**HW #2 - due 10/25/2019**

The goal of this homework is to illustrate applications of the singular value decomposition (SVD) and Tikhonov regularization to image deblurring.

## Mathematical Background:

Consider a linear system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{d} \tag{1}$$

where $\mathbf{d} \in \mathbb{R}^n$ is a given vector (observed/received data), $\mathbf{x} \in \mathbb{R}^n$ is an unknown vector (transmitted data), and $\mathbf{A} \in \mathbb{R}^{n\times n}$ is a given nonsingular matrix (transformation of data).
In many applications the observed/received data is corrupted by unknown noise or measurement errors, such that we only have access to

$$\hat{\mathbf{d}} = \mathbf{d} + \boldsymbol{\xi}$$

where $\xi \in \mathbb{R}^n$ is an unknown noise vector. The problem is then formulated as follows:

*Given the matrix* $\mathbf{A}$ *and a vector* $\hat{\mathbf{d}}$*, provide an approximation of the unknown vector* $\mathbf{x}$.

Practical difficulties arise when $\mathbf{A}$ is ill-conditioned. In such case, an attempt to provide an approximation $\hat{\mathbf{x}} \approx \mathbf{x}$ by solving

$$\mathbf{A}\hat{\mathbf{x}} = \hat{\mathbf{d}} \tag{2}$$

will not work since $\hat{\mathbf{x}}$ will be drastically corrupted by noise. Regularization techniques are required to address this issue.

**Truncated SVD.** Consider the singular value decomposition of the matrix $A$

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}} \in \mathbb{R}^{n\times n} \tag{3}$$

where $\mathbf{U} = [\mathbf{u}_1\,\mathbf{u}_2 \ldots \mathbf{u}_n], \mathbf{S} = diag(\sigma_1, \sigma_2, \ldots \sigma_n), \mathbf{V} = [\mathbf{v}_1\,\mathbf{v}_2 \ldots \mathbf{v}_n]$. The solution to the system (2) is expressed as

$$\hat{\mathbf{x}} = [\mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}]^{-1}\hat{\mathbf{d}} = \mathbf{V}\mathbf{S}^{-1}\mathbf{U}^{\mathrm{T}}\hat{\mathbf{d}} = \sum_{i=1}^{n} \frac{\mathbf{u}_i^{\mathrm{T}}\hat{\mathbf{d}}}{\sigma_i}\mathbf{v}_i \tag{4}$$

For very small $\sigma_i$, the terms $(\mathbf{u}_i^{\mathrm{T}}\hat{\mathbf{d}})/\sigma_i$ will result in noise amplification.

In the *truncated SVD* regularization, an approximation $\hat{\mathbf{x}}_k$ to the solution is obtained by truncating the sum in (4) at a selected index $k$,

$$\widehat{\mathbf{x}}_k = \sum_{i=1}^{k} \frac{\mathbf{u}_i^{\mathrm{T}}\hat{\mathbf{d}}}{\sigma_i}\mathbf{v}_i \tag{5}$$

The main difficulty is to find an appropriate truncation index $k$ such that $\widehat{\mathbf{x}}_k$ is a good approximation to the true value $\mathbf{x}$.

**Tikhonov Regularization.** In this approach, an approximation to $\mathbf{x}$ is obtained as the solution to the least squares minimization problem:

$$\min_{\widehat{x}\in\mathbb{R}^n} J(\widehat{\mathbf{x}}), \quad where \;\; J(\widehat{\mathbf{x}}) \stackrel{def}{=} \|\mathbf{A}\widehat{\mathbf{x}} - \widehat{\mathbf{d}}\|^2 + \lambda^2\|\widehat{\mathbf{x}}\|^2 \tag{6}$$

In (6), $\|\cdot\|$ denotes the Euclidean vector norm and $\lambda$ is a scalar *regularization parameter* that controls the *smoothness* of the solution. If $\lambda = 0$ then no regularization is applied and the solution is (4). If $\lambda$ is large, then $\lambda^2\|\widehat{\mathbf{x}}\|^2$ has a significant contribution to the cost (6) and the solution $\widehat{\mathbf{x}}$ can not be a good approximation of $\mathbf{x}$. We view the solution to (6) as a function of the parameter $\lambda$ and denote it $\widehat{\mathbf{x}}_\lambda$. The main difficulty is to find an appropriate value of the regularization parameter $\lambda$ such that $\widehat{\mathbf{x}}_\lambda$ is a good approximation to the true value $\mathbf{x}$.

The solution $\widehat{\mathbf{x}}_\lambda$ to the minimization problem (6) is obtained by solving the linear system

$$\left(\mathbf{A}^{\mathrm{T}}\mathbf{A} + \lambda^2\mathbf{I}\right)\widehat{\mathbf{x}}_\lambda = \mathbf{A}^{\mathrm{T}}\widehat{\mathbf{d}} \tag{7}$$

and may be expressed using the SVD (3) of $\mathbf{A}$ as

$$\widehat{\mathbf{x}}_\lambda = \sum_{i=1}^{n} f_i(\lambda)\frac{\mathbf{u}_i^{\mathrm{T}}\widehat{\mathbf{d}}}{\sigma_i}\mathbf{v}_i \tag{8}$$

where the filter factors $f_i(\lambda)$ are defined as

$$f_i(\lambda) = \frac{\sigma_i^2}{\sigma_i^2 + \lambda^2}, \;\; i = 1, 2, \ldots n \tag{9}$$

The main difficulty in this approach is to select an appropriate value of the *regularization parameter* $\lambda$.

## Homework content

Consider an image represented as a matrix $\mathbf{X} \in \mathbb{R}^{n\times m}$ and a blurring process represented by the matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$. Whereas the true image is the solution to the matrix equation

$$\mathbf{AX} = \mathbf{D}$$

we want to reconstruct an approximation $\widehat{\mathbf{X}}$ to the image $\mathbf{X}$ given the blurring matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ and a matrix of received noisy data

$$\widehat{\mathbf{D}} = \mathbf{AX} + \boldsymbol{\xi}$$

The regularization procedure (5) or (8-9) is used to find an approximate solution $\widehat{\mathbf{X}}$, one column at a time:

Algorithm

$[U, S, V] = svd(A)$

for $i = 1 : m$

$\hat{d} = \hat{D}(:, i)$

evaluate $\hat{x}_i$ from (5) - TSVD solution *or* from (8-9) - Tikhonov solution

$\hat{X} = [\hat{x}_1\, \hat{x}_2 \ldots \hat{x}_m]$

For this assignment $n = m = 128$, and $X$ is a $128 \times 128$ matrix representing the image of a coin. The blurring operation is represented as follows: Consider a $128 \times 128$ symmetric tridiagonal matrix $B$ with entries [1]

$$B(i,i) = 1 - 2L, \;\; i = 1, 2, \dots n$$

$$B(i,i+1) = L, \;\; i = 1, 2, \dots n-1; \quad B(i+1,i) = L, \;\; i = 1, 2, \dots n-1$$

where $L = 0.45$. Then the blurring operator is

$$\mathbf{A} = \mathbf{B}^{10}$$

The file "hw2data.m" represents the $128 \times 128$ noisy data matrix $\hat{\mathbf{D}}$, obtained as

$$\hat{\mathbf{D}} = \mathbf{A}\mathbf{X} + \boldsymbol{\xi}$$

where $\boldsymbol{\xi}$ is an unknown $128 \times 128$ noise matrix. To visualize $\hat{\mathbf{D}}$, in MATLAB you may execute:

load hw2data.m; D = hw2data; imagesc(D); colormap(gray)

**Jour job:** Implement the TSVD and the Tikhonov regularization procedures to reconstruct an approximation $\hat{\mathbf{X}}$ of the unknown image $\mathbf{X}$ such that the year and the inscriptions on the coin can be read.

- (50 points) Implement a function $\hat{X} = regularize(A, \hat{D}, method, p)$ that takes as input the noisy data matrix $\hat{D}$, the blurring matrix $A$ and the regularization parameter $p$, and returns $\hat{X}$, the regularization approximation to $X$. The input value *method* is used to distinguish between the TSVD solution (5) in which case $p$ is the truncation index and the Tikhonov solution (8-9) in which case $p$ is the regularization parameter $\lambda$. Alternatively, you may write a function for each method *tsvd* and *tikhonov*, respectively.

- (20 points) Find values of $k$ and $\lambda$ such that from $\hat{\mathbf{X}}$ the year and the inscriptions on the coin can be read. Plot the $l - curve$.

**Things to hand in:** listing of the *tsvd* and *tikhonov* functions, value of index $k$ (tsvd) and parameter $\lambda$ (Tikhonov), reconstructed image: year, inscriptions; a plot of the singular values $\sigma_i$ on a $\log_{10}$ scale and the filter factors $f_i$; a plot of the $l - curve$.

[1] such matrix results from discretization of the 1-D heat equation $u_t - k u_{xx} = 0$ with $L = k\Delta t/(\Delta x)^2$.