Formulations of Matrix Products from Vector Products

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Introduction

The equations for matrix-vector and matrix-matrix products can be formulated and understood in terms of vector-scalar and vector-vector products.

Notation

To emphasize the connection between the vector and matrix products, we use:

$A \in \mathbb{R}^{m \times n}$	
$a_j \in \mathbb{R}^n$	jth column vector
$a_k^T \in \mathbb{R}^{1 \times m}$	kth row vector

Formula Table

Table 1: Primitive Operations: Vector {Scalar, Vector} Products

Formula	Name
$u^T k = k u^T = \begin{bmatrix} ku_1 & \dots & ku_n \end{bmatrix}$	Scalar row-vector product $\in \mathbb{R}^{1 \times m}$
$kv = vk = \begin{bmatrix} kv_1 \\ \vdots \\ kv_n \end{bmatrix}$	Scalar col-vector product $\in \mathbb{R}^n$
$u^T v = \begin{bmatrix} u_1 & \dots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + \cdots + u_n v_n$	Inner Product $\in \mathbb{R}$
$uv^T = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix} = \begin{bmatrix} (u_1 v_1) & \dots & (u_1 v_n) \\ \vdots & \ddots & \vdots \\ (u_m v_1) & \dots & (u_m v_n) \end{bmatrix}$	Outer Product $\in \mathbb{R}^{m \times n}$

Table 2: Matrix Vector Products

Formula	Name
$Ax = \underbrace{\begin{bmatrix} \vert & & \vert \\ a_1 & \dots & a_n \\ \vert & & \vert \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}}_{u^T v = \text{ inner product}} = \underbrace{x_1 a_1}_{\substack{\text{column} \\ \text{vector}}} + \cdots + x_n a_n$	Linear combination of columns: $\in \mathbb{R}^n$ $\substack{\text{inner} \\ \text{product}} \longrightarrow \substack{\text{column vector} \\ \text{terms}}$
$yA = \underbrace{\begin{bmatrix} y_1 & \dots & y_m \end{bmatrix} \begin{bmatrix} -a_1^T- \\ \vdots \\ -a_m^T- \end{bmatrix}}_{u^T v = \text{ inner product}} = \underbrace{y_1 a_1^T}_{\substack{\text{row} \\ \text{vector}}} + \cdots + y_m a_m^T$	Linear combination of rows: $\in \mathbb{R}^{1 \times m}$ $\substack{\text{inner} \\ \text{product}} \longrightarrow \substack{\text{row vector} \\ \text{terms}}$
$Ax = \underbrace{\begin{bmatrix} -a_1^T- \\ \vdots \\ -a_m^T- \end{bmatrix}}_{\vec{v}k = \text{ scalar product}} x = \begin{bmatrix} [-a_1^T-]x \\ \vdots \\ \underbrace{[-a_m^T-]x}_{\substack{u^T v = \\ \text{inner product}}} \end{bmatrix} = \underbrace{\begin{bmatrix} [-a_1^T-] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \\ \vdots \\ [-a_m^T-] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \end{bmatrix}}_{\text{scalar}}$	Scalar products with rows: $\in \mathbb{R}^n$ $\substack{\text{scalar} \\ \text{product}} \longrightarrow \substack{\text{inner} \\ \text{product}} \longrightarrow \substack{\text{scalar} \\ \text{terms}}$
$yA = \underbrace{y \begin{bmatrix} \vert & & \vert \\ a_1 & \dots & a_n \\ \vert & & \vert \end{bmatrix}}_{k\vec{v} = \text{scalar product}} = \begin{bmatrix} \vert & & \vert \\ ya_1 & \dots & ya_n \\ \underbrace{\vert}_{\substack{u^T v = \\ \text{inner} \\ \text{product}}} & & \vert \end{bmatrix} = \begin{bmatrix} \underbrace{\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}^T a_1}_{\text{scalar}} & \dots & \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}^T a_n \end{bmatrix}$	Scalar products with columns: $\in \mathbb{R}^{1 \times m}$ $\substack{\text{scalar} \\ \text{product}} \longrightarrow \substack{\text{inner} \\ \text{product}} \longrightarrow \substack{\text{scalar} \\ \text{terms}}$

Table 3: Matrix-Matrix Products

Formula	Name
$AB = \underbrace{A \begin{bmatrix} \vert & & \vert \\ b_1 & \dots & b_p \\ \vert & & \vert \end{bmatrix}}_{k\vec{v}=\text{ scalar product}} = \begin{bmatrix} \underbrace{\begin{matrix} \vert \\ Ab_1 \\ \vert \end{matrix}}_{\substack{A\vec{x}=\vec{b} \\ \text{matrix-vector} \\ \text{product, column} \\ \text{vector}}} & \dots & \begin{matrix} \vert \\ Ab_p \\ \vert \end{matrix} \end{bmatrix}$	(matrix) Scalar products with columns: $\text{scalar product} \rightarrow \text{matrix-col-vector product} \rightarrow \text{column vector terms}$
$AB = \underbrace{\begin{bmatrix} -a_1^T- \\ \vdots \\ -a_m^T- \end{bmatrix} B}_{\vec{v}k=\text{ scalar product}} = \begin{bmatrix} [-a_1^T-]B \\ \vdots \\ \underbrace{[-a_m^T-]B}_{\substack{yA= \\ \text{linear combination} \\ \text{of rows}}} \end{bmatrix}$ $= \underbrace{\begin{bmatrix} [-a_1^T-] \begin{bmatrix} -b_1- \\ \vdots \\ -b_n- \end{bmatrix} \\ \vdots \\ [-a_m^T-] \begin{bmatrix} -b_1- \\ \vdots \\ -b_n- \end{bmatrix} \end{bmatrix}}_{\substack{u^Tv= \\ \text{inner product,} \\ \text{row vector}}}$	(matrix) Scalar products with rows: $\text{scalar product} \rightarrow \text{linear combination of rows} \rightarrow \text{row vector terms}$
$AB = \underbrace{\begin{bmatrix} -a_1^T- \\ \vdots \\ -a_m^T- \end{bmatrix} \begin{bmatrix} \vert & & \vert \\ b_1 & \dots & b_p \\ \vert & & \vert \end{bmatrix}}_{uv^T=\text{ outer product}}$ $= \begin{bmatrix} (a_1^T b_1) & \dots & (a_1^T b_p) \\ \vdots & \ddots & \vdots \\ \underbrace{(a_m^T b_1)}_{\substack{u^Tv= \\ \text{inner} \\ \text{product}}} & \dots & (a_m^T b_p) \end{bmatrix}$	Outer Products: Dot products: $\text{outer product} \rightarrow \text{inner product} \rightarrow \text{scalar terms}$
$AB = \underbrace{\begin{bmatrix} \vert & & \vert \\ a_1 & \dots & a_n \\ \vert & & \vert \end{bmatrix} \begin{bmatrix} -b_1^T- \\ \vdots \\ -b_n^T- \end{bmatrix}}_{u^Tv=\text{inner product}} = \underbrace{a_1 b_1^T}_{\substack{uv^T= \\ \text{outer} \\ \text{product}}} + \dots + a_n b_n^T$	Linear Combination of Rank-1 Matricies $\text{inner product} \rightarrow \text{outer product} \rightarrow \text{matrix terms}$