Math 2120B Assignment 5

1. For the following matrices, find the eigenvalues, a basis for each eigenspace and the algebraic and geometric multiplicities of each eigenvalue. Determine whether or not the matrix is diagonalisable, and if so, find an invertible matrix P such that $P^{-1}AP$ is diagonal.

$$\text{(a) } A = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & -2 \\ -1 & 0 & -2 \end{bmatrix} \qquad \text{(b) } A = \begin{bmatrix} i & 1 \\ 2 & -i \end{bmatrix}$$

2. For the following linear transformations, find the eigenvalues, a basis for each eigenspace and the algebraic and geometric multiplicities of each eigenvalue. Determine whether or not the linear map is diagonalisable, and if so, find a basis β for V such that $[T]_\beta$ is diagonal.

   (a) $T : \mathbf{P}_3(\mathbb{R}) \to \mathbf{P}_3(\mathbb{R}), T(f(x)) = f(x) + f'(x)$.

   (b) $T : M_{22}(\mathbb{R}) \to M_{22}(\mathbb{R}), T(A) = A^T - \mathrm{tr(A)I_2}$

3. (a) Prove that the characteristic polynomial of $A \in M_{nn}(F)$ takes the form

$$c_A(t) = (a_{11} - t)(a_{22} - t) \cdots (a_{nn} - t) + q(t)$$

   where $q(t)$ is a polynomial with degree at most $n - 2$. (Apply induction on n.)

   (b) Show that the leading coefficient of $c_A(t)$ is $(-1)^n$ and its degree is n. Show that $\mathrm{det(A) = c_A(0)}$ is the constant coefficient of $c_A(t)$. Show that the $n-1$st coefficient of $c_A(t)$ is $(-1)^{n-1}\mathrm{tr(A)}$.

4. (a) Suppose $A \in M_{nn}(F)$ is similar to an upper triangular matrix and A has distinct eigenvalues $\lambda_1, \dots, \lambda_k$ of algebraic multiplicities $m_1, \dots, m_k$. Show that

$$\mathrm{det(A)} = \prod_{\mathrm{i=1}}^{\mathrm{k}} \lambda_{\mathrm{i}}^{\mathrm{m_i}}, \mathrm{tr(A)} = \sum_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{m_i}\lambda_{\mathrm{i}}$$

   (b) Suppose $A \in M_{44}(\mathbb{R})$ is a diagonalisable matrix. If $A^2 = A$ and $\mathrm{tr(A)} = 2$, what are the eigenvalues of A and their algebraic multiplicities?

   (c) Suppose $A \in M_{22}(\mathbb{R})$ is similar to an upper triangular matrix. If $\mathrm{tr(A) = tr(A^2)} = 0$ show that $A^2 = 0$.

5. Let $T : V \to W$ be a linear transformation where V is an n dimensional F vector space and W is an m dimensional F vector space. Let β be a basis of V and γ a basis of W. Show that $\phi_\gamma(R(T)) = R([T]_\beta^\gamma)$ where $\phi_\gamma : W \to F^m$ is the coordinate map of γ. Deduce that

$$\phi_\gamma|_{R(T)} : R(T) \to R([T]_\beta^\gamma)$$

   is an isomorphism.