundant models,
  2. stationary AR models that depend on the future, and
  3. MA models that are not unique.
- To overcome these problems, we will require some additional restrictions on the model parameters.
- The **AR and MA polynomials** are defined as

$$\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p, \quad \phi_p \neq 0 \tag{30}$$
$$\theta(z) = 1 + \theta_1 z + \cdots + \theta_q z^q, \quad \theta_q \neq 0 \tag{31}$$

respectively, where z is a complex number.

- We require that $\phi(z)$ and $\theta(z)$ have no common factors.

Autoregressive Moving Average Models

- An $ARMA(p,q)$ model is said to be **causal**, if the time series $\{x_t; t = 0, \pm 1, \cdots\}$ can be written as a one-sided linear process

$$x_t = \sum_{j=0}^{\infty} \psi_j w_{t-j} = \psi(B) w_t \tag{32}$$

where $\psi(B) = \sum_{j=0}^{\infty} \psi_j B^j$, and $\sum_{j=0}^{\infty} |\psi_j| < \infty$; we set $\psi_0 = 1$.

- An $ARMA(p,q)$ model is causal if and only if $\phi(z) \neq 0$ for $|z| \leq 1$. The coefficients of the linear process can be determined by solving

$$\psi(z) = \sum_{j=0}^{\infty} \psi_j z^j = \frac{\theta(z)}{\phi(z)}, \quad |z| \leq 1 \tag{33}$$

Autoregressive Moving Average Models

- An $ARMA(p, q)$ model is said to be **invertible**, if the time series $\{x_t; t = 0, \pm 1, \cdots\}$ can be written as

$$\pi(B)x_t = \sum_{j=0}^{\infty} \pi_j x_{t-j} = w_t \tag{34}$$

where $\pi(B) = \sum_{j=0}^{\infty} \pi_j B^j$, and $\sum_{j=0}^{\infty} |\pi_j| < \infty$; we set $\pi_0 = 1$.

- An $ARMA(p, q)$ model is invertible if and only if $\theta(z) \neq 0$ for $|z| \leq 1$. The coefficients of $\pi(B)$ can be determined by solving

$$\pi(z) = \sum_{j=0}^{\infty} \pi_j z^j = \frac{\phi(z)}{\theta(z)}, \quad |z| \leq 1 \tag{35}$$

Autoregressive Moving Average Models

- For an AR(1) model, $(1-\phi B)x_t = w_t$, to be causal, the root of $\phi(z) = 1 - \phi z$ must lie outside of the unit circle. That is, $|\phi| < 1$.
- Consider the AR(2) model, $(1-\phi_1 B - \phi_2 B^2)x_t = w_t$. the causal condition requires that the two roots of $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$ lie outside of the unit circle. That is $\left|\frac{\phi_1 \pm \sqrt{\phi_1^2 + 4\phi_2}}{-2\phi_2}\right| > 1$, which is equivalent to

$$\phi_1 + \phi_2 < 1, \quad \phi_2 - \phi_1 < 1, \quad |\phi_2| < 1 \tag{36}$$