Math 2120B Assignment 1

1. Prove that $V = \mathbb{R}^2$ with addition defined as

$$(x_1, y_1) \oplus (x_2, y_2) \equiv (x_1 + x_2, y_1 + y_2 + 1), (x_1, y_1), (x_2, y_2) \in V$$

and scalar multiplication defined as

$$a \odot (x, y) \equiv (ax, ay + a - 1), (x, y) \in V, a \in \mathbb{R}$$

is a real vector space under these operations.

2. Which of the following sets W are subspaces of the given vector space V over the field F? Support your answer.

   (a) $V = \mathbb{R}^3$, $F = \mathbb{R}$

   $$W = \{(a, b, c) \in \mathbb{R}^3, |a^2 + b^2 = c^2\}$$

   (b) $V = \mathcal{F}(\mathbb{R}, \mathbb{R})$, $F = \mathbb{R}$

   $$W = \{f | f(x + y) = f(x) + f(y) \text{ for all } x, y \in \mathbb{R}\}$$

3. Let V be an F vector space. Let W_1 and W_2 be subspaces of V. A vector space W is called the *direct sum* of W_1 and W_2 if $W_1 \cap W_2 = \{0\}$ and $W = W_1 + W_2$ where

$$W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 | \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}$$

We denote this direct sum by $W = W_1 \oplus W_2$.

   (a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

   (b) Prove that any subspace of V that contains both W_1 and W_2 must contain $W_1 + W_2$.

   (c) Show that a vector space W is a direct sum of subspaces W_1 and W_2 if and only if each vector in W can be written uniquely as $\mathbf{x}_1 + \mathbf{x}_2$ where $\mathbf{x}_1 \in W_1, \mathbf{x}_2 \in W_2$.

4. A function $g \in \mathcal{F}(\mathbb{R}, \mathbb{R})$ is called an *even function* if $g(-t) = g(t)$ for all $t \in \mathbb{R}$ and is called an *odd function* if $g(-t) = -g(t)$.

(a) Prove that $\mathcal{F}_-(\mathbb{R})$, the set of odd functions in $\mathcal{F}(\mathbb{R},\mathbb{R})$, is a subspace of $\mathcal{F}(\mathbb{R},\mathbb{R})$. Note that we proved that $\mathcal{F}_+(\mathbb{R})$, the set of even functions in $\mathcal{F}(\mathbb{R},\mathbb{R})$, is a subspace in $\mathcal{F}(\mathbb{R},\mathbb{R})$ in class.

(b) Prove that $\mathcal{F}(\mathbb{R},\mathbb{R}) = \mathcal{F}_+(\mathbb{R}) \oplus \mathcal{F}_-(\mathbb{R})$.

5. (Problem 31 in section 1.3 of text) Let W be a subspace of a vector space V over a field F. For any $\mathbf{v} \in V$ the set $\{\mathbf{v}\} + W = \{\mathbf{v} + \mathbf{w} | \mathbf{w} \in W\}$ is called the *coset* of W *containing* $\mathbf{v}$. It is customary to denote this coset by $\mathbf{v} + W$ rather than $\{\mathbf{v}\} + W$.

   (a) Prove that $\mathbf{v} + W$ is a subspace of V if and only if $\mathbf{v} \in W$.

   (b) Prove that $\mathbf{v}_1 + W = \mathbf{v}_2 + W$ if and only if $\mathbf{v}_1 - \mathbf{v}_2 \in W$.

   (c) Addition and scalar multiplication by scalars of F can be defined in the collection $S = \{\mathbf{v} + W | \mathbf{v} \in V\}$ of all cosets of W as follows:

$$(\mathbf{v}_1 + W) + (\mathbf{v}_2 + W) = (\mathbf{v}_1 + \mathbf{v}_2) + W$$

   for all $\mathbf{v}_1, \mathbf{v}_2 \in V$ and

$$a(\mathbf{v} + W) = a\mathbf{v} + W$$

   for all $\mathbf{v} \in V$ and $a \in F$.

   Prove that the preceding operations are well defined; that is, show that if $\mathbf{v}_1 + W = \mathbf{v}'_1 + W$ and $\mathbf{v}_2 + W = \mathbf{v}'_2 + W$, then

$$(\mathbf{v}_1 + W) + (\mathbf{v}_2 + W) = (\mathbf{v}'_1 + W) + (\mathbf{v}'_2 + W)$$

   and

$$a(\mathbf{v}_1 + W) = a(\mathbf{v}'_1 + W)$$

   for all $a \in F$.

   (d) Prove that the set S is a vector space with the operations defined in (c). This vector space is called the *quotient space of V modulo W* and is denoted by V/W.