Ramanujan Graphs and Interlacing Families

Comprehensive Examination Part II Western Mathematics Department

Andrew W. Herring

April 12, 2019

Abstract

Ramanujan graphs are an important class of graphs in combinatorics and number theory. In a certain sense they are optimal expander graphs (graphs which are sparse, yet highly connected), and are thus naturally of interest to network scientists. The original constructions of Ramanujan graphs come from number theory. Until recently it was merely conjectured that for an arbitrary fixed degree of regularity, there exist an infinite number of Ramanujan graphs of that degree. In [10] Marcus, Spielman, and Srivastava proved the conjecture using degree 2 covering graphs and a new technique they call interlacing families of polynomials. We give a self contained walk through of this important development.

1 Introduction

Let G be a d-regular graph. Let $A(G)$ denote the (vertex) adjacency matrix of G, and $\mathrm{Spec}(A(G)) = \{\text{eigenvalues of } A(G)\}$ (considered as a multi-set). One can show that $d \in \mathrm{Spec}(A(G))$ (because G is d-regular), and $-d \in \mathrm{Spec}(A(G))$ if and only if G is bipartite [10]. These will be called the **trivial eigenvalues** of $A(G)$. Following the tradition of Lubotzky, Phillips, and Sarnak [9], we say that G is **Ramanujan** if all the non-trivial eigenvalues of $A(G)$ live in the interval $(-2\sqrt{d-1}, 2\sqrt{d-1})$. Lubtozky, et al. gave the first constructions of Ramanujan graphs by considering Cayley graphs of certain arithmetic groups [9]. Until very recently however, we were left with the following open question: *for each integer d, are there an infinite number of Ramanujan graphs of degree d?* (Lubotzky et al. did construct infinite families of Ramanujan graphs, but only for degree $q+1$ where q is a prime power) [10].

Let $\hat{G}$ be a 2-lift of G (a 2-lift is the same as a topological cover of degree 2, although there is a purely combinatorial description in Section 2). In [1], Bilu and Linial showed that some of the eigenvalues of $A(\hat{G})$ are inherited from $A(G)$ whereas some are "new." With this fact in mind, Bilu and Linial suggest the following tactic for constructing an infinite family of Ramanujan graphs of degree d: start with a d-regular Ramanujan graph (eg. the d-regular complete graph is Ramanujan) and try to construct successive 2-lifts so that the "new" eigenvalues are always confined to the Ramanujan interval $(-2\sqrt{d-1}, 2\sqrt{d-1})$. This was precisely the strategy employed by Marcus, Spielman, and Srivastava in [10] where they answered the question in the affirmative: for each d there are an infinite number of d-regular (bipartite) Ramanujan graphs of degree d (see Theorem 6.6).

This existence result was greeted with much acclaim when it was published in 2015, but we feel that the new heuristic developed en route to solving the problem will pay dividends beyond the realm of Ramanujan graphs. To be brief, Marcus, Spielmann, and Srivastava realized that proving that the average of some family of objects has some nice property $\mathcal{P}$ suffices to show that some member of the family has $\mathcal{P}$. In their own words: *"We do this by proving that the roots of the expected characteristic polynomial of a randomly signed adjacency matrix lie in this range. In general, a statement like this is useless, as the roots of a sum of polynomials do not necessarily have anything to do with the roots of the polynomials in the sum. However, there seem to be many sums of combinatorial polynomials for which this intuition is wrong."* [10]. See section 5 for more on this theme.
In this project, we give a completely self contained survey of this influential work of Marcus, et al. [10]. In section 2, we consider the objects of study: 2-lifts and signings. In section 3, we lay the groundwork for proving an important bound on the roots of the matching polynomial of a graph due to Heilmann and Lieb [8]. In section 4, we prove the bound of Heilmann and Lieb. In section 5 we study the main contribution of Marcus, Spielman, and Srivastava: the theory of interlacing families of polynomials. In section 6 we apply interlacing families to derive the main result: Theorem 6.6. In section 7, we establish the real rootedness of a family of polynomials using the theory of real stable polynomials (that polynomials in this family have only real roots is used in an essential way in section 6).

2 2 -lifts and Signings

Let $G = (V, E)$ be a graph. By this we mean that V will be a set of vertices and E will be some subset of $(V \times V)/\sim$ where $(u, v) \sim (v, u)$ for all $u, v \in V$: in other words, G is an undirected simple graph. We'll still allow (u, v) to denote the class of (u, v) in $(V \times V)/\sim$.

Definition 2.1. *A* 2***-lift of*** G *is a graph* $\hat{G} = (\hat{V}, \hat{E})$ *which has a pair of vertices* $\{v_0, v_1\}$ *for every* $v \in V$*. The pair* $\{v_0, v_1\}$ *is called the* ***fiber of*** v*. Each edge in* G *corresponds to a pair of edges in* $\hat{G}$ *as follows: for* $(u, v) \in E$*, let* $\{u_0, u_1\}$ *and* $\{v_0, v_1\}$ *be the respective fibers of* u *and* v*. Then* $\hat{E}$ *contains one of the following pairs of edges:*

$$\{(u_0, v_0), (u_1, v_1)\}, \text{ or} \tag{1}$$

$$\{(u_0, v_1), (u_1, v_0)\}. \tag{2}$$

In either case, we refer to the pair of edges $\{(u_0, v_0), (u_1, v_1)\}$ *or* $\{(u_0, v_1), (u_1, v_0)\}$ *as* ***the edges above*** (u, v)*. A* ***signing of*** G *is a function* $s : E \to \{\pm 1\}$*.*

Lemma 2.2. *There's a one-to-one correspondence*

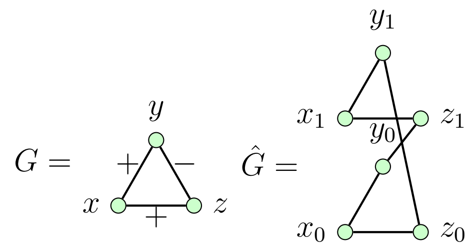
$$\{\textit{signings on } G\} \leftrightarrows \{2\textit{-lifts of } G\}$$

Proof. We define a signing

$$s(u, v) = \begin{cases} 1, & \text{if the edges above } (u, v) \text{ are of type (1);} \\ -1, & \text{if the edges above } (u, v) \text{ are of type (2).} \end{cases}$$

for all $(u, v) \in E$, and this assignment is clearly reversible. $\square$

Example 2.3. Here is an example of a graph G, a 2-lift $\hat{G}$ and the corresponding signing s.



Let G be a graph with adjacency matrix A, and let s be a signing on G. We define the **signed adjacency matrix of** (G, s), denoted A_s, to be the matrix obtained from A by replacing each non-zero entry $A_{(u,v)}$ with $s(u, v)$. We define the **signed characteristic polynomial**, denoted $f_s(x)$, to be the characteristic polynomial of A_s:

$$f_s(x) = \det(xI - A_s)$$

Example 2.4. Let $(G, \hat{G}, s)$ be as in Example 2.3. Then we have

$$A_s = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

and $f_s(x) = x^3 - 3x + 2 = (x + 2)(x - 1)^2$. Note that in order to write down the signed adjacency matrix we have to choose an ordering on the vertices; we've chosen the ordering (x, y, z).

Given a graph, signing, and corresponding 2-lift $(G, s, \hat{G})$, some of the eigenvalues of $A(\hat{G})$ are inherited from $A(G)$: these will be called the "old eigenvalues". Thus assuming that $\mathrm{Spec}(A(G))$ has some nice property $\mathcal{P}$, to prove that $\mathrm{Spec}(A(\hat{G}))$ has $\mathcal{P}$, it suffices to show that those eigenvalues of $A(\hat{G})$ not inherited from $A(G)$ (the "new eigenvalues") have $\mathcal{P}$. The utility of the signed adjacency matrix A_s is that $\mathrm{Spec}(A_s)$ consists of precisely the new eigenvalues, as Lemma 2.5 demonstrates.

Lemma 2.5. *[Lemma 3.1 of Bilu-Linial [1]] Let G be a graph with adjacency matrix A, let s be a signing on G, let A_s be the signed adjacency matrix of (G, s), let $\hat{G}$ be the 2-lift of G associated to s, and let $\hat{A}$ be the adjacency matrix of $\hat{G}$. Then*

$$\mathrm{Spec}(\hat{A}) = \mathrm{Spec}(A) \cup \mathrm{Spec}(A_s)$$

as mult-sets. Furthermore, the multiplicty of each eigenvalue of $\hat{A}$ is the sum of its multiplicities in A and A_s.

Proof. It's easy to see that

$$\hat{A} = \begin{pmatrix} A_1 & A_2 \\ A_2 & A_1 \end{pmatrix}, \quad \text{where } A_1 = A(V, s^{-1}(1)) \text{ and } A_2 = A(V, s^{-1}(-1))$$

Given an eigenvector v of A with eigenvalue μ, it's immediate from the above block decomposition of $\hat{A}$ that $\hat{v} = (v, v)^T$ is an eigenvector of $\hat{A}$ with value μ. Similarly, if u is an eigenvector of A_s with value λ, then $\hat{u} = (u, -u)^T$ is an eigenvector of $\hat{A}$ with value λ.
As the $\hat{v}$ and $\hat{u}$ are perpendicular and $2n$ in number, they span all of the eigenvectors of $\hat{A}$. $\square$

2.1 Matching polynomial of a graph

The core of our task is putting bounds on roots of various polynomials. Of essential import is a bound (found in 3.10) on the roots of the so called matching polynomial of G.

Definition 2.6. *A* ***matching*** *on $G = (V, E)$ is a subset $M \subseteq E$ of edges with the property that every $v \in V$ is incident to at most one $e \in M$. Let $a(G; i)$ denote the number of matchings M on G with $|M| = i$ (we assign the value $a(G; 0) = 1$). Then we define the* ***matching polynomial of*** *G, denoted $\mu_G(x)$, as*

$$\mu_G(x) := \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a(G; i) x^{n-2i}$$

where $n = |V|$.

For example, if G is the cycle graph from Example 2.3, then $\mu_G(x) = x^3 - 3x$. The next Theorem illustrates why we care about matching polynomials in this context.

Theorem 2.7. *[Corollary 2.2 of Godsil and Gutman [6]] We have*

$$\mathbb{E}_{s \in \{\pm 1\}^m}[f_s(x)] = \mu_G(x)$$

Proof. Let $|V| = n$. We begin by expanding the determinant over permutations:

$$\begin{aligned}
\mathbb{E}_s[\det(xI - A_s)] &= \mathbb{E}_s \left[\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} (xI - A_s)_{i,\sigma(i)} \right] \\
&= \sum_{k=0}^{n} (-1)^k x^{n-k} \sum_{S \subset [n], |S| = k} \ \sum_{\pi \in \mathrm{Sym}(S)} \mathbb{E}_s \left[\operatorname{sgn}(\pi) \prod_{i \in S} (A_s)_{i,\pi(i)} \right] \\
&= \sum_{k=0}^{n} (-1)^k x^{n-k} \sum_{S \subset [n], |S| = k} \ \sum_{\pi \in \mathrm{Sym}(S)} \mathbb{E}_s \left[\operatorname{sgn}(\pi) \prod_{i \in S} s_{i,\pi(i)} \right]
\end{aligned}$$

Since the s_{ij} are independent with $\mathbb{E}[s_{ij}] = 0$, the only products which survive in the last line are those that contain even (0 or 2) powers of the s_{ij}. Thus we can restrict our attention to those permutations π with only orbits of size 2. Such permutations correspond bijectively to perfect matchings on S (bearing in mind that in order for s_{ij} to appear in the product on the last line, we need i and j incident). When $|S|$ is odd, there are no perfect matchings. When $|S|$ is even, each perfect matching corresponds to a product of $|S|/2$ disjoint transpositions. As $\mathbb{E}[s_{ij}^2] = 1$, we have

$$\mathbb{E}_s[\det(xI - A_s)] = \sum_{k=0,\ k \text{ even}}^{n} \ \sum_{|S| = k} \ \sum_{\text{matchings } \pi \text{ on } S} (-1)^{|S|/2} = \mu_G(x)$$

□

Heilmann and Lieb introduced the matching polynomial in [8] and they also proved a few important statements about the roots of $\mu_G(x)$: for out purposes the most notable of these are Theorems 3.10 and 3.9

3 Toward Heilmann-Lieb

In this section and the one which follows, we prove Heilmann and Lieb's bound on the roots of $\mu_G(x)$. All of the arguments are take from Godsil's excellent book [5].
There's an important recurrence relation satisfied by the matching polynomial $\mu(G)$ involving deleting certain vertices from G. We fix some notation: we'll write $w \sim v$ in order to record the fact that $(u, v) \in E$. For any $v \in V$, we write $G - v$ to denote the induced subgraph of G with vertex set $V \setminus \{v\}$.

Lemma 3.1. *For any graph G and any $v \in V$ we have the following recurrence relation (which will henceforth be referred to as the 3* ***term recurrence****):*

$$\mu(G, x) = x\mu(G - v, x) - \sum_{w \sim v} \mu(G - v - w, x)$$

Proof. Fix $v \in V$. There are two possible cases for an i-matching of G: either the matching contains an edge incident to v or it does not. If the matching has no edge incident to v, then we are really interested in i-matchings on the graph $G - v$. If the matching does have an edge incident to v, then we sum the number of $i-1$-matchings on the graphs $G - v - w$ where w is incident to v. This argument gives us the relationship on the coefficients of the matching polynomials:

$$a(G; i) = a(G - v; i) + \sum_{v \sim w} a(G - v - w; i - 1)$$

Substituting this into the definition of the matching polynomial gives the following:

$$\begin{aligned}
\mu(G, x) &= \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a(G; i) x^{n-2i} \\
&= \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \left[a(G - v; i) + \sum_{w \sim v} a(G - v - w; i - 1) \right] x^{n-2i} \\
&= \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a(G - v; i) x^{n-2i} + \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \left[\sum_{w \sim v} a(G - v - w; i - 1) \right] x^{n-2i} \\
&= x \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i a(G - v; i) x^{(n-1)-2i} - \sum_{w \sim v} \left[\sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j a(G - v - w; j) \right] x^{(n-2)-2j} \\
&= x\mu(G - v, x) - \sum_{w \sim v} \mu(G - v - w, x)
\end{aligned}$$

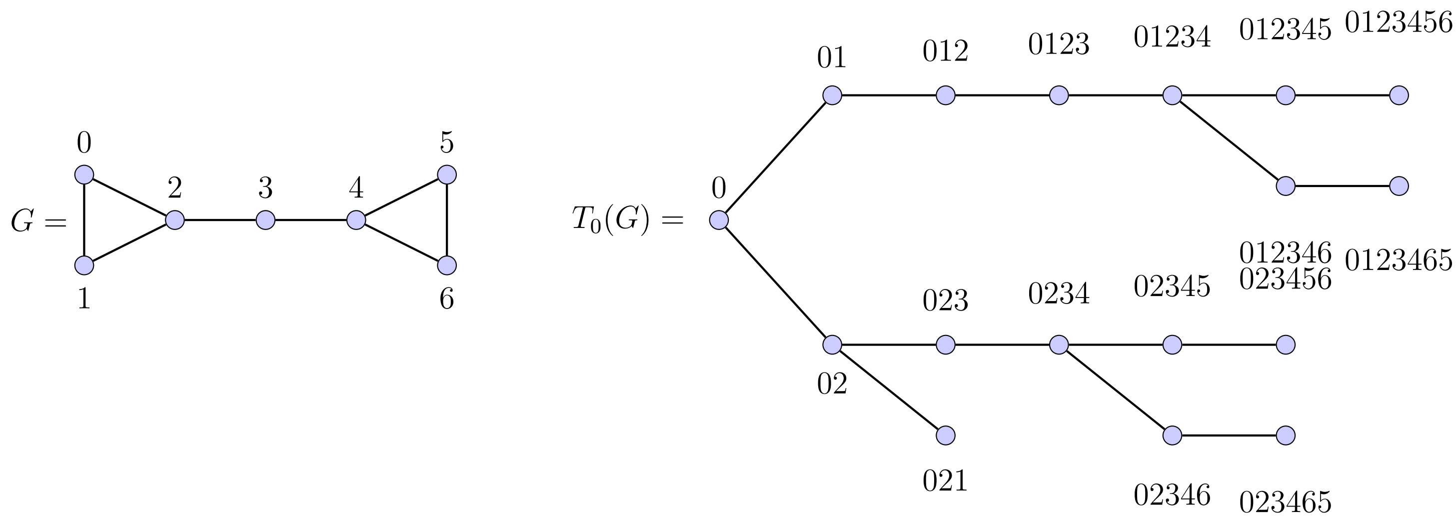
□

Toward proving Theorem 3.10, we introduce a tree which is associated to G and which captures much of the spectral data of G.

Definition 3.2. *Let G be a graph and u a vertex of G. Then the* ***path tree of G rooted at u*** *is the graph $T_u(G)$ defined as follows: the vertices of $T_u(G)$ are given by non repeating (digit-wise) sequences of vertices, and two such sequences are connected by an edge whenever one sequence is obtained from the other by concatenation of a single additional vertex.*

Put differently, vertices of $T_u(G)$ can be thought of as non-bactracking, simple walks on G and two such are connected whenever one is obtained from the other by taking an additional step.

Example 3.3. Here we see an example of G and $T_0(G)$.



Remark 3.4. As indicated by Example 3.3, we consider the sequence with a single entry u (equivalently a path of length 0 starting at u) and label the corresponding vertex of $T_u(G)$ also by u. If we delete u from $T_u(G)$, then the result is a forest with one component for each neighbor of u. If v is a neighbor of u, then the component of $T_u(G) - u$ containing the path uv is isomorphic to $T_v(G - u)$.

Proposition 3.5. *Let G be a graph and $a \in V(G)$. Then*

$$\frac{\mu(G, x)}{\mu(G - a, x)} = \frac{\mu(T_a(G), x)}{\mu(T_a(G) - a, x)}$$

and consequently $\mu(G, x)$ divides $\mu(T_a(G), x)$.

Proof. We begin by observing that if G is a tree then the statement trivially holds since in this case $G = T_u(G)$ for all $u \in V(G)$. As the only graphs on less than 3 vertices are trees, the statement holds for all graphs on at most 2 vertices. We proceed by induction on the number of vertices of G. By the 3 term recurrence and induction, we have

$$\begin{aligned}
\frac{\mu(G, x)}{\mu(G - a, x)} &= \frac{x\mu(G - a, x) - \sum_{b\sim a} \mu(G - a - b, x)}{\mu(G - a, x)} \\
&= x - \sum_{b\sim a} \frac{\mu(G - a - b, x)}{\mu(G - a, x)} \\
&= x - \sum_{b\sim a} \frac{\mu(T_b(G - a) - b, x)}{\mu(T_b(G - a), x)} = (*)
\end{aligned}$$

Observe the following relations:

$$T_b(G-a)-b = \coprod_{c\sim b, c\neq a} T_c(G-a-b)$$

$$T_a(G)-a = \coprod_{c\sim a} T_c(G-a)$$

The second of these implies that

$$\mu(T_a(G)-a,x) = \prod_{c\sim a} \mu(T_c(G-a),x)$$

Now let ab be the vertex in $T_a(G)$ corresponding to the length one path in G from a to b. Then we have

$$\begin{aligned} T_a(G)-a-ab &= \left(\coprod_{c\sim a, c\neq b} T_c(G-a)\right) \coprod \left(\coprod_{c\sim b, c\neq a} T_c(G-a-b)\right) \\ &= \left(\coprod_{c\sim a, c\neq b} T_c(G-a)\right) \coprod T_b(G-a)-b \end{aligned}$$

from which it follows that

$$\mu(T_a(G)-a-ab,x) = \left(\prod_{c\sim a, c\neq b} \mu(T_c(G-a),x)\right) \mu(T_b(G-a)-b,x)$$

Thus we have

$$\frac{\mu(T_a(G)-a-ab,x)}{\mu(T_a(G)-a,x)} = \frac{\left(\prod_{c\sim a, c\neq b} \mu(T_c(G-a),x)\right) \mu(T_b(G-a)-b,x)}{\prod_{c\sim a} \mu(T_c(G-a),x)} = \frac{\mu(T_b(G-a)-b,x)}{\mu(T_b(G-a),x)}$$

If we plug this back into $(*)$ we see that

$$\begin{aligned} \frac{\mu(G,x)}{\mu(G-a,x)} = (*) &= x - \sum_{b\sim a} \frac{\mu(T_b(G-a)-b,x)}{\mu(T_b(G-a),x)} \\ &= x - \sum_{b\sim a} \frac{\mu(T_a(G)-a-ab,x)}{\mu(T_a(G)-a,x)} = \frac{x\mu(T_a(G)-a,x) - \sum_{b\sim a} \mu(T_a(G)-a-ab,x)}{\mu(T_a(G)-a,x)} \\ &= \frac{\mu(T_a(G),x)}{\mu(T_a(G)-a,x)} \end{aligned}$$

by the 3 term recurrence. This proves the first statement.
Since $T_b(G-a)$ is isomorphic to a component of $T_a(G)-a$, we have

$$\mu(T_b(G-a),x) \mid \mu(T_a(G)-a,x)$$

By the induction hypothesis we also have that

$$\mu(G-a,x) \mid \mu(T_b(G-a),x)$$

Thus by the first statement we have

$$\mu(T_a(G),x)\frac{\mu(T_b(G-a),x)}{q(x)} = \mu(G,x)p(x)\mu(T_b(G-a),x)$$

for some polynomials $p(x)$ and $q(x)$. This proves that

$$\mu(G,x) \mid \mu(T_a(G),x)$$

as desired. $\square$

Corollary 3.6. *$Z(\mu(G,x)) \subseteq Z(\mu(T_a(G),x))$.*

This next Lemma allows us to deduce that the matching polynomial of a tree has only real roots because the characteristic polynomial of its adjacency matrix does.

Lemma 3.7. *Let T be a tree and let A_T be its adjacency matrix. Then $\mu(T,x) = \phi(A_T,x)$.*

Proof. We begin by expanding the determinant over permutations:

$$\phi(A_T,x) = \sum_{\pi \in S_n} (-1)^{\mathrm{sgn}(\pi)} x^{|\{a:\pi(a)=a\}|} \prod_{a:\pi(a)\neq a} (-A_T(a,\pi(a)))$$

We claim that the only permutations which contribute non-zero terms to the above sum are involutions. Suppose π is a permutation with $\pi(\pi(a)) \neq a$. Then there exist $a = a_1, \ldots, a_k$ with $\pi(a_i) = a_{i+1}$ for all $i = 1, \ldots, k-1$ and $\pi(a_k) = a_1$ with $k > 2$. In order for π to contribute to the sum, it must be that

$$A_T(a_i, a_{i+1}) = 1 = A_T(a_k, a_1)$$

for all $i = 1, \ldots, k-1$. This means that T has a cycle of length k; a contradiction since T is a tree and $k > 2$.
Thus the only terms which contribute are involutions which have only 2-cycles and fixed points. The number of permutations with k many cycles of length 2 is precisely $a(T,k)$: two edges with a vertex in common would correspond to a product of non-disjoint 2 cycles which is not a 2 cycle. The sign of such an involution is $(-1)^k$ and $|\{a : \pi(a) = a\}| = n - 2k$. $\square$

Definition 3.8. *Let A be a matrix, and let* $\mathrm{spec}(A)$ *denote the (multi)-set of a eigenvalues of A. The **spectral radius** of A, denoted $\rho(A)$, is defined by*

$$\rho(A) := \max\{|\lambda| : \lambda \in \mathrm{spec}(A)\}$$

As we will see, much of our remaining task will involve coming up with bounds on $\rho(A)$ for various matrices A. The bulk of the effort in this direction comes from proving the following crucial result.

Theorem 3.9 (Heilmann-Lieb [8])**.** *Let T be a tree with maximum degree $\Delta > 1$. Then $\rho(A_T) < 2\sqrt{\Delta - 1}$.*

We'll prove Theorem 3.9 in the next section. For now, if we assume Theorem 3.9 holds, we can prove the following:

Theorem 3.10. *Let $Z(f)$ denote the (multi)-set of zeros of a polynomial f. Let G be a graph with maximum degree Δ. Then we have*

(1) $Z(\mu(G,x)) \subset \mathbb{R}$;

(2) $Z(\mu(G,x)) \subset (-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1})$.

Proof. (1) The characteristic polynomial of a symmetric matrix has only real roots. Thus the characteristic polynomial of an adjacency matrix of a graph has only real roots. By Corollary 3.6 and Lemma 3.7, we have

$$Z(\mu(G,x) \subset Z(\mu(T_u(G),x)) = Z(\phi(A_{T_u(G)},x)) \subset \mathbb{R}$$

for any $u \in V$.

(2) Fix $u \in V$, let $T := T_u(G)$, and Δ' denote the maximum degree of T. By construction of T, we have $\Delta' \leq \Delta$. Now consider $\alpha \in Z(\mu(T,x))$; by Lemma 3.7, $\alpha \in (Z(\phi(A_T,x)))$. By definition of spectral radius, we have $|\alpha| \leq \rho(A_T)$. By Heilmann-Lieb 3.9, we have

$$|\alpha| \leq \rho(A_T) \leq 2\sqrt{\Delta'-1} \leq 2\sqrt{\Delta-1}$$

We're done by Corollary 3.6 as

$$Z(\mu(G,x)) \subseteq Z(\mu(T,x)) \subset (-2\sqrt{\Delta-1}, 2\sqrt{\Delta-1})$$

□

4 Heilmann-Lieb Proved

In this section, we prove Theorem 3.9. We'll need some matrix theory.

4.1 Perron-Frobenius

Given a matrix C, let $|C|$ denote the matrix with (i,j) entry $|C_{i,j}|$. Given a second matrix D, we'll write $D \leq C$ whenever $C - D$ exists and is non-negative. By the **underlying graph** of C, we mean the directed graph whose adjacency matrix is given by replacing each non-zero entry of C with 1.

Theorem 4.1 (Perron-Frobenius, Theorem 2.6.1 in [5])**.** *Let C be a non-negative $n \times n$ matrix whose underlying graph G is strongly connected (i.e. for all vertices x and y, if there is a path from x to y, then there is a path from y to x). Then*

(a) $\rho(C)$ is a simple, non-zero eigenvalue of C, and the corresponding eigenvector may be taken to be positive.

(b) Let $\rho_1, \dots, \rho_m \in \operatorname{spec}(C)$ such that $|\rho_i| = \rho$ for all $i = 1, \dots, m$. Then $m > 1$ if and only if all closed walks in G have length divisible by m. For each $i = 1, \dots, m$, $\frac{\rho_i}{\rho}$ is an m-th root of 1.

(c) Let D be an $n \times n$ matrix with $|D| \leq C$. Then $\rho(D) \leq \rho(C)$ with equality if and only if $D = \pm C$.

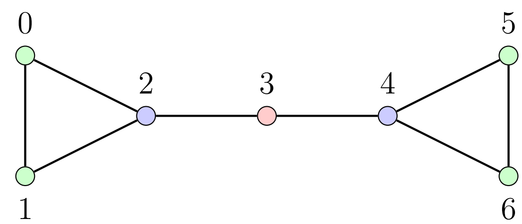
This is a standard result in any basic course on matrix theory; we omit the proof.

Remark 4.2. If H is a spanning (i.e. $V(G) = V(H)$) subgraph of G, then Theorem 4.1 (c) tells us that $\rho(H) \leq \rho(G)$ with equality if and only if $H = G$. Any subgraph can be extended to a spanning subgraph by including isolated vertices; such an extension does not change spectral data. Thus the same statement as above holds with "spanning subgraph" replaced with "subgraph."

4.2 Equitable Partitions

Let G be a graph, and consider a partition π of V. The elements of the partition are called **cells** so that if $\pi = (C_1, \dots, C_k)$ then π has k cells, the i-th of which is C_i. π is **equitable** if for all $i, j \in \{1, \dots, k\}$, the number of neighbors which a vertex in C_i has in C_j is independent of the choice of vertex in C_i.

Example 4.3. (1) Let G be the bow tie graph from Example 3.3 and let $\pi = \{C_0, C_1, C_2\}$ where $C_0 = \{0\}$, $C_1 = \{2, 4\}$, and $C_2 = \{0, 1, 5, 6\}$

$G =$ 

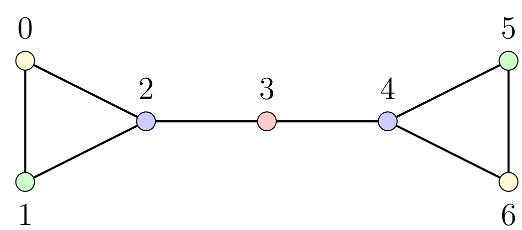
Then π is evidently equitable.

(2) Let G be connected. For any vertex $u \in V(G)$ we can form the **distance partition at** u, denoted π_u by taking

$$\pi_u = \{C_0, C_1, \dots, C_k\}$$

where C_i is the collection of vetices at distance i from u. Thus in (1) above we have $\pi = \pi_3$.

(3) We can generate lots of equitable partitions: let $\Gamma := \operatorname{Aut}(G)$ be the automorphism group of G. Then $V(G)$ is partitioned into the orbits under the action of Γ. Indeed, if u and v belong to the same cell (i.e. the same Γ-orbit), then there is an automorphism σ sending u to v. Since σ maps each cell onto itself, we see that u and v have the same number of neighbors in each cell, so the partition is equitable. Here is an example of such an orbit partition:

$G =$ 

Explicitly, $\pi = \{C_1, C_2, C_3, C_4\}$ where

$$C_1 = \{0, 6\}, \quad C_2 = \{2, 4\}, \quad C_3 = \{3\}, \quad C_4 = \{1, 5\}.$$

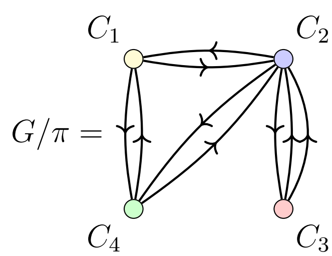
Notice that this partition is not a distance partition for any vertex u.

4.3 Quotients by Equitable Partitions

Let $\pi = (C_1, \ldots, C_k)$ be an equitable partition on G. Let $c_{i,j}$ denote the number of vertices in C_j to which a given vertex in C_i is incident. The **quotient** G/π is defined to be the directed graph with vertices the cells of π and with $c_{i,j}$ directed edges joining C_i to C_j. For the remainder of this section, we adopt a slight notational change: $A(G)$ (as opposed to A_G) will denote the adjacency matrix of a graph G. Thus we can consider the adjacency matrix of the quotient G/π:

$$A(G/\pi)_{i,j} = c_{i,j}, \text{ for } 1 \leq i, j \leq n$$

Example 4.4. Let π be the orbit partition from E_2 on the bow tie graph G. Then



Notice that in general G/π can be quite complicated with multi-edges and multiple self loops.

As we see below, the quotient of a graph G by an equitable partition retains much of the important spectral data of G.

Definition 4.5. *Let $\pi = (C_1, \ldots, C_k)$ be a partition of $V = \{v_1, \ldots, v_n\}$. The* ***characteristic matrix*** *$P = P(\pi)$ of π is the $n \times k$ matrix given by*

$$(P(\pi))_{i,j} = \begin{cases} 1, & v_i \in C_j; \\ 0, & \textit{else}. \end{cases}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, k$.

Lemma 4.6. *Let π be a partition of V with characteristic matrix P. If π is equitable, then $A(G)P = PA(G/\pi)$.*

Proof. Let $A := A(G)$. The (i, j) entry of AP is equal to the number of vertices in C_j adjacent to the i-th vertex v_i. π is equitable if and only if this number depends on the cell in which v_i lives, but not on v_i itself. Thus if π is equitable, then the columns of AP are constant on the cells of π. Thus the columns of AP are linear combinations of characteristic functions on the cells of π i.e. linear combinations of the columns of P. Thus there exists a matrix B such that

$$AP = PB$$

and one easily shows that $B = A(G/\pi)$. $\square$

Our reason for considering quotients by equitable partitions is the following Proposition: the quotient captures the extremal spectral data of G, and as we'll see in Lemmas 4.10 and 4.11 it can be easier to bound $\rho(A(G/\pi))$.

Proposition 4.7. *Let G be connected and π and equitable partition on G. Then $\rho(A(G)) = \rho(A(G/\pi))$.*

Toward proving Proposition 4.7, we have the following intermediate results.

Lemma 4.8. *Let π be an equitable partition on G with c cells. Let $P = P(\pi)$, $A = A(G)$, and $B = A(G/\pi)$. We have*

(a) if $BX = \theta x$, then $APx = \theta Px$;

(b) if $Ay = \theta y$ then $y^T PB = \theta y^T P$;

(c) $\phi(B) \mid \phi(A)$.

Proof. By Lemma 4.6, if $Bx = \theta x$ then

$$APx = PBx = P\theta x = \theta Px$$

Similarly, if $Ay = \theta y$ then

$$\theta y^T P = y^T AP = y^T PB$$

As the columns of P are linearly independent, there's a basis $p_1, \dots, p_n$ for $\mathbb{R}^n$ with $p_1, \dots, p_c$ the columns of P. With respect to this basis, A has the form

$$A = \begin{pmatrix} B & X \\ 0 & Y \end{pmatrix}$$

for some X and Y. Written in this way, it's clear that $\phi(B) \mid \phi(A)$.

□

4.4 Eigenvectors as vertex weight functions

Here is a useful way to think of eigenvectors of $A(G)$. Let $A = A(G)$ and suppose that z is an eigenvector associated to $\theta \in \mathrm{Spec}(A)$. Since the entries of A are all 0 or 1, $Az = \theta z$ is equivalent to the equations

$$\theta z_i = \sum_{j \sim i} z_j$$

for $i = 1, \dots, n$. Now we view z as a function $V(G) \to \mathbb{R}$ which assigns value z_i to vertex i. With such a viewpoint, the above equations imply that θ times the value on vertex i is equal to the sum of the values on neighbors of vertex i. Conversely, any function on $V(G)$ with this property can be seen to be an eigenvector. We'll regard the values assigned to the vertices by z as "weights."
Now, Lemma 4.6 shows that if x is an eigenvector of B, then Px is an eigenvector of A (associated to the same eigenvalue) which is constant on the cells of π. On the other hand, if y is an eigenvector of A, then $y^T P$ is a left eigenvector of B (associated to the same eigenvalue) if and only if it's non-zero. As a

function on $V(G)$, y assigns weights to each vertex of G, and $y^T P = 0$ if and only if the sum of weights of vertices in any cell C_i is zero.
Since $\mathrm{Spec}(B) \subseteq \mathrm{Spec}(A)$, we have that $|\mu| \le \rho(A)$ for all $\mu \in \mathrm{Spec}(B)$ and thus $\rho(B) \le \rho(A)$. If G is connected, then Perron-Frobenius 4.1 implies that the eigenvector x belonging to the spectral radius $\rho(A)$ has all entries positive. Hence $x^T P$ cannot possibly be zero in this case, and therefore $\rho(A) \in \mathrm{Spec}(B)$. But then

$$\rho(A) = |\rho(A)| \in \{|\mu| : \mu \in \mathrm{Spec}(B)\}$$

and so $\rho(A) \le \rho(B)$. This proves Proposition 4.7.

4.5 The spectral radius of a tree

Our last step in proving Heilmann-Lieb 3.9, is to compute the spectral radius of a tree.

Definition 4.9. *A tree T is* ***centrally symmetric with center*** *u if and only if for any vertices x and y of T which are the same distance from u, there exists an automorphism σ of T with $\sigma(u) = u$ and $\sigma(x) = y$.*

The two Lemmas which follow are quite easy; their proofs are omitted.

Lemma 4.10. *Let T be a tree with maximum degree Δ. Then T is an induced subgraph of a centrally symmetric tree T_Δ with the property that every vertex of T_Δ has degree Δ or degree 1.*

Lemma 4.11. *A tree is centrally symmetric with respect to the vertex u if and only if the distance partition with respect to u is equitable.*

Combining Lemma 4.10 with our Remark 4.2 regarding Perron-Frobenius and subgraphs, we see that one strategy for bounding $\rho(A(T))$ is to bound $\rho(A(T_\Delta))$. If π is the distance partition with respect to the center of T_Δ (which exists by Lemma 4.10), then Corollary 4.7 gives that $\rho(A(T_\Delta)) = \rho(A(T_\Delta)/\pi)$. Let $B = A(T_\Delta/\pi)$. It's easily verified that

$$B = \begin{pmatrix} 0 & \Delta & 0 & & \\ 1 & 0 & \Delta - 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 0 & \Delta - 1 \\ & & & 1 & 0 \end{pmatrix}$$

Let D be the diagonal matrix with the same size as B, but with i-th diagonal entry $(\sqrt{\Delta - 1})^{i-1}$. Let $\delta = \sqrt{\Delta - 1}$ and we find that

$$DBD^{-1} = \begin{pmatrix} 0 & \Delta/\delta & 0 & & \\ \delta & 0 & \delta & & \\ & \ddots & \ddots & \ddots & \\ & & \delta & 0 & \delta \\ & & & \delta & 0 \end{pmatrix} = \tilde{B}$$

Proof. **of Heilmann-Lieb 3.9:** As previously mentioned, Perron-Frobenius 4.1 tells us that $\rho(A(T)) \leq \rho(A(T_\Delta))$, and Corollary 4.7 implies $\rho(A(T_\Delta)) = \rho(\tilde{B})$ with $\tilde{B}$ as above. Now if $\tilde{B}x = \lambda x$, then

$$|\lambda||x_i| = |\lambda x_i| = \left|\sum_j (\tilde{B})_{i,j} x_j\right| \leq \sum_j |(\tilde{B})_{i,j}||x_j|$$

Now take i so that $|x_i|$ is maximal. Then

$$|\lambda||x_i| \leq \sum_j |(\tilde{B})_{i,j}||x_j| \leq \sum_j |(\tilde{B})_{i,j}||x_i| = \left(\sum_j |(\tilde{B})_{i,j}|\right)|x_i|$$

In the sum $\sum_j |(\tilde{B})_{i,j}|$ we are summing the entries in the i-th row of $\tilde{B}$. Every row has the form $(0, \Delta/\delta, 0, \ldots, 0)$, or $(0, \ldots, 0, \delta, 0)$, or has two entries which are δ and the rest zero. Thus the possible row sums we can have are Δ/δ, 2δ, or δ. Each of these quantities is at most $2\sqrt{\Delta - 1}$ (we have $\Delta/\delta = \frac{\Delta}{\sqrt{\Delta-1}} \leq 2\sqrt{\Delta - 1}$ since $\Delta \geq 2$). Therefore,

$$|\lambda||x_i| \leq (2\sqrt{\Delta - 1})|x_i|$$

So we've shown that for all $\lambda \in \operatorname{Spec}(\tilde{B})$,

$$|\lambda| \leq 2\sqrt{\Delta - 1}$$

from which it immediately follows that $\rho(\tilde{B}) \leq 2\sqrt{\Delta - 1}$. In summary,

$$\rho(A(T)) \leq \rho(A(T_\Delta)) = \rho(\tilde{B}) \leq 2\sqrt{\Delta - 1}$$

where the first of these inequalities is an equality if and only if $T = T_\Delta$ by Remark 4.2. □

5 Interlacing Families

As alluded to, the magic of [10] is a principle we like to call "just bound the average." Up to this point we've established two important facts which deserve recalling: (1) $\mathbb{E}_{s \in \{\pm 1\}^m}[f_s(x)] = \mu_G(x)$, and (2) if G is d-regular, the roots of $\mu_G(x)$ are bounded in absolute value by $2\sqrt{d-1}$. Of course our goal is to prove that there is some signing s on G so that the roots of $f_s(x)$ have this property. What is truly remarkable is that Marcus, Spielman, and Srivastava proved that the bound of $2\sqrt{d-1}$ on the absolute value of the roots of $\mathbb{E}_{s \in \{\pm 1\}^m}[f_s(x)]$ guarantees the existence of just such a signing s on G. The key ingredient is the theory of interlacing families of polynomials which we present in this section.

Definition 5.1. *We say that a polynomial $g(x) = \prod_{i=1}^{n-1}(x - \alpha_i)$* ***interlaces*** *a polynomial $f(x) = \prod_{i=1}^{n}(x - \beta_i)$ if*

$$\beta_1 \leq \alpha_1 \leq \beta_2 \leq \alpha_2 \leq \ldots \leq \alpha_{n-1} \leq \beta_n$$

We say that polynomials $f_1, \ldots, f_k$ have a ***common interlacing*** *if there's a single polynomial g which interlaces each of the f_i.*

Let $\beta_{i,j}$ be the j-th smallest root of f_i. Then an equivalent characterization of $f_1, \ldots, f_k$ having a common interlacing is the existence of numbers $\alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n$ such that

$$\beta_{i,j} \in [\alpha_{j-1}, \alpha_j]$$

for all i and j (indeed, then one can just take the α_j to be the roots of a common interlacer g).
Here is the first example of the utility of common interlacing.

Lemma 5.2. *Let $f_1, \ldots, f_k$ be degree n polynomials with real roots and positive leading coefficients. Define $f_\emptyset = \sum_{i=1}^{k} f_i$. If $f_1, \ldots f_k$ have a common interlacing, then there exists some $i \in \{1, \ldots, k\}$ such that the largest root of f_i is at most the largest root of $f_\emptyset$.*

Proof. Let g be a polynomial which interlaces all of the f_i and let α_{n-1} be the largest root of g. As each f_j has positive leading coefficient, it is eventually positive. As each f_j has exactly one root that is at least α_{n-1}, it is non-positive at α_{n-1}. By these observations, $f_\emptyset$ is non-positive at α_{n-1} and is eventually positive. Thus $f_\emptyset$ has a root that is at least α_{n-1} and so it's largest root is at least α_{n-1}: call this root β_n.
Since $f_\emptyset$ is the sum of the f_j, there must be some $i \in \{1, \ldots, k\}$ with $f_i(\beta_n) \geq 0$. As f_i has at most one root which is at least α_{n-1}, and $f_i(\alpha_{n-1}) \leq 0$, the largest root of f_i is at least α_{n-1} and at most β_n. $\square$

Definition 5.3. *Let $S_1, \ldots, S_m$ be finite sets, and for every assignment $(s_1, \ldots, s_m) \in S_1 \times \ldots \times S_m$, let $f_{s_1,\ldots,s_m}(x)$ be a real rooted, degree n polynomial with positive leading coefficient. Given a partial assignment $(s_1, \ldots s_k) \in S_1 \times \ldots \times S_k$ with $k < m$, define*

$$f_{s_1,\ldots,s_k} = \sum_{(s_{k+1},\ldots,s_m) \in S_{k+1} \times \ldots \times S_m} f_{s_1,\ldots,s_k,s_{k+1},\ldots,s_m}$$

and also

$$f_\emptyset = \sum_{(s_1,\ldots,s_m) \in S_1 \times \ldots \times S_m} f_{s_1,\ldots,s_m}$$

*Then we say that the polynomials $\{f_{s_1,\ldots,s_m}\}_{s_1,\ldots s_m}$ form an **interlacing family** if for all $k = 0, \ldots, m-1$ and all $(s_1, \ldots, s_k) \in S_1 \times \ldots \times S_k$, the polynomials*

$$\{f_{s_1,\ldots,s_k,t}\}_{t \in S_{k+1}}$$

have a common interlacing.

Example 5.4. Here we demonstrate in a generic example, what is required for a family of polynomials to form an interlacing family. Let $m = 2$ and $S_1 = S_2 = \mathbb{Z}/2\mathbb{Z}$. Then the family has four polynomials:

$$\{f_{0,0}(x), f_{0,1}(x), f_{1,0}(x), f_{1,1}(x)\}$$

This family forms an interlacing family provided that each of the following sets has a common interlacing:

$$\{f_{0,0}(x), f_{0,1}(x)\}$$

$$\{f_{1,0}(x), f_{1,1}(x)\}$$

$$\{f_{0,0}(x) + f_{0,1}(x),\ f_{1,0}(x) + f_{1,1}(x)\}$$

Theorem 5.5. *Let $S_1, \dots, S_m$ be finite sets, and let $\{f_{s_1,\dots,s_m}\}$ be an interlacing family. Then there exists some assignment $(s_1, \dots, s_m) \in S_1 \times \dots \times S_m$ so that the largest root of $f_{s_1,\dots,s_m}$ is at most the largest root of $f_\emptyset$.*

Proof. The proof is by induction on m. From the definition of interlacing family, the polynomials $\{f_t\}_{t \in S_1}$ have a common interlacing, and their sum is $f_\emptyset$:

$$\sum_{t \in S_1} f_t = f_\emptyset$$

So by Lemma 5.2, there is some $s_1 \in S_1$ such that f_{s_1} has largest root which is at most the largest root of $f_\emptyset$.
Now for any $(s_1, \dots, s_k) \in S_1 \times \dots \times S_k$, the polynomials $\{f_{s_1,\dots,s_k,t}\}_{t \in S_{k+1}}$ have a common interlacing, and their sum is $f_{s_1,\dots,s_k}$. So by induction, there is some choice of t (say $t = s_{k+1}$), so that the largest root of $f_{s_1,\dots,s_{k+1}}$ is at most the largest root of $f_{s_1,\dots,s_k}$. $\square$

We will show in Section 7 that the polynomials $\{f_s\}_{s \in \{\pm 1\}^m}$ form an interlacing family; to this end we'll need to establish the existence of certain common interlacings. It turns out that this is equivalent to giving certain real rootedness statements such as the following due to Dedieu [4].

Lemma 5.6. *Let $f_1, \dots, f_k$ be polynomials of the same degree with positive leading coefficients. Then $f_1, \dots, f_k$ have a common interlacing if and only if $\sum_{i=1}^k \lambda_i f_i$ is real rooted for all $\lambda_1, \dots, \lambda_k$ non-negative which satisfy $\sum_{i=1}^k \lambda_i = 1$.*

6 Main Result

Toward showing that $\{f_s\}_{s \in \{\pm 1\}^m}$ forms an interlacing family, the bulk of the work will be in proving the following.

Theorem 6.1. *The polynomial*

$$\sum_{s \in \{\pm 1\}^m} \left(\prod_{i: s_i = 1} p_i \right) \left(\prod_{i: s_i = -1} (1 - p_i) \right) f_s(x)$$

is real rooted for all $p_1, \dots, p_m \in [0, 1]$.

Theorem 6.1 is proved using the theory of real stable polynomials. We give a proof in Section 7.

Theorem 6.2. *$\{f_s\}_{s \in \{\pm 1\}^m}$ is an interlacing family.*

To prove Theorem 6.2, we must show that for every $k = 0, \dots, m-1$ and every $(s_1, \dots, s_k) \in \{\pm 1\}^k$, the set

$$\{f_{s_1,\dots,s_k,1}(x) \ , \ f_{s_1,\dots,s_k,-1}(x)\}$$

has a common interlacing. By Lemma 5.6, it suffices to prove that for every $\mu \in [0, 1]$

$$\mu f_{s_1,\dots,s_k,1}(x) + (1 - \mu) f_{s_1,\dots,s_k,-1}(x)$$

is real rooted. To do this, we introduce a few preliminary results.

Proposition 6.3. *For all k with $0 \leq k \leq m-1$, the polynomial*

$$g_{s_1,\dots,s_k}(\lambda) := \sum_{s_{k+1},\dots,s_m} \left(\prod_{\substack{i:s_i=1 \\ k+1 \leq i \leq m}} \lambda_i \right) \left(\prod_{\substack{i:s_i=-1 \\ k+1 \leq i \leq m}} (1-\lambda_i) \right) f_{s_1,\dots,s_m}$$

is real rooted for all $(s_1,\dots,s_k) \in \{\pm 1\}^k$ and all $\lambda = (\lambda_1,\dots,\lambda_m) \in [0,1]^m$.

Proof. The proof is by induction on k. When $k=0$, this is exactly Theorem 6.1.
So suppose inductively that $g_{s_1,\dots,s_{k-1}}(\lambda)$ is real rooted for all λ and all $s_1,\dots,s_{k-1}$. We claim that

$$g_{s_1,\dots,s_{k-1}}(\lambda) = \lambda_k g_{s_1,\dots,s_{k-1},1}(\lambda) + (1-\lambda_k) g_{s_1,\dots,s_{k-1},-1}(\lambda)$$

Indeed, we compute

$$\begin{aligned} g_{s_1,\dots,s_{k-1}}(\lambda) &= \sum_{s_k,\dots,s_m} \left(\prod_{\substack{i:s_i=1 \\ k \leq i \leq m}} \lambda_i \right) \left(\prod_{\substack{i:s_i=-1 \\ k \leq i \leq m}} (1-\lambda_i) \right) f_{s_1,\dots,s_m} \\ &= \lambda_k \sum_{s_{k+1},\dots,s_m} \left(\prod_{\substack{i:s_i=1 \\ k+1 \leq i \leq m}} \lambda_i \right) \left(\prod_{\substack{i:s_i=-1 \\ k+1 \leq i \leq m}} (1-\lambda_i) \right) f_{s_1,\dots,s_{k-1},1,s_{k+1},\dots,s_m} \\ &\qquad + (1-\lambda_k) \sum_{s_{k+1},\dots,s_m} \left(\prod_{\substack{i:s_i=1 \\ k+1 \leq i \leq m}} \lambda_i \right) \left(\prod_{\substack{i:s_i=-1 \\ k+1 \leq i \leq m}} (1-\lambda_i) \right) f_{s_1,\dots,s_{k-1},-1,s_{k+1},\dots,s_m} \\ &= \lambda_k g_{s_1,\dots,s_{k-1},1}(\lambda) + (1-\lambda_k) g_{s_1,\dots,s_{k-1},-1}(\lambda) \end{aligned}$$

Thus our inductive hypothesis can be rephrased as

$$\lambda_k g_{s_1,\dots,s_{k-1},1}(\lambda) + (1-\lambda_k) g_{s_1,\dots,s_{k-1},-1}(\lambda)$$

is real rooted for all λ and all $s_1,\dots,s_{k-1}$. In particular, it's real rooted for those special λ which have $\lambda_k = 0$ and $\lambda_k = 1$. For such special λ, $g_{s_1,\dots,s_{k-1},1}(\lambda)$ and $g_{s_1,\dots,s_{k-1},-1}(\lambda)$ are real rooted. But then this shows that $g_{s_1,\dots,s_{k-1},1}(\lambda)$ and $g_{s_1,\dots,s_{k-1},-1}(\lambda)$ are real rooted for all λ (and for all $s_1,\dots,s_{k-1}$) since neither depends on λ_k. But then this proves that $g_{s_1,\dots,s_k}(\lambda)$ is real rooted for all λ and all $(s_1,\dots,s_k) \in \{\pm 1\}^k$, so we're done by induction. $\square$

Corollary 6.4. *For all $(s_1,\dots,s_k) \in \{\pm 1\}^k$ and all $\mu \in [0,1]$, the polynomial*

$$\mu f_{s_1,\dots,s_k,1}(x) + (1-\mu) f_{s_1,\dots,s_k,-1}(x)$$

is real rooted.

Proof. Let $\mu \in [0,1]$ be arbitrary and let $\lambda = (1, \ldots, 1, \mu, \frac{1}{2}, \ldots, \frac{1}{2}) \in [0,1]^m$ where μ appears in position $k+1$. By Proposition 6.3, $g_{s_1,\ldots,s_k}(\lambda)$ is real rooted for all $(s_1, \ldots, s_k) \in \{\pm 1\}^k$. We have

$$\begin{aligned} g_{s_1,\ldots,s_k}(\lambda) &= \sum_{s_{k+1},\ldots,s_m} \left(\prod_{\substack{i:s_i=1 \\ k+1 \leq i \leq m}} \lambda_i \right) \left(\prod_{\substack{i:s_i=-1 \\ k+1 \leq i \leq m}} (1-\lambda_i) \right) f_{s_1,\ldots,s_m} \\ &= \sum_{s_{k+2},\ldots,s_m} \left(\mu \left(\frac{1}{2}\right)^{\ell} f_{s_1,\ldots,s_k,1,s_{k+2},\ldots,s_m} + (1-\mu) \left(\frac{1}{2}\right)^{\ell} f_{s_1,\ldots,s_k,-1,s_{k+2},\ldots,s_m} \right) \\ &= \left(\frac{1}{2}\right)^{\ell} (\mu f_{s_1,\ldots,s_k,1}(x) + (1-\mu) f_{s_1,\ldots,s_k,-1}(x)) \end{aligned}$$

which proves the result. □

Notice that Corollary 6.4 also proves Theorem 6.2.

Theorem 6.5. *Let G be a graph with adjacency matrix A and universal cover T. Then there exists a signing s of A so that all the eigenvalues of A_s are at most $\rho(T)$. In particular, if G is d-regular, there is a signing s so that the eigenvalues of A_s are at most $2\sqrt{d-1}$.*

Proof. By Theorem 2.7, we have

$$\mathbb{E}_{s \in \{\pm 1\}^m}[f_s(x)] = \mu_G(x)$$

By Theorem 6.2, $\{f_s\}_{s \in \{\pm 1\}^m}$ is an interlacing family. Finally we observe that, in the language of interlacing families,

$$\sum_{(s_1,\ldots,s_m) \in \{\pm 1\}^m} f_{s_1,\ldots,s_m} = f_{\emptyset} = \mathbb{E}_{s \in \{\pm 1\}^m}[f_s(x)] = \mu_G(x)$$

By Theorem 5.5, there is a signing $(s_1, \ldots, s_m) \in \{\pm 1\}^m$ so that the largest root of $f_{s_1,\ldots,s_m}$ is at most the largest root of $f_{\emptyset} = \mu_G(x)$. But by Heilmann and Lieb **??**, we know that the largest root of $\mu_G(x)$ is bounded in absolute value by $\rho(T)$. In the special case where G is d-regular, we saw that $\rho(T) \leq 2\sqrt{d-1}$. □

Theorem 6.6. *For every $d \geq 3$, there exists an infinite sequence of d-regular bipartite Ramanujan graphs.*

Proof. It's easy to see that the complete bipartite graph of degree d is Ramanujan. By Theorem 6.5, for every d-regular bipartite Ramanujan graph G, there's a 2-lift in which every non-trivial eigenvalue is at most $2\sqrt{d-1}$. As the 2-lift of any bipartite is bipartite, and the eigenvalues of bipartite graphs are symmetric about 0, this 2-lift is also d-regular bipartite Ramanujan. □

7 Real Stability

In this section we prove Theorem 6.1 using results coming from the theory of real stable polynomials. Real stability is a multivariate generalization of real rootedness as we will see.

Definition 7.1. *$f \in \mathbb{R}[z_1, \ldots, z_n]$ is **real stable** (or $\mathbb{R}$-**stable**) iff $f \equiv 0$ or if*

$$f(z_1, \ldots, z_n) \neq 0$$

whenever $\mathrm{Im}(z_i) > 0$ for each $i = 1, \ldots, n$. The collection of all real stable polynomials in n variables will be denoted $\mathcal{H}_n(\mathbb{R})$: that is,

$$\mathcal{H}_n(\mathbb{R}) := \{f \in \mathbb{R}[z_1, \ldots, z_n] : f \text{ is } \mathbb{R}\text{-stable}\}$$

The first easy observation is that $f \in \mathcal{H}_1(\mathbb{R})$ if and only if f is real rooted. Much of what is known about real stable polynomials is due to Borcea and Brändén including the following result from [3].

Lemma 7.2. *Let $A_1, \ldots, A_m$ be positive semidefinite matrices. Then*

$$\det(z_1 A_1 + \ldots + z_m A_m)$$

is $\mathbb{R}$-stable.

It's easy to see that if f and g are $\mathbb{R}$-stable, then so is the product fg. What's less obvious, but true is that if $f(x_1, \ldots, x_k) \in \mathcal{H}_k(\mathbb{R})$, then $f(x_1, \ldots, x_{k-1}, c) \in \mathcal{H}_{k-1}(\mathbb{R})$ for all $c \in \mathbb{R}$ (see [11] Lemma 2.4 for a proof.) For a variable x_i, let Z_{x_i} be the operator which acts on polynomials by setting x_i to 0.
In [2], Borcea and Brändén characterize a class of $\mathbb{R}$-stability preserving differential operators. We fix some notation: let ∂_{z_i} denote the operation of partial differentiation with respect to z_i. For $\alpha, \beta \in \mathbb{N}^n$ let

$$z^\alpha := \prod_{i=1}^n z_i^{\alpha_i}, \quad \partial^\beta := \prod_{i=1}^n (\partial_{z_i})^{\beta_i}$$

Theorem 7.3. *Let $T : \mathbb{R}[z_1, \ldots, z_n] \to \mathbb{R}[z_1, \ldots, z_n]$ be an operator of the form*

$$F = \sum_{\alpha, \beta \in \mathbb{N}^n} c_{\alpha,\beta} z^\alpha \partial^\beta$$

where $c_{\alpha,\beta} \in \mathbb{R}$ and $c_{\alpha,\beta} = 0$ for all but finitely many α, β. Define

$$F_T(z, w) := \sum_{\alpha,\beta} c_{\alpha,\beta} z^\alpha w^\beta$$

Then $T : \mathcal{H}_n(\mathbb{R}) \to \mathcal{H}_n(\mathbb{R})$ if and only if $F_T(z, -w)$ is $\mathbb{R}$-stable.

We need just a special case whose proof is an easy application of Theorem 7.3:

Corollary 7.4. *For all $a, b \geq 0$ and variables x, y the operator $T = 1 + a\partial_x + b\partial_y$ preserves $\mathbb{R}$-stability.*

Lemma 7.5. *For an invertible matrix A, vectors u, v and $p \in [0, 1]$ we have*

$$Z_x Z_y (1 + p\partial_x + (1-p)\partial_y) \det(A + xuu^T + yvv^T)$$
$$= p \det(A + uu^T) + (1-p) \det(A + vv^T)$$

Proof. The matrix determinant lemma (which can be found in many places, for example in [7]) says that for invertible A,

$$\det(A + tuu^T) = \det(A)(1 + tu^T A^{-1} u)$$

In particular, this implies Jacobi's formula for the derivative of the determinant:

$$\partial_t \det(A + tuu^T) = \det(A)(u^T A^{-1} u)$$

We remark that each of Z_x and Z_y commute with the determinant [eg. $Z_x Z_y(\det(A + xuu^T + yvv^T)) = \det(A)$.] What's more, we have that for any differentiable function $r(x, y)$

$$Z_y(\partial_x(r(x, y))) = \partial_x(Z_y(r(x, y))), \quad Z_x(\partial_y(r(x, y))) = \partial_y(Z_x(r(x, y)))$$

By these remarks and Jacobi's formula, we see

$$\begin{aligned} Z_x Z_y(1 + p\partial_x + (1-p)\partial_y)\det(A + xuu^T + yvv^T) &= \det(A) + p\det(A)(u^t A^{-1} u) + (1-p)\det(A)(v^T A^{-1} v) \\ &= p\det(A)(1 + u^T A^{-1} u) + (1-p)\det(A)(1 + v^T A^{-1} v) \end{aligned}$$

Note that we used the fact that $\det(A) = p\det(A) + (1-p)\det(A)$. By the matrix determinant lemma again, this equals

$$p\det(A + uu^T) + (1-p)\det(A + vv^T)$$

□

Now our first important real rootedness result.

Theorem 7.6. *Let $u_1, \ldots, u_m, v_1, \ldots, v_m \in \mathbb{R}^n$, let $p_1, \ldots, p_m \in [0, 1]$, and let D be a positive semi-definite matrix. Then*

$$P(x) := \sum_{S \subseteq [m]} \left(\prod_{i \in S} p_i\right) \left(\prod_{i \notin S} (1 - p_i)\right) \det\left(xI + D + \sum_{i \in S} u_i u_i^T + \sum_{i \notin S} v_i v_i^T\right)$$

is real rooted.

Proof. Let $y_1, \ldots, y_m, z_1, \ldots, z_m$ be formal variables and consider

$$Q(x, y_1, \ldots, y_m, z_1, \ldots, z_m) := \det\left(xI + D + \sum_i y_i u_i u_i^T + \sum_i z_i v_i v_i^T\right)$$

As each $u_i u_i^T$ and each $v_i v_i^T$ is positive semi-definite, Lemma 7.2 together with the fact that specializing a variable to a real constant preserves $\mathbb{R}$-stability show that Q is $\mathbb{R}$-stable. We claim that

$$P(x) = \left(\prod_{i=1}^m Z_{y_i} Z_{z_i} T_i\right) Q(x, y_1, \ldots, y_m, z_1, \ldots, z_m)$$

where $T_i = 1 + p_i \partial_{y_i} + (1 - p_i)\partial_{z_i}$. To prove this, we show by induction on k that

$$\left(\prod_{i=1}^k Z_{y_i} Z_{z_i} T_i\right) Q(x, y_1, \ldots, y_m, z_1, \ldots, z_m)$$

equals

$$\sum_{S\subseteq[k]}\left(\prod_{i\in S}p_i\right)\left(\prod_{i\in[k]\setminus S}(1-p_i)\right)\det\left(xI+D+\sum_{i\in S}u_iu_i^T+\sum_{i\in[k]\setminus S}v_iv_i^T+\sum_{i>k}y_iu_iu_i^T+z_iv_iv_i^T\right)$$

The case $k=0$ follows by definition of Q. One can use Lemma 7.5 to prove the inductive step. The case $k=m$ is exactly the identity we are interested in. So since

$$P(x)=\left(\prod_{i=1}^m Z_{y_i}Z_{z_i}T_i\right)Q(x,y_1,\dots,y_m,z_1,\dots,z_m)$$

and since each T_i and each Z_{y_i} and Z_{z_i} is $\mathbb{R}$-stable preserving for all $i=1,\dots,m$, the $\mathbb{R}$-stability of Q implies the $\mathbb{R}$-stability of P. But since P has only one variable x, P is $\mathbb{R}$-stable implies P is real rooted. $\square$

Finally we reap the fruits of our labor: we prove Theorem 6.1.

Proof. For $u\in V$, let d_u denote its degree, and let $d=\max\{d_u : u\in V\}$. We show that the polynomial

$$\sum_{s\in\{\pm1\}^m}\left(\prod_{i:s_i=1}p_i\right)\left(\prod_{i:s_i=-1}(1-p_i)\right)\det(xI-A_s)$$

is real rooted. In fact we show that

$$\sum_{s\in\{\pm1\}^m}\left(\prod_{i:s_i=1}p_i\right)\left(\prod_{i:s_i=-1}(1-p_i)\right)\det(xI+dI-A_s)=(*)$$

is real rooted. This is equivalent to real rootedness of the first polynomial since the roots of the two polynomials differ by $d\in\mathbb{R}$.
For each $(u,v)\in E$, define the rank-1 matrices

$$L^1_{u,v}=(e_u-e_v)(e_u-e_v)^T,\quad L^{-1}_{u,v}=(e_u+e_v)(e_u+e_v)^T$$

where e_u is the standard unit vector in $\mathbb{R}^{|E|}$ pointing in the direction of u. Consider a signing s on G and let $s_{u,v}$ denote $s((u,v))$. We show that

$$dI-A_s=\sum_{(u,v)\in E}L^{s_{u,v}}_{u,v}+D$$

where D is the diagonal matrix with entries

$$D_{u,v}=\begin{cases} d-d_u, & u=v;\\ 0, & u\neq v.\end{cases}$$

Indeed, it's easy to see that

$$\left(\sum_{(u,v)\in E}L^{s_{u,v}}_{u,v}\right)_{i,j}=\begin{cases} d_u, & i=j=u;\\ -s_{u,v}, & i=u,j=v,i\neq j.\end{cases}$$

Since D is diagonal with non-negative diagonal entries, it's positive semi-definite. Let $a_{u,v} := (e_u - e_v)$ and $b_{u,v} := (e_u + e_v)$. Then we have

$$(*) = \sum_{\{\pm 1\}^m} \left(\prod_{i:s_i=1} p_i \right) \left(\prod_{i:s_i=-1} (1-p_i) \right) \det \left(xI + D + \sum_{s_{u,v}=1} a_{u,v} a_{u,v}^T + \sum_{s_{u,v}=-1} b_{u,v} b_{u,v}^T \right)$$

and this polynomial is real rooted by Theorem 7.6. □

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