Math 2120B Assignment 3

1. (a) Is $T : M_{nn}(\mathbb{R}) \to \mathbb{R}$, $T(A) = \det(A)$ a linear transformation? If so, prove it. If not, give an explicit numerical counterexample.

   (b) Repeat part (a) for $T : \mathcal{F}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ given by $T(f) = f(1)$.

   (c) Suppose $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ is a linear map with $T(x + 1) = x$, $T(1) = 1$ and $T(x^2 + x) = x^2 + 1$. Find $T(a + bx + cx^2)$.

2. Let $T : F^n \to F^m$ be a linear map (with vectors written as columns). Show that there exists an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. [Hint: take the jth column of A to be the vector $T(\mathbf{e}_j) \in \mathbb{R}^n$ where $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ is the standard basis for F^n.] Note that this implies that any linear transformation from F^n to F^m takes the form $L_A : F^n \to F^m$ for some $A \in M_{mn}(F)$.

3. In each case, find a basis of $N(T)$ and a basis for $R(T)$ and determine the rank and nullity of the linear transformation T. Is T 1-1, onto or neither?

   (a) $T : M_{22} \to M_{22}; T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+b & b+c \\ c+d & d+a \end{bmatrix}$

   (b) $T : M_{22} \to M_{22}; T(X) = XA - AX$ where $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

4. Show that every polynomial $f(x) \in P_{n-1}(F)$ can be written as $f(x) = p(x + 1) - p(x)$ for some polynomial $p(x) \in P_n(F)$. [Hint: Define $T : P_n(F) \to P_{n-1}(F)$ as $T(p(x)) = p(x + 1) - p(x)$. Show that T is a linear transformation and find $N(T)$ and $R(T)$.]

5. Let V be a vector space and W a subspace of V. Let $q : V \to V/W$ be defined by $q(\mathbf{v}) = \mathbf{v} + W$ for $\mathbf{v} \in V$.

   (a) Prove that $q : V \to V/W$ is a linear transformation which is onto and show that $N(q) = W$.

   (b) Suppose that V is finite dimensional. Use (a) and the dimension theorem to derive a formula relating dim(V), dim(W) and dim(V/W).

   (c) Alternatively, let $\{\mathbf{w}_1, \dots, \mathbf{w}_k\}$ be a basis for W. and let $\{\mathbf{w}_1, \dots, \mathbf{w}_k, \mathbf{v}_1, \dots, \mathbf{v}_{n-k}\}$ be a basis for V extended from the basis for W. Show that $\{\mathbf{v}_1+W, \dots, \mathbf{v}_{n-k}+W\}$ is a basis for V/W. Note that this should confirm your answer in (b).