Lecture 9 State Space Models

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Introduction to State Space Models

- It was introduced by Kalman (1960) and Kalman and Bucy (1961) in the space tracking setting.
- It consists of a state equation for the motion of x_t and an observation equation for y_t based on the location x_t.
- The most basic special case is linear (state) Gaussian (observation) model, a.k.a. dynamic linear model.
- It has been applied to modeling data from economics, medicine, climate modeling, and engineering.

Introduction to State Space Models

In this lecture, we focus primarily on linear Gaussian state space models. We will

- present various forms of the model
- introduce the concepts of prediction, filtering and smoothing
- perform maximum likelihood estimation
- present special topics such as hidden Markov models (HMM), stochastic volatility
- discuss a Bayesian approach

Introduction to State Space Models

- In general, there are two principles assumed in state space models:
  1. The state process $\{x_t\}$ is Markovian, i.e. the future $x_s : s > t$ and the past $x_s : s < t$ are independent conditioned on the present x_t;
  2. The observations $\{y_t\}$ are independent given the states x_t.

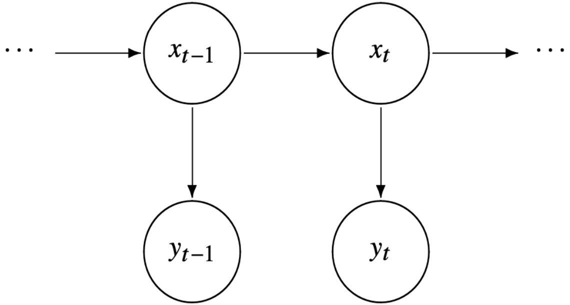

***Fig. 6.1.** Diagram of a state space model.*

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Linear Gaussian Model

- The liner Gaussian model is the most fundamental state-space model which takes the following form:

$$x_t = \Phi x_{t-1} + w_t, \quad w_t \overset{iid}{\sim} N_p(0, Q) \tag{1}$$

$$y_t = A_t x_t + v_t, \quad v_t \overset{iid}{\sim} N_q(0, R) \tag{2}$$

where we assume $x_0 \sim N_p(\mu_0, \Sigma_0)$, and A_t is a $q \times p$ measurement/observation matrix.

Linear Gaussian Model

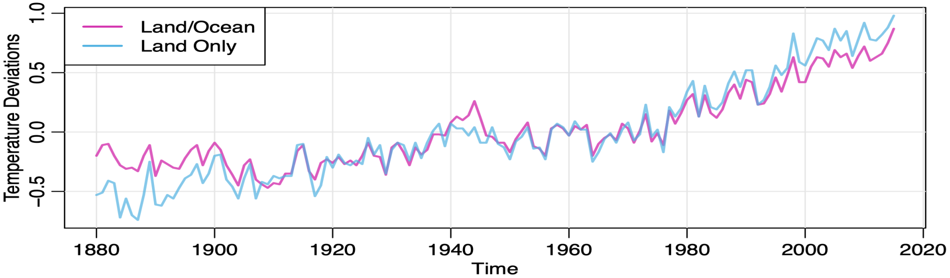

Fig. 6.3. *Annual global temperature deviation series, measured in degrees centigrade, 1880–2015. The series differ by whether or not ocean data is included.*

- Two time series of temperature $\{y_{t1}, y_{t2}\}$ are modeled as a linear Gaussian state-space model

$$\begin{bmatrix} y_{t1} \\ y_{t_2} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} x_t + \begin{bmatrix} v_{t1} \\ v_{t_2} \end{bmatrix}, \quad R = \mathrm{Var} \begin{bmatrix} v_{t1} \\ v_{t_2} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} \tag{3}$$

$$x_t = \delta + x_{t-1} + w_t, \quad Q = \mathrm{Var}(w_t)$$

Linear Gaussian Model

- Let's consider a univariate state-space model

$$\begin{aligned} y_t &= x_t + v_t, \quad v_t \overset{iid}{\sim} N(0, \sigma_v^2) \\ x_t &= \phi x_{t-1} + w_t, \quad w_t \overset{iid}{\sim} N(0, \sigma_w^2) \end{aligned} \tag{4}$$

  where $x_0 \sim N(0, \sigma_w^2/(1-\phi^2))$, and x_0, $\{v_t\}$, $\{w_t\}$ are independent.
- For AR(1) process, we could compute the autocovariance function of x_t

$$\gamma_x(h) = \frac{\sigma_w^2}{1-\phi^2}\phi^h, \quad h = 0, 1, 2, \cdots \tag{5}$$

- Based on the observation equation and independence assumption, we could compute the autocovariance function of y_t

$$\gamma_y(h) = \mathrm{Cov}(x_t + v_t, x_{t-h}, + v_{t-h}) = \begin{cases} \frac{\sigma_w^2}{1-\phi^2} + \sigma_v^2, & h = 0 \\ \gamma_x(h), & h \geq 1 \end{cases} \tag{6}$$

Linear Gaussian Model

- Therefore, the ACF of the observations y_t is

$$\rho_y(h) = \frac{\gamma_y(h)}{\gamma_y(0)} = \left(1 + \frac{\sigma_v^2}{\sigma_w^2}(1 - \phi^2)\right)^{-1} \phi^h \tag{7}$$

- Note, this is NOT the autocorrelation structure of AR(1) unless $\sigma_v^2 = 0$.
- In general, this is ACF of ARMA(1,1), if we identify the following model

$$y_t = \phi y_{t-1} + \theta u_{t-1} + u_t \tag{8}$$

with $u_t \overset{iid}{\sim} N(0, \sigma_u^2)$ for σ_u^2, θ to be determined.

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Filtering, Smoothing and Forecasting

- The objective of the analysis involving the state-space model is to produce estimators for the underlying unobserved (latent) process x_t, given the data $y_{1:s} = \{y_1, \cdots, y_s\}$ up to time s.
- Based on the relationship between t and s, the problems can be divided into 3 types:
  1. *smoothing*: $t < s$
  2. *filtering*: $t = s$
  3. *forecasting or prediction*: $t > s$
- We fix some notations

$$x_t^s = \mathrm{E}(x_t | y_{1:s}), \quad P_{t_1,t_2}^s = \mathrm{E}\{(x_{t_1} - x_{t_1}^s)(x_{t_2} - x_{t_2}^s)'\} \tag{9}$$

The Kalman Filter

- First, we present the Kalman filter for the filtering and forecasting problems.
- Consider a linear filter of x_t^t in terms of $y_{1:t}$, $x_t = \sum_{s=1}^t B_s y_s$, for the state-space equation with initial condition $x_0^0 = \mu_0$ and $P_0^0 = \Sigma_0$

$$x_t = \Phi x_{t-1} + \Upsilon u_t + w_t, \quad w_t \overset{iid}{\sim} N_p(0, Q) \tag{10}$$

$$y_t = A_t x_t + \Gamma u_t + v_t, \quad v_t \overset{iid}{\sim} N_q(0, R) \tag{11}$$

- The Kalman filter consists of the following two steps:
  1. *predict*:
  $$x_t^{t-1} = \Phi x_{t-1}^{t-1} + \Upsilon u_t, \quad P_t^{t-1} = \Phi P_{t-1}^{t-1} \Phi' + Q \tag{12}$$
  2. *update*:
  $$x_t^t = x_t^{t-1} + K_t(y_t - A_t x_t^{t-1} - \Gamma u_t), \quad P_t^t = [I - K_t A_t] P_t^{t-1} \tag{13}$$

  where $K_t = P_t^{t-1} A_t' [A_t P_t^{t-1} A_t' + R]^{-1}$ is called the *Kalman gain*.

The Kalman Filter

- The prediction for $t > n$ can be obtained from the *predict* step with initial conditions x_n^n and P_n^n.
- The innovations (prediction errors) are

$$\epsilon_t = y_t - \mathrm{E}(y_t | y_{1:t-1}) = y_t - A_t x_t^{t-1} - \Gamma u_t \tag{14}$$

$$\Sigma_t = \mathrm{Var}(\epsilon_t) = \mathrm{Var}[A_t(x_t - x_t^{t-1}) + v_t] = A_t P_t^{t-1} A_t' + R \tag{15}$$

- What are the possible issues?

The Kalman Filter (density point of view)

- Let's take a density point of view. The two assumptions (Markovianity and conditional independence) translate into

$$p_\theta(x_t|x_{0:t-1}) = p_\theta(x_t|x_{t-1}), \quad p_\theta(y_{1:n}|x_{1:n}) = \prod_{t=1}^n p_\theta(y_t|x_t) \tag{16}$$

- For the linear Gaussian model, we have

$$p_\theta(x_t|x_{t-1}) = \phi(x_t; \Phi x_{t-1}, Q), \quad p_\theta(y_t|x_t) = \phi(y_t; A_t x_t, R) \tag{17}$$

The Kalman Filter (density point of view)

- For the *prediction* step, we could obtain

$$\begin{aligned} p_\theta(x_t|y_{1:t-1}) &= \int_{\mathbb{R}^p} p_\theta(x_t, x_{t-1}|y_{1:t-1}) dx_{t-1} \\ &= \int_{\mathbb{R}^p} \phi(x_t; \Phi x_{t-1}, Q)\phi(x_{t-1}; x_{t-1}^{t-1}, P_{t-1}^{t-1}) dx_{t-1} \\ &= \phi(x_t; x_t^{t-1}, P_t^{t-1}) \end{aligned} \tag{18}$$

- For the *update* step, we have

$$\begin{aligned} p_\theta(x_t|y_{1:t}) = p_\theta(x_t|y_t, y_{1:t-1}) &\propto p_\theta(y_t|x_t)p_\theta(x_t|y_{1:t-1}) \\ &= \phi(y_t; A_t x_t, R)\phi(x_t; x_t^{t-1}, P_t^{t-1}) \\ &\propto \phi(x_t; x_t^t, P_t^t) \end{aligned} \tag{19}$$

The Kalman Smoother

- For the state-space model (10)(11) with initial conditions x_n^n and P_n^n, the Kalman Smoother is

$$
\begin{aligned}
x_{t-1}^n &= x_{t-1}^{t-1} + J_{t-1}(x_t^n - x_t^{t-1}) && (20)\\
P_t^n &= P_{t-1}^{t-1} + J_{t-1}(P_t^n - P_t^{t-1})J_{t-1}' && (21)
\end{aligned}
$$

for $t = n, n-1, \cdots, 1$, where $J_{t-1} = P_{t-1}^{t-1}\Phi'[P_t^{t-1}]^{-1}$.

- In the following, let's investigate a simple univariate state-space model, the *local level model*.

$$
\begin{aligned}
y_t &= \mu_t + v_t, \quad v_t \overset{iid}{\sim} N(0, \sigma_v^2)\\
\mu_t &= \mu_{t-1} + w_t, \quad w_t \overset{iid}{\sim} N(0, \sigma_w^2)
\end{aligned}
\tag{22}
$$

The Local Level Model

- Using the density argument or formulae, we obtain the prediction

$$\mu_t | y_{1:t-1} \sim N(\mu_t^{t-1}, P_t^{t-1}) \tag{23}$$

$$\mu_t^{t-1} = u_{t-1}^{t-1}, \quad P_t^{t-1} = P_{t-1}^{t-1} + \sigma_w^2 \tag{24}$$

- And we can obtain the update (filter)

$$\mu_t | y_{1:t} \sim N(\mu_t^t, P_t^t) \tag{25}$$

$$\mu_t^t = \mu_t^{t-1} + K_t(y_t - \mu_t^{t-1}), \quad K_t = \frac{P_t^{t-1}}{P_t^{t-1} + \sigma_v^2}, \quad P_t^t = (1 - K_t)P_t^{t-1} \tag{26}$$

- We note that $P_t^{t-1} \geq P_t^t \geq P_t^n$ for $t = n, n-1, \cdots, 1$.

The Local Level Model

- We could use R function `Ksmooth0` (which calls `Kfilter0` for the filtering) to obtain prediction x_t^{t-1} (`xp`), filter x_t^t (`xf`) and smooth x_t^n (`xs`) respectively.
- Their error bounds are $x_t^{t-1} \pm 2\sqrt{P_t^{t-1}}$ (`Pp`), $x_t^t \pm 2\sqrt{P_t^t}$ (`Pf`), and $x_t^n \pm 2\sqrt{P_t^n}$ (`Ps`) respectively.

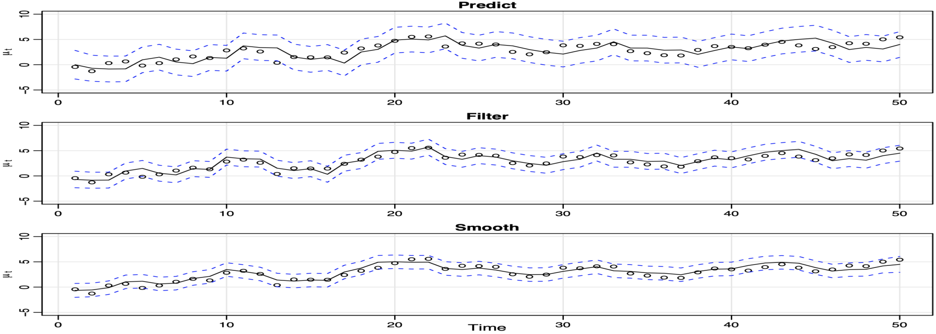

Fig. 6.4. *Displays for Example 6.5. The simulated values of μ_t, for $t = 1, \ldots, 50$, given by (6.51) are shown as points. The top shows the predictions μ_t^{t-1} as a line with $\pm 2\sqrt{P_t^{t-1}}$ error bounds as dashed lines. The middle is similar, showing $\mu_t^t \pm 2\sqrt{P_t^t}$. The bottom shows $\mu_t^n \pm 2\sqrt{P_t^n}$.*

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Maximum Likelihood Estimation

- Recall the state space model with initial condition $x_0^0 = \mu_0$ and $P_0^0 = \Sigma_0$

$$x_t = \Phi x_{t-1} + \Upsilon u_t + w_t, \quad w_t \overset{iid}{\sim} N_p(0, Q)$$
$$y_t = A_t x_t + \Gamma u_t + v_t, \quad v_t \overset{iid}{\sim} N_q(0, R)$$

- The estimation of parameters $\Theta = \{\mu_0, \Sigma_0, \Phi, Q, R, \Upsilon, \Gamma\}$ is quite involved.
- The likelihood is computed using the *innovations* $\{\epsilon_t\}_{t=1}^n$ and their convariances

$$\epsilon_t = y_t - A_t x_t^{t-1} - \Gamma u_t, \quad \Sigma_t = A_t P_t^{t-1} A_t' + R \tag{27}$$

- Then the likelihood, $L_Y(\Theta)$, can be written (up to a constant)

$$-\log L_Y(\Theta) = \frac{1}{2}\sum_{t=1}^n \log|\Sigma_t(\Theta)| + \frac{1}{2}\sum_{t=1}^n \epsilon_t(\Theta)'\Sigma_t(\Theta)^{-1}\epsilon_t(\Theta) \tag{28}$$

Maximum Likelihood Estimation

We could perform a Newton-Raphson estimation procedure:

1. Select initial values for the parameters, say, $\Theta^{(0)}$.
2. Run the Kalman filter, using the initial parameter values, to obtain a set of innovations and error covariances $\{\epsilon_t^{(0)}\}_{t=1}^n$ and $\{\Sigma_t^{(0)}\}_{t=1}^n$.
3. Run one iteration of a Newton-Raphson optimization on $-\log L_Y(\Theta)$ to obtain a new set of estimates $\Theta^{(1)}$.
4. Iterate between (2) and (3) for step j to obtain $\{\epsilon_t^{(j)}\}_{t=1}^n$ and $\{\Sigma_t^{(j)}\}_{t=1}^n$ until the estimates or the likelihood stabilize, i.e. their consecutive difference fall below some threshold.

Maximum Likelihood Estimation

- Recall univariate state-space model

$$\begin{aligned} y_t &= x_t + v_t, \quad v_t \overset{iid}{\sim} N(0, \sigma_v^2) \\ x_t &= \phi x_{t-1} + w_t, \quad w_t \overset{iid}{\sim} N(0, \sigma_w^2) \end{aligned}$$

  where $x_0 \sim N(0, \sigma_w^2/(1-\phi^2))$, and x_0, $\{v_t\}$, $\{w_t\}$ are independent.
- We generate $y_{1:n}$ for $n = 100$ with true values $\phi = 0.8$, $\sigma_w^2 = \sigma_v^2 = 1$.
- Now we use Newton-Raphson procedure (`optim` function) to obtain the MLE of $(\phi, \sigma_w^2, \sigma_v^2)$ given data.
- To start, we initialize parameters using what we calculated before

$$\phi^{(0)} = \frac{\hat{\rho}_y(2)}{\hat{\rho}_y(1)}, \quad (\sigma_w^2)^{(0)} = \frac{1-(\phi^{(0)})^2}{\phi^{(0)}}\hat{\gamma}_y(1), \quad (\sigma_w^2)^{(0)} = \hat{\gamma}_y(0) - \frac{(\sigma_w^2)^{(0)}}{1-(\phi^{(0)})^2} \tag{29}$$

- See more details in R codes.

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Hidden Markov Models

- *Hidden Markov models (HMM)* were developed parallel to state-space models in Goldfeld and Quandt (1973) and Lindgren (1978). A good summary is Rabiner and Juang (1986).
- Recall the first principle of state-space model is latent process x_t being Markovian. If the states x_t are a discrete valued Markov chain, then the state-space model is called *hidden Markov model (HMM)*.
- Assume the states, x_t, are Markov chain taking values in $1, \cdots, m$, with the following stationary distribution and transition probabilities

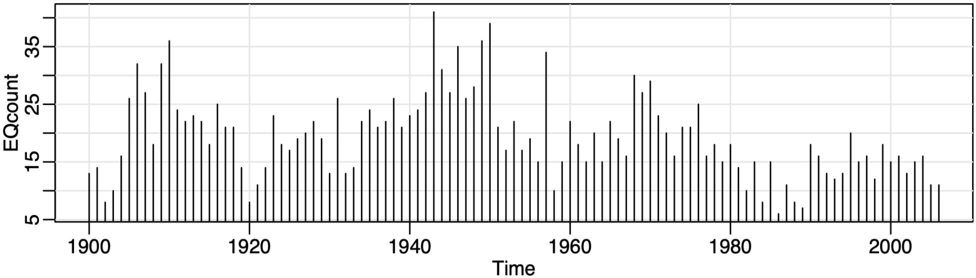
$$\pi_j = \Pr(x_t = j), \quad \pi_{ij} = \Pr(x_{t+1} = j | x_t = i) \tag{30}$$

for $t = 0, 1, 2, \cdots$ and $i, j = 1, \cdots, m$.

- We then need to specify the observation model:

$$p_j(y_t) = \Pr(y_t | x_t = j) \tag{31}$$

Poisson - HMM

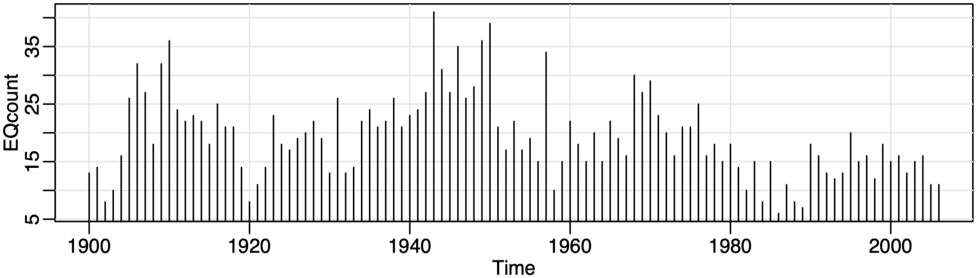


- Consider a Poisson-HMM model for the number of major earthquakes, y_t.
- Assume x_t to be a stationary two-state Markov chain taking values in $\{1, 2\}$:

$$\pi_1 = \frac{\pi_{21}}{\pi_{12} + \pi_{21}}, \quad \pi_2 = \frac{\pi_{12}}{\pi_{12} + \pi_{21}} \tag{32}$$

- The number of major earthquakes given $j \in \{1, 2\}$ follows a Poisson distribution: $p_j(y) = \frac{\lambda_j^y e^{-\lambda_j}}{y!}$, $y = 1, 2, \cdots$

Poisson - HMM



- The marginal distribution of y_t becomes a mixture of Poissons

$$p_\Theta(y_t) = \pi_1 p_1(y_t) + \pi_2 p_2(y_t), \quad \Theta = (\lambda_1, \lambda_2) \tag{33}$$

- The mean is $\mathrm{E}(y_t) = \pi_1\lambda_1 + \pi_2\lambda_2$ and the variance is $\mathrm{Var}(y_t) = \mathrm{E}(y_t) + \pi_1\pi_2(\lambda_2 - \lambda_1)^2 \geq \mathrm{E}(y_t)$.
- The autocovariance is

$$\gamma_y(h) = \sum_{i=1}^{2}\sum_{j=1}^{2} \pi_i(\pi_{ij}^h - \pi_j)\lambda_i\lambda_j = \pi_1\pi_2(\lambda_2 - \lambda_1)^2(1 - \pi_{12} - \pi_{21})^h \tag{34}$$

HMM Filter

- Denote $\pi_j(t|s) = \Pr(x_t = j|y_{1:s})$.
- *HMM Filter*: For $t = 1, \cdots, n$,

$$\pi_j(t|t-1) = \sum_{i=1}^{m} \pi_i(t-1|t-1)\pi_{ij} \tag{35}$$

$$\pi_j(t|t) = \frac{\pi_j(t)p_j(y_t)}{\sum_{i=1}^{m} \pi_i(t)p_i(y_t)} \tag{36}$$

with initial condition $\pi_j(1|0) = \pi_j$.

HMM Smoother

- Denote $\varphi_j(t) = p(y_{t+1:n}|x_t = j)$.
- *HMM Smoother*: For $t = n-1, \cdots, 0$,

$$\pi_j(t|n) = \frac{\pi_j(t|t)\varphi_j(t)}{\sum_{i=1}^m \pi_i(t|t)\varphi_j(t)} \tag{37}$$

$$\pi_{ij}(t|n) = \pi_i(t|n)\pi_{ij}p_j(y_{t+1})\frac{\varphi_j(t+1)}{\varphi_i(t)} \tag{38}$$

$$\varphi_i(t) = \sum_{i=1}^m \pi_{ij}p_i(y_{t+1})\varphi_j(t+1) \tag{39}$$

where $\varphi_j(n) = 1$ for $j = 1, \cdots, m$.

Poisson - HMM

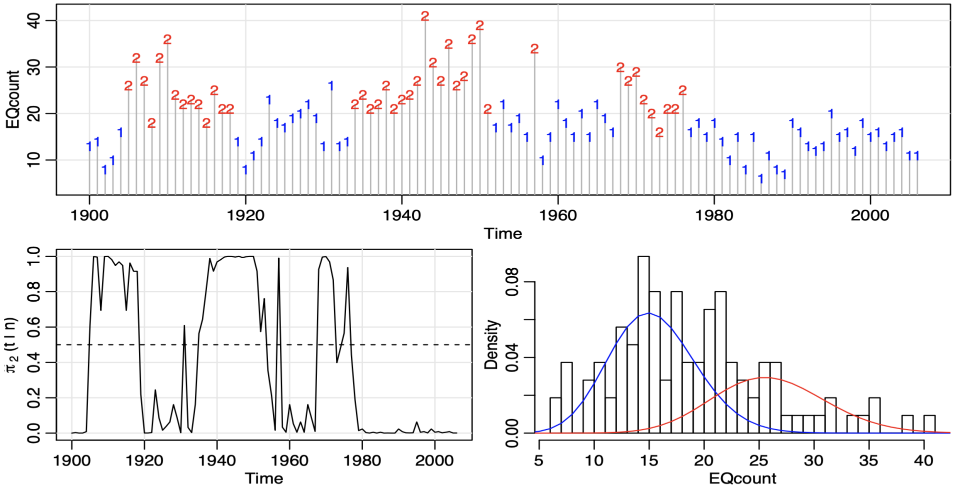


Fig. 6.13. *Top: Earthquake count data and estimated states. Bottom left: Smoothing probabilities. Bottom right: Histogram of the data with the two estimated Poisson densities superimposed (solid lines).*

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Bayesian Analysis of State Space Models

- Now we consider some Bayesian approaches for the linear-Gaussian state space model

$$x_t = \Phi x_{t-1} + w_t, \quad w_t \overset{iid}{\sim} N_p(0, Q)$$
$$y_t = A_t x_t + v_t, \quad v_t \overset{iid}{\sim} N_q(0, R)$$

- The main objective is to obtain the posterior density of the parameters $p(\Theta|y_{1:n})$ or $p(x_{0:n}|y_{1:n})$ if the states are meaningful.
- We could apply the following Gibbs sampler for state space models
  1. Draw $\theta' \sim p(\Theta|x_{0:n}, y_{1:n})$
  2. Draw $x'_{0:n} \sim p(x_{0:n}|\Theta', y_{1:n})$

Bayesian Analysis of State Space Models

- For the first step $\theta' \sim p(\Theta|x_{0:n}, y_{1:n})$, we could use conjugate priors or Metropolis-Hastings algorithm in sampling the posterior

$$p(\Theta|x_{0:n}, y_{1:n}) \propto \pi(\Theta)p(x_0|\Theta)\prod_{t=1}^{n} p(x_t|x_{t-1}, \Theta)p(y_t|x_t, \Theta) \tag{40}$$

where $\pi(\Theta)$ is the prior on the parameters.

- For the second step $x'_{0:n} \sim p(x_{0:n}|\Theta', y_{1:n})$, it amounts to sampling from the joint smoothing distribution of the latent state sequence:

$$p_\Theta(x_{0:n}|y_{1:n}) = p_\Theta(x_n|y_{1:n})p_\Theta(x_{n-1}|x_n, y_{1:n})\cdots p_\Theta(x_0|x_1) \tag{41}$$

- Note for each component as above, Frühwirth-Schnatter (1994) use the forward-filtering, backward- sampling (FFBS) algorithm

$$p_\Theta(x_t|x_{t+1}, y_{1:t}) \propto p_\Theta(x_t|y_{1:y})p_\Theta(x_{t+1}|x_t) \tag{42}$$

where in the linear-Gaussian case, $x_t|y_{1:t} \sim N_p^\Theta(x_t^t, P_t^t)$ and $x_{t+1}|x_t \sim N_p^\Theta(\Phi x_t, Q)$.

Bayesian Analysis of State Space Models

- Let's revisit the local level model

$$\begin{aligned} y_t &= \mu_t + v_t, \quad v_t \overset{iid}{\sim} N(0, \sigma_v^2) \\ \mu_t &= \mu_{t-1} + w_t, \quad w_t \overset{iid}{\sim} N(0, \sigma_w^2) \end{aligned} \tag{43}$$

- We generate $y_{1:n}$ for $n = 100$ with true values $\sigma_w^2 = 0.5$ and $\sigma_v^2 = 1$.
- We adopt inverse gamma priors for variance parameters

$$\sigma_w^2 \sim \Gamma^{-1}(a_0/2, b_0/2), \quad \sigma_v^2 \sim \Gamma^{-1}(c_0/2, d_0/2) \tag{44}$$

where we set $a_0 = b_0 = c_0 = d_0 = 0.02$.

- We run MCMC to obtain 1010 samples and burn in the first 10.

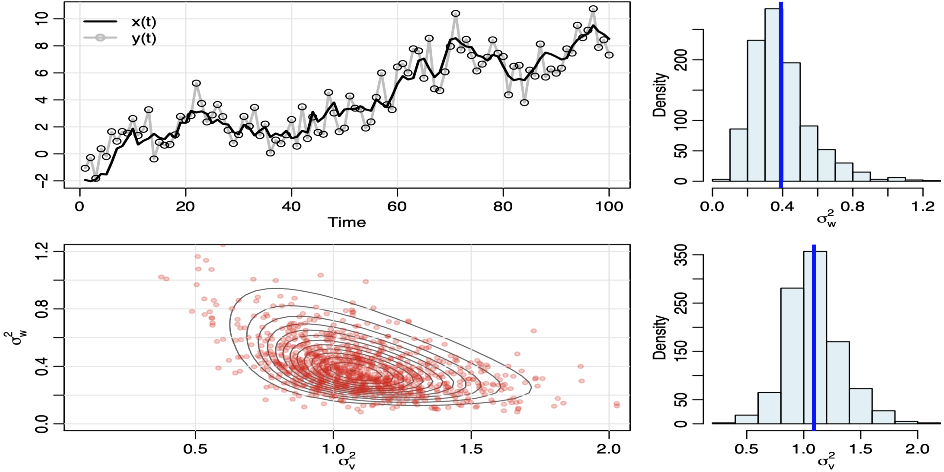

Fig. 6.20. *Display for Example 6.26: Left: Generated states, x_t and data y_t. Contours of the likelihood (solid line) of the data and sampled posterior values as points. Right : Marginal sampled posteriors and posterior means (vertical lines) of each variance component. The true values are $\sigma_w^2 = .5$ and $\sigma_v^2 = 1$.*