Genus Bounds for Some Dynatomic Modular Curves

UWO Final Thesis Exam Public Lecture

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Overview

1. Introduction
2. Function Fields and Curves
3. Flavors of Maximal Subgroups and Genus Bounds

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- Define $\mathrm{Per}_n(g) := \{\alpha \in \bar{F} : \alpha$ has period $n\}$;
- For any subset $S \subseteq \bar{F}$, define $\mathrm{Per}_n(g; S) := \mathrm{Per}_n(g) \cap S$

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- $n = 6^*$; [9]

*Conditional on Birch and Swinnerton-Dyer Conjecture

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- Hence, $\mathrm{Per}_2(f_c; \mathbb{Q}) = \{0, -1\}$.

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Thus F.E.(n) holds for every $n \geq 5$ except possibly for $n = 8$

Dynatomic Polynomials

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Example:

$$\Phi_1(x) = x^2 - x + t; \;\; \Phi_2(x) = x^2 + x + t + 1$$

$$\Phi_3(x) = x^6 + x^5 + (3t+1)x^4 + (2t+1)x^3 + \left(3t^2 + 3t + 1\right)x^2 + \left(t^2 + 2t + 1\right)x + t^3 + 2t^2 + t + 1$$

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4. $\Phi_n(x)$ divides $\Phi_n(f(x))$. [7]

Corollary

Let $r := \frac{\deg \Phi_n(x)}{n}$. Then

$$Z = \sqcup_{i=1}^{r} A_i$$

where $A_i = \{\alpha_i, f(\alpha_i), \ldots, f^{n-1}(\alpha_i)\}$ for some $\alpha_i \in L_0$ for every $i = 1, \ldots r$.

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Each set A_i is an f-orbit.

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- We have $\mathrm{Per}_n(f_c) \subseteq Z_c$, but equality need not hold.

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D_n is the zero set of the discriminant of Φ_n, and is therefore finite.

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Remark

By Hilbert's Irreducibility Theorem, E_n is known to be "thin." Thin sets are small (in a suitable sense), but can still be infinite.

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- The curve W corresponding to $K_0 = \mathbb{Q}(t)$ is $\mathbb{P}^1_{\mathbb{Q}}$, the projective line defined over $\mathbb{Q}$.

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- Any curve of the form X_H is called a dynatomic modular curve.

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By considering C over $\mathbb{C}$, we obtain a compact Riemann surface X; then the genus $g(C)$ counts the number of "holes/handles" of X.

Genus of a curve

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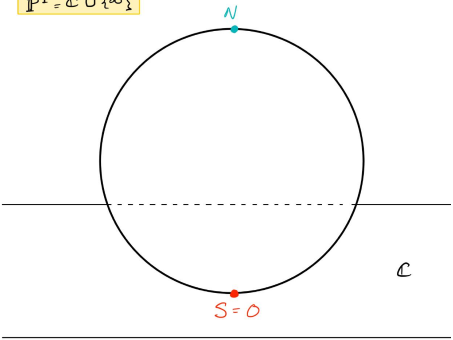

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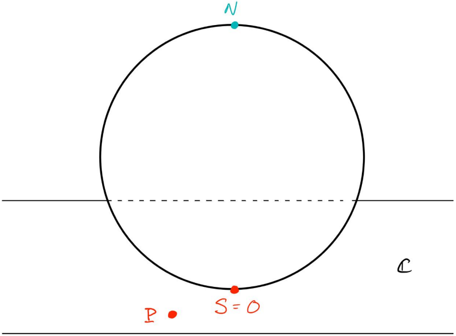

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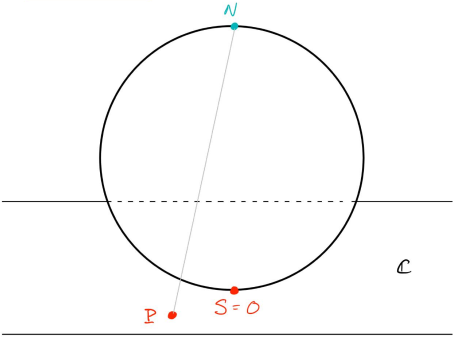

" Stereographic Projection "

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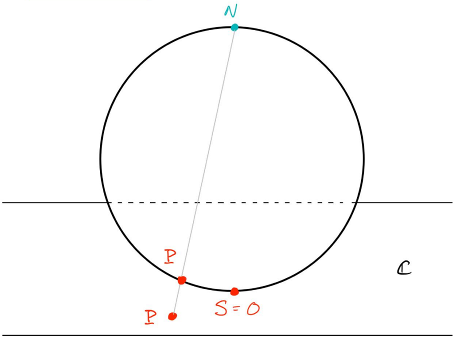

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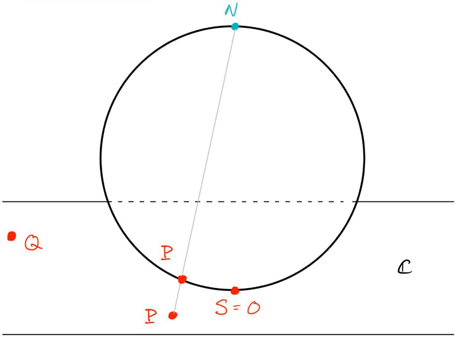

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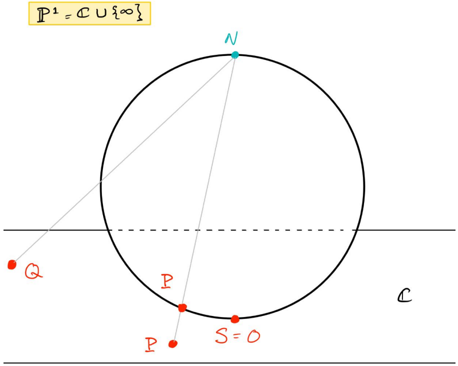

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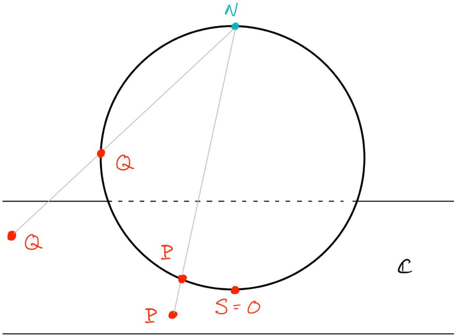

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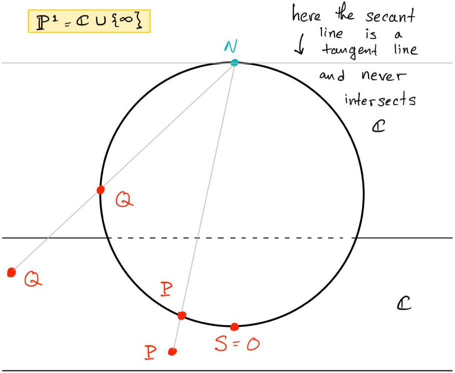

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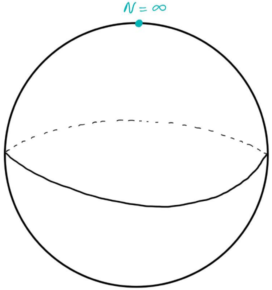

Thus, $g(\mathbb{P}^1) = g($  $) = 0$

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" elliptic curves "

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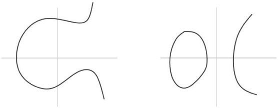

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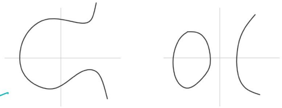

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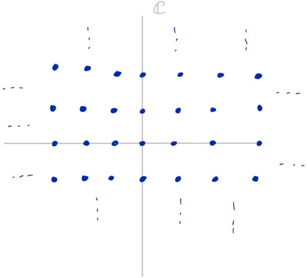

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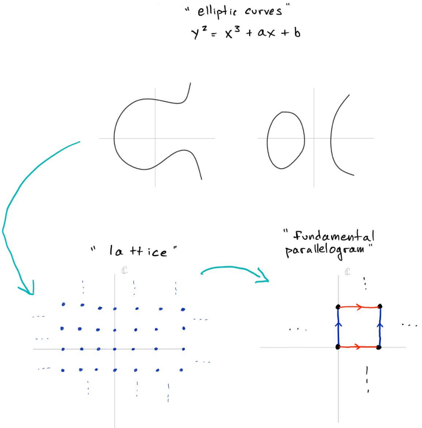

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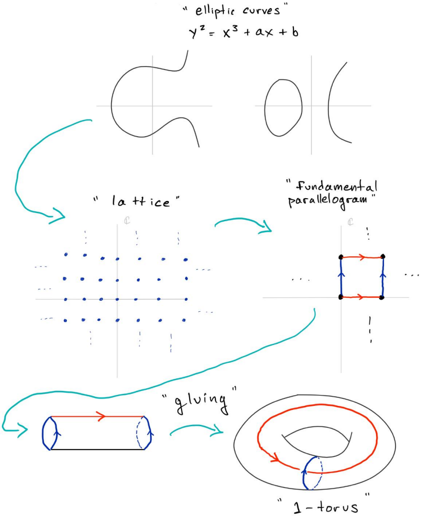

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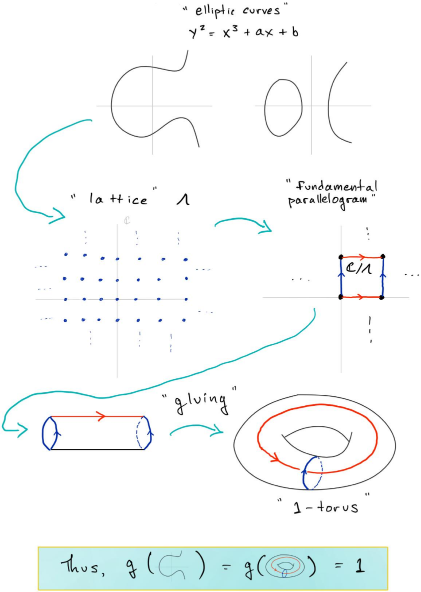

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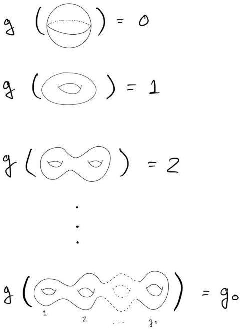

Rational Points and Exceptions

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Theorem: (see Propositions 3.3.1 and 3.3.5 of [8])

Let $c \in \mathbb{Q} - D_n$. Then $c \in E_n$ if and only if $c \in \pi_H(X_H(\mathbb{Q}))$ for some proper subgroup $H \lneq G$.

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3. $M \supseteq H \iff L^M \subseteq L^H \iff X_H \to X_M$, and genus is non-decreasing along curve morphisms: $g(X_H) \geq g(X_M)$.

Flavors of Maximal Subgroups and Genus Bounds

Semi-Direct Product Structure

Recall our earlier

Corollary

$Z = \sqcup_{i=1}^{r} A_i$, where $A_i = \{\alpha_i, f(\alpha_i), \ldots, f^{n-1}(\alpha_i)\}$ is the i-th f-orbit for every $i \in [r] := \{1, \ldots, r\}$.

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- Get a permutation representation $\phi : G \to \Gamma$; let $N := \ker(\phi)$.

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4. The semi-direct product $N \rtimes \Gamma = (\mathbb{Z}/n)^r \rtimes S_r$ is defined, and in fact $G \cong N \rtimes \Gamma$ [1].

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2. For $n \geq 8$, we can use a powerful theorem of Guralnick and Shareshian ([4]) to show that $g(X_M) \geq 2$ for every chocolate maximal M.

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5. In these cases we consider a second type of ramified point corresponding to prime divisors p of n other than 2, i.e., $p = 5$ for $n = 10$ and $p = 3$ for $n = 12$.

[1] Thierry Bousch. “Sur quelques problemes de dynamique holomorphe”. PhD thesis. Paris 11, 1992.

[2] G. Faltings. “Endlichkeitssätze für abelsche Varietäten über Zahlkörpern”. In: *Invent. Math.* 73.3 (1983), pp. 349–366.

[3] E. V. Flynn, Bjorn Poonen, and Edward F. Schaefer. “Cycles of quadratic polynomials and rational points on a genus-2 curve”. In: *Duke Math. J.* 90.3 (1997), pp. 435–463.

[4] Robert M. Guralnick and John Shareshian. "Symmetric and alternating groups as monodromy groups of Riemann surfaces. I. Generic covers and covers with many branch points". In: *Mem. Amer. Math. Soc.* 189.886 (2007), pp. vi+128.

[5] David Krumm. "A finiteness theorem for specializations of dynatomic polynomials". In: *Algebra & Number Theory* 13.4 (2019), pp. 963–993.

[6] Patrick Morton. "Arithmetic properties of periodic points of quadratic maps, II". In: *Acta Arithmetica* 87.2 (1998), pp. 89–102.

[7] Patrick Morton and Pratiksha Patel. "The Galois theory of periodic points of polynomial maps". In: *Proc. London Math. Soc. (3)* 68.2 (1994), pp. 225–263. ISSN: 0024-6115.

[8] Jean-Pierre Serre. *Topics in Galois theory*. Second. Vol. 1. Research Notes in Mathematics. A K Peters, Ltd., Wellesley, MA, 2008.

[9] Michael Stoll. "Rational 6-cycles under iteration of quadratic polynomials". In: *LMS Journal of Computation and Mathematics* 11 (2008), pp. 367–380.

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