Math 2120B Assignment 2

1. (a) Extend the linearly independent set

$$I = \left\{ \begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} \right\}$$

   to a basis for $M_{22}(\mathbb{R})$.

   (b) Find a basis contained in the spanning set

$$G = \{x^2 + 3, x + 2, x^2 - 2x - 1, x^2 + x\}$$

   for $P_2(\mathbb{R})$.

   (c Are the following subsets of V linearly independent or linearly dependent? If dependent, find a linear dependence relation.

   - $V = \mathcal{F}([0,1], \mathbb{R}); S = \{\frac{1}{x^2+x-6}, \frac{1}{x^2-5x+6}, \frac{1}{x^2-9}\}$
   - $V = \mathcal{F}(\mathbb{R}, \mathbb{R}); S = \{x, e^x, e^{2x}\}$.

2. Find a basis and calculate the dimension of the following subspaces W of V:

   (a) $V = P_3(\mathbb{R})$, $W = \{p(x) \in P_3(\mathbb{R}) | p(2) = 0\}$.

   (b) $V = M_{22}(\mathbb{R})$, $W = \{A \in M_{22}(\mathbb{R}) | AB = BA\}$ where

$$B = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

3. If $A = \{a_1, \ldots, a_n\}$ is a set then $\mathcal{F}(A, \mathbb{R})$ is a $\mathbb{R}$ vector space as we proved in class. For each $a \in A$ let $f_a \in \mathcal{F}(A, \mathbb{R})$ be the function with

$$f_a(b) = \delta_{a,b}, a, b \in A$$

   where $\delta_{a,b}$ is the Kronecker delta function. Show that $\{f_a | a \in A\}$ is a basis for the real vector space $\mathcal{F}(A, \mathbb{R})$.

4. (a) Prove that if W_1 and W_2 are finite-dimensional subspaces of a vector space V, then the subspace $W_1 + W_2$ is finite dimensional, and $\dim(\mathrm{W}_1 + \mathrm{W}_2) = \dim(\mathrm{W}_1) + \dim(\mathrm{W}_2) - \dim(\mathrm{W}_1 \cap \mathrm{W}_2)$. [Hint: Start with a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for $W_1 \cap W_2$ and extend this set to a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{v}_1, \ldots, \mathbf{v}_m\}$ for W_1 and to a basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, \mathbf{w}_1, \ldots \mathbf{w}_p\}$ for W_2.]

   (b) Let W_1 and W_2 be finite-dimensional subspaces of a vector space V and let $V = W_1 + W_2$. Deduce that V is the direct sum of W_1 and W_2 if and only if $\dim(\mathrm{V}) = \dim(\mathrm{W}_1) + \dim(\mathrm{W}_2)$.

5. Let $V = M_{22}(\mathbb{R})$,

$$W_1 = \left\{ \begin{bmatrix} a & b \\ c & a \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \qquad W_2 = \left\{ \begin{bmatrix} 0 & a \\ -a & b \end{bmatrix} : a, b \in \mathbb{R} \right\}$$

   Give a basis for W_1 and W_2. Find $W_1 + W_2$ and $W_1 \cap W_2$ and a basis for each. What are the dimensions of each of W_1, W_2, $W_1 + W_2$, and $W_1 \cap W_2$?