Groups and Covers of Graphs

MS in Mathematics (Plan B) Thesis Defense

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Category of Directed Graphs

Covers

Category of Even Graphs

$\pi_1(G, v_0)$

Two Categories

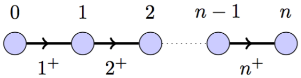
The Fiber Functor

Objects

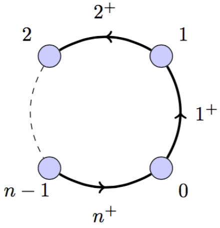
A **directed graph G** is a triple $(V(G), E(G), E(G) \xrightarrow{(t,h)} V(G)^2)$ with

- $V(G)$ a set consisting of the **vertices** of G;
- $E(G) \subseteq V(G)^2 := V(G) \times V(G)$ consisting of the **edges** of G
- A pair of maps $(t, h) : E(G) \to V(G)^2$ which are called the head and tail maps respectively. $t(e)$ is called the **tail** of e and $h(e)$ is called the **head** of e.

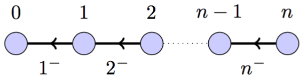
Objects (Examples)



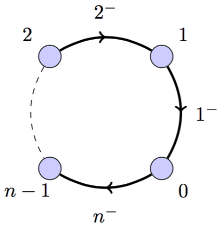
(a) Path_n^+



(b) Cycle_n^+



(a) Path_n^-



(b) Cycle_n^-

Morphisms

Given two directed graphs

- $G_1 = (V(G_1), E(G_1), E(G_1) \xrightarrow{(t_{G_1}, h_{G_1})} V(G_1)^2)$,
- $G_2 = (V(G_2), E(G_2), E(G_2) \xrightarrow{(t_{G_2}, h_{G_2})} V(G_2)^2)$,

a morphism $G_1 \xrightarrow{f} G_2$ is a pair of maps $V(G_1) \xrightarrow{f_V} V(G_2)$ and $E(G_1) \xrightarrow{f_E} E(G_2)$ such that the following diagram commutes:

$$\begin{tikzcd}
E(G_1) \arrow[r, "f_E"] \arrow[d, "(t_{G_1}, h_{G_1})"'] & E(G_2) \arrow[d, "(t_{G_2}, h_{G_2})"] \\
V(G_1)^2 \arrow[r, "f_v^2"'] & V(G_2)^2
\end{tikzcd}$$

$$\mathrm{Hom}(G_1, G_2) = \{\text{morphisms } f : G_1 \to G_2\}$$

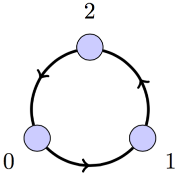
Morphisms (Examples)

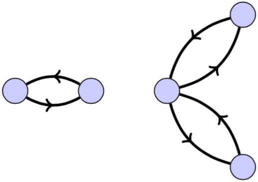
(1) The identity morphism $I \in \mathrm{Hom}(G, G)$: $I_V = \mathrm{Id}_{\mathrm{V(G)}}$, $I_E = \mathrm{Id}_{\mathrm{E(G)}}$.

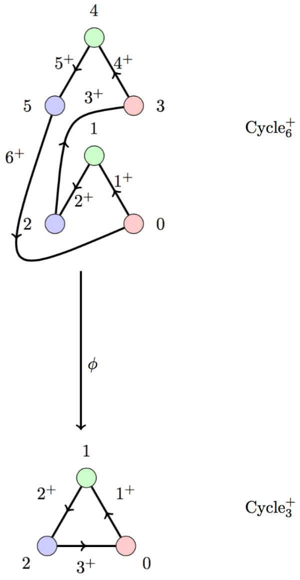
(2) Quotient morphisms $\psi^{\pm} \in \mathrm{Hom}(\mathsf{Path}_n^{\pm}, \mathsf{Cycle}_n^{\pm})$ which identifies the vertices 0 and n in $\mathsf{Path}_n^{\pm}$.

Connectedness

A directed graph G is **connected** if for any $v, w \in V(G)$ we can travel along edges (possibly in the wrong direction) to get from v to w.



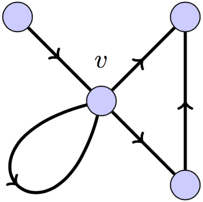




A cover of a graph G is a graph H which is "locally isomorphic" to G.

Edge Neighborhood

$G = (V(G), E(G), E(G) \xrightarrow{(t_G, h_G)} V(G)^2)$, $v \in V(G)$. The **(edge) neighborhood** N_v of v consists of one-third of each edge in $t_G^{-1}(v) \cup h_G^{-1}(v)$.



(Directed) Cover

Let G be a directed graph. A **cover** of G is a pair (H, ϕ) such that:

(1) $H = (V(H), E(H), E(H) \xrightarrow{(t_H, h_H)} V(H)^2)$ is a directed graph,

(2) $\phi \in \mathrm{Hom}(H, G)$ is a surjective morphism,

(3) for each $\hat{w} \in V(H)$, $\phi_E : N_{\hat{w}} \to N_{\phi_V(\hat{w})}$ is a bijection.

(H, ϕ) is a **finite cover** if and only if H is a finite directed graph.

Examples

(1) $G \xrightarrow{\mathrm{Id}_G} G$

(2) $\mathrm{Cycle}^{\pm}_{dn} \xrightarrow{\pi} \mathrm{Cycle}^{\pm}_{n}$, where π reduces things modulo n

(3) $\coprod_{i=1}^{d} G \to G$

Degree

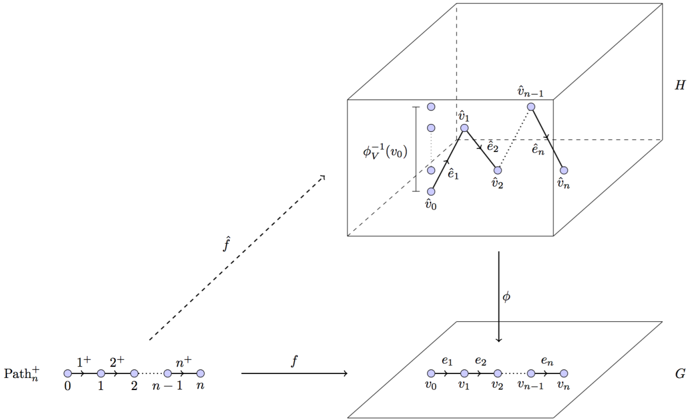
Let G be finite, connected and (H, ϕ) a finite cover. The degree of H over G is defined to be $|\phi^{-1}(v)|$ where $v \in V(G)$.

Path Lifting

Let (G, v_0) be a connected pointed graph, let (H, ϕ) be a cover, and let $f \in \mathrm{Hom}(\mathrm{Path}_n^+, G)$ satisfy $f_V(0) = v_0$. Then for each vertex $\hat{v}_0 \in \phi_V^{-1}(v_0)$ there exists a unique morphism $\hat{f} \in \mathrm{Hom}(\mathrm{Path}_n^+, H)$ such that $\hat{f}_V(0) = \hat{v}_0$ and $\phi \circ \hat{f} = f$. Such an $\hat{f}$ is called a **lift** of f.

Reversed Path Lifting

Let (G, v_0) be a connected pointed graph and suppose that $f \in \mathrm{Hom}(\mathrm{Path}_n^-, G)$ satisfies $f_V(n) = v_0$. Then for every $\hat{v}_0 \in \phi_V^{-1}(v_0)$ there is a unique $\hat{f} \in \mathrm{Hom}(\mathrm{Path}_n^-, H)$ such that $\hat{f}_V(n) = \hat{v}_0$ and $\phi \circ \hat{f} = f$.



A meaningless picture.

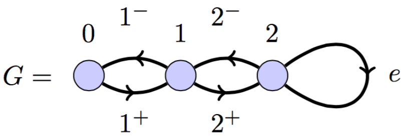
Transposition on a Directed Graph

Let $G = (V, E, E \xrightarrow{(t,h)} V^2)$ be a directed graph. A **transposition** on G is a permutation $\tau_G \in \mathrm{Sym}(E)$ which satisfies

(1) $\tau_G^2 = \mathrm{Id_E}$

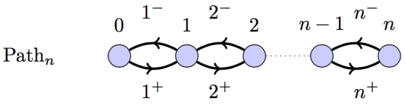
(2) $t(\tau_G(e)) = h(e)$ and $h(\tau_G(e)) = t(e)$

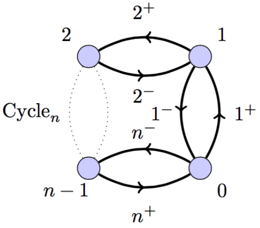
In other words a transposition on G associates to each edge an edge running in the opposite direction.



Even Graphs

A directed graph $G = (V, E, E \xrightarrow{(t,h)} V^2)$ is **even** if and only if there exists a transposition τ_G on G which is **fixed point free**. In other words $\tau(e) \neq e$ for every $e \in E(G)$.





Even Graph Morphisms

An **even morphism** from (G, τ_G) to (H, τ_H) is a directed graph morphism $\phi \in \mathrm{Hom}(G, H)$ such that the following diagram commutes:

$$\begin{tikzcd} E(G) \arrow[r, "\phi_E"] \arrow[d, "\tau_G"'] & E(H) \arrow[d, "\tau_H"] \\ E(G) \arrow[r, "\phi_E"'] & E(H) \end{tikzcd}$$

The set of even morphisms from (G, τ_G) to (H, τ_H) will be denoted $\mathrm{Hom}_\tau((G, \tau_G), (H, \tau_H))$.

(Even) Covers

Let (G, τ_G) be an even graph. A **(even) cover of** G is a pair $((H, \tau_H), \phi)$ which satisfies the following conditions:

(1) $H = (V(H), E(H), E(H) \xrightarrow{(t_H, h_H)} V(H)^2)$ is a directed graph;

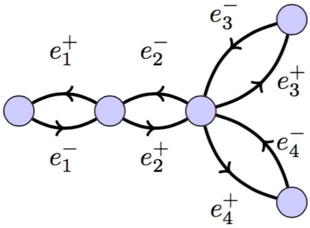
(2) (H, τ_H) is an even graph;

(3) $\phi \in \mathrm{Hom}_\tau((H, \tau_H), (G, \tau_G))$ is surjective;

(4) for every $\hat{w} \in V(H)$, $\phi_E : N_{\hat{w}} \to N_{\phi_V(\hat{w})}$ is a bijection.

Orientations

Let (G, τ_G) be an even graph. Then $\langle \tau_G \rangle \leq \mathrm{Sym}(E(G))$ acts on $E(G)$. Since τ_G is fixed point free, every orbit $\{e, \tau_G(e)\}$ has exactly two elements. Pick one arbitrarily and call it e^+, call the other e^-. Thus $E(G) = E^+(G) \coprod E^-(G)$.

$G =$ 

$$E(G) = \{e_1^+, e_2^+, e_3^+, e_4^+\} \coprod \{e_1^-, e_2^-, e_3^-, e_4^-\}$$

Cycles

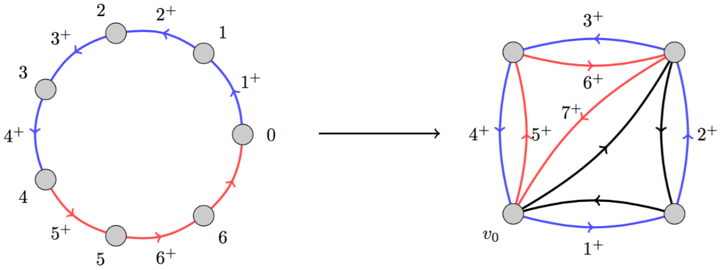
$\mathrm{Cycles}(G, v_0) :=$
$\cup_{n \geq 0}\{\text{even morphisms } \mathrm{Cycle}_n \xrightarrow{f} G \text{ with } f(0) = v_0\}$.

DCycles

$f \in \mathrm{Cycles}(G, v_0)$ yields a pair of morphisms: $\mathrm{Cycle}_n^+ \xrightarrow{f^+} G$ and $\mathrm{Cycle}_n^- \xrightarrow{f^-} G$. The resulting set of "directed cycles" is denoted $\mathrm{DCycles}(G, v_0)$.

Composing Cycles

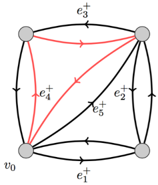
Given $f, g \in \mathrm{DCycles}(G, v_0)$, we define a composition law so that $f \cdot g$ means "first follow f, then follow g."



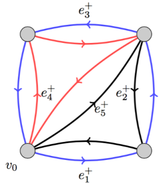
First blue, then red.
$\mathrm{DCycles}(G, v_0)$ is closed under this composition!

free : DCycles$(G, v_0) \to F(E^+(G))$

Let $F(E^+(G))$ be the free group on $E^+(G)$. The inverse symbol to e^+ is e^-. $f \in \mathrm{DCycles}(G, v_0)$, then $\mathrm{free}(f) \in F(E^+(G))$: write down the sequence of edges $f \in \mathrm{DCycles}(G, v_0)$ visits in order, get a word in $F(E^+(G))$.

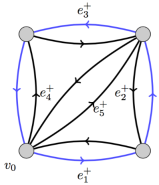


$\mathrm{free}(f^+) = e_4^+ e_3^- e_5^-$



$\mathrm{free}(f^+ \cdot g^-) = e_4^+ e_3^- e_5^- e_1^+ e_2^- e_3^+ e_4^-$

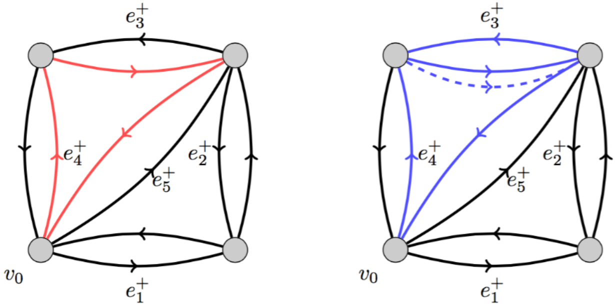
$\mathrm{free}(g^- \cdot f^+) = e_1^+ e_2^- e_3^+ e_4^- e_4^+ e_3^- e_5^-$



$\mathrm{free}(g^-) = e_1^+ e_2^- e_3^+ e_4^-$

Homotopy Equivalence

Introduce an equivalence relation $\sim$ on $\mathrm{DCycles}(G, v_0)$:
$f \sim g$ if and only if $\mathrm{free}(f) = \mathrm{free}(g)$.



$[f] = \{g \in \mathrm{DCycles}(G, v_0) : g \sim f\}$

The Fundamental Group: $\pi_1(G, v_0)$

- $\pi_1(G, v_0) := \{[f] : f \in \mathrm{DCycles}(G, v_0)\}$
- $[f][g] = [f \cdot g]$
- $[f^{\pm}]^{-1} = [f^{\mp}]$
- $[I]$ is the identity where I is the trivial cycle

We get a group, I promise

$\mathbf{Cov}(G)$

Fix G, finite, even, connected graph, and fix $v_0 \in V(G)$. We define a category $\mathbf{Cov}(G)$ which has objects which are finite, even covers $((H, \tau_H), \phi)$ of G.
Given $((H_1, \tau_{H_1}), \phi_1), ((H_2, \tau_{H_2}), \phi_2) \in \mathbf{Cov}(G)$, a morphism between them is a map f which satisfies:

(1) $f \in \mathrm{Hom}_\tau((H_1, \tau_{H_1}), (H_2, \tau_{H_2}))$;

(2) $\phi_2 = \phi_1 \circ f$

$\pi_1\mathbf{Set}(G)$

We define a category $\pi_1\mathbf{Set}(G)$ which has as objects finite $\pi_1(G, v_0)$-sets: i.e. finite sets on which $\pi_1(G, v_0)$ acts. In other words $F \in \pi_1\mathbf{Set}(G)$ if and only if there exists a group homomorphism $\Phi : \pi_1(G, v_0) \to \mathrm{Sym}(F)$, called the permutation representation.

Simplifying Notation: $[C] \cdot x := \Phi([C])(x)$,

$F_1, F_2 \in \pi_1\mathbf{Set}(G)$ with perm. reps. Φ_1 and Φ_2. Then f is a morphism from F_1 to F_2 if and only if the following hold:

(1) $f : F_1 \to F_2$ is a set map;

(2) $f([C] \cdot x) = [C] \cdot f(x)$ for every $[C] \in \pi_1(G, v_0)$ and for every $x \in F_1$.

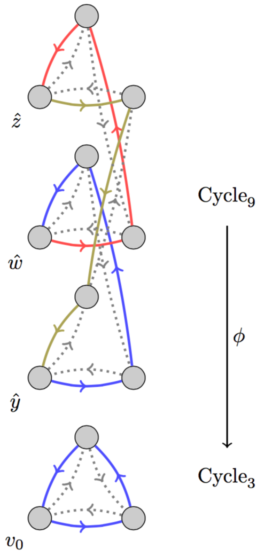
We define a (covariant) functor $\mathcal{F} : \mathbf{Cov}(G) \to \pi_1\mathbf{Set}(G)$ called **the fiber functor**.

"You tell me a finite even cover, I'll tell you a $\pi_1(G, v_0)$*-set."*

Early punchline: $\mathcal{F}(((H, \tau_H), \phi)) = \phi^{-1}(v_0)$.

Must therefore demonstrate $\Phi : \pi_1(G, v_0) \to \mathrm{Sym}(\phi^{-1}(v_0))$ a group homomorphism.

(1) Cycles Lift to Paths



Let $f^+ \in \text{DCycles}(G, v_0)$. Define

$$L_{f^+} : \phi_V^{-1}(v_0) \to \phi_V^{-1}(v_0)$$

as follows:

(1) $f^- : \text{Path}_n^- \to \text{Cycle}_n^- \to G$ satisfies $f^-(n) = v_0$

(2) Fix $\hat{y} \in \phi^{-1}(v_0)$.

(3) By Path Lifting, there is a unique morphism $\hat{f}_{\hat{y}}^- \in \mathrm{Hom}(\text{Path}_n^-, H)$ with $\hat{f}_{\hat{y}}^-(n) = \hat{y}$ and $f^- = \phi \circ \hat{f}_{\hat{y}}^-$.

(4) $L_{f^+}(\hat{y}) := \hat{f}_{\hat{y}}^-(0)$.

Fancy pants way of saying the following: there is a unique lift of f^+ which **ends** at $\hat{y}$: send $\hat{y}$ to where that unique lift began.

Let $g^- \in \text{DCycles}(G, v_0)$. Define

$$L_{g^-} : \phi_V^{-1}(v_0) \to \phi_V^{-1}(v_0)$$

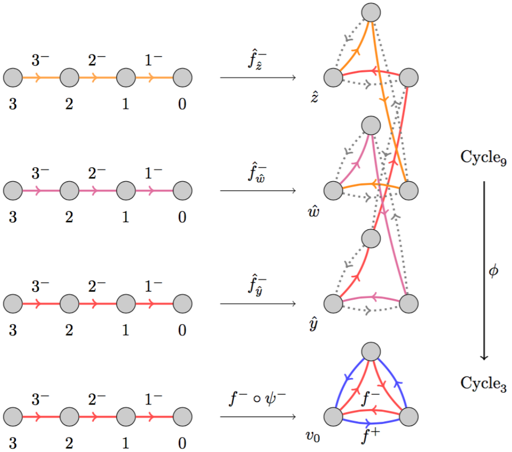
as follows:

(1) $g^+ : \text{Path}_n^+ \to \text{Cycle}_n^+ \to G$ satisfies $g^+(0) = v_0$

(2) Fix $\hat{y} \in \phi^{-1}(v_0)$.

(3) By Path Lifting, there is a unique morphism $\hat{g}_{\hat{y}}^+ \in \mathrm{Hom}(\text{Path}_n^+, H)$ with $\hat{g}_{\hat{y}}^+(0) = \hat{y}$ and $g^+ = \phi \circ \hat{g}_{\hat{y}}^+$.

(4) $L_{g^-}(\hat{y}) := \hat{g}_{\hat{y}}^+(n)$.



$$L_{f^+} = (\hat{y}\hat{z}\hat{w}).$$

I Promise

Here is a list of things which I promise are true but which I won't go into now:

(1) $(L_{f^\pm})^{-1} = L_{f^\mp}$ for every $f^\pm \in \mathrm{DCycles}(G, v_0)$. **Hence** $L_{f^+} \in \mathrm{Sym}(\phi^{-1}(v_0))$

(2) If $[f] = [g] \in \pi_1(G, v_0)$, then $L_f = L_g \in \mathrm{Sym}(\phi^{-1}(v_0))$.

(3) Let $\Phi : \pi_1(G, v_0) \to \mathrm{Sym}(\phi^{-1}(v_0))$ be given by

$$\Phi([f]) = L_f$$

Then Φ is a group homomorphism.

Therefore $\phi^{-1}(v_0)$ is a $\pi_1(G, v_0)$ -set!

$\mathcal{F}$ on Morphisms

Let h be a morphism from $((H_1, \tau_{H_1}), \phi_1)$ to $((H_2, \tau_{H_2}), \phi_2)$ in $\mathbf{Cov}(G)$. Recall that this means

(1) $h \in \mathrm{Hom}_\tau((G_1, \tau_{G_1}), (H_2, \tau_{H_2}))$;

(2) $\phi_1 = \phi_2 \circ h$

By definition:

(1) $\mathcal{F}(((H_1, \tau_{H_1}), \phi_1)) = \phi_1^{-1}(v_0)$

(2) $\mathcal{F}(((G_2, \tau_{G_2}), \phi_2)) = \phi_2^{-1}(v_0)$

So we need to define $\mathcal{F}(h)$ which is a $\pi_1\mathbf{Set}(G)$ morphism from $\phi_1^{-1}(v_0)$ to $\phi_2^{-1}(v_0)$.

$\mathcal{F}$ on Morphisms

Let $\hat{w} \in \phi_1^{-1}(v_0)$. Then

$$\mathcal{F}(h)(\hat{w}) = h(\hat{w}) \in \phi_2^{-1}(v_0)$$

Prove that

$$\mathcal{F}(h)([C] \cdot \hat{w}) = [C] \cdot (\mathcal{F}(h)(\hat{w}))$$

for every $[C] \in \pi_1(G, v_0)$.

KEY: Uniqueness of lifts!

Thanks:

- Dr. Chris Hall
- Dr. John Hitchcock
- Dr. Tyrrell McAllister
- Lori Dockter
- Department of Mathematics
- Viewers like YOU

Questions

Now we describe a functor $\mathcal{G} : \pi_1\mathbf{Set}(G) \to \mathbf{Cov}(G)$ called the reverse functor.

"You give me a $\pi_1(G, v_0)$-set, I'll give you a finite even cover"

First fix a spanning tree T for G. Let $X(G) := E(G) \setminus E(T)$ called the set of excess edges. Since G has an orientation, so does $X(G) = X^+(G) \coprod X^-(G)$. A few preliminaries:

Unique Cycles Through Excess Edges

For every $x^\pm \in X(G)$, there is a unique homotopy class $[C_{x^\pm}]$ of cycles which pass through $x^\pm$

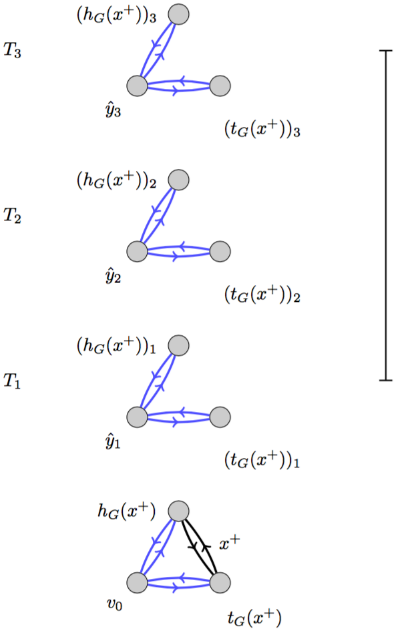
Free Generators of $\pi_1(G, v_0)$

Let $X(G) = \{x_1^+, \ldots, x_g^+\} \coprod \{x_1^-, \ldots, x_g^-\}$. Then $[C_{x_1^+}], \ldots, [C_{x_g^+}]$ and $[C_{x_1^-}], \ldots, [C_{x_g^-}]$ freely generate $\pi_1(G, v_0)$.

Let F be a finite $\pi_1(G, v_0)$-set with $|F| = d$. After relabeling, $F = \{1, \ldots, d\}$. Let Φ be the permutation representation for $\pi_1(G, v_0)$ acting on F.
Notation: Allow $[C_{x_j^{\pm}}](\ell)$ to denote $\Phi([C_{x_j^{\pm}}])(\ell)$.
Start by defining

$$\tilde{H} = \coprod_{i=1}^{d} T_i$$

where T_i is isomorphic to T for every $i = 1, \ldots, d$:
for every $i = 1, \ldots, d$ there exists $\varphi_i \in \mathrm{Hom}(T_i, T)$ which is a bijection.
Further Notation: $\hat{v}_i := \varphi_i^{-1}(v)$ for $v \in V(G)$ for every $i = 1, \ldots, d$.

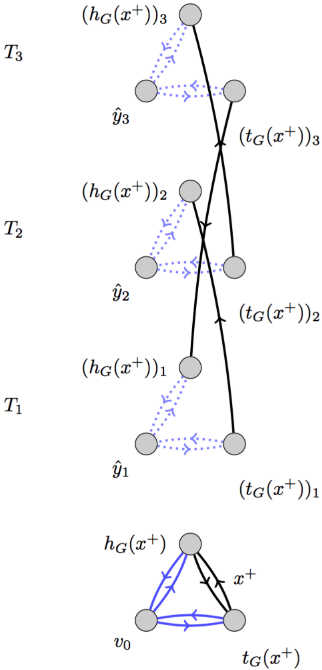


$$\tilde{H} = T_1 \coprod T_2 \coprod T_3$$

We finish by forming the edges above $x_j^{\pm}$ for every $x_j^{\pm} \in X(G)$: for every $\ell \in \{1, \ldots, d\}$ we'll form $(\hat{x}_j^{\pm})_\ell$ with

$$t_H((\hat{x}_j^{\pm})_\ell) = (t_G(x_j^{\pm}))_\ell$$
$$h_H((\hat{x}_j^{\pm})_\ell) = (h_G(x_j^{\pm}))_{[C_{x_j^{\pm}}](\ell)}$$

For the example which follows, suppose that $\Phi([C_{x^+}]) = (123)$.


$(h_G(x^+))_3$
T_3
$\hat{y}_3$
$(t_G(x^+))_3$
$(h_G(x^+))_2$
T_2
$\hat{y}_2$
$(t_G(x^+))_2$
$(h_G(x^+))_1$
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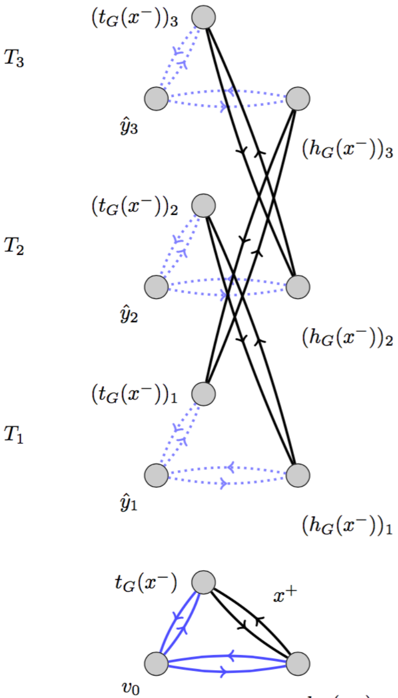
x^+
v_0
$t_G(x^+)$


$(t_G(x^-))_3$
T_3
$\hat{y}_3$
$(h_G(x^-))_3$
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