5 December, 2019

Final Exam Review Problems:

Calc 1000A - 003

I. $\boxed{\text{Functions:}}$

Review concepts:

- Definition of function.
- Domain and range;
- Inverses;
- even/odd functions.
- one-to-one functions;

Problems:

(1) Find domain and range for:

- $f(x) = \dfrac{2}{3x-1}$;
- $g(x) = \sqrt{16-x^4}$;
- $h(x) = \ln(x+6)$.

(2) Is the function even, odd, or neither?

- $h(x) = 2x - x^2$;
- $g(x) = \cos(x^3)$;
- $F(x) = x^5 + x$.

(3) ~~Find the~~

(a) Find the inverse function

For $F(x) = \dfrac{x+1}{2x+1}$.

(b) If $g(t) = 2t + \ln(t)$, what is $g^{-1}(2)$?

II. $\boxed{\text{Limits}}$

Review concepts:

- Definition of a limit of a function;
- right/left sided limits;
- limits at ∞;
- infinite limits;
- vertical/horizontal asymptotes;
- limit laws;
- Squeeze Theorem.
- continuity;
- Intermediate Value Theorem;

Problems:

(4) Explain in words what each of the following mathematical statements means and illustrate the statement with a sketch:

- $\lim\limits_{x \to a} f(x) = L$;
- $\lim\limits_{x \to a^+} f(x) = L$;
- $\lim\limits_{x \to a^-} f(x) = L$;
- $\lim\limits_{x \to a} f(x) = \infty$;
- $\lim\limits_{x \to a} f(x) = -\infty$;
- $\lim\limits_{x \to \infty} f(x) = L$;
- $\lim\limits_{x \to -\infty} f(x) = L$.

(5) What does the Squeeze Theorem say? (Use precise mathematical language).

(6) Find the limit:

- $\lim\limits_{x \to 1} e^{x^3 - x}$; $\quad$ • $\lim\limits_{x \to -3} \dfrac{x^2 - 9}{x^2 + 2x - 3}$

- $\lim_{v \to 4^+} \dfrac{4-v}{|4-v|}$;

- $\lim_{x \to -\infty} \dfrac{1 - 2x^2 - x^4}{5 + x - 3x^4}$;

- $\lim_{x \to 0^+} \tan^{-1}(1/x)$.

(7) Prove that $\lim_{x \to 0} x^2 \cos\left(\frac{1}{x^2}\right) = 0$.

[Hint: Squeeze Theorem ...]

(8) What does it mean that a function $f(x)$ is continuous at $x = a$?

(9) Let

$$f(x) = \begin{cases} \sqrt{-x}, & \text{for } x < 0 ; \\ 3 - x, & \text{for } 0 \le x < 3 ; \\ (x-3)^2, & \text{for } x > 3 . \end{cases}$$

At which points of $\mathbb{R}$ is $f(x)$ discontinuous?

(10) Show that $\cos\sqrt{x} = e^x - 2$ has a solution on $(0, 1)$.

III. $\boxed{\text{Derivatives}}$

Review Concepts:

- Limit definition of derivative;
- differentiability definition;
- differentiation rules;
- Chain rule;
- implicit differentiation;
- logarithmic differentiation;

Problems:

(11) Use precise mathematical language to express what it means that a function $f(x)$ is differentiable at $x = a$.

(12) Find a function $f(x)$ and a number a so that

$$\lim_{h \to 0} \frac{(2+h)^6 - 64}{h} = f'(a).$$

(13) Find equations of tangent lines to the curve $y = \frac{2}{1-3x}$ at points with x-coordinates 0 and -1.

(14) Calculate y':

- $y = \frac{e^{1/x}}{x^2}$; • $y = x \cos^{-1} x$; • $y = (\arcsin(2x))^2$

- $y + x \cos y = x^2 y$; $y = 3^{x \ln(x)}$;

- $x e^y = y \sin x$; • $y = \frac{(x^2+1)^4}{(2x+1)^3 (3x-1)^5}$.

(15) Find equation for tangent line to the curve $x^2 + 4xy + y^2 = 13$ at $(2,1)$.

(16) What is $\lim_{h \to 0} \frac{\sqrt[4]{16+h} - 2}{h}$?

IV. Applications of Derivatives

Review Concepts:

- Definitions of local/absolute minimum and maximum values;
- Connection between a function being increasing/decreasing and the derivative;
- Concavity;
- L'Höpital's rule and the various indeterminate forms;
- Definition of an anti-derivative;

Problems:

(17) The volume of a cube is increasing at a rate of 10 cm^3/min. How fast is the surface area increasing when

(17 cont'd) the edge is 30 cm?

(18) Find the local and absolute extreme values of

$$f(x) = x^2 e^{-x} \quad \text{on} \quad [-1, 3].$$

(19) Evaluate the limits:

- $\lim\limits_{x \to 0} \dfrac{\tan(4x)}{x + \sin(2x)}$;
- $\lim\limits_{x \to 0} \dfrac{e^x - 1}{\tan x}$;
- $\lim\limits_{x \to -\infty} (x^2 - x^3) e^{2x}$;
- $\lim\limits_{x \to (\pi/2)^-} \left[(\tan x)^{\cos x} \right]$.

(20) Let $f(x) = \dfrac{(x-1)^3}{x^2}$. Describe the intervals where f is increasing and decreasing. Find all local extreme values. Find all inflection points. Describe the intervals where f is concave up and down.

(21) Find two positive integers so that the sum of the first with four times the second is 1,000, and the product of the two is as large as possible.

(22) Find ~~a fun~~ the unique function $f(x)$ which satisfies the conditions:

$$f''(x) = 1 - 6x + 48x^2, \quad f(0) = 1, \quad f'(0) = 2.$$

(23) State (using precise mathematical language) what it means that $F(x)$ is an anti-derivative of $f(x)$.

V. $\boxed{\text{Integration:}}$

Review Concepts:

- Definitions of L_n and R_n, the respective left and right n^{th} Riemann sums of function $f(x)$ on $[a,b]$;

- Limit definition of $\int_a^b F(x)\,dx$;
- $\sum$ summation notation and those closed form expressions for the special sums $\sum_{i=1}^{n} i$, $\sum_{i=1}^{n} i^2$, $\sum_{i=1}^{n} i^3$ we discussed;
- Telescoping sums;
- Properties of the ~~D~~ definite integral;
- Fundamental Theorem of Calculus I & II;
- Indefinite integrals;
- Substitution;

Problems:

(24) Write down the sums L_5 and R_5 for $f(x) = x^2 - x$ on $[0,2]$.

(25) Evaluate:

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} \frac{\pi}{n}\cdot \sin\left(\frac{i\pi}{n}\right)\right)$$

(26) If $\int_0^6 F(x)\,dx = 10$, $\int_0^4 F(x)\,dx = 7$,

what is $\int_4^6 F(x)\,dx$?

(27) State (using precise mathematical language)

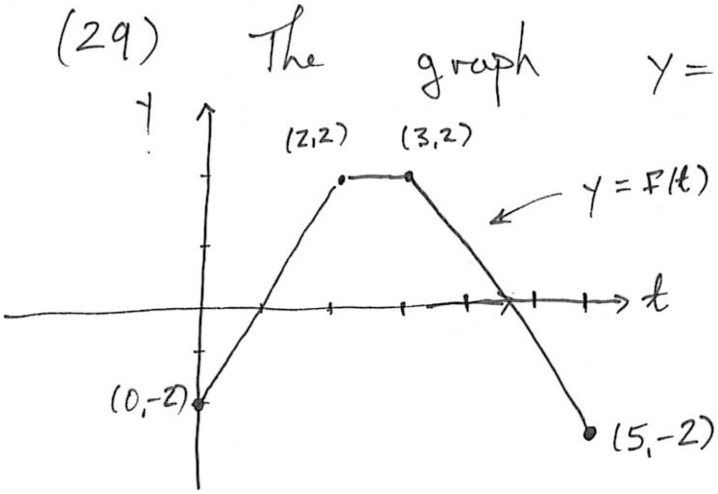
the Fundamental Theorem of Calculus

(both I and II).

(28) Evaluate:

- $\int_0^1 \frac{d}{dx}\left(e^{\arctan x}\right) dx$;
- $\frac{d}{dx}\left(\int_0^x e^{\arctan t}\, dt\right)$.
- $\frac{d}{dx} \int_0^1 e^{\arctan x}\, dx$;

(29) The graph $y = F(t)$ is shown:



Let $g(x) = \int_0^x F(t)\,dt$.

Find $g(4)$ and $g'(4)$.

(30) Evaluate the integrals:

- $\int_1^2 8x^3 + 3x^2 \, dx$;
- $\int_0^1 (1-x)^9 \, dx$;
- $\int_1^9 \frac{\sqrt{u} - 2u^2}{u} \, du$;
- $\int_0^2 y^2 \sqrt{1+y^3} \, dy$;
- $\int_1^{10} \frac{x}{x^2-4} \, dx$.
- $\int \frac{\sin(\ln(x))}{x} \, dx$;
- $\int \frac{\sec\theta \tan\theta}{1+\sec\theta} \, d\theta$.

(31) Find the derivative for each of the following:

- $F(x) = \int_0^x \frac{t^2}{1+t^3} \, dt$;
- $y = \int_{\sqrt{x}}^x \frac{e^t}{t} \, dt$;
- $g(x) = \int_{\sin x}^1 \frac{1-t^2}{1+t^4} \, dt$.

(32) If F is continuous on $[0,1]$, prove that $\int_0^1 F(x)\,dx = \int_0^1 F(1-x)\,dx$.

VI. Applications of Integration

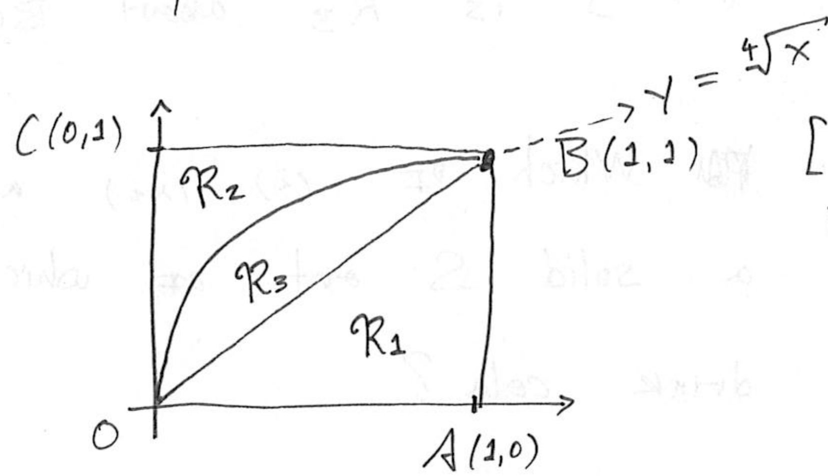
Review Concepts:

- Area, net-area, and integrals;

Problems: (33) Sketch the region enclosed by the curves, and calculate its area:

- $y = 4 - x^2$, $y = 0$;
- $y = x^3$, $x = -1$, $x = 2$;
- $y = \sqrt{x-1}$, $x - y = 1$;
- $y = \cos x$, $y = 2 - \cos x$, $x = 0$, $x = 2\pi$ ~~$0 \le x \le 2\pi$~~;
- $y = x^3$, $y = x$.

(34) Consider the picture which shows three regions $\mathcal{R}_1$, $\mathcal{R}_2$, $\mathcal{R}_3$ in the x,y - plane :



[The curved piece is the graph of $y = \sqrt[4]{x}$].

For each solid S, compute $Vol(S)$, the volume of S :

(i) S is $\mathcal{R}_1$ rotated about OA ;

(ii) S is $\mathcal{R}_1$ rotated about AB ;

(iii) S is $\mathcal{R}_1$ rotated about BC ;

(iv) S is $\mathcal{R}_1$ rotated about BC ;

(v) S is $\mathcal{R}_2$ about OA ;

(vi) S is $\mathcal{R}_2$ about OC ;

(vii) S is $\mathcal{R}_2$ about AB ;

(34 cont'd) (viii) S is $\mathcal{R}_3$ about OA;

(ix) S is $\mathcal{R}_3$ about OC;

(x) S is $\mathcal{R}_3$ about AB;

(xi) S is $\mathcal{R}_3$ about BC;

Bonus: Which of (i) – (xi) above gives a solid S out of which you can drink cola?

Remember: Studying in many shorter sessions is more effective than studying in fewer long sessions. — Psychology

Good luck! — A.