HW #4 - Kalman filter for a mass-spring system

Due on November 25, in class

Consider the mass-spring system given by the second order differential equation

$$x''(t) + x(t) = 0 \tag{1}$$

with the initial conditions

$$x(0) = x_0, \quad x'(0) = v_0 \tag{2}$$

Introducing the velocity variable $v(t) = x'(t)$, an equivalent formulation to (1-2) is

$$\begin{bmatrix} x' \\ v' \end{bmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{bmatrix} x \\ v \end{bmatrix}, \qquad \begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ v_0 \end{bmatrix} \tag{3}$$

The solution to the initial-value problem (3) is expressed as

$$x(t) = v_0 \sin(t) + x_0 \cos(t), \qquad v(t) = v_0 \cos(t) - x_0 \sin(t) \tag{4}$$

The solution (4) is denoted in vector format as

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ v(t) \end{bmatrix} \tag{5}$$

and we refer to $\mathbf{x}(t) \in \mathbb{R}^n$ (here n=2) as the *true state* of the dynamical system. The initial condition is specified as

$$\mathbf{x}(0) = \begin{bmatrix} x(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{6}$$

Henceforth, it is assumed that we can obtain information on the true state $\mathbf{x}(t)$ through state measurements and a time discrete model. Observations are represented as

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \epsilon_o \in \mathbb{R}^m, \quad (m = 1 \textit{ or } m = 2) \tag{7}$$

where $\mathbf{H} \in \mathbb{R}^{m \times n}$ denotes the observation matrix and $\epsilon_o \in \mathbb{R}^m$ is a random vector of observational errors.

A time-discrete model to the true dynamics is obtained by applying the Euler's method to the system (3) with a time step Δt

$$\hat{\mathbf{x}}_{k+1} = \mathbf{M}\hat{\mathbf{x}}_k \tag{8}$$

where

$$\mathbf{M} = \begin{pmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{pmatrix} \tag{9}$$

The evolution of the true state $\mathbf{x}_k = \mathbf{x}(t_k)$ in (5) is given by

$$\mathbf{x}_{k+1} = \mathbf{M}\mathbf{x}_k + \mathbf{w}_k \tag{10}$$

where $\mathbf{w}_k \in \mathbb{R}^n$ is an unknown vector consisting of numerical discretization errors.

Our goal is to implement the Kalman filter to estimate and predict the true state $\mathbf{x}_k$ based on the model (8) and observations (7). The setup is as follows.

- A prior guess to the true (unknown) initial state at $t = 0$ is prescribed as

$$\hat{\mathbf{x}}_0 = \mathbf{x}(0) + \epsilon_b$$

where the initial condition error in each component is $\epsilon_b \sim N(0, \sigma_b^2)$ with $\sigma_b = 1$.

- The numerical discretization time step is taken $\Delta t = 1$.
- The observational operator is defined as
  1. Case 1 (both state components are observed)

$$\mathbf{H} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{11}$$

  2. Case 2 (only first state component is observed)

$$\mathbf{H} = [1 \;\; 0] \tag{12}$$

- The observational error is specified as $\epsilon_o \sim N(0, \sigma_o^2)$ where $\sigma_o = 0.1$.
- Each component of the model error is specified as $w_k \sim N(0, \sigma_q^2)$ where $\sigma_q = \Delta t = 1$
- The time interval for the analysis it $0 \leq t \leq t_N$, that is $k = 0 : N - 1$ in (8).

The KF algorithm is formulated as follows:

$x_0^a = x(0) + \epsilon_b$ % true initial state plus random error taken from $N(0, 1)$
$\mathbf{P}^a = \mathbf{I}$ % initial analysis error covariance matrix is set to identity

for $k = 0 : 999$

% forecast step

$x_{k+1}^f = \mathbf{M} x_k^a$

$\mathbf{P}^f = \mathbf{M}\mathbf{P}^a\mathbf{M}^{\mathrm{T}} + \mathbf{Q}$ % forecast error covariance where $\mathbf{Q} = \sigma_q^2 \mathbf{I}_{n \times n}$

% generate observation(s) at t_{k+1}

$y = \mathbf{H}x(t_{k+1}) + \epsilon_o$ % generate observation(s) at t_{k+1} of true state plus random errors taken from $N(0, \sigma_o^2)$

% analysis step
$x_{k+1}^a = x_{k+1}^f + \mathbf{K}(y - \mathbf{H}x_{k+1}^f)$ % where $\mathbf{K}$ is the Kalman gain matrix

$\mathbf{P}^a = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}^f$ % analysis error covariance at t_{k+1}
end % end for loop

Your job:

(10 points) Investigate the performance of the discrete model in pure forecast mode.

(30 points) Implement the KF algorithm, as described above.

(30 points) Provide a qualitative study of the analysis and forecast errors in the time interval $0 \leq t \leq 1000$, that is for $N = 1000$ time steps.