Stationary spatial process models
Generalized linear spatial process modeling
Areal data models
Generalized linear areal data modeling

Lecture 5 Hierarchical modeling of spatial data

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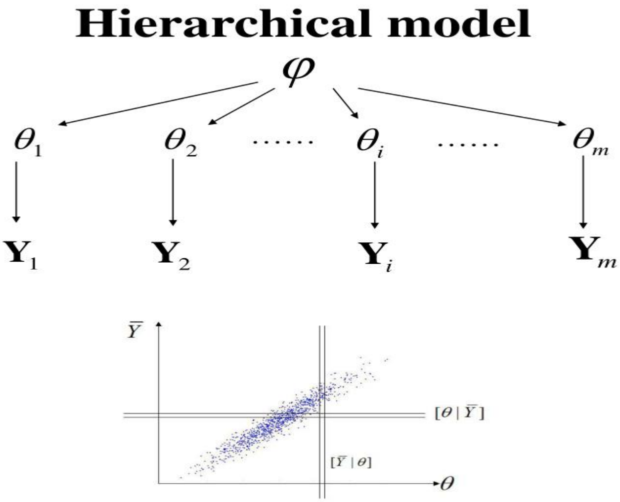
Bayesian hierarchical modeling



Stationary spatial process models

- The basic model is
$$Y(\mathrm{s}) = \mu(\mathrm{s}) + w(\mathrm{s}) + \epsilon(\mathrm{s}) \tag{1}$$
- The mean structure $\mu(\mathrm{s}) = \mathrm{x}^T \boldsymbol{\beta}$.
- The residual is partitioned into two parts: the spatial part $w(\mathrm{s})$ and the non-spatial errors $\epsilon(\mathrm{s})$.
- Recall that the $w(\mathrm{s})$ introduces the partial sill (σ^2) and the range ϕ; while the $\epsilon(\mathrm{s})$ adds the nugget τ^2.
- Assume stationarity. Pure error $\epsilon(\mathrm{s})$ vs spatial error $w(\mathrm{s})$. We further assume
  - $w(\mathrm{s}+\mathrm{h}) - w(\mathrm{s}) \to 0$ as $\mathrm{h} \to 0$
  - $[w(\mathrm{s}+\mathrm{h}) + \epsilon(\mathrm{s}+\mathrm{h})] - [w(\mathrm{s}) + \epsilon(\mathrm{s})] \not\to 0$ as $\mathrm{h} \to 0$
- *Microscale* view: $eps(\mathrm{s})$ is a spatial process with very rapid decay in association, and only matters at high resolution.

Isotropic models

- Suppose we have data $Y(\mathrm{s}_i)$, $i = 1, \cdots, n$, and let $\mathrm{Y} = (Y(\mathrm{s}_1), \cdots, Y(\mathrm{s}_n))^T$. In the basic Gaussian isotropic kriging model, we assume

$$\Sigma = \sigma^2 H(\phi) + \tau^2 I \tag{2}$$

where H is correlation matrix with $H_{ij} = \rho(\mathrm{s}_i - \mathrm{s}_j; \phi)$ and ρ is a valid isotropic correlation function on $\mathbb{R}^2$, e.g. $\rho(\mathrm{s}_i - \mathrm{s}_j; \phi) = \exp(-\phi \|\mathrm{s}_i - \mathrm{s}_j\|)$

- Collecting all model parameters into a vector $\boldsymbol{\theta} = (\boldsymbol{\beta}, \sigma^2, \tau^2, \phi)$, we have

$$p(\boldsymbol{\theta}|\mathrm{y}) \propto f(\mathrm{y}|\boldsymbol{\theta}) p(\boldsymbol{\theta}) \tag{3}$$

where

$$\mathrm{Y}|\boldsymbol{\theta} \sim N(X\boldsymbol{\beta}, \sigma^2 H(\phi) + \tau^2 I) \tag{4}$$

Prior specification

- Typically, independent priors are chosen for the different parameters

$$p(\boldsymbol{\theta}) = p(\boldsymbol{\beta})p(\sigma^2)p(\tau^2)p(\phi) \tag{5}$$

- We usually adopt the following priors

$$\begin{aligned} \beta &\sim N_{p+1}(\mu_0, \Lambda_0) && (6) \\ \sigma^2 &\sim \Gamma^{-1}(a_1, b_1) && (7) \\ \tau^2 &\sim \Gamma^{-1}(a_2, b_2) && (8) \\ \phi &\sim \Gamma(a_3, b_3) && (9) \end{aligned}$$

- In Matérn class, the product $\sigma^2\phi^{2\nu}$ can be identified but not the individuals (Zhang, 2004). Often, a very informative prior is imposed for ϕ (e.g. uniform over interval) and a relatively vague prior is used for σ^2.

Prior specification

- When $\tau^2 \equiv 0$, with the exponential covariance, Berger et al. (2001) impose the *objective* priors of the form

$$p(\boldsymbol{\beta}, \sigma^2, \phi) \propto \frac{p(\phi)}{(\sigma^2)^\alpha} \tag{10}$$

- This implies a flat prior for $\boldsymbol{\beta}$. With a uniform prior for ϕ, it can be shown that an improper posterior arises for $\alpha < 2$.
- If $\sigma^2 \sim \Gamma^{-1}(\epsilon, \epsilon)$, then $\alpha = 1 + \epsilon$ getting an improper posterior for small ϵ.
- If $\sigma^2 \sim \Gamma^{-1}(a_1, b_1)$, $a_1 \geq 1$ is recommended $\alpha \geq 2$.
- Closed form of *marginal* posterior of $\boldsymbol{\beta}$ may be available (e.g. with Gaussian likelihood).

Hierarchical modeling

- The hierarchical modeling shares the following generic format

$$[\text{data}|\text{process}, \text{parameters}]\ [\text{process}|\text{parameters}]\ [\text{parameters}] \tag{11}$$

- The previous spatial process model (4) can be recast as a hierarchical model with two stages

$$\mathsf{Y}|\boldsymbol{\theta}, \mathsf{W} \sim N(X\boldsymbol{\beta} + \mathsf{W}, \tau^2 I) \tag{12}$$
$$\mathsf{W}|\sigma^2, \phi \sim N(0, \sigma^2 H(\phi) + \tau^2 I) \tag{13}$$

where $\mathsf{W} = (w(\mathsf{s}_1), \cdots, w(\mathsf{s}_n))^T$ is the spatial random effect that captures spatial dependence.

- The parameter space is now augmented from $\boldsymbol{\theta}$ to $(\boldsymbol{\theta}, \mathsf{W})$, with the dimension increased by n.

Hierarchical modeling

- The resulting $p(\boldsymbol{\theta}|\mathrm{y})$ is the same, but we have the choice of using MCMC to fit either $f(\mathrm{y}|\boldsymbol{\theta})p(\boldsymbol{\theta})$ or $f(\mathrm{y}|\boldsymbol{\theta},\mathrm{W})p(\mathrm{W}|\boldsymbol{\theta})p(\boldsymbol{\theta})$.
- Interest is often in the spatial surface $\mathrm{W}|\mathrm{y}$ as well as prediction for $W(\mathrm{s}_0)|\mathrm{y}$ for various choices of s_0.
- Note $p(\mathrm{W}|\mathrm{y})$ can be recovered from $p(\boldsymbol{\theta}|\mathrm{y})$:

$$p(\mathrm{W}|\mathrm{y}) = \int p(\mathrm{W}|\boldsymbol{\theta},\mathrm{y})p(\boldsymbol{\theta}|\mathrm{y})d\boldsymbol{\theta} \tag{14}$$

which can be sampled by one for one composition of posterior sampling: $\mathrm{W}^{(g)} \sim p(\mathrm{W}|\boldsymbol{\theta}^{(g)},\mathrm{y})$ with $\boldsymbol{\theta}^{(g)} \sim p(\boldsymbol{\theta}|\mathrm{y})$.

Hierarchical modeling

- Bayesian kriging involves prediction of $Y_0 \equiv Y(\mathrm{s}_0)$ at a new location s_0 with associated covariate $\mathrm{x}(\mathrm{s}_0)$:

$$p(y_0|\mathrm{y}, X, \mathrm{x}_0) = \int p(y_0, \boldsymbol{\theta}|\mathrm{y}, X, \mathrm{x}_0) d\boldsymbol{\theta} = \int p(y_0|\mathrm{y}, \boldsymbol{\theta}, \mathrm{x}_0) p(\boldsymbol{\theta}|\mathrm{y}, X) d\boldsymbol{\theta} \quad (15)$$

- In practice, we use MCMC to obtain posterior samples $\{\boldsymbol{\theta}^{(g)}\}_{g=1}^G$ with $\boldsymbol{\theta}^{(g)} \sim p(\boldsymbol{\theta}|\mathrm{y}, X)$, and approximate the above predictive distribution with

$$\hat{p}(y_0|\mathrm{y}, X, \mathrm{x}_0) = \frac{1}{G} \sum_{g=1}^G p(y_0|\mathrm{y}, \boldsymbol{\theta}^{(g)}, \mathrm{x}_0) \quad (16)$$

- Replicates of prediction $y_0^{(g)}$ can be obtained using the composition sampling.
- Multi-output prediction for $\mathrm{Y}_0 = (Y(\mathrm{s}_{01}), \cdots, Y(\mathrm{s}_{0m}))^T$ is also available

$$p(\mathrm{y}_0|\mathrm{y}, X, \mathrm{x}_0) = \int p(\mathrm{y}_0|\mathrm{y}, \boldsymbol{\theta}, \mathrm{x}_0) p(\boldsymbol{\theta}|\mathrm{y}, X) d\boldsymbol{\theta} \approx \frac{1}{G} \sum_{g=1}^G p(\mathrm{y}_0|\mathrm{y}, \boldsymbol{\theta}^{(g)}, \mathrm{x}_0) \quad (17)$$

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Hierarchical Model

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Stationary spatial process models

Generalized linear spatial process modeling

Areal data models

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Generalized linear spatial process modeling

- In some point-referenced data sets, measurements $Y(\mathbf{s})$ are not naturally modeled as a normal distribution. e.g. binary data, and counting data.
- The observations $Y(\mathbf{s}_i)$ are modeled independent conditioned on $\boldsymbol{\beta}$ and $w(\mathbf{s}_i)$ within the class of exponential family:

$$p(y(\mathbf{s}_i)|\boldsymbol{\beta}, w(\mathbf{s}_i), \gamma) = h(y(\mathbf{s}_i), \gamma) \exp\{\gamma[y(\mathbf{s}_i)\eta(\mathbf{s}_i) - \psi(\eta(\mathbf{s}_i))]\} \tag{18}$$

where $g(\eta(\mathbf{s}_i)) = \mathbf{x}^T(\mathbf{s}_i)\boldsymbol{\beta} + w(\mathbf{s}_i)$ for some link function g, and γ is a dispersion parameter.

- We presume the $w(\mathbf{s}_i)$ to be the spatial random effect coming from a Gaussian process, i.e. $\mathbf{W} \sim N(0, \sigma^2 H(\phi) + \tau^2 I)$.

Generalized linear spatial process modeling

- Using conditional independence, we have the joint distribution

$$f(y(\mathrm{s}_1), \cdots, y(\mathrm{s}_n)|\boldsymbol{\beta}, \sigma^2, \phi, \gamma) = \int \left(\prod_{i=1}^{n} f(y(\mathrm{s}_i)|\boldsymbol{\beta}, w(\mathrm{s}_i), \gamma) \right) p(\mathrm{W}|\sigma^2, \phi) d\mathrm{W} \tag{19}$$

- We presume the $w(\mathrm{s}_i)$ to be the spatial random effect coming from a Gaussian process, i.e. $\mathrm{W} \sim N(0, \sigma^2 H(\phi) + \tau^2 I)$. Note W may not be marginalized out in general.
- Binary data. We model $Y(\mathrm{s})$ through the latent process $Z(\mathrm{s}) = \mathrm{x}(\mathrm{s})^T \boldsymbol{\beta} + w(\mathrm{s}) + \epsilon(\mathrm{s})$:

$$\Pr(Y(\mathrm{s}) = 1) = \Pr(Z(\mathrm{s}) \geq 0) = g^{-1}(\mathrm{x}(\mathrm{s})^T \boldsymbol{\beta} + w(\mathrm{s})) \tag{20}$$

for some link function $g(\cdot)$ such that g^{-1} takes $p(\mathrm{s}) := \Pr(Y(\mathrm{s}) = 1)$ to $\mathbb{R}^1$.

- Possible choices: the logit $g(x) = \log \frac{x}{1-x}$ and the probit $g(x) = \Phi^{-1}(x)$.

Generalized linear spatial process modeling

- Consider real estate data at 50 locations in Baton Rouge, LA.
- $Y(s) = 1$ indicates that the price of the property at location s is "high" (above the median for the region); $Y(s) = 0$ indicates that the price is "low".
- Covariates: the house's age, total living area, and other area in the property.
- We fit the model with the logit link, assuming vague priors for $\boldsymbol{\beta}$, a $\mathrm{unif}(0, 10)$ prior for ϕ and a $\Gamma^{-1}(0.1, 0.1)$ prior for σ^2.

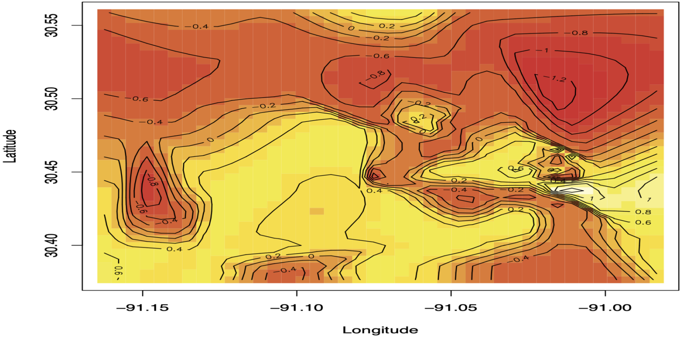

Figure 6.7 Image plot of the posterior median surface of the latent spatial process w(s), binary

Generalized linear spatial process modeling

- Counting data. We model $Y(\mathrm{s})$ through the latent process $Z(\mathrm{s}) = \mathrm{x}(\mathrm{s})^T\boldsymbol{\beta} + w(\mathrm{s}) + \epsilon(\mathrm{s})$:

$$Y(\mathrm{s}) \sim \mathrm{pois}(g^{-1}(Z(\mathrm{s}))) \tag{21}$$

for some link function $g(\cdot)$, e.g. the canonical link $g(x) = \log(x)$.

- This is related to the log-Gaussian Cox Process (LGCP) model.

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Disease mapping

- An area of strong biostatistical and epidemiological interest is that of *disease mapping*.
- We typically have count data of the following:

$$\begin{aligned} Y_i &= \text{observed number of cases of disease in county } i, i = 1, \cdots, I \\ E_i &= \text{expected number of cases of disease in county } i, i = 1, \cdots, I \end{aligned}$$

  where E_i are thought of as fixed and known functions of n_i, the number of persons at risk for the disease in county i.
- One can simply assume

$$E_i = n_i \bar{r} = n_i \frac{\sum_i y_i}{\sum_i n_i} = \sum_i y_i \frac{n_i}{\sum_i n_i} \tag{22}$$

  i.e. $\bar{r}$ is the overall disease rate in the entire study region.

Disease mapping

- However, such *internal standardization* is “cheating”. And one might modify it by age-adjusted rates for the disease

$$E_i = n_{ij} r_j \tag{23}$$

where n_{ij} is the person-years at risk in area i for age group j, and $\bar{r}_j$ is the disease rate in age group j. This process is called *external standardization*.

- The usual model for Y_i is the Poisson model

$$Y_i | \eta_i \sim \mathrm{Pois}(E_i \eta_i) \tag{24}$$

where η_i is the true *relative risk* of disease in region i.

- The maximum likelihood estimate (MLE) of η_i is

$$\hat{\eta}_i = \mathrm{SMR}_i = \frac{Y_i}{E_i} \tag{25}$$

- $\mathrm{Var}(\mathrm{SMR}_i) = \mathrm{Var}(Y_i)/E_i^2 = \eta_i/E_i$. $\widehat{\mathrm{Var}}(\mathrm{SMR}_i) = \hat{\eta}_i/E_i = Y_i/E_i^2$.

Hierarchical Bayesian methods

- For detecting extra-Poisson variability (overdispersion) in the observed rates, we seek *random effects* model for η_i through hierarchical Bayesian modeling.
- Poisson-gamma model

$$Y_i|\eta_i \overset{ind}{\sim} \mathrm{pois}(E_i\eta_i), \quad i = 1, \cdots, I \tag{26}$$

$$\eta_i \overset{iid}{\sim} \Gamma(a, b) \tag{27}$$

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