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Homework 3: Induction & Recursion

8. Let A = [1, 1; 1, 0], show that $A^n = [f_{n+1}, f_n; f_n, f_{n-1}]$ where f_n is nth Fibonacci number
    a. Base Case: n = 1
        i. $A^1 = [f_{1+1}, f_1; f_1, f_{1-1}] = [f_2, f_1; f_1, f_0] = [1, 1; 1, 0] = A$
    b. Assume that for all integers k > 1, the matrix for A^k is valid.
    c. Prove that if A^k is valid, then A^{k+1} is valid and equals $[f_{k+1+1}, f_{k+1}; f_{k+1}, f_{k-1+1}]$.
        i. $A^{k+1} = A * A^k$
        ii. $= [1, 1; 1, 0] * [f_{k+1}, f_k; f_k, f_{k-1}]$
        iii. $= [(f_{k+1} + f_k) , (f_k + f_{k-1}) ; f_{k+1}, f_k]$
        iv. $= [f_{k+2}, f_{k+1}; f_{k+1}, f_k]$
    d. The matrix returned by A^{k+1} is equal to the original matrix, so the proof is complete.
9. Recursive definition of the set of bit strings that are palindromes
    a. Basis: A bit string consisting of 0 or 1 bits is a palindrome.
    b. Recursive Rule: If a bit string S is a palindrome, then a new bit string S' can be formed by adding any bit B to the beginning and end of S to form the string BSB. Since S is a palindrome, BSB, and therefore S', is also a palindrome.
10. Full Binary Trees: $l(T)$, number of leaves, is 1 more than i(T), number of internal vertices
    a. Base Case: i = 1 (tree with a single vertex)
        i. A tree with a single vertex has no branches, so the single vertex must act as a leaf. $l(T) = 1$ and $i(T) = 0$.
    b. Assume that for all full binary trees T such that i < 1, $l(T) = 1 + i(T)$.
        i. $l(T_1) = 1 + i(T_1)$, and $l(T_2) = 1 + i(T_2)$
    c. Prove that $l(T_1 + T_2) = 1 + i(T_1 + T_2)$
        i. T_1 and T_2 are connected with a root connected to the roots of the two individual trees.
        ii. No new leaves are added, so number of leaves in the new tree is sum of leaves in two individual trees
            1. $l(T_1 + T_2) = l(T_1) + l(T_2)$
        iii. One new vertex is added to connect the two roots, so number of vertices is the sum of the vertices of the original trees plus one.
            1. $i(T_1 + T_2) = i(T_1) + i(T_2) + 1$
        iv. Substitute $l(T_1) = 1 + i(T_1)$ and $l(T_2) = 1 + i(T_2)$
            1. $l(T_1 + T_2) = 1 + i(T_1) + 1 + i(T_2)$
            2. $l(T_1 + T_2) = (i(T_1) + i(T_2) + 1) + 1$
            3. $l(T_1 + T_2) = i(T_1 + T_2) + 1$
    d. Therefore, for any full binary tree T, $l(T) = 1 + i(T)$.