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A cover of a graph G is a graph H which is "locally isomorphic" to G: if we zoom in to the edge neighborhood around any vertex in H, the corresponding edge neighborhood of G looks identical. For a fixed cover H of G there is a natural action of $\pi_1(G, v_0)$, the fundamental group of G, on the vertices in H over v_0. Conversely, given an action of $\pi_1(G, v_0)$ on a finite set F we can construct a finite cover of G. We will elaborate on both correspondences and show that they induce an "equivalence of categories."

GROUPS AND COVERS OF GRAPHS

by

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To Simon.

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Andrew W. Herring

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Chapter 1

Introduction

1.1 Covering Spaces in Algebraic Topology

An important tool in Algebraic Topology is that of a covering space. In imprecise terms, a covering space of a topological space X is a topological space $\tilde{X}$ together with a continuous map $\phi : \tilde{X} \to X$ which is a "local isomorphism." Whereas X and $\tilde{X}$ may look quite different from a global perspective, the local isomorphism property dictates that if we zoom in close enough, the local pictures of X and $\tilde{X}$ are identical.

As an example of their utility, one can use covering spaces to compute the fundamental group of the circle S^1 [1]. As another example, one proves the Bruower Fixed Point Theorem using covering spaces and derives as a ridiculous yet mathematically valid consequence the following: at any particular moment in time, there exist antipodal points P and Q on the Earth's surface such that the wind speed and temperature at P are precisely the wind speed and temperature at Q [1].

1.1.1 Classifying Covering Spaces

Isomorphism classes of covering spaces of a fixed topological space X are classified by actions of $\pi_1(X, y)$ (the first fundamental group of X with base point y) on the set of points in the covering spaces which correspond to y [2]. To a fixed cover $\tilde{X}$, there is a natural group action of $\pi_1(X, y)$ on $\phi^{-1}(y)$, called the **Monodromy Action**, which has been studied extensively. Conversely, if we start with an arbitrary group action of $\pi_1(X, y)$ on $\phi^{-1}(y)$, we can construct a covering space $\tilde{X}$ of X from this data [2]. This pair of processes taking us between the realms of covering spaces and sets on which

$\pi_1(X, y)$ acts is an example of an "equivalence of categories." Roughly speaking, when two categories are equivalent, they may be regarded as essentially the same [3]

1.2 Present Work

In this thesis we hope to describe an analogous equivalence of categories in the particular context of graphs. To this end we will need to precisely pin down those graphs amenable to this goal (Chapters 2 and 3), we will say what we mean by the fundamental group of a graph (Chapter 4), we will say what is precisely meant by an equivalence of categories (Chapter 5), we will define a functor constituting "one half" of the categorical equivalence (Chapter 6), we will define a functor constituting the "second half" (Chapter 7), and we will finally prove that the pair of functors satisfy the necessary conditions to induce an equivalence of categories (Chapter 8). Wherever possible, we will comment on parallels between our development and the development in the more widely studied context of Algebraic Topology.

Chapter 2

Directed Graphs

2.1 Category of Directed Graphs

We wish to define a category of directed graphs in which to work. In imprecise terms, a category is a collection of objects and for every pair of objects A, B a set $\mathrm{Hom}(A, B)$ of maps called morphisms. We also need a composition law whenever composing morphisms makes sense. Consider a few examples: there's the category **Set** whose objects are sets and whose morphisms are set maps, there's the category **Group** whose objects are groups and whose morphisms are group homomorphisms, the last example we consider is the category **Top** whose objects are topological spaces and whose morphisms are continuous maps. First we define the objects of our category: directed graphs

Definition 2.1.1. A **directed graph G** is a triple $(V(G), E(G), E(G) \xrightarrow{(t,h)} V(G)^2)$ with $V(G)$ a set consisting of the **vertices** of G, $E(G)$ a subset of $V(G)^2 := V(G) \times V(G)$ consisting of the **edges** of G, and the pair of maps $(t, h)E(G) \to V(G)^2$ which are called the head and tail maps respectively. Given $e \in E(G)$, $t(e)$ is called the **tail** of e and $h(e)$ is called the **head** of e.

We must also define the morphisms in our category of directed graphs:

Definition 2.1.2. Given two directed graphs

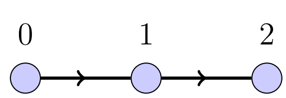
- $G_1 = (V(G_1), E(G_1), E(G_1) \xrightarrow{(t_{G_1}, h_{G_1})} V(G_1)^2)$,
- $G_2 = (V(G_2), E(G_2), E(G_2) \xrightarrow{(t_{G_2}, h_{G_2})} V(G_2)^2)$,

a morphism $G_1 \xrightarrow{f} G_2$ is a pair of maps $V(G_1) \xrightarrow{f_V} V(G_2)$ and $E(G_1) \xrightarrow{f_E} E(G_2)$ such that the following diagram commutes:

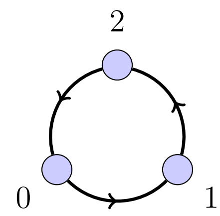
$$\begin{tikzcd} E(G_1) \arrow[r, "f_E"] \arrow[d, "(t_{G_1}, h_{G_1})"'] & E(G_2) \arrow[d, "(t_{G_2}, h_{G_2})"] \\ V(G_1)^2 \arrow[r, "f_V^2"'] & V(G_2)^2 \end{tikzcd}$$

where $f_V^2(v, w) = (f_V(v), f_V(w))$ for $v, w \in V_1$. If we compose right to left so that $f \circ g$ means "first g, then f," we have $(t_{G_2}, h_{G_2}) \circ f_E = f_V^2 \circ (t_{G_1}, h_{G_1})$.
When we say that a morphism $f = (f_E, f_V)$ is injective (respectively surjective), we mean that both of the maps f_V and f_E are injective (respectively surjective).
We'll use $\mathrm{Hom}(G_1, G_2)$ to denote the set of all (directed graph) morphisms from G_1 to G_2. We will use $\mathrm{Inj}(G_1, G_2)$ to denote the set of all injective (directed graph) morphisms from G_1 to G_2.

On occasion we'll drop the subscripts which indicate whether we're talking about f_V or f_E: if it's clear from the context that $v \in V(G)$ and $e \in E(G)$ then we'll write $f(v)$ to mean $f_V(v)$ and $f(e)$ to mean $f_E(e)$.
We comment that our definition of a morphism between directed graphs preserves all relevant graph theoretic data: namely adjacency of vertices via edges. This should be regarded as analogous to the case in Topology: there we study continuous maps between spaces which preserve all relevant topological data.
The picture of a directed graph will be constructed as follows: every vertex $v \in V(G)$ will be drawn as a circle and an edge $e \in E(G)$ will be drawn as a an arrow extending from $t(e)$ to $h(e)$. Consider the example of a graph P with $V(P) = \{0, 1, 2\}$, $E(P) = \{(0, 1), (1, 2)\}$ and where $(t, h)(0, 1) = (t(0, 1), h(0, 1)) = (0, 1)$ and $(t, h)(1, 2) = (1, 2)$. Then according to our rules for drawing we would have



Consider a second example of a directed graph C with $V(C) = \{0, 1, 2\}$, $E(C) = \{(0, 1), (1, 2), (2, 0)\}$ and with $(t, h)(0, 1) = (0, 1)$, $(t, h)(1, 2) = (1, 2)$, and $(t, h)(2, 0) = (2, 0)$. Then this graph looks like



Each of the above two examples belongs to a class of directed graphs.

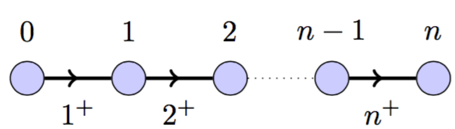
Definition 2.1.3. Let n be an arbitrary. fixed, non-negative integer. Then the directed graph $\mathbf{Path}_n^+$ is given by

- $V(\text{Path}_n^+) = \{0, 1, 2, \ldots, n-1, n\}$;
- $E(\text{Path}_n^+) = \{(i-1, i) : 1 \leq i \leq n\}$;
- $(t, h)(i-1, i) = (i-1, i)$ for $1 \leq i \leq n$.

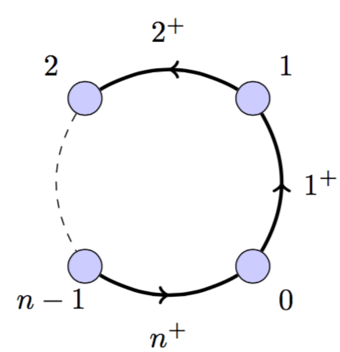
The directed graph $\mathbf{Cycle}_n^+$ is given by

- $V(\text{Cycle}_n^+) = \{0, 1, 2, \ldots, n-1\}$;
- $E(\text{Cycle}_n^+) = \{(i-1, i) : 1 \leq i \leq n-1\} \cup \{(n-1, 0)\}$;
- $(t, h)(i, i+1) = (i, i+1)$ for $0 \leq i \leq n-2$ and $(t, h)(n-1, 0) = (n-1, 0)$.

For notational simplicity, we'll denote the edge $(i-1, i)$ in Path_n^+ by i^+ for $1 \leq i \leq n$. We'll similarly allow i^+ to denote the edge $(i-1, i)$ in Cycle_n^+ for every $1 \leq i \leq n-1$ and n^+ will denote the edge $(n-1, 0)$. In particular the first two examples given were Path_2^+



(a) Path_n^+



(b) Cycle_n^+

and Cycle_3^+ respectively.

We now give some examples of morphisms.

(1) For any directed graph G there is the identity morphism $\mathrm{Id}_{\mathrm{G}} \in \mathrm{Hom}(G, G)$ which acts as the identity (set) map on both $E(G)$ and $V(G)$. Given any category and any object G, the identity morphism $\mathrm{Id}_{\mathrm{G}} \in \mathrm{Hom}(G, G)$ is characterized by the following property: for any other object H and morphisms $f \in \mathrm{Hom}(G, H)$ and $g \in \mathrm{Hom}(H, G)$ we have $\mathrm{Id}_{\mathrm{G}} \circ f = f$ and $g \circ \mathrm{Id}_{\mathrm{G}} = g$.

(2) We define a **quotient morphism** f in the category of directed graphs to be any morphism which is surjective (which by our convention means that f_V and f_E are each surjective). We give an example of a quotient morphism $\psi^+ : \mathrm{Path}_n^+ \to \mathrm{Cycle}_n^+$. We define

- $\psi_V^+(j) = j$ for $j = 0, 1, \dots, n-1$ and $\psi_V^+(n) = 0$;
- $\psi_E^+(j^+) = j^+$ for $j = 1, 2, \dots, n$.

We prove that $\psi^+ = (\psi_V^+, \psi_E^+)$ is a morphism of directed graphs. For simplicity, let's adopt the notation that $P = \mathrm{Path}_n^+$ and $C = \mathrm{Cycle}_n^+$. Let $j^+ \in E(P)$ be arbitrary. Then we compute:

$$((t_C, h_C) \circ \psi_E^+)(j^+) = (t_C, h_C)(j^+) = \begin{cases} (j-1, j) & \text{if (j = 1 , \dots, n-1);} \\ (n-1, 0) & \text{if } j = n. \end{cases}$$

On the other hand,

$$\begin{aligned} ((\psi_V^+)^2 \circ (t_P, h_P))(j^+) &= (\psi_V^+)^2((j-1, j)) \\ &= (\psi_V^+(j-1), \psi_V^+(j)) \\ &= \begin{cases} (j-1, j) & \text{if (j = 1 , \dots, n-1);} \\ (n-1, 0) & \text{if } j = n. \end{cases} \end{aligned}$$

Therefore we have that $((\psi_V^+)^2 \circ (t_P, h_P)) = ((t_C, h_C) \circ \psi_E^+)$ and this is precisely what it means for $\psi \in \mathrm{Hom}(P, C)$. Essentially the morphism ψ^+ identifies 0 and n in $V(\mathrm{Path}_n^+)$ and the resulting directed graph is none other than Cycle_n^+.

2.2 Subgraphs

Definition 2.2.1. Let $G = (V(G), E(G), E(G) \xrightarrow{(t,h)} (V(G))^2)$ be a directed graph. A directed graph $H = (V(H), E(H), E(H) \xrightarrow{(t_H, h_H)} (V(H)^2)$ is a **subgraph of** G if and only if the following hold:

(1) $V(H) \subseteq V(G)$,

(2) $E(H) \subseteq (t,h)^{-1}((V(H))^2)$,

(3) $(t_H, h_H)(e) = (t,h)(e)$ for every $e \in E(H)$.

The subgraph H is an **induced subgraph of** G if $E(H) = (t,h)^{-1}((V(H))^2)$. In this case we will write $H = G|_{V(H)}$.

Consider the examples depicted in Figure (2.1)

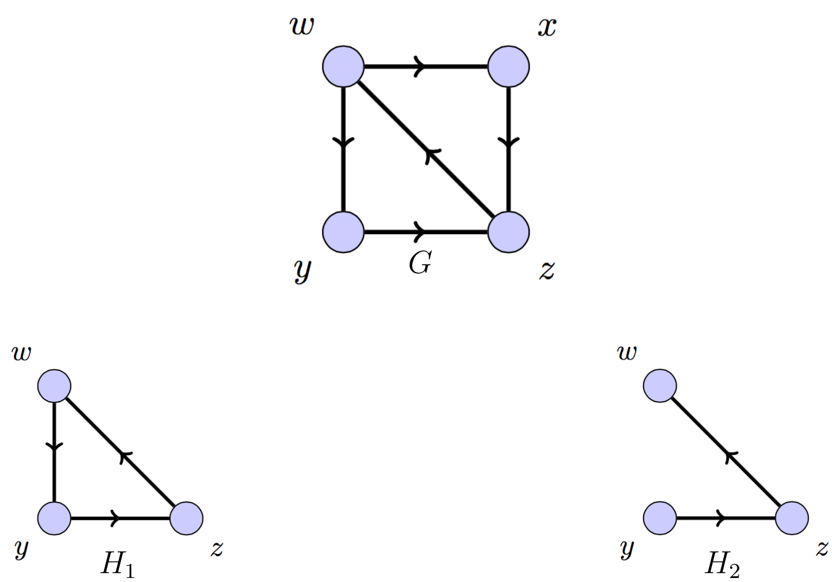


Figure 2.1: H_1 and H_2 are both subgraphs of G. H_1 is induced, whereas H_2 is not.

2.3 Connectedness

Here we define what it means for a directed graph to be connected and define the connected components of a graph. To do so, let n be a fixed, non-negative integer and let $\epsilon \in \{+,-\}^n$, for example if $n = 3$ then such an ϵ is $\epsilon = (+,-,+)$. We'll let ϵ_i denote the i-th coordinate of ϵ so that in our example $\epsilon_1 = + = \epsilon_3$ and $\epsilon_2 = -$. Then we define the directed graph Path_n^ϵ by

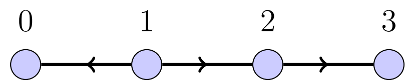
- $V(\mathrm{Path}_n^\epsilon) = \{0, 1, \ldots, n\}$;

- $E(\mathrm{Path}_n^\epsilon) = \{(i-1, i) : \epsilon_i = +\} \cup \{(i, i-1) : \epsilon_i = -\}$;

- $(t,h)((i-1,i)) = (i-1,i)$ and $(t,h)((i,i-1)) = (i,i-1)$.

For example,

$$\mathrm{Path}_3^{(-,+,+)} =$$



Now given a directed graph G, we define a relation on the vertex set $V(G)$ as follows: given $v, w \in V(G)$ we say $v \sim_n w$ if and only if there exists $\epsilon \in \{+,-\}^n$ and an injective morphism $f \in \mathrm{Inj}(\mathrm{Path}_n^{\epsilon}, G)$ such that $f(0) = v$ and $f(n) = w$. Furthermore, given $v, w \in V(G)$ we define $v \approx w$ if and only if there exists an $n \geq 0$ for which $v \sim_n w$. It's not difficult to verify that $\approx$ defines an equivalence relation on $V(G)$. Therefore if we let $[v] = \{w \in V(G) : w \approx v\}$ denote the equivalence class containing v, then we can consider the quotient set $(V/\approx) := \{[v] : v \in V(G)\}$ consisting of all the equivalence classes under $\approx$ in $V(G)$.

Definition 2.3.1. The **connected component** of G containing $v \in V(G)$, denoted $\mathbf{comp_G(v)}$, is defined to be the induced subgraph of G with vertex set $[v]$. G is **connected** if and only if it has a single connected component.

We briefly comment that our notion of connectedness ought to be regarded as analogous to the topological property of a space being path connected. Indeed, a path connected topological space is one in which any two points of the space can be connected by a path contained entirely within the space. In the topological case, every path connected space is connected but the converse need not hold (see [1] P. 90).

The connected components of a directed graph G partition G into disjoint induced subgraphs in a sense which will be made precise by the following lemma.

Lemma 2.3.1. *Let $v, w \in V(G)$. Then exactly one of the following holds:*

(i) $\mathrm{comp}_G(v) = \mathrm{comp}_G(w)$*;*

(ii) for any $v' \in V(\mathrm{comp}_G(v))$ and any $w' \in V(\mathrm{comp}_G(w))$, we have $v' \not\approx w'$.

Proof. Let's first assume that (i) does not hold. Since $\mathrm{comp}_G(v)$ and $\mathrm{comp}_G(w)$ are induced subgraphs of G, then the only possibility is that $[v] = V(\mathrm{comp}_G(v)) \neq V(\mathrm{comp}_G(w)) = [w]$. Now, let $v' \in V(\mathrm{comp}_G(v))$ and $w' \in V(\mathrm{comp}_G(w))$ be arbitrary. Since $[v] \neq [w]$ it follows that $v' \not\approx w'$.

Now we assume that (i) holds and prove that (ii) does not. If $\mathrm{comp}_G(v) = \mathrm{comp}_G(w)$, then in particular $[v] = V(\mathrm{comp}_G(v)) = V(\mathrm{comp}_G(w)) = [w]$ and $v \approx w$. Therefore in this case (ii) fails.

Finally we assume (ii) and prove that (i) does not hold. We certainly have that $v \in V(\mathrm{comp}_G(v))$ and $w \in V(\mathrm{comp}_G(w))$. Then by hypothesis $v \not\approx w$, so that $V(\mathrm{comp}_G(v)) = [v] \neq [w] = V(\mathrm{comp}_G(w))$, hence $\mathrm{comp}_G(v) \neq \mathrm{comp}_G(w)$, which completes the proof.

□

It follows from the lemma that for any directed graph G, we can write

$$G = \coprod_{[v]\in(V/\approx)} \mathrm{comp}_G(v)$$

2.4 Category of Pointed Graphs

A second category of interest to us is the Category of Pointed Graphs. An object in this category is a pair (G, W) where G is a directed graph and W is a subset of $V(G)$: we call such an object a **pointed graph**. Given two pointed graphs (G, W) and (H, Y), **a morphism of pointed graphs** is a map $f = (f_V, f_E)$ for which the following hold:

(1) $f \in \mathrm{Hom}(G, H)$;

(2) $f_V(W) \subseteq Y$;

(3) $(f_V)|_W$ is injective.

Of particular interest will be the case where the subset $W \subseteq V(G)$ contains a single vertex. It's clear from the definitions that if $(G, \{v\})$ and $(H, \{w\})$ are two such pointed graphs, then any pointed graph morphism f between them must satisfy $f_V(v) = w$.

2.5 Operations on Directed Graphs

Here we describe a few procedures for creating new directed graphs from old ones. One such way is to take the transpose of a directed graph:

Definition 2.5.1. Given a directed graph $G = (V(G), E(G), E(G) \xrightarrow{(t,h)} V(G)^2)$ we define the **graph transpose**, denoted G^T, by

$$G^T = (V(G^T), E(G^T), E(G^T) \xrightarrow{(t_{G^T}, h_{G^T})} V(G^T)^2)$$

where

- $V(G^T) = V(G)$;
- $E(G^T) = E(G)$;
- $(t_{G^T}, h_{G^T})(e) = (h(e), t(e))$ for $e \in E(G^T)$.

In other words, to obtain G^T from G we simply switch the head and tail of every single edge in G. This allows us to define two new directed graphs: let $\text{Path}_n^- := (\text{Path}_n^+)^T$ and $\text{Cycle}_n^- := (\text{Cycle}_n^+)^T$. We'll adopt similar notations as before so that in Path_n^-, j^- will denote the edge $(j, j-1)$ for every $j = 1, \ldots, n$ and so that in Cycle_n^-, j^- will denote the edge $(j, j-1)$ for every $j = 1, \ldots, n-1$ and n^- will denote the edge $(0, n-1)$.

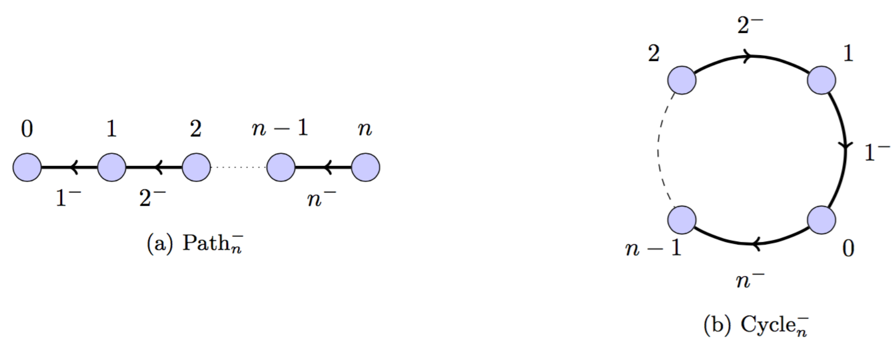


Figure 2.2: Two more directed graphs.

The next construction details how to "glue" two directed graphs along a common directed subgraph. We first describe the construction in the category of sets. As such, let S_1 and S_2 be two sets. We define the **disjoint union**, $S_1 \coprod S_2$, by

$$S_1 \coprod S_2 := \bigcup_{j=1}^{2} \{(s_j, j) : s_j \in S_j\}$$

An immediate consequence of the definition is that we have a pair of bijective (set) maps $\iota_1 : S_1 \to S_1 \coprod S_2$ and $\iota_2 : S_2 \to S_1 \coprod S_2$ defined by

$$\iota_j(s_j) = (s_j, j)$$

for $j = 1, 2$.

The disjoint union is precisely the coproduct in the category of sets. By this we mean that $S_1 \coprod S_2$ satisfies the following universal property: given a set Y and a pair of injections

$f_1 : S_1 \to Y$ and $f_2 : S_2 \to Y$, there is a unique map $f : S_1 \coprod S_2 \to Y$ such that the following diagram commutes:

$$\begin{tikzcd}
S_1 \arrow[dr, hook, "\iota_1"'] \arrow[drr, "f_1"] & & \\
& S_1 \coprod S_2 \arrow[r, dashed, "f"] & Y \\
S_2 \arrow[ur, hook, "\iota_2"] \arrow[urr, "f_2"'] & &
\end{tikzcd}$$

Indeed, suppose that injections $f_1 : S_1 \to Y$ and $f_2 : S_2 \to Y$ exist. Given $(s_j, j) \in S_1 \coprod S_2$, then we define $f(s_j, j) = f_j(s_j)$. To prove that f as defined is unique, suppose that we had another map $g : S_1 \coprod S_2 \to Y$ which made the diagram

$$\begin{tikzcd}
S_1 \arrow[dr, hook, "\iota_1"'] \arrow[drr, "f_1"] & & \\
& S_1 \coprod S_2 \arrow[r, dashed, "g"] & Y \\
S_2 \arrow[ur, hook, "\iota_2"] \arrow[urr, "f_2"'] & &
\end{tikzcd}$$

commute. Then for any $(s_j, j) \in S_1 \coprod S_2$, we would have

$$\begin{aligned}
g((s_j, j)) &= (g \circ \iota_j)(s_j) \\
&= f_j(s_j) \\
&= f((s_j, j))
\end{aligned}$$

Now suppose that we have a set I and a pair of injective maps $f_1 : I \to S_1$ and $f_2 : I \to S_2$. We define a relation $\sim$ on $S_1 \coprod S_2$: we say that $(s_1, 1) \sim (s_2, 2)$ if and only if there exits $\alpha \in I$ such that $s_1 = f_1(\alpha)$ and $s_2 = f_2(\alpha)$ and we also impose reflexivity so that $(s_j, j) \sim (s_j, j)$ for any $(s_j, j) \in S_1 \coprod S_2$.

Lemma 2.5.1. *The relation $\sim$ on $S_1 \coprod S_2$ is an equivalence relation.*

Proof. First we observe that by definition of $\sim$, if $(s_j, j) \sim (t_j, j)$ then it must be that $s_j = t_j$. Notice also that symmetry of $\sim$ is immediate. We prove that $\sim$ is transitive so that it is an equivalence relation. Let $(s_j, j), (t_k, k), (r_\ell, \ell) \in S_1 \coprod S_2$ and suppose that $(s_j, j) \sim (t_k, k)$ and $(t_k, k) \sim (r_\ell, \ell)$. First, if $j = k = \ell$ then we have $s_j = t_k = r_\ell$ and $(s_j, j) \sim (r_\ell, \ell)$. If $j = k$ and $k \neq \ell$, then $s_j = t_k$ and there exists some $\alpha \in I$ such that $f_k(\alpha) = t_k$ and $f_\ell(\alpha) = r_\ell$. But then since $j = k$ and $s_j = t_k$ we have $f_j(\alpha) = s_j$ and $f_\ell(\alpha) = r_\ell$. The case $j \neq k$ and $k = \ell$ is handled symmetrically as the previous. Suppose now that $j \neq k$ and $k \neq \ell$. Since $j, k, \ell \in \{1, 2\}$ we must have $j = \ell$. Then there exist $\alpha_1, \alpha_2 \in I$ such that

$$f_j(\alpha_1) = s_j \;\; f_k(\alpha_2) = t_k$$
$$f_k(\alpha_1) = t_k \;\; f_\ell(\alpha_2) = r_\ell$$

But then since f_k is injective, we see that $\alpha_1 = \alpha_2$. Then, since $j = \ell$

$$\begin{aligned} r_\ell &= f_\ell(\alpha_2) \\ &= f_j(\alpha_1) \\ &= s_j \end{aligned}$$

So $(s_j, j) = (r_\ell, \ell)$ and thus $(s_j, j) \sim (r_\ell, \ell)$. Therefore $\sim$ is transitive, hence an equivalence relation on $S_1 \coprod S_2$. $\square$

Let $[s_j, j] := \{(t_k, k) \in S_1 \coprod S_2 : (t_k, k) \sim (s_j, j)\}$ denote the equivalence class containing (s_j, j). We can therefore define the quotient set

$$S_1 \bigcup_I S_2 := (S_1 \coprod S_2)/\sim \, = \{[s_j, j] : (s_j, j) \in S_1 \coprod S_2\}$$

and we allow

$$\pi : S_1 \coprod S_2 \to S_1 \bigcup_I S_2$$

to denote the quotient map which sends (s_j, j) to its equivalence class $[s_j, j]$. According to the diagram

$$
\begin{tikzcd}
I \arrow[rrr, hook, "f_1"] \arrow[ddd, hook, "f_2"'] \arrow[dr, equal] & & & S_1 \arrow[ddd, hook, dashed, "\tilde{f}_1 = \pi\circ\iota_1"] \arrow[dl, equal] \\
& I \arrow[r, hook, "f_1"] \arrow[d, hook, "f_2"'] & S_1 \arrow[d, hook, two heads, "\iota_1"] & \\
& S_2 \arrow[r, hook, two heads, "\iota_2"'] \arrow[dl, equal] & S_1 \coprod S_2 \arrow[dr, two heads, "\pi"] & \\
S_2 \arrow[rrr, hook, dashed, "\tilde{f}_2 = \pi\circ\iota_2"'] & & & S_1 \bigcup_I S_2
\end{tikzcd}
$$

we obtain maps $\tilde{f}_1 := \pi \circ \iota_1 : S_1 \to S_1 \bigcup_I S_2$ and $\tilde{f}_2 := \pi \circ \iota_2 : S_2 \to S_1 \bigcup_I S_2$.

Lemma 2.5.2. *$\tilde{f}_1$ and $\tilde{f}_2$ are injective.*

Proof. Suppose that $\tilde{f}_1(s_1) = \tilde{f}_1(t_1)$ for some $s_1, t_1 \in S_1$. Then

$$
\begin{aligned}
[s_1, 1] &= \pi((s_1, 1)) \\
&= \pi(\iota_1(s_1)) \\
&= \tilde{f}_1(s_1) \\
&= \tilde{f}_1(t_1) \\
&= \pi(\iota_1(t_1)) \\
&= \pi((t_1, 1)) \\
&= [t_1, 1]
\end{aligned}
$$

Thus $(s_1, 1) \sim (t_1, 1)$ and so we must have $s_1 = t_1$ which proves that $\tilde{f}_1$ is injective. $\square$

It's interesting to note that the "inner square" of the diagram does not commute ($\iota_1 \circ f_1 \neq \iota_2 \circ f_2$) whereas the "outer square" does ($\tilde{f}_1 \circ f_1 = \tilde{f}_2 \circ f_2$). Indeed, let $\alpha \in I$ be arbitrary. Then

$$(\tilde{f}_1 \circ f_1)(\alpha) = (\pi \circ \iota_1 \circ f_1)(\alpha) = \pi((f_1(\alpha), 1)) = [f_1(\alpha), 1]$$

On the other hand

$$(\tilde{f}_2 \circ f_2)(\alpha) = (\pi \circ \iota_2 \circ f_2)(\alpha) = \pi((f_2(\alpha), 2)) = [f_2(\alpha), 2]$$

But $(f_1(\alpha), 1) \sim (f_2(\alpha), 2)$ so we have that $[f_1(\alpha), 1] = [f_2(\alpha), 2]$, as desired.

Philosophically, we think about two sets S_1 and S_2 and I their common intersection (indeed this is the case if f_1 and f_2 are inclusion maps). Then in the quotient $S_1 \bigcup_I S_2$ we "glue" the copies S_1 and S_2 in $S_1 \coprod S_2$ along I.

We're now prepared to describe the construction for graphs. First off, if we're given two directed graphs

- $G_1 = (V_1, E_1, E_1 \xrightarrow{(t_1,h_1)} V_1^2)$;
- $G_2 = (V_2, E_2, E_2 \xrightarrow{(t_2,h_2)} V_2^2)$,

then we define the disjoint union, $G_1 \coprod G_2$ to be the directed graph with

- $V(G_1 \coprod G_2) = V_1 \coprod V_2$;
- $E(G_1 \coprod G_2) = E_1 \coprod E_2$;
- $E(G_1 \coprod G_2) \xrightarrow{(t_{\coprod},h_{\coprod})} (V(G_1 \coprod G_2))^2$,

where $(t_{\coprod}, h_{\coprod})(e_j, j) = ((t_j(e_j), j), (h_j(e_j), j))$.

Notice that as a consequence of the definition, we have injective maps $(\iota_j)_V : V_j \to V(G_1 \coprod G_2)$ and $(\iota_j)_E : E_j \to E(G_1 \coprod G_2)$ for $j = 1, 2$. One easily shows that $\iota_1 := ((\iota_1)_V, (\iota_2)_E) \in \mathrm{Hom}(G_1, G_1 \coprod G_2)$ and $\iota_2 := ((\iota_2)_V, (\iota_2)_E) \in \mathrm{Hom}(G_2, G_1 \coprod G_2)$.

Theorem 2.5.1. *$G_1 \coprod G_2$ as defined is the coproduct in the category of directed graphs. That is, given a third directed graph $H = (V(H), E(H), E(H) \xrightarrow{(t_H,h_H)} V(H)^2)$ and injective morphisms $f_1 \in \mathrm{Hom}(G_1, H)$ and $f_2 \in \mathrm{Hom}(G_2, H)$, there exists a unique morphism $f \in \mathrm{Hom}(G_1 \coprod G_2, H)$ such that the following diagram commutes:*

$$\begin{tikzcd}
G_1 \arrow[dr, hook, "\iota_1"'] \arrow[drr, hook, "f_1"] & & \\
& G_1 \coprod G_2 \arrow[r, dashed, "f"] & H \\
G_2 \arrow[ur, hook, "\iota_2"] \arrow[urr, hook, "f_2"'] & &
\end{tikzcd}$$

Proof. Assume that there exist injective morphisms $f_1 \in \mathrm{Hom}(G_1, H)$ and $f_2 \in \mathrm{Hom}(G_2, H)$. By the Universal Property for the Coproduct of Sets, there exists a unique (set) map $f_V : V_1 \coprod V_2 \to V(H)$ such that $(f_1)_V = f_V \circ (\iota_1)_V$ and $(f_2)_V = f_V \circ (\iota_2)_V$ defined by $f_V(v_j, j) = (f_j)_V(v_j)$. By the same reasoning, there exists a unique (set) map f_E :

$E_1 \coprod E_2 \to E(H)$ which satisfies $(f_1)_E = f_E \circ (\iota_1)_E$ and $(f_2)_E = f_E \circ (\iota_2)_E$ defined by $f_E(e_j, j) = (f_j)_E(e_j)$. We prove that $f := (f_V, f_E) \in \mathrm{Hom}(G_1 \coprod G_2, H)$. This entails showing that the diagram

$$\begin{tikzcd} E_1 \coprod E_2 \arrow[r, "f_E"] \arrow[d, "(t_{\coprod}, h_{\coprod})"'] & E(H) \arrow[d, "(t_H, h_H)"] \\ (V_1 \coprod V_2)^2 \arrow[r, "f_V^2"'] & V(H)^2 \end{tikzcd}$$

commutes. To that end, let $(e_j, j) \in E_1 \coprod E_2$. Then

$$(t_H, h_H)(f_E(e_j, j)) = (t_H, h_H)(f_j)_E(e_j)$$

On the other hand,

$$\begin{aligned} (f_V)^2((t_{\coprod}, h_{\coprod})(e_j, j)) &= (f_V)^2((t_j(e_j), j), (h_j(e_j), j)) \\ &= (f_V(t_j(e_j), j), f_V(h_j(e_j), j)) \\ &= ((f_j)_V(t_j(e_j)), (f_j)_V(h_j(e_j))) \end{aligned}$$

But we assumed that $f_j \in \mathrm{Hom}(G_j, H)$. Therefore the diagram

$$\begin{tikzcd} E_j \arrow[r, "(f_j)_E"] \arrow[d, "(t_j, h_j)"'] & E(H) \arrow[d, "(t_H, h_H)"] \\ (V_j)^2 \arrow[r, "(f_j)_V^2"'] & V(H)^2 \end{tikzcd}$$

commutes: given $e_j \in E_j$ we have

$$\begin{aligned} (t_H, h_H)(f_j)_E(e_j) &= (f_j)_V^2(t_j(e_j), h_j(e_j)) \\ &= ((f_j)_V(t_j(e_j)), (f_j)_V(h_j(e_j))) \end{aligned}$$

This proves that $(t_H, h_H)(f_E(e_j, j)) = (f_V)^2((t_{\coprod}, h_{\coprod})(e_j, j))$ and hence that $f \in \mathrm{Hom}(G_1 \coprod G_2, H)$. The uniqueness of f is an immediate consequence of the uniqueness of the maps f_V and f_E. $\square$

Now suppose that we have another graph directed $H = (V, E, E \xrightarrow{(t,h)} V^2)$ and a pair of injective directed graph morphisms $f_1 \in \mathrm{Hom}(H, G_1)$ and $f_2 \in \mathrm{Hom}(H, G_2)$ then we define $G_1 \bigcup_H G_2$ to be the directed graph with

- $V(G_1 \bigcup_H G_2) = V_1 \bigcup_V V_2$;

- $E(G_1 \bigcup_H G_2) = E_1 \bigcup_E E_2$;

- $E(G_1 \bigcup_H G_2) \xrightarrow{(t_{\bigcup_H}, h_{\bigcup_H})} (V(G_1 \bigcup_H G_2))^2$,

where $(t_{\bigcup_H}, h_{\bigcup_H})([e_j, j]) = ([t_j(e_j), j], [h_j(e_j), j])$. We show that the head and tail maps $t_{\bigcup_H}$ and $h_{\bigcup_H}$ are well defined. Suppose that $[e_j, j] = [e_k, k] \in E_1 \bigcup_E E_2$. If $j = k$, then we must have $e_j = e_k$, and hence

$$\begin{aligned}(t_{\bigcup_H}, h_{\bigcup_H})([e_j, j]) &= ([t_j(e_j), j], [h_j(e_j), j]) \\ &= ([t_k(e_k), k], [h_k(e_k), k]) \\ &= (t_{\bigcup_H}, h_{\bigcup_H})([e_k, k])\end{aligned}$$

So assume that $j \neq k$. Because $(e_j, j) \sim (e_k, k) \in E_1 \coprod E_2$, there exists $e \in E$ such that $e_j = (f_j)_E(e)$ and $e_k = (f_k)_E(e)$. Recalling that $f_j \in \mathrm{Hom}(H, G_j)$ for $j = 1, 2$, we have that each square in the diagram

$$\begin{tikzcd}
E_1 \arrow[d, "{(t_1,h_2)}"'] & E \arrow[l, "(f_1)_E"'] \arrow[r, "(f_2)_E"] \arrow[d, "{(t,h)}"] & E_2 \arrow[d, "{(t_2,h_2)}"] \\
V_1^2 & V^2 \arrow[l, "(f_1)_V^2"] \arrow[r, "(f_2)_V^2"'] & V_2^2
\end{tikzcd}$$

commutes. Therefore we have

$$\begin{aligned}(t_j, h_j)(e_j) &= (t_j, h_j) \circ (f_j)_E(e) \\ &= (f_j)_V^2 \circ (t, h)(e)\end{aligned}$$

and

$$\begin{aligned}(t_k, h_k)(e_k) &= (t_k, h_k) \circ (f_k)_E(e) \\ &= (f_k)_V^2 \circ (t, h)(e)\end{aligned}$$

In particular we obtain a system of equations

$$\begin{cases} t_j(e_j) = (f_j)_V(t(e)) \\ h_j(e_j) = (f_j)_V(h(e)) \\ t_k(e_k) = (f_k)_V(t(e)) \\ h_k(e_k) = (f_k)_V(h(e)) \end{cases}$$

The first and third of these equations implies that $(t_j(e_j), j) \sim (t_k(e_k), k) \in V_1 \coprod V_2$ whence $[t_j(e_j), j] = [t_k(e_k), k] \in V_1 \bigcup_V V_2$. The second and fourth of the equations implies that $(h_j(e_j), j) \sim (h_k(e_k), k) \in V_1 \coprod V_2$ so that $[h_j(e_j), j] = [h_k(e_k), k] \in V_1 \bigcup_V V_2$. Therefore

$$\begin{aligned}(t_{\bigcup_H}, h_{\bigcup_H})[e_j, j] &= ([t_j(e_j), j], [h_j(e_j), j]) \\ &= ([t_k(e_k), k], [h_k(e_k), k]) \\ &= (t_{\bigcup_H}, h_{\bigcup_H})[e_k, k]\end{aligned}$$

Therefore the head and tail maps on $G_1 \bigcup_H G_2$ are well defined.

Finally we come to our first example of gluing graphs together. Let $H = \mathrm{Path}_3^+$, let $G_1 = \mathrm{Cycle}_5^+$ and let $G_2 = \mathrm{Cycle}_6^+$. Refer to Figure (2.3) for the names of vertices and edges in each.

Then we have injective morphisms $f_1 \in \mathrm{Hom}(H, G_1)$ and $f_2 \in \mathrm{Hom}(H, G_2)$. $f_1 = ((f_1)_V, (f_1)_E)$ is given by $(f_1)_V(j) = v_j$ for $j = 0, 1, 2$ and $(f_1)_E(j^+) = e_j^+$ for $j = 1, 2$. $f_2 = ((f_2)_V, (f_2)_E)$ is defined by $(f_2)_V(j) = w_j$ for $j = 0, 1, 2$ and $(f_2)_E(j^+) = f_j^+$ for $j = 1, 2$. It should be clear that f_1 and f_2 respect the head and tail maps.

We consider two more important examples of graph gluing. First, let H be the directed graph with vertex set $\{0, 1, \ldots, n\}$ and with empty edge set. We let $G_1 = \mathrm{Path}_n^+$ and $G_2 = \mathrm{Path}_n^-$. It's clear that we have injective morphisms $f_1 \in \mathrm{Hom}(H, G_1)$ and $f_2 \in \mathrm{Hom}(H, G_2)$. Then we define $\mathbf{Path}_n := G_1 \bigcup_H G_2$, the (symmetric) path on n-vertices (see Figure (2.4)). For the next example let H be the directed graph $V(H) = \{0, 1, \ldots, n-1\}$ and with empty vertex set, let $G_1 = \mathrm{Cycle}_n^+$ and let $G_2 = \mathrm{Cycle}_n^-$ with the obvious injective morphisms $f_1 \in \mathrm{Hom}(H, G_1)$ and $\mathrm{Hom}(H, G_2)$. Then we define $\mathrm{Cycle}_n := G_1 \bigcup_H G_2$ to be the (symmetric) cycle with n-edges (see Figure (2.5)).

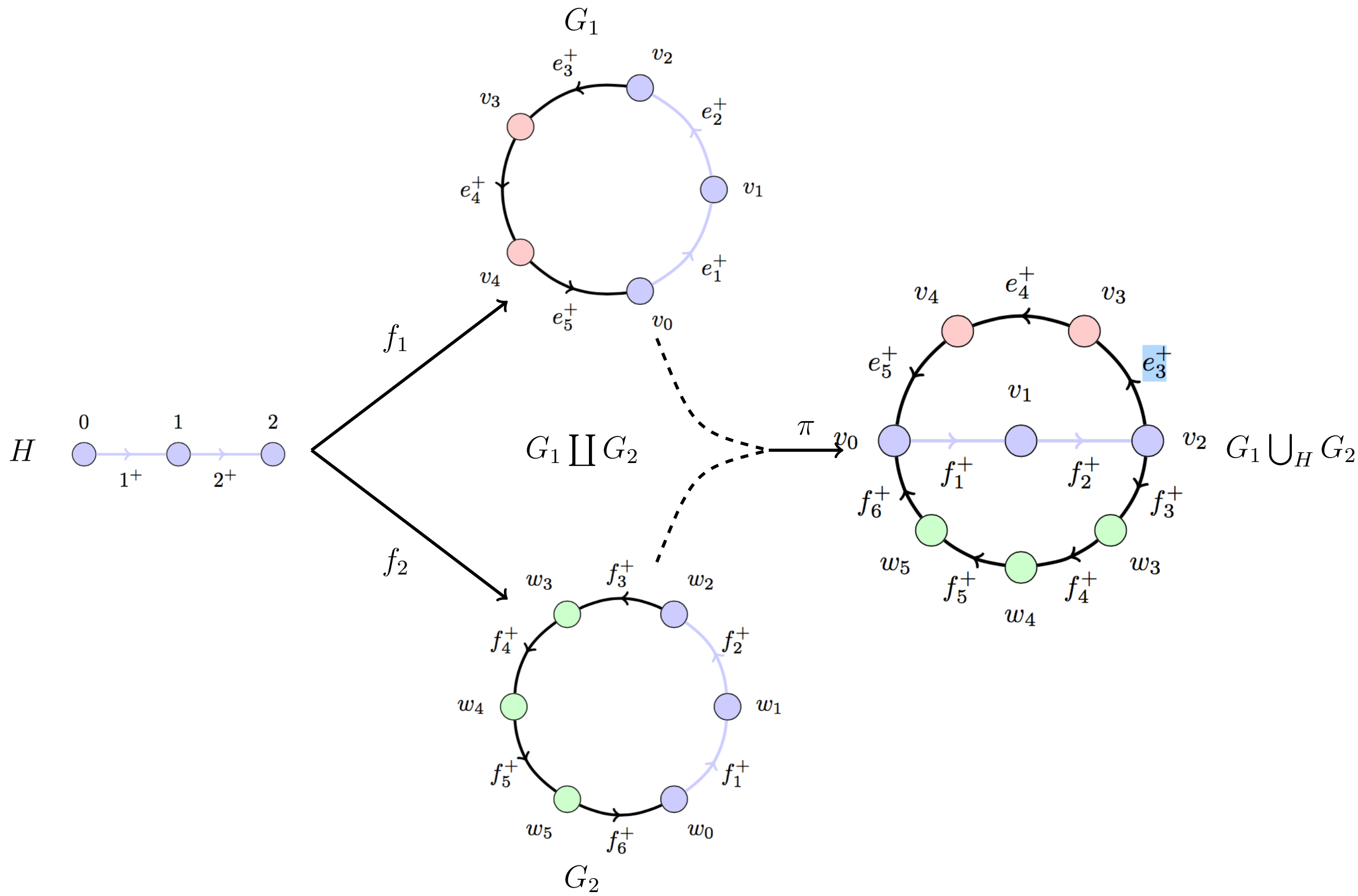


Figure 2.3: Two graphs glued along a common subgraph.

2.6 Covers

Covers

We have arrived at one of the most important ideas of the present work: that of a cover. Before defining covers, we need to define the (edge) neighborhood of a vertex.

Definition 2.6.1. Let $G = (V(G), E(G), E(G) \xrightarrow{(t_G, h_G)} V(G)^2)$ be a connected graph and $v \in V(G)$ an arbitrary vertex. The **(edge) neighborhood** of v, denoted N_v, consists of one-third of each edge in $t_G^{-1}(v) \cup h_G^{-1}(v)$. Figure (2.6) gives an example.

Although strictly speaking neighborhoods consist only of one-third edges, we will frequently employ the abuse of notation $e \in N_v$ to mean that $e \in t_G^{-1}(v) \cup h_G^{-1}(v)$.

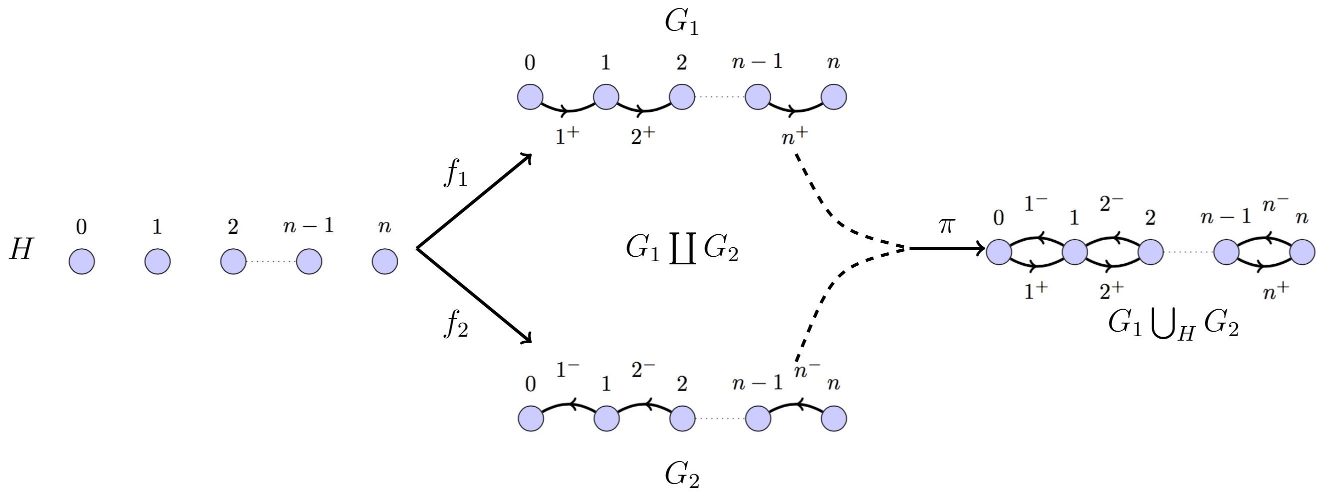


Figure 2.4: Path_n is Path_n^+ and Path_n^- glued along vertices.

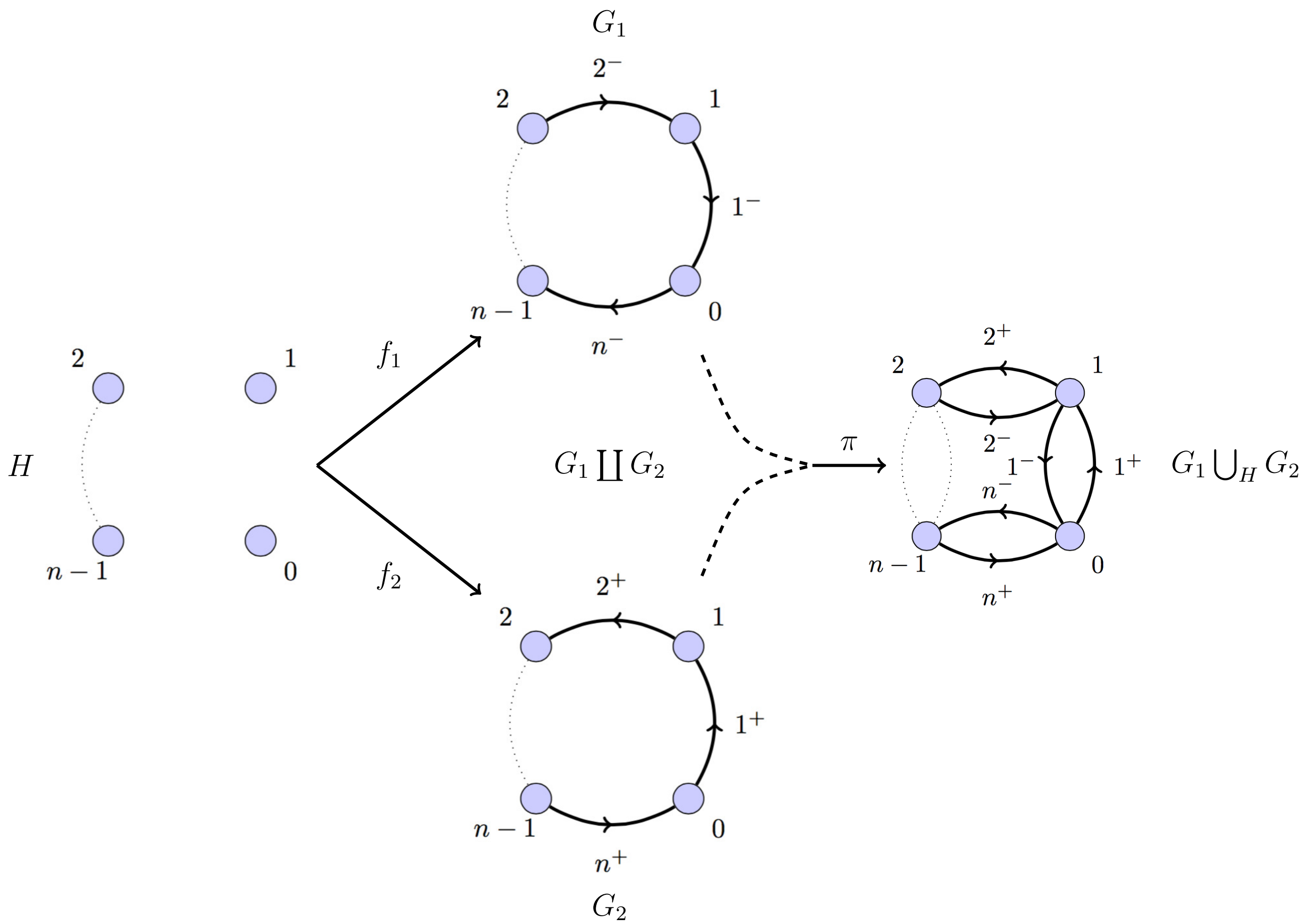


Figure 2.5: Cycle_n is Cycle_n^+ and Cycle_n^- glued along vetices

We can now define covers.

Definition 2.6.2. Let G be a directed graph. A **cover** of G is a pair (H, ϕ) such that the following hold:

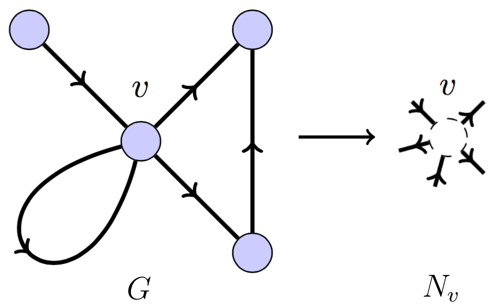


Figure 2.6: (edge) neighborhood of a vertex

(1) $H = (V(H), E(H), E(H) \xrightarrow{(t_H, h_H)} V(H)^2)$ is a directed graph,

(2) $\phi \in \mathrm{Hom}(H, G)$ is a surjective morphism,

(3) for each $\hat{w} \in V(H)$, $\phi_E : N_{\hat{w}} \to N_{\phi_V(\hat{w})}$ is a bijection.

(H, ϕ) is a **finite cover** if and only if H is a finite directed graph.

Given a directed graph G and a cover (H, ϕ) it may be the case that from a global perspective G and H look very different. Even so, in any neighborhood of a vertex in H, ϕ is an isomorphism so that locally H and G look the same. This closely mirrors the development in Algebraic Topology where a covering map is a map which preserves topological data and is a local isomorphism (see [2] P. 56). We hope that the examples of covers given will illustrate this philosophy. Before giving examples however, we need to define an important invariant, called the degree, of a finite cover (H, ϕ). That the degree is well defined follows from the following important lemma:

Lemma 2.6.1 (Edge Lifting)**.** *Let G be a directed graph, (H, ϕ) a cover, and $v_0 \in V(G)$. Then for every $\hat{v}_0 \in \phi_V^{-1}(v_0)$ and every $e_1 \in N_{v_0}$ with $t_G(e_1) \neq h_G(e_1)$, there exists a unique edge $\hat{e}_1 \in N_{\hat{v}_0}$ such that $\phi_E(\hat{e}_1) = e_1$ and*

$$\begin{cases} t_H(\hat{e}_1) = \hat{v}_0 \text{ and } \phi_V(h_H(\hat{e}_1)) = h_G(e_1) & \text{if } v_0 = t_G(e_1); \\ h_H(\hat{e}_1) = \hat{v}_0 \text{ and } \phi_V(t_H(\hat{e}_1)) = t_G(e_1) & \text{if } v_0 = h_G(e_1). \end{cases}$$

Proof. Let $e_1 \in N_{v_0}$ with $t_G(e_1) = v_0$ and let $\hat{v}_0 \in \phi_V^{-1}(v_0)$. Then $\phi_E : N_{\hat{v}_0} \to N_{v_0}$ is a bijection, hence there exists a unique edge $\hat{e}_1 \in N_{\hat{v}_0}$ such that $\phi_E(\hat{e}_1) = e_1$. Since $\hat{e}_1 \in N_{\hat{v}_0}$ it must be the case that $t_H(\hat{e}_1) = \hat{v}_0$ or $h_H(\hat{e}_1) = \hat{v}_0$. Recalling that $\phi \in \mathrm{Hom}(H, G)$, the diagram

$$\begin{tikzcd} E(H) \arrow[r, "\phi_E"] \arrow[d, "(t_H,h_H)"'] & E(G) \arrow[d, "(t_G,h_G)"] \\ V(H)^2 \arrow[r, "\phi_V^2"'] & V(G)^2 \end{tikzcd}$$

must commute. If $h_H(\hat{e}_1) = \hat{v}_0$ then we would have

$$v_0 = \phi_V(h_H(\hat{e}_1)) = h_G(\phi_E(\hat{e}_1)) = h_G(e_1)$$

But we assumed that $t_G(e_1) = v_0$ and that $t_G(e_1) \neq h_G(e_1)$, so it must be the case that $t_H(\hat{e}_1) = \hat{v}_0$. Also by commutativity of the diagram, we have

$$\phi_V(h_H(\hat{e}_1)) = h_G(\phi_E(\hat{e}_1)) = h_G(e_1).$$

The case in which $h_G(e_1) = v_0$ is proved symmetrically. $\square$

We note that in the above lemma, we assumed that e_1 was not a self loop at the vertex v_0. If e_1 were a self loop at v_0, there is still guaranteed to exist a unique edge $\hat{e_1} \in N_{\hat{v}_0}$ and it's easy to prove that $t_H(\hat{e}_1), h_H(\hat{e}_1) \in \phi_V^{-1}(v_0)$: indeed at least one of $t_H(\hat{e}_1)$ and $h_H(\hat{e}_1)$ equals $\hat{v}_0$. The issue is that either of these may hold (or even both) and we cannot say which.

The Edge Lifting Lemma should be compared with the "path lifting property" presented in [2] on page 60.

The Edge Lifting Lemma allows us to prove the following:

Lemma 2.6.2. *Let G be a finite directed graph and (H, ϕ) a finite cover. If $y \sim_1 z \in V(G)$, then $|\phi_V^{-1}(y)| = |\phi_V^{-1}(z)|$.*

Proof. Assume that $y \sim_1 z \in V(G)$. Then this implies that there exists an edge $e \in E(G)$ such that $t(e) = y$ and $h(e) = z$ or vice versa. Let's assume that $t(e) = y$ and $h(e) = z$ (the case where $t(e) = z$ and $h(e) = y$ is proved symmetrically). We show that $|\phi_V^{-1}(y)| = |\phi_V^{-1}(z)|$ by associating to each vertex in $\phi_V^{-1}(y)$ a unique vertex in $\phi_V^{-1}(z)$ and vice versa. Given $\hat{y} \in \phi_V^{-1}(y)$, the Edge Lifting Lemma implies that there exists a unique edge $\hat{e} \in N_{\hat{y}}$ such that $\phi_E(\hat{e}) = e$ and $\phi_V(h_H(\hat{e})) = h_G(e) = v_1$. Thus $h_H(\hat{e}) \in \phi_V^{-1}(z)$ is a unique vertex associated to $\hat{y}$.

Conversely, given $\hat{z} \in \phi_V^{-1}(z)$, the Edge Lifting Lemma gives a unique edge $\hat{f} \in N_{\hat{z}}$ such that $\phi_E(\hat{f})$ and $\phi_V(t_H(\hat{f})) = t_G(e) = y$. Thus $t_H(\hat{f}) \in \phi_V^{-1}(y)$ is a unique vertex associated to $\hat{z}$. Therefore we have that $|\phi_V^{-1}(y)| = |\phi_V^{-1}(z)|$, as desired. $\square$

Corollary 2.6.1. *Let G be a finite directed graph and (H, ϕ) a finite cover. Then for every $v \in V(G)$ and every $y, z \in V(\text{comp}_G(v))$, we have $|\phi_V^{-1}(y)| = |\phi_V^{-1}(z)|$.*

Proof. Since $y, z \in V(\text{comp}_G(v))$ there exists an integer $n \geq 0$, some $\epsilon \in \{+, -\}^n$, and an injective morphism $f \in \text{Hom}(\text{Path}_n^\epsilon, G)$ such that $f_V(0) = y$ and $f_V(n) = z$. We let v_j denote $f_V(j)$ for every $j = 0, \ldots, n$. Notice that $v_0 = y$ and $v_n = z$. The key observation is that $v_j \sim_1 v_{j+1}$ for each $j = 0, \ldots, n-1$. Therefore by the previous lemma we have

$$|\phi_V^{-1}(y)| = |\phi_V^{-1}(v_0)| = |\phi_V^{-1}(v_1)| = \ldots = |\phi_V^{-1}(v_n)| = |\phi_V^{-1}(z)|$$

as desired. $\square$

If G is connected, then $G = \text{comp}_G(v)$ for any $v \in V(G)$ and hence $|\phi_V^{-1}(y)| = |\phi_V^{-1}(z)|$ for every $y, z \in V(G)$.

Definition 2.6.3 (Degree of a Cover)**.** Let G be a finite connected directed graph and (H, ϕ) a finite cover. Then the **degree of the cover** (H, ϕ) is the number $d := |\phi_V^{-1}(y)|$ where y is any vertex of G.

Our definition matches the definition given by [2] on page 56. In particular, we point out that the structures of the arguments are essentially the same: in [2] it is stated (without proof) that the degree of a cover is "locally constant" and so if the base space is connected, it is constant. This is the same structural outline of the argument given above.

Now we give a few examples of covers.

(1) For any directed graph G, there is the trivial cover: (G, Id_G) where Id_G is the identity morphism on the graph G. It's immediate from the definitions that (G, Id_G) is a cover of G of degree one.

(2) Let $n \geq 2$ and $d \geq 1$ be integers. Then $(\text{Cycle}_{dn}^+, \phi)$ is a degree d cover of Cycle_n^+ where ϕ is defined as follows: we let $\bar{j}$ denote the remainder upon dividing j by n and define $\phi_V(j) = \bar{j}$ for every $j = 0, \ldots, dn-1$ and $\phi_E(j^+) = (\bar{j})^+$ for $j = 1, \ldots, dn$.

(3) We can take the disjoint union of an arbitrary number k of copies of a directed graph

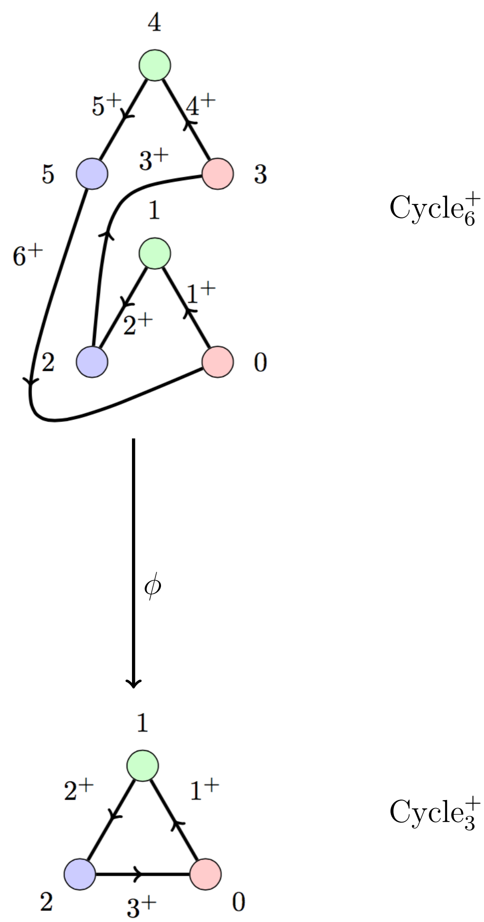


Figure 2.7: Cycle$_6^+$ is a degree two cover of Cycle$_3^+$.

G as we describe inductively:

$$\coprod_{i=1}^{2} G = G \coprod G$$
$$\coprod_{i=1}^{k} G = \left(\coprod_{i=1}^{k-1} G \right) \coprod G$$

For a fixed $d \geq 2$, $(\coprod_{i=1}^{d} G, \phi)$ is a degree d cover of G where

$$\phi_V((v, i)) = v$$
$$\phi_E((e, i)) = e$$

for every $v \in V(G)$ and $e \in E(G)$.

2.7 Lifting Theorems

Now we prove, by applying the Edge Lifting Lemma inductively, that we can lift paths.

Theorem 2.7.1 (Path Lifting)**.** *Let (G, v_0) be a connected pointed graph, let (H, ϕ) be a cover, and let $f \in \mathrm{Hom}(Path_n^+, G)$ satisfy $f_V(0) = v_0$. Then for each vertex $\hat{v}_0 \in \phi_V^{-1}(v_0)$ there exists a unique morphism $\hat{f} \in \mathrm{Hom}(Path_n^+, H)$ such that $\hat{f}_V(0) = \hat{v}_0$ and $\phi \circ \hat{f} = f$.*

Proof. First of all, let v_j denote $f_V(j) \in V(G)$ for each $j = 0, 1, \ldots, n$ and let e_j denote $f_E(j^+) \in E(G)$ for each $j = 1, 2, \ldots, n$. Let $\hat{v}_0 \in \phi_V^{-1}(v_0)$ be arbitrary. Since f is a morphism of directed graphs and as such preserves head and tail maps, we know that $t_G(e_1) = v_0$, thus $e_1 \in N_{v_0}$. Therefore, by the Edge Lifting Lemma, there exists a unique edge $\hat{e}_1 \in N_{\hat{v}_0}$ such that $\phi_E(\hat{e}_1) = e_1$ and $\phi_V(h_H(\hat{e}_1)) = h_G(e_0) = v_1$: let $\hat{v}_1 := h_H(\hat{e}_1)$. We define $\hat{f}_V(0) := \hat{v}_0$, $\hat{f}_E(1^+) = \hat{e}_1$, and $\hat{f}_V(1) = \hat{v}_1$. Then we have

$$\phi_V(\hat{f}_V(0)) = \phi_V(\hat{v}_0) = v_0 = f_V(0)$$
$$\phi_V(\hat{f}_V(1)) = \phi_V(\hat{v}_1) = v_1 = f_V(1)$$
$$\phi_E(\hat{f}_E(1^+)) = \phi_E(\hat{e}_1) = e_1 = f_E(1^+)$$

Thus on the induced subgraph $\mathrm{Path}_n^+|_{\{0,1\}}$ we have $\phi \circ \hat{f} = f$.
Now, we assume inductively that we've defined $\hat{f}_V(\ell) = \hat{v}_\ell$ for every $\ell = 0, \ldots, j-1$ and $\hat{f}_E(\ell^+) = \hat{e}_\ell$ for every $\ell = 1, \ldots, j-1$ with $(t_H, h_H)(\hat{e}_\ell) = (\hat{v}_{\ell-1}, \hat{v}_\ell)$ for every $\ell = 1, \ldots, j-1$ and that $\phi \circ \hat{f} = f$ on the induced subgraph $\mathrm{Path}_n^+|_{\{0,\ldots,j-1\}}$. We have that

$$v_{j-1} = f_V(j-1) = \phi_V(\hat{f}_V(j-1)) = \phi_V(\hat{v}_{j-1})$$

thus $\hat{v}_{j-1} \in \phi_V^{-1}(v_{j-1})$. Therefore by the Edge Lifting Lemma, there exists a unique edge $\hat{e}_j \in N_{\hat{v}_{j-1}}$ such that $\phi_E(\hat{e}_j) = e_j$. Let $\hat{v}_j := h_H(\hat{e}_j)$. We have

$$\begin{aligned}\phi_V(\hat{v}_j) &= \phi_V(h_H(\hat{e}_j)) \\ &= h_G(\phi_E(\hat{e}_j)) \\ &= h_G(e_j) \\ &= v_j\end{aligned}$$

We define $\hat{f}_V(j) := \hat{v}_j$ and $\hat{f}_E(j^+) := \hat{e}_j$. Then we have

$$\phi_V(\hat{f}_V(j)) = \phi_V(\hat{v}_j) = v_j = f_V(j)$$
$$\phi_E(\hat{f}_E(j^+)) = \phi_E(\hat{e}_j) = e_j = f_E(j^+)$$

Therefore on the induced subgraph $\mathrm{Path}_n^+|_{\{0,\ldots,j\}}$ we have $\phi\circ\hat{f} = f$. Therefore by induction we have a map $\hat{f} = (\hat{f}_V, \hat{f}_E)$ defined by $\hat{f}_V(j) = \hat{v}_j$ for every $j = 0, 1, \ldots, n$ and $\hat{f}_E(j^+) = \hat{e}_j$ for every $j = 1, \ldots, n$ with the properties that $(t_H, h_H)(\hat{e}_j) = (\hat{v}_{j-1}, \hat{v}_j)$ for every $j = 1, \ldots, n$ and $\phi \circ \hat{f} = f$. We prove that $\hat{f} \in \mathrm{Hom}(\mathrm{Path}_n^+, H)$. Let $j^+ \in E(\mathrm{Path}_n^+)$ be arbitrary. On the one hand

$$\begin{aligned}(t_H, h_H)(\hat{f}_E(j^+)) &= (t_H, h_H)(\hat{e}_j)\\ &= (\hat{v}_{j-1}, \hat{v}_j)\end{aligned}$$

On the other

$$\begin{aligned}(\hat{f}_V)^2((t_{\mathrm{Path}_n^+}, h_{\mathrm{Path}_n^+})(j^+)) &= (\hat{f}_V)^2(j-1, j)\\ &= (\hat{v}_{j-1}, \hat{v}_j)\end{aligned}$$

Therefore the diagram

$$\begin{tikzcd}
E(\mathrm{Path}_n^+) \arrow[r, "f_E"] \arrow[d, "(t_{\mathrm{Path}_n^+}, h_{\mathrm{Path}_n^+})"'] & E(H) \arrow[d, "(t_H, h_H)"] \\
V(\mathrm{Path}_n^+)^2 \arrow[r, "f_V^2"'] & V(H)^2
\end{tikzcd}$$

commutes and $\hat{f} \in \mathrm{Hom}(\mathrm{Path}_n^+, H)$. The uniqueness of $\hat{f}$ follows immediately from the uniqueness of each edge produced by the Edge Lifting Lemma. □

Corollary 2.7.1. *Let (G, v_0) be a connected pointed graph and suppose that $f \in \mathrm{Hom}(Path_n^-, G)$ satisfies $f_V(n) = v_0$. Then for every $\hat{v}_0 \in \phi_V^{-1}(v_0)$ there is a unique $\hat{f} \in \mathrm{Hom}(Path_n^-, H)$ such that $\hat{f}_V(n) = \hat{v}_0$ and $\phi \circ \hat{f} = f$.*

2.8 Inherited Eigenvalues

Many properties of a directed graph G can be deduced by examining the eigenvalues of the so called adjacency matrix of G. First we define the adjacency matrix of a directed graph and then we prove that for a finite cover (H, ϕ), some number of the eigenvalues of the adjacency matrix for H are "inherited" from G.

Definition 2.8.1. Let $G = (V(G), E(G), E(G) \xrightarrow{(t_G, h_G)} V(G)^2)$ be a finite directed graph and let $v, w \in V(G)$. Then we denote $E_{v,w} = \{e \in E(G) : t(e) = v,\ h(e) = w\}$. Let

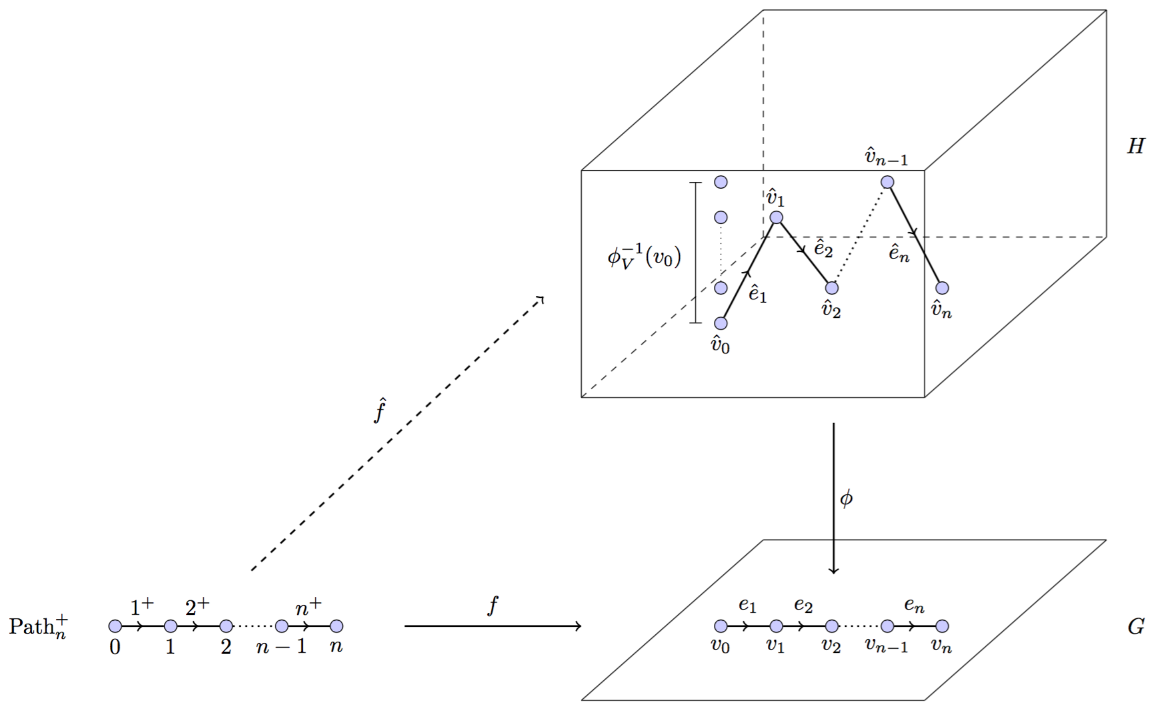


Figure 2.8: path lifting

$n := |V(G)|$. The **adjacency matrix of** G, denoted $A(G)$ is the $n \times n$ integer matrix whose rows and columns are each indexed by $V(G)$ given by $A(G) = (a_{v,w})_{v,w \in V(G)}$ where $a_{v,w} = |E_{v,w}|$.

Notice that we cannot write down the matrix $A(G)$ until we've fixed some order on $A(G)$ but that any two such adjacency matrices subject to different orders on $V(G)$ will be the same up to a change of basis. For example, if $G = \mathrm{Cycle}_3^+$ and we've chosen the natural increasing order on $V(G) = \{0, 1, 2\}$ then we have

$$A(G) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

Definition 2.8.2. The **characteristic polynomial of (the adjacency matrix of)** G, denoted $\mathrm{char}_G(z)$, is given by $\mathrm{char}_G(z) = \det(zI - A(G))$ where I is the $|V(G)| \times |V(G)|$ identity matrix. λ is an **eigenvalue of** $A(G)$ if and only if $\mathrm{char}_G(\lambda) = 0$.

Continuing with the previous example of $G = \mathrm{Cycle}_3^+$, we have $\mathrm{char}_G(z) = z^3 - 1$. It follows that $A(G)$ has a single real eigenvalue, $\lambda_1 = 1$ and a pair of non-real eigenvalues $\lambda_2 = e^{2\pi i/3}$ and $\lambda_3 = e^{-2\pi i/3}$.

We consider a second example: let $H = \text{Cycle}_6^+$ with vertices ordered in increasing order. Then we have

$$A(H) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\text{char}_H(z) = z^6 - 1 = (z^3 - 1)(z^3 + 1)$.

Recall that there exists $\phi \in \text{Hom}(H, G)$ such that (H, ϕ) is a degree two cover of G. We further observe that $\text{char}_G(z)$ divides $\text{char}_H(z)$. Notice that this is equivalent to the statement that every eigenvalue of $A(G)$ is an eigenvalue of $A(H)$. This phenomenon is not happenstance as we now prove.

Definition 2.8.3. Let G be a directed graph. Then we define $\mathcal{L}^2(G) := \{f : V(G) \to \mathbb{C} : \sum_{v \in V(G)} |f(v)|^2 < \infty\}$. The **adjacency operator of** G, denoted $\mathcal{A}_G : \mathcal{L}^2(G) \to \mathcal{L}^2(G)$, is defined by

$$(\mathcal{A}_G(f))(v) = \sum_{e \in t_G^{-1}(v)} f(h(e))$$

for every $v \in V(G)$. $f \in \mathcal{L}^2(G)$ is an **eigenfunction of** G **with eigenvalue** λ if and only if

$$(\mathcal{A}_G f)(v) = \lambda f(v)$$

for every $v \in V(G)$.

Let $f, g \in \mathcal{L}^2(G)$ and let $\alpha \in \mathbb{C}$. If we define $f + g$ and αf to be the functions given by

$$(f + g)(v) = f(v) + g(v), \ (\alpha f)(v) = \alpha(f(v))$$

then $\mathcal{L}^2(G)$ has the structure of a $\mathbb{C}$-vector space.

Lemma 2.8.1. *Let G be a finite directed graph. For every $v \in V(G)$ we define $[v] \in \ell^2(G)$,* ***the characteristic function of*** *v by*

$$[v](w) = \begin{cases} 1, & \text{if } w = v; \\ 0, & \text{if } v \neq w. \end{cases}$$

Then $\{[v]\}_{v \in V(G)}$ is a basis for $\mathcal{L}^2(G)$.

Proof. Let $f \in \mathcal{L}^2(G)$. We claim that $f = \sum_{v \in V(G)} f(v)[v]$. Indeed for each $w \in V(G)$ we have $f(w) = \sum_{v \in V(G)} f(v)[v](w)$. This proves that $\{[v]\}_{v \in V(G)}$ spans $\mathcal{L}^2(G)$. Assume that we had $\sum_{v \in V(G)} c_v[v] = 0 \in \mathcal{L}^2(G)$. This means that

$$\sum_{v \in V(G)} c_v[v](w) = 0$$

for every $w \in V(G)$. By definition, the only term in the above sum which survives is that which corresponds to $v = w$. In this case $[v](w) = 1$ and we see that $c_w = 0$. Therefore $\{[v]\}_{v \in V(G)}$ is linearly independent. This proves the lemma. □

In the case when G is a finite directed graph, we see that the matrix representation of $\mathcal{A}_G$ with respect to the canonical basis $\{[v]\}_{v \in V(G)}$ is precisely $A(G)$. Indeed,

$$(\mathcal{A}_G[v])(w) = \sum_{e \in t_G^{-1}(w)} [v](h_G(e)) = |E_{w,v}|$$

so that $\mathcal{A}_G[v] = \sum_{w \in V(G)} |E_{w,v}|[w] = \sum_{w \in V(G)} a_{w,v}[w]$.

Theorem 2.8.1. *Let G be a finite directed graph and (H, ϕ) a finite cover. If f is an eigenfunction of G with eigenvalue λ, then $\hat{f}$ is an eigenfunction of H with eigenvalue λ where $\hat{f} : V(H) \to \mathbb{C}$ is defined by $\hat{f}(\hat{w}) = f(\phi_V(\hat{w}))$.*

Proof. We show that $(\mathcal{A}_H \hat{f})(\hat{w}) = \lambda \hat{f}(\hat{w})$ for every $\hat{w} \in V(H)$. By definition of $\hat{f}$ and the fact that $\phi \in \mathrm{Hom}(H, G)$, we have that

$$\begin{aligned}
(\mathcal{A}_H \hat{f})(\hat{w}) &= \sum_{\hat{e} \in t_H^{-1}(\hat{w})} \hat{f}(h_H(\hat{e})) \\
&= \sum_{\hat{e} \in t_H^{-1}(\hat{w})} f(\phi_V(h_H(\hat{e}))) \\
&= \sum_{\hat{e} \in t_H^{-1}(\hat{w})} f(h_G(\phi_E(\hat{e})))
\end{aligned}$$

for every $\hat{w} \in V(H)$. Now, let $e := \phi_E(\hat{e})$. Then $\hat{e} \in t_H^{-1}(\hat{w})$ if and only if $t_H(\hat{e}) = \hat{w}$ if and only if

$$\begin{aligned}
\phi_V(\hat{w}) &= \phi_V(t_H(\hat{e})) \\
&= t_G(\phi_E(\hat{e})) \\
&= t_G(e)
\end{aligned}$$

which holds if and only if $e \in t_G^{-1}(\phi_V(\hat{w}))$. Therefore we have

$$\begin{aligned}
(\mathcal{A}_H \hat{f})(\hat{w}) &= \sum_{\hat{e} \in t_H^{-1}(\hat{w})} f(h_G(\phi_E(\hat{e}))) \\
&= \sum_{e \in t_G^{-1}(\phi_V(\hat{w}))} f(h_G(e)) \\
&= (\mathcal{A}_G f)(\phi_V(\hat{w})) \\
&= \lambda f(\phi_V(\hat{w})) \\
&= \lambda \hat{f}(\hat{w})
\end{aligned}$$

This proves that $\hat{f}$ is an eigenfunction of H with eigenvalue λ, as desired.

$\square$

Chapter 3

Even Graphs

3.1 Symmetric Graphs

In this section we define a subcategory of the category of directed graphs whose objects are symmetric graphs. Morally, a symmetric graph has the property that for every edge, there corresponds an edge running the opposite direction. A subtlety in the definition of symmetric graphs will lead us to refine our notion to that of an even graph.

Definition 3.1.1. A **transposition** on a directed graph $G = (V, E, E \xrightarrow{(t,h)} V^2)$ is a permutation $\tau_G \in \mathrm{Sym}(E)$ such that

(1) $\tau_g^2 = \mathrm{Id_E}$

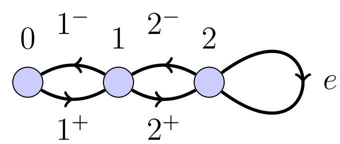
(2) $t(\tau_g(e)) = h(e)$ and $h(\tau(e)) = t(e)$

Property (2) above states precisely that a transposition τ_G swaps the head and tail of an edge. Note that graphs need not possess transpositions: try as we might, we cannot find a transposition on Path_n^+ for any $n \geq 1$. On the other hand, Cycle_n and Path_n possess transpositions τ_{Cycle_n} and τ_{Path_n} defined by: $\tau_{\mathrm{Cycle}_n}(j^+) = \tau_{\mathrm{Path}_n}(j^+) = j^-$ for $j = 1, \ldots, n$.

Definition 3.1.2. A directed graph G is **symmetric** if there exists a transposition τ_G on G. The **category of symmetric graphs** is the category whose objects are symmetric graphs and whose morphisms are directed graph morphisms.

In the technical language, the category of symmetric graphs is a full subcategory of the category of directed graphs. This means that given two symmetric graphs G and H, the set of (directed graph) morphisms from G to H coincides with the set of symmetric graph morphisms from G to H. Put another way, we don't "lose" any morphisms when we consider G and H in the a priori "smaller" category of directed graphs. As such there is no danger in using $\mathrm{Hom}(G, H)$ to denote the set of symmetric graph morphisms from G to H.

To reiterate, the motivation for considering symmetric graphs was to obtain graphs such that to every edge there corresponds an edge running the opposite direction. Symmetric graphs do not quite possess this property as we illustrate with the following example: consider the graph

$G =$ 

Then τ_G defined by $\tau_G(1^+) = 1^-$, $\tau_G(2^+) = 2^-$ and $\tau_G(e) = e$ is a transposition on G so that G is symmetric. But the loop edge e does not have a corresponding edge running in the "opposite" direction. One solution to this obstruction is to insist that transpositions fix no edges.

3.2 Even Graphs

Definition 3.2.1. A symmetric graph G is **even** if and only if there exists a transposition τ_G on G which is fixed point free. The **category of even graphs** has as objects pairs (G, τ_G) where G is an even graph and τ_G is a fixed point free transposition on G. A morphism from (G, τ_G) to (H, τ_H) in this category is a directed graph morphism $\phi \in \mathrm{Hom}(G, H)$ with the additional property that the following diagram commutes:

$$\begin{tikzcd}
E(G) \arrow[r, "\phi_E"] \arrow[d, "\tau_G"'] & E(H) \arrow[d, "\tau_H"] \\
E(G) \arrow[r, "\phi_E"'] & E(H)
\end{tikzcd}$$

Given two directed graphs (G, τ_G) and (H, τ_H), we'll denote the set of even graph morphisms from (G, τ_G) to (H, τ_H) by $\mathrm{Hom}_\tau((G, \tau_G), (H, \tau_H))$.

It is immediate that the pairs $(\mathrm{Path}_n^+, \tau_{\mathrm{Path}_n})$ and $(\mathrm{Cycle}_n^+, \tau_{\mathrm{Cycle}_n})$ are even graphs with τ_{Path_n} and τ_{Cycle_n} as defined in Section (2.9).

Definition 3.2.2. Let (G, τ_G) be an even graph. (H, τ_H) is a **sub-even graph** if and only if the following hold:

(1) H is a subgraph of G in the directed graph sense;

(2) $\tau_H = \tau_G|_{E(H)}$.

We say that (H, τ_H) is an **induced sub-even graph** if and only if H is induced in the directed graph sense.

For even graphs, self loops come in pairs as we now prove.

Lemma 3.2.1. *Let $((V, E, E \xrightarrow{(t,h)} V^2), \tau_G)$ be an even graph and $v \in V$. Then the number of self loops at v is even.*

Proof. Suppose that $e \in E$ is a self loop at v: i.e. $t(e) = h(e) = v$. We show that there exists another self loop at v which is paired with e under τ_G. Indeed, let $f := \tau_G(e)$. Since τ_G is assumed to be fixed point free it must be that $f \neq e$. But $t(f) = t(\tau_G(e)) = h(e) = v$ and $h(f) = h(\tau_G(e)) = t(e) = v$, so that f is a self loop at v as well. $\square$

3.3 Pointed Even Graphs

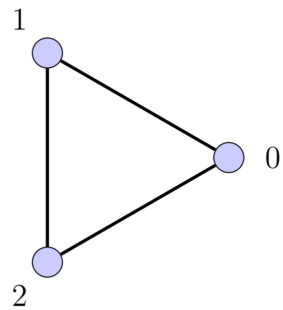
Definition 3.3.1. A **(singly) pointed even graph** is a pair $((G, \tau_G), v)$ where (G, τ_G) is an even graph and $v \in V(G)$. Given $((G_1, \tau_{G_1}), v_1)$ and $((G_2, \tau_{G_2}), v_2)$ two even pointed graphs, a **morphism of even pointed graphs** is an even graph morphism $f \in \mathrm{Hom}_\tau((G_1, \tau_{G_1}), (G_2, \tau_{G_2}))$ with the additional property that $f(v_1) = v_2$. The collection of all even pointed graph morphisms from $((G_1, \tau_{G_1}), v_1)$ to $((G_2, \tau_{G_2}), v_2)$ will be denoted $\mathrm{Hom}_\tau(((G_1, \tau_{G_1}), v_1), ((G_2, \tau_{G_2}), v_2))$.

3.4 Undirected Graphs

We feel compelled at this stage to provide some justification for the slightly non-standard manner in which we have proceeded. In this section we will define undirected graphs and will show how to obtain an undirected graph from an even graph. There is also a manner in which an even graph can be associated to an undirected graph, however the association is only unique up to isomorphism in the category of directed graphs. Given that undirected graphs and even graphs are so closely related, it's reasonable to ask: why even graphs and not undirected graphs? In this section we hope to answer the question.

Definition 3.4.1. An **undirected graph** G is a pair (V, E) where V is a set and $E \subseteq \{\{v, w\} : v, w \in V(G)\}$ where we consider $\{\{v, w\} : v, w \in V(G)\}$ as a multi-set.

We remark that if $\{v, v\} = \{v\} \in E$ then our convention will be to give G a self loop at v. Consider the example $\mathrm{UCycle}_3 = (\{0, 1, 2\}, \{\{0, 1\}, \{1, 2\}, \{2, 0\}\})$:

$\mathrm{UCycle}_3 =$ 

Now, let (G, τ_G) be an even graph with $G = (V(G), E(G), E(G) \xrightarrow{(t,h)} V(G)^2)$. We define $\tilde{E}(G) = E(G)/\langle\tau_G\rangle$ to be the set of orbits of $E(G)$ under the group action of $\langle\tau_G\rangle$. Thus an element $\tilde{e} \in \tilde{E}(G)$ is a multi-set $\tilde{e} = \{e, \tau_G(e)\}$ for some $e \in E(G)$. We remark that the expressions $t(\tilde{e})$ and $h(\tilde{e})$ are nonsensical if $\tilde{e} \in \tilde{E}(G)$. That being said, given $\tilde{e} = \{e, \tau_G(e)\} \in \tilde{E}(G)$ we'll use $\{t(\tilde{e}), h(\tilde{e})\}$ to denote $\{t(e), h(e)\}$. To the even graph (G, τ_G) we associate the undirected graph

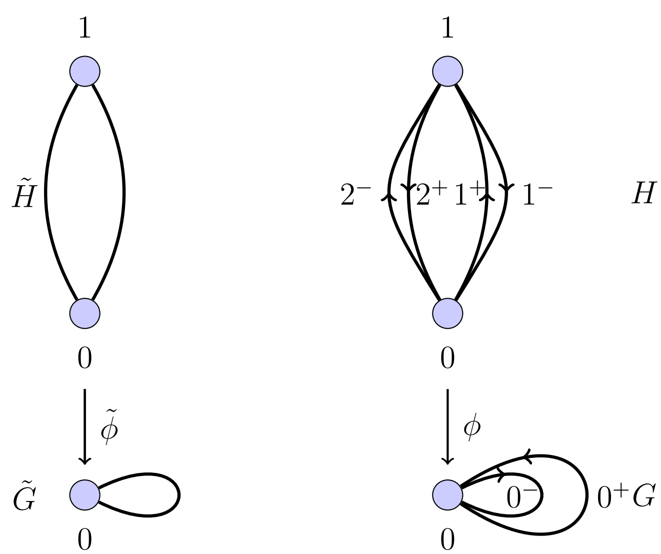
$$\tilde{G} = (V(G), \{\{t(e), h(e)\} : e \in \tilde{E}(G)\})$$

In the reverse direction, given an undirected graph $\tilde{G}$ one can obtain an even graph by "blowing up" each (undirected) edge $\{v, w\}$ to a pair of (directed) edges e and f with $t(e) = h(f) = v$ and $h(e) = t(f) = w$. While this process certainly produces an even graph there may be (especially in the presence of multiple edges between a pair of vertices in the undirected graph) many fixed point free transpositions from which to choose.

Let $\tilde{G}$ be an undirected graph. Then we define $E_{v,w} = \{\{v, w\}\} \subseteq E(\tilde{G})$ as a multi-set; i.e. $E_{v,w}$ consists of all undirected edges connecting v and w. Mirroring the development in the directed graph case, we define $A(\tilde{G})$, the adjacency matrix of the undirected graph to be the integer matrix given by $A(\tilde{G}) = (a_{v,w})_{v,w \in V(\tilde{G})}$ where $a_{v,w} = |E_{v,w}|$. With this definition we may consider the characteristic polynomial and hence eigenvalues of $\tilde{G}$.

Because we will ultimately work with even graphs and will develop the precise language there, we elect not to define in precise terms the notion of a cover for undirected graphs. We hope however that the reader will not feel morally compromised in considering the pair $(\tilde{H}, \tilde{\phi})$ to be a degree two cover (in the undirected graph sense) of $\tilde{G}$ where $\tilde{H} = (\{0,1\}, \{\{0,1\}, \{0,1\}\})$, where $\tilde{G} = (\{0\}, \{\{0,0\}\})$ and where $\tilde{\phi}_V(0) = \tilde{\phi}_V(1) = 0$ and $\tilde{\phi}_E(\{0,1\}) = \{0,0\}$.

The corresponding example in the category of even graphs consists of G and (H, ϕ) (forgetting for a moment about the precise fixed point free transpositions but rather observing that at least one certainly exists for each of G and H) where $G = \text{Cycle}_1$, where $H = \text{Cycle}_2$ and where $\phi_V(0) = \phi_V(1) = 0$, $\phi_E(1^+) = \phi_E(2^+) = 0^+$, and $\phi_E(1^-) = \phi_E(2^-) = 0^-$.



Let us consider the adjacency matrices and characteristic polynomials of $\tilde{H}$, $\tilde{G}$, H, and G:

Graph	Adjacency Matrix	Characteristic Polynomial
$\tilde{H}$	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	$z^2 - 4$
$\tilde{G}$	?	?
H	$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$	$z^2 - 4$
G	(2)	$z - 2$

We are of the opinion that one most naturally takes $A(\tilde{G})$ to be (1): indeed there is exactly one edge connecting 0 to itself. In this case $\mathrm{char}_{\tilde{G}}(z) = z-1$. But this conclusion disagrees slightly with a theme which we would like to hold universally: that the characteristic polynomial of a graph should divide the characteristic polynomial of a cover. Now, many graph theorists will retort that it is simply convention that $A(\tilde{G}) = (2)$ (although not one which is universally accepted!) In this case we have $\mathrm{char}_{\tilde{G}}(z) = z - 2 \mid z^2 - 4 = \mathrm{char}_{\tilde{H}}(z)$. The point is that we need not appeal to any convention in the even graph case. There is no ambiguity that $A(G) = (2)$ and that the divisibility property for characteristic polynomials of graphs and their covers holds.

3.5 Girth and Trees

Definition 3.5.1. Let G be an even graph. Then

$$\mathrm{Girth}(G) := \min\{\{n \geq 1 : \mathrm{Inj}_\tau((\mathrm{Cycle}_n, \tau_{\mathrm{Cycle}_n}), (G, \tau_G)) \neq \emptyset\} \cup \{+\infty\}\}$$

We say that G is a **forest** if and only if $\mathrm{Girth}(G) = +\infty$. G is a **tree** if and only if G is a connected forest.

It follows that every connected component of a forest is a tree and as such every forest can be written as a disjoint union of trees. One easily shows that if G is a forest, then G has no multi-edges (i.e. for all $v, w \in V(G)$ $|E_{v,w}| \leq 1$) and that G has no self loops.

Trees should be regarded as analogies to “simply connected” spaces. According to [1], a space is simply connected if it is path connected and has trivial fundamental group. As we stated, a tree is certainly (path) connected and, after our discussion of the fundamental group of a graph, we will show that trees have trivial fundamental groups.

A crucially important notion that we will make use of is that of a spanning tree.

Definition 3.5.2. Let (G, τ_G) be an even graph with $G = (V, E, E \xrightarrow{(t,h)} V^2)$. A **spanning tree of** G is a sub-even graph (T, τ_T) with $T = (V(T), E(T), E(T) \xrightarrow{(t_T,h_T)} V(T)^2)$ such that

(1) T is a tree;

(2) for every $v \in V(G)$, $v \in t_T(E(T)) \cup h_T(E(T))$

In other words, a spanning tree is a tree which "visits" every vertex of G. Notice that every connected, even graph has a spanning tree.

3.6 Even Covers

Definition 3.6.1. Let (G, τ_G) be an even graph. A **(even) cover of** G is a pair $((H, \tau_H), \phi)$ which satisfies the following conditions:

(1) $H = (V(H), E(H), E(H) \xrightarrow{(t_H, h_H)} V(H)^2)$ is a directed graph;

(2) (H, τ_H) is an even graph;

(3) $\phi \in \mathrm{Hom}_\tau((H, \tau_H), (G, \tau_G))$ is surjective;

(4) for every $\hat{w} \in V(H)$, $\phi_E : N_{\hat{w}} \to N_{\phi_V(\hat{w})}$ is a bijection.

We observe that the above definition is just a specialized notion of a cover of directed graphs: we've simply adjusted the requirements so that objects and morphisms belong properly to the category of even graphs. In particular, any cover in the even graph sense is a cover in the directed graph sense. The first result that we prove is an updated version of the Edge Lifting Lemma

Lemma 3.6.1 ((Even) Edge Lifting Lemma)**.** *Let (G, τ_G) be a directed graph, $((H, \tau_H), \phi)$ a cover, and $v_0 \in V(G)$. Then for every $\hat{v}_0 \in \phi_V^{-1}(v_0)$ and every $e_1 \in N_{v_0}$ with $t_G(e_1) \neq h_G(e_1)$, there exists a unique edge $\hat{e}_1 \in N_{\hat{v}_0}$ such that $\phi_E(\hat{e}_1) = e_1$ and*

$$\begin{cases} t_H(\hat{e}_1) = \hat{v}_0 \text{ and } \phi_V(h_H(\hat{e}_1)) = h_G(e_1) & \text{if } v_0 = t_G(e_1); \\ h_H(\hat{e}_1) = \hat{v}_0 \text{ and } \phi_V(t_H(\hat{e}_1)) = t_G(e_1) & \text{if } v_0 = h_G(e_1). \end{cases}$$

Furthermore there exists a second edge $\sigma(\hat{e}_1) \in N_{\hat{v}_0}$ (distinct from $\hat{e}_1$) which satisfies:

$$\begin{cases} t_H(\sigma(\hat{e}_1)) = h_H(\hat{e}_1) \\ h_H(\sigma(\hat{e}_1)) = t_H(\hat{e}_1) \\ \phi_E(\sigma(\hat{e}_1)) = \tau(e_1) \end{cases}$$

Proof. The statement of the lemma which proceeds the word "furthermore" is exactly the Edge Lifting Lemma and has already been proved. The existence and distinctness from $\hat{e}_1$ of $\tau_H(\hat{e}_1)$ follows immediately from the fact that (H, τ_H) is an even graph: $\tau_H \in \mathrm{Sym}(E(H))$ fixes no edges. Since τ_H is a transposition on H, it swaps heads and tails:

$t_H(\tau_H(\hat{e}_1)) = h_H(\hat{e}_1)$ and $h_H(\tau_H(\hat{e}_1)) = t_H(\hat{e}_1)$. Finally we recall that ϕ is a morphism of even graphs and as such the following diagram commutes:

$$\begin{tikzcd} E(H) \arrow[r, "\phi_E"] \arrow[d, "\tau_H"'] & E(G) \arrow[d, "\tau_G"] \\ E(H) \arrow[r, "\phi_E"'] & E(G) \end{tikzcd}$$

Thus, $\phi_E(\tau_H(\hat{e}_1)) = \tau_G(\phi_E(\hat{e}_1)) = \tau_G(e_1)$. □

We prove that we can lift even paths as well. First we recall that $(\mathrm{Path}_n, \tau_{\mathrm{Path}_n})$ is an even graph where $\tau_{\mathrm{Path}_n}(j^+) = j^-$ for every $j = 1, \ldots, n$.

Theorem 3.6.1 ((Even) Path Lifting)**.** *Let (G, τ_G) be an even graph, $((H, \tau_H), \phi)$ an even cover and fix some $v_0 \in V(G)$. Suppose that $f \in \mathrm{Hom}_\tau((Path_n, \tau_{Path_n}, (G, \tau_G))$ satisfies $f_V(0) = v_0$. Then for every $\hat{v}_0 \in \phi_V^{-1}(v_0)$ there exists a unique morphism $\hat{f} \in \mathrm{Hom}_\tau((Path_n, \tau_{Path_n}), (H \tau_H))$ such that $\hat{f}_V(0) = \hat{v}_0$ and $\phi \circ \hat{f} = f$.*

Proof. Let $v_0 \in V(G)$ and suppose that $f \in \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (G, \tau_G))$ satisfies $f_V(0) = v_0$. Before we proceed, we fix some notation:

$$\begin{aligned} v_j &:= f_V(j) && \text{for } j = 0, \ldots, n; \\ e_j^+ &:= f_E(j^+) && \text{for } j = 1, \ldots, n; \\ e_j^- &:= f_E(j^-) && \text{for } j = 1, \ldots, n. \end{aligned}$$

Since f is a morphism of the even graphs $(\mathrm{Path}_n, \tau_{\mathrm{Path}_n}) \to (G, \tau_G)$, f_E commutes with the transpositions τ_G and τ_{Path_n}. Hence, $\tau_G(e_j^+) = \tau_G(f_E(j^+)) = f_E(\tau_{\mathrm{Path}_n}(j^+)) = f_E(j^-) = e_j^-$.

We observe that there is an "inclusion" morphism $\iota^+ \in \mathrm{Hom}(\mathrm{Path}_n^+, \mathrm{Path}_n)$ given by $\iota_V^+(j) = j$ for $j = 0, \ldots, n$ and $\iota_E^+(j^+) = j^+$ for $j = 1, \ldots, n$. Importantly ι^+ is merely a morphism of directed graphs. It is not incorrect to record the fact that $f \in \mathrm{Hom}(\mathrm{Path}_n, G)$ since every morphism of even graphs is, in particular, a morphism of directed graphs. As such we obtain a directed graph morphism $f^+ := f \circ \iota^+ \in \mathrm{Hom}(\mathrm{Path}_n^+, G)$. Then for a fixed, arbitrary $\hat{v}_0 \in \phi_V^{-1}(v_0)$, the (directed) Path Lifting Theorem gives a unique morphism $\hat{f}^+ \in \mathrm{Hom}(\mathrm{Path}_n^+, H)$ satisfying $\hat{f}_V^+(0) = \hat{v}_0$ and $\phi \circ \hat{f}^+ = f^+$. We further define

$$\begin{aligned} \hat{v}_j &:= \hat{f}_V^+(j) && \text{for } j = 0, \ldots, n; \\ \hat{e}_j^+ &:= \hat{f}_E^+(j^+) && \text{for } j = 1, \ldots, n; \end{aligned}$$

Since (H, τ_H) is even, we produce a second set of edges $\hat{e}_j^- := \tau_H(\hat{e}_j^+)$ for $j = 1, \ldots, n$ which satisfy $\hat{e}_j^- \neq \hat{e}_j^+$ and $t_H(\hat{e}_j^-) = \hat{v}_j$ and $h_H(\hat{e}_j^-) = \hat{v}_{j-1}$.
From the fact that $\phi \circ \hat{f}^+ = f^+$, it follows that

$$\begin{aligned} \phi_V(\hat{v}_j) &= \phi_V(\hat{f}_V^+(j)) = f_V^+(j) = v_j && \text{for } j = 0, \ldots, n; \\ \phi_E(\hat{e}_j^+) &= \phi_E(\hat{f}_E^+(j^+)) = f_E^+(j^+) = e_j^+ && \text{for } j = 1, \ldots, n; \end{aligned}$$

Since $\phi \in \mathrm{Hom}_\tau((H, \tau_H), (G, \tau_G))$, we see that $\phi_E(\hat{e}_j^-) = \phi_E(\tau_H(\hat{e}_j^+)) = \tau_G(\phi_E(\hat{e}_j^+)) = \tau_G(e_j^+) = e_j^-$. We define a map $\hat{f} = (\hat{f}_V, \hat{f}_E) : \mathrm{Path}_n \to H$ by

$$\begin{aligned} \hat{f}_V(j) &= \hat{v}_j && \text{for } j = 0 \ldots, n; \\ \hat{f}_E(j^+) &= \hat{e}_j^+ && \text{for } j = 1, \ldots, n; \\ \hat{f}_E(j^-) &= \hat{e}_j^- && \text{for } j = 1, \ldots, n. \end{aligned}$$

First we prove that $\hat{f} \in \mathrm{Hom}(\mathrm{Path}_n, H)$, i.e. that the diagram

$$\begin{tikzcd} E(\mathrm{Path}_n) \arrow[r, "\hat{f}_E"] \arrow[d, "(t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n})"'] & E(H) \arrow[d, "(t_H, h_H)"] \\ V(\mathrm{Path}_n)^2 \arrow[r, "\hat{f}_V^2"'] & V(H)^2 \end{tikzcd}$$

commutes. So let $j^+ \in E(\mathrm{Path}_n)$. Then we have

$$\begin{aligned} ((t_H, h_H) \circ \hat{f}_E)(j^+) &= (t_H, h_H)(\hat{e}_j^+) \\ &= (\hat{v}_{j-1}, \hat{v}_j) \end{aligned}$$

On the other hand,

$$\begin{aligned} (\hat{f}_V^2 \circ (t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n}))(j^+) &= \hat{f}_V^2((j-1, j)) \\ &= (\hat{v}_{j-1}, \hat{v}_j) \end{aligned}$$

Alternatively, let $j^- \in E(\mathrm{Path}_n)$. Then

$$\begin{aligned} ((t_H, h_H) \circ \hat{f}_E)(j^-) &= (t_H, h_H)(\hat{e}_j^-) \\ &= (\hat{v}_j, \hat{v}_{j-1}) \end{aligned}$$

On the other hand

$$\begin{aligned} (\hat{f}_V^2 \circ (t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n}))(j^-) &= \hat{f}_V^2(j, j-1) \\ &= (\hat{v}_j, \hat{v}_{j-1}) \end{aligned}$$

Therefore the above diagram commutes and we have $\hat{f} \in \mathrm{Hom}(\mathrm{Path}_n, H)$, as desired.

To show that in fact $\hat{f} \in \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (H, \tau_H))$, we need to show that the diagram

$$\begin{tikzcd}
E(\mathrm{Path}_n) \arrow[r, "\hat{f}_E"] \arrow[d, "\tau_{\mathrm{Path}_n}"'] & E(H) \arrow[d, "\tau_H"] \\
E(\mathrm{Path}_n) \arrow[r, "\hat{f}_E"'] & E(H)
\end{tikzcd}$$

commutes. So let $j^+ \in E(\mathrm{Path}_n)$. Then we have

$$\begin{aligned}
\tau_H(\hat{f}_E(j^+)) &= \tau_H(\hat{e}_j^+) \\
&= \hat{e}_j^- \\
&= \hat{f}_E(j^-) \\
&= \hat{f}_E(\tau_{\mathrm{Path}_n}(j^+))
\end{aligned}$$

Alternatively, given $j^- \in E(\mathrm{Path}_n)$ we see that

$$\begin{aligned}
\tau_H(\hat{f}_E(j^-)) &= \tau_H(\hat{e}_j^-) \\
&= \hat{e}_j^+ \\
&= \hat{f}_E(j^+) \\
&= \hat{f}_E(\tau_{\mathrm{Path}_n}(j^-))
\end{aligned}$$

This proves that the second diagram commutes and hence that $\hat{f} \in \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (H, \tau_H))$.

Finally we prove that $\phi \circ \hat{f} = f$ by which we mean that $\phi_V \circ \hat{f}_V = f_V$ and $\phi_E \circ \hat{f}_E = f_E$.

For the first of these, let $j \in V(\mathrm{Path}_n)$. Then

$$\begin{aligned}
(\phi_V \circ \hat{f}_V)(j) &= \phi_V(\hat{v}_j) \\
&= v_j \\
&= f_V(j)
\end{aligned}$$

Also, given $j^+ \in E(\mathrm{Path}_n)$ we have

$$\begin{aligned}
(\phi_E \circ \hat{f}_E)(j^+) &= \phi_E(\hat{e}_j^+) \\
&= e_j^+ \\
&= f_E(j^+)
\end{aligned}$$

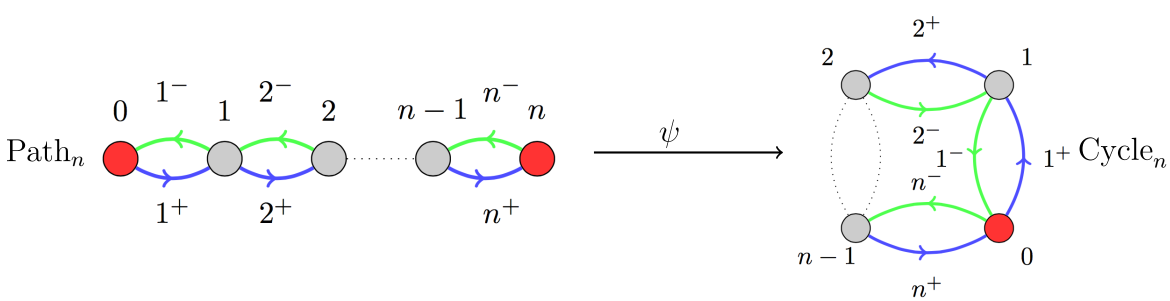
For $j^- \in E(\text{Path}_n)$ we see that

$$\begin{aligned}(\phi_E \circ \hat{f}_E)(j^-) &= \phi_E(\hat{e}_j^-) \\ &= e_j^- \\ &= f_E(j^-)\end{aligned}$$

These computations prove that $\phi \circ \hat{f} = f$, as desired. $\square$

Once again, Lemma 3.6.1 and Theorem 3.6.1 should be regarded as analogies to the path lifting property in [2]. We give an example of a quotient morphism $\psi \in \text{Hom}_\tau((\text{Path}_n, \tau_{\text{Path}_n}), (\text{Cycle}_n, \tau_{\text{Cycle}_n}))$. We define $\psi_V(j) = j$ for $j = 0, \ldots, n-1$ and $\psi_V(n) = 0$ and $\psi_E(j^\pm) = j^\pm$ for $j = 1, \ldots, n$. Essentially we identify the vertices 0 and n in Path_n and doing so yields Cycle_n. The morphism ψ splits into a pair of directed graph morphisms $\psi^+ \in \text{Hom}(\text{Path}_n^+, \text{Cycle}_n^+)$ and $\psi^- \in \text{Hom}(\text{Path}_n^-, \text{Cycle}_n^-)$ given by

$$\begin{aligned}\psi^+ &= \psi|_{\text{Path}_n^+} \\ \psi^- &= \psi|_{\text{Path}_n^-}\end{aligned}$$



In the figure the two red vertices 0 and n in Path_n are identified to yield the single red vertex 0 in Cycle_n. The blue edges represent the directed morphism ψ^+ and the green edges represent ψ^-.

3.7 Tree Lifting

Here we prove an important result about an even cover of a tree. Namely, that any even cover of a tree is a forest, hence a disjoint union of trees. We take a step further and prove that an even cover of a tree is a disjoint union of trees each of which is isomorphic to the original.

Lemma 3.7.1. *Let (G, τ_G) be an even graph and $((H, \tau_H), \phi)$ an even cover. Then* $\mathrm{Girth}(H) < \infty$ *implies* $\mathrm{Girth}(G) < \infty$.

Proof. If $\mathrm{Girth}(H) < \infty$, then by definition there exists some $n \geq 1$ and some $f \in \mathrm{Hom}_\tau((\mathrm{Cycle}_n, \tau_{\mathrm{Cycle}_n}), (H, \tau_H))$. Then composing ϕ with f we have $\phi \circ f \in \mathrm{Hom}_\tau((\mathrm{Cycle}_n, \tau_{\mathrm{Cycle}_n}), (G, \tau_G))$. But from this morphism we can extract an injective, even morphism from Cycle_m to G for some m (possibly smaller than n): if $\phi \circ f$ fails to be injective, then the only possibility is that $\phi \circ f$ successively backtracks at some finite number of edges in G. After deleting any possible successive backtracking, we obtain an injective morphism from some Cycle_m to G, whence $\mathrm{Girth}(G) < \infty$. $\square$

By taking the contrapositive, we obtain the following Corollary.

Corollary 3.7.1. *Let (G, τ_G) be an even graph and $((H, \tau_H), \phi)$ an even cover. If G is a tree, then H is a forest.*

Suppose now that G is a tree and that $((H, \tau_H), \phi)$ is a finite, even cover of G. By the corollary, there exists some $d \geq 1$ such that $H = \coprod_{i=1}^d T_i$ where T_i is a tree for every $i = 1, \ldots, d$. In fact we can show that each of the T_i is isomorphic to G. For this we need an intermediate result.

Lemma 3.7.2. *Let (G, τ_G) be a connected, even graph and $((H, \tau_H), \phi)$ an even cover. Then every connected component of H is, in its own right, an even cover of (G, τ_G).*

Now we can prove the desired result.

Lemma 3.7.3. *Let (T, τ_T) be a tree and let $((H, \tau_H), \phi)$ be a finite, even cover of T. Then*

$$H = \coprod_{i=1}^{d} T_i$$

where d is the degree of the cover (H, ϕ) and each T_i is isomorphic to T.

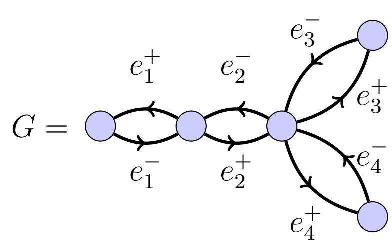
Proof. By the corollary, we know that H is a disjoint union of trees. It's immediate that these trees are precisely the connected components of H. By the previous lemma, each of these connected components of H is itself a cover of T. But any connected cover of T must be isomorphic to T. Therefore every connected component, T_i, of H is isomorphic to T. Now let d be the degree of the cover H over G. Each T_i contains one copy of the vertex set of T and therefore, in order that the degree of H over T is d, we must have d many of the T_i. This proves that $H = \coprod_{i=1}^d T_i$. $\square$

Chapter 4

Fundamental Group of a Graph

4.1 Orientations on a Even Graph

Let (G, τ_G) be a finite, even graph. Then the subgroup $\langle \tau_G \rangle \leq \mathrm{Sym}(E(G))$ generated by the transposition τ_G acts on $E(G)$. The orbits of $E(G)$ under this action are of the form $\{e, \tau_G(e)\}$ for $e \in E(G)$. It follows from the definition that G is even if and only if the orbit $\{e, \tau_G(e)\}$ has exactly two elements for every $e \in E(G)$. It follows that $|E(G)| = 2m$ for some $m \geq 1$. For every $e \in E(G)$, we arbitrarily choose one element from the orbit containing e which we designate as e^+. The resulting set of representatives will be denoted $E^+(G)$; i.e. $E^+(G) = \{e^+ : e \in E(G)\} \subset E(G)$. Given $e^+ \in E^+(G)$, we'll allow e^- to denote $\tau_G(e)$ the remaining element in the orbit containing e. Then we'll define $E^-(G) := \{e^- : e \in E(G)\} \subset E(G)$. We have thus partitioned $E(G)$ as $E(G) = E^+(G) \coprod E^-(G)$. Such a decomposition will be called an **orientation** on G. For example we have standard orientations on Path_n and Cycle_n: $E(\mathrm{Cycle}_n) = \{1^+, 2^+, \ldots, n^+\} \coprod \{1^-, 2^-, \ldots, n^-\} = E(\mathrm{Path}_n)$.



The figure shows an example of an arbitrary orientation $E(G) = \{e_1^+, e_2^+, e_3^+, e_4^+\} \coprod \{e_1^-, e_2^-, e_3^-, e_4^-\}$.

Fundamental Group of a Graph

Let $((G, \tau_G), v_0)$ be a finite, connected, even, pointed graph together with an orientation $E(G) = E^+(G) \coprod E^-(G)$.

Definition 4.1.1. Let

$$\mathrm{Cycles}(G, v_0) := \bigcup_{n \geq 0} \{f \in \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (G, \tau_G)) : f(0) = f(n) = v_0\}$$

Such an $f \in \mathrm{Cycles}(G, v_0)$ is called an **cycle in G based at** v_0. Given $f \in \mathrm{Cycles}(G, v_0)$, we write $|f| = n$ whenever $f \in \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (G, \tau_G))$. There is a distinguished element, $I \in \mathrm{Cycles}(G, v_0)$ with $I \in \mathrm{Hom}_\tau((\mathrm{Path}_0, \tau_{\mathrm{Path}_0}), (G, \tau_G))$ with $I_V(0) = v_0$ and empty edge map called the **identity cycle**.

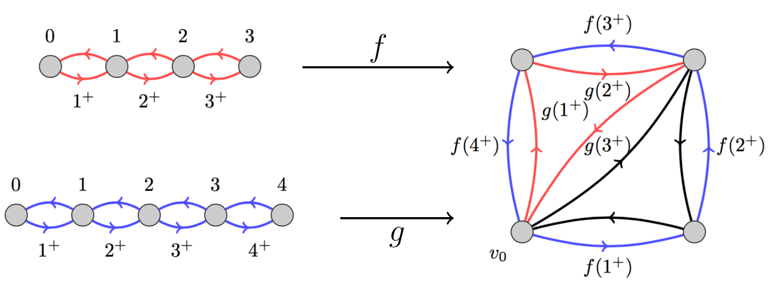
Given two cycles $f, g \in \mathrm{Cycles}(G, v_0)$, we define a composition law $\cdot$ which produces $f \cdot g \in \mathrm{Cycles}(G, v_0)$. The image of$f \cdot g$ in G will correspond to first following the image of f, then the image of g:

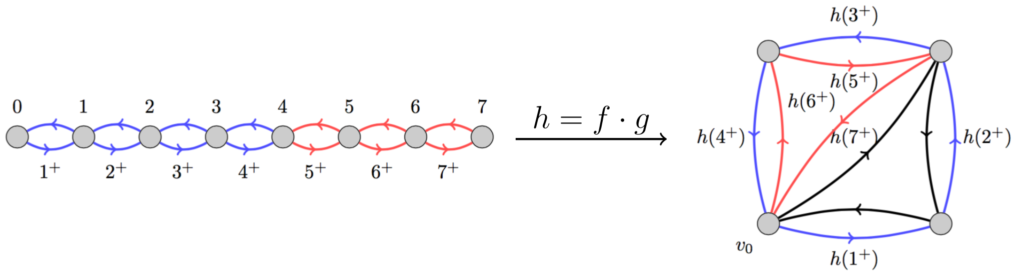
$$(f \cdot g)_V(j) = \begin{cases} f_V(j) & \text{for } j = 0, \ldots, |f| - 1; \\ g_V(j - |f|) & \text{for } j = |f|, \ldots, |f| + |g|. \end{cases}$$

$$(f \cdot g)_E(j^+) = \begin{cases} f_E(j^+) & \text{for } j = 1, \ldots, |f|; \\ g_E((j - |f|)^+) & \text{for } j = |f| + 1, \ldots, |f| + |g|. \end{cases}$$

$$(f \cdot g)_E(j^-) = \tau_G((f \cdot g)_E(j^+)) \quad \text{for } j = 1, \ldots, |f| + |g|.$$

We remark further that the identity cycle I is distinguished by the fact that for any $f \in \mathrm{Cycles}(G, v_0)$ we have $f \cdot I = f = I \cdot f$. An example of cycle composition is presented in the figures which follow.





While our composition law for cycles is necessarily discrete, it closely mirrors in philosophy the composition law for cycles as presented in [1] on page 89: if β and γ are two continuous paths into a space X then $\beta \cdot \gamma$ corresponds to first following β at "double speed" and then following γ at double speed. This is essentially our definition in the graph setting.

Free Groups

Let S be a set of formal symbols. We suppose that for every $s \in S$ that there corresponds an inverse symbol s^{-1} in some set S^{-1}. A **word in** S is a concatenation of elements from $S \cup S^{-1}$. Then the **free group generated by** S, denote $F(S)$, is the set of all words in S with multiplication given by "concatenation of words." The identity element in $F(S)$ is the empty word, which we'll denote by 1. The inverse of s will be the symbol s^{-1}. The only relations which must be satisfied in $F(S)$ is that $ss^{-1} = s^{-1}s = 1$ and $s1 = 1s = s$ for every $s \in S$. We observe that this is the minimal number of relations which must be satisfied by elements of a set in order that the set forms a group.

We give an example: let $S = \{a, b\}$, $S^{-1} = \{a^{-1}, b^{-1}\}$. Then, for example we have words $abba^{-1}b^{-1} \in F(S)$ and $abbaa^{-1}a^{-1}a^{-1}b^{-1} \in F(S)$. Notice that

$$\begin{aligned} abbaa^{-1}a^{-1}b^{-1} &= abb1a^{-1}b^{-1} \\ &= abba^{-1}b^{-1} \end{aligned}$$

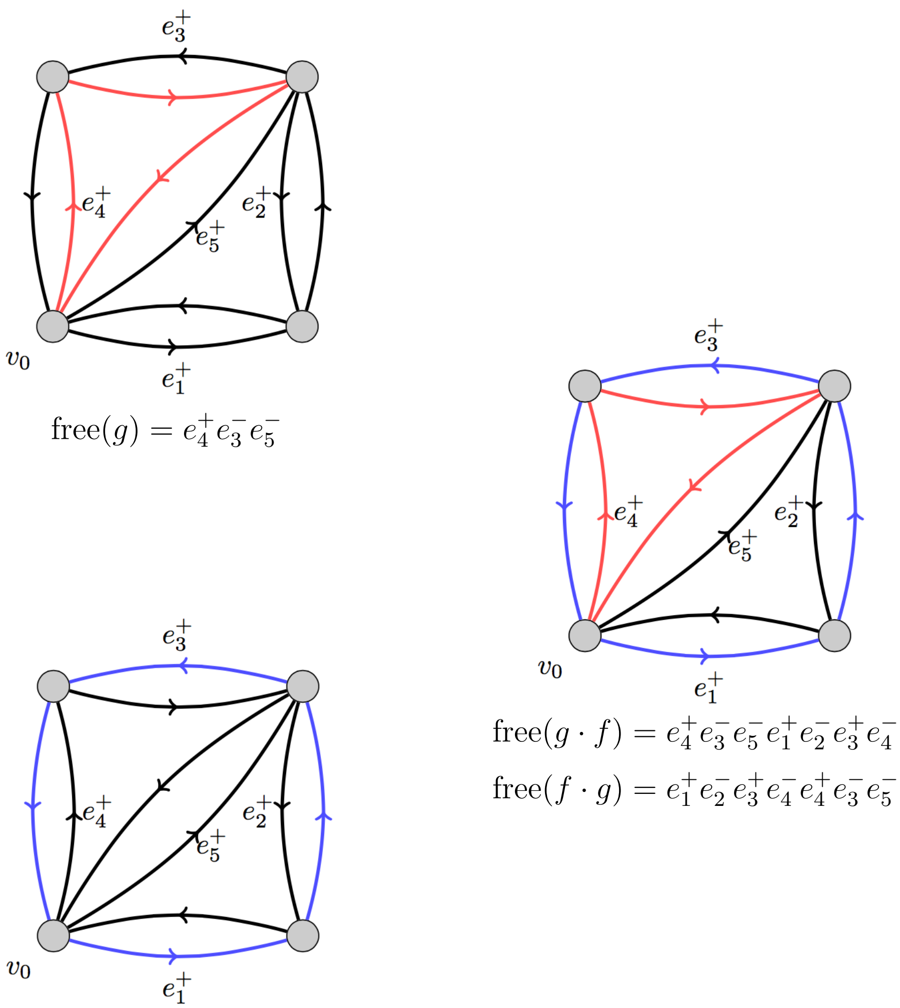
The first word $abba^{-1}b^{-1}$ is an example of what is called a **reduced word** in that no application of the relations in $F(S)$ can produce an equivalent word with a smaller length. The second example $abbaa^{-1}a^{-1}a^{-1}b^{-1}$ is not a reduced word: as we saw, by applying the relations in the group $abbaa^{-1}a^{-1}a^{-1}b^{-1} = abba^{-1}b^{-1}$ and the second of these words is shorter than the original word. It is a fact that for any set S, any word in $F(S)$ is equal to a unique reduced word.

free : **Cycles**$(G, v_0) \to F(E^+(G))$

We associate to every cycle in G based at v_0, a word in the free group on $E^+(G)$. The inverse symbol to $e^+ \in E^+(G)$ will be $e^- \in E^-(G)$ and vice versa.

Definition 4.1.2. Let $f \in \text{Cycles}(G, v_0)$. Then we define

$$\begin{aligned} \text{free}(f) &= f_E(1^+) f_E(2^+) \dots f_E(|f|^+) \\ &= \prod_{j=1}^{|f|} f_E(j^+) \in F(E^+(G)) \end{aligned}$$



$\text{free}(g) = e_4^+ e_3^- e_5^-$

$\text{free}(g \cdot f) = e_4^+ e_3^- e_5^- e_1^+ e_2^- e_3^+ e_4^-$

$\text{free}(f \cdot g) = e_1^+ e_2^- e_3^+ e_4^- e_4^+ e_3^- e_5^-$

$\text{free}(f) = e_1^+ e_2^- e_3^+ e_4^-$

Figure 4.1: free : DCycles$(G, v_0) \to F(E^+(G))$ is a monoid homomorphism.

Lemma 4.1.1. free : *Cycles*$(G, v_0) \to F(E^+(G))$ *respects the composition law defined on* *Cycles*(G, v_0). *That is, for any* $f, g \in$ *Cycles*(G, v_0) *we have*

$$\text{free}(f \cdot g) = \text{free}(f)\text{free}(g) \in F(E^+(G))$$

Proof. By definition,

$$\begin{aligned}\mathrm{free}(f \cdot g) &= \prod_{j=1}^{|f \cdot g|} (f \cdot g)_E(j^+) \\ &= \prod_{j=1}^{|f|} f_E(j^+) \prod_{j=|f|+1}^{|f|+|g|} g_E((j-|f|)^+)\end{aligned}$$

Now let $k = j - |f|$. Then we have

$$\begin{aligned}\prod_{j=1}^{|f|} f_E(j^+) \prod_{j=|f|+1}^{|f|+|g|} g_E((j-|f|)^+) &= \prod_{j=1}^{|f|} f_E(j^+) \prod_{k=1}^{|g|} g_E(k^+) \\ &= \mathrm{free}(f)\mathrm{free}(g)\end{aligned}$$

This proves that $\mathrm{free}(f \cdot g) = \mathrm{free}(f)\mathrm{free}(g)$, as desired. □

We refer the reader to Figure (4.1) for a concrete presentation of the result of the lemma.

Now we define an equivalence relation on $\mathrm{Cycles}(G, v_0)$ as follows: given $f, g \in \mathrm{Cycles}(G, v_0)$ we define $f \sim g$ if and only if $\mathrm{free}(f) = \mathrm{free}(g) \in F(E^+(G))$. That $\sim$ defines an equivalence relation follows from the fact that equality is an equivalence relation on $F(E^+(G))$. We call the equivalence relation $\sim$ **homotopy equivalence** and if $f \sim g$ we will say that f and g are **homotopic cycles** or that they are **homotopically equivalent**.

In the example presented in Figure (4.2) the cycles f and g are homotopic. First a word of explanation on the figure: the second dashed edge is not to indicate the presence of a second edge next to e_3^-, but rather that the cycle g passes over e_3^- a second time. To see that f and g are homotopic, we observe

$$\begin{aligned}\mathrm{free}(g) &= e_4^+ e_3^- e_3^+ e_3^- e_5^- \\ &= e_4^+ e_3^- e_5^- \\ &= \mathrm{free}(f)\end{aligned}$$

Thus $[f] = [g] \in \pi_1(G, v_0)$. In fact for any two homotopic cycles, one may be obtained from the other by deleting a number of "successive backtracks." A successive backtrack is when a cycle visits an edge, then the edge running the opposite direction, then the original edge once again. This behavior is reflected by having pairs of the form $e^{\pm}e^{\mp}$ appearing in the free word associated to the cycle and this observation substantiates our claim.

Cycle homotopy in our context of graphs is much less interesting than what can happen in the topological space context. In topology two cycles are homotopic if one can be continuously deformed into the other (see [1] P. 113). This is essentially our situation as well, except that the rigidity of graphs forces our hand: the only meaningful “continuous deformation” from one cycle to another is the deletion or insertion of successive back tracking edges.

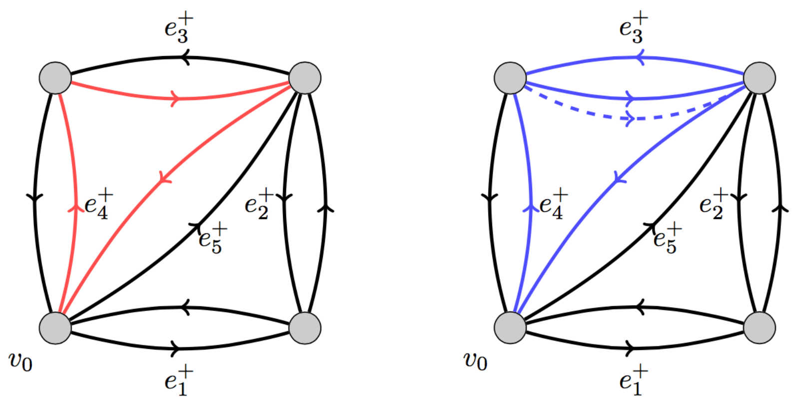


Figure 4.2: Homotopic cycles in G. f is shown in red and g in blue. We have $[f] = [g] \in \pi_1(G, v_0)$.

$\pi_1(G, v_0)$

Let $[f]$ denote the equivalence class with respect to homotopy equivalence which contains f. Explicitly,

$$[f] = \{g \in \text{Cycles}(G, v_0) : g \sim f\} \subseteq \text{Cycles}(G, v_0)$$

Now we define a set, $\pi_1(G, v_0)$, which we will see has a group structure. Together with its group structure, $\pi_1(G, v_0)$ will be known as the **fundamental group of G (based at v_0)**.

$$\begin{aligned}\pi_1(G, v_0) &= \text{Cycles}(G, v_0)/\sim \\ &= \{[f] : f \in \text{Cycles}(G, v_0)\}\end{aligned}$$

We define a multiplication on $\pi_1(G, v_0)$: given $[f], [g] \in \pi_1(G, v_0)$ we define

$$[f][g] = [f \cdot g]$$

This is identical to the multiplication defined in the topological context ([1] P. 115).

Lemma 4.1.2. *The multiplication on $\pi_1(G, v_0)$ given by $[f][g] = [f \cdot g]$ is well defined.*

Proof. Suppose that $[f_1] = [f_2] \in \pi_1(G, v_0)$ and $[g_1] = [g_2] \in \pi_1(G, v_0)$. Then by definition this means that $\mathrm{free}(f_1) = \mathrm{free}(f_2)$ and $\mathrm{free}(g_1) = \mathrm{free}(g_2)$. We observe that

$$\begin{aligned}
\mathrm{free}(f_1 \cdot g_1) &= \mathrm{free}(f_1)\mathrm{free}(g_1) \\
&= \mathrm{free}(f_2)\mathrm{free}(g_2) \\
&= \mathrm{free}(f_2 \cdot g_2)
\end{aligned}$$

It follows by definition then that $[f_1 \cdot g_1] = [f_2 \cdot g_2]$ which proves that our multiplication on $\pi_1(G, v_0)$ is well defined. □

It's immediate that $[I] \in \pi_1(G, v_0)$ gives the identity element in $\pi_1(G, v_0)$ since

$$[f][I] = [f \cdot I] = [f] = [I \cdot f] = [I][f]$$

In order to conclude that $\pi_1(G, v_0)$ is a group, we need to show that every $[f] \in \pi_1(G, v_0)$ has an inverse $[f]^{-1} \in \pi_1(G, v_0)$: i.e. an element $[f]^{-}1$ such that

$$[f][f]^{-1} = [I] = [f]^{-1}[f]$$

Given $f \in \mathrm{Cycles}(G, v_0)$ we define a map $f^{-1} : \mathrm{Path}_n \to G$ as follows:

$$\begin{aligned}
&(f^{-1})_V(j) = f_V(|f| - j) && \text{for } j = 0, \ldots, |f|; \\
&(f^{-1})_E(j^+) = \tau_G(f_E((|f| + 1 - j)^+)) && \text{for } j = 1, \ldots, |f|; \\
&(f^{-1})_E(j^-) = \tau_G((f^{-1})_E(j^+)) && \text{for } j = 1, \ldots, |f|.
\end{aligned}$$

Lemma 4.1.3. *Let $f \in$ Cycles(G, v_0), then f^{-1} as described above is also in Cycles(G, v_0).*

Proof. First we must prove that the diagram

$$\begin{tikzcd}
E(\mathrm{Path}_n) \arrow[r, "(f^{-1})_E"] \arrow[d, "(t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n})"'] & E(G) \arrow[d, "(t_G, h_G)"] \\
V(\mathrm{Path}_n)^2 \arrow[r, "(f^{-1})_V^2"'] & V(G)^2
\end{tikzcd}$$

commutes. To that end, let $j \in \{1, \ldots, |f|\}$ be arbitrary. Then

$$\begin{aligned}
(t_G, h_G)((f^{-1})_E(j^+)) &= (t_G, h_G)\left(\tau_G(f_E((|f| + 1 - j)^+))\right) \\
&= \left(h_G(f_E((|f| + 1 - j)^+)), t_G(f_E((|f| + 1 - j)^+))\right)
\end{aligned}$$

On the other hand,

$$\begin{aligned}(f^{-1})^2_V((t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n})(j^+)) &= (f^{-1})^2_V(j-1, j) \\ &= ((f^{-1})_V(j-1), (f^{-1})_V(j)) \\ &= (f_V(|f|-(j-1)), f_V(|f|-j)) \\ &= (f_V(|f|+1-j), f_V(|f|-j))\end{aligned}$$

Now we invoke that the assumption that $f \in \mathrm{Cycles}(G, v_0)$. In particular this means that the following diagram must commute:

$$\begin{tikzcd} E(\mathrm{Path}_n) \arrow[r, "f_E"] \arrow[d, "(t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n})"'] & E(G) \arrow[d, "(t_G, h_G)"] \\ V(\mathrm{Path}_n)^2 \arrow[r, "f_V^2"'] & V(G)^2 \end{tikzcd}$$

Now,

$$\begin{aligned}(t_G, h_G)(f_E((|f|+1-j)^-)) &= (t_G(f_E((|f|+1-j)^-)), h_G(f_E((|f|+1-j)^-))) \\ &= (t_G(f_E(\tau_{\mathrm{Path}_n}((|f|+1-j)^+))).h_G(f_E(\tau_{\mathrm{Path}_n}((|f|+1-j)^+)))) \\ &= (t_G(\tau_G(f_E((|f|+1-j)^+))), h_G(\tau_G(f_E((|f|+1-j)^+)))) \\ &= (h_G(f_E((|f|+1-j)^+)), t_G(f_E((|f|+1-j)^+)))\end{aligned}$$

But if we follow the diagram the other way around then the above must equal

$$\begin{aligned}(f_V^2)((t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n})((|f|+1-j)^-))) &= (f_V^2)(|f|+1-j, |f|-j) \\ &= (f_V(|f|+1-j), f_V(|f|-j))\end{aligned}$$

Therefore $(f_V(|f|+1-j), f_V(|f|-j)) = (h_G(f_E((|f|+1-j)^+)), t_G(f_E((|f|+1-j)^+)))$ and thus

$$(f^{-1})^2_V((t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n})(j^+)) = (t_G, h_G)((f^{-1})_E(j^+))$$

We also have that by definition

$$\begin{aligned}(f^{-1})_V((t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n})(j^-)) &= (f^{-1})_V(j, j-1) \\ &= (f_V(|f|-j), f_V(|f|-(j-1))) \\ &= (f_V(|f|-j), f_V(|f|+1-j))\end{aligned}$$

and

$$\begin{aligned}(t_G, h_G)((f^{-1})_E(j^-)) &= (t_G, h_G)(\tau_G((f^{-1})_E(j^+))) \\ &= (t_G, h_G)(f_E((|f|+1-j)^+))\end{aligned}$$

Once again invoking commutativity of the diagram for f, we have

$$\begin{aligned}(t_G, h_G)(f_E((|f|+1-j)^+)) &= (f_V^2)((t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n})((|f|+1-j)^+)) \\ &= f_V^2((|f|+1-j-1, |f|+1-j)) \\ &= (f_V(|f|-j), f_V(|f|+1-j))\end{aligned}$$

Therefore

$$(f^{-1})_V((t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n})(j^-)) = (t_G, h_G)((f^{-1})_E(j^-))$$

and the diagram commutes. Therefore we see that $f^{-1} \in \mathrm{Hom}(\mathrm{Path}_n, G)$. To show that f^{-1} is a morphism of even graphs, we need to show that the diagram

$$\begin{tikzcd} E(\mathrm{Path}_n) \arrow[r, "(f^{-1})_E"] \arrow[d, "\tau_{\mathrm{Path}_n}"'] & E(G) \arrow[d, "\tau_G"] \\ E(\mathrm{Path}_n) \arrow[r, "(f^{-1})_E"'] & E(G) \end{tikzcd}$$

commutes. So let $j \in \{1, \ldots, |f|\}$ be arbitrary. Then we have

$$\begin{aligned}\tau_G((f^{-1})_E(j^+)) &= \tau_G(\tau_G(f_E((|f|+1-j)^+))) \\ &= f_E((|f|+1-j)^+)\end{aligned}$$

On the other hand,

$$\begin{aligned}(f^{-1})_E(\tau_{\mathrm{Path}_n}(j^+)) &= (f^{-1})_E(j^-) \\ &= \tau_G((f^{-1})_E(j^+)) \\ &= \tau_G(\tau_G(f_E((|f|+1-j)^+)) \\ &= f_E((|f|+1-j)^+)\end{aligned}$$

Notice that the expressions match. Furthermore

$$\begin{aligned}\tau_G((f^{-1})_E(j^-)) &= \tau_G(\tau_G((f^{-1})_E(j^+))) \\ &= (f^{-1})_E(j^+)\end{aligned}$$

but

$$(f^{-1})_E(\tau_{\mathrm{Path}_n}(j^-)) = (f^{-1})_E(j^+)$$

This proves that the second diagram commutes and hence that $f^{-1} \in \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (G, \tau_G))$, as desired. $\square$

It follows from the definition that the inverse of a cycle visits the same vertices but in reverse order: Figure (4.3) illustrates this point.

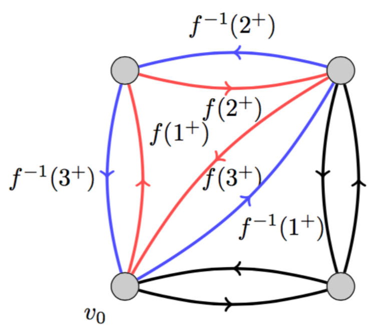


Figure 4.3: A cycle and its inverse.

Lemma 4.1.4. *Let* $[f] \in \pi_1(G, v_0)$*. Then* $[f][f^{-1}] = [f \cdot f^{-1}] = [I] = [f^{-1} \cdot f] = [f^{-1}][f]$*.*

Proof. We observe that

$$\begin{aligned}
\mathrm{free}(f \cdot f^{-1}) &= \mathrm{free}(f)\mathrm{free}(f^{-1}) \\
&= \prod_{j=1}^{|f|} f_E(j^+) \prod_{k=1}^{|f^{-1}|} (f^{-1})_E(k^+) \\
&= \prod_{j=1}^{|f|} f_E(j^+) \prod_{k=1}^{|f|} \tau_G(f_E((|f|+1-k)^+))
\end{aligned}$$

Now we simply observe that the first term when $j = |f|$ in the first product and $k = 1$ in the second product we are multiplying $f_E(|f|^+)$ and $\tau_G(f_E(|f|^+))$ consecutively. But by definition of our orientation, the edges $f_E(|f|^+)$ and $\tau_G(f_E(|f|^+))$ are inverses in $F(E^+(G))$. Therefore these terms cancel one another out. Next, when $j = |f| - 1$ in the first product and $k = 2$ in the second product we are multiplying $f_E((|f|-1)^+)$ and $\tau_G(f_E((|f|-1)^+))$ consecutively. But then these cancel one another out and so on. For this reason we see that $\mathrm{free}(f \cdot f^{-1}) = \mathrm{free}(I)$, the empty word. Therefore $[f \cdot f^{-1}] = [I]$, by definition. An identical argument proves that $[f^{-1} \cdot f] = [I]$. $\square$

The lemma therefore shows that $[f]^{-1} = [f^{-1}] \in \pi_1(G, v_0)$. We are therefore able to conclude that $\pi_1(G, v_0)$ is a group, the **fundamental group** of our graph G.
Let T be a finite tree and let $v_0 \in V(G)$ be an arbitrary, fixed base point. Then $\pi_1(T, v_0) = \langle [I] \rangle$, the trivial group. Indeed, by definition T has no non-trivial cycles. This substantiates our earlier claim that trees should be thought of as "simply connected" graphs.

Path Composition and Homotopy

Definition 4.1.3. Let

$$\mathrm{Paths}(G) := \bigcup_{n \geq 0} \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (G, \tau_G))$$

Given $p \in \mathrm{Paths}(G)$, we say that $|p| = n$ whenever $p \in \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (G, \tau_G))$. For every $w \in V(G)$ there is a distinguished element, $I_w \in \mathrm{Paths}(G)$ with $I_w \in \mathrm{Hom}_\tau((\mathrm{Path}_0, \tau_{\mathrm{Path}_0}), (G, \tau_G))$ with $(I_w)_V(0) = w$ and empty edge map called the **identity path at** w**.**

Given $p, q \in \mathrm{Paths}(G)$ which satisfy $p_V(|p|) = q_v(0) \in V(G)$ then we define a composition $p \cdot q$ which corresponds to first following p and then following q as follows:

$$(p \cdot q)_V(j) = \begin{cases} p_V(j) & \text{for } j = 0, \ldots, |p| - 1; \\ q_V(j - |p|) & \text{for } j = |p|, \ldots, |p| + |q|. \end{cases}$$

$$(p \cdot q)_E(j^+) = \begin{cases} p_E(j^+) & \text{for } j = 1, \ldots, |p|; \\ q_E((j - |p|)^+) & \text{for } j = |p| + 1, \ldots, |p| + |q|. \end{cases}$$

$$(p \cdot q)_E(j^-) = \ \tau_G((p \cdot q)_E(j^+)) \quad \text{for } j = 1, \ldots, |p| + |q|.$$

Let $w \in V(G)$ and suppose that $p, q \in \mathrm{Paths}(G)$ satisfy $p(|p|) = w$ and $q(0) = w$. Then evidently we have

$$p \cdot I_w = p, \ I_w \cdot q = q$$

As before, let $F(E^+(G))$ denote the free group on the set $E(G) = E^+(G) \coprod E^-(G)$ of (oriented) edges of G. Then we define a map

$$\mathrm{free} : \mathrm{Paths}(G) \to F(E^+(G))$$
$$p \mapsto \prod_{j=1}^{|p|} p(j^+)$$

We define an equivalence relation $\sim$ on $\mathrm{Paths}(G)$ as follows: $p \sim q$ if and only if $\mathrm{free}(p) = \mathrm{free}(q) \in F(E^+(G))$. If $p \sim q$ then we say that p and q are **homotopic paths** or that they are **homotopically equivalent**. We remark that if p and q are homotopic paths, then they start and end at the same vertices. Finally, let $[p]$ denote the equivalence class of paths in $\mathrm{Paths}(G)$ under $\sim$:

$$[p] = \{q \in \mathrm{Paths}(G) : q \sim p\}$$

Group of Permutations on a Set

Definition 4.1.4. Given a finite set X, we define $\mathrm{Sym}(X)$ to be the group of **right** permutations (bijective maps $f : X \to X$) of X. Thus given $x \in X$ and $f \in \mathrm{Sym}(X)$ we write x^f for the image of x under f.

Notice that given $f, g \in \mathrm{Sym}(X)$ we have

$$x^{fg} = (x^f)^g$$

so that by writing the composition fg we mean **"first apply f, then apply g"**.

Chapter 5

Equivalent Categories

Two Categories

In this section we define two categories related to a finite, connected even pointed graph $((G, \tau_G), v_0)$: the category $\mathbf{Cov}(G)$ of finite even covers of G and the category $\pi_\mathbf{1}(\mathbf{G}) - \mathbf{Set}$ of finite $\pi_1(G, v_0)$-sets. An object in the category of $\mathbf{Cov}(G)$ is a finite even cover $((H, \tau_H), \phi)$. If $((H_1, \tau_{H_1}), \phi_1)$ and $((H_2, \tau_{H_2}), \phi_2)$ are two such finite even covers of G, then a map $f : ((H_1, \tau_{H_1}), \phi_1) \to ((H_2, \tau_{H_2}), \phi_2)$ is a morphism of finite even covers if and only if the following hold:

(1) $f \in \mathrm{Hom}_\tau((H_1, \tau_{H_1}), (H_2, \tau_{H_2}))$;

(2) $\phi_1 = \phi_2 \circ f$.

The analogy to the development in Algebraic Topology is a good one: there a morphism between covering spaces is a continuous map which for which property (2) above holds ([2] P. 67).

Objects in the category $\pi_\mathbf{1}(\mathbf{G}) - \mathbf{Set}$ are finite sets F on which the group $\pi_1(G, v_0)$ acts. We recall that $\pi_1(G, v_0)$ acts on F if and only if there is a group homomorphism $\Phi : \pi_1(G, v_0) \to \mathrm{Sym}(F)$ (such a group homomorphism is referred to as the permutation representation of the action of $\pi_1(G, v_0)$ on F). If F_1 and F_2 are two $\pi_1(G, v_0)$-sets with associated permutation representations $\Phi_1 : \pi_1(G, v_0) \to \mathrm{Sym}(F_1)$ and $\Phi_2 : \pi_1(G, v_0) \to \mathrm{Sym}(F_2)$, then a map $t : F_1 \to F_2$ is a morphism of $\pi_1(G, v_0)$-sets if and only if the following hold:

(1) $t : F_1 \to F_2$ is a map of sets;

(2) $t(x^{\Phi_1([C])}) = (t(x))^{\Phi_2([C])}$ for every $[C] \in \pi_1(G, v_0)$ and for every $x \in F_1$.

The interested reader should compare the above with the definition of "isomorphism of sets with $\pi_1(X, x_0)$-action" given in the middle of page 70 in [2].

There is a convenient shorthand which we will frequently employ. Let $F \in \boldsymbol{\pi_1}(\mathbf{G}) - \mathbf{Set}$ with permutation representation $\Phi : \pi_1(G, v_0) \to \mathrm{Sym}(F)$. Then given $[C] \in \pi_1(G, v_0)$ and $x \in F$ we'll write $x^{[C]}$ in place of the cumbersome $x^{\Phi([C])}$. With respect to this notation, property (2) above becomes

$$t(x^{[C]}) = (t(x))^{[C]}$$

A map t satisfying either formulation of property (2) is said to be $\pi_1(G, v_0)$**-equivariant**.

An Equivalence of Categories

Let $((G, \tau_G), v_0)$ be a finite, connected even pointed graph. The remainder of the paper is devoted to showing that the category $\mathbf{Cov}(G)$ is "equivalent" to the category $\boldsymbol{\pi_1}(\mathbf{G}) - \mathbf{Set}$. We will define a pair of functors

$$\mathcal{F} : \mathbf{Cov}(G) \to \boldsymbol{\pi_1}(\mathbf{G}) - \mathbf{Set}$$

and

$$\mathcal{G} : \boldsymbol{\pi_1}(\mathbf{G}) - \mathbf{Set} \to \mathbf{Cov}(G)$$

When we compose these two functors in either order we will get a functor which is "equivalent," in a sense which will be made precise, to the identity functor on each category (depending on the order of composition.) We will call $\mathcal{F}$ the **fiber functor** and $\mathcal{G}$ the **reverse functor**.

Chapter 6

The Fiber Functor

The Fiber Functor

In this section we define a functor

$$\mathcal{F} : \mathbf{Cov}(G) \to \pi_{\mathbf{1}}(\mathbf{G}) - \mathbf{Set}$$

which associates to each finite, even cover $((H, \tau_H), \phi_H)$ of (G, τ_G) a finite $\pi_1(G, v_0)$-set. We'll associate to $((H, \tau_H), \phi_H)$ the set $\phi_H^{-1}(v_0)$:

$$\mathcal{F}(((H, \tau_H), \phi_H)) = \phi_H^{-1}(v_0)$$

Notice that [2] also associates the set of points above the base point to a covering space (P. 69).

It is our task to prove that $\phi_H^{-1}(v_0)$ is a finite set on which $\pi_1(G, v_0)$ acts.

Let (G, τ_G) be a fixed finite, even connected graph and let $((H, \tau_H), \phi_H)$ be a finite even cover of (G, τ_G). Let $f \in \text{Cycles}(G, v_0)$ and let $\hat{y} \in \phi_H^{-1}(v_0)$ be arbitrary. By the (even) Path Lifting Lemma, there is a unique morphism

$$\hat{f}_{\hat{y}} \in \text{Hom}_\tau((\text{Path}_n, \tau_{\text{Path}_n}), (G, \tau_G))$$

which satisfies

(1) $\hat{f}_{\hat{y}}(0) = \hat{y}$;

(2) $f = \phi_H \circ \hat{f}_{\hat{y}}$.

Recall that such a morphism $\hat{f}_{\hat{y}}$ is called a **lift** of f. Figure (6.1) shows a cycle f and its three H-lifts, $\hat{f}_{\hat{y}}$, $\hat{f}_{\hat{w}}$, and $\hat{f}_{\hat{z}}$.
This matches the development in [2] where they consider unique lifts of a cycle starting at various fiber elements (P. 69).

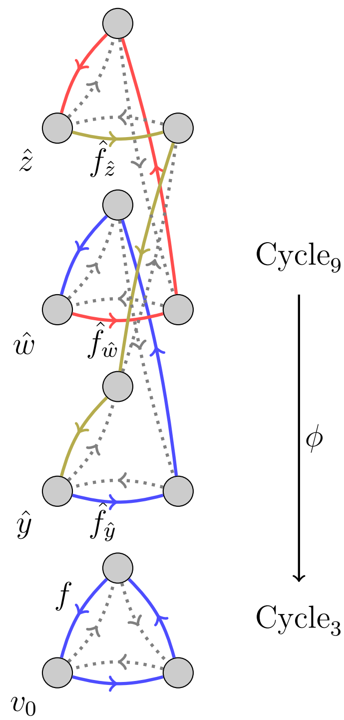


Figure 6.1: Cycles lift to paths.

Then we define a map

$$L_f : \phi_H^{-1}(v_0) \to \phi_H^{-1}(v_0)$$

by

$$\hat{y} \mapsto \hat{y}^{L_f} := \hat{f}_{\hat{y}}(|f|)$$

By property (2) above we have that $\phi_H(\hat{f}_{\hat{y}}(|f|)) = f(|f|) = v_0$ which shows that $\hat{y}^{L_f} \in \phi_H^{-1}(v_0)$.
Here differ in an important way from the development presented in [2] on page 69: given a fiber element $\hat{y}$, there is a unique lift of a given cycle which *ends* at $\hat{y}$. Then they send $\hat{y}$ to the point at which that unique cycle began. The reason for this subtle difference is that we compose permutations in the opposite order than does Hatcher.

Lemma 6.0.5. *For every $f \in Cycles(G, v_0)$ the map*

$$L_f : \phi_H^{-1}(v_0) \to \phi_H^{-1}(v_0)$$

is a bijection with inverse $L_{f^{-1}}$. This implies that $L_f \in \mathrm{Sym}(\phi_H^{-1}(v_0))$.

Proof. We prove that $(L_f)^{-1} = L_{f^{-1}}$. Let $\hat{y} \in \phi_H^{-1}(v_0)$. Then by definition

$$(\hat{h}^{L_f})^{L_{f^{-1}}} = (\hat{f}_{\hat{y}})^{L_{f^{-1}}}$$

Let $\hat{w} := \hat{f}_{\hat{y}}(|f|) = \hat{y}^{L_f} \in \phi_H^{-1}(v_0)$. Then our aim is to prove that

$$\hat{w}^{L_{f^{-1}}} = \hat{y}$$

By the (Even) Path Lifting Lemma, there is a unique morphism $\widehat{f^{-1}}_{\hat{w}} \in \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (H, \tau_H))$ which satisfies

(1) $\widehat{f^{-1}}_{\hat{w}}(0) = \hat{w}$;

(2) $f^{-1} = \phi_H \circ \widehat{f^{-1}}_{\hat{w}}$.

By definition

$$\hat{w}^{L_{f^{-1}}} = \widehat{f^{-1}}_{\hat{w}}(|f^{-1}|) = \widehat{f^{-1}}_{\hat{w}}(|f|)$$

Thus we must show that

$$\widehat{f^{-1}}_{\hat{w}}(|f|) = \hat{y} = \hat{f}_{\hat{y}}(0)$$

We prove the slightly more general statement that

$$(\widehat{f^{-1}}_{\hat{w}})_V(j) = \hat{f}_{\hat{y}}(|f| - j)$$

for every $j = 0, \dots, |f|$. We begin by observing that

$$\hat{f}_{\hat{y}}(|f|) = \hat{w} = \widehat{f^{-1}}_{\hat{w}}(0)$$

so that the statement holds for $j = 0$.

We first show that $\tau_H(\widehat{f^{-1}}_{\hat{w}}(1^+)) = \hat{f}_{\hat{y}}(|f|^+)$. First observe that by definition of f^{-1} we have

$$f^{-1}(1^+) = \tau_G(f(|f|^+))$$

Then since $\widehat{f^{-1}}_{\hat{w}}$ is a lift of f^{-1}, since $\hat{f}_{\hat{y}}$ is a lift of f, and since all the relevant morphism are even graph morphisms, we have

$$\begin{aligned}
\phi_H((\widehat{f^{-1}}_{\hat{w}})(1^+)) &= f^{-1}(1^+) \\
&= \tau_G(f(|f|^+)) \\
&= f(\tau_{\mathrm{Path}_n}(|f|^+)) \\
&= \phi_H(\hat{f}_{\hat{y}}(\tau_{\mathrm{Path}_n}(|f|^+))) \\
&= \phi_H(\tau_H(\hat{f}_{\hat{y}}(|f|^+)))
\end{aligned}$$

Thus $\widehat{f^{-1}}_{\hat{w}}(1^+)$ and $\tau_H(\hat{f}_{\hat{y}}(|f|^+))$ are mapped by ϕ_H to the same edge in G. But we also have that

$$\widehat{f^{-1}}_{\hat{w}}(1^+),\ \tau_H(\hat{f}_{\hat{y}}(|f|^+)) \in N_{\hat{w}}$$

We recall that ϕ_H is a bijection on edge neighborhoods. Therefore it must be the case that

$$\widehat{f^{-1}}_{\hat{w}}(1^+) = \tau_H(\hat{f}_{\hat{y}}(|f|^+))$$

But then we have that

$$\begin{aligned} h_H(\widehat{f^{-1}}_{\hat{w}}(1^+)) &= h_H(\tau_H(\hat{f}_{\hat{y}}(|f|^+))) \\ &= t_H(\hat{f}_{\hat{y}}(|f|^+)) \end{aligned}$$

Recalling that both $\widehat{f^{-1}}_{\hat{w}}$ and $\hat{f}_{\hat{y}}$ are directed graph morphisms from Path_n to H, both squares of the following diagram must commute:

$$\begin{tikzcd} E(\mathrm{Path}_n) \arrow[r, "(\widehat{f^{-1}}_{\hat{w}})_E"] \arrow[d, "(t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n})"'] & E(H) \arrow[d, "(t_H, h_H)"] & E(\mathrm{Path}_n) \arrow[l, "(\hat{f}_{\hat{y}})_E"'] \arrow[d, "(t_{\mathrm{Path}_n}, h_{\mathrm{Path}_n})"] \\ V(\mathrm{Path}_n)^2 \arrow[r, "(\widehat{f^{-1}}_{\hat{w}})_V^2"'] & V(H)^2 & V(\mathrm{Path}_n)^2 \arrow[l, "(\hat{f}_{\hat{y}})_V^2"] \end{tikzcd}$$

By commutativity of the left square we have that

$$\begin{aligned} h_H(\widehat{f^{-1}}_{\hat{w}}(1^+)) &= (\widehat{f^{-1}}_{\hat{w}})_V(h_{\mathrm{Path}_n}(1^+)) \\ &= (\widehat{f^{-1}}_{\hat{w}})_V(1) \end{aligned}$$

and by commutativity of the right square we see that

$$\begin{aligned} t_H(\hat{f}_{\hat{y}}(|f|)^+)) &= \hat{f}_{\hat{y}}(t_{\mathrm{Path}_n}(|f|^+)) \\ &= \hat{f}_{\hat{y}}(|f|-1) \end{aligned}$$

Putting it all together, we see that

$$(\hat{f}_{\hat{y}})_V(|f|-1) = t_H(\hat{f}_{\hat{y}}(|f|)^+)) = h_H(\widehat{f^{-1}}_{\hat{w}}(1^+)) = (\widehat{f^{-1}}_{\hat{w}})_V(1)$$

as desired. It follows by induction that

$$(\widehat{f^{-1}}_{\hat{w}})_V(j) = \hat{f}_{\hat{y}}(|f|-j)$$

for every $j = 0, \ldots, |f|$. In particular the statement holds for $j = |f|$:

$$\widehat{f^{-1}}_{\hat{w}}(|f|) = \hat{y} = \hat{f}_{\hat{y}}(0)$$

As we previously remarked, this is equivalent to

$$(\hat{y}^{L_f})^{L_{f^{-1}}} = \hat{y}$$

This proves that $(L_f)^{-1} = L_{f^{-1}}$. Thus $L_f \in \mathrm{Sym}(\phi_H^{-1}(v_0))$, as desired. $\square$

In Figure (6.2) we illustrate a cycle, its inverse, and the lifts appearing in the proof of the Lemma.

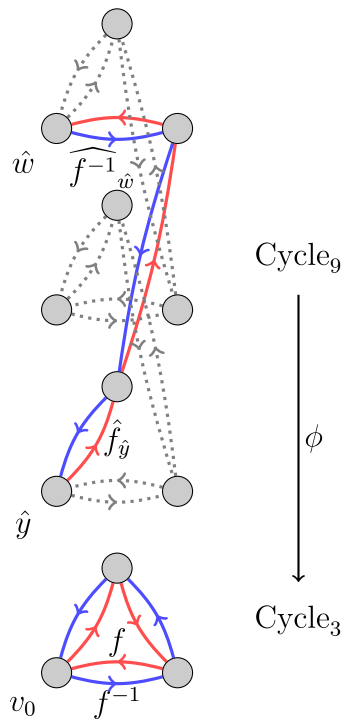


Figure 6.2: A cycle, its inverse, and their lifts.

Lemma 6.0.6. *The map*

$$\mathit{Cycles}(G, v_0) \to \mathrm{Sym}(\phi_H^{-1}(v_0))$$
$$f \mapsto L_f$$

induces a homomorphism $\Phi : \pi_1(G, v_0) \to \mathrm{Sym}(\phi_H^{-1}(v_0))$. *By this we mean that* $f \mapsto L_f$ *factors through the quotient*

$$\mathit{Cycles}(G, v_0) \to \pi_1(G, v_0)$$
$$f \mapsto [f]$$

In other words the diagram

$$\begin{tikzcd} Cycles(G, v_0) \arrow[r, "f \mapsto [f]"] \arrow[dr, "f \mapsto L_f"'] & \pi_1(G, v_0) \arrow[d, "\Phi:[f] \mapsto L_f"] \\ & \mathrm{Sym}(\phi_H^{-1}(v_0)) \end{tikzcd}$$

commutes and

$$\Phi([f][g]) = \Phi([f])\Phi([g])$$

for every $[f], [g] \in \pi_1(G, v_0)$*.*

Proof. Note that the diagram trivially commutes. In order to show that Φ is a homomorphism, let $\hat{y} \in \mathrm{Sym}(\phi_H^{-1}(v_0))$ be arbitrary. Then we have

$$\begin{aligned} \hat{y}^{L_{f \cdot g}} &= \widehat{f \cdot g}_{\hat{y}}(|f \cdot g|) \\ &= \hat{g}_{\hat{y}^{L_f}}(|g|) \\ &= (\hat{y}^{L_f})^{L_g} \end{aligned}$$

Since $\hat{y}$ was arbitrary we conclude that $L_{f \cdot g} = L_f L_g$ and thus

$$\begin{aligned} \Phi([f][g]) &= \Phi([f \cdot g]) \\ &= L_{f \cdot g} \\ &= L_f L_g \\ &= \Phi([f])\Phi([g]) \end{aligned}$$

This proves that $\Phi : \pi_1(G, v_0) \to \mathrm{Sym}(\phi_H^{-1}(v_0))$ is a group homomorphism, as desired. $\square$

We've therefore shown that $\mathcal{F}(((H, \tau_H), \phi_H)) = \phi_H^{-1}(v_0)$ is a finite $\pi_1(G, v_0)$-set, as desired.

Now we need to say how $\mathcal{F} : \mathbf{Cov}(G) \to \boldsymbol{\pi_1}(\mathbf{G}) - \mathbf{Set}$ acts on morphisms in $\mathbf{Cov}(G)$. To this end, let

$$((H_1, \tau_{H_1}), \phi_{H_1}),\ ((H_2, \tau_{H_2}), \phi_{H_2}) \in \mathbf{Cov}(G)$$

and let

$$f \in \mathrm{Hom}_{\mathbf{Cov}(G)}(((H_1, \tau_{H_1}), \phi_{H_1}), ((H_2, \tau_{H_2}), \phi_{H_2}))$$

Recall that by definition this means that

(1) $f \in \mathrm{Hom}_\tau((H_1, \tau_{H_1}), (H_2, \tau_{H_2}))$;

(2) $\phi_{H_1} = \phi_{H_2} \circ f$.

By definition

$$\mathcal{F}((((H_1, \tau_{H_1}), \phi_{H_1})) = \phi_{H_1}^{-1}(v_0) \in \pi_{\mathbf{1}}(\mathbf{G}) - \mathbf{Set}$$

and

$$\mathcal{F}((((H_2, \tau_{H_2}), \phi_{H_2})) = \phi_{H_2}^{-1}(v_0) \in \pi_{\mathbf{1}}(\mathbf{G}) - \mathbf{Set}$$

So let $\hat{y} \in \phi_{H_1}^{-1}(v_0)$. We define

$$\mathcal{F}(f)(\hat{y}) = f(\hat{y})$$

By property (2) above we have that

$$v_0 = \phi_{H_1}(\hat{y}) = \phi_{H_2}(f(\hat{y}))$$

from which it follows that

$$\mathcal{F}(f)(\hat{y}) = f(\hat{y}) \in \phi_{H_2}^{-1}(v_0)$$

Thus $\mathcal{F}(f) : \phi_{H_1}^{-1}(v_0) \to \phi_{H_2}^{-1}(v_0)$.
Now we must show that $\mathcal{F}(f)$ as defined is a morphism of the $\pi_1(G, v_0)$ sets $\phi_{H_1}^{-1}(v_0)$ and $\phi_{H_2}^{-1}(v_0)$.

Lemma 6.0.7. *Let $\Phi_1 : \pi_1(G, v_0) \to \mathrm{Sym}(\phi_{H_1}^{-1}(v_0))$ and $\Phi_2 : \pi_1(G, v_0) \to \mathrm{Sym}(\phi_{H_2}^{-1}(v_0))$ denote the respective permutation representations of the $\pi_1(G, v_0)$-actions on $\phi_{H_1}^{-1}(v_0)$ and $\phi_{H_2}^{-1}(v_0)$. Then we have*

$$\mathcal{F}(f)(\Phi_1([g])(\hat{w})) = \Phi_2([g])((\mathcal{F}(f))(\hat{w}))$$

For every $[g] \in \pi_1(G, v_0)$ and every $\hat{w} \in \phi_{H_1}^{-1}(v_0)$. In other words $\mathcal{F}(f)$ is a morphism of $\pi_1(G, v_0)$-sets.

Proof. According to our shorthand, the equation in the statement of the lemma may be written

$$\mathcal{F}(f)([g](\hat{w})) = [g]((\mathcal{F}(f))(\hat{w}))$$

By definition of $\mathcal{F}(f)$ this is equivalent to showing that

$$f([g](\hat{w})) = [g](f(\hat{w}))$$

By definition

$$[g](\hat{w}) = \hat{w}^{L_g} = \hat{g}_{\hat{w}}(|g|)$$

where $\hat{g}_{\hat{w}}$ is the unique lift of g which starts at $\hat{w}$. Thus we must show that

$$\hat{g}_{f(\hat{w})}(|g|) = f(\hat{w})^{L_g} = f(\hat{w}^{L_g}) = f(\hat{g}_{\hat{w}}(|g|))$$

We begin by observing that

$$\hat{g}_{\hat{w}} \in \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (H_1, \tau_{H_1}))$$

and

$$f \in \mathrm{Hom}_\tau((H_1, \tau_{H_1}), (H_2, \tau_{H_2}))$$

Therefore the composition

$$f \circ \hat{g}_{\hat{w}} \in \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (H_2, \tau_{H_2}))$$

On the other hand $\hat{g}_{f(\hat{w})}$ is a lift in H_2 of g which starts at $f(\hat{w})$. Our strategy will be to prove that $f \circ \hat{g}_{\hat{w}}$ is also a lift in H_2 of g which starts at $f(\hat{w})$ and then exploit uniqueness of lifts. First we observe that

$$(f \circ \hat{g}_{\hat{w}})(0) = f(\hat{g}_{\hat{w}}(0)) = f(\hat{w}) \in \phi_{H_2}^{-1}(v_0)$$

so that $f \circ \hat{g}_{\hat{w}}$ starts at $f(\hat{w})$. In order to show that $f \circ \hat{g}_{\hat{w}}$ is an H_2-lift of g we first recall that

$$\phi_{H_1} = \phi_{H_2} \circ f$$

since f is a morphism in $\mathbf{Cov}(G)$ from $H_1 \to H_2$ and that

$$g = \phi_{H_1} \circ \hat{g}_{\hat{w}}$$

since $\hat{g}_{\hat{w}}$ is an H_1-lift of g. This pair of facts implies that

$$\begin{aligned} \phi_{H_2} \circ f \circ \hat{g}_{\hat{w}} &= \phi_{H_1} \circ \hat{g}_{\hat{w}} \\ &= g \end{aligned}$$

This is precisely what it means for $f \circ \hat{g}_{\hat{w}}$ to be an H_2-lift of g. But then by uniqueness of lifts, we have that

$$f \circ \hat{g}_{\hat{w}} = \hat{g}_{f(\hat{w})} \in\in \mathrm{Hom}_\tau((\mathrm{Path}_n, \tau_{\mathrm{Path}_n}), (H_2, \tau_{H_2}))$$

In particular we have that

$$f(\hat{g}_{\hat{w}}(|g|)) = \hat{g}_{f(\hat{w})}(|g|)$$

as desired. This proves that

$$\mathcal{F}(f) \in \mathrm{Hom}_{\boldsymbol{\pi_1}(\mathbf{G})-\mathbf{Set}}(\phi_{H_1}^{-1}(v_0), \phi_{H_2}^{-1}(v_0))$$

as desired. $\square$

Lemma 6.0.8. *Let $((H_1, \tau_{H_1}), \phi_1)$, $((H_2, \tau_{H_2}), \phi_2)$ and $((H_3, \tau_{H_3}), \phi_3)$ be three finite, even covers of G and let $f_1 : ((H_1, \tau_{H_1}), \phi_1) \to ((H_2, \tau_{H_2}), \phi_2)$ and $f_2 : ((H_2, \tau_{H_2}), \phi_2) \to ((H_3, \tau_{H_3}), \phi_3)$ be two morphisms. Then $\mathcal{F}(f_2 \circ f_1) = \mathcal{F}(f_2) \circ \mathcal{F}(f_1)$.*

Proof. Let $\hat{w} \in \phi_1^{-1}(v_0)$ be arbitrary. Then

$$\mathcal{F}(f_2 \circ f_1)(\hat{w}) = (f_2 \circ f_1)(\hat{w}) = f_2(f_1(\hat{w}))$$

On the other hand,

$$\begin{aligned}(\mathcal{F}(f_2) \circ \mathcal{F}(f_1))(\hat{w}) &= \mathcal{F}(f_2)(\mathcal{F}(f_1)(\hat{w})) \\ &= \mathcal{F}(f_2)(f_1(\hat{w})) \\ &= f_2(f_1(\hat{w})\end{aligned}$$

Thus $\mathcal{F}(f_2 \circ f_1) = \mathcal{F}(f_2) \circ \mathcal{F}(f_1)$, as desired. $\square$

Lemma 6.0.9. *Let $((H_1, \tau_{H_1}), \phi_1)$ be a finite, even cover of G and let $\mathrm{Id}_{\mathrm{H}_1}$ be the identity morphism on H_1. Then $\mathcal{F}(\mathrm{Id}_{\mathrm{H}_1}) = \mathrm{Id}_{\phi_1^{-1}(\mathrm{v}_0)} = \mathrm{Id}_{\mathcal{F}(\mathrm{H}_1)}$.*

Proof. Let $\hat{w} \in \phi_1^{-1}(v_0)$ be arbitrary. Then

$$\mathcal{F}(\mathrm{Id}_{\mathrm{H}_1})(\hat{w}) = \mathrm{Id}_{\mathrm{H}_1}(\hat{w}) = \mathrm{Id}_{\phi_1^{-1}(\mathrm{v}_0)}(\hat{w})$$

Therefore we have $\mathcal{F}(\mathrm{Id}_{\mathrm{H}_1}) = \mathrm{Id}_{\phi_1^{-1}(\mathrm{v}_0)} = \mathrm{Id}_{\mathcal{F}(\mathrm{H}_1)}$, as desired. $\square$

Corollary 6.0.1. *$\mathcal{F} : \mathbf{Cov}(G) \to \boldsymbol{\pi_1}(\mathbf{G}) - \boldsymbol{Set}$ is a (covariant) functor.*

Chapter 7

The Reverse Functor

The Reverse Functor

In this section we describe a functor $\mathcal{G}$ from the category of finite $\pi_1(G, v_0)$-sets to the category of finite even covers of G. Let F be a finite $\pi_1(G, v_0)$-set with associated permutation representation $\Phi : \pi_1(G, v_0) \to \mathrm{Sym}(F)$. We show how to obtain from this data, a finite even cover $\mathcal{G}(F) = ((H, \tau_H), \phi)$ of (G, τ_G). First we fix some notation. Let $((G, \tau_G), v_0)$ be a finite, even, connected pointed graph and $T \subseteq G$ a spanning tree of G. We define $X(G) := E(G) \setminus E(T)$, the **excess edges of** G. Let $|X| := g$. We say that G has genus g. Finally, let $E(G) = E^+(G) \coprod E^-(G)$ be an orientation on G. As such we inherit a partition $X(G) = X^+(G) \coprod X^-(G) = \{x_1^+, \ldots, x_g^+\} \coprod \{x_1^-, \ldots, x_g^-\}$.

Lemma 7.0.10. *For every $x^\pm \in X(G)$, there exists a unique homotopy class $[C_{x^\pm}] \in \pi_1(G, v_0)$ such that every $C \in [C_{x^\pm}]$ satisfies $C_E(j^\pm) = x^\pm$ for some $j^\pm \in \{1^\pm, \ldots, |C|^\pm\}$ and for every $k^+ \in \{1^+, \ldots, |C|^+\}$ we have $C_E(k^+) \neq y^\pm$ for every $y^\pm \in X(G) \setminus \{x^\pm\}$.*

Put differently, the Lemma says that for every excess edge $x^\pm$ there is a unique homotopy class each of whose elements passes through the edge $x^\pm$ and no other excess edge.

Lemma 7.0.11. *If $X(G) = \{x_1^+, \ldots, x_g^+\} \coprod \{x_1^-, \ldots, x_g^-\}$, then $\pi_1(G, v_0)$ is freely generated by*

$$[C_{x_1^+}], \ldots, [C_{x_g^+}], [C_{x_1^-}], \ldots, [C_{x_g^-}]$$

Furthermore we have $[C_{x_j^\pm}]^{-1} = [C_{x_j^\mp}]$ for every $j \in \{1, \ldots, g\}$.

For proofs of the two Lemmas above, we refer the reader to [4] page 13.

Now, let F be a finite $\pi_1(G, v_0)$-set with $|F| = d$. Since $|F| = d$, then we have $\mathrm{Sym}(F) \cong S_d$, the symmetric group on d elements. Therefore we'll relabel, if necessary, so that $F = \{1, \ldots, d\}$. Let $\Phi : \pi_1(G, v_0) \to \mathrm{Sym}(F)$ be the permutation representation associated to the action of $\pi_1(G, v_0)$ on F. To simplify notation down the line, we'll allow $[C_{x_j^\pm}](\ell)$ to denote $\Phi([C_{x_j^\pm}])(\ell)$ for $\ell \in \{1, \ldots, d\}$. We'll write a given permutation in $\mathrm{Sym}(F)$ in disjoint cycle notation. For example if $F = \{1, 2, 3, 4\}$ after relabeling, then the map $\sigma \in \mathrm{Sym}(F)$ given by

$$1^\sigma = 2,\ 2^\sigma = 1,\ 3^\sigma = 4,\ 4^\sigma = 3$$

will be written as $\sigma = (12)(34)$.

From all this data, we'll show how to construct a finite, even cover (H, τ_H) of (G, τ_G). First we define

$$\tilde{H} = \coprod_{i=1}^{d} T_i$$

where T_i is isomorphic (as a directed graph) to T for every $i = 1, \ldots, d$. More precisely, for every $i = 1, \ldots, d$ there exists $\varphi_i \in \mathrm{Hom}(T_i, G)$ which is a bijection. We label the unique vertex in T_ℓ which corresponds to v_0 under φ_ℓ with $\hat{y}_\ell$, i.e., $\varphi_i(\hat{y}_\ell) = v_0$. Additionally, if $v_k \in V(G) \setminus \{v_0\}$ then we'll let $(\hat{v}_k)_\ell$ denote the unique vertex in T_ℓ with $\varphi_\ell((\hat{v}_k)_\ell) = v_k$. The notation for edges $\hat{e} \in E(\tilde{H})$ is similar: for such an $\hat{e}$ there exist $e \in E(T)$ and $\ell \in \{1, \ldots, d\}$ such that $\hat{e} = \varphi_\ell^{-1}(e)$. In this case we write $\hat{e} = \varphi_\ell^{-1}(e) := \hat{e}_\ell$.

We define the head and tail maps on $\tilde{H}$. If $\hat{e} \in E(\tilde{H})$ then $\hat{e} \in E(T_\ell)$ for some $\ell\{1, \ldots, d\}$. Then we define

$$(t_{\tilde{H}}, h_{\tilde{H}})(\hat{e}) = ((t_T(\varphi_\ell(\hat{e})))_\ell, (h_T(\varphi_\ell(\hat{e})))_\ell)$$

With the information given so far $\tilde{H}$ is a directed graph. We show that $\tilde{H}$ is an even graph by defining a fixed point free transposition $\tau_{\tilde{H}}$ on $\tilde{H}$: once again, if $\hat{e} \in E(\tilde{H})$ then $\hat{e} \in E(T_\ell)$ for some $\ell \in \{1, \ldots, d\}$ and we define

$$\tau_{\tilde{H}}(\hat{e}) = \varphi_\ell^{-1}(\tau_T(\varphi_\ell(\hat{e}))) \in E(T_\ell)$$

Lemma 7.0.12. *$\tau_{\tilde{H}}$ is a fixed point free transposition on the directed graph $\tilde{H}$ so that $\tilde{H}$ is an even graph.*

Proof. Let $\hat{e} \in E(\tilde{H})$ so that in particular $\hat{e} \in E(T_\ell)$ for some $\ell \in \{1, \ldots, d\}$. Then since

τ_T swaps heads and tails in T,

$$\begin{aligned} t_{\tilde{H}}(\tau_{\tilde{H}}(\hat{e})) &= ((t_T(\varphi_\ell(\varphi_\ell^{-1}(\tau_T(\varphi_\ell(\hat{e})))))))_\ell \\ &= ((t_T(\tau_T(\varphi_\ell(\hat{e})))))_\ell \\ &= (h_T(\varphi_\ell(\hat{e})))_\ell \\ &= (h_{\tilde{H}})(\hat{e}) \end{aligned}$$

A similar computation shows that $h_{\tilde{H}}(\tau_{\tilde{H}}(\hat{e})) = t_{\tilde{H}}(\hat{e})$.

Observe that also $\tau_{\tilde{H}}$ satisfies $\tau_{\tilde{H}}^2 = \mathrm{Id}_{\mathrm{E(\tilde{H})}}$:

$$\begin{aligned} \tau_{\tilde{H}}(\tau_{\tilde{H}}(\hat{e})) &= \tau_{\tilde{H}}(\varphi_\ell^{-1}(\tau_T(\varphi_\ell(\hat{e})))) \\ &= \varphi_\ell^{-1}(\tau_T(\varphi_\ell(\varphi_\ell^{-1}(\tau_T(\varphi_\ell(\hat{e})))))) \\ &= \varphi_\ell^{-1}(\varphi_\ell(\hat{e})) \\ &= \hat{e} \end{aligned}$$

Notice that we used the fact that $\tau_T^2 = \mathrm{Id_T}$.

We show, by contradiction, that $\tau_{\tilde{H}}$ fixes no edges of $\hat{H}$. If we had

$$\tau_{\tilde{H}}(\hat{e}) = \hat{e}$$

then by definition

$$\varphi_\ell^{-1}(\tau_T(\varphi_\ell(\hat{e}))) = \hat{e}$$

Applying φ_ℓ to each side yields

$$\tau_T(\varphi_\ell(\hat{e})) = \varphi_\ell(\hat{e})$$

a contradiction since τ_T fixes no edges.

Therefore $\tilde{H}$ is an even graph. □

Now we define a pair of maps $\phi_V : V(\tilde{H}) \to V(T)$ and $\phi_E : E(\tilde{H}) \to E(T)$ and prove that $\phi = (\phi_V, \phi_E) \in \mathrm{Hom}_\tau((\tilde{H}, \tau_{\tilde{H}}), (T, \tau_T))$. Let $\hat{v} \in V(\tilde{H})$. Then $\hat{v} \in V(T_\ell)$ for some $\ell \in \{1, \ldots, d\}$. By our convention then we write $\hat{v} = (\hat{v})_\ell$. We define

$$\phi_V(\hat{v}) = \phi((\hat{v})_\ell) = \varphi_\ell((\hat{v})_\ell) = v \in V(T)$$

If $\hat{e} \in E(\tilde{H})$, then $\hat{e} \in E(T_\ell)$ for some $\ell \in \{1, \ldots, d\}$ and we define

$$\phi_E(\hat{e}) = \varphi_\ell(\hat{e})$$

Proposition 7.0.1. $\phi \in \mathrm{Hom}_\tau((\tilde{H}, \tau_{\tilde{H}}), (T, \tau_T))$.

Proof. First we prove that $\phi \in \mathrm{Hom}(\tilde{H}, T)$. This entails showing that the diagram

$$\begin{tikzcd} E(\tilde{H}) \arrow[r, "\phi_E"] \arrow[d, "(t_{\tilde{H}}, h_{\tilde{H}})"'] & E(T) \arrow[d, "(t_T, h_T)"] \\ V(\tilde{H})^2 \arrow[r, "\phi_V^2"'] & V(T)^2 \end{tikzcd}$$

To that end, let $\hat{e} \in E(\tilde{H})$ hence in $E(T_\ell)$ for some $\ell \in \{1, \ldots, d\}$. On one hand

$$(t_T, h_T)(\phi_E(\hat{e})) = (t_T, h_T)(\phi_\ell(\hat{e}))$$

On the other,

$$\begin{aligned} \phi_V^2((t_{\tilde{H}}, h_{\tilde{H}})(\hat{e})) &= \phi_V^2((t_T(\varphi_\ell(\hat{e})))_\ell, (h_T(\varphi_\ell(\hat{e})))_\ell) \\ &= (t_T(\varphi_\ell(\hat{e})), h_T(\varphi_\ell(\hat{e}))) \\ &= (t_T, h_T)(\varphi_\ell(\hat{e})) \end{aligned}$$

So the diagram commutes and hence $\phi \in \mathrm{Hom}(\tilde{H}, T)$. To show that $\phi \in \mathrm{Hom}_\tau((\tilde{H}, \tilde{H}), (T, \tau_T))$ we must show that the following diagram commutes:

$$\begin{tikzcd} E(\tilde{H}) \arrow[r, "\phi_E"] \arrow[d, "\tau_{\tilde{H}}"'] & E(T) \arrow[d, "\tau_T"] \\ E(\tilde{H}) \arrow[r, "\phi_E"'] & E(T) \end{tikzcd}$$

So let $\hat{e} \in E(\tilde{H})$ hence $\hat{e} \in E(T_\ell)$ for some $\ell \in \{1, \ldots, d\}$. Then on one hand

$$\tau_T(\phi_E(\hat{e})) = \tau_T(\varphi_\ell(\hat{e}))$$

On the other,

$$\begin{aligned} \phi_E(\tau_{\tilde{H}}(\hat{e})) &= \phi_E(\varphi_\ell^{-1}(\tau_T(\varphi_\ell(\hat{e})))) \\ &= \varphi_\ell(\varphi_\ell^{-1}(\tau_T(\varphi_\ell(\hat{e})))) \\ &= \tau_T(\varphi_\ell(\hat{e}) \end{aligned}$$

Therefore the diagram commutes and $\phi \in \mathrm{Hom}_\tau((\tilde{H}, \tau_{\tilde{H}}), (T, \tau_T))$, as desired. $\square$

In fact $((\tilde{H}, \tau_{\tilde{H}}), \phi)$ is an even cover of (T, τ_T): ϕ is clearly surjective and for any $\hat{v} \in V(\tilde{H})$, ϕ_E is a bijection on $N_{\hat{v}}$ since φ_i is for every $i = 1, \ldots, d$. Notice that by definition $\phi(\hat{y}_i) = v_0$ for every $i = 1, \ldots, d$ so that $\phi^{-1}(v_0) = \{\hat{y}_1, \ldots, \hat{y}_d\}$ which is isomorphic (as a set) to $F = \{1, \ldots, d\}$.

Consider the example presented on the left side in Figure (7.1). The spanning tree in $G = \text{Cycle}_3$ is depicted in blue and $X(G) = \{x^+\} \coprod \{x^-\}$ and $F = \{1, 2, 3\}$ which is isomorphic (as a set) to $\{\hat{y}_1, \hat{y}_2, \hat{y}_3\}$.

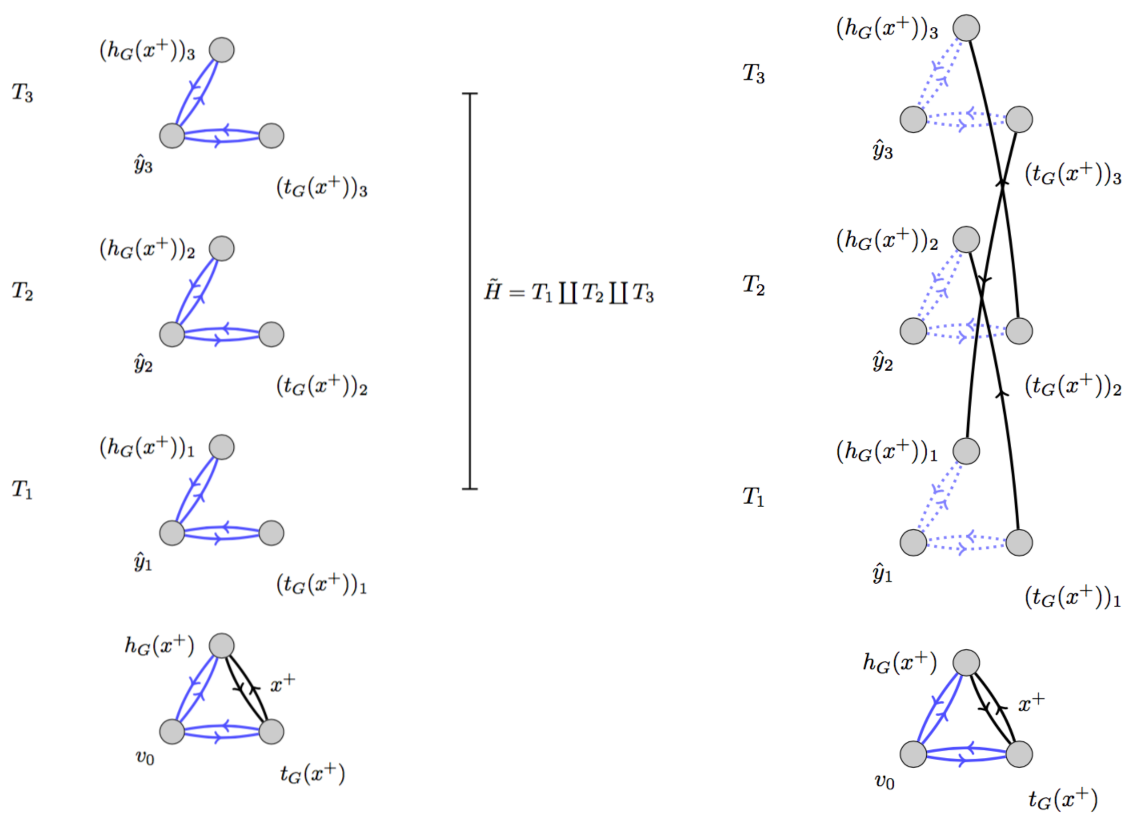


Figure 7.1: Constructing a cover. The spanning tree in $G = \text{Cycle}_3$ is depicted in blue. In the figure on the left we construct $\tilde{H}$. In the figure on the right, having constructed $\tilde{H}$ we form the edges above $x^+ \in X(G)$.

Now we complete $\tilde{H}$ to a finite even cover H of G by constructing the $2d$ edges $(\hat{x}_j^{\pm})_\ell$ and $(\hat{x}_j^{\mp})_\ell$ for $\ell = 1, \ldots, d$ which will correspond to $x_j^{\pm} \in X(G)$ for every $j = 1, \ldots, g$. Then for every $\ell \in \{1, \ldots, d\}$ we'll form $(\hat{x}_j^{\pm})_\ell$ with

$$t_H((\hat{x}_j^{\pm})_\ell) = (t_G(x_j^{\pm}))_\ell$$
$$h_H((\hat{x}_j^{\pm})_\ell) = (h_G(x_j^{\pm}))_{\ell^{[C_{x_j^{\pm}}]}}$$

Let's continue with the example presented in Figure (7.1). We had $F = \{1, 2, 3\}$ and $X =$

$\{x^+\}\coprod\{x^-\}$. Suppose now that $\Phi : \pi_1(G, v_0) \to \mathrm{Sym}(F) \equiv S_3$ satisfies $\Phi([C_{x^+}]) = (123)$. Then this informs how we form the edges above x^+. According to our definitions $(\hat{x})_1$, $(\hat{x})_2$, and $(\hat{x})_3$ should satisfy

$$\begin{aligned}
t_H((\hat{x}^+)_1) &= (t_G(x^+))_1 \\
h_H((\hat{x}^+)_1) &= (h_G(x^+))_{1^{[C_{x^+}]}} \\
&= (h_G(x^+))_2 \\
t_H((\hat{x}^+)_2) &= (t_G(x^+))_2 \\
h_H((\hat{x}^+)_2) &= (h_G(x^+))_{2^{[C_{x^+}]}} \\
&= (h_G(x^+))_3 \\
t_H((\hat{x}^+)_3) &= (t_G(x^+))_3 \\
h_H((\hat{x}^+)_3) &= (h_G(x^+))_{3^{[C_{x^+}]}} \\
&= (h_G(x^+))_1
\end{aligned}$$

To finish the example, we construct the edges above $x^- \in X(G)$: the resulting directed graph will be called H. Recalling that $\Phi : \pi_1(G, v_0) \to \mathrm{Sym}(F)$ is a homomorphism and that $[C_{x^+}]^{-1} = [C_{x^-}] \in \pi_1(G, v_0)$ we have

$$\Phi([C_{x^-}]) = \Phi([C_{x^+}]^{-1}) = (\Phi([C_{x^+}]))^{-1} = (123)^{-1} = (132) \in \mathrm{Sym}(F)$$

So $(\hat{x}^-)_1$, $(\hat{x}^-)_2$, and $(\hat{x}^-)_3$ must satisfy

$$\begin{aligned}
t_H((\hat{x}^-)_1) &= (t_G(x^-))_1 \\
h_H((\hat{x}^-)_1) &= (h_G(x^-))_{1^{[C_{x^-}]}} \\
&= (h_G(x^-))_3 \\
t_H((\hat{x}^-)_2) &= (t_G(x^-))_2 \\
h_H((\hat{x}^-)_2) &= (h_G(x^-))_{2^{[C_{x^-}]}} \\
&= (h_G(x^-))_1 \\
t_H((\hat{x}^-)_3) &= (t_G(x^-))_3 \\
h_H((\hat{x}^-)_3) &= (h_G(x^-))_{3^{[C_{x^-}]}} \\
&= (h_G(x^-))_2
\end{aligned}$$

Then Figure (7.2) shows the finished product. Of course we need to prove that the directed

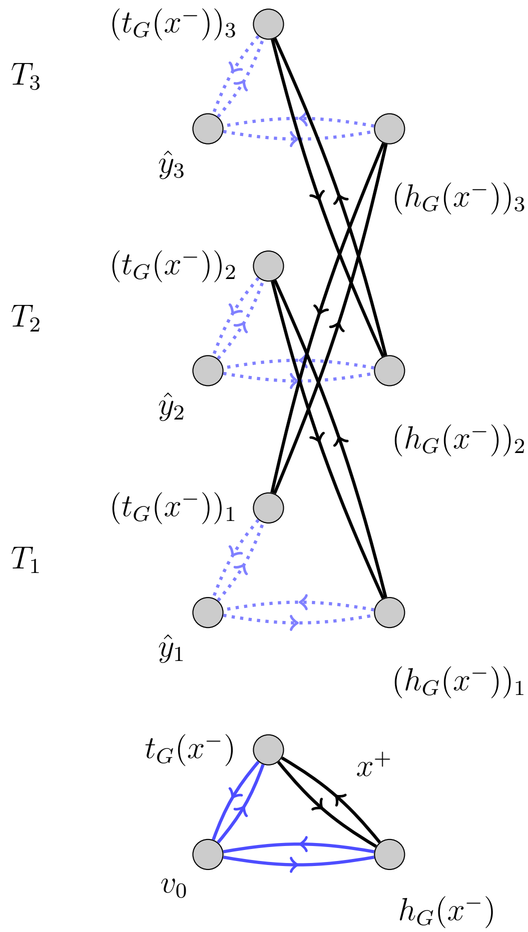


Figure 7.2: A finished cover H of G. The dotted blue edges in T_1, T_2, and T_3 are so drawn purely for aesthetics.

graph we've constructed is a finite, even cover of G. First we define $\tau_H(\hat{e}) = \tau_{\tilde{H}}(\hat{e})$ if $\hat{e} \in E(\tilde{H})$ and $\tau_H((\hat{x}_j^{\pm})_\ell) = (\hat{x}_j^{\mp})_{\ell^{[C_{x_j^{\pm}}]}}$ for every $j = 1, \ldots, g$ and every $\ell = 1, \ldots, d$.

Lemma 7.0.13. *τ_H is a fixed point free transposition on H so that H is a finite even graph.*

Proof. We've already shown that τ_H is a fixed point free transposition on $\tilde{H}$. We must therefore prove that τ_H is a fixed point free transposition on the edges above those edges in $X(G)$. First we observe that τ_H has order 2:

$$\begin{aligned}\tau_H(\tau_H((\hat{x}_j^{\pm})_\ell)) &= \tau_H((\hat{x}_j^{\mp})_{\ell^{[C_{x_j^{\pm}}]}}) \\ &= ((\hat{x}_j^{\pm})_{\ell^{[C_{x_j^{\pm}}][C_{x_j^{\mp}}]}}) \\ &= (\hat{x}_j^{\pm})_\ell\end{aligned}$$

Now we show that τ_H swaps heads and tails:

$$\begin{aligned}t_H(\tau_H((\hat{x}_j^{\pm})_\ell)) &= t_H((\hat{x}_j^{\mp})_{\ell^{[C_{x_j^{\pm}}]}}) \\ &= (t_G(x_j^{\mp}))_{\ell^{[C_{x_j^{\pm}}]}} \\ &= (h_G(x_j^{\pm}))_{\ell^{[C_{x_j^{\pm}}]}} \\ &= h_H((\hat{x}_j^{\pm})_\ell)\end{aligned}$$

A symmetric calculation shows that $h_H(\tau_H((\hat{x}_j^{\pm})_\ell) = t_H((\hat{x}_j^{\pm})_\ell)$.

To show that τ_H is fixed point free, we argue by contradiction. Suppose that

$$\tau_H((\hat{x}_j^{\pm})_\ell) = (\hat{x}_j^{\pm})_\ell$$

for some $j \in \{1, \ldots, g\}$ and some $\ell \in \{1, \ldots, d\}$. Then by definition

$$(\hat{x}_j^{\mp})_{\ell^{[C_{x_j^{\pm}}]}} = \tau_H((\hat{x}_j^{\pm})_\ell) = (\hat{x}_j^{\pm})_\ell$$

We apply h_H to each side of the above equation to obtain

$$\begin{aligned}(h_G(x_j^{\pm}))_{\ell^{[C_{x_j^{\pm}}]}} &= h_H((\hat{x}_j^{\pm})_\ell) \\ &= h_H((\hat{x}_j^{\mp})_{\ell^{[C_{x_j^{\pm}}]}}) \\ &= (h_G(x_j^{\mp}))_{\ell^{[C_{x_j^{\pm}}][C_{x_j^{\mp}}]}} \\ &= (h_G(x_j^{\mp}))_\ell\end{aligned}$$

By our construction, the only way for two vertices $\hat{v}_\ell$ and $\hat{w}_k$ in H to be equal is if $\ell = k$ and $v = w \in V(G)$. Therefore the above would imply that $\ell^{[C_{x_j^{\pm}}]} = \ell$ and

$$h_G(x_j^{\pm}) = h_G(x_j^{\mp})$$

This is a contradiction since $\tau_G(x_j^{\pm}) = x_j^{\mp}$ and τ_G is supposed to swap heads and tails. Therefore τ_H is a fixed point free transposition on H and H is therefore a finite, even graph. $\square$

Finally we start with ϕ which was defined on $\tilde{H}$ and extend to H by defining $\phi((\hat{x}_j^{\pm})_\ell) = x_j^{\pm}$ for every $\ell = 1, \ldots, d$ and every $j = 1, \ldots, g$.

Theorem 7.0.1. *$((H, \tau_H), \phi)$ as defined is a finite, even cover of (G, τ_G).*

Proof. We already proved that $((\tilde{H}, \tau_{\tilde{H}}), \phi|_{\tilde{H}})$ is a finite even cover of (T, τ_T). As such we need to argue that the desired properties hold for the edges $(\hat{x}_j^{\pm})_\ell$ for every $j = 1, \ldots, g$ and $\ell = 1, \ldots, d$. First we prove that ϕ is a directed graph morphism which entails showing that the following diagram commutes:

$$\begin{tikzcd} E(H) \arrow[r, "\phi_E"] \arrow[d, "(t_H, h_H)"'] & E(G) \arrow[d, "(t_G, h_G)"] \\ V(H)^2 \arrow[r, "\phi_V^2"'] & V(G)^2 \end{tikzcd}$$

So let $\ell \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, g\}$ be arbitrary. Then

$$(t_G, h_G)(\phi_E((\hat{x}_j^{\pm})_\ell)) = (t_G, h_G)(x_j^{\pm})$$

On the other hand

$$\begin{aligned} \phi_V^2((t_H, h_H)((\hat{x}_j^{\pm})_\ell)) &= \phi_V^2(((t_G(x_j^{\pm}))_\ell), (h_G(x_j^{\pm}))_{\ell^{[C_{x_j^{\pm}}]}})) \\ &= (t_G(x_j^{\pm}), h_G(x_j^{\pm})) \\ &= (t_G, h_G)(x_j^{\pm}) \end{aligned}$$

This proves that the diagram commutes. To prove that $\phi \in \mathrm{Hom}_\tau((H, \tau_H), (G, \tau_G))$ we need to show that the diagram

$$\begin{tikzcd} E(H) \arrow[r, "\phi_E"] \arrow[d, "\tau_H"'] & E(G) \arrow[d, "\tau_G"] \\ E(H) \arrow[r, "\phi_E"'] & E(G) \end{tikzcd}$$

commutes. On one hand,

$$\tau_G(\phi_E((\hat{x}_j^{\pm})_\ell)) = \tau_G(x_j^{\pm}) = x_j^{\mp}$$

On the other,

$$\begin{aligned} \phi_E(\tau_H((\hat{x}_j^{\pm})_\ell)) &= \phi_E((\hat{x}_j^{\mp})_{\ell^{[C_{x_j^{\pm}}]}}) \\ &= x_j^{\mp} \end{aligned}$$

Therefore the second diagram commutes and we conclude that $\phi \in \mathrm{Hom}_\tau((H,\tau_H),(G,\tau_G))$. To finally conclude that $((H,\tau_H),\phi)$ is a finite even cover of (G,τ_G), we need to show that for every $\hat{v} \in V(H)$, $\phi_E : N_{\hat{v}} \to N_{\phi_V(\hat{v})}$ is a bijection. We recall that ϕ had this property on $\tilde{H}$. In extending $\tilde{H}$ to H however, we introduced new edges. As such we will be concerned with proving the statement for those vertices in H which are adjacent to the edges $(\hat{x}_j^{\pm})_\ell$. So let $\hat{v} \in V(H)$ satisfy $\phi(\hat{v}) = t_G(x_j^{\pm})$ or $\phi(\hat{v}) = h_G(x_j^{\pm})$ for some $j \in \{1,\ldots,g\}$. Then we have $x_j^{\pm}, x_j\mp \in N_{\phi(\hat{v})}$. We prove that there are unique edges in $N_{\hat{v}}$ which are mapped by ϕ to $x_j^{\pm}$ and $x_j^{\mp}$. Since $\hat{v} \in V(H)$ then $\hat{v} \in V(T_\ell)$ for some $\ell \in \{1,\ldots,d\}$. We assume that $\phi(\hat{v}) = t_G(x_j^+)$ and $\phi(\hat{v}) = h_G(x_j^-)$ (the case where $\phi(\hat{v}) = h_G(x_j^+)$ and $\phi(\hat{v}) = t_G(x_j^-)$ is proved in the same way). Then by definition

$$\begin{aligned}\hat{v} &= (t_G(x_j^+))_\ell \\ &= t_H((\hat{x}_j^+)_\ell) \\ \hat{v} &= (h_G(x_j^-))_\ell \\ &= h_H((\hat{x}_j^-)_{\ell^{[C_{x_j^+}]}})\end{aligned}$$

Then we have $(\hat{x}_j^+)_\ell \in N_{\hat{v}}$ and $(\hat{x}_j^-)_{\ell^{[C_{x_j^+}]}} \in N_{\hat{v}}$ and by definition

$$\begin{aligned}\phi((\hat{x}_j^+)_\ell) &= x_j^+ \\ \phi((\hat{x}_j^-)_{\ell^{[C_{x_j^+}]}}) &= x_j^-\end{aligned}$$

So we simply need to show that $(\hat{x}_j^+)_\ell \neq (\hat{x}_j^-)_{\ell^{[C_{x_j^+}]}}$. But this is immediate since

$$\tau_H((\hat{x}_j^+)_\ell) = (\hat{x}_j^-)_{\ell^{[C_{x_j^+}]}}$$

and we saw that τ_H doesn't fix any edges. Therefore we've found unique edges in $N_{\hat{v}}$ which are mapped to x_j^+ and x_j^- respectively and therefore $\phi : N_{\hat{v}} \to N_{\phi_V(\hat{v})}$ is a bijection. All of this together shows that $((H,\tau_H),\phi)$ is a finite even cover of (G,τ_G), as desired. □

Given F, a finite $\pi_1(G,v_0)$-set, we therefore define $\mathcal{G}(F) = ((H,\tau_H),\phi)$ as above. The Theorem tells us that $\mathcal{G}(F)$ is a finite even cover of (G,τ_G). The reverse functor $\mathcal{G}$ is therefore defined on objects.

While the analogy to the development in [2] on this point of constructing a cover from a fundamental group action (P. 69) is less clear than previous analogies, parallels may still be drawn. Hatcher begins by constructing the "universal cover" of the base space X as the collection of all homotopy classes of lifts of cycles in the base space. The universal cover of a space X is a covering space $\tilde{X}_0$ of X with the property that for every covering space $\tilde{X}$ of X, $\tilde{X}_0$ is a covering space of $\tilde{X}$. $\tilde{X}_0$ is extremal amongst all covering spaces of X in this sense. In our context of graphs, we also have such an extremal object $\tilde{T}_0$ called the "universal covering tree of G" . It is an infinite tree (unless G is a finite tree) which is a cover of G and which is a cover of every other cover of G. While $\tilde{T}_0$ was not obviously useful in our development it certainly represents a clear analogy of $\tilde{X}_0$.

Hatcher further uses the fundamental group action to determine how to connect "sheets" of the covering space under construction. This is precisely how we proceeded: we used the $\pi_1(G, v_0)$-action to inform how we connected the tree components in the spanning forest for our cover.

Now we need to say how $\mathcal{G}$ acts on morphisms. Toward this goal, let F_1 and F_2 be two finite $\pi_1(G, v_0)$-sets with respective permutation representations $\Phi_1 : \pi_1(G, v_0) \to \mathrm{Sym}(F_1)$ and $\Phi_2 : \pi_1(G, v_0) \to \mathrm{Sym}(F_2)$. Suppose that we have a morphism f from F_1 to F_2 which we recall is a set map $f : F_1 \to F_2$ with the additional property that

$$f(\ell^{[C]}) = (f(\ell))^{[C]}$$

for every $[C] \in \pi_1(G, v_0)$ and every $\ell \in F_1$. Let $((H_1, \tau_{H_1}), \phi_1) = \mathcal{G}(F_1)$ and $((H_2, \tau_{H_2}), \phi_2) = \mathcal{G}(F_2)$. We need to define a morphism of finite even covers $\mathcal{G}(f) : ((H_1, \tau_{H_1}), \phi_1) \to ((H_2, \tau_{H_2}), \phi_2)$. Recall that in order for $\mathcal{G}(f)$ to be such a morphism we must have

(1) $\mathcal{G}(f) \in \mathrm{Hom}_\tau((H_1, \tau_{H_1}), (H_2, \tau_{H_2}))$

(2) $\phi_1 = \phi_2 \circ \mathcal{G}(f)$

After relabeling, suppose that $F_1 = \{1, \ldots, d_1\}$ and $F_2 = \{1, \ldots, d_2\}$. According to our construction of the objects $\mathcal{G}(F_1) = ((H_1, \tau_{H_1}), \phi_1)$ and $\mathcal{G}(F_2) = ((H_2, \tau_{H_2}), \phi_2)$ we started by defining

$$\tilde{H}_1 = \coprod_{i=1}^{d_1} T_i^{(1)} \subseteq H_1, \ \tilde{H}_2 = \coprod_{k=1}^{d_2} T_k^{(2)} \subseteq H_2$$

with isomorphisms $\varphi_i^{(1)} : T_i^{(1)} \cong T$ and $\varphi_k^{(2)} : T_k^{(2)} \cong T$ for every $i = 1, \ldots, d_1$ and every $k = 1, \ldots, d_2$. We completed $\tilde{H}_1$ to H_1 by forming the edges $((\hat{x}_j^{\pm})_i^{(1)}$ for every $j = 1, \ldots, g$ and every $i = 1, \ldots d_1$ and completed $\tilde{H}_2$ to H_2 by forming the edges $(\hat{x}_j^{\pm})_k^{(2)}$ for every $j = 1, \ldots, g$ and every $k = 1, \ldots, d_2$. We also recall the notation valid for $n = 1, 2$: if

$\hat{e}^{(n)} \in E(\tilde{H}_n) \subseteq E(H_n)$ then there exists some $\ell \in \{1, \ldots, d_n\}$ for which $\hat{e}^{(n)} \in E(T_\ell^{(n)})$ and as such $\hat{e}^{(n)} = \hat{e}_\ell^{(n)} = (\varphi_\ell^{(n)})^{-1}(e)$ for some $e \in E(T)$. Then we define

$$\mathcal{G}(f)(\hat{e}_\ell^{(1)}) = \mathcal{G}(f)((\varphi_\ell^{(1)})^{-1}(e)) = (\varphi_{f(\ell)}^{(2)})^{-1}(e) = \hat{e}_{f(\ell)}^{(2)}$$

Recall the notation for vertices valid for $n = 1, 2$: if $\hat{v}^{(n)} \in V(H_n)$ then there exists some $\ell \in \{1, \ldots, d_n\}$ for which $\hat{v}^{(n)} \in V(T_\ell^{(n)})$ and so we write $\hat{v}^{(n)} = \hat{v}_\ell^{(n)} = (\varphi_\ell^{(n)})^{-1}(v)$ for $v \in V(G)$. Then we define

$$\mathcal{G}(f)(\hat{v}_\ell^{(1)}) = \mathcal{G}(f)((\varphi_\ell^{(1)})^{-1}(v)) = (\varphi_{f(\ell)}^{(2)})^{-1}(v) = \hat{v}_{f(\ell)}^{(2)}$$

Finally we define

$$\mathcal{G}(f)((\hat{x}_j^{\pm})_\ell^{(1)}) = (\hat{x}_j^{\pm})_{f(\ell)}^{(2)}$$

for $j = 1, \ldots, g$ and $\ell = 1, \ldots, d_1$.

Theorem 7.0.2. *$\mathcal{G}(f)$ as defined above is an even graph morphism from (H_1, τ_{H_1}) to (H_2, τ_{H_2}). In other words $\mathcal{G}(f) \in \mathrm{Hom}_\tau((H_1, \tau_{H_1}), (H_2, \tau_{H_2}))$, and $\phi_2 = \phi_1 \circ \mathcal{G}(f)$.*

Proof. First we prove that $\mathcal{G}(f) \in \mathrm{Hom}(H_1, H_2)$ which entails showing that the diagram

$$\begin{tikzcd}
E(H_1) \arrow[r, "\mathcal{G}(f)_E"] \arrow[d, "(t_{H_1}, h_{H_1})"'] & E(H_2) \arrow[d, "(t_{H_2}, h_{H_2})"] \\
V(H_1)^2 \arrow[r, "\mathcal{G}(f)_V^2"'] & V(H_2)^2
\end{tikzcd}$$

commutes. So let $\hat{e}^{(1)} \in E(H_1)$. We consider two cases: suppose first that $\hat{e}^{(1)} \in E(\tilde{H}_1)$. Then $\hat{e}^{(1)} = \hat{e}_\ell^{(1)} = (\varphi_\ell^{(1)})^{-1}(e)$ for some $\ell \in \{1, \ldots, d_1\}$ and some $e \in E(T)$. Then according to our definition

$$\mathcal{G}(f)(\hat{e}^{(1)}) = \hat{e}_{f(\ell)}^{(2)} = (\varphi_{f(\ell)}^{(2)})^{-1}(e)$$

and by definition of the head and tail maps on $\tilde{H}$

$$(t_{H_2}, h_{H_2})(\mathcal{G}(f)(\hat{e}^{(1)})) = \left((t_T(e))_{f(\ell)}^{(2)}, (h_T(e))_{f(\ell)}^{(2)}\right)$$

On the other hand

$$\begin{aligned}
(\mathcal{G}(f)_V^2)(((t_{H_1}, h_{H_1})(\hat{e}^{(1)}))) &= (\mathcal{G}(f)_V^2)\left((t_T(e))_\ell^{(1)}, (h_T(e))_\ell^{(1)}\right) \\
&= \left((t_T(e))_{f(\ell)}^{(2)}, (h_T(e))_{f(\ell)}^{(2)}\right)
\end{aligned}$$

Now we consider the case where $\hat{e}^{(1)} = (\hat{x}_j^{\pm})_\ell^{(1)}$ for some $j \in \{1, \ldots, g\}$ and some $\ell \in \{1, \ldots, d_1\}$. Then

$$\begin{aligned}(t_{H_2}, h_{H_2})(\mathcal{G}(f)_E((\hat{x}_j^{\pm})_\ell^{(1)})) &= (t_{H_2}, h_{H_2})((\hat{x}_j^{\pm})_{f(\ell)}^{(2)}) \\ &= \left((t_G(x_j^{\pm}))_{f(\ell)}^{(2)}, (h_G(x_j^{\pm}))^{(2)}_{f(\ell)^{[C_{x_j^{\pm}}]}} \right)\end{aligned}$$

But since f is a morphism of $\pi_1(G, v_0)$-sets, $\Phi_2([C_{x_j^{\pm}}])(f(\ell)) = f(\Phi_1([C_{x_j^{\pm}}])(\ell))$, hence

$$(t_{H_2}, h_{H_2})(\mathcal{G}(f)_E((\hat{x}_j^{\pm})_\ell^{(1)})) = \left((t_G(x_j^{\pm}))_{f(\ell)}^{(2)}, (h_G(x_j^{\pm}))^{(2)}_{f(\ell^{[C_{x_j^{\pm}}]})} \right)$$

On the other hand

$$\begin{aligned}\mathcal{G}(f)_V^2 \left((t_{H_1}, h_{H_1})(\hat{x}_j^{\pm})_\ell^{(1)} \right) &= \mathcal{G}(f)_V^2 \left((t_G(x_j^{\pm}))_\ell^{(1)}, (h_G(x_j^{\pm}))^{(1)}_{\ell^{[C_{x_j^{\pm}}]}} \right) \\ &= \left((t_G(x_j^{\pm}))_{f(\ell)}^{(2)}, (h_G(x_j^{\pm}))^{(2)}_{f(\ell^{[C_{x_j^{\pm}}]})} \right)\end{aligned}$$

Miraculously the two expressions match. Therefore $\mathcal{G}(f)$ is a directed graph morphism: $\mathcal{G}(f) \in \mathrm{Hom}(H_1, H_2)$. To show that in fact $\mathcal{G}(f)$ is a morphism of even graphs, we must show that the following diagram commutes:

$$\begin{tikzcd} E(H_1) \arrow[r, "\mathcal{G}(f)_E"] \arrow[d, "\tau_{H_1}"'] & E(H_2) \arrow[d, "\tau_{H_2}"] \\ E(H_1) \arrow[r, "\mathcal{G}(f)_E"'] & E(H_2) \end{tikzcd}$$

Again we split the proof into two cases: first suppose that $\hat{e}^{(1)} \in E(\tilde{H}_1)$. As before, there exists some $\ell \in \{1, \ldots, d_1\}$ so that

$$\hat{e}^{(1)} = \hat{e}_\ell^{(1)} = (\varphi_\ell^{(1)})^{-1}(e)$$

for some $e \in E(T)$. Then in this case,

$$\begin{aligned}\tau_{H_2}(\mathcal{G}(f)_E(\hat{e}^{(1)})) &= \tau_{H_2}(\hat{e}_{f(\ell)}^{(2)}) \\ &= \tau_{H_2}((\varphi_{f(\ell)}^{(2)})^{-1}(e)) \\ &= ((\varphi_{f(\ell)}^{(2)})^{-1} \circ \tau_T \circ \varphi_{f(\ell)}^{(2)})((\varphi_{f(\ell)}^{(2)})^{-1}(e)) \\ &= ((\varphi_{f(\ell)}^{(2)})^{-1} \circ \tau_T)(e)\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\mathcal{G}(f)_E \circ \tau_{H_1})(\hat{e}^{(1)}) &= \mathcal{G}(f)_E \left(\tau_{H_1}((\varphi_\ell^{(1)})^{-1}(e))\right) \\
&= \mathcal{G}(f)_E \left((\varphi_\ell^{(1)})^{-1} \circ \tau_T \circ \varphi_\ell^{(1)})((\varphi_\ell^{(1)})^{-1}(e))\right) \\
&= \mathcal{G}(f)_E \left((\varphi_\ell^{(1)})^{-1}(\tau_T(e))\right) \\
&= (\varphi_{f(\ell)}^{(2)})^{-1}(\tau_T(e))
\end{aligned}$$

Now we instead assume that $\hat{e}^{(1)} = (\hat{x}_j^{\pm})_\ell^{(1)}$ for some $j \in \{1, \ldots, g\}$ and some $\ell \in \{1, \ldots, d_1\}$. In the computation that follows we make use once again of the fact that $f(\ell)^{[C_{x_j^{\pm}}]} = f(\ell^{[C_{x_j^{\pm}}]})$r.

$$\begin{aligned}
\tau_{H_2}(\mathcal{G}(f)_E(\hat{x}_j^{\pm})_\ell^{(1)}) &= \tau_{H_2}((\hat{x}_j^{\pm})_{f(\ell)}^{(2)}) \\
&= (\hat{x}_j^{\mp})^{(2)}_{f(\ell)^{[C_{x_j^{\pm}}]}} \\
&= (\hat{x}_j^{\mp})^{(2)}_{f(\ell^{[C_{x_j^{\pm}}]})}
\end{aligned}$$

On the other hand

$$\begin{aligned}
(\mathcal{G}(f)_E \circ \tau_{H_1})((\hat{x}_j^{\pm})_\ell^{(1)}) &= \mathcal{G}(f)_E(\hat{x}_j^{\mp})^{(1)}_{\ell^{[C_{x_j^{\pm}}]}} \\
&= (\hat{x}_j^{\mp})^{(2)}_{f(\ell^{[C_{x_j^{\pm}}]})}
\end{aligned}$$

Therefore in both of the possible cases, the second diagram commutes and we conclude that $\mathcal{G}(f) \in \mathrm{Hom}_\tau((H_1, \tau_{H_1}), (H_2, \tau_{H_2}))$, as desired.

What remains to be shown is that $\phi_2 = \phi_2 \circ \mathcal{G}(f)$. We need to verify that $(\phi_1)_E = (\phi_2)_E \circ (\mathcal{G}(f))_E$ and that $(\phi_1)_V = (\phi_2)_V \circ (\mathcal{G}(f))_V$. For the first of these statements our argument splits into two cases. For the first case suppose that $\hat{e}^{(1)} \in E(\tilde{H}_1)$. Then there is some $\ell \in \{1, \ldots, d_1\}$ and some $e \in E(T)$ for which $\hat{e}^{(1)} = (\varphi_\ell^{(1)})^{-1}(e)$. Then by definition $\phi_1(\hat{e}^{(1)}) = e$. On the other hand

$$\begin{aligned}
\phi_2(\mathcal{G}(f)(\hat{e}^{(1)})) &= \phi_2((\varphi_{f(\ell)}^{(2)})^{-1}(e)) \\
&= e
\end{aligned}$$

If instead we assume that $\hat{e}^{(1)} = (\hat{x}_j^{\pm})_\ell^{(1)}$ for some $\ell \in \{1, \ldots, d_1\}$ and some $j \in \{1, \ldots, g\}$, then

$$\phi_1((\hat{x}_j^{\pm})_\ell^{(1)}) = x_j^{\pm}$$

On the other hand

$$\begin{aligned}\phi_2(\mathcal{G}(f)(\hat{x}_j^{\pm})_\ell^{(1)}) &= \phi_2((\hat{x}_j^{\pm})_{f(\ell)}^{(2)}) \\ &= x_j^{\pm}\end{aligned}$$

This proves that $(\phi_1)_E = (\phi_2)_E \circ (\mathcal{G}(f))_E$. For the second statement, let $\hat{v}^{(1)} \in V(H_1)$. Then there exists some $\ell \in \{1, \ldots, d_1\}$ and some $v \in V(G)$ such that $\hat{v}^{(1)} = (\varphi_\ell^{(1)})^{-1}(v)$. Then we have

$$\phi_1(\hat{v}^{(1)}) = \phi_1((\varphi_\ell^{(1)})^{-1}(v)) = v$$

On the other hand we have

$$\begin{aligned}\phi_2(\mathcal{G}(f)\hat{v}^{(1)}) &= \phi_2(\mathcal{G}(f)((\varphi_\ell^{(1)})^{-1}(v))) \\ &= \phi_2((\varphi f(\ell)^{(2)})^{-1}(v)) \\ &= v\end{aligned}$$

This proves that $(\phi_1)_V = (\phi_2)_V \circ (\mathcal{G}(f))_V$. Therefore we have $\phi_1 = \phi_2 \circ \mathcal{G}(f)$, as desired. $\square$

We've shown that whenever $f : F_1 \to F_2$ is a morphism of finite $\pi_1(G, v_0)$-sets $\mathcal{G}(f)$ as we've defined it is a morphism of finite even covers $\mathcal{G}(f) : \mathcal{G}(F_1) \to \mathcal{G}(F_2)$. To finally conclude that $\mathcal{G}$ is a (covariant) functor from $\boldsymbol{\pi_1}(\mathbf{G}) - \mathbf{Set}$ to $\mathbf{Cov}(G)$, we prove two lemmas.

Lemma 7.0.14. *Let F_1, F_2, and F_3 be three finite $\pi_1(G, v_0)$-sets and let $\mathcal{G}(F_i) = ((H_i, \tau_{H_i}), \phi_i)$ for $i = 1, 2, 3$. If we have morphisms*

$$F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} F_3$$

then $\mathcal{G}(f_2 \circ f_1) = \mathcal{G}(f_2) \circ \mathcal{G}(f_1)$ so that $\mathcal{G}$ respects composition.

Proof. Suppose that after relabeling, $F_i = \{1, \ldots, d_i\}$ for $i = 1, 2, 3$. Let $\hat{v}^{(1)} \in V(H_1)$. Then there exists some $\ell \in \{1, \ldots, d_1\}$ and some $v \in V(G)$ for which $\hat{v}^{(1)} = (\varphi_\ell^{(1)})^{-1}(v)$. Then we have

$$\mathcal{G}(f_2 \circ f_1)(\hat{v}^{(1)}) = (\varphi^{(3)}_{(f_2 \circ f_1)(\ell)})^{-1}(v)$$

and

$$\begin{aligned}(\mathcal{G}(f_2) \circ \mathcal{G}(f_1))(\hat{v}^{(1)}) &= \mathcal{G}(f_2)((\varphi^{(2)}_{f_1(\ell)})^{-1}(v)) \\ &= (\varphi^{(3)}_{f_2(f_1(\ell))})^{-1}(v)\end{aligned}$$

Now suppose that $\hat{e}^{(1)} \in E(\tilde{H}_1) \subseteq E(H_1)$. Then there exist $\ell \in \{1, \ldots, d_1\}$ and $e \in E(T)$ such that $\hat{e}_1 = (\varphi_\ell^{(1)})^{-1}(e)$. Then we have

$$\mathcal{G}(f_2 \circ f_1)(\hat{e}_1) = (\varphi^{(3)}_{(f_2 \circ f_1)(\ell)})^{-1}(e)$$

and

$$\begin{aligned}(\mathcal{G}(f_2) \circ \mathcal{G}(f_1))(\hat{e}^{(1)}) &= \mathcal{G}(f_2)((\varphi^{(2)}_{f_1(\ell)})^{-1}(e)) \\ &= (\varphi^{(3)}_{f_2(f_1(\ell))})^{-1}(e)\end{aligned}$$

Finally we suppose that $\hat{e}^{(1)} = (\hat{x}_j^{\pm})_\ell^{(1)}$ for some $j \in \{1, \ldots, g\}$ and some $\ell \in \{1, \ldots, d_1\}$. Then we have

$$\mathcal{G}(f_2 \circ f_1)((\hat{x}_j^{\pm})_\ell^{(1)}) = (\hat{x}_j^{\pm})_{(f_2 \circ f_1)(\ell)}$$

and

$$\begin{aligned}(\mathcal{G}(f_2) \circ \mathcal{G}(f_1))((\hat{x}_j^{\pm})_\ell^{(1)}) &= \mathcal{G}(f_2)((\hat{x}_j^{\pm})_{f_1(\ell)}) \\ &= (\hat{x}_j^{\pm})_{f_2(f_1(\ell))}\end{aligned}$$

Therefore we see that $\mathcal{G}(f_2 \circ f_1) = \mathcal{G}(f_2) \circ \mathcal{G}(f_1)$, as desired. □

Lemma 7.0.15. *Let F be a finite $\pi_1(G, v_0)$-set and let $\mathrm{Id_F} : F \to F$ denote the identity morphism. Suppose that $\mathcal{G}(F) = ((H, \tau_H), \phi)$. Then $\mathcal{G}(\mathrm{Id_F}) = \mathrm{Id}_{\mathcal{G}(\mathrm{F})} = \mathrm{Id_H} \in \mathrm{Hom}_\tau((H, \tau_H), (H, \tau_H))$.*

Proof. Suppose that after relabeling $F = \{1, \ldots, d\}$. Then by definition $\mathrm{Id_F}(\ell) = \ell$ for every $\ell = 1, \ldots, d$. On the other hand

$$\mathrm{Id}_{\mathcal{G}(\mathrm{F})} = \mathrm{Id_H} : \begin{cases} \hat{v} \mapsto \hat{v} \\ \hat{e} \mapsto \hat{e} \end{cases}$$

for every $\hat{v} \in V(H)$ and every $\hat{e} \in E(H)$. So let $\hat{v} \in V(H)$. Then there exist some $\ell \in \{1, \ldots, d\}$ and $v \in V(G)$ such that $\hat{v} = \varphi_\ell^{-1}(v)$. Then we have

$$\begin{aligned}\mathcal{G}(\mathrm{Id_F})(\hat{v}) &= \mathcal{G}(f)(\varphi_\ell^{-1}(v)) \\ &= \varphi^{-1}_{\mathrm{Id_F}(\ell)}(v) \\ &= \varphi_\ell^{-1}(v) \\ &= \hat{v}\end{aligned}$$

To prove the statement for the edge maps, we consider two cases separately. First suppose that $\hat{e} \in E(\tilde{H}) \subseteq E(H)$. Then $\hat{e} = \varphi_\ell^{-1}(e)$ for some $\ell \in \{1, \dots, d\}$ and some $e \in E(T)$. Then

$$\begin{aligned}
\mathcal{G}(\mathrm{Id}_\mathrm{F})(\hat{e}) &= \varphi^{-1}_{\mathrm{Id}_\mathrm{F}(\ell)}(e) \\
&= \varphi_\ell^{-1}(e) \\
&= \hat{e}
\end{aligned}$$

If instead we assumed that $\hat{e} = (\hat{x}_j^\pm)_\ell$ for some $j \in \{1, \dots, g\}$ and some $\ell \in \{1, \dots, d\}$ then we would have

$$\begin{aligned}
\mathcal{G}(\mathrm{Id}_\mathrm{F})(\hat{x}_j^\pm)_\ell &= (\hat{x}_j^\pm)_{\mathrm{Id}_\mathrm{F}(\ell)} \\
&= (\hat{x}_j^\pm)_\ell
\end{aligned}$$

We've therefore shown that $\mathcal{G}(\mathrm{Id}_\mathrm{F}) = \mathrm{Id}_\mathrm{H} = \mathrm{Id}_{\mathcal{G}(\mathrm{F})}$, as desired. $\square$

The Theorem together with the two lemmas prove that $\mathcal{G}$ is a (covariant) functor from the category of finite $\pi_1(G, v_0)$-sets to the category of finite even covers of (G, τ_G).

Chapter 8

Equivalence of $\mathbf{Cov}(G)$ and $\pi_1\mathbf{Set}(G)$

Equivalent Categories

In this final section we prove the main result of the work. There is an extremely strong sense in which two categories $\mathfrak{C}$ and $\mathfrak{D}$ can be considered to be the same. Let $I_{\mathfrak{C}}$ and $I_{\mathfrak{D}}$ denote the respective identity functors on $\mathfrak{C}$ and $\mathfrak{D}$. If there exist functors $\mathcal{F} : \mathfrak{C} \to \mathfrak{D}$ and $\mathcal{G} : \mathfrak{D} \to \mathfrak{C}$ with the property that $\mathcal{F} \circ \mathcal{G} = I_{\mathfrak{D}}$ and $\mathcal{G} \circ \mathcal{D} = I_{\mathfrak{C}}$, then we say that $\mathfrak{C}$ and $\mathfrak{D}$ are isomorphic categories. In practice this is frequently too much to ask. As such we have a weaker sense in which $\mathfrak{C}$ and $\mathfrak{D}$ can be considered to be the same. This leads to the notion of "equivalent categories." $\mathfrak{C}$ and $\mathfrak{D}$ are equivalent categories if there exist functors $\mathcal{F} : \mathfrak{C} \to \mathfrak{D}$ and $\mathcal{G} : \mathfrak{D} \to \mathfrak{C}$ which satisfy properties (1) and (2) which follow:

(1) For every $X \in \mathfrak{C}$ there exists an isomorphism $\eta_X : (\mathcal{G} \circ \mathcal{F})(X) \to X$ for which the following diagram

$$\begin{tikzcd} (\mathcal{G} \circ \mathcal{F})(X) \arrow[r, "\eta_X"] \arrow[d, "(\mathcal{G}\circ\mathcal{F})(f)"'] & X \arrow[d, "f"] \\ (\mathcal{G} \circ \mathcal{F})(Y) \arrow[r, "\eta_Y"'] & Y \end{tikzcd}$$

commutes for every morphism $f : X \to Y$

(2) For every $X \in \mathfrak{D}$ there exists an isomorphism $\eta_X : (\mathcal{F} \circ \mathcal{G})(X) \to X$ for which the diagram

$$\begin{tikzcd} (\mathcal{F} \circ \mathcal{G})(X) \arrow[r, "\eta_X"] \arrow[d, "(\mathcal{F}\circ\mathcal{G})(f)"'] & X \arrow[d, "f"] \\ (\mathcal{F} \circ \mathcal{G})(Y) \arrow[r, "\eta_Y"'] & Y \end{tikzcd}$$

commutes for every morphism $f : X \to Y$.

Note that this is a special application of the general definition of a "natural transformation" given in [3] on page 51. To get a sense of how this abstract definition guarantees preservation of categorical structure we highlight the important consequence: suppose that $\mathcal{F} : \mathfrak{C} \to \mathfrak{D}$ and $\mathcal{G} : \mathfrak{D} \to \mathfrak{C}$ induce an equivalence of categories as defined above. Then if $\mathcal{F}(X_1)$ is isomorphic to $\mathcal{F}(X_2)$ in $\mathfrak{D}$, we can conclude that X_1 and X_2 are isomorphic in $\mathfrak{C}$. Likewise if $\mathcal{G}(Y_1)$ is isomorphic to $\mathcal{G}(Y_2)$ in $\mathfrak{C}$, then it is because Y_1 and Y_1 are isomorphic in $\mathfrak{D}$.

In this section we prove that the categories $\mathbf{Cov}(G)$ (the category of finite, even covers of G) and the category $\pi_\mathbf{1}(\mathbf{G}) - \mathbf{Set}$ (the category of finite $\pi_1(G, v_0)$-sets) are equivalent and that our functors $\mathcal{F} : \mathbf{Cov}(G) \to \pi_\mathbf{1}(\mathbf{G}) - \mathbf{Set}$ and $\mathcal{G} : \pi_\mathbf{1}(\mathbf{G}) - \mathbf{Set} \to \mathbf{Cov}(G)$ induce the equivalence. Fix $((G, \tau_G), v_0)$ a finite, even, connected, pointed graph and also fix $\mathfrak{C} = \mathbf{Cov}(G)$ and $\mathfrak{D} = \pi_\mathbf{1}(\mathbf{G}) - \mathbf{Set}$. Let $\mathcal{F}$ and $\mathcal{G}$ be the fiber functor and reverse functor respectively. Then property (2) in the definition of equivalent categories is extremely easy to prove. Since $\mathcal{F} \circ \mathcal{G} : \pi_\mathbf{1}(\mathbf{G}) - \mathbf{Set} \to \pi_\mathbf{1}(\mathbf{G}) - \mathbf{Set}$, let $X \in \pi_\mathbf{1}(\mathbf{G}) - \mathbf{Set}$ be arbitrary. We recall that $\mathcal{G}(X)$ is a finite even cover $((H, \tau_H), \phi)$ with the property that $\phi^{-1}(v_0) = X$, and that if $\mathcal{F}(((H, \tau_H), \phi)) = \phi^{-1}(v_0)$. Then

$$(\mathcal{F} \circ \mathcal{G})(X) = \mathcal{F}(((H, \tau_H), \phi)) = \phi^{-1}(v_0) = X$$

Therefore for any $X \in \pi_\mathbf{1}(\mathbf{G}) - \mathbf{Set}$ we can take $\eta_X = \mathrm{Id_X}$, the identity morphism on X, which is certainly an isomorphism. In this case the diagram (2) will certainly commute. Thus our task is to prove condition (1). Once again, let $\mathfrak{C} = \mathbf{Cov}(G)$ and $\mathfrak{D} = \pi_\mathbf{1}(\mathbf{G}) - \mathbf{Set}$ and let $\mathcal{F}$ and $\mathcal{G}$ denote the fiber functor and the reverse functor respectively. Given $((H, \tau_H), \phi_H) \in \mathbf{Cov}(G)$, we need to define an isomorphism (in the category $\mathbf{Cov}(G)$)

$$\eta_H : (\mathcal{G} \circ \mathcal{F})(((H, \tau_H), \phi_H)) \to ((H, \tau_H), \phi_H)$$

Let $((H', \tau_{H'}), \phi_{H'}) = (\mathcal{G} \circ \mathcal{F})(((H, \tau_H), \phi_H))$. Then by definition such an η_H must satisfy the following conditions:

(1) $\eta_H \in \mathrm{Hom}_\tau((H', \tau_{H'}), (H, \tau_H))$;

(2) $\phi_{H'} = \phi_H \circ \eta_H$;

(3) η_H is invertible.

We recall how the finite even degree d cover $((H', \tau_{H'}), \phi_{H'}) = (\mathcal{G} \circ \mathcal{F})(((H, \tau_H), \phi_H))$ is constructed. Suppose that (after possible relabeling) $\phi_H^{-1}(v_0) = \{1, \ldots, d\}$. First of all, $\mathcal{F}(((H, \tau_H), \phi_H)) = \phi_H^{-1}(v_0)$. Let Φ denote the permutation representation of the action of $\pi_1(G, v_0)$ on $\phi_H^{-1}(v_0)$. Then upon applying the reverse functor $\mathcal{G}$, we construct $((H', \tau_{H'}), \phi_{H'})$ as follows: we fix a spanning tree $T \subseteq G$ and an orientation on $E(G)$. This determines a set $X_T(G) = \{x_1^+, \ldots, x_g^+\} \coprod \{x_1^-, \ldots, x_g^-\}$ of excess edges in G for some $g \in \mathbb{Z}_{\geq 0}$ and also a set $[C_{x_1^+}], \ldots, [C_{x_g^+}], [C_{x_1^-}], \ldots, [C_{x_g^-}]$ of free generators of $\pi_1(G, v_0)$. Then we begin by defining

$$\tilde{H}' = \coprod_{i=1}^{d} T_i'$$

where each T_i' is isomorphic to T. By this we mean that for each $i = 1, \ldots, d$ there exists a morphism $\varphi_i' \in \mathrm{Hom}(T_i', T)$ which is a bijection. Since T is a spanning tree in G, $\tilde{H}'$ will be a spanning forest in H'. Then we adopted the following notation: given $\hat{v}' \in V(\tilde{H}')$ there exists some $\ell \in \{1, \ldots, d\}$ and some $v \in V(T) = V(G)$ for which $\hat{v}' = (\varphi_\ell')^{-1}(v)$. In this case we write $\hat{v}' = \hat{v}_\ell'$. Note that depending on the particular context we will use both notations so that $\hat{v}_\ell' = \hat{v}' = (\varphi_\ell')^{-1}(v)$. Our notation for edges in the spanning forest of H' is the same: given $\hat{e}' \in E(\tilde{H}')$, there exist $\ell \in \{1, \ldots, d\}$ and $e \in E(T)$ such that $\hat{e}' = (\varphi_\ell')^{-1}(e)$. In this case we'll write $\hat{e}_\ell' = \hat{e}' = (\varphi_\ell')^{-1}(e)$. Notice that according to our definition, the vertex $i \in \{1, \ldots, d\} = \phi_H^{-1}(v_0)$ is in $V(T_i')$.

Now let $j \in \{1, \ldots, g\}$ be arbitrary. Then we form the d edges $\{(\hat{x}_j^{\pm})_\ell'\}_{\ell=1,\ldots d}$ according to the rules:

$$t_H((\hat{x}_j^{\pm})_\ell') = (t_G(x_j^{\pm}))_\ell' \in V(T_\ell')$$
$$h_H((\hat{x}_j^{\pm})_\ell') = (h_G(x_j^{\pm}))'_{\ell^{[C_{x_j^{\pm}}]}} \in V(T'_{\ell^{[C_{x_j^{\pm}}]}})$$

We also recall how the covering map $\phi_{H'}$ was defined: given $(\varphi_\ell')^{-1}(v) \in V(\tilde{H}')$,

$$\phi_{H'}((\varphi_\ell')^{-1}(v)) = v$$

For $(\varphi_\ell')^{-1}(e) \in E(\tilde{H}')$ we had

$$\phi_{H'}((\varphi_\ell')^{-1}(e)) = e$$

Finally

$$\phi_{H'}((\hat{x}_j^{\pm})_\ell') = x_j^{\pm}$$

We adopt matching notation for the original cover $((H, \tau_H), \phi_H)$ as we now describe: we proved that

$$\phi_H^{-1}(T) = \coprod_{i=1}^{d} T_i$$

where for each $i = 1, \ldots, d$ there exist morphisms $\varphi_i \in \mathrm{Hom}(T_i, T)$ which are bijections. The T_i will be labeled so that $i \in \{1, \ldots, d\} = \phi_H^{-1}(v_0)$ lives in $V(T_i)$. Let $j \in \{1, \ldots, g\}$ be arbitrary. Then we'll let

$$\phi_H^{-1}(x_j^{\pm}) = \{(\hat{x}_j^{\pm})_i\}_{i=1,\ldots,d}$$

Furthermore, we'll let $(\hat{x}_j^{\pm})_\ell$ be the unique edge in $\phi_H^{-1}(x_j^{\pm})$ which satisfies

$$t_H((\hat{x}_j^{\pm})_\ell) \in V(T_\ell)$$

With this definition it follows that

$$h_H((\hat{x}_j^{\pm})_\ell) \in V(T_{\ell^{[C_{x_j^{\pm}}]}})$$

since the unique lift of $C_{x_j^{\pm}}$ in H beginning at ℓ ends at $\ell^{[C_{x_j^{\pm}}]} \in V(T_{\ell^{[C_{x_j^{\pm}}]}})$. According to our notation for (H, τ_H), we need to know how τ_H acts on edges above excess edges.

Lemma 8.0.16. *Let (H, τ_H) be a finite, even cover of (G, τ_G) together with the notation as described above. Then for every $j \in \{1, \ldots, g\}$ and every $\ell \in \{1, \ldots, d\}$ we have*

$$\tau_H((\hat{x}_j^{\pm})_\ell) = (\hat{x}_j^{\mp})_{\ell^{[C_{x_j^{\pm}}]}}$$

Proof. Let $\hat{y} = \tau_H((\hat{x}_j^{\pm})_\ell)$. Then

$$\begin{aligned} t_H(\hat{y}) &= h_H((\hat{x}_j^{\pm})_\ell) \\ &= (h_G(x_j^{\pm}))_{\ell^{[C_{x_j^{\pm}}]}} \\ &= (t_G(x_j^{\mp}))_{\ell^{[C_{x_j^{\pm}}]}} \end{aligned}$$

and

$$\begin{aligned} h_H(\hat{y}) &= t_H((\hat{x}_j^{\pm})_\ell) \\ &= (t_G(x_j^{\pm}))_\ell \\ &= (h_G(x_j^{\mp}))_\ell \end{aligned}$$

On the other hand we have

$$t_H\left((\hat{x}_j^{\mp})_{\ell^{[C_{x_j^{\pm}}]}}\right) = (t_G(x_j^{\mp}))_{\ell^{[C_{x_j^{\pm}}]}}$$

and

$$\begin{aligned} h_H\left((\hat{x}_j^{\mp})_{\ell^{[C_{x_j^{\pm}}]}}\right) &= (h_G(x_j^{\mp}))_{[\ell^{[C_{x_j^{\mp}}]}]^{[C_{x_j^{\pm}}]}} \\ &= (h_G(x_j^{\mp}))_{\ell} \end{aligned}$$

Therefore it follows that $\hat{y} = (\hat{x}_j^{\mp})_{\ell^{[C_{x_j^{\pm}}]}}$, as desired. $\square$

In Figure (8.1), we illustrate out labeling convention for a cover H of G according to the choice of a spanning tree T of G. Now we define the map

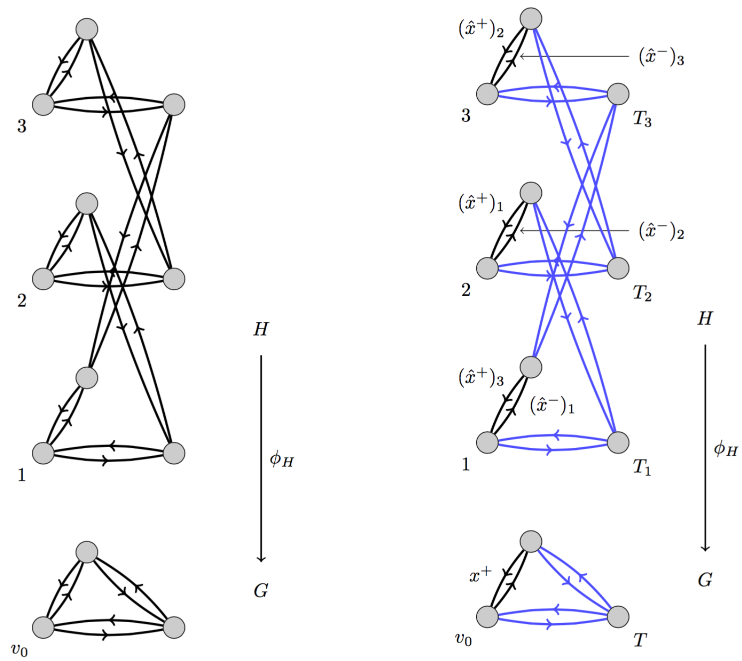


Figure 8.1: Illustration of our labeling convention for a cover H of G according to a choice of spanning tree T of G.

$$\eta_H : ((H', \tau_{H'}), \phi_{H'}) \to ((H, \tau_H), \phi_H)$$

as follows: for every $i = 1, \ldots, d$ we send $\eta_H : T_i' \mapsto T_i$. More explicitly, given $\hat{v}' \in V(T_i')$ there exist $\ell \in \{1, \ldots, d\}$ and some $v \in V(T)$ for which $\hat{v}' = (\varphi_\ell')^{-1}(v)$ and we define

$$\eta_H(\hat{v}') = \eta_H((\varphi_\ell')^{-1}(v)) = \varphi_\ell^{-1}(v) \in V(T_\ell)$$

Given $\hat{e}' \in E(T'_\ell)$ we have $\hat{e}' = (\varphi'_\ell)^{-1}(e)$ for some $\ell \in \{1, \ldots, d\}$ and some $e \in E(T)$. Then we define

$$\eta_H(\hat{e}') = \eta_H((\varphi'_\ell)^{-1}(e)) = \varphi_\ell^{-1}(e) \in E(T_\ell)$$

Finally we define

$$\eta_H((\hat{x}_j^{\pm})'_\ell) = (\hat{x}_j^{\pm})_\ell$$

for every $\ell \in \{1, \ldots, d\}$ and every $j \in \{1, \ldots, g\}$.

With η_H defined, we prove that it is an isomorphism of finite even covers of G.

Lemma 8.0.17. *Let η_H be as defined above. Then $\eta_H \in \mathrm{Hom}_\tau((H', \tau_{H'}), (H, \tau_H))$.*

Proof. First we prove that $\eta_H \in \mathrm{Hom}(H', H)$. To this end, we must show that the diagram

$$\begin{array}{ccc} E(H') & \xrightarrow{(\eta_H)_E} & E(H) \\ {\scriptstyle (t_{H'}, h_{H'})}\downarrow & & \downarrow{\scriptstyle (t_H, h_H)} \\ V(H')^2 & \xrightarrow[(\eta_H)_V^2]{} & V(H)^2 \end{array}$$

commutes. First let $\hat{e}' \in E(\tilde{H}')$. Then $\hat{e}' = (\varphi'_\ell)^{-1}(e)$ for some $\ell \in \{1, \ldots, d\}$ and some $e \in E(T)$. Then we have

$$\begin{aligned} (t_H, h_H)((\eta_H)_E((\varphi'_\ell)^{-1}(e))) &= (t_H, h_H)(\varphi_\ell^{-1}(e)) \\ &= ((t_G(e))_\ell, (h_G(e))_\ell) \end{aligned}$$

On the other hand,

$$\begin{aligned} (\eta_H)_V^2((t_{H'}, h_{H'})((\varphi'_\ell)^{-1}(e))) &= (\eta_H)_V^2((t_G(e))'_\ell, (h_G(e))'_\ell) \\ &= (\eta_H)_V^2((\varphi'_\ell)^{-1}(t_G(e)), (\varphi'_\ell)^{-1}(h_G(e))) \\ &= (\varphi_\ell^{-1}(t_G(e)), \varphi_\ell^{-1}(h_G(e))) \\ &= ((t_G(e))_\ell, (h_G(e))_\ell) \end{aligned}$$

Now let $j \in \{1, \ldots, g\}$ and $\ell \in \{1, \ldots, d\}$ be arbitrary. Then

$$\begin{aligned} (t_H, h_H)((\eta_H)_E(\hat{x}_j^{\pm})'_\ell) &= (t_H, h_H)(\hat{x}_j^{\pm})_\ell \\ &= \left((t_G(x_j^{\pm}))_\ell, (h_G(x_j^{\pm}))_{\ell^{[C_{x_j^{\pm}}]}}\right) \end{aligned}$$

On the other hand,

$$\begin{aligned} (\eta_H)_V^2((t_{H'}, h_{H'})((\hat{x}_j^{\pm})'_\ell)) &= (\eta_H)_V^2\left((t_G(x_j^{\pm}))'_\ell, (h_G(x_j^{\pm}))'_{\ell^{[C_{x_j^{\pm}}]}}\right) \\ &= \left((t_G(x_j^{\pm}))_\ell, (h_G(x_j^{\pm}))_{\ell^{[C_{x_j^{\pm}}]}}\right) \end{aligned}$$

This proves that η_H is a directed graph morphism: $\eta_H \in \mathrm{Hom}(H', H)$.

To prove that in fact, $\eta_H \in \mathrm{Hom}_\tau((H', \tau_{H'}), (H, \tau_H))$, we need to prove that the following diagram commutes:

$$\begin{tikzcd} E(H') \arrow[r, "(\eta_H)_E"] \arrow[d, "\tau_{H'}"'] & E(H) \arrow[d, "\tau_H"] \\ E(H') \arrow[r, "(\eta_H)_E"'] & E(H) \end{tikzcd}$$

To that end, let $\hat{e}' \in E(\tilde{H}')$. Then $\hat{e}' = (\varphi'_\ell)^{-1}(e)$ for some $\ell \in \{1, \ldots, d\}$ and some $e \in E(T)$. Then

$$\begin{aligned} \tau_H((\eta_H)_E(\hat{e}')) &= \tau_H(\varphi_\ell^{-1}(e)) \\ &= \varphi_\ell^{-1}(\tau_G(e)) \end{aligned}$$

On the other hand

$$\begin{aligned} (\eta_H)_E(\tau_{H'}(\hat{e}')) &= (\eta_H)_E(\tau_{H'}((\varphi'_\ell)^{-1}(e))) \\ &= (\eta_H)_E((\varphi'_\ell)^{-1}(\tau_G(e))) \\ &= \varphi_\ell^{-1}(\tau_G(e)) \end{aligned}$$

Let $\ell \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, g\}$ be arbitrary. Then we have

$$\begin{aligned} \tau_H((\eta_H)_E((\hat{x}_j^{\pm})'_\ell)) &= \tau_H((\hat{x}_j^{\pm})_\ell) \\ &= (\hat{x}_j^{\mp})_{[C_{x_j^{\pm}}](\ell)} \end{aligned}$$

and

$$\begin{aligned} (\eta_H)_E(\tau_{H'}((\hat{x}_j^{\pm})'_\ell)) &= (\eta_H)_E\left((\hat{x}_j^{\mp})'_{[C_{x_j^{\pm}}](\ell)}\right) \\ &= (\hat{x}_j^{\mp})_{[C_{x_j^{\pm}}](\ell)} \end{aligned}$$

Therefore the diagram commutes and we've shown that $\eta_H \in \mathrm{Hom}_\tau((H', \tau_{H'}), (H, \tau_H))$, as desired. $\square$

Lemma 8.0.18. *Let* $((H, \tau_H), \phi_H)$ *and* $((H', \tau_{H'}), \phi_{H'})$ *be as described above. Then we have*

$$\phi_{H'} = \phi_H \circ \eta_H$$

Proof. First, let $\hat{v}' \in V(H')$ so that $\hat{v}' = (\varphi'_\ell)^{-1}(v)$ for some $\ell \in \{1, \dots, d\}$ and some $v \in V(G)$. Then by definition,

$$\begin{aligned} \phi_{H'}(\hat{v}') &= \phi_{H'}((\varphi'_\ell)^{-1}(v)) \\ &= v \\ &= \phi_H(\varphi_\ell^{-1}(v)) \\ &= \phi_H((\eta_H)_V(\hat{v}')) \end{aligned}$$

Let $\hat{e}' \in E(\tilde{H}')$. Then $\hat{e}' = (\varphi'_\ell)^{-1}(e)$ for some $\ell \in \{1, \dots, d\}$ and some $e \in E(T)$. By definition,

$$\phi_{H'}(\hat{e}') = e$$

whereas

$$(\phi_H \circ \eta_H)(\hat{e}') = \phi_H(\varphi_\ell^{-1}(e)) = e$$

Finally, let $j \in \{1, \dots, g\}$ and $\ell \in \{1, \dots, d\}$ be arbitrary. Then

$$\phi_{H'}((\hat{x}_j^{\pm})'_\ell) = x_j^{\pm}$$

and

$$(\phi_H \circ \eta_H)((\hat{x}_j^{\pm})'_\ell) = \phi_H((\hat{x}_j^{\pm})_\ell) = x_j^{\pm}$$

Therefore $\phi_{H'} = \phi_H \circ \eta_H$, as desired. □

Theorem 8.0.3. *Let $((H, \tau_H), \phi_H)$, $((H', \tau_{H'}), \phi_{H'})$, and $\eta_H : ((H', \tau_{H'}), \phi_{H'}) \to ((H, \tau_H), \phi_H)$ be as described above. Then η_H is an isomorphism in the category $\mathbf{Cov}(G)$.*

Proof. The two lemmas establish that $\eta_H : ((H', \tau_{H'}), \phi_{H'}) \to ((H, \tau_H), \phi_H)$ is a morphism in the category $\mathbf{Cov}(G)$. It is easy to see that η_H is in fact a bijection as one easily writes down an inverse morphism. Therefore η_H is an isomorphism from $((H', \tau_{H'}), \phi_{H'})$ to $((H, \tau_H), \phi_H)$. □

We have yet to prove the commutativity of the diagram (1). Let $((H_1, \tau_{H_1}), \phi_{H_1})$ and $((H_2, \tau_{H_2}), \phi_{H_2})$ be two finite even covers of (G, τ_G) of respective degrees d_1 and d_2 and let

$$f \in \mathrm{Hom}_{\mathbf{Cov}(G)}(((H_1, \tau_{H_1}), \phi_{H_1}), ((H_2, \tau_{H_2}), \phi_{H_2}))$$

be a morphism between them. Furthermore, let

$$((H'_i, \tau_{H'_i}), \phi_{H'_i}) := (\mathcal{G} \circ \mathcal{F})(((H_i, \tau_{H_i}), \phi_{H_i}))$$

for $i = 1, 2$. Then we must show that the following diagram commutes:

$$\begin{tikzcd}
((H_1', \tau_{H_1'}), \phi_{H_1'}) \arrow[r, "\eta_{H_1}"] \arrow[d, "(\mathcal{G}\circ\mathcal{F})(f)"'] & ((H_1, \tau_{H_1}), \phi_{H_1}) \arrow[d, "f"] \\
((H_2', \tau_{H_2'}), \phi_{H_2'}) \arrow[r, "\eta_{H_2}"'] & ((H_2, \tau_{H_2}), \phi_{H_2})
\end{tikzcd}$$

In other words we must prove that

$$f \circ \eta_{H_1} = \eta_{H_2} \circ ((\mathcal{G} \circ \mathcal{F})(f))$$

as morphisms in $\mathbf{Cov}(G)$. This entails showing that $f \circ \eta_{H_1}$ and $\eta_{H_2} \circ ((\mathcal{G} \circ \mathcal{F})(f))$ act the same on vertices and on edges in $(H_1', \tau_{H_1'})$. First we make sense of the morphism $(\mathcal{G} \circ \mathcal{F})(f) : ((H_1', \tau_{H_1'}), \phi_{H_1'}) \to ((H_2', \tau_{H_2'}), \phi_{H_2'})$. We recall that $f \in \mathrm{Hom}_{\mathbf{Cov}(G)}(((H_1, \tau_{H_1}), \phi_{H_1}), ((H_2, \tau_{H_2}), \phi_{H_2}))$ is equivalent to the pair of conditions:

(1) $f \in \mathrm{Hom}_\tau((H_1, \tau_{H_1}), (H_2, \tau_{H_2}))$;

(2) $\phi_{H_1} = \phi_{H_2} \circ f$.

By definition of the fiber functor $\mathcal{F}$, we have

$$\mathcal{F}(((H_i, \tau_{H_i}), \phi_{H_i})) = \phi_{H_i}^{-1}(v_0)$$

for $i = 1, 2$. Furthermore, given $\ell \in \phi_{H_1}^{-1}(v_0)$ we defined $(\mathcal{F}(f))(\ell) = f(\ell) \in \phi_{H_2}^{-1}(v_0)$. It follows from the definition of the reverse functor $\mathcal{G}$ then that

$$(\mathcal{G} \circ \mathcal{F})(f) : \begin{cases} ((\varphi_\ell^{(1)})')^{-1}(v) \mapsto ((\varphi_{f(\ell)}^{(2)})')^{-1}(v) \\ ((\varphi_\ell^{(1)})')^{-1}(e) \mapsto ((\varphi_{f(\ell)}^{(2)})')^{-1}(e) \\ ((\hat{x}_j^{\pm})_\ell^{(1)})' \mapsto ((\hat{x}_j^{\pm})_{f(\ell)}^{(2)})' \end{cases}$$

for every $v \in V(T)$, every $e \in E(T)$, every $\ell \in \{1, \ldots, d_1\}$ and every $j \in \{1, \ldots, g\}$. With regards to the more convenient notation, the first two rules above can be rewritten as

$$(\mathcal{G} \circ \mathcal{F})(f) : \begin{cases} (\hat{v}_\ell^{(1)})' \mapsto (\hat{v}_{f(\ell)}^{(2)})' \\ (\hat{e}_\ell^{(1)})' \mapsto (\hat{e}_{f(\ell)}^{(2)})' \end{cases}$$

We are therefore ready to state and prove the final result of the work.

Theorem 8.0.4. *Let $((H_i, \tau_{H_i}), \phi_{H_i})$ and $((H_i', \tau_{H_i'}), \phi_{H_i'})$ be as described above for $i = 1, 2$. Then the diagram*

$$\begin{tikzcd} ((H_1', \tau_{H_1'}), \phi_{H_1'}) \arrow[r, "\eta_{H_1}"] \arrow[d, "(\mathcal{G}\circ\mathcal{F})(f)"'] & ((H_1, \tau_{H_1}), \phi_{H_1}) \arrow[d, "f"] \\ ((H_2', \tau_{H_2'}), \phi_{H_2'}) \arrow[r, "\eta_{H_2}"'] & ((H_2, \tau_{H_2}), \phi_{H_2}) \end{tikzcd}$$

commutes.

Proof. Let $(\hat{v}^{(1)})' \in V(\tilde{H}_1')$. Then there exist $v \in V(T)$ and $\ell \in \{1, \ldots, d_1\}$ for which $(\hat{v}^{(1)})' = ((\varphi_\ell^{(1)})')^{-1}(v) = (\hat{v}_\ell^{(1)})'$. Then by definition

$$\begin{aligned}(f \circ \eta_{H_1})((\hat{v}^{(1)})') &= (f \circ \eta_{H_1})((\hat{v}_\ell^{(1)})') \\ &= f(\hat{v}_\ell^{(1)})\end{aligned}$$

On the other hand,

$$\begin{aligned}\eta_{H_2}\left(((\mathcal{G}\circ\mathcal{F})(f))((\hat{v}^{(1)})')\right) &= \eta_{H_2}\left(((\mathcal{G}\circ\mathcal{F})(f))(\hat{v}_\ell^{(1)})')\right) \\ &= \eta_{H_2}((\hat{v}_{f(\ell)}^{(2)})') \\ &= \hat{v}_{f(\ell)}^{(2)}\end{aligned}$$

Therefore we must prove that $f(\hat{v}_\ell^{(1)}) = \hat{v}_{f(\ell)}^{(2)}$. By definition $\phi_{H_2}(\hat{v}_{f(\ell)}^{(2)}) = v$, so that $\hat{v}_{f(\ell)}^{(2)} \in \phi_{H_2}^{-1}(v)$. But also, since f is a morphism of covers we have

$$\phi_{H_2}(f(\hat{v}_\ell^{(1)})) = \phi_{H_1}(\hat{v}_\ell^{(1)}) = v$$

Therefore $f(\hat{v}_\ell^{(1)}) \in \phi_{H_2}^{-1}(v)$. We observe that ℓ and $\hat{v}_\ell^{(1)}$ are both in $V(T_\ell^{(1)})$ and that $f(\ell)$ is in $T_{f(\ell)}^{(2)}$. But since $T_\ell^{(1)}$ is a tree, there is a unique path (contained entirely) in $T_\ell^{(1)}$ from ℓ to $\hat{v}_\ell^{(1)}$. It follows then that there is a unique path (contained entirely) in $T_{f(\ell)}^{(2)}$ from $f(\ell)$ to $f(\hat{v}_\ell^{(1)})$. We conclude that $f(\hat{v}_\ell^{(1)}) \in V(T_{f(\ell)}^{(2)})$. It's true by definition that $\hat{v}_{f(\ell)}^{(2)} \in V(T_{f(\ell)}^{(2)})$. But then it must be that $f(\hat{v}_\ell^{(1)}) = \hat{v}_{f(\ell)}^{(2)}$: because $\varphi_{f(\ell)}^{(2)} : T_{f(\ell)}^{(2)} \to T$ is a bijection and as such there is only one vertex in $V(T_{f(\ell)}^{(2)}) \cap \phi_{H_2}^{-1}(v)$.
Now, let $(\hat{e}^{(1)})' \in E(\tilde{H}_1'$. Then $(\hat{e}^{(1)})' = ((\varphi_\ell^{(1)})')^{-1}(e) = (\hat{e}_\ell^{(1)})'$ for some $e \in E(T)$ and some $\ell \in \{1, \ldots, d_1\}$. So we have

$$(f \circ \eta_{H_1})((\hat{e}_\ell^{(1)})') = f(\hat{e}_\ell^{(1)})$$

On the other hand,

$$\begin{aligned}(\eta_{H_2} \circ ((\mathcal{G} \circ \mathcal{F})(f)))((\hat{e}_\ell^{(1)})') &= \eta_{H_2}((\hat{e}_{f(\ell)}^{(2)})') \\ &= \hat{e}_{f(\ell)}^{(2)}\end{aligned}$$

So we must show that $f(\hat{e}_\ell^{(1)}) = \hat{e}_{f(\ell)}^{(2)}$. First we prove that they're in the same fiber: we have

$$\phi_{H_2}(f(\hat{e}_\ell^{(1)})) = \phi_{H_1}(\hat{e}_\ell^{(1)}) = e$$

so that $f(\hat{e}_\ell^{(1)}) \in \phi_{H_2}^{-1}(e)$. But we also have $\phi_{H_2}(\hat{e}_{f(\ell)}^{(2)}) = e$ so that $\hat{e}_{f(\ell)}^{(2)} \in \phi_{H_2}^{-1}(e)$.
Next we show that $f(\hat{e}_\ell^{(1)})$ and $\hat{e}_{f(\ell)}^{(2)}$ are in the same component of $\tilde{H}_2$. We begin by observing that

(1) $t_{H_1}(\hat{e}_\ell^{(1)})$ and $h_{H_1}(\hat{e}_\ell^{(1)})$ are both in $V(T_\ell^{(1)})$;

(2) $t_{H_2}(\hat{e}_{f(\ell)}^{(2)})$ and $h_{H_2}(\hat{e}_{f(\ell)}^{(2)})$ are both in $V(T_{f(\ell)}^{(2)})$.

The first of these observations implies that there are unique paths (contained entirely) in $T_\ell^{(1)}$ from $t_{H_1}(\hat{e}_\ell^{(1)})$ to ℓ and from $h_{H_1}(\hat{e}_\ell^{(1)})$ to ℓ. There are therefore unique paths (contained entirely) in $T_{f(\ell)}^{(2)}$ from $f(t_{H_1}(\hat{e}_\ell^{(1)}))$ to $f(\ell)$ and from $f(h_{H_1}(\hat{e}_\ell^{(1)}))$ to $f(\ell)$. Therefore we have that both $f(t_{H_1}(\hat{e}_\ell^{(1)}))$ and $f(h_{H_1}(\hat{e}_\ell^{(1)}))$ are in $V(T_{f(\ell)}^{(2)})$. But f is a morphism of even graphs, so we have

$$\begin{aligned}t_{H_2}(f(\hat{e}_\ell^{(1)})) &= f(t_{H_1}(\hat{e}_\ell^{(1)})) \in V(T_{f(\ell)}^{(2)}) \\ h_{H_2}(f(\hat{e}_\ell^{(1)})) &= f(h_{H_1}(\hat{e}_\ell^{(1)})) \in V(T_{f(\ell)}^{(2)})\end{aligned}$$

Since both its head and tail live in $V(T_{f(\ell)}^{(2)})$, it must be the case that $f(\hat{e}_\ell^{(1)}) \in E(T_{f(\ell)}^{(2)})$. Once again, because $(\varphi_{f(\ell)}^{(2)})^{-1}$ is a bijection on $T_{f(\ell)}^{(2)}$, and because $f(\hat{e}_\ell^{(1)})$ and $\hat{e}_{f(\ell)}^{(2)}$ are in the same fiber (above e), they are forced to coincide.
Finally, let $j \in \{1, \ldots, g\}$ be arbitrary. Then we have

$$(f \circ \eta_{H_1})(((\hat{x}_j^{\pm})_\ell^{(1)})') = f((\hat{x}_j^{\pm})_\ell^{(1)})$$

On the other hand,

$$\begin{aligned}(\eta_{H_2} \circ ((\mathcal{G} \circ \mathcal{F})(f)))(((\hat{x}_j^{\pm})_\ell^{(1)})') &= \eta_{H_2}\left(((\hat{x}_j^{\pm})_{f(\ell)}^{(2)})'\right) \\ &= (\hat{x}_j^{\pm})_{f(\ell)}^{(2)}\end{aligned}$$

So we must prove that $f((\hat{x}_j^{\pm})^{(1)}_{\ell}) = (\hat{x}_j^{\pm})^{(2)}_{f(\ell)}$. As before, because of the covering map property satisfied by f, i.e. that $\phi_{H_1} = \phi_{H_2} \circ f$, we have that both $f((\hat{x}_j^{\pm})^{(1)}_{\ell})$ and $(\hat{x}_j^{\pm})^{(2)}_{f(\ell)}$ are in the fiber $\phi_{H_2}^{-1}(x_j^{\pm})$.

Next we observe that $t_{H_2}((\hat{x}_j^{\pm})^{(2)}_{f(\ell)} \in V(T^{(2)}_{f(\ell)})$ and $h_{H_2}((\hat{x}_j^{\pm})^{(2)}_{f(\ell)}) \in V(T^{(2)}_{f(\ell)^{[C_{x_j^{\pm}}]}})$. Since f is, in particular, a directed graph morphism, we have

$$t_{H_2}(f((\hat{x}_j^{\pm})^{(1)}_{\ell})) = f(t_{H_1}((\hat{x}_j^{\pm})^{(1)}_{\ell}))$$

Since $t_{H_1}((\hat{x}_j^{\pm})^{(1)}_{\ell} \in V(T^{(1)}_{\ell})$, we have that $f(t_{H_1}((\hat{x}_j^{\pm})^{(1)}_{\ell})) \in V(T^{(2)}_{f(\ell)})$, and thus $t_{H_2}(f((\hat{x}_j^{\pm})^{(1)}_{\ell})) \in V(T^{(2)}_{f(\ell)})$. Because $h_{H_1}((\hat{x}_j^{\pm})^{(1)}_{\ell}) \in V(T^{(1)}_{\ell^{[C_{x_j^{\pm}}]}})$ we have

$$f(h_{H_1}((\hat{x}_j^{\pm})^{(1)}_{\ell})) \in V(T^{(2)}_{f(\ell^{[C_{x_j^{\pm}}]})})$$

But since f is, in particular, a morphism of directed graphs, we have

$$h_{H_2}(f((\hat{x}_j^{\pm})^{(1)}_{\ell})) = f(h_{H_1}((\hat{x}_j^{\pm})^{(1)}_{\ell}))$$

Therefore we conclude that

$$h_{H_2}(f((\hat{x}_j^{\pm})^{(1)}_{\ell})) \in V(T^{(2)}_{f(\ell^{[C_{x_j^{\pm}}]})})$$

Recall that $f(\ell)^{[C_{x_j^{\pm}}]} = f(\ell^{[C_{x_j^{\pm}}]})$ because $\mathcal{F}(f)$ is a morphism in $\boldsymbol{\pi_1}(\mathbf{G}) - \mathbf{Set}$ and $\mathcal{F}(f)(\ell) = f(\ell)$. Therefore $T^{(2)}_{f(\ell^{[C_{x_j^{\pm}}]})} = T^{(2)}_{f(\ell)^{[C_{x_j^{\pm}}]}}$. Therefore $(\hat{x}_j^{\pm})^{(2)}_{f(\ell)}$ and $f((\hat{x}_j^{\pm})^{(1)}_{\ell})$ have the same heads and tails and are therefore the same edge.

We have therefore shown that

$$\eta_{H_2} \circ ((\mathcal{G} \circ \mathcal{F})(f)) = f \circ \eta_{H_1}$$

so the diagram commutes and we are finished.

$\square$

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