Math 2120B Assignment 4

1. For $M_{22}(\mathbb{R}) \xrightarrow{T} M_{22}(\mathbb{R}) \xrightarrow{S} P_2(\mathbb{R})$ where $T(A) = A^T$ and

$$S\begin{bmatrix} a & b \\ c & d \end{bmatrix} = b + (a+d)x + cx^2$$

   Find $S \circ T : M_{22}(\mathbb{R}) \to P_2(\mathbb{R})$ and verify that

$$[S \circ T]_\beta^\delta = [S]_\gamma^\delta [T]_\beta^\gamma$$

   where $\beta = \gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and $\delta = \{1, x, x^2\}$. Verify also that $[S(A)]_\delta = [S]_\gamma^\delta [A]_\gamma$ where

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

2. Show that each linear transformation is bijective. Find the matrix $[T]_\beta^\gamma$ of $T : V \to W$ corresponding to the bases β of V and γ of W. In each case, show that $[T]_\beta^\gamma$ is invertible and use the fact that $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$ to determine the action of T^{-1}.

   (a) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$, $T(p(x)) = p(x+1)$, $\beta = \gamma = \{1, x, x^2\}$.

   (b) $T : M_{22}(\mathbb{R}) \to P_3(\mathbb{R})$,

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a+b+c) + (b+c)x + cx^2 + dx^3$$

$$\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}, \gamma = \{1, x, x^2, x^3\}$$

3. Let $T : V \to W$ be a linear transformation. Show that there exists a basis β of V and a basis γ of W such that

$$[T]_\beta^\gamma = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

   where r is the rank of T. [Hint: Let $\beta_{N(T)} = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a basis for $N(T)$ and extend it to a basis $\beta = \{\mathbf{v}_1, \ldots, \mathbf{v}_{n-k}, \mathbf{u}_1, \ldots, \mathbf{u}_k\}$ for V. Show that $\beta_{R(T)} = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_{n-k})\}$ is a basis for $R(T)$. Extend $\beta_{R(T)}$ to a basis

$$\gamma = \{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_{n-k}), \mathbf{w}_1, \ldots, \mathbf{w}_{m-n+k}\}$$

   of W.]

4. Recall that

$$E^{ij}E^{kl} = \delta_{jk}E^{il} \text{ for all } 1 \le i, j, k, l \le n$$

   where

$$\beta = \{E^{ij} | 1 \le i, j \le n\}$$

is the standard basis for $M_{nn}(\mathbb{R})$. Recall also that the trace function is defined by

$$\mathrm{tr} : \mathrm{M_{nn}}(\mathbb{R}) \to \mathbb{R}, \mathrm{tr}(\mathrm{X}) = \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{x_{ii}}$$

Note that tr is a linear transformation.

(a) For each (i, j) with $1 \leq i \neq j \leq n$, find $A, B \in M_{nn}(\mathbb{R})$ such that $E^{ij} = AB - BA$ and find $C, D \in M_{nn}(\mathbb{R})$ such that $E^{ii} - E^{jj} = CD - DC$.

(b) Suppose $T : M_{nn}(\mathbb{R}) \to \mathbb{R}$ is linear. Prove that the following statements are equivalent:

(1) $T(AB) = T(BA)$ for all $A, B \in M_{nn}(\mathbb{R})$

(2) T is a scalar multiple of the trace function.

Hint: Use (a) to show 1 implies 2.

(c) Find a basis for the subspace

$$W = \{X \in M_{nn}(\mathbb{R}) | \mathrm{tr}(\mathrm{X}) = 0\}$$

of $M_{nn}(\mathbb{R})$.

(d) Use (a) and (c) to show that $X \in M_{nn}(\mathbb{R})$ can be written as a sum of matrices of the form $AB - BA$ if and only if $\mathrm{tr}(\mathrm{X}) = 0$.

5. A $n \times n$ matrix A is called *strictly upper triangular* if $(A)_{ij} = 0$ for all i, j satisfying $i \geq j$.

(a) Let A be a $n \times n$ strictly upper triangular matrix. Prove that, for $k \geq 1$, the matrix A^k has the property that $(A^k)_{ij} = 0$ for all (i, j) with $j - i < k$. [Hint: you will want to argue by induction. For the $k + 1$th term, you will need to split the sum into two parts. Experiment with $n = 2$ and $n = 3$ to see where you need to split up the sum.]

(b) Using the previous part, show that $A^n = 0$ for any $n \times n$ strictly upper triangular matrix. [So, strictly upper triangular matrices are nilpotent.]