# Ramanujan Graphs and Interlacing Families

Andrew W. Herring

Department of Mathematics
Western University

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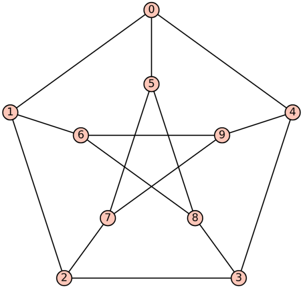
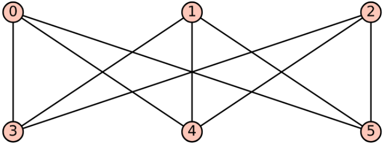
# Ramanujan graphs

- Ramanujan graphs are important to combinatorialists and (some) number theorists.
- They are "optimal expander graphs:" simultaneously sparse yet highly connected.
- They satisfy a Riemann Hypothesis.
- The original constructions come from number theory.



# Definitions

- $G = (V, E)$ a graph
- $A = A(G)$ the adjacency matrix of $G$: it's a $V \times V$ matrix with $(A(G))_{i,j} = 1$ if $(i, j) \in E$ and 0 otherwise.
- Since we consider undirected (symmetric) graphs here, $A$ is a symmetric matrix, hence has real spectrum (real eigenvalues).
- If $G$ is $d$-regular, then $d \in \mathrm{Spec}(A) = \{\text{eigenvalues of } A\}$. The smallest eigenvalue is at least $-d$ with equality $\iff$ $G$ is bipartite. These will be called the **trivial eigenvalues**.
- Let $\lambda(G)$ denote the largest absolute value of the non-trivial eigenvalues. $\lambda(G)$ is an estimate of the expansion properties of $G$; the smaller, the better.

# Definitions

- But $\lambda(G)$ cannot be made arbitrarily small. Alon-Boppana [8] says: $\forall \varepsilon > 0$, every infinite sequence of $d$-regular graphs contains a graph with non-trivial eigenvalue of absolute value at least $2\sqrt{d-1} - \varepsilon$.
- Thus infinite families of $d$-regular graphs all of whose non-trivial eigenvalues lie in $(-2\sqrt{d-1}, 2\sqrt{d-1})$ are "best possible."
- A $d$-regular graph $G$ is **Ramanujan** if all non-trivial eigenvalues lie in $(-2\sqrt{d-1}, 2\sqrt{d-1})$.

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- Marcus, Spielman, Srivastava [7]: "k."

# 2-lifts

- A 2-**lift of** $G$ is a graph $\hat{G} = (\hat{V}, \hat{E})$ which has a pair of vertices $\{v_0, v_1\}$ for every $v \in V$. The pair $\{v_0, v_1\}$ is called the **fiber of** $v$.

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- Each edge in $G$ corresponds to a pair of edges in $\hat{G}$: for $(u, v) \in E$, let $\{u_0, u_1\}$ and $\{v_0, v_1\}$ be the fibers of $u$ and $v$. Then $\hat{E}$ contains one of the following pairs:

$$\{(u_0, v_0), (u_1, v_1)\}, \text{ or} \tag{1}$$

$$\{(u_0, v_1), (u_1, v_0)\}. \tag{2}$$

In either case, we refer to the pair of edges $\{(u_0, v_0), (u_1, v_1)\}$ or $\{(u_0, v_1), (u_1, v_0)\}$ as **the edges above** $(u, v)$.

# signings

- A **signing of** $G$ is a function $s : E \to \{\pm 1\}$.
- There's a one-to-one correspondence

$$\{\text{signings on } G\} \leftrightarrows \{\text{2-lifts of } G\}$$

- We define a signing

$$s(u,v) = \begin{cases} 1, & \text{edges above } (u,v) \text{ are of type } (1); \\ -1, & \text{edges above } (u,v) \text{ are of type } (2). \end{cases}$$

for all $(u,v) \in E$.

## signings

- $(G, A, s)$ graph, adjacency matrix, signing. Define the **signed adjacency matrix** $A_s$ to be the matrix obtained from $A$ by replacing each non-zero entry $A_{(u,v)}$ with $s(u, v)$.
- define the **signed characteristic polynomial**, $f_s(x)$, to be the characteristic polynomial of $A_s$:

$$f_s(x) = \det(xI - A_s)$$

# signings





### Lemma (Bilu-Linial [1])

*(G,A,s) triple as before, $A_s$ the signed adjacency matrix of $(G,s)$, $\hat{G}$ the 2-lift of $G$ associated to $s$, and let $\hat{A}$ be the adjacency matrix of $\hat{G}$. Then*

$$\mathrm{Spec}(\hat{A}) = \mathrm{Spec}(A) \cup \mathrm{Spec}(A_s)$$

*as mult-sets. The multiplicty of each eigenvalue of $\hat{A}$ is the sum of its multiplicities in $A$ and $A_s$.*

**So the only new spectral data from a 2-lift comes from $A_s$; try and control these "new" eigenvalues.**

# signings

## Proof.

- Notice
$$\hat{A} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}$$
where $A_1 = A(V, s^{-1}(1))$ and $A_2 = A(V, s^{-1}(-1))$.
- If $v$ is an eigenvector of $A$ with eigenvalue $\mu$, then $\hat{v} = (v, v)^T$ is an eigenvector of $\hat{A}$ with value $\mu$.
- Similarly, if $u$ is an eigenvector of $A_s$ with value $\lambda$, then $\hat{u} = (u, -u)^T$ is an eigenvector of $\hat{A}$ with value $\lambda$.

$\square$

# matching polynomial

- $G = (V, E)$
- A **matching** on $G$ is a subset $M \subseteq E$ of edges with the property that every $v \in V$ is incident to at most one $e \in M$.
- Let $a(G; i)$ denote the number of matchings $M$ on $G$ with $|M| = i$ (we assign the value $a(G; 0) = 1$).
- Then we define the **matching polynomial of** $G$, denoted $\mu_G(x)$, as

$$\mu_G(x) := \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a(G; i) x^{n-2i}$$

where $n = |V|$.

# why we care

- We have (Godsil-Gutman [4])

$$\mathbb{E}_{s\in\{\pm 1\}^m}[f_s(x)] = \mu_G(x)$$

- (Heilmann-Lieb [5]) For every graph $G$, the zeros of $\mu_G(x)$ are real. If $G$ has maximum degree $d$, then the roots of $\mu_G(x)$ are bounded in absolute value by $2\sqrt{d-1}$.

# interlacing polynomials

- We say that a polynomial $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$ **interlaces** a polynomial $f(x) = \prod_{i=1}^{n}(x - \beta_i)$ if

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \ldots \leq \alpha_{n-1} \leq \beta_n$$

- We say that polynomials $f_1, \ldots, f_k$ have a **common interlacing** if there's a single polynomial $g$ which interlaces each of the $f_i$.
- Let $\beta_{i,j}$ be the $j$-th smallest root of $f_i$. Then an equivalent characterization of $f_1, \ldots, f_k$ having a common interlacing is the existence of numbers $\alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n$ such that

$$\beta_{i,j} \in [\alpha_{j-1}, \alpha_j]$$

for all $i$ and $j$.

# just bound the average

## Lemma (M.S.S. [7] Lemma 4.2)

*Let $f_1, \ldots, f_k$ be degree $n$ polynomials with real roots and positive leading coefficients. Define*

$$f_\emptyset = \sum_{i=1}^{k} f_i$$

*If $f_1, \ldots f_k$ have a common interlacing, then there exists some $i \in \{1, \ldots, k\}$ such that the largest root of $f_i$ is at most the largest root of $f_\emptyset$.*

## Proof.

Proof by picture. □

# interlacing families

Let $S_1, \ldots, S_m$ be finite sets, and for every assignment $(s_1, \ldots, s_m) \in S_1 \times \ldots \times S_m$, let $f_{s_1,\ldots,s_m}(x)$ be a real rooted, degree $n$ polynomial with positive leading coefficient. Given a partial assignment $(s_1, \ldots s_k) \in S_1 \times \ldots \times S_k$ with $k < m$, define

$$f_{s_1,\ldots,s_k} = \sum_{(s_{k+1},\ldots,s_m)\in S_{k+1}\times\ldots\times S_m} f_{s_1,\ldots,s_k,s_{k+1},\ldots,s_m}$$

and also

$$f_\emptyset = \sum_{(s_1,\ldots,s_m)\in S_1\times\ldots\times S_m} f_{s_1,\ldots,s_m}$$

Then we say that the polynomials $\{f_{s_1,\ldots,s_m}\}_{s_1,\ldots s_m}$ form an **interlacing family** if for all $k = 0, \ldots, m-1$ and all $(s_1, \ldots, s_k) \in S_1 \times \ldots \times S_k$, the polynomials

$$\{f_{s_1,\ldots,s_k,t}\}_{t\in S_{k+1}}$$

have a common interlacing.

# example

Let $m = 2$ and $S_1 = S_2 = \mathbb{Z}/2\mathbb{Z}$. Then the family has four polynomials:

$$\{f_{0,0}(x), f_{0,1}(x), f_{1,0}(x), f_{1,1}(x)\}$$

This family forms an interlacing family provided that each of the following sets has a common interlacing:

$$\{f_{0,0}(x), f_{0,1}(x)\}$$

$$\{f_{1,0}(x), f_{1,1}(x)\}$$

$$\{f_{0,0}(x) + f_{0,1}(x),\ \ f_{1,0}(x) + f_{1,1}(x)\}$$

# just bound the average: remix

### Lemma (M.S.S. [7] Lemma 4.4)

*Let $S_1, \ldots, S_m$ be finite sets, and let $\{f_{s_1,\ldots,s_m}\}$ be an interlacing family. Then there exists some assignment $(s_1, \ldots, s_m) \in S_1 \times \ldots \times S_m$ so that the largest root of $f_{s_1,\ldots,s_m}$ is at most the largest root of $f_\emptyset$.*

### Proof.

Induction on $k$; $k = 0$ is the base case and is handled by Lemma 4.2. $\square$

# real stability

We'll require the real rootedness of certain polynomials. These facts are established using ideas from the theory of "real stable polynomials."

$f \in \mathbb{R}[z_1, \ldots, z_n]$ is **real stable** if it is the zero polynomial or if

$$f(z_1, \ldots, z_n) \neq 0$$

whenever $\mathrm{Im}(z_i) > 0$ for all $i = 1, \ldots, n$.

Using ideas from real stability, Dedieu proved

## Lemma (Dedieu [3])

*Let $f_1, \ldots, f_k$ be polynomials of the same degree with positive leading coefficients. Then $f_1, \ldots, f_k$ have a common interlacing if and only if $\sum_{i=1}^{k} \lambda_i f_i$ is real rooted for all $\lambda_1, \ldots, \lambda_k$ non-negative which satisfy $\sum_{i=1}^{k} \lambda_i = 1$.*

# real stability

- Borcea and Branden [2] characterized linear operators on multivariate polynomials which preserve real stability.
- Using their characterization, M.S.S. proved

**Theorem (M.S.S. [7] Theorem 5.1)**

*The polynomial*

$$\sum_{s\in\{\pm 1\}^m} \left( \prod_{i:s_i=1} p_i \right) \left( \prod_{i:s_i=-1} (1-p_i) \right) f_s(x)$$

*is real rooted for all* $p_1, \ldots, p_m \in [0,1]$.

## the payoff

### Theorem (M.S.S. [7] Theorem 5.2)

$\{f_s\}_{s\in\{\pm1\}^m}$ *is an interlacing family.*

- By definition this requires showing that for every $k = 0, \ldots, m-1$ and every $(s_1, \ldots, s_k) \in \{\pm1\}^k$, the set
$$\{f_{s_1,\ldots,s_k,1}(x)\ ,\ f_{s_1,\ldots,s_k,-1}(x)\}$$
has a common interlacing.
- By Dedieu, it suffices to prove that for every $\mu \in [0,1]$
$$\mu f_{s_1,\ldots,s_k,1}(x) + (1-\mu) f_{s_1,\ldots,s_k,-1}(x)$$
is real rooted.

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  is real rooted.
- This is done by assigning special values to the $p_i$ in Theorem 5.1.

## the payoff

### Theorem (M.S.S. [7] Theorem 5.3)

*Let $G$ be a $d$-regular graph with adjacency matrix $A$. Then there exists a signing $s$ of $A$ so that all of the eigenvalues of $A_s$ are at most $2\sqrt{d-1}$.*

### Proof.

- $\mathbb{E}_{s\in\{\pm 1\}^m}[f_s(x)] = \mu_G(x)$
- $\{f_s\}_{s\in\{\pm 1\}^m}$ is an interlacing family.
- $\sum_{(s_1,\ldots,s_m)\in\{\pm 1\}^m} f_{s_1,\ldots,s_m} = f_\emptyset = \mathbb{E}_{s\in\{\pm 1\}^m}[f_s(x)]$
- since it's an interlacing family, there's a signing $(s_1,\ldots,s_m)$ so that the largest root of $f_{s_1,\ldots,s_m}$ is at most the largest root of $f_\emptyset = \mu_G(x)$.
- but the largest root of $\mu_G(x)$ is at most $2\sqrt{d-1}$

$\square$

## the payoff

**Theorem (M.S.S. [7] Theorem 5.5)**

*For every $d \geq 3$ there exists an infinite sequence of $d$-regular bipartite Ramanujan graphs.*

**Proof.**

- easy to see that the complete bipartite graph of degree $d$ is Ramanujan;
- by previous theorem, for every $d$-regular bipartite Ramanujan graph $G$, there's a 2-lift in which every non-trivial eigenvalue is at most $2\sqrt{d-1}$.
- as the 2-lift of any bipartite is bipartite, and the eigenvalues of bipartite graphs are symmetric about 0, this 2-lift is also $d$-regular bipartite Ramanujan.

□

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