Determinant Formula for the Ihara Zeta Function of a Graph

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In this report we study the proof that the Ihara Zeta function of a locally finite graph has a convenient formula given in terms of a determinant of certain "Hecke Operators" on the graph. The formula shows, in particular, that the zeta function is the inverse of a polynomial-a fact which is highly non-obvious from first definitions. Where possible, we point out similarities to the development given by Bass to the presentation given in Math 9148 A-Expander Graphs.

1 Graphs

A **Graph** X is a set also denoted by X of **vertices**, together with a set EX of (oriented) **edges**, **point maps**

$$\partial_0,\ \partial_1 : EX \to X$$

and an **orientation reversal** map

$$J : EX \to EX$$

i.e., a map $J : EX \to EX$ which satisfies

(1) $J(e) := \bar{e}$;

(2) $\partial_i \circ J = \partial_{1-i}$ for $i = 0, 1$;

(3) $\bar{\bar{e}} = e \neq \bar{e}$ for every $e \in EX$.

In other words, ∂_0 and ∂_1 are the respective head and tail maps as defined in class, and J is our fixed point free transposition. All this together is to say that for the purpose of this document, a "graph" is what we (in class) might call a *symmetric* graph.
Given $x \in X$, we define for $j = 0, 1$

$$E_j(x) = \{e \in EX : \partial_j(e) = x\}$$

Then we see that $JE_j(x) = E_{1-j}(x)$ and we therefore define

$$d(x) := q(x) + 1 := |E_0(x)| = |E_1(x)|$$

the **degree** of $x \in X$. We'll assume henceforth that X is **locally finite**; i.e., that $d(x) < \infty$ for every $x \in X$.

1.1 Paths

An (edge) path is a sequence $c = (e_1, \dots, e_m)$ of edges satisfying

$$\partial_0(e_i) = \partial_1(e_{i-1})$$

for each $1 < i \leq m$. We say that c is a path of **length** m extending from $a := \partial_0 e_1$ to $b := \partial_1 e_m$. The path $\bar{c} = (\bar{e_m}, \dots m\bar{e_1})$ extends from b to a. c is **reduced** if $e_i \neq \bar{e_{i-1}}$ for all $1 < i \leq m$. c is **closed** if $a = b$.

2 Hecke Operators

Let R be a ring and S a set. We'll allow $R^{(S)}$ to denote the free R-module with basis S.
Let X be a graph with data as described in the previous section. We define

$$C_0 := \mathbb{C}^{(X)},\ C_1 := \mathbb{C}^{(EX)}$$

If we let $G := \mathrm{Aut}(X)$ be the group of automorphisms of X, then C_0 and C_1 are naturally regarded as modules over the group algebra $\mathbb{C}[G]$. A **Hecke Operator** on X is a $\mathbb{C}[G]$-linear map between C_0 and C_1. Those which will be of interest to us may be represented schematically by

$$C_1 \xrightarrow{J,T} C_1 \underset{\sigma_0,\sigma_1}{\overset{\partial_0,\partial_1}{\rightleftarrows}} C_0 \xrightarrow{Q,\delta_m,\Delta} C_0$$

We defined ∂_0, ∂_1, and J in the previous section. Define

$$\sigma_j(x) = \sum_{e \in E_j(x)} e$$

for every $x \in X$ and $j = 0, 1$. Thus we see that $\partial_j J = \partial_{1-j}$ but also that $J\sigma_j = \sigma_{1-j}$ for $j = 0, 1$.
For $e \in EX$, define

$$T(e) = \sum_{\substack{(e,e_1): \\ \text{red.}}} e_1$$

where we sum over all *reduced* paths (e, e_1). We similarly define

$$T^m(e) = \sum_{\substack{(e,e_1,\ldots,e_m) \\ red.}} e_m$$

where again we sum over reduced paths only.
For $x \in X$, define

$$\delta_m(x) = \sum_{\substack{(e_1,\ldots,e_m) \\ \text{red.},\ \partial_0 e_1 = x}} \partial_1 e_m$$

where here we sum over all reduced paths $(e_1, \ldots, e_m)$ which start at x. Intuitively, $\delta_m(x)$ encodes the possible vertices at which we may end up after embarking on an m step non-backtracking walk beginning from x. Of special import are the cases $\delta_0 = I$ and $\delta_1 := \delta$; plain old δ is called the **adjacency operator**. We finally define $Q(x) := q(x)x$ where $d(x) = q(x)+1$ is the degree of x as we defined in the first section. Thus we arrive at the first of several functional equations.

Proposition 1. *The following functional equations hold*

$$\delta^2 = \delta_2 + Q + I;$$
$$\delta\delta_m = \delta_{m+1} + Q\delta_{m-1},$$

for all $m \geq 2$.

Proof. We'll prove the first; the second follows an identical line of reasoning. Let $x \in X$. Then

$$\begin{aligned}\delta^2(x) &= \delta\left(\sum_{\substack{e_1:\\ \partial_0 e_1 = x}} \partial_1 e_1\right)\\ &= \sum_{\substack{e_1:\\ \partial_0 e_1 = x}} \delta(\partial_1 e_1)\\ &= \sum_{\substack{e_1:\\ \partial_0 e_1 = x}} \left(\sum_{\substack{e_1':\\ \partial_0 e_1' = \partial_1 e_1}} \partial_1 e_1'\right)\end{aligned}$$

Thus there are two possibilities: in moving from e_1 to $e_1\prime$, either we backtrack or we don't. For every backtracking edge, we generate x in the final sum and there are $d(x)$ many edges which return to x from $\partial_1 e_1$. Thus the above sum is equal to

$$d(x)x + \sum_{\substack{(e_1, e_1'):\\ \text{red., } \partial_0 e_1 = x}} \partial_1 e_1' = d(x)x + \delta_2(x)$$

We're finished upon recalling that $d(x)x = (q(x)+1)x = Q(x) + I(x)$. □

2.1 Laplacian

Let u be an indeterminate and define

$$\Delta(u) := I - \delta u + Q u^2$$

At the value $u = 1$ we obrain the **Laplacian** of X:

$$\Delta = I + Q - \delta$$

Applying the formulas in Proposition 1, we obtain the following result:

Claim 1.

$$\Delta(u)\left(\sum_{m \geq 0} \delta_m u^m\right) = (1 - u^2)I$$

2.2 Several Functional Equations

Here we prove several functional equations involving the Hecke operators represented in our shcematic.

Proposition 2. *The following hold:*

$$\begin{aligned}\partial_0\sigma_0 = \partial_1\sigma_1 = Q + I,\\ \partial_1\sigma_0 = \partial_0\sigma_1 = \delta.\end{aligned}$$

Proof. We'll prove the first of each of these, claiming that the second follows an identical line of reasoning. Let $x \in X$.

$$\begin{aligned}
(\partial_0 \sigma_0)(x) &= \partial_0 \left(\sum_{e \in E_0(x)} e \right) \\
&= \sum_{e \in E_0(x)} \partial_0 e \\
&= |E_{0(x)}| x \\
&= d(x)x = (Q + I)(x)
\end{aligned}$$

The essential points being to recall that, by definition, $E_j(x) = \partial_j^{-1}(x)$ and $Q(x) = q(x)x = (d(x)+1)(x)$. Similarly,

$$\begin{aligned}
(\partial_1 \sigma_0)(x) &= \partial_1 \left(\sum_{e \in E_0(x)} e \right) \\
&= \sum_{e \in E_0(x)} \partial_1(e) \\
&= \sum_{\substack{e: \\ \partial_0 e = x}} \partial_1 x = \delta(x)
\end{aligned}$$

□

Proposition 3. *The following holds:*

$$\sigma_0 \partial_1 = T + J$$

Proof.

$$\begin{aligned}
(\sigma_0 \partial_1) e &= \sum_{e' \in E_0(\partial_1 e)} e' \\
&= \sum_{\substack{e': \\ \partial_0 e' = \partial_1 e}} e' \\
&= \sum_{(e, e')} e' \\
&= \bar{e} + \sum_{\substack{(e,e'): \\ red.}} e' \\
&= Je + Te
\end{aligned}$$

Here we'll make the simple observation that a path (e, e') can be of two essential types: one which backtracks (i.e. one for which $e' = \bar{e}$) and one which does not (one for which (e, e') is reduced). □

Corollary 1. *The following holds:*

$$\sigma_0 \partial_0 = TJ + I$$

Proof.

$$\sigma_0 \partial_0 = \sigma_0 \partial_1 J = (T + J)J = TJ + J^2 = TJ + I$$

By Proposition 3, the fact that J "swaps heads and tails:" $\partial_0 = \partial_1 J$, and the fact that J is an involution: $J^2 = I$. □

Recall that u is a complex indeterminate. We define

$$\begin{aligned}\partial(u) &= \partial_0 u - \partial_1 \\ \sigma(u) &= \sigma_0 u\end{aligned}$$

Proposition 4. *We have*

$$\partial(u)\sigma(u) = \Delta(u) - (1-u^2)I$$

Proof. First, Proposition 2 gives us that

$$\begin{aligned}\partial(u)\sigma_0 &= (\partial_0 u - \partial_1)\sigma_1 \\ &= \partial_0\sigma_0 u - \partial_1\sigma_0 \\ &= (Q+I)u + \delta\end{aligned}$$

Then by the definition of Δ,

$$\Delta(u) = I - \delta u + Qu^2$$

we see

$$\begin{aligned}\partial(u)\sigma(u) &= ((Q+I)u - \delta)u \\ &= (Q+I)u^2 - \delta u \\ &= Qu^2 + Iu^2 - \delta u \\ &= \Delta(u) - (1-u^2)I\end{aligned}$$

$\square$

Corollary 2. *The following holds:*

$$\sigma_0\partial(u) = (T+J)(Ju - I)$$

Proof. This follows immediately from Corollary 1 and Proposition 3:

$$\begin{aligned}\sigma_0\partial(u) &= \sigma_0(\partial_0 u - \partial_1) \\ &= \sigma_0\partial_0 u - \sigma_0\partial_1 \\ &= (TJ+I)u - T + J \\ &= (T+J)(Ju - I)\end{aligned}$$

$\square$

3 Matrices of Endomorphisms

Let

$$C := C_0 \oplus C_1$$

Then we have a decomposition of $E := \mathrm{End}_{\mathbb{C}[G]}(C)$ as

$$E = \begin{bmatrix} E_0 & H_1 \\ H_0 & E_1 \end{bmatrix}$$

where

$$E_i := \mathrm{End}_{\mathbb{C}[G]}(C_i), \quad H_i := \mathrm{Hom}_{\mathbb{C}[G]}(C_i, C_{1-i})$$

for $i = 0, 1$. This decomposition in mind, we consider L and M elements of $E[u]$ given by

$$L := \begin{bmatrix} (1-u^2)I_0 & \partial(u) \\ 0 & I_1 \end{bmatrix}$$

$$M := \begin{bmatrix} I_0 & -\partial(u) \\ \sigma(u) & (1-u^2)I_1 \end{bmatrix}$$

Theorem 1. *The following holds:*

$$LM = \begin{bmatrix} \Delta(u) & 0 \\ \sigma(u) & (1-u^2)I_1 \end{bmatrix} \in E[u]$$

Proof.

$$\begin{aligned} LM &= \begin{bmatrix} (1-u^2)I_0 & \partial(u) \\ 0 & I_1 \end{bmatrix} \begin{bmatrix} I_0 & -\partial(u) \\ \sigma(u) & (1-u^2)I_1 \end{bmatrix} \\ &= \begin{bmatrix} (1-u^2)I_0 + \partial(u)\sigma(u) & 0 \\ \sigma(u) & (1-u^2)I_1 \end{bmatrix} \\ &= \begin{bmatrix} \Delta(u) & 0 \\ \sigma(u) & (1-u^2)I_1 \end{bmatrix} \end{aligned}$$

where the final equality follows from Proposition 4. □

Theorem 2. *The following holds:*

$$ML = \begin{bmatrix} (1-u^2)I_0 & 0 \\ \sigma(u)(1-u^2) & (I_1 - Tu)(I_1 - Ju) \end{bmatrix}$$

Proof.

$$\begin{aligned} ML &= \begin{bmatrix} I_0 & -\partial(u) \\ \sigma(u) & (1-u^2)I_1 \end{bmatrix} \begin{bmatrix} (1-u^2)I_0 & \partial(u) \\ 0 & I_1 \end{bmatrix} \\ &= \begin{bmatrix} (1-u^2)I_0 & 0 \\ \sigma(u)(1-u^2)I_0 & \sigma(u)\partial(u) + (1-u^2)I_1 \end{bmatrix} \end{aligned}$$

We'll examine just the lower right entry above. By Corollary 2 and the fact that $J^2 = I_1$, we have

$$\begin{aligned} \sigma(u)\partial(u) + (1-u^2)I_1 &= (T+J)(Ju - I_1)u + (1-u^2)I_1 \\ &= TJu^2 - Tu + I_1u^2 - Ju + I_1 - I_1u^2 \\ &= (I_1 - Tu)(I_1 - Ju) \end{aligned}$$

This shows that ML has the form claimed in the Theorem. □

4 Determinants

Claim 2. *From the fact that $L, LM, ML \in I + uE$, it follows that M is invertible in $E[[u]]$.*

But then Claim 2 tells us something very useful: the calculation

$$M^{-1}(ML)M = LM$$

shows that ML and LM have the same determinant. So we begin to compute:

$$\begin{aligned}
\det(ML) &= \det \begin{bmatrix} (1-u^2)I_0 & 0 \\ \sigma(u)(1-u^2) & (I_1 - Tu)(I_1 - Ju) \end{bmatrix} \\
&= \det((1-u^2)I_0)\det((I_1 - Tu)(I_1 - Ju)) \\
&= (1-u^2)^{|X|}\det(I_1 - Tu)\det(I_1 - Ju)
\end{aligned}$$

We recall that I_0 is the identity operator on $C_0 = \mathbb{C}^{(X)}$, the free $\mathbb{C}$-module on X which clearly has rank $|X|$. This explains why $\det((1-u^2)I_0) = (1-u^2)^{|X|}$.
We consider the factor $\det(I_1 - Ju)$. The Hecke Operator $I_1 - Ju$ lives in

$$E_1 = \mathrm{End}_{\mathbb{C}[G]}(C_1) = \mathrm{End}_{\mathbb{C}[G]}(\mathbb{C}^{(EX)})$$

Using a similar idea as before, we'll decompose E_1 according to the corresponding decomposition of $EX = EX^+ \coprod EX^-$ into edges and their inverses (e and $\bar{e} = J(e)$). Thus,

$$E_1 = \begin{bmatrix} EX^{++} & EX^{+-} \\ EX^{-+} & EX^{--} \end{bmatrix}$$

where EX^{++} and EX^{--} consist of endomorphisms on EX^+ and EX^- respectively, and EX^{+-} (respectively EX^{-+}) consist of those morphisms from $EX^+ \to EX^-$ (respectively $EX^- \to EX^+$). With respect to this decomposition, we can write

$$I_1 - Ju = \begin{bmatrix} I & -Iu \\ -Iu & I \end{bmatrix}$$

and apply a neat trick: since

$$\det \begin{bmatrix} I & Iu \\ 0 & I \end{bmatrix} = 1$$

We have that

$$\begin{aligned}
\det(I_1 - Ju) &= \det\left(\begin{bmatrix} I & Iu \\ 0 & I \end{bmatrix} \begin{bmatrix} I & -Iu \\ -Iu & I \end{bmatrix} \right) \\
&= \det \begin{bmatrix} I - Iu^2 & 0 \\ -Iu & I \end{bmatrix} \\
&= \det(I(1-u^2)) = (1-u^2)^{|EX|/2}
\end{aligned}$$

Incorporating this with what we found previously, we see that

$$\det(ML) = (1-u^2)^{|X|}(1-u^2)^{|EX|/2}\det(I_1 - Tu)$$

On the other hand, we have

$$\det(LM) = \det \begin{bmatrix} \Delta(u) & 0 \\ \sigma(u) & (1-u^2)I_1 \end{bmatrix}$$

$$= \det(\Delta(u))\det((1-u^2)I_1)$$

$$= \det(\Delta(u))(1-u^2)^{|EX|}$$

Recall that I_1 is a $\mathbb{C}[G]$-endomorphism of $C_1 = \mathbb{C}^{(EX)}$ which has rank $|EX|$. But we said that $\det(ML) = \det(LM)$. Thus our results are summarized in the following Theorem.

Theorem 3. *The following holds*

$$\det(I_1 - Tu) = (1-u^2)^{|EX|/2-|X|}\det(\Delta(u))$$

5 Ihara Zeta Function of a Graph

A path $c = (e_1, \ldots, e_m)$ is said to have a **tail** if $e_m = \bar{e_1}$. The closed path c is called a **primitive** if it is reduced, has no tail, and $c \neq d^k$ for $k > 1$. The last of these requirements enforces that primes are traversed only once. Finally, we'll identify all primitive paths which use the same sequence of edges, albeit in different orders (so that, for example $(e_1, e_2, \ldots, e_m)$ is identified with $(e_2, e_3, \ldots, e_m, e_1)$, etc.) This effectively removes the dependence on a particular starting point. A **prime** in X is an equivalence class $[p]$ of primitive closed paths. Recall that if $p = (e_1, \ldots, e_m)$, the length $\nu(p) = \nu([p]) = m$.
The **Ihara Zeta Function**, $\zeta_X(u)$ of a locally finite, connected graph X is defined by

$$\zeta_X(u) = \zeta(u, X) = \prod_{[p]} (1 - u^{\nu(p)})^{-1}$$

This brings us to the remarkable Theorem which is the subject of our study.

Theorem 4 (Ihara)**.** *Let X be a locally finite, connected graph. Let $r := \mathrm{rank}(\pi_1(X)) = |EX|/2 - |X| + 1$ be the rank of the fundamental group $\pi_1(X)$. Then we have the following:*

$$\zeta_X(u)^{-1} = (1-u^2)^{r-1}\det(I - \delta u + Qu^2)$$

Along the way to proving this Theorem is an important, but straightforward intermediate result.

Claim 3. *We have*

$$\zeta_X(u)^{-1} = \det(I_1 - Tu)$$

Toward proving Claim 3, one considers the "edge zeta function" of X.
Finally we are finished: Ihara's Theorem follows as an immediate corollary of Claim 3 and Theorem 3.

References

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